# WIGGLYHEDRA

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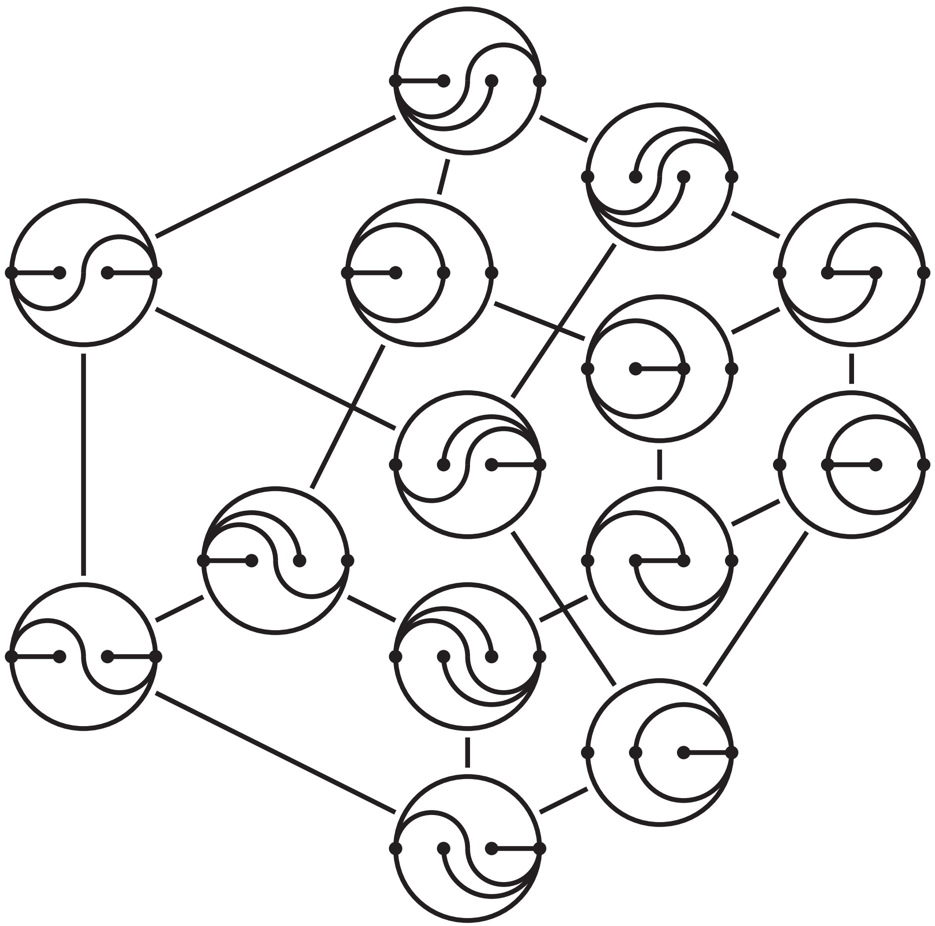
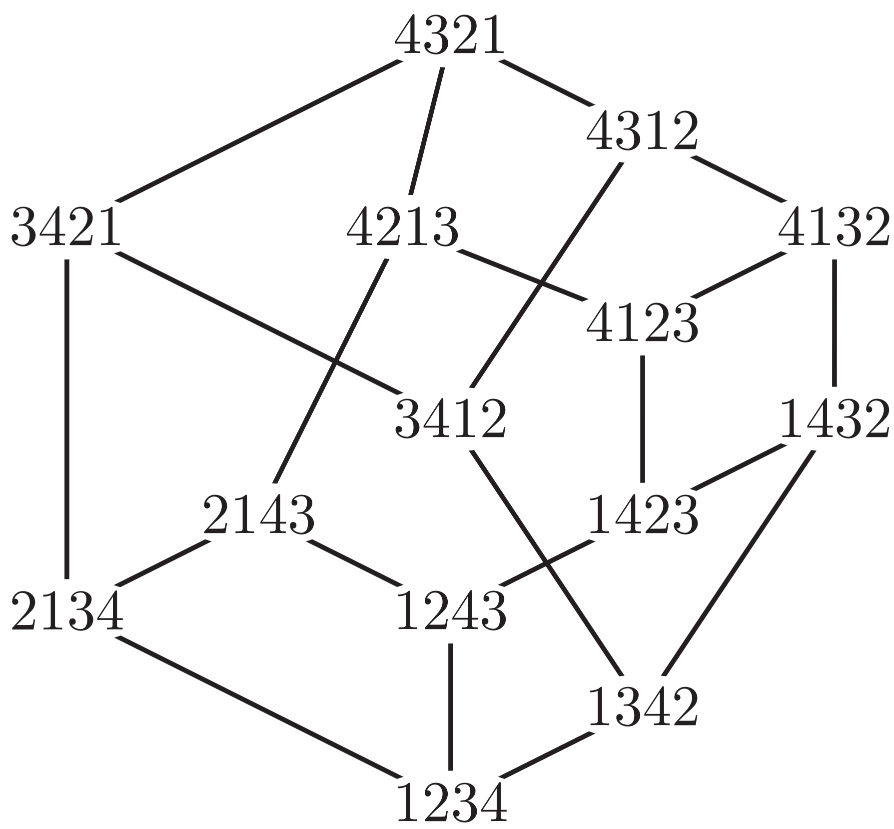
V. PILAUD (Universitat de Barcelona)

arxiv:2407.11632



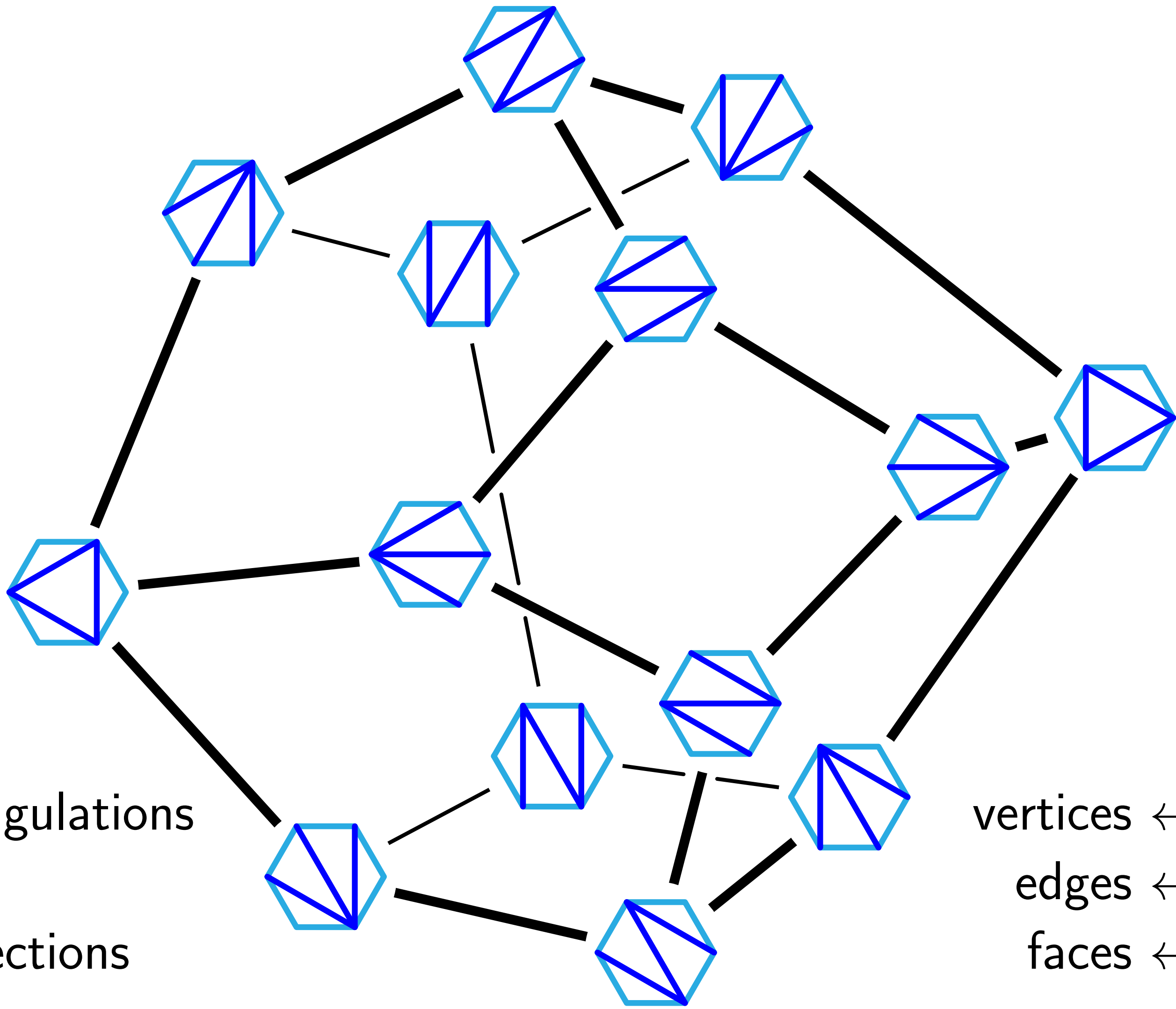



DMG Seminar Berlin — October 9, 2024

# TRIANGULATIONS & ASSOCIAHEDRA

# ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

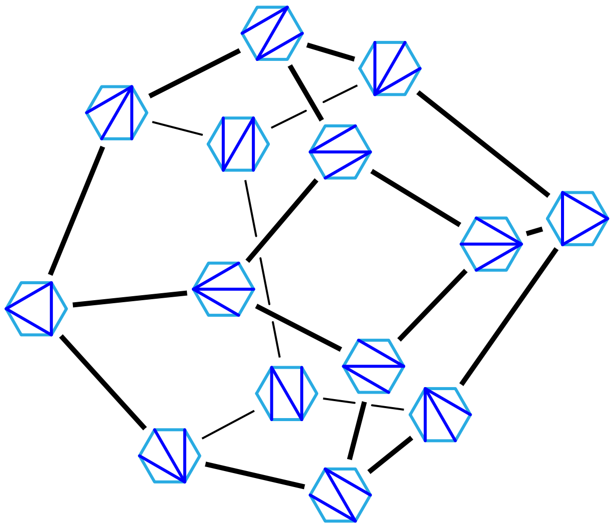


vertices ↔ triangulations
edges ↔ flips
faces ↔ dissections

vertices ↔ binary trees
edges ↔ rotations
faces ↔ Schröder trees

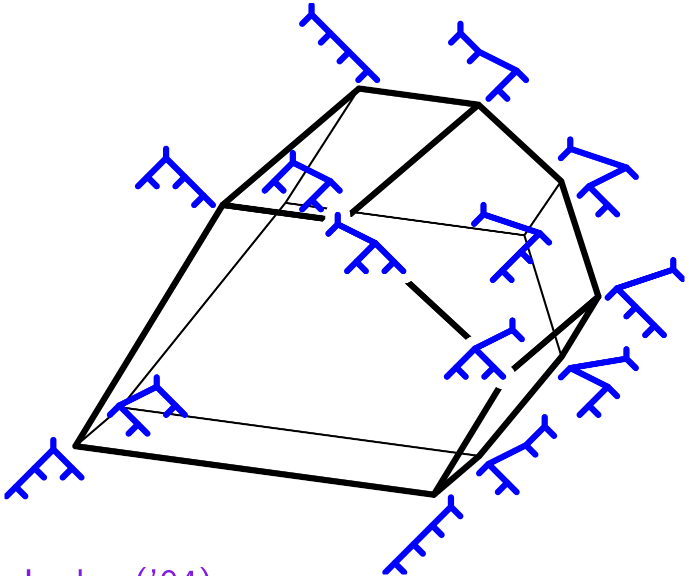
# THREE FAMILIES OF REALIZATIONS

## SECONDARY POLYTOPE



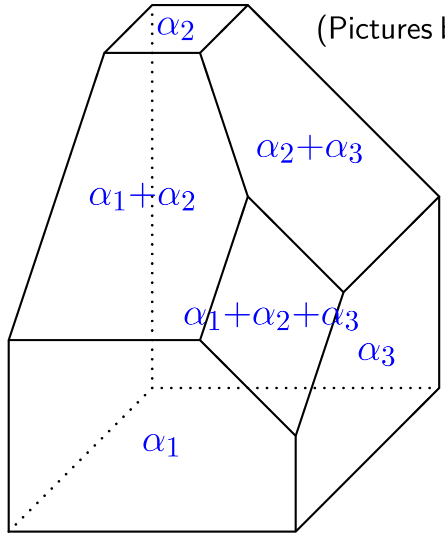
Gelfand–Kapranov–Zelevinsky ('94)
Billera–Filliman–Sturmfels ('90)

## LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg–Lange ('07)
Hohlweg–Lange–Thomas ('12)
Hohlweg–Pilaud–Stella ('18)
Pilaud–Santos–Ziegler ('24)

## CHAP.–FOM.–ZEL.'S ASSOCIAHEDRON

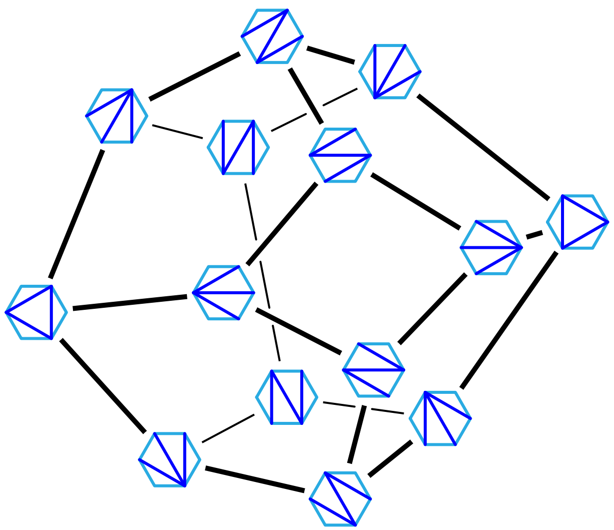



(Pictures by CFZ)

Chapoton–Fomin–Zelevinsky ('02)
Ceballos–Santos–Ziegler ('11)

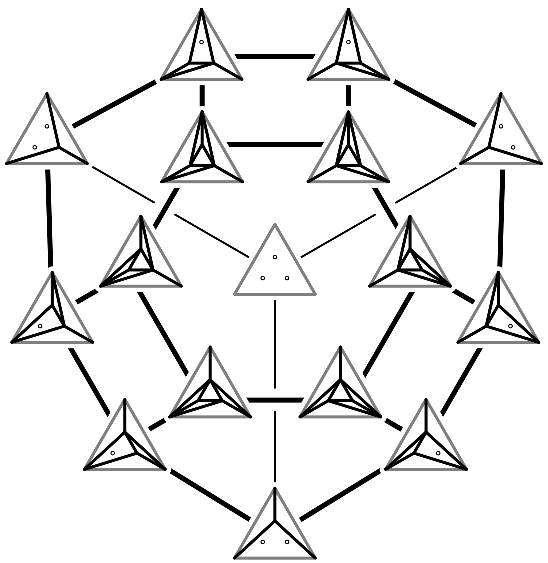
# THREE FAMILIES OF REALIZATIONS

## SECONDARY POLYTOPE

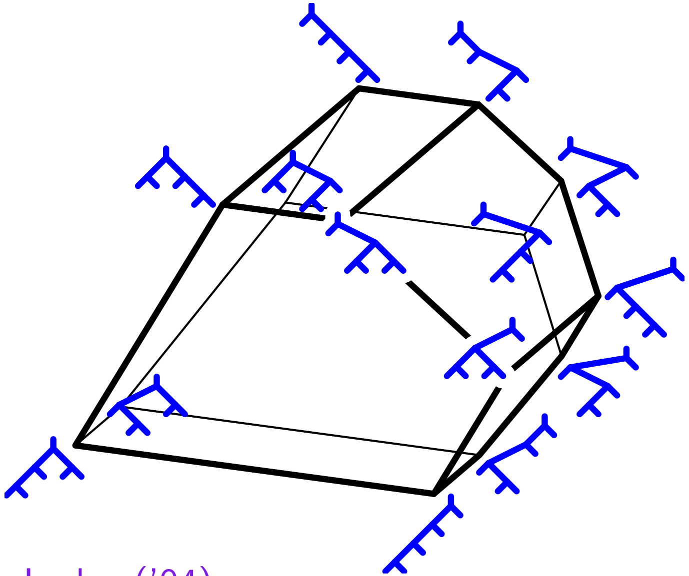


Gelfand–Kapranov–Zelevinsky ('94)

Billera–Filliman–Sturmfels ('90)



## LODAY'S ASSOCIAHEDRON



Loday ('04)

Hohlweg–Lange ('07)

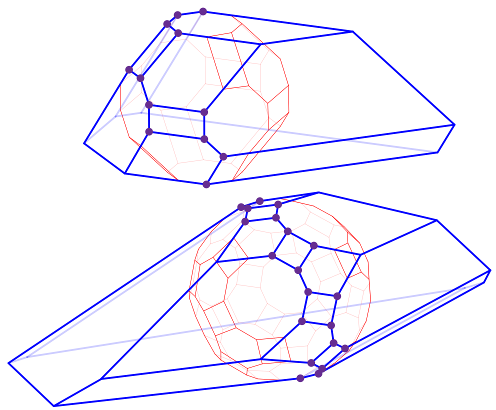
Hohlweg–Lange–Thomas ('12)

Hohlweg–Pilaud–Stella ('18)

Pilaud–Santos–Ziegler ('24)

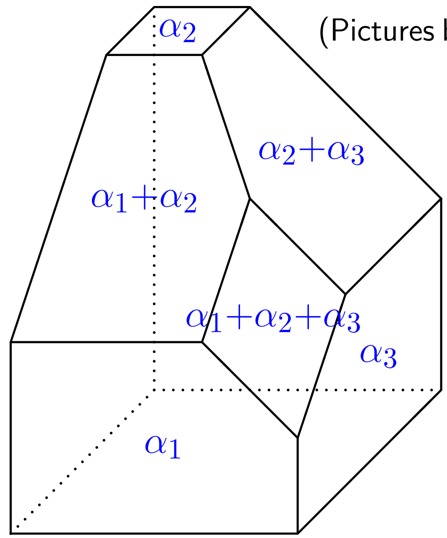
Hopf algebra

Cluster algebras



## CHAP.–FOM.–ZEL.'S ASSOCIAHEDRON

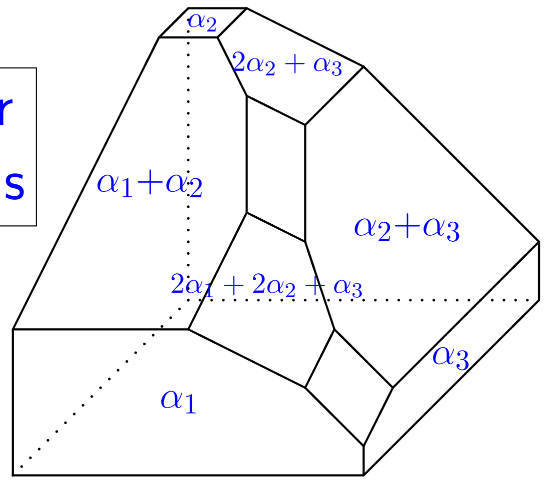
(Pictures by CFZ)




