Shard polytopes

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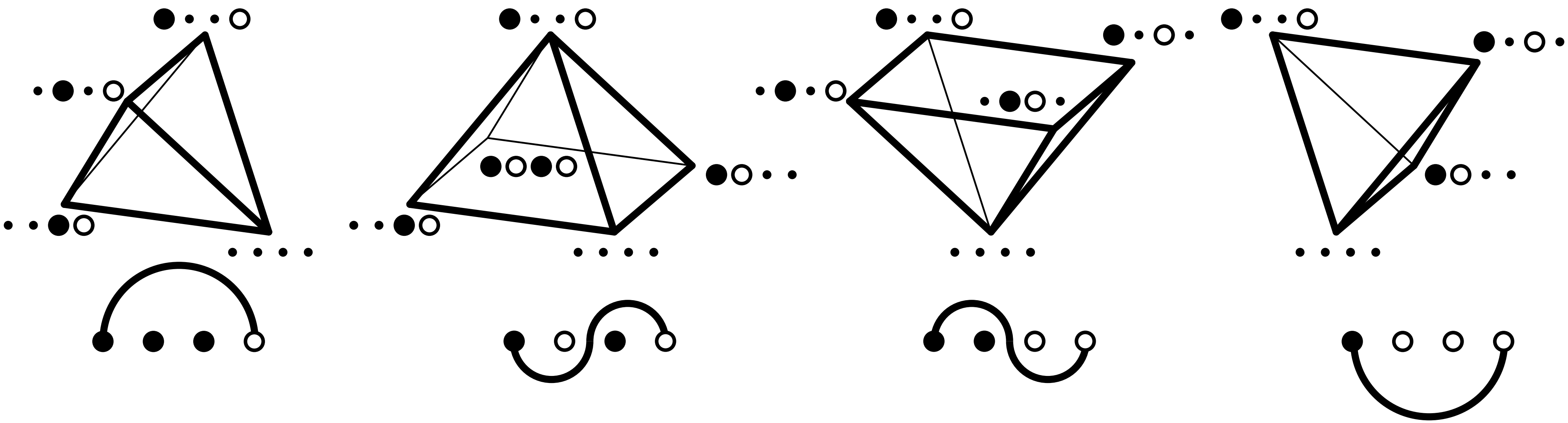

Séminaire AGATA, Montpellier
Thursday December 10th, 2020

slides available at: http://www.lix.polytechnique.fr/~pilaud/documents/presentations/shardPolytopes.pdf
preprint available at: http://www.arxiv.org/abs/2007.01008

TWO CLASSICAL LATTICES AND POLYTOPES

LATTICES: WEAK ORDER AND TAMARI LATTICE

<ins>lattice</ins> = partially ordered set L where any $X \subseteq L$ admits a <ins>meet</ins> $\bigwedge X$ and a <ins>join</ins> $\bigvee X$

<ins>lattice congruence</ins> = equivalence relation on L compatible with meets and joins

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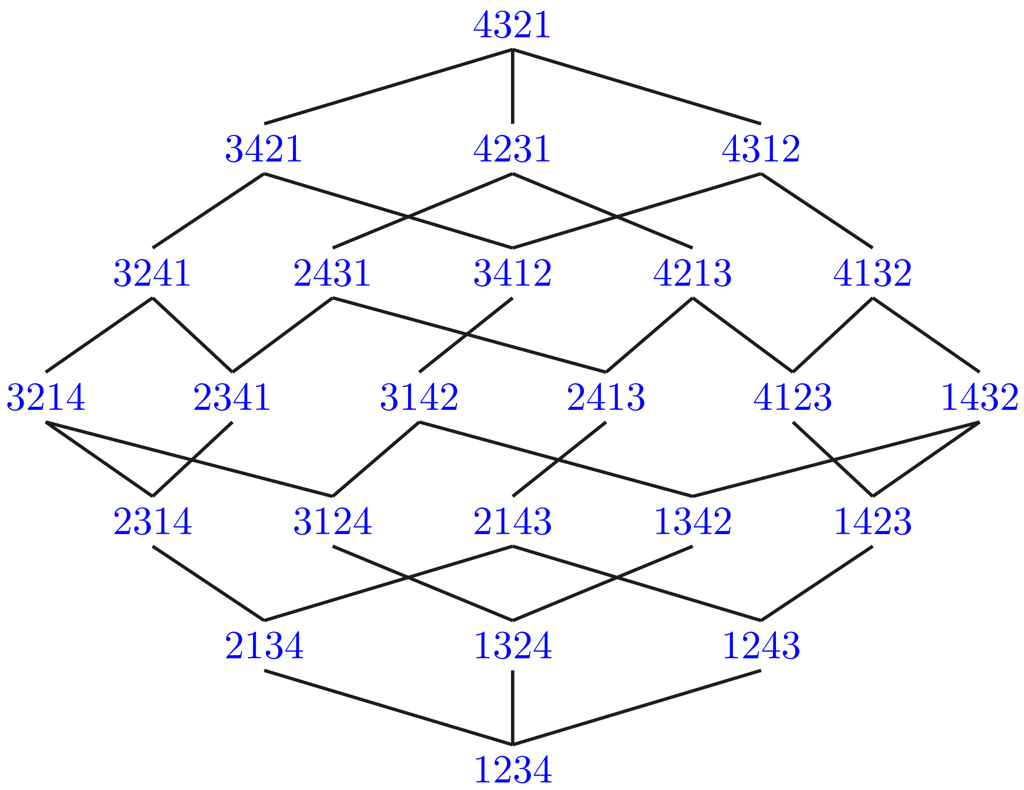

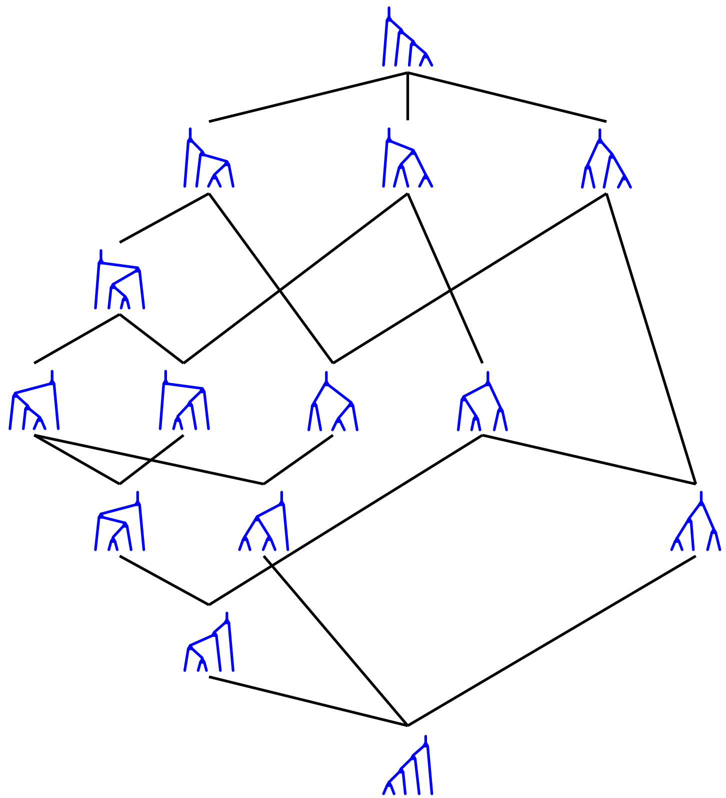

weak order = permutations of $\mathfrak{S}_n$
ordered by inclusion of inversion sets