Chapoton–Fomin–Zelevinsky ('02)

Ceballos–Santos–Ziegler ('11)

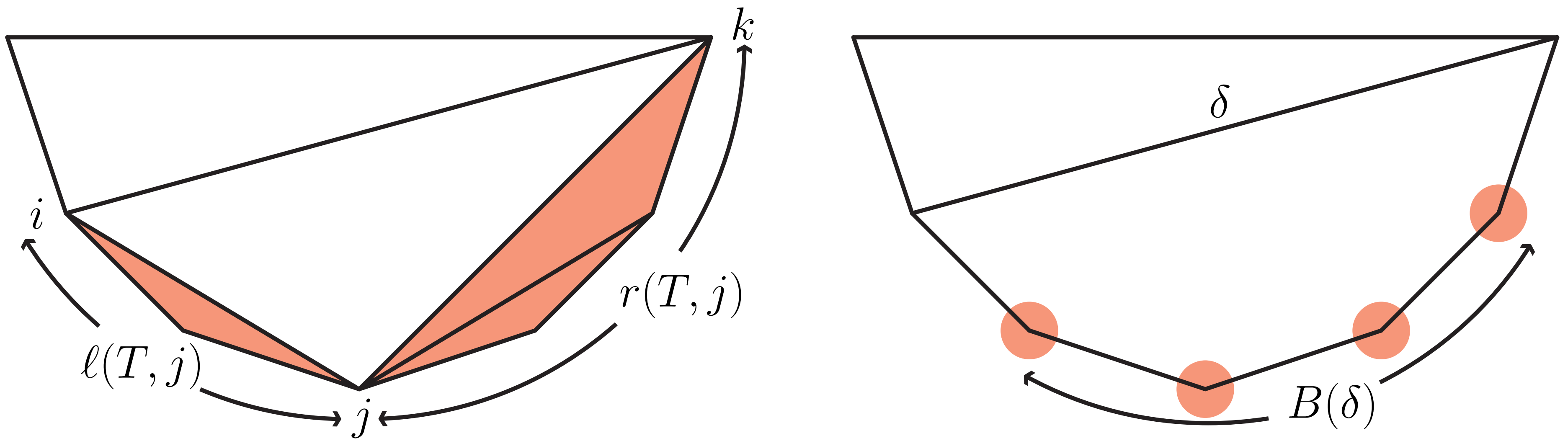
Cluster algebras



# LODAY'S ASSOCIAHEDRON

Loday's associahedron $= \operatorname{conv}\{L(T) \mid T \text{ triangulation of the } (n+3)\text{-gon}\}$

$$= \mathbb{H} \cap \bigcap_{\substack{\delta \text{ diagonal} \\ \text{of the } (n+3)\text{-gon}}} \boldsymbol{H}^{\geq}(\delta)$$

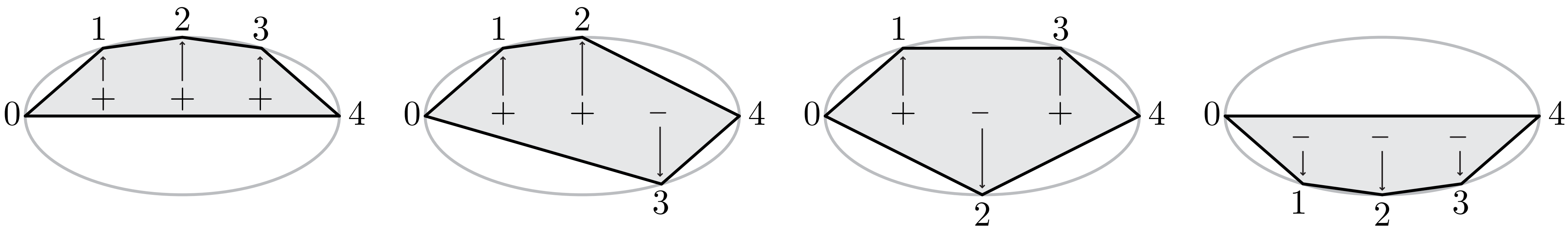



$$L(T) = \big(\ell(T,j)\cdot r(T,j)\big)_{j\in[n+1]} \qquad \boldsymbol{H}^{\geq}(\delta) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j\in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

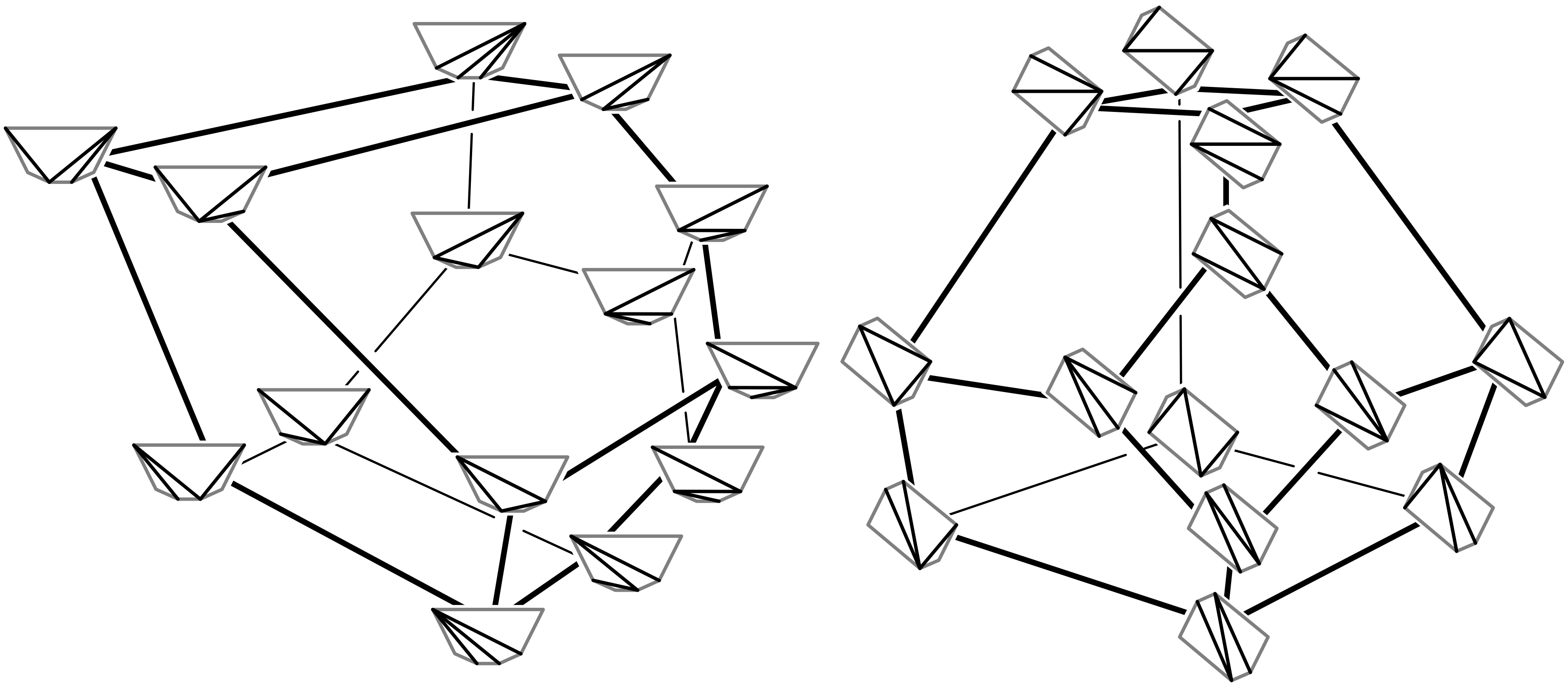
Loday ('04)

# HOHLWEG & LANGE'S ASSOCIAHEDRA

Can also replace Loday's $(n+3)$-gon by others...



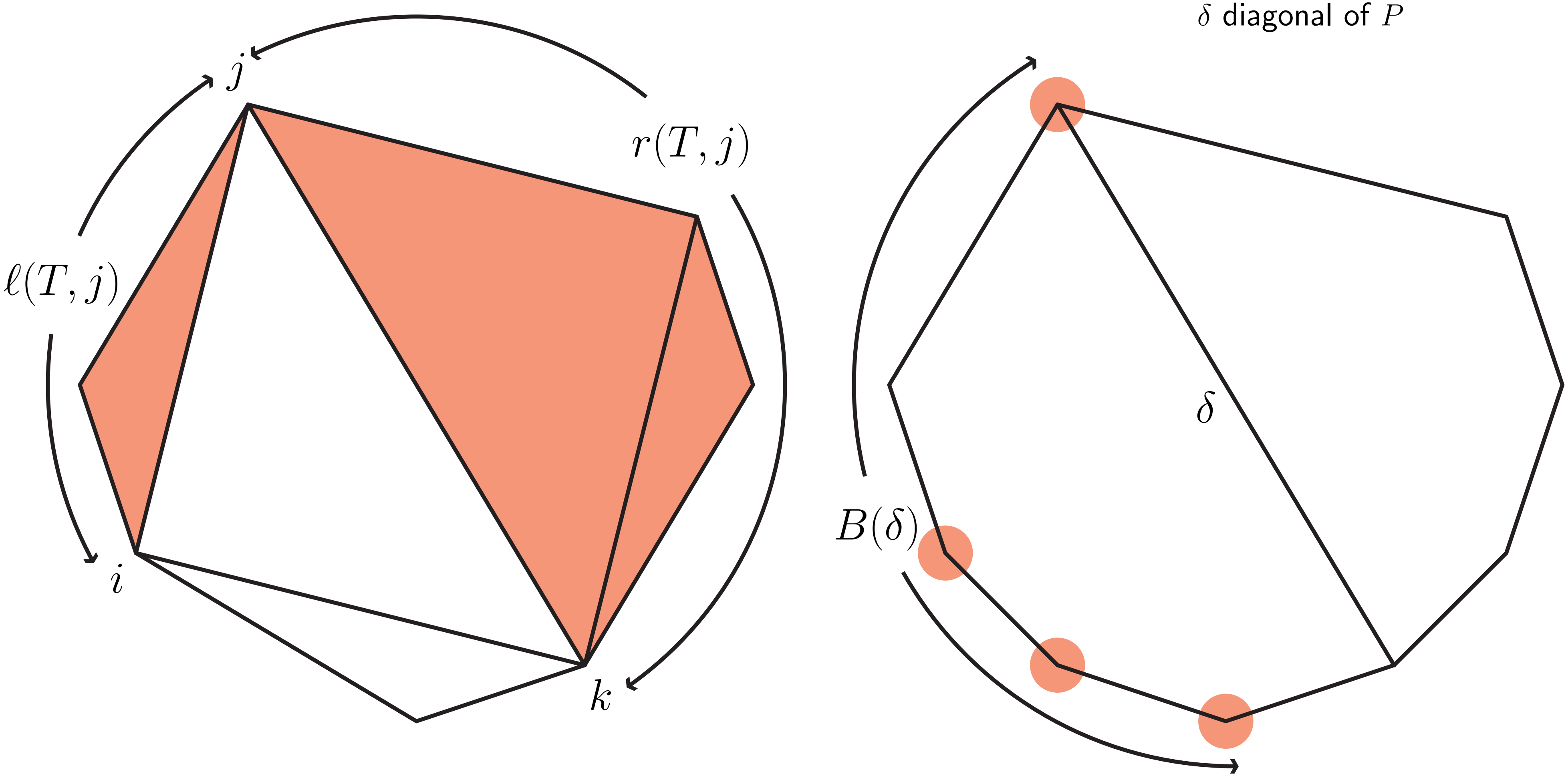

...to obtain different realizations of the associahedron



Hohlweg–Lange ('07)

# HOHLWEG & LANGE'S ASSOCIAHEDRA

$$\mathbb{A}\mathsf{sso}(P) \;=\; \operatorname{conv}\{HL(T) \mid T \text{ triangulation of } P\} \;=\; \mathbb{H} \cap \bigcap_{\delta \text{ diagonal of } P} \boldsymbol{H}^{\geq}(\delta)$$



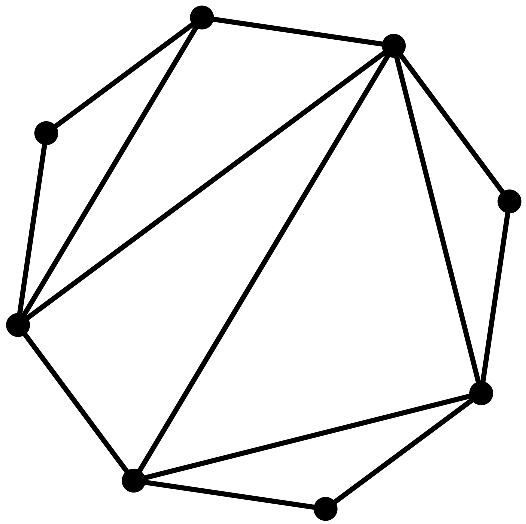
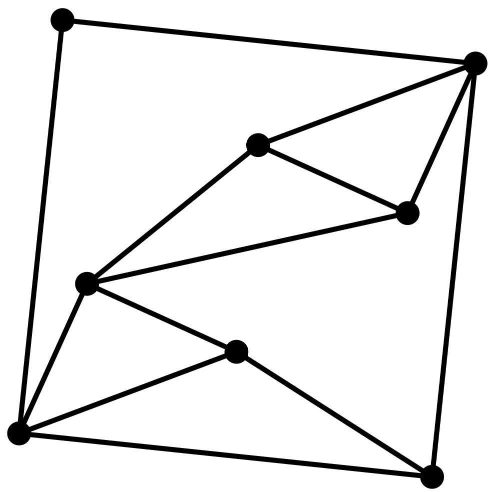
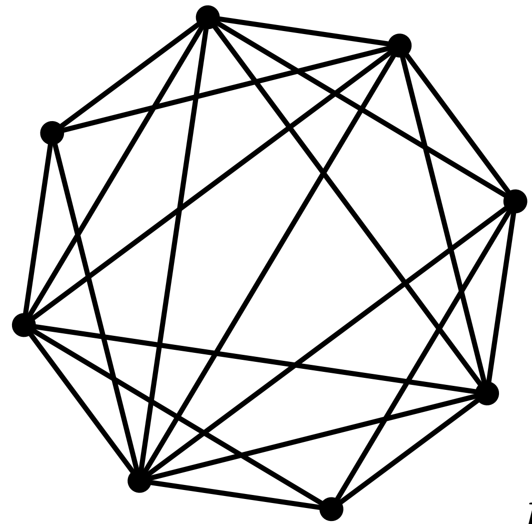

$$HL(T)_j = \begin{cases} \ell(T,j)\cdot r(T,j) & \text{if } j \text{ down} \\ n+2-\ell(T,j)\cdot r(T,j) & \text{if } j \text{ up} \end{cases}$$

$$\boldsymbol{H}^{\geq}(\delta) = \left\{ \boldsymbol{x} \;\middle|\; \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

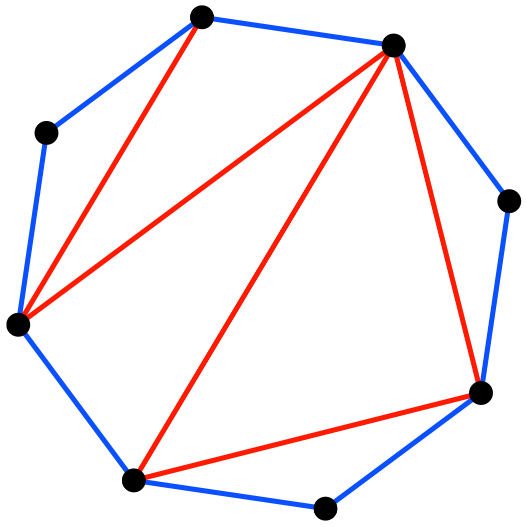
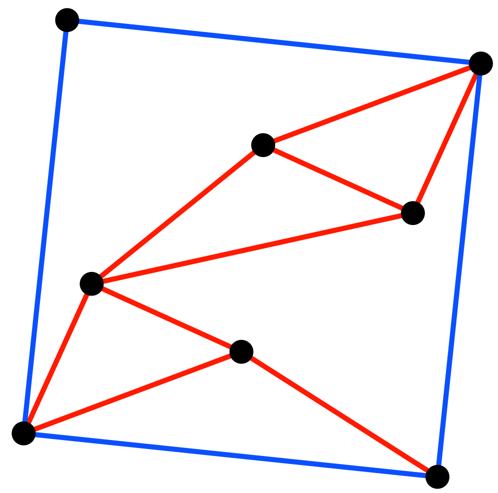
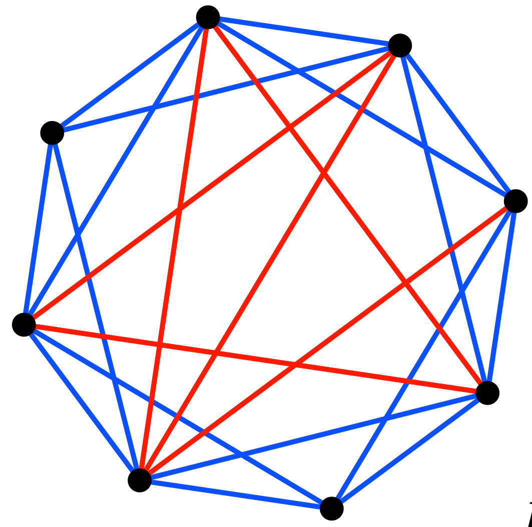
Hohlweg–Lange ('07)

# PSEUDOTRIANGULATIONS & PPT POLYTOPE

# THREE GEOMETRIC STRUCTURES

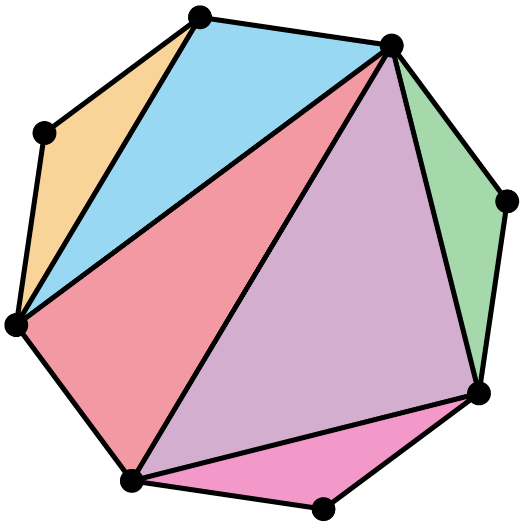
| triangulations | pseudotriangulations | multitriangulations |
|---|---|---|
|  |  |  $k = 2$ |
| crossing-free | crossing-free pointed | $(k+1)$-crossing-free |
| | Pocchiola–Vegter ('96)<br>Rote–Santos–Streinu ('08) | Capoyleas–Pach ('92)<br>Jonsson ('05) |