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

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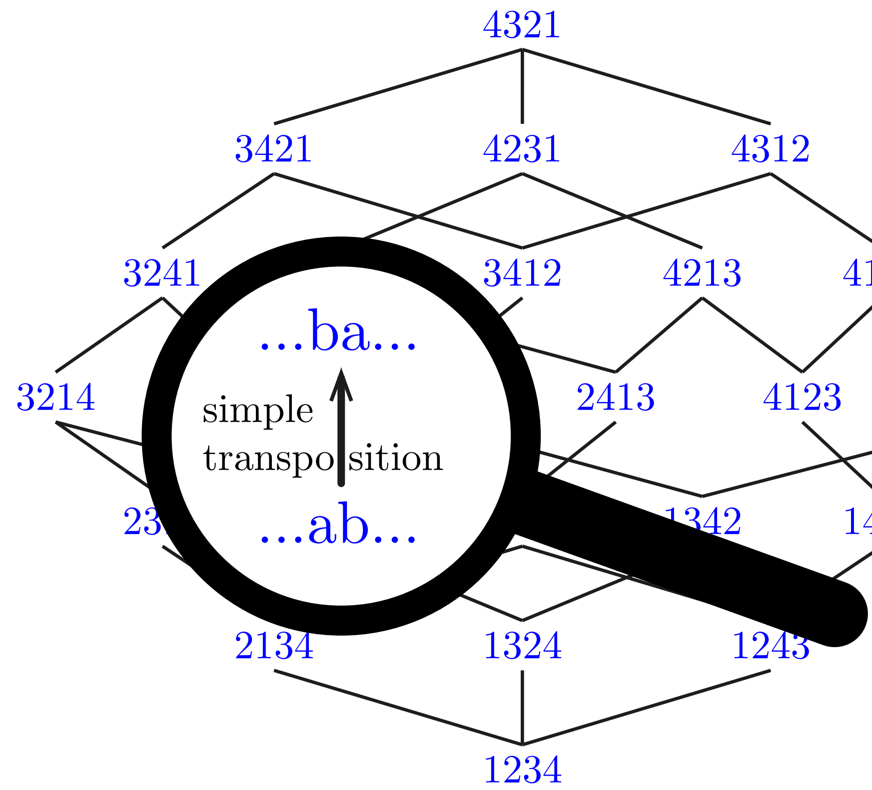

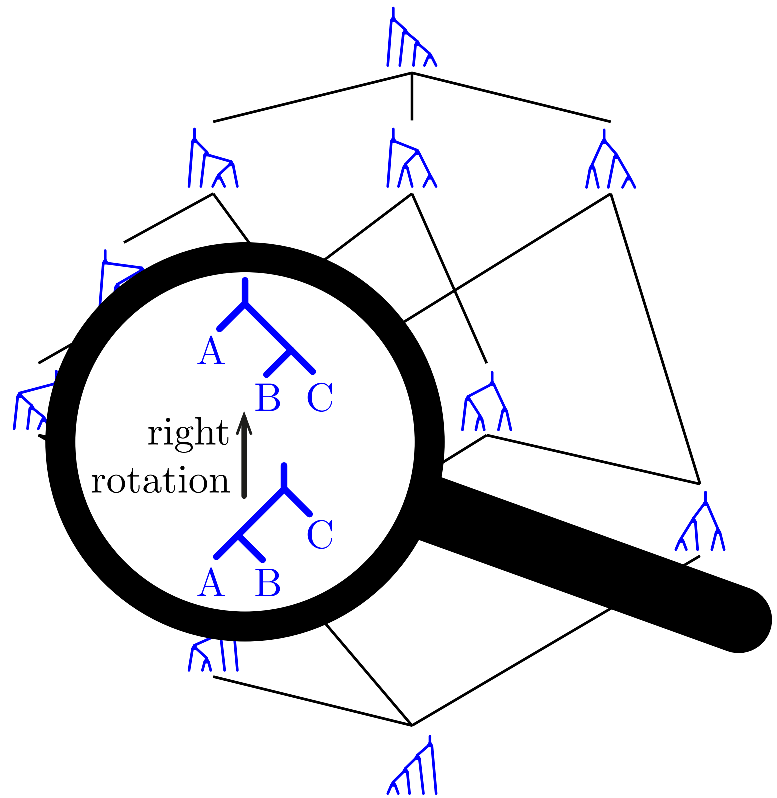

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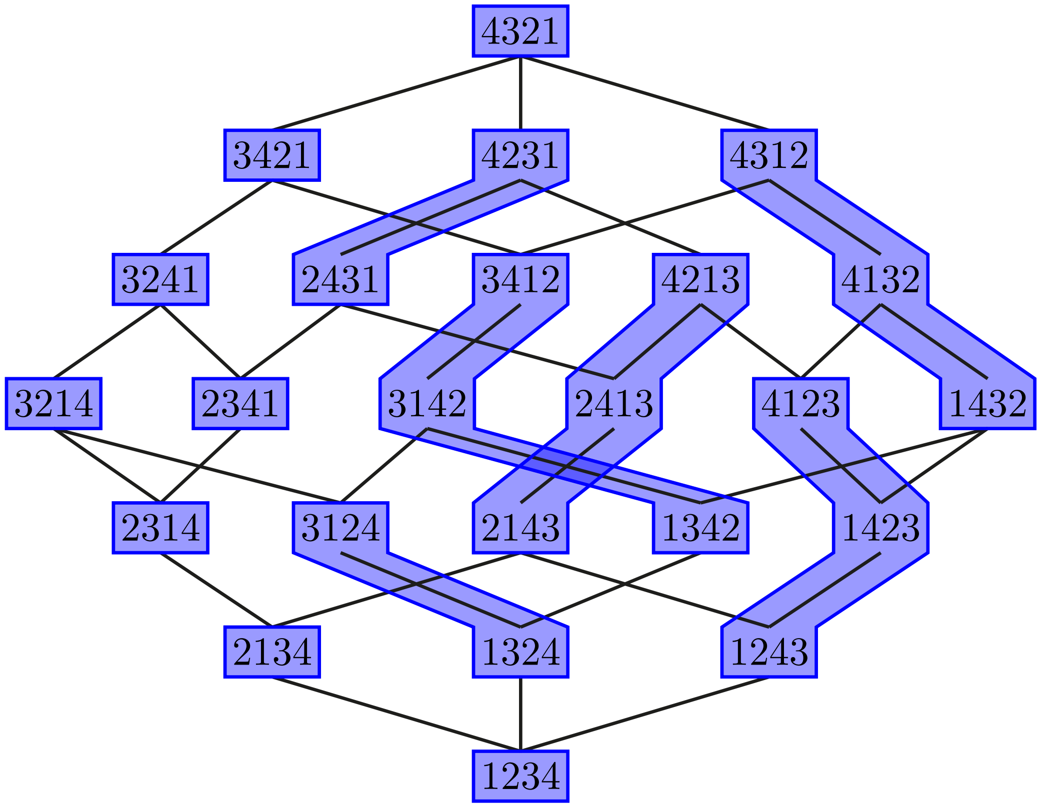

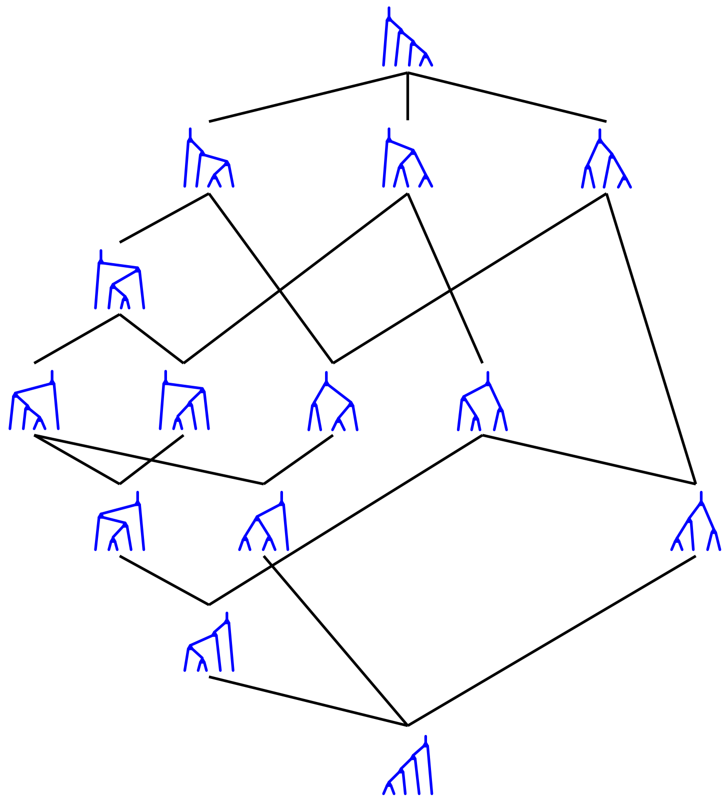

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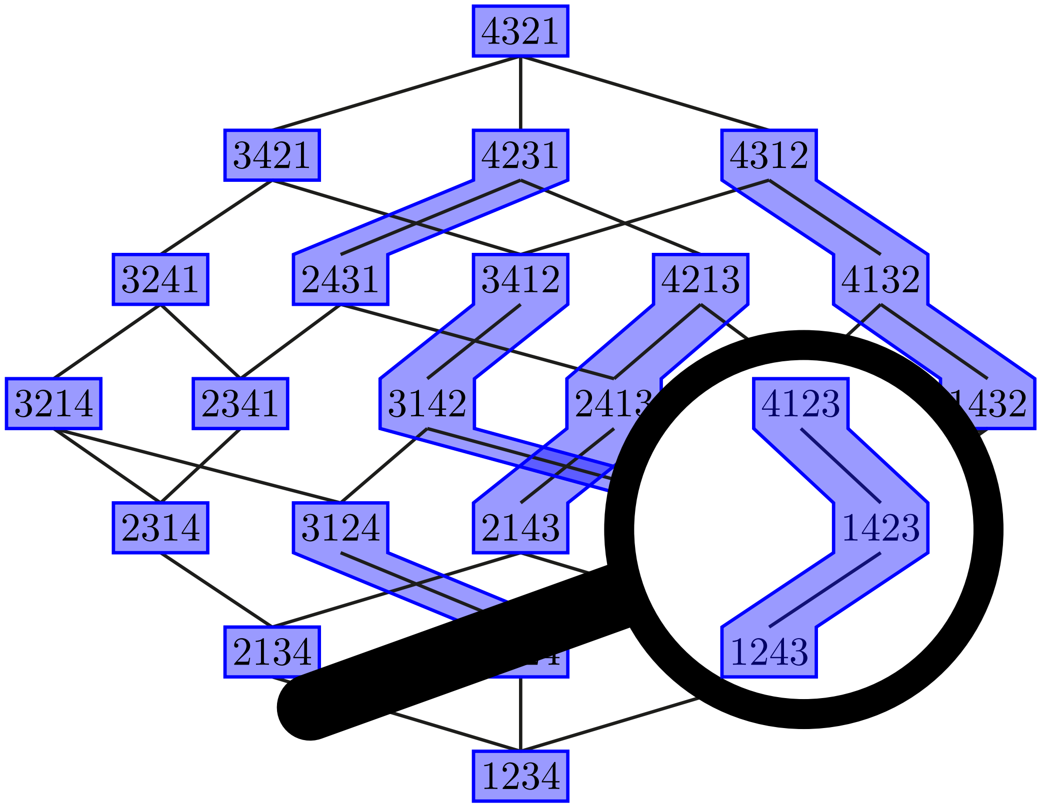

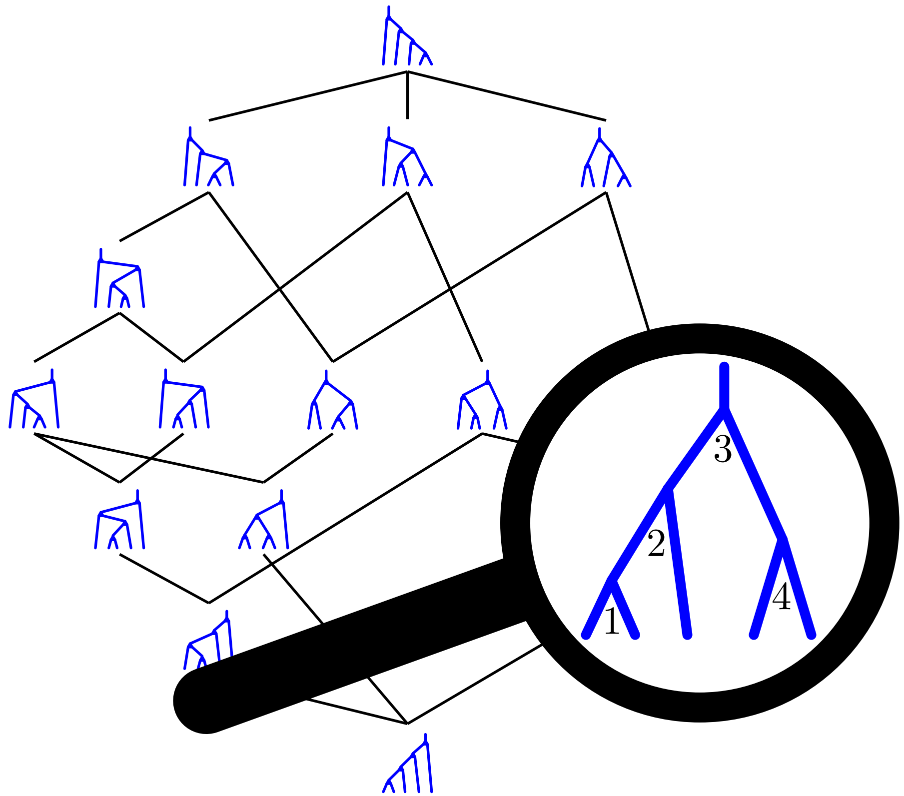

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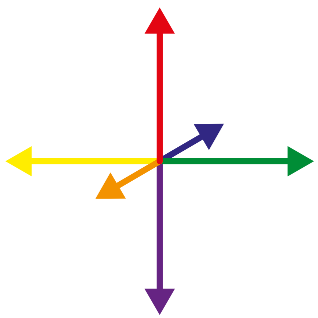
POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

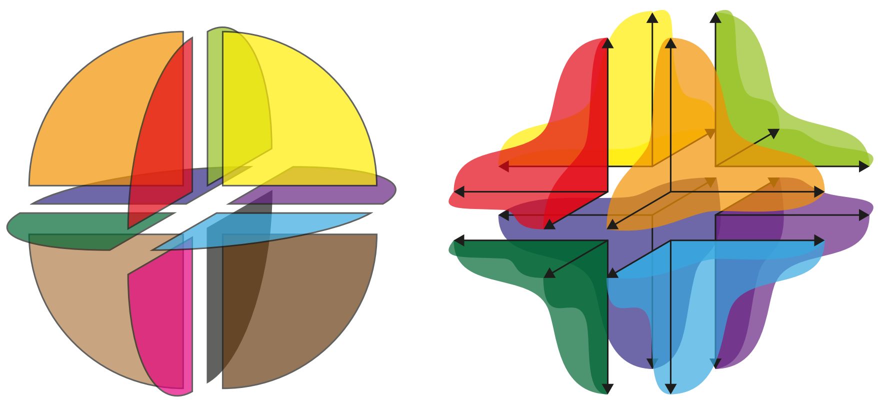
fan = collection of polyhedral cones closed by faces and intersecting along faces