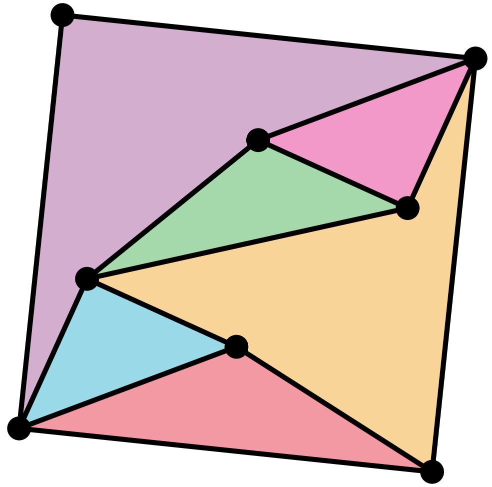
# THREE GEOMETRIC STRUCTURES

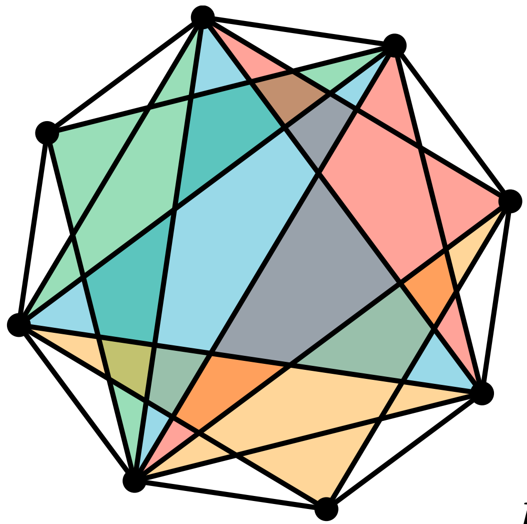
| triangulations | pseudotriangulations | multitriangulations |
| --- | --- | --- |
|  |  | <br> |
| crossing-free | crossing-free pointed | $(k+1)$-crossing-free |
| | Pocchiola–Vegter ('96)<br>Rote–Santos–Streinu ('08) | Capoyleas–Pach ('92)<br>Jonsson ('05) |

# THREE GEOMETRIC STRUCTURES

triangulations | pseudotriangulations | multitriangulations







$k = 2$

crossing-free

crossing-free pointed

Pocchiola–Vegter ('96)
Rote–Santos–Streinu ('08)

$(k+1)$-crossing-free

Capoyleas–Pach ('92)
Jonsson ('05)

triangles

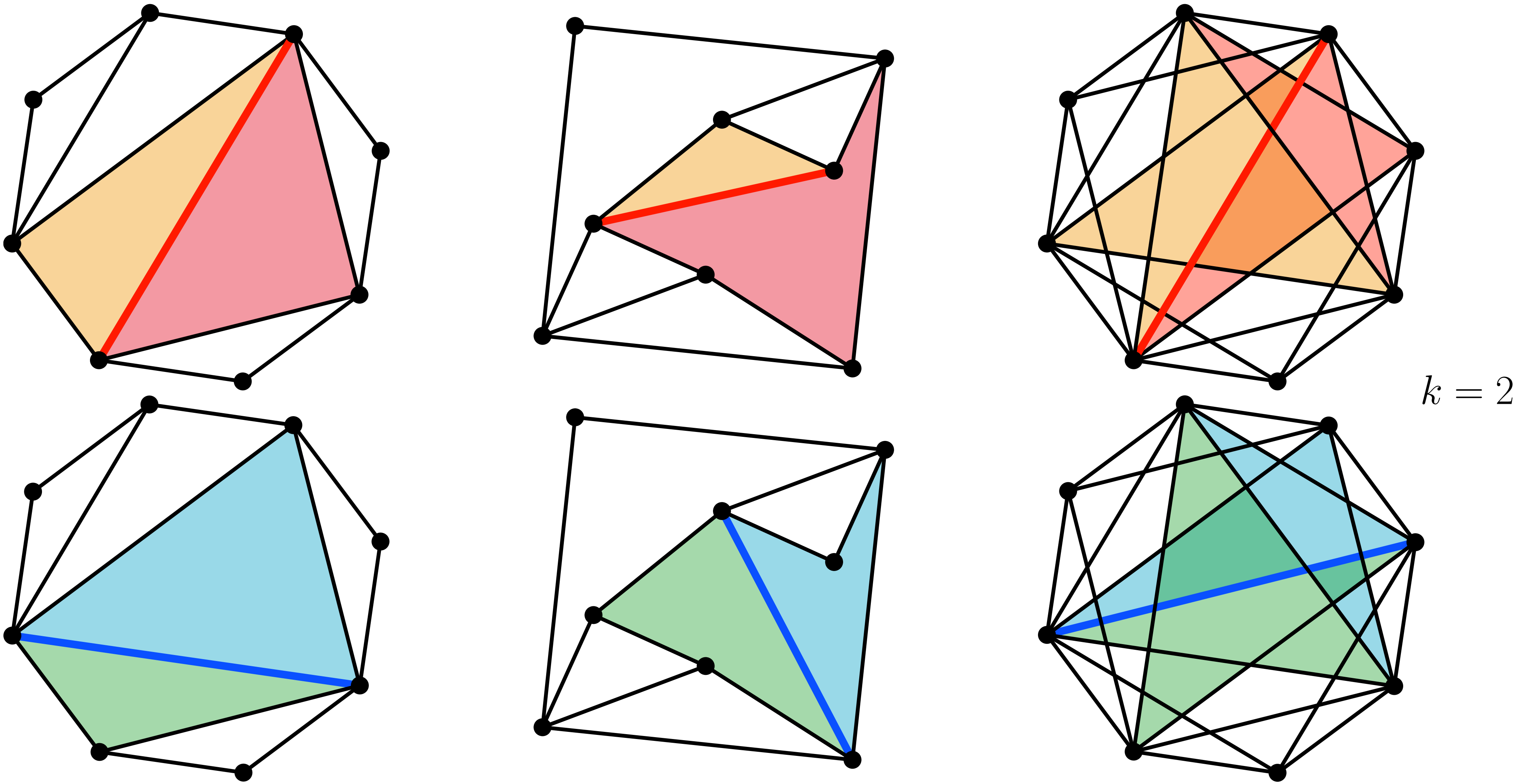
pseudotriangles

$k$-stars

P.–Santos ('09)

# THREE GEOMETRIC STRUCTURES

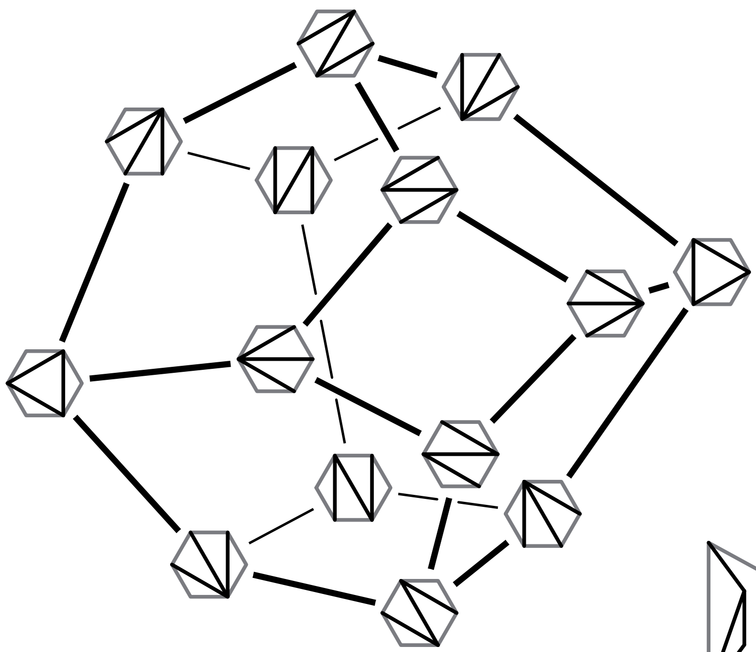
triangulations | pseudotriangulations | multitriangulations

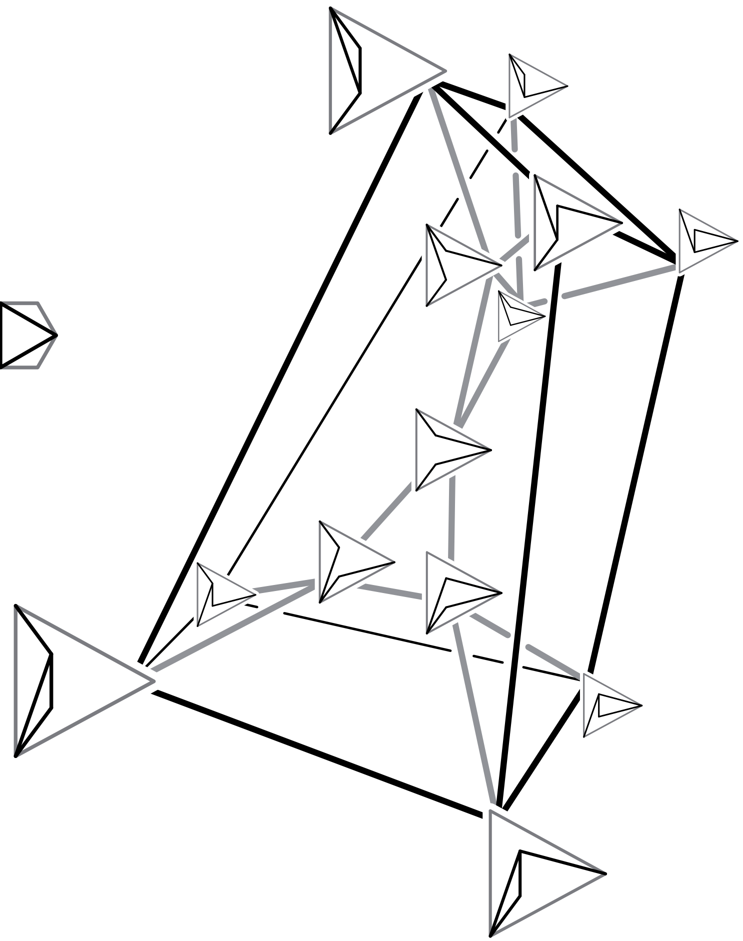


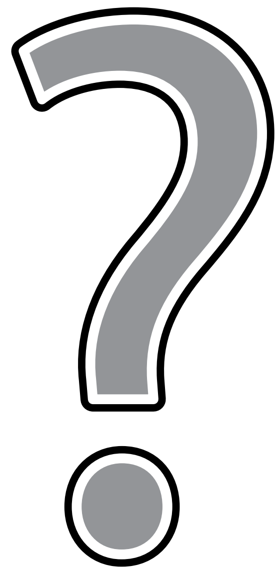

flip = exchange an internal edge with the common bisector of the two adjacent cells

# THREE GEOMETRIC STRUCTURES

triangulations | pseudotriangulations | multitriangulations





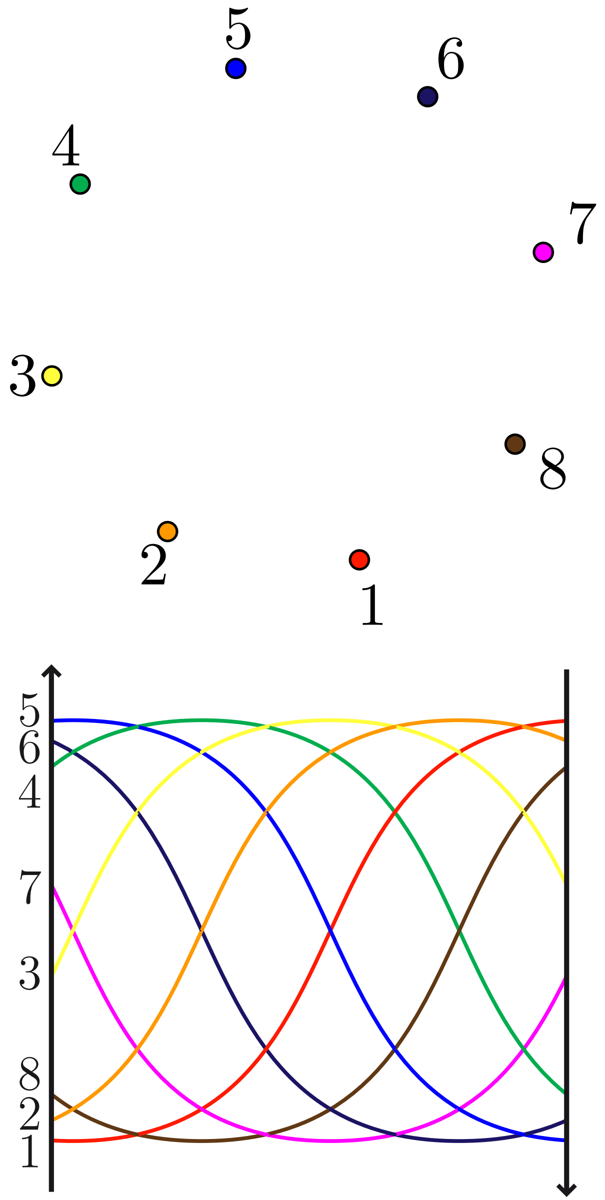


associahedron | pseudotriangulation polytope | multiassociahedron

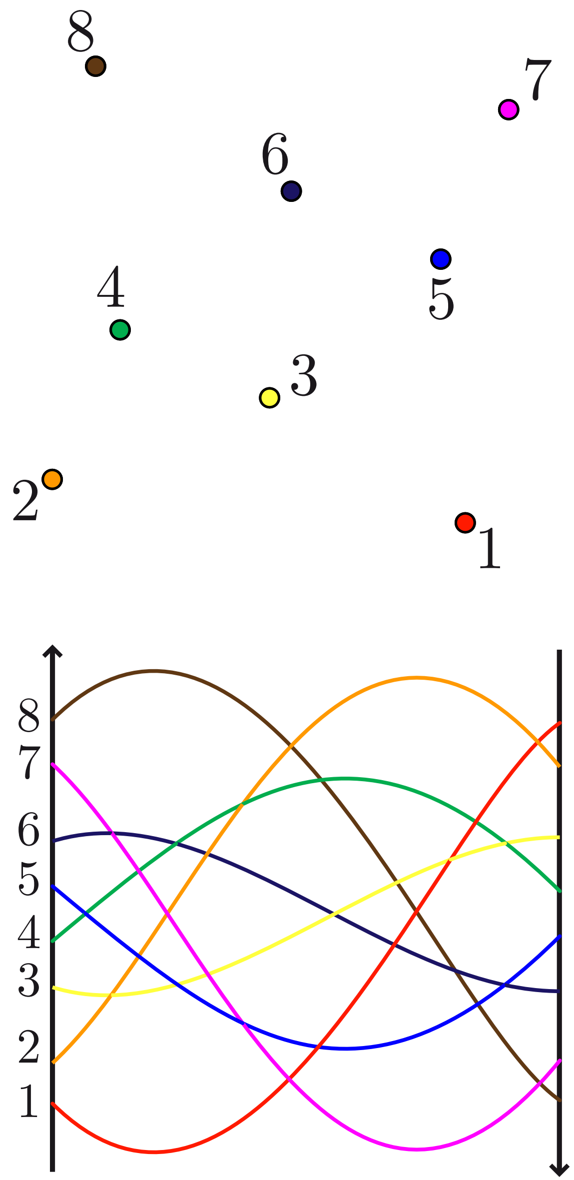
Rote–Santos–Streinu ('03)