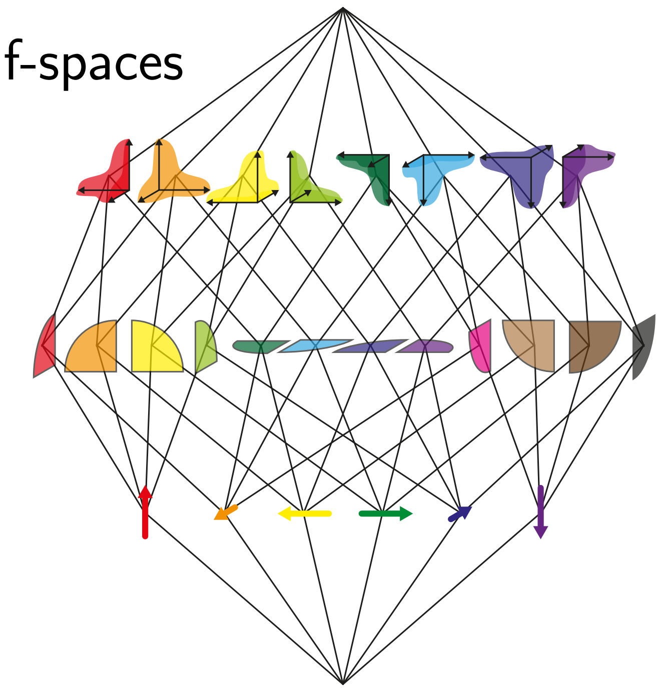
polytope = convex hull of a finite set = intersection of finitely many affine half-space polyhedral cone = positive span of a finite set of $\mathbb{R}^n$

= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces and where any two cones intersect along a face





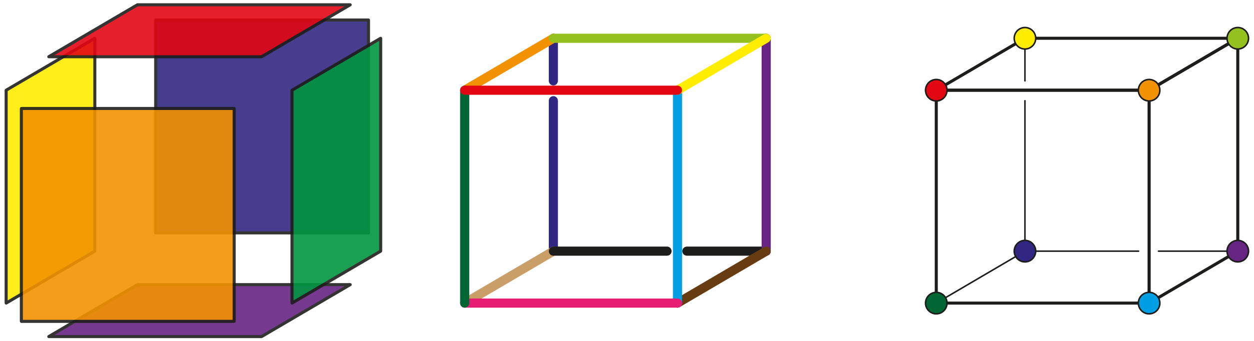


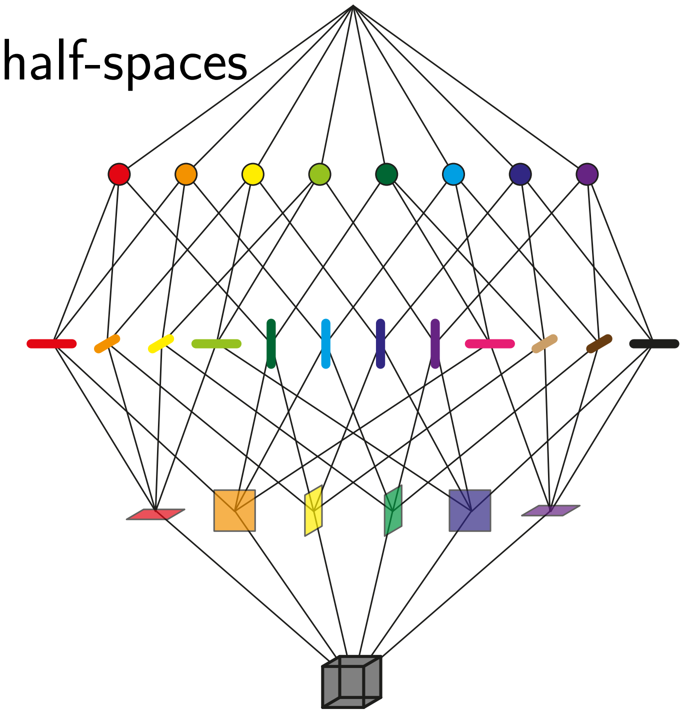
POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

polytope $=$ convex hull of a finite set of $\mathbb{R}^n$
$=$ bounded intersection of finitely many affine half-spaces

face $=$ intersection with a supporting hyperplane

face lattice $=$ all the faces with their inclusion relations





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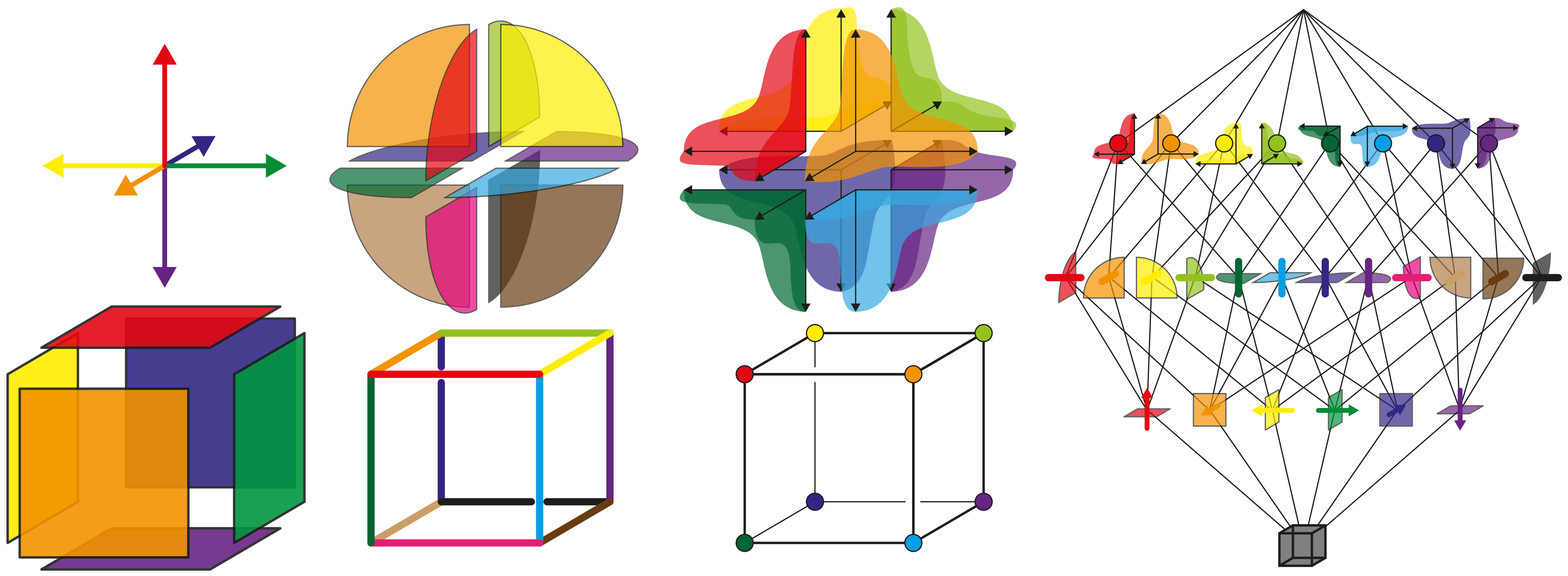

face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = $\{$ normal cone of $\mathbb{F}$ $\mid$ $\mathbb{F}$ face of $\mathbb{P}$ $\}$

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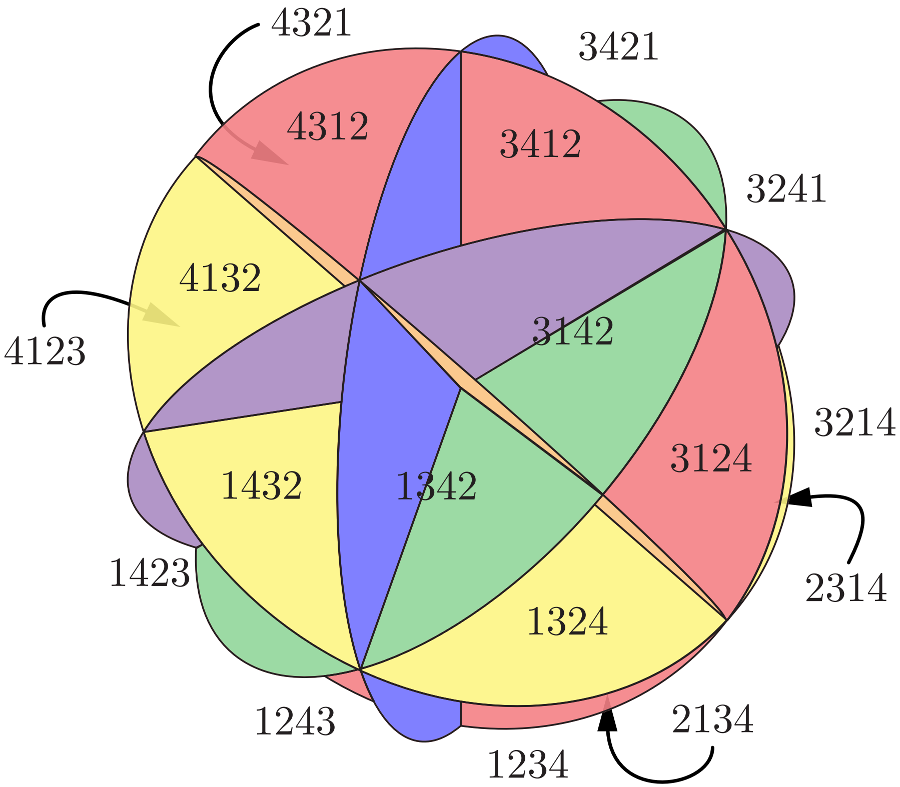
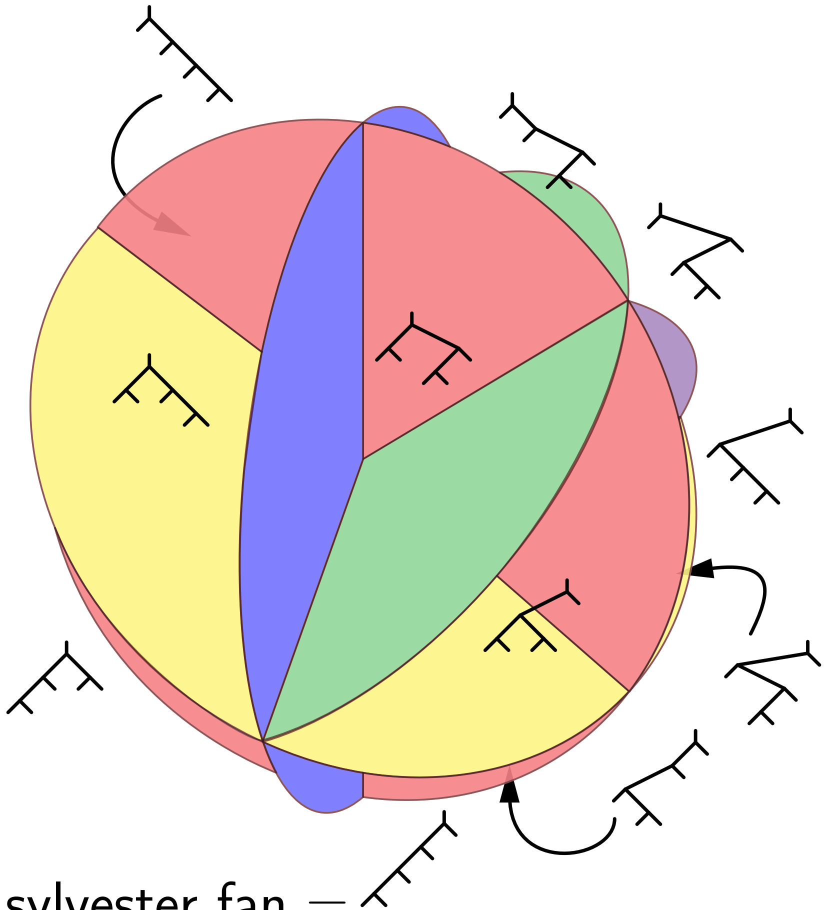


braid fan =
$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$



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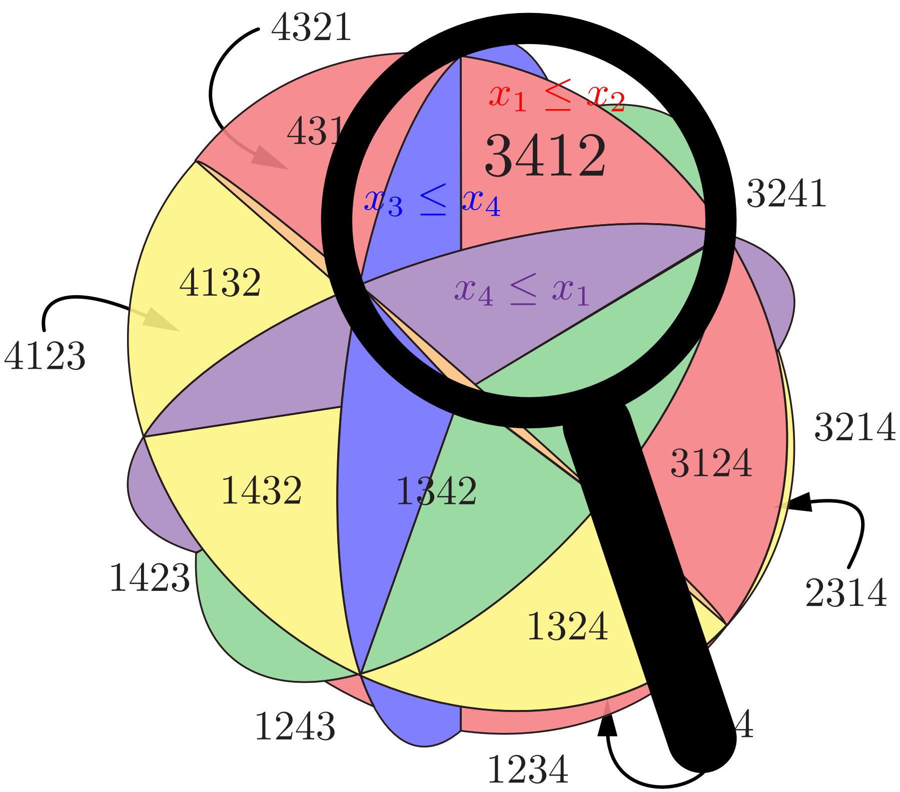