# DUALITY

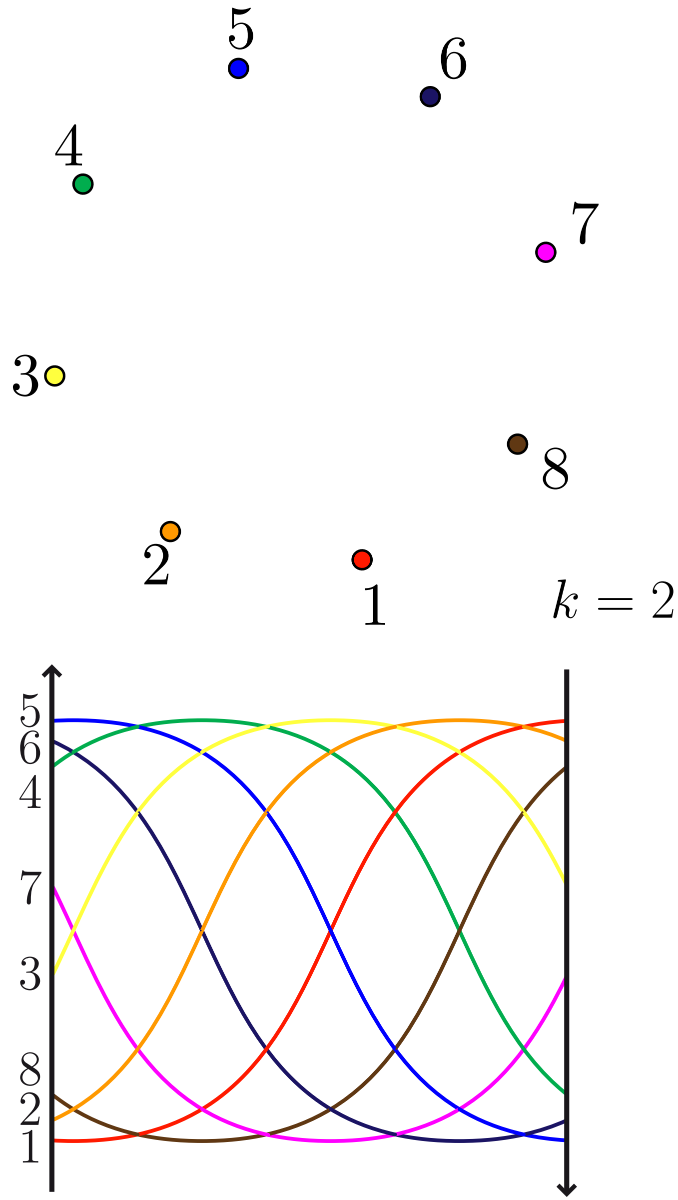
## triangulations




## pseudotriangulations



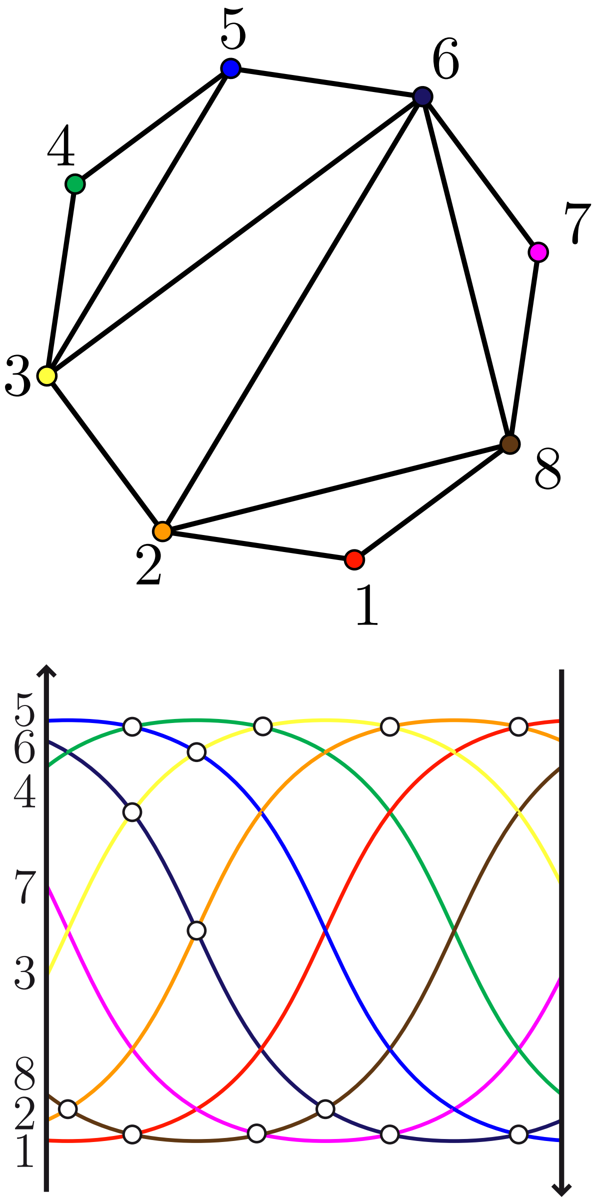

## multitriangulations



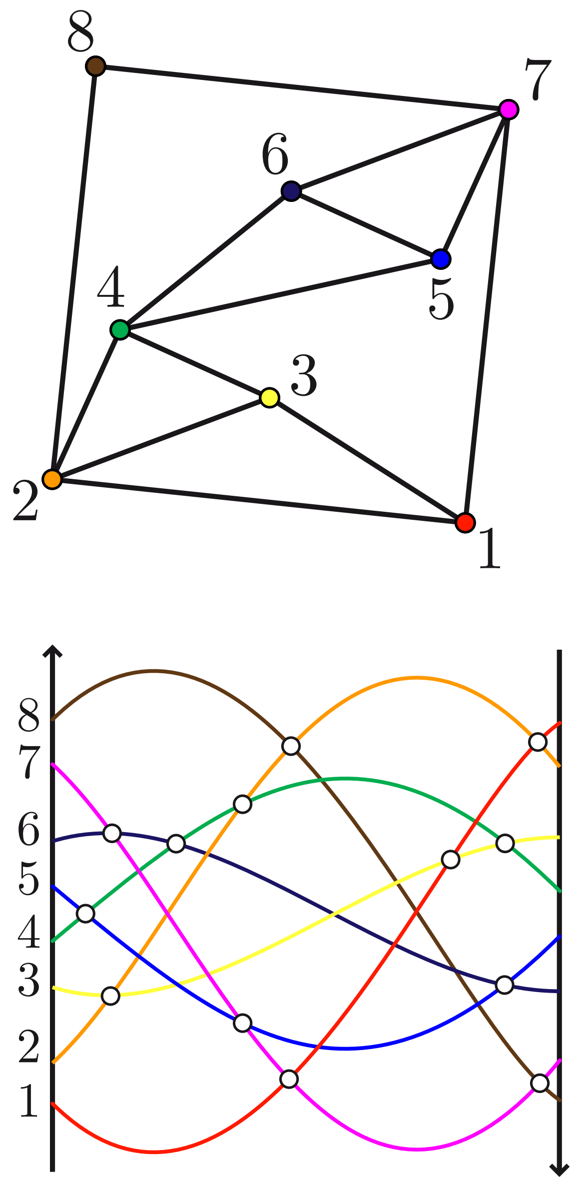

P.–Pocchiola ('12)

# DUALITY

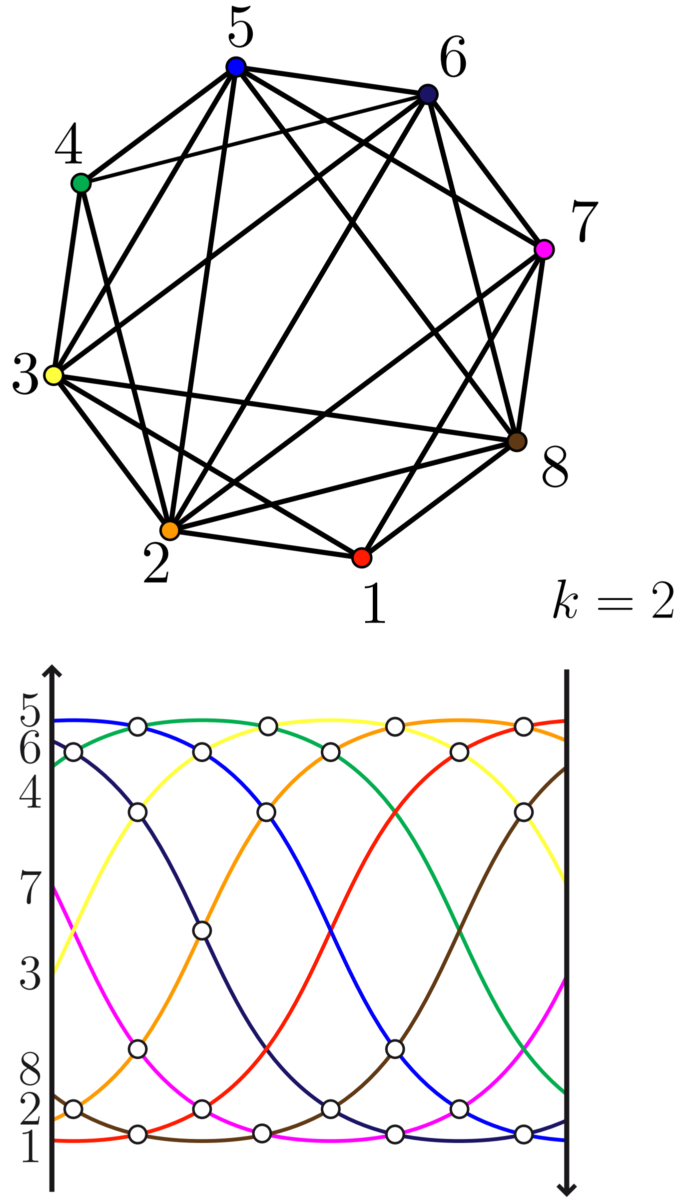
## triangulations




## pseudotriangulations



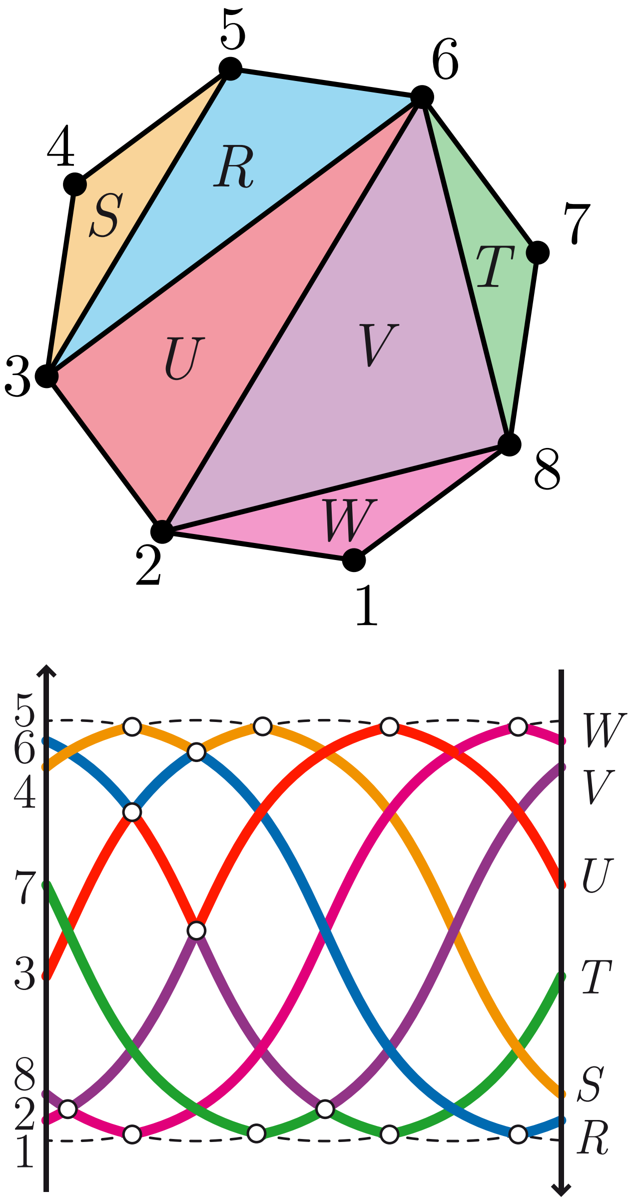

## multitriangulations



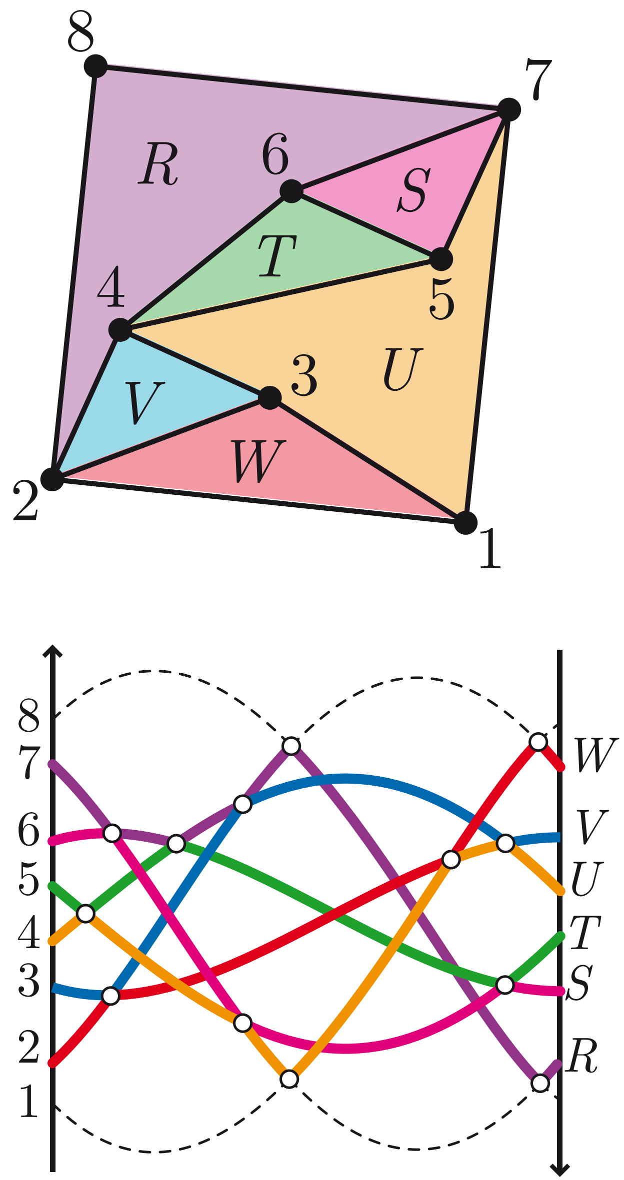

P.–Pocchiola ('12)

# DUALITY

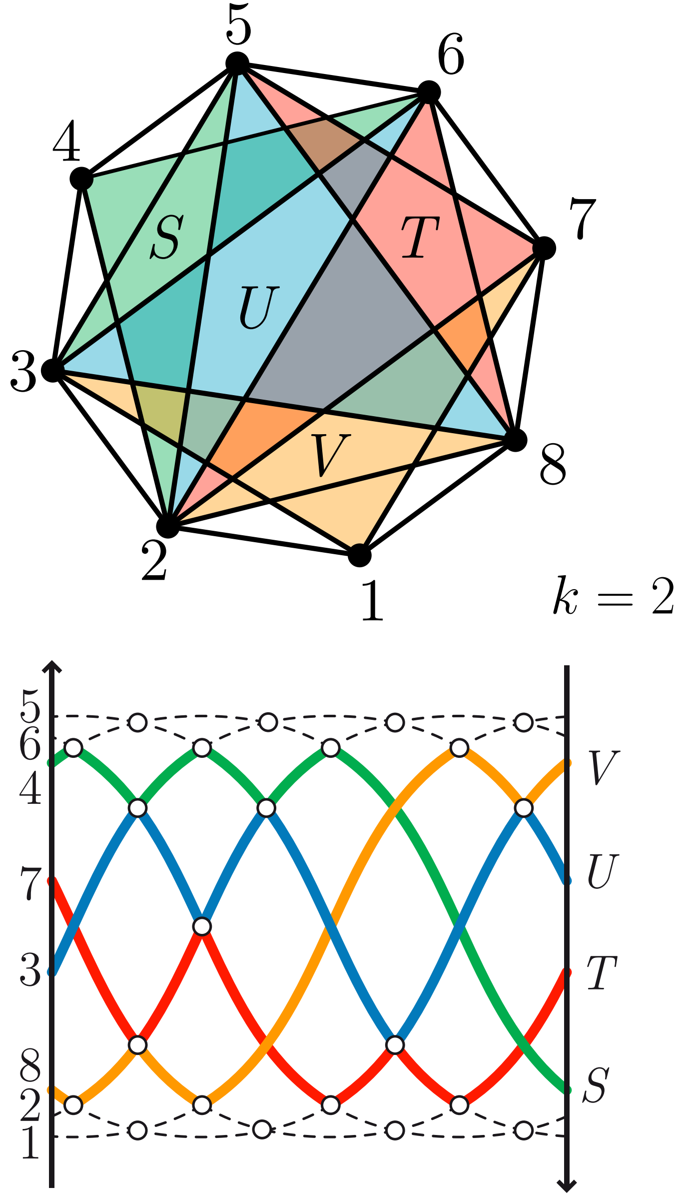
triangulations




pseudotriangulations




multitriangulations



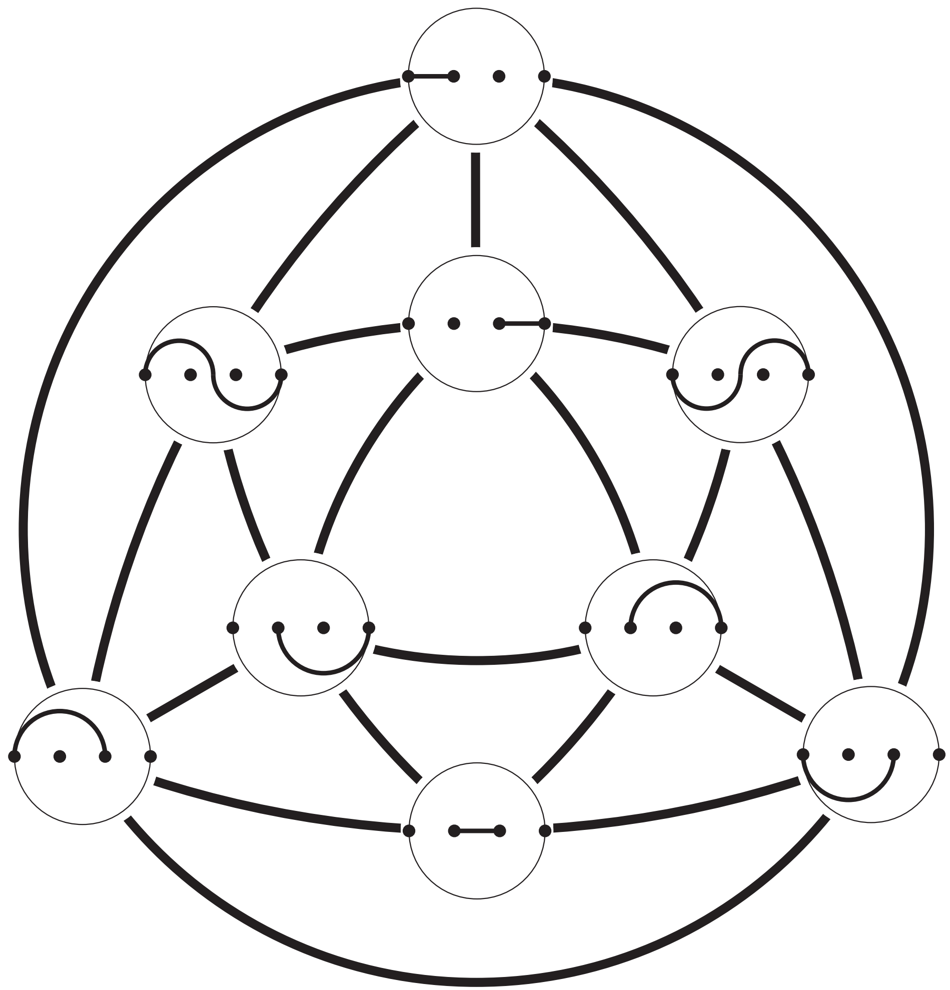

P.–Pocchiola ('12)

# WIGGLY PSEUDOTRIANGULATIONS & WIGGLY COMPLEX

# WIGGLY COMPLEX

wiggly dissection = set of pairwise non-crossing and pointed wiggly arcs on $n + 2$ points

wiggly complex $\mathrm{WC}_n$ = simplicial complex of wiggly dissections



# WIGGLY COMPLEX

wiggly dissection = set of pairwise non-crossing and pointed wiggly arcs on $n+2$ points

wiggly complex $\mathrm{WC}_n$ = simplicial complex of wiggly dissections

$$f(\mathrm{WC}_1) = (1, 2)$$
$$f(\mathrm{WC}_2) = (1, 9, 21, 14)$$
$$f(\mathrm{WC}_3) = (1, 24, 154, 396, 440, 176)$$
$$f(\mathrm{WC}_4) = (1, 55, 729, 4002, 10930, 15684, 11312, 3232)$$
$$f(\mathrm{WC}_5) = (1, 118, 2868, 28110, 140782, 400374, 673274, 662668, 352728, 78384)$$

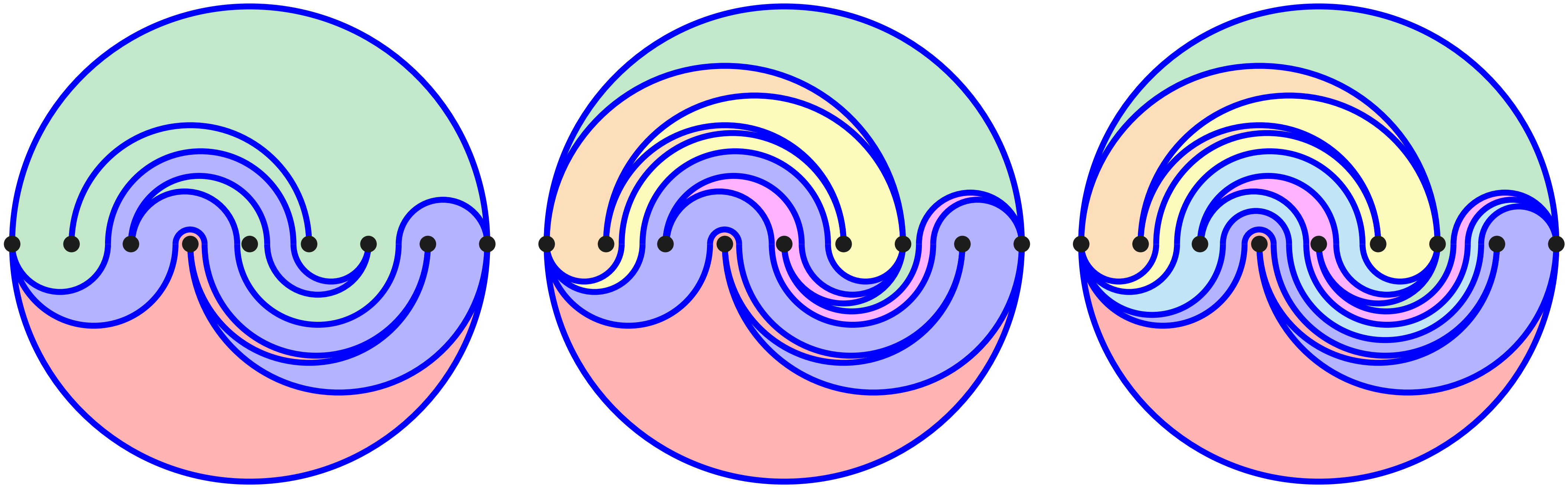
$$h(\mathrm{WC}_1) = (1, 1)$$
$$h(\mathrm{WC}_2) = (1, 6, 6, 1)$$
$$h(\mathrm{WC}_3) = (1, 19, 68, 68, 19, 1)$$
$$h(\mathrm{WC}_4) = (1, 48, 420, 1147, 1147, 420, 48, 1)$$
$$h(\mathrm{WC}_5) = (1, 109, 1960, 11254, 25868, 25868, 11254, 1960, 109, 1)$$

# WIGGLY PSEUDOTRIANGULATIONS

$c$ cell in a wiggly dissection with boundary $\partial_c$

degree $\delta_c = 1/2\,\#\text{arcs on } \partial_c + 2\,\#\text{connected components of } \partial_c - 1$

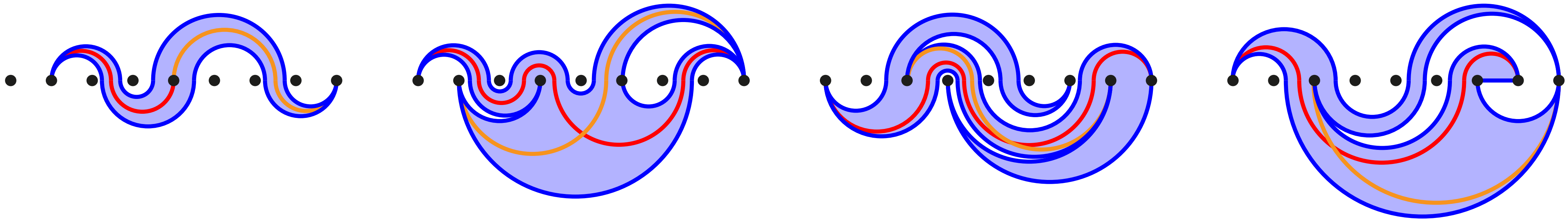
pseudotriangle = cell of degree $3$ pseudoquadragle = cell of degree $4$



PROP. The inclusion maximal wiggly pseudodissections are the pseudotriangulations, and contain $2n-1$ internal arcs and $n$ cells. Bapat–P. (24$^+$)

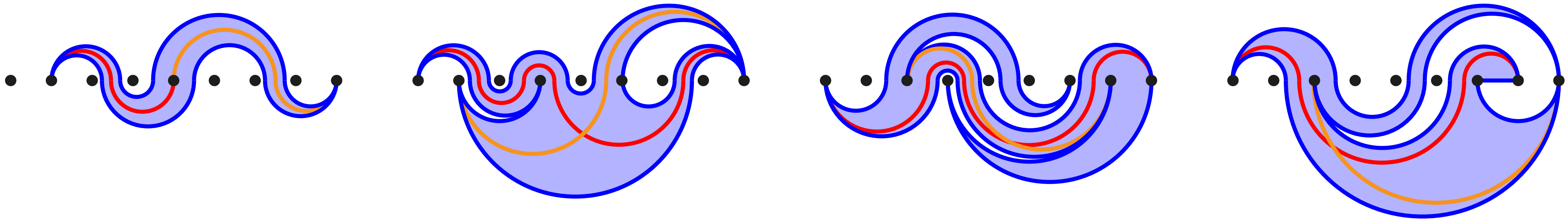
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|
| $wp_n$ | 2 | 14 | 176 | 3232 | 78384 | 2366248 | 85534176 | 3602770400 | ... |

PROP. Any wiggly pseudoquadrangle has exactly two wiggly diagonals, and they either cross or are non pointed. Bapat–P. ($24^+$)

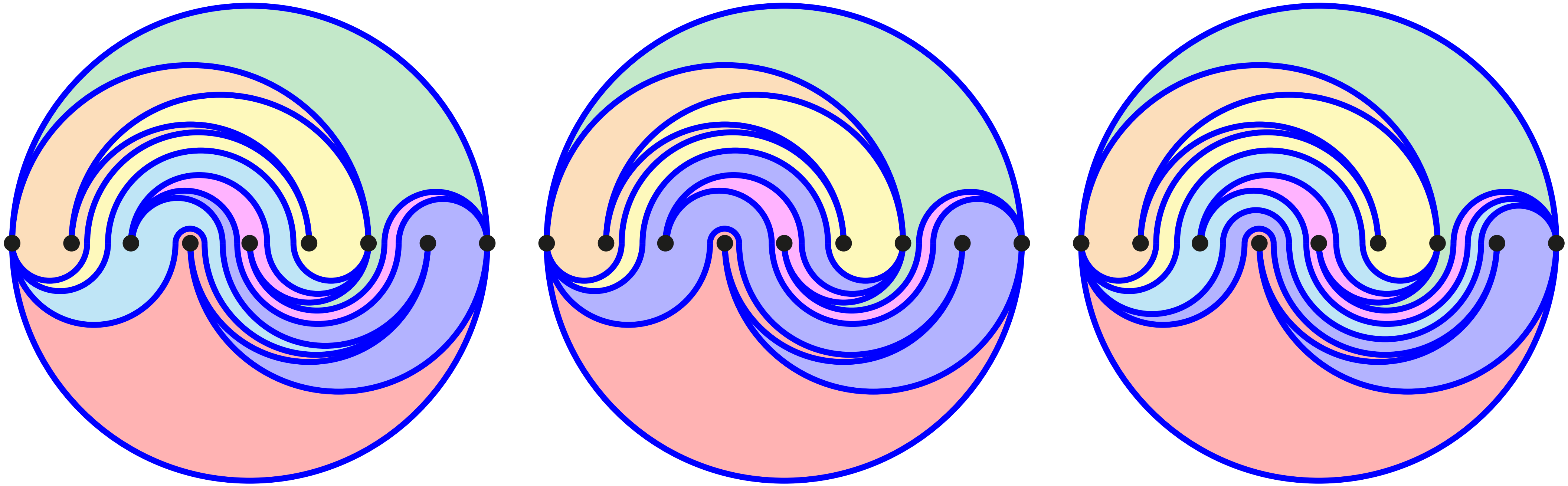


# WIGGLY FLIP GRAPH

PROP. Any wiggly pseudoquadrangle has exactly two wiggly diagonals, and they either cross or are non pointed. Bapat–P. ($24^+$)

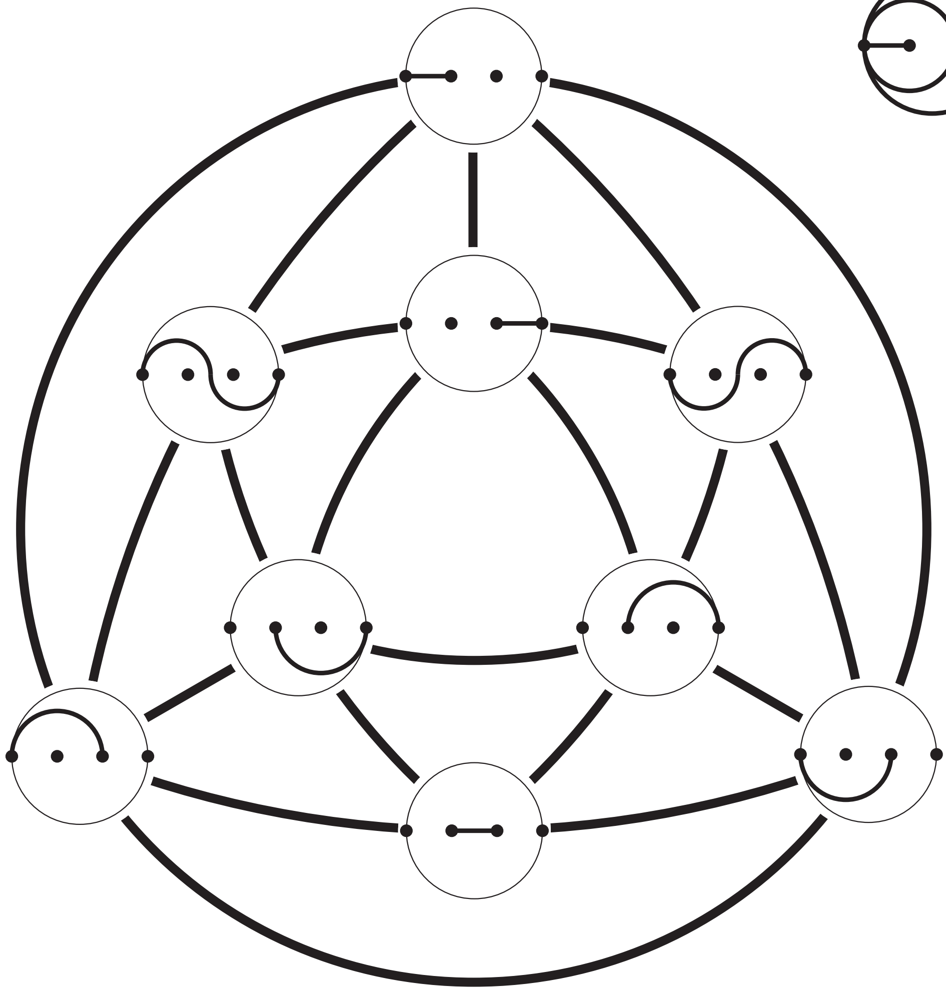


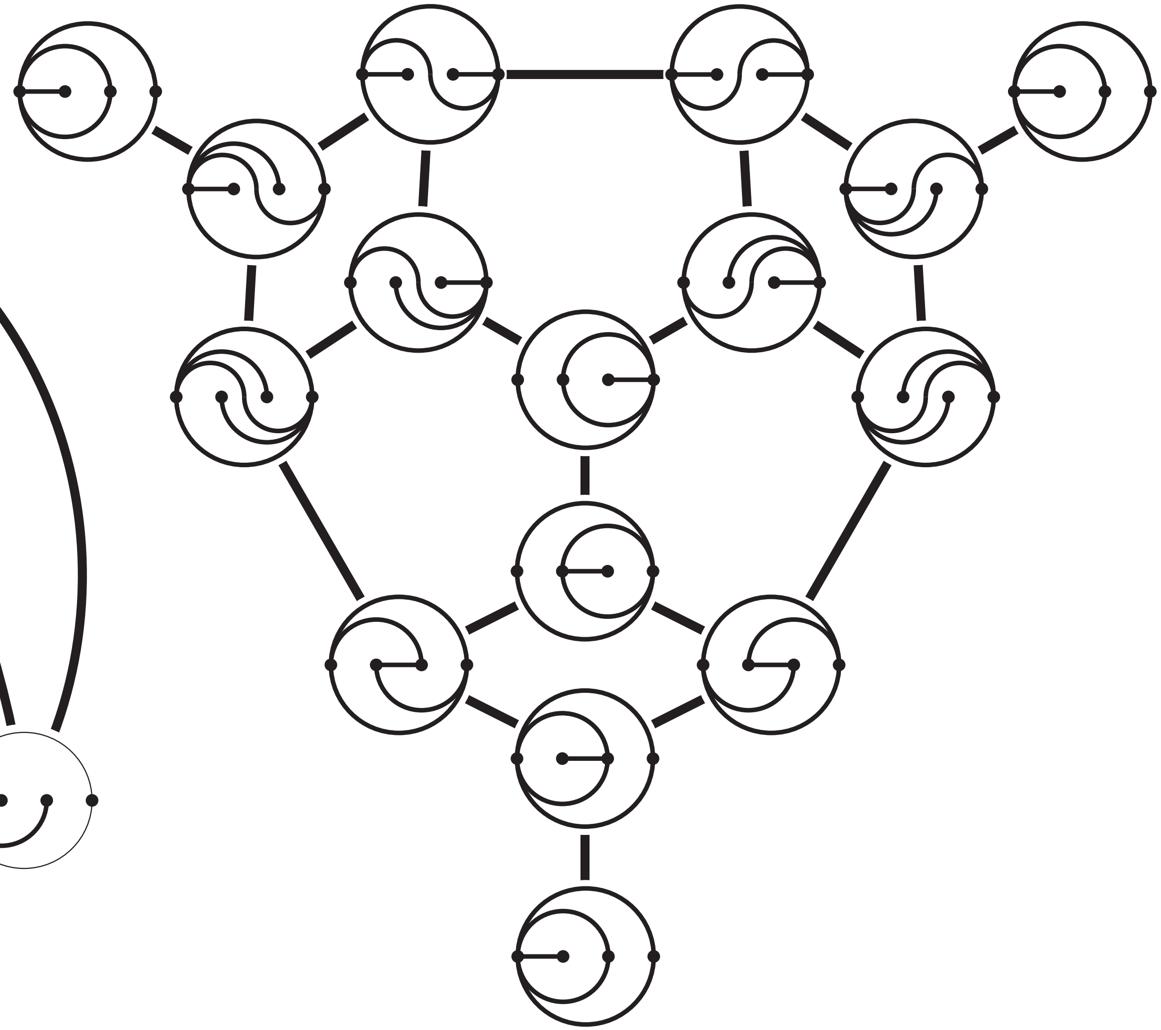
wiggly flip graph $\mathrm{WFG}_n$ = a vertex for each wiggly pseudotriangulation; an edge between $T$ and $T'$ if $T \smallsetminus \{\alpha\} = T' \smallsetminus \{\alpha'\}$



# WIGGLY FLIP GRAPH

PROP. The wiggly flip graph $\mathrm{WFG}_n$ is $(2n-1)$-regular and connected. Bapat–P. ($24^{+}$)



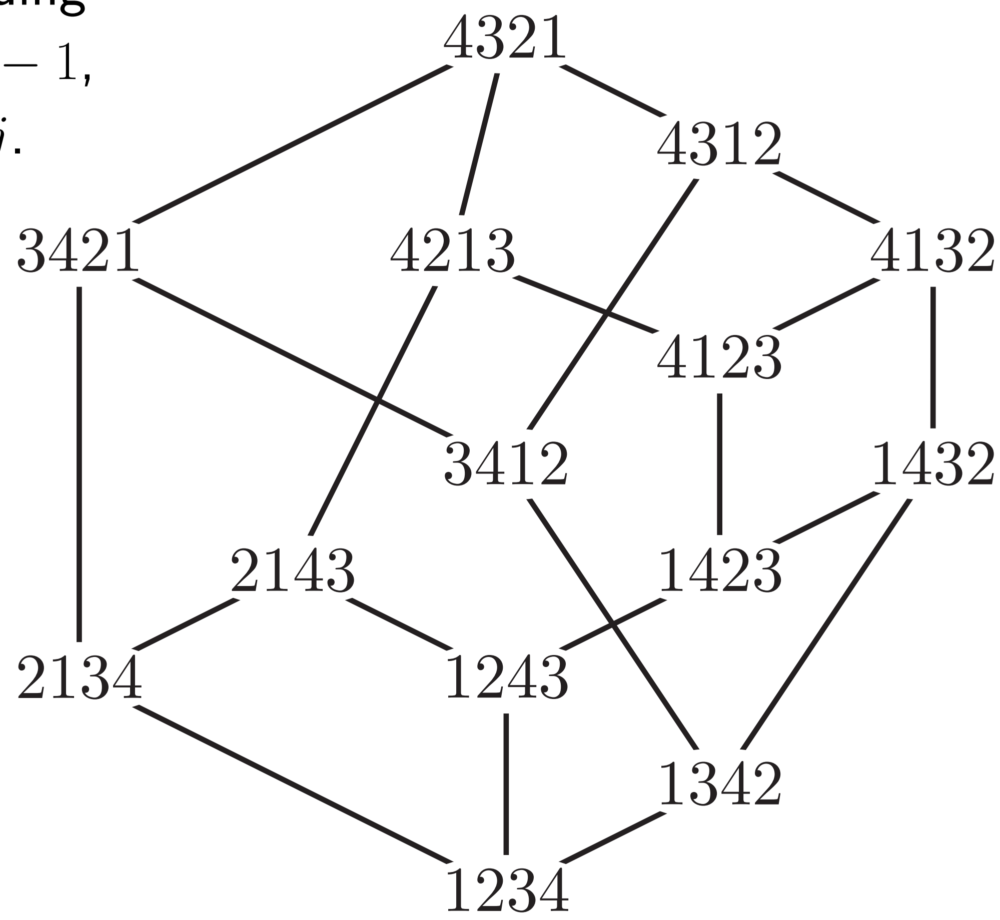


# WIGGLY PERMUTATIONS & WIGGLY LATTICE

# WIGGLY PERMUTATIONS

wiggly permutation = permutation of $2n$ avoiding

- $(2j-1) \cdots i \cdots (2j)$ for $j \in [n]$ and $i < 2j-1$,
- $(2j) \cdots k \cdots (2j-1)$ for $j \in [n]$ and $k > 2j$.




PROP. The wiggly permutations induce a sublattice $\mathrm{WL}_n$ of the weak order on $\mathfrak{S}_{2n}$.

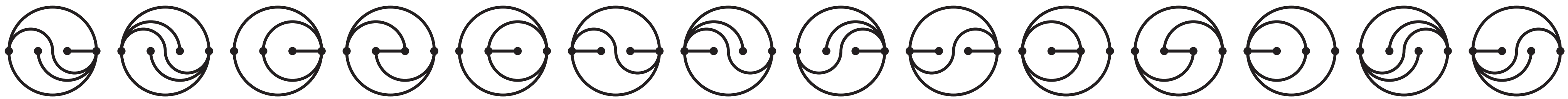
Bapat–P. (24[+])

PROP. The cover graph of the lattice $\mathrm{WL}_n$ is $(2n-1)$-regular and connected.