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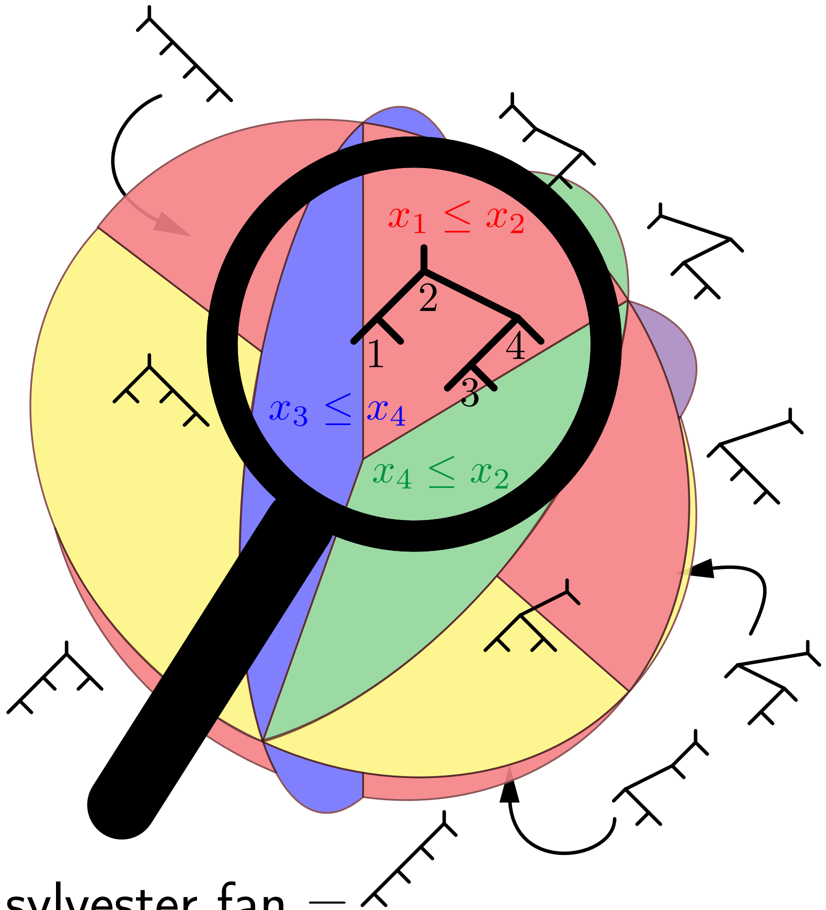

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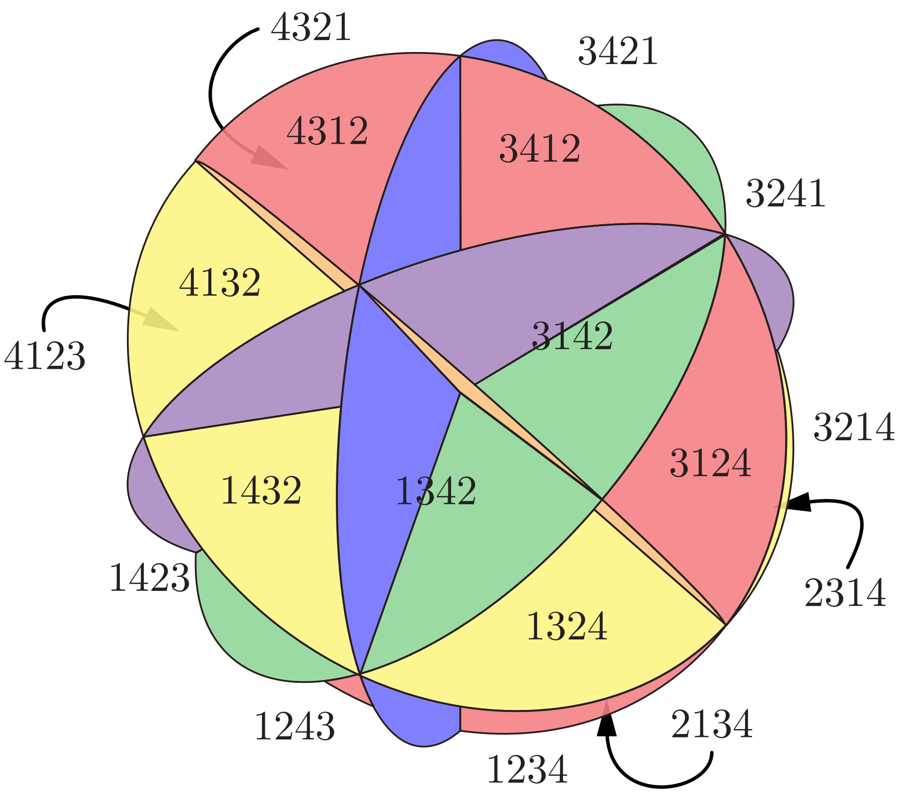

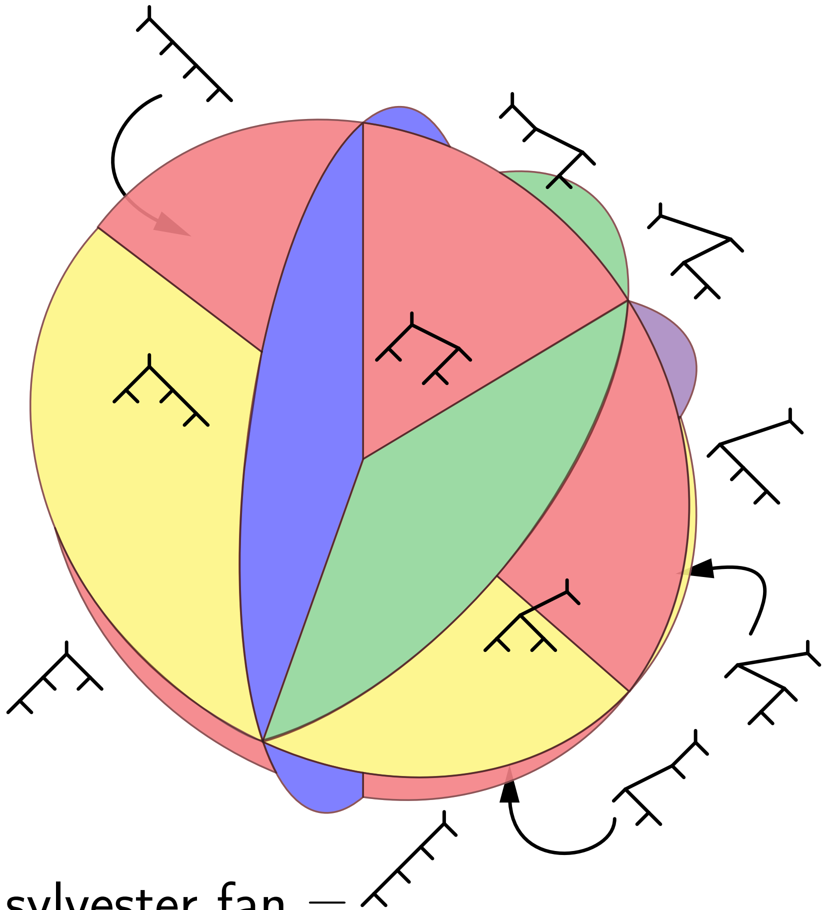

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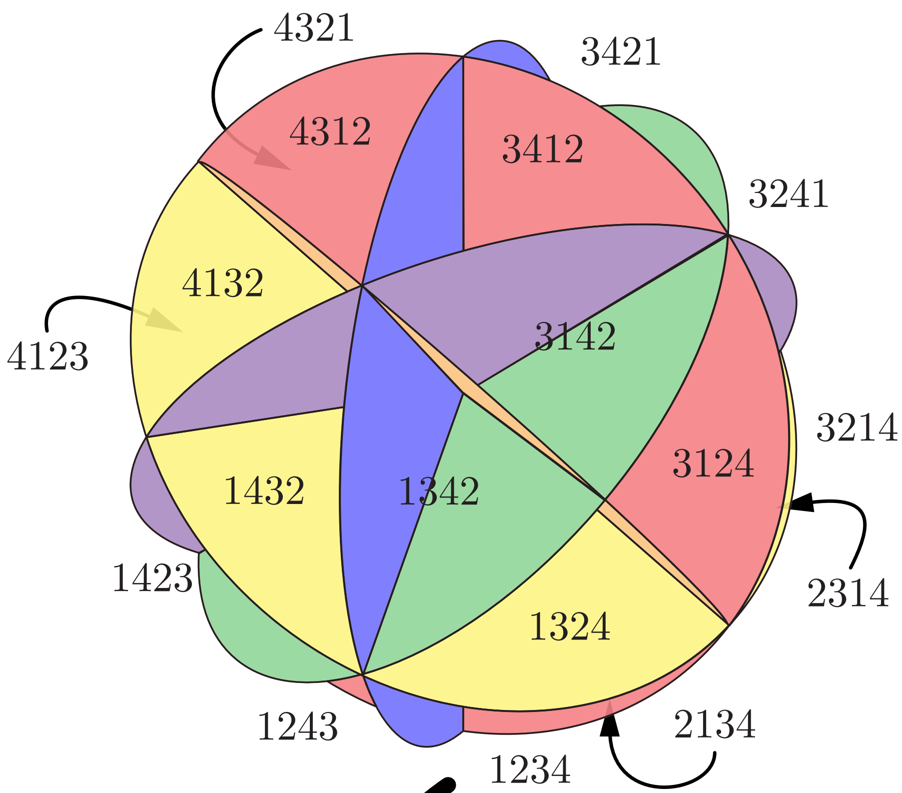
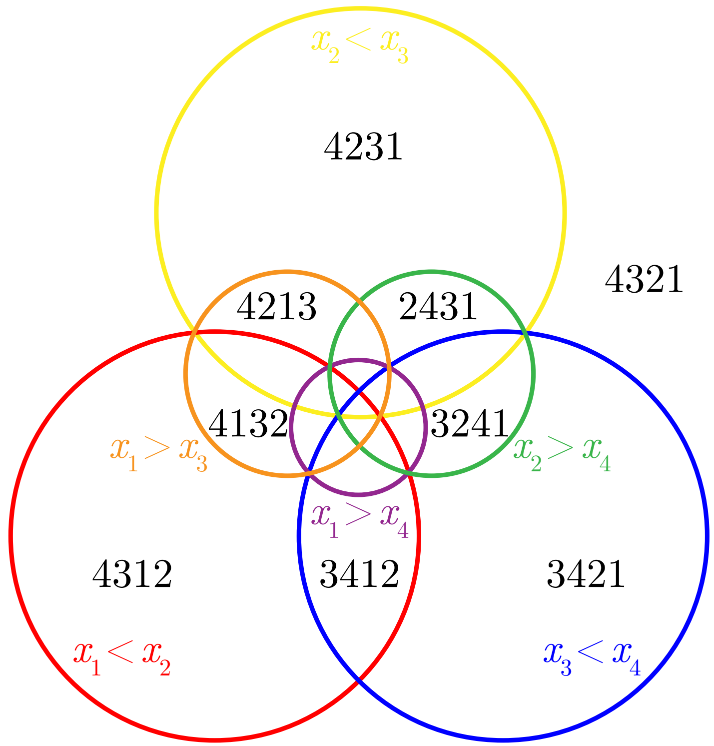
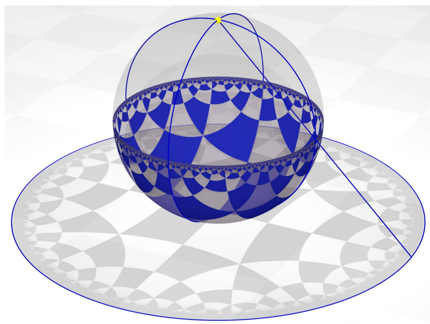




stereographic projection =



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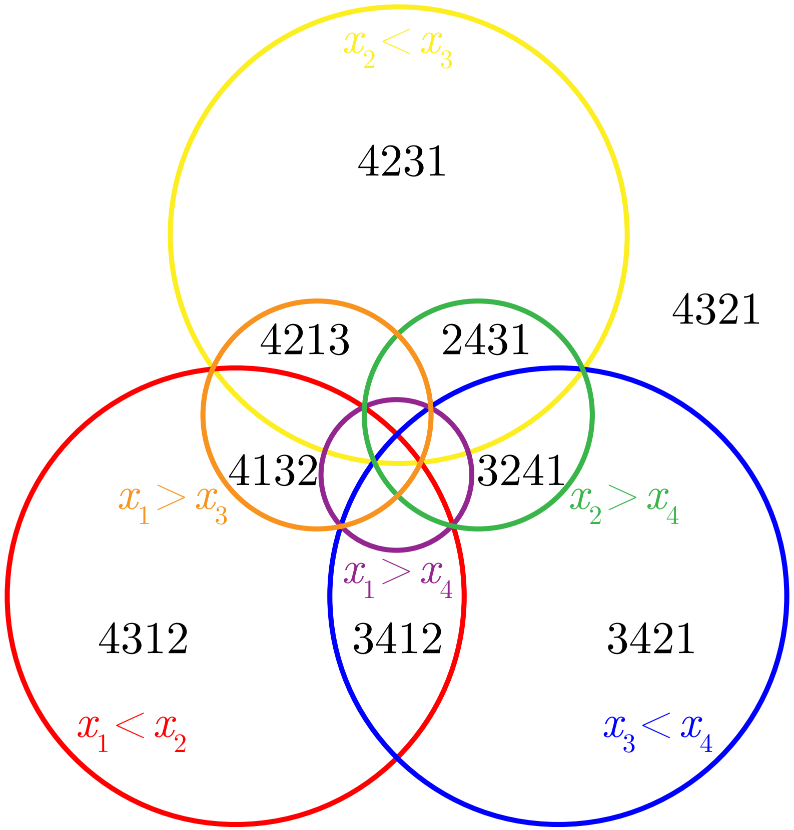