Bapat–P. (24[+])

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|
| $wp_n$ | 2 | 14 | 176 | 3232 | 78384 | 2366248 | 85534176 | 3602770400 | ... |

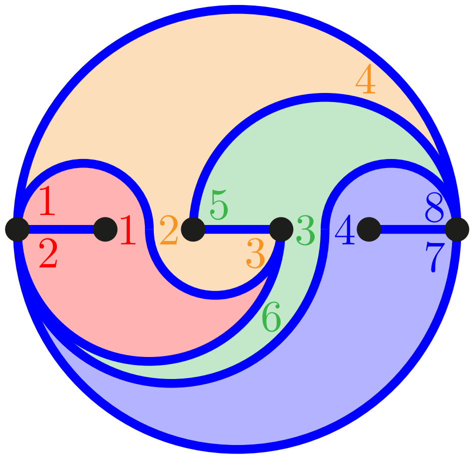
# WIGGLY PSEUDOTRIANGULATIONS $\longleftrightarrow$ WIGGLY PERMUTATIONS



1234 1243 1342 1423 1432 2134 2143 3412 3421 4123 4132 4213 4312 4321

PROP. The wiggly pseudotriangulations and wiggly permutations are in bijection.

Bapat–P. ($24^+$)

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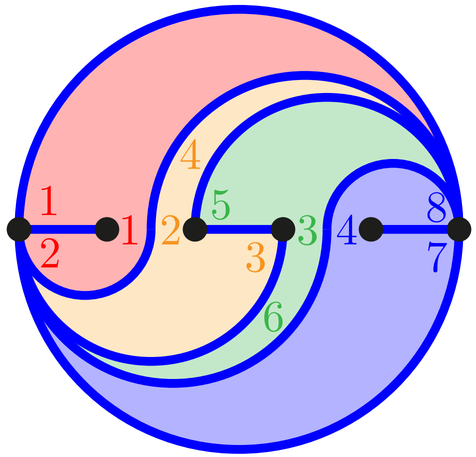


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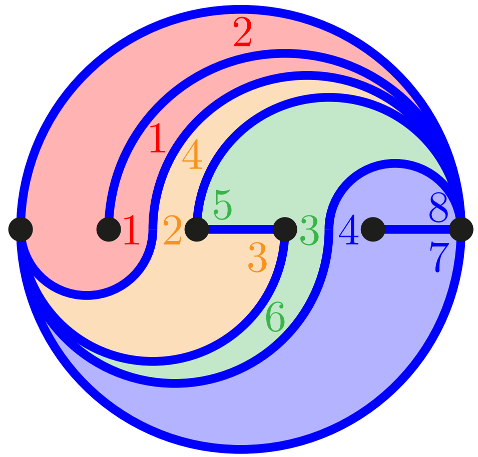


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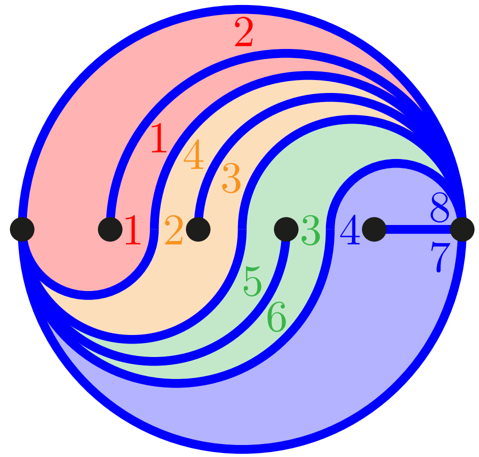



78653412

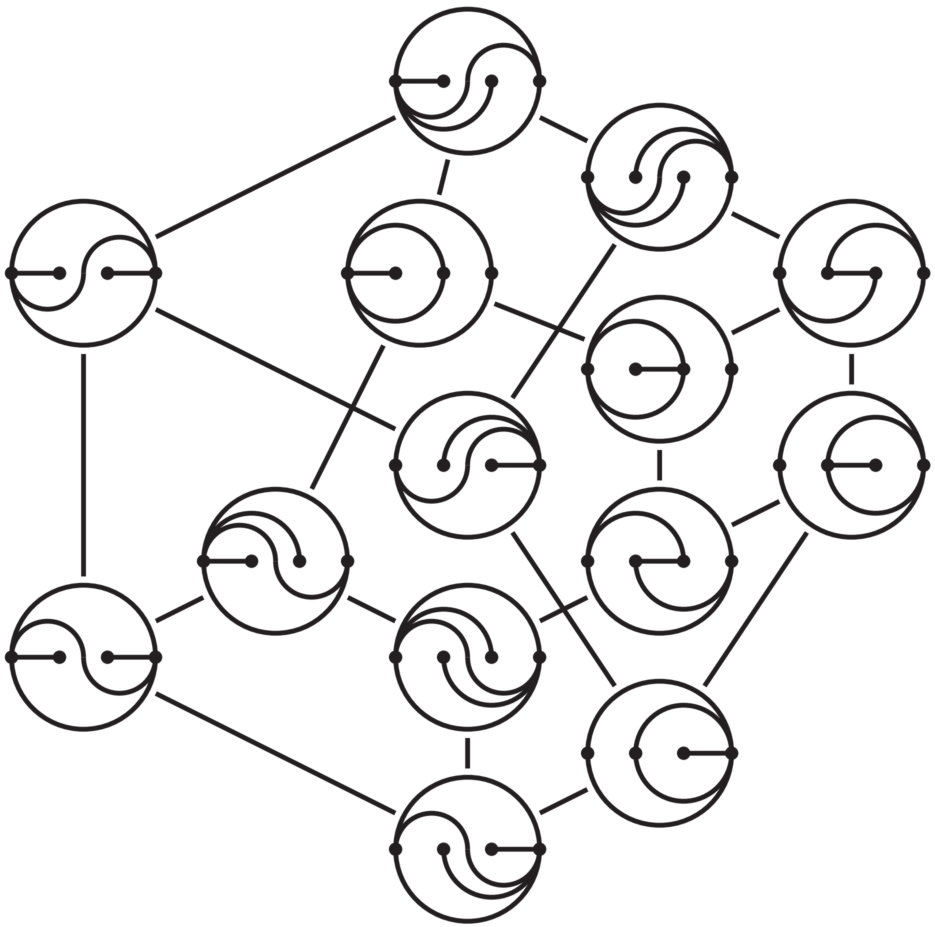
permutation of $2n$ avoiding $(2j-1)\cdots i \cdots (2j)$ for $j \in [n]$ and $i < 2j-1$

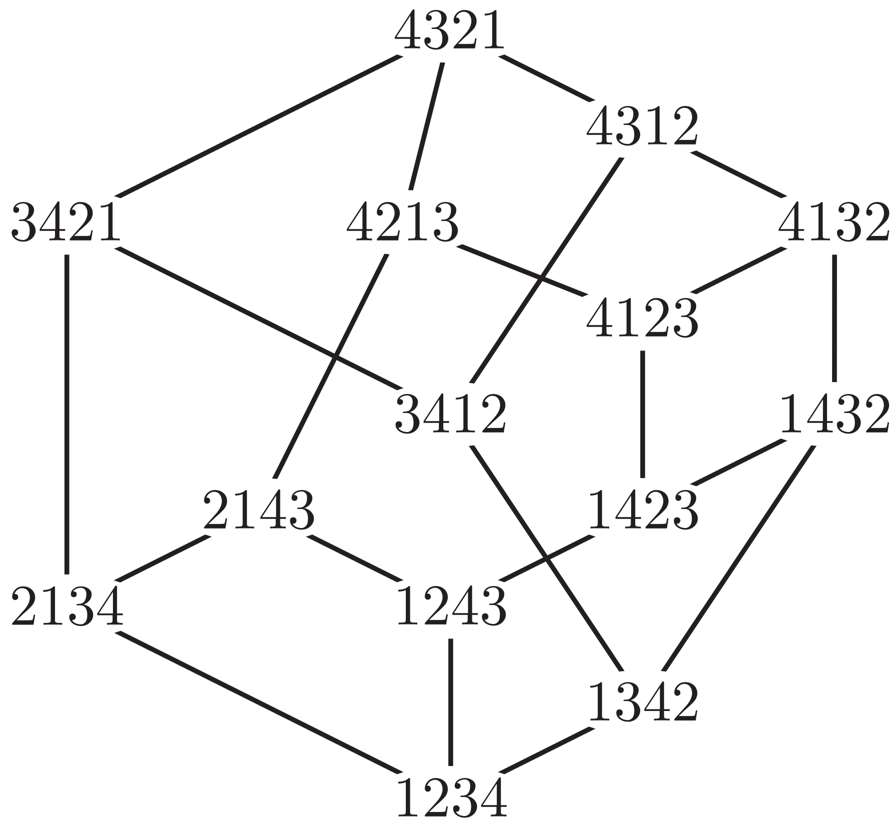
$(2j)\cdots k \cdots (2j-1)$ for $j \in [n]$ and $k > 2j$

# WIGGLY PSEUDOTRIANGULATIONS ⟷ WIGGLY PERMUTATIONS

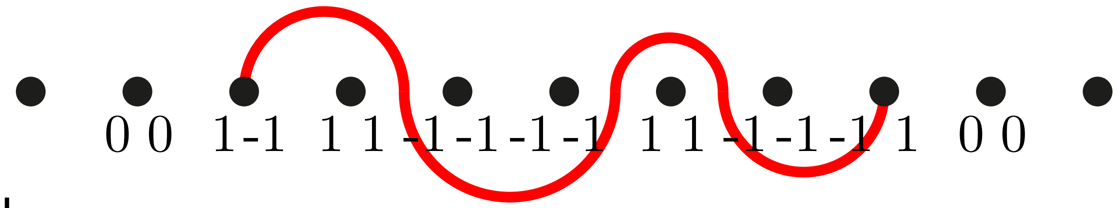
PROP. This bijection induces a directed graph isomorphism between
- the wiggly increasing flip graph on wiggly pseudotriangulations,
- the Hasse diagram of the wiggly lattice on wiggly permutations.

Bapat–P. ($24^+$)



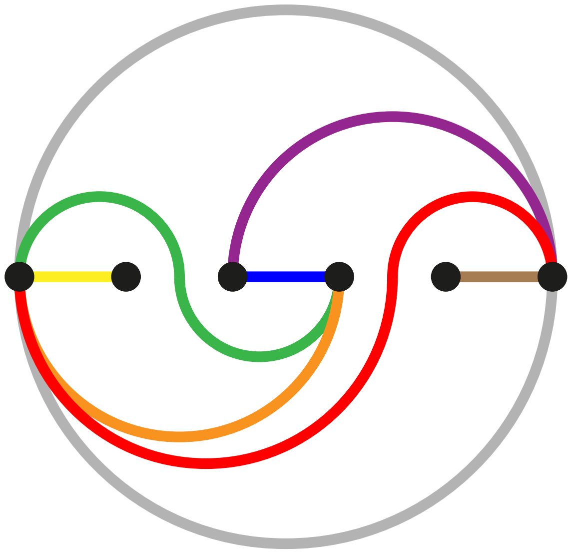


# WIGGLY FAN

$\boldsymbol{g}$-vector of $\alpha$ = proj. on $\sum_{i=1}^{2n} x_i = 0$ of 


$\boldsymbol{c}$-vector of $\alpha \in T$ = you don't want to know...

PROP. For any wiggly pseudotriangulation $T$, the $\boldsymbol{g}$-vectors $\{\boldsymbol{g}(\alpha) \mid \alpha \in T^\circ\}$ and the $\boldsymbol{c}$-vectors $\{\boldsymbol{c}(\alpha, T) \mid \alpha \in T^\circ\}$ form dual bases. Bapat–P. (24⁺)



$$\begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\end{matrix}\begin{pmatrix} 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

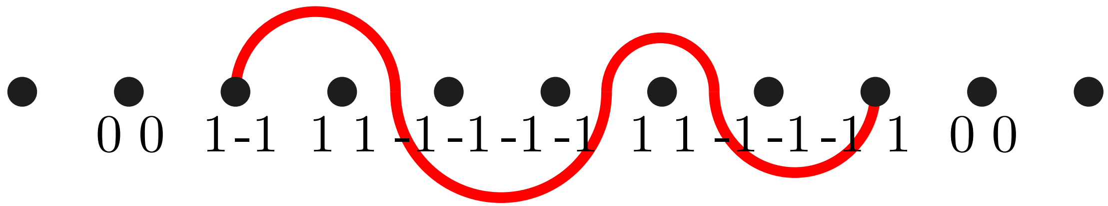
$\hat{\boldsymbol{g}}(T)$

$$\begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\end{matrix}\begin{pmatrix} 0 & 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & -1 & 0 & 0 & 0 & -2 & 2 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$4\boldsymbol{c}(T)$

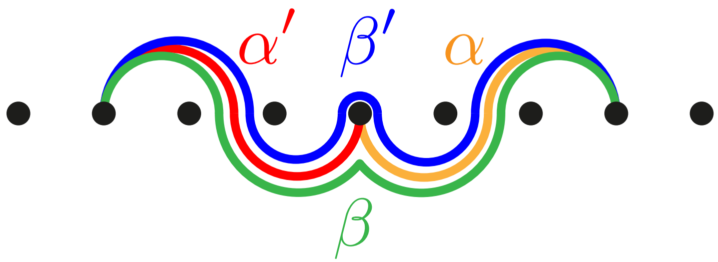
# WIGGLY FAN

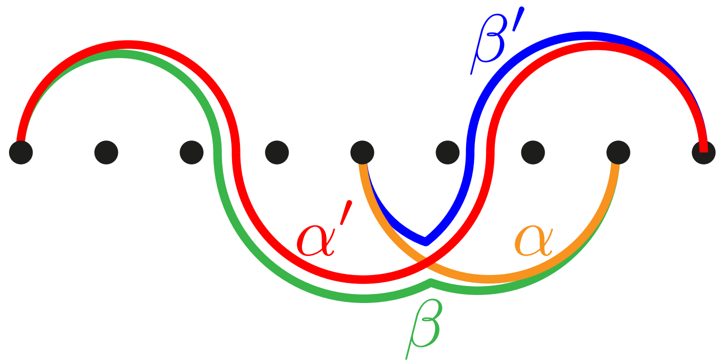
<u>$\boldsymbol{g}$-vector</u> of $\alpha$ = proj. on $\sum_{i=1}^{2n} x_i = 0$ of




THM. The cones $\langle \boldsymbol{g}(\alpha) \mid \alpha \in D \rangle$ for all wiggly dissections $D$ form a complete simplicial fan $\mathrm{WF}_n$ (in $\sum_{i=1}^{2n} x_i = 0$). Bapat–P. ($24^+$)

Main observation:




$\boldsymbol{g}(\alpha) + \boldsymbol{g}(\alpha') = \big(\boldsymbol{g}(\beta) + \boldsymbol{g}(\beta')\big)/2$

$\boldsymbol{g}(\alpha) + \boldsymbol{g}(\alpha') = \boldsymbol{g}(\beta) + \boldsymbol{g}(\beta')$

# WIGGLYHEDRON

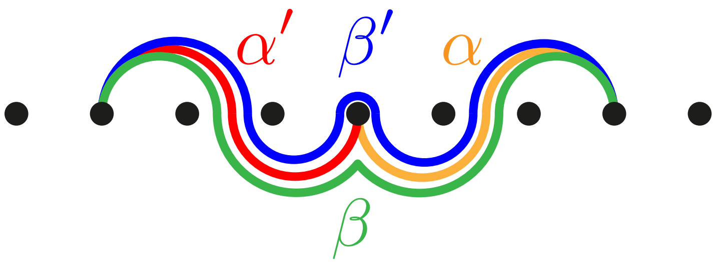
# WIGGLYHEDRON

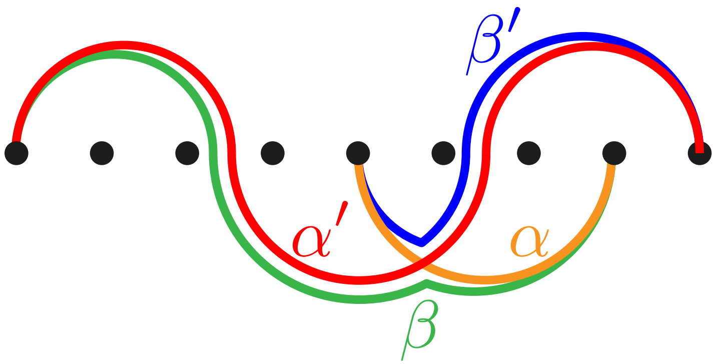
incompatibility degree $\delta(\alpha, \alpha') =$

- $0$ if $\alpha$ and $\alpha'$ are pointed and non-crossing,
- $1$ is $\alpha$ and $\alpha'$ are not pointed,
- the number of crossings of $\alpha$ and $\alpha'$ if they are crossing.

$\kappa(\alpha) =$ incompatibility number of $\alpha = \sum_{\alpha'} \delta(\alpha, \alpha')$.

Main observation:




$\kappa(\alpha) + \kappa(\alpha') > \big(\kappa(\beta) + \kappa(\beta')\big)/2$ $\qquad$ $\kappa(\alpha) + \kappa(\alpha') > \kappa(\beta) + \kappa(\beta')$

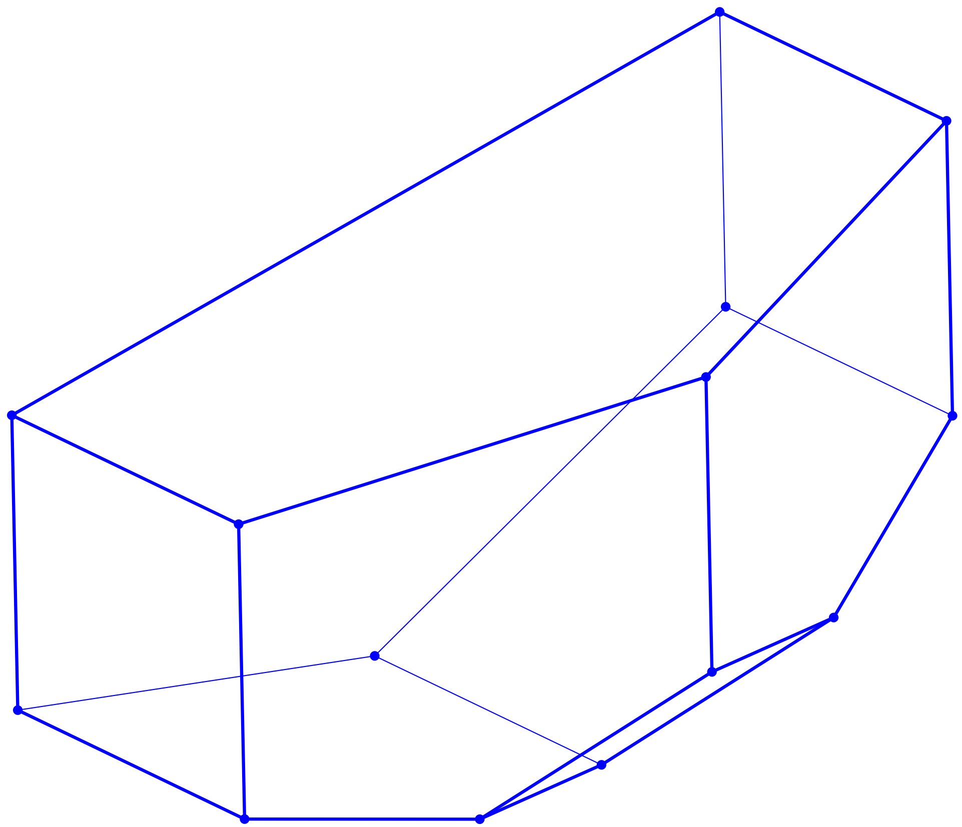
Hence, $\kappa$ satisfies all wall-crossing inequalities of the wiggly fan...

# WIGGLYHEDRON

THM. The wiggly fan $\mathrm{WF}_n$ is the normal fan of a simplicial $(2n-1)$-dimensional polytope, called the wigglyhedron $\mathbb{W}_n$, and defined equivalently as

- intersection of the halfspaces $\left\{ \boldsymbol{x} \in \mathbb{R}^{2n} \;\middle|\; \langle\, \boldsymbol{g}(\alpha) \mid \boldsymbol{x} \,\rangle \le \kappa(\alpha) \right\}$ for all wiggly arcs $\alpha$,
- convex hull of $\boldsymbol{p}(T) := \sum_{\alpha \in T} \kappa(\alpha)\, \boldsymbol{c}(\alpha, T)$ for all wiggly pseudotriangulations $T$.

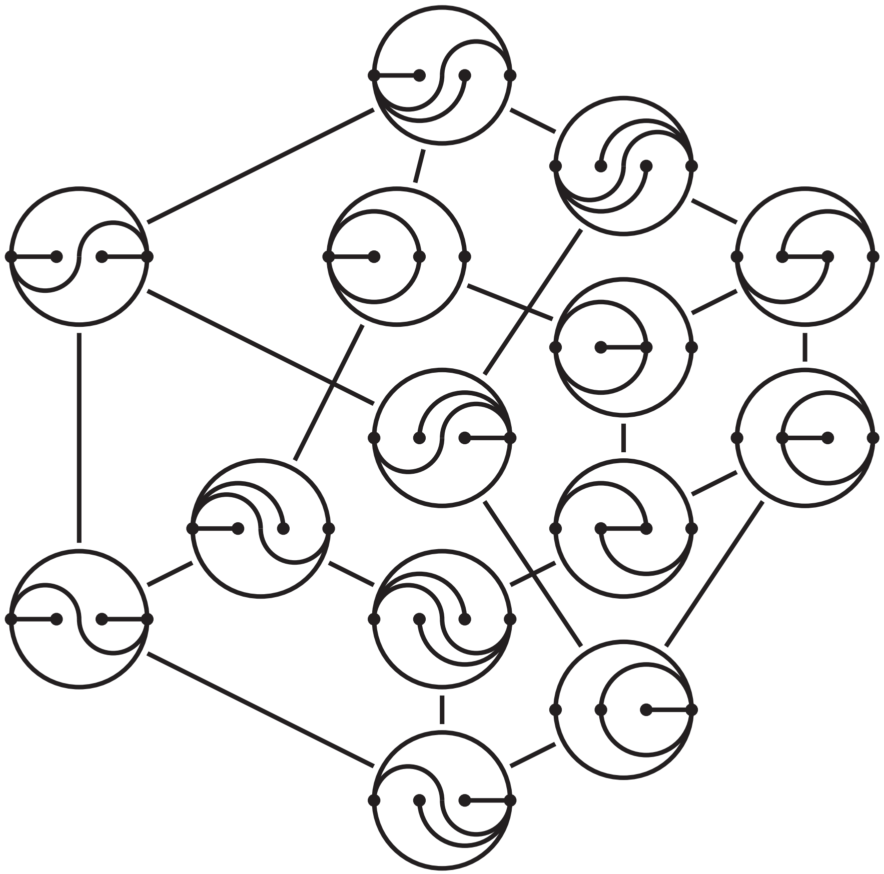
Bapat–P. (24⁺)

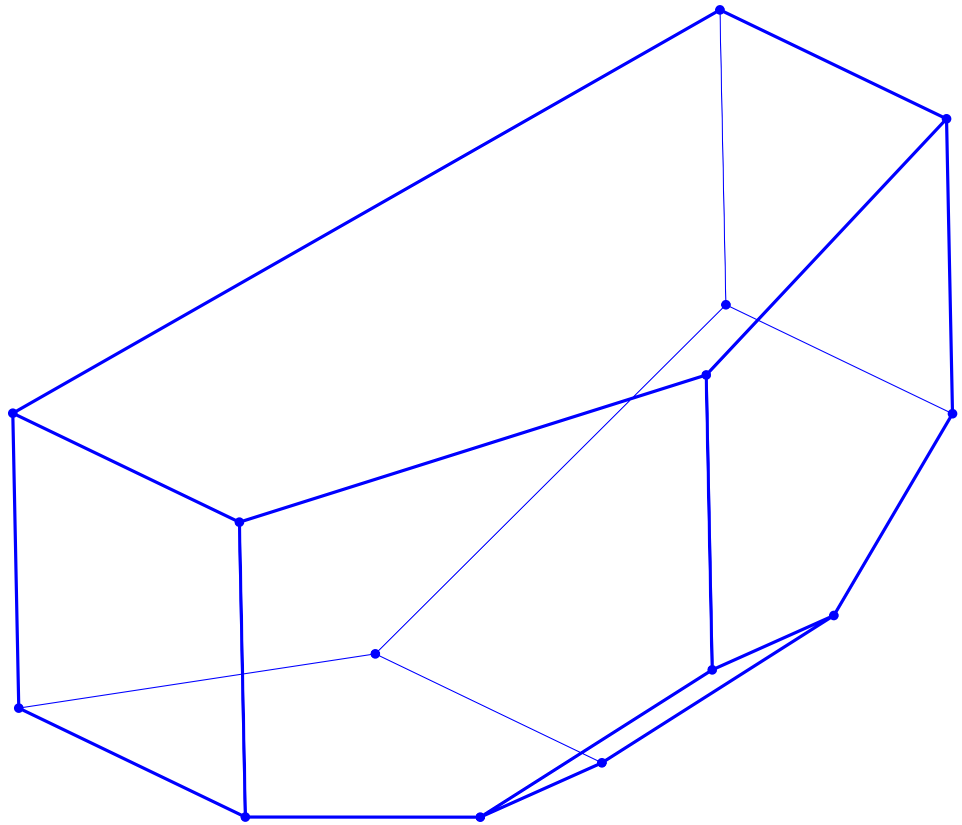


# WIGGLYHEDRON

THM. The wigglyhedron $\mathbb{W}_n$ is a simple $(2n-1)$-dimensional polytope such that
- the wiggly complex $\mathrm{WC}_n$ is the boundary complex of the polar of $\mathbb{W}_n$,
- the Hasse diagram of the wiggly lattice is a linear orientation of the graph of $\mathbb{W}_n$.

Bapat–P. ($24^+$)





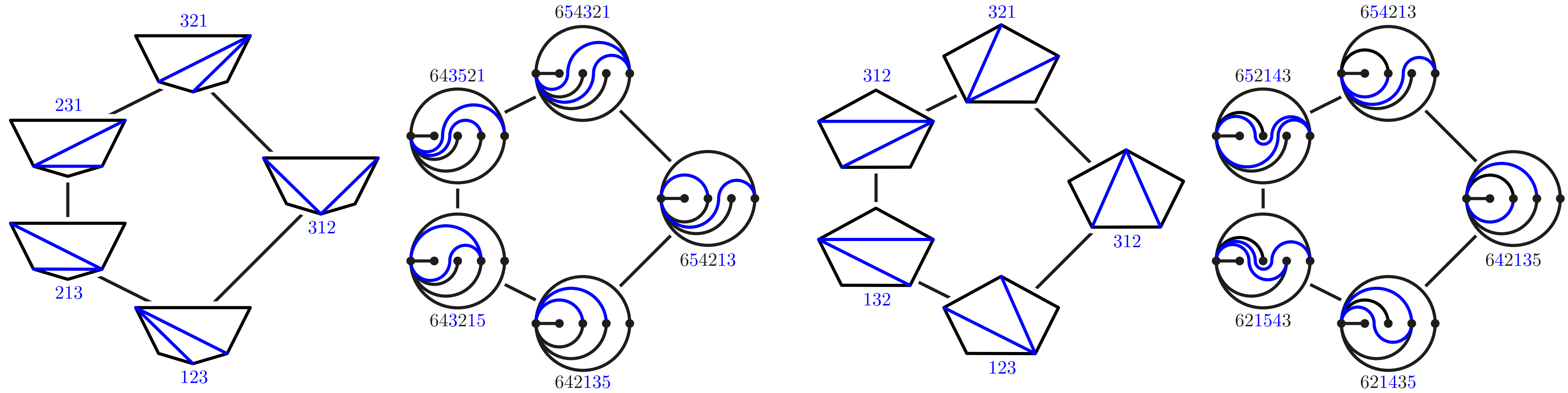
# CAMBRIAN CONSIDERATIONS

# CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+,-\}^n$, there are lattice isomorphisms between

- $\delta$-triangulations = triangulation of the $\delta$-gon, whose vertex at abscissa $i$ has ordinate positive if $\delta_j = +$ and negative if $\delta_j = -$
- $\delta$-permutations = permutation of $[n]$ avoiding for $i < j < k$ $\cdots j \cdots ki \cdots$ if $\delta_j = +$ and $\cdots ik \cdots j \cdots$ if $\delta_j = -$
- $\delta$-wiggly pseudotriangulations = wiggly pseudotriangulation containing the arcs $(0, j, [1, j[, \varnothing)$ for $\delta_j = +$ and $(0, j, \varnothing, [1, j[)$ for $\delta_j = -$
- $\delta$-wiggly permutations = wiggly permutation $\sigma$ of $[2n]$ such that $\delta_j = + \implies \sigma^{-1}(i) \le \sigma^{-1}(2j-1)$ and $\delta_j = - \implies \sigma^{-1}(2j) \le \sigma^{-1}(i)$

Bapat–P. (24[+])

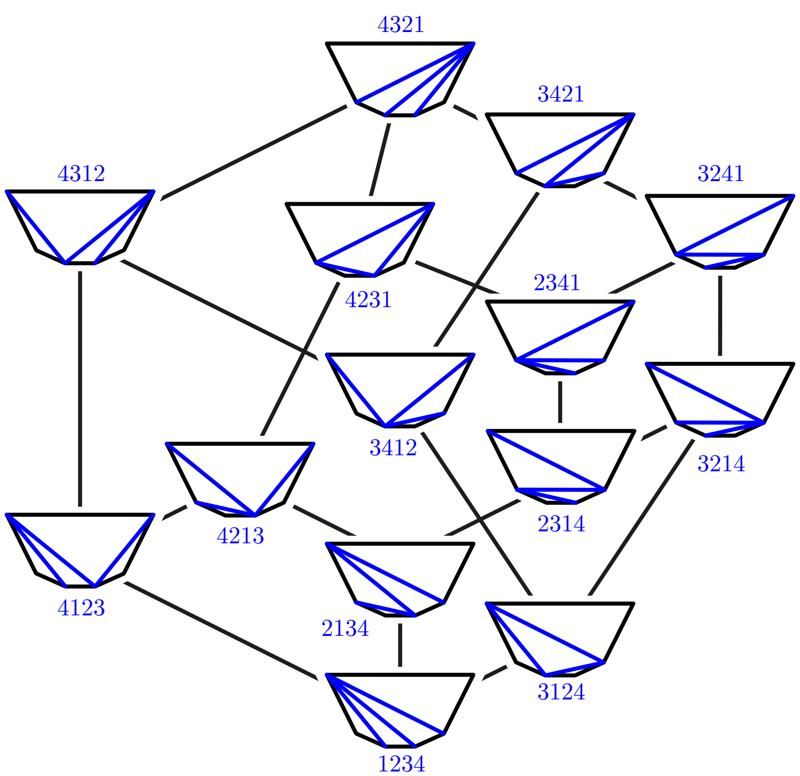


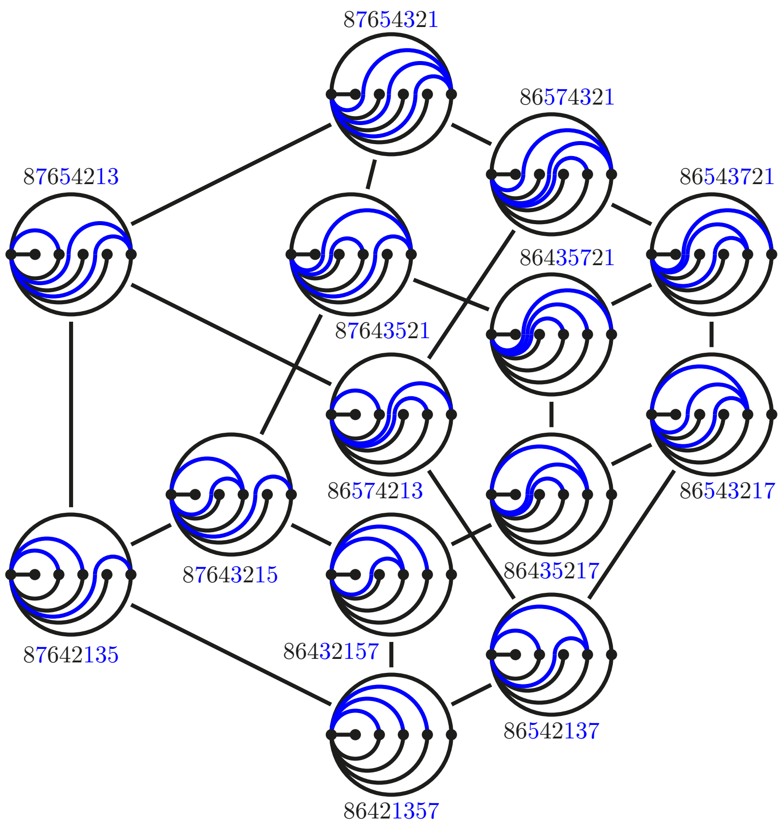
# CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+,-\}^n$, there are lattice isomorphisms between

- $\delta$-triangulations
- $\delta$-permutations
- $\delta$-wiggly pseudotriangulations
- $\delta$-wiggly permutations

Bapat–P. ($24^+$)



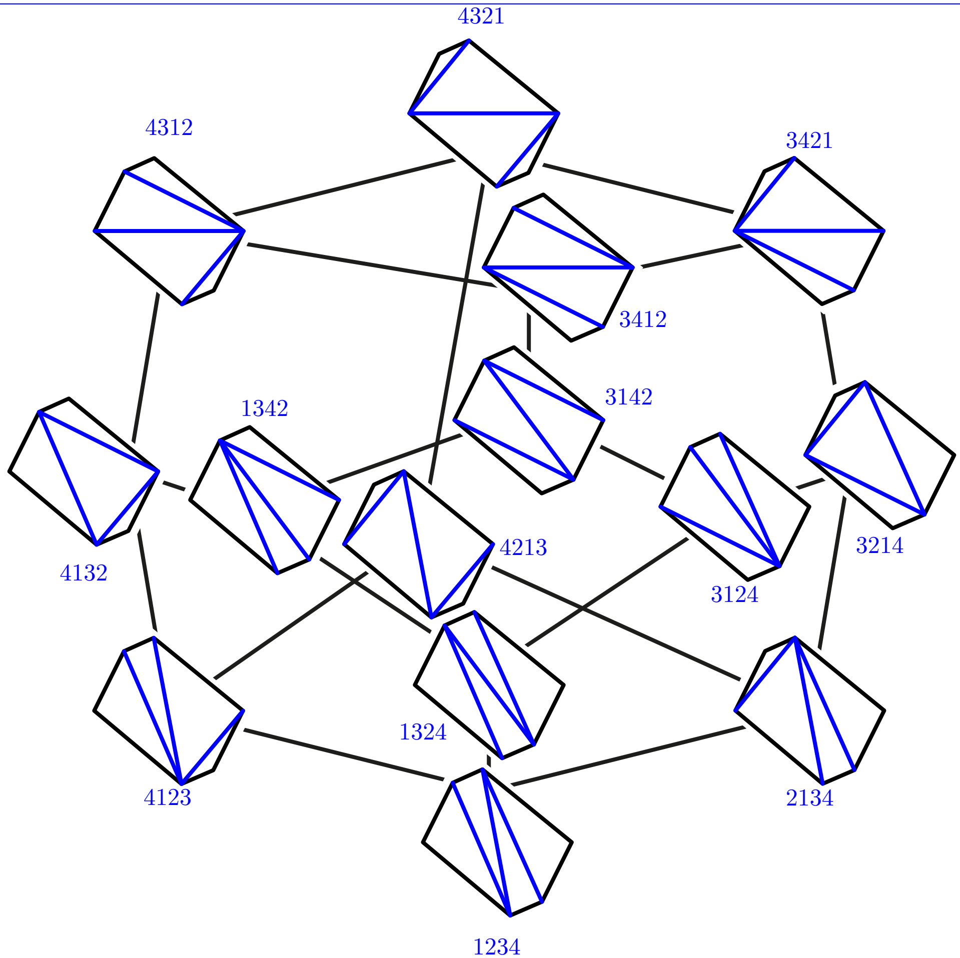



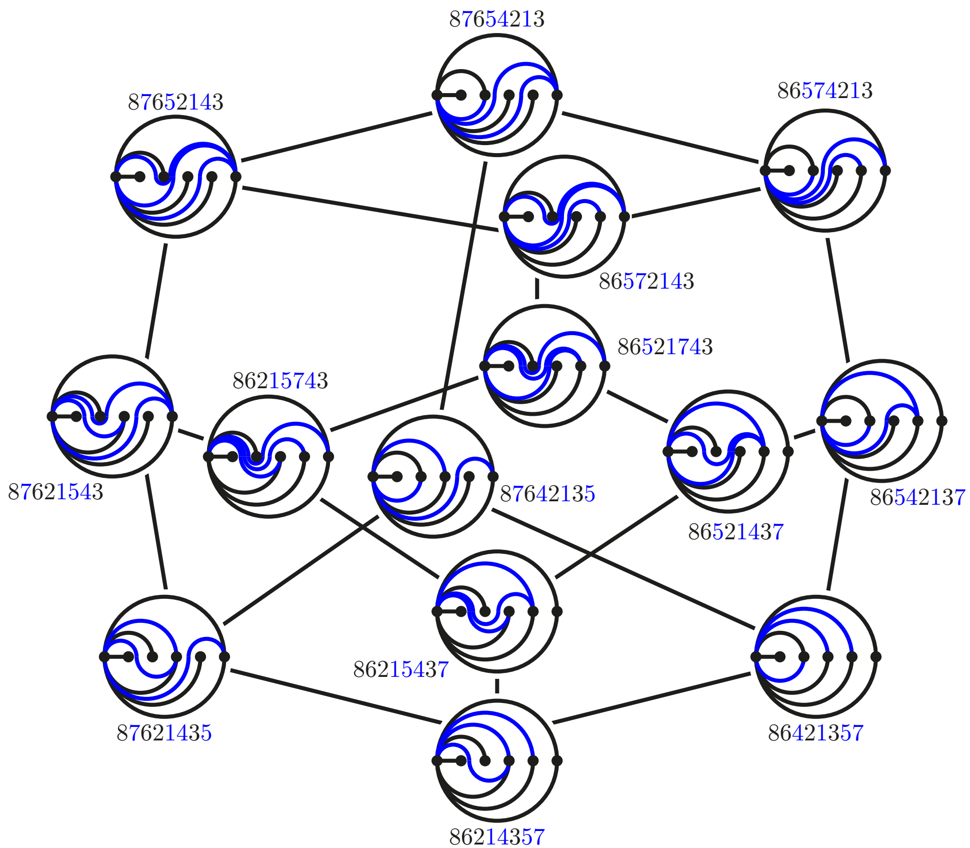
# CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+, -\}^n$, there are lattice isomorphisms between