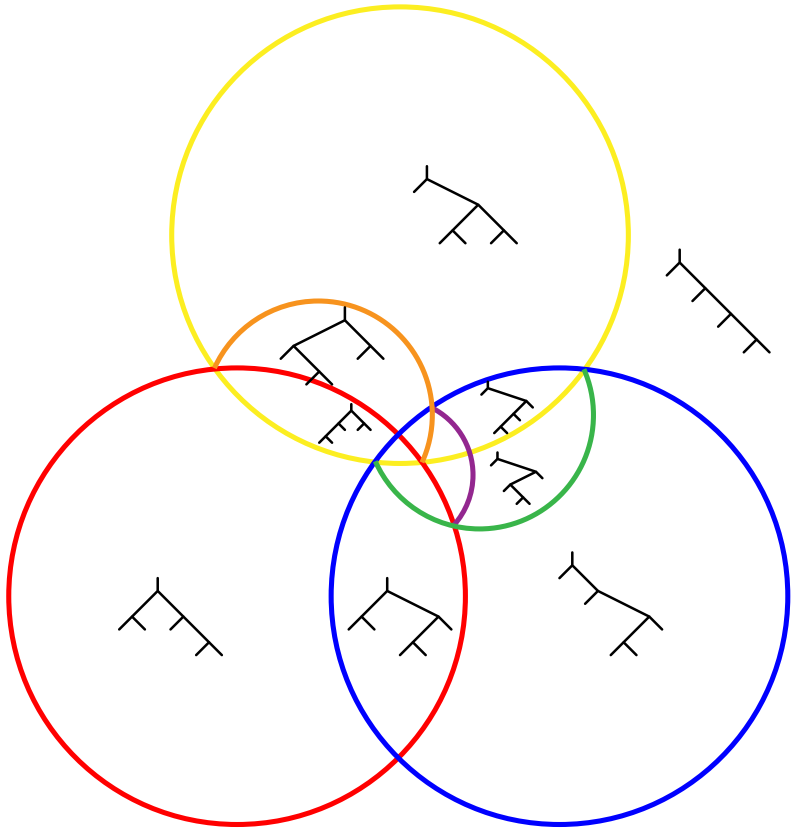

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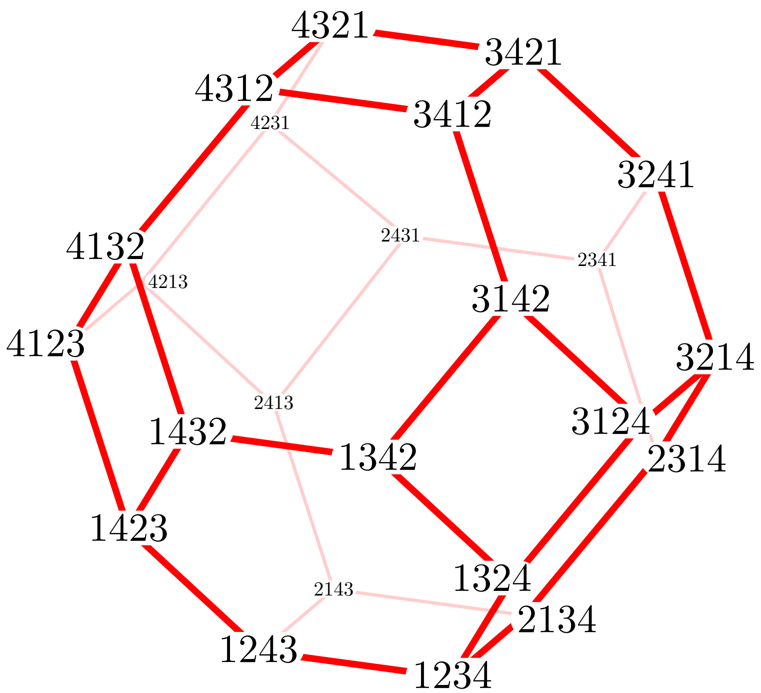

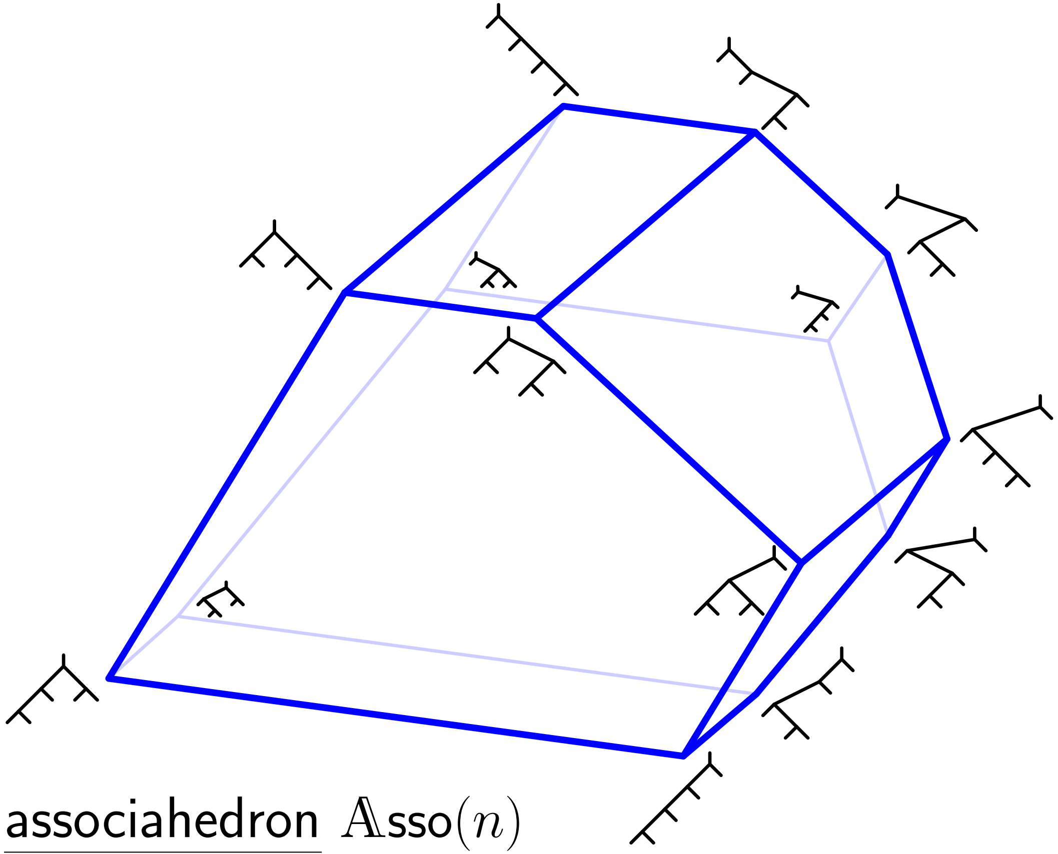

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$$= \mathrm{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \geq \binom{|J|+1}{2} \right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$$= \mathrm{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; \mathrm{T} \text{ binary tree} \right\}$$

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Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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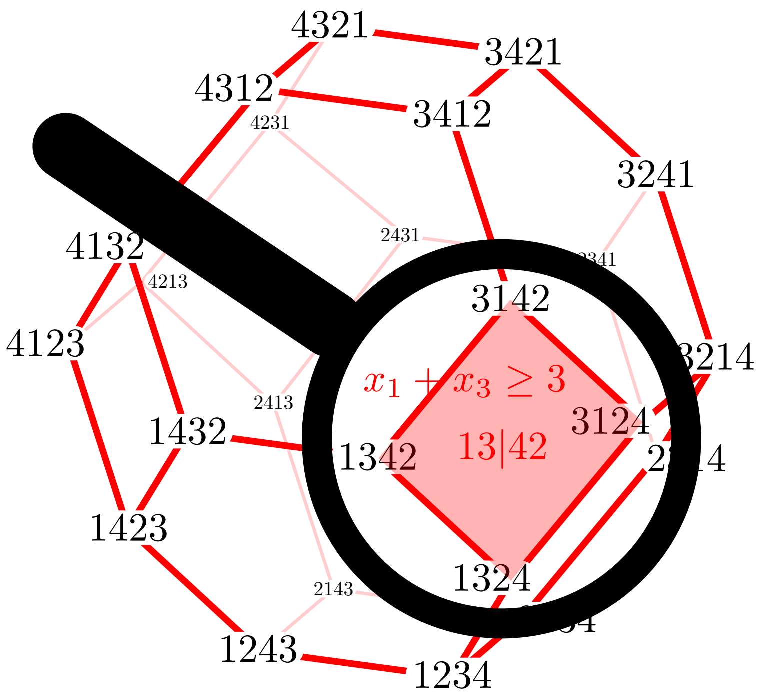