- $\delta$-triangulations
- $\delta$-permutations
- $\delta$-wiggly pseudotriangulations
- $\delta$-wiggly permutations

Bapat–P. ($24^+$)



# CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+, -\}^n$, there are lattice isomorphisms between

- $\delta$-triangulations
- $\delta$-permutations
- $\delta$-wiggly pseudotriangulations
- $\delta$-wiggly permutations

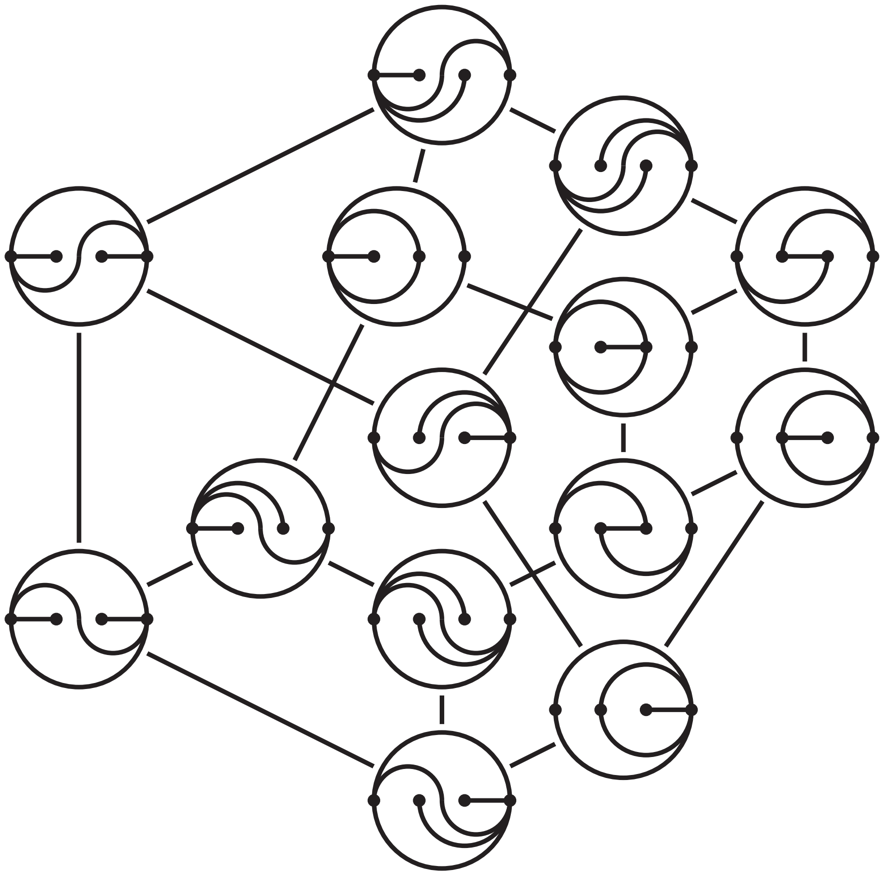
Bapat–P. ($24^+$)

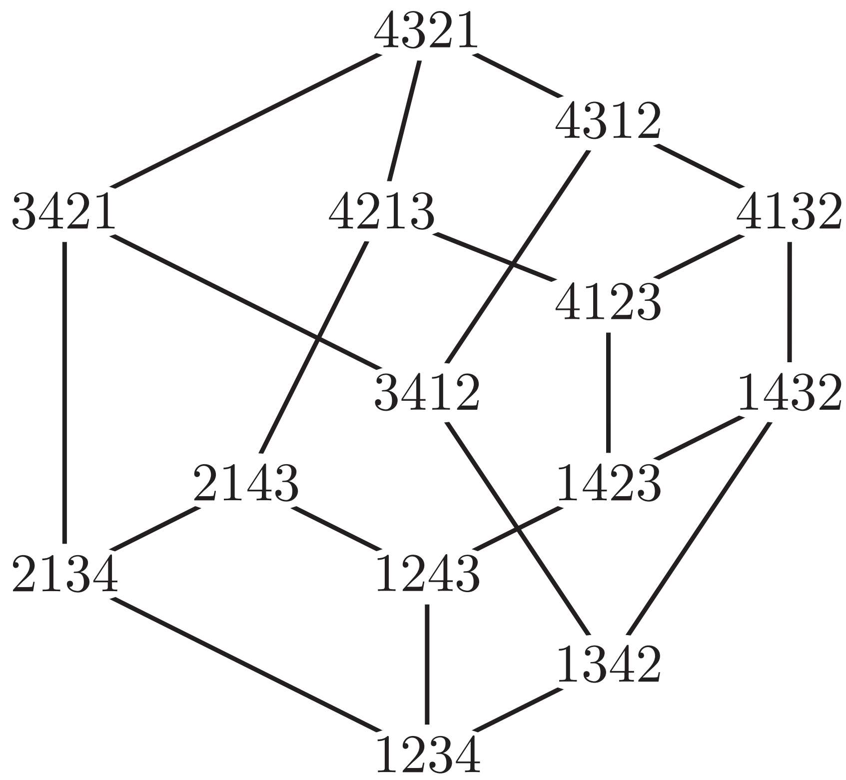
PROP. The $\delta$-associahedron $\mathbb{A}\mathrm{sso}_\delta$ is normally equivalent to the face of the wigglyhedron $\mathbb{W}_n$ corresponding to the wiggly pseudodissection formed by the $\delta$-wiggly arcs.

Bapat–P. ($24^+$)

# SOME OPEN PROBLEMS

# OPEN PROBLEM 1: GRAPH PROPERTIES OF WIGGLYHEDRON



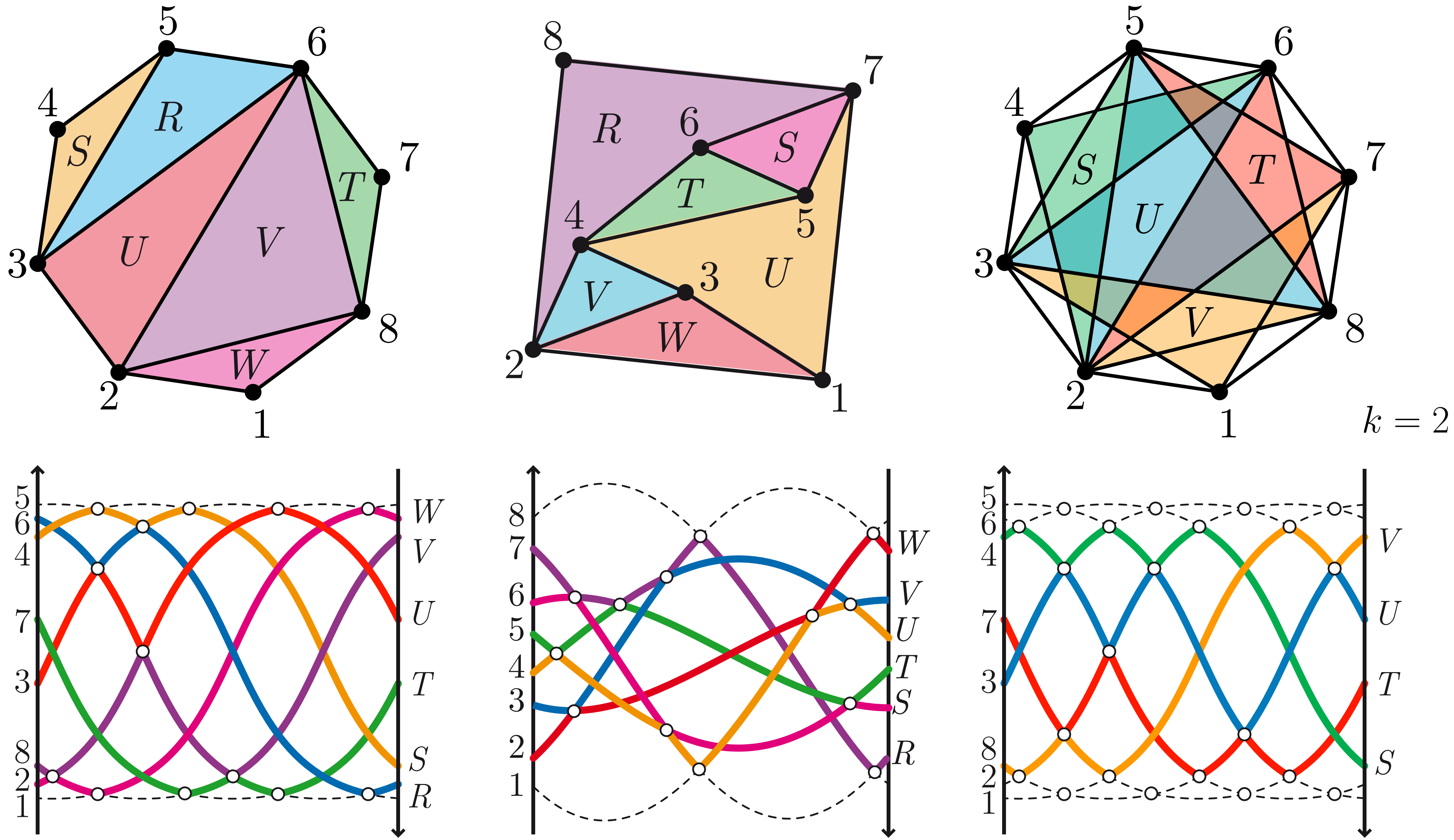



Q1a. Is the wiggly flip graph Hamiltonian?

[NB: wiggly permutations do not form a zigzag language]

Q1b. What is the diameter of the wiggly flip graph?

# OPEN PROBLEM 2: WIGGLY PSEUDOTRIANGULATIONS AND DUALITY



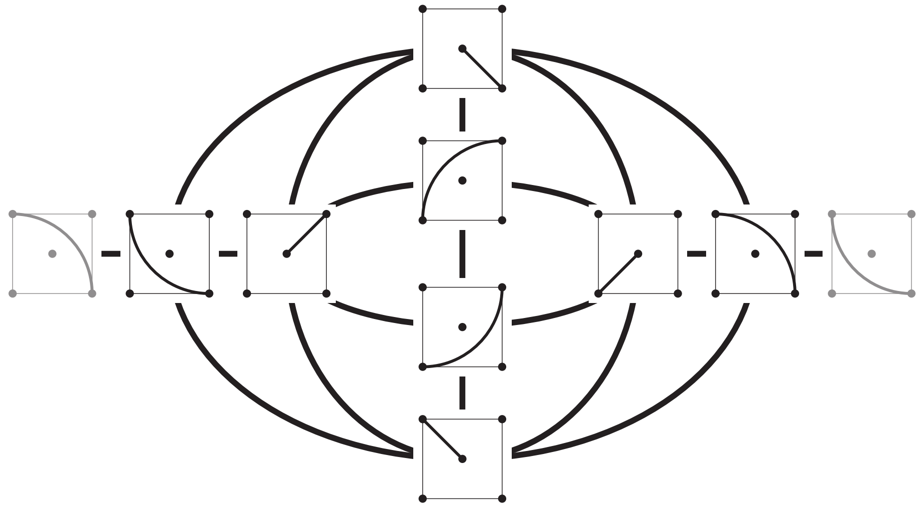

Q2. Is there a dual interpretation of wiggly pseudotriangulations?

# OPEN PROBLEM 3: WIGGLY PSEUDOTRIANGULATIONS OF POINT SETS

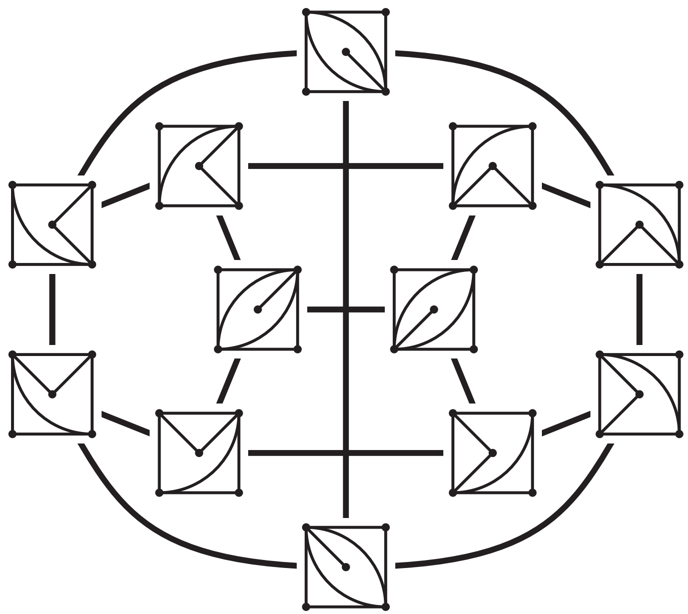
$\boldsymbol{P}$ arbitrary point set in the plane

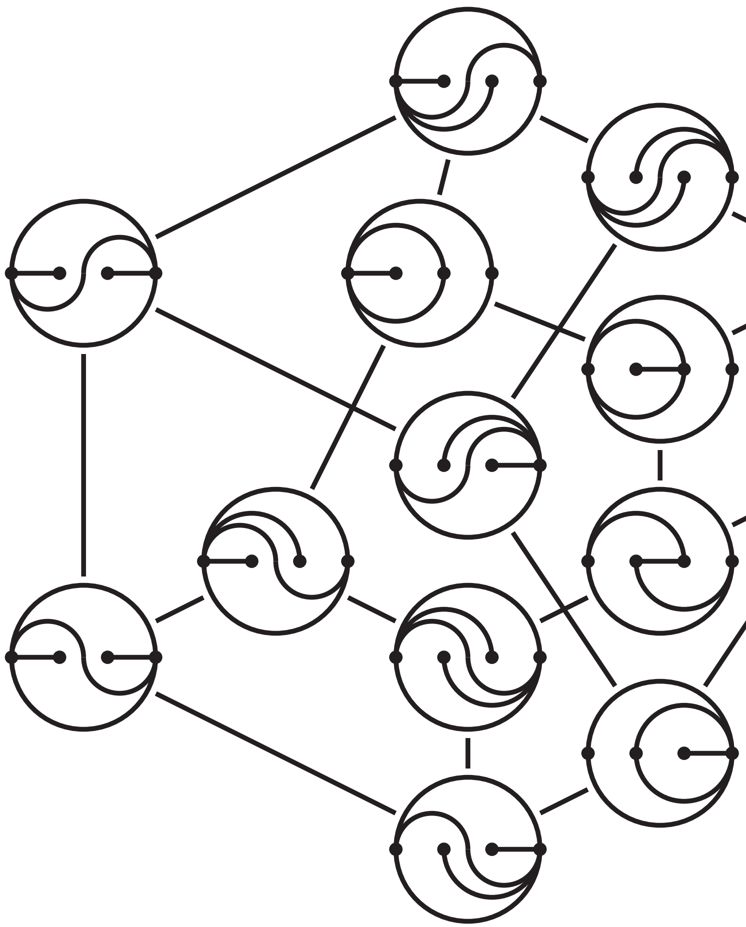
<u>wiggly complex</u> $\mathrm{WC}_{\boldsymbol{P}}$ = simplicial complex of non-crossing and pointed wiggly edges

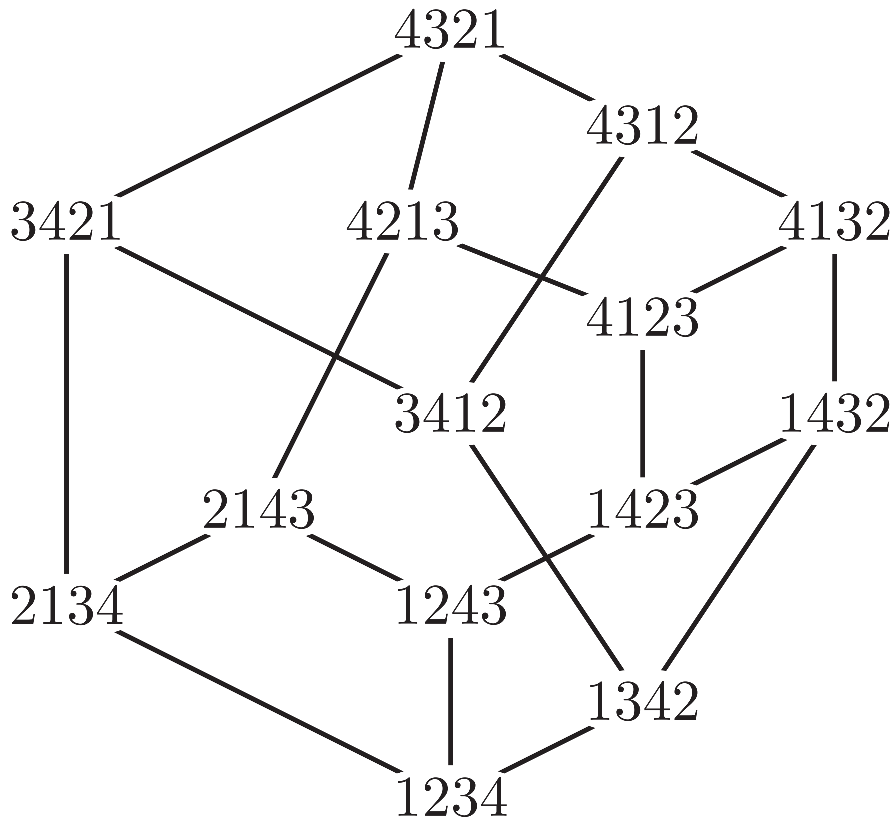
Q3a. Is $\mathrm{WC}_{\boldsymbol{P}}$ the boundary complex of a simplicial polytope?
[NB: aligned points ⇒ wigglyhedron
general position ⇒ Rote–Santos–Streinu]



Q3b. Is the graph of $\mathrm{WC}_{\boldsymbol{P}}$ Hamiltonian?



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# *THANK YOU*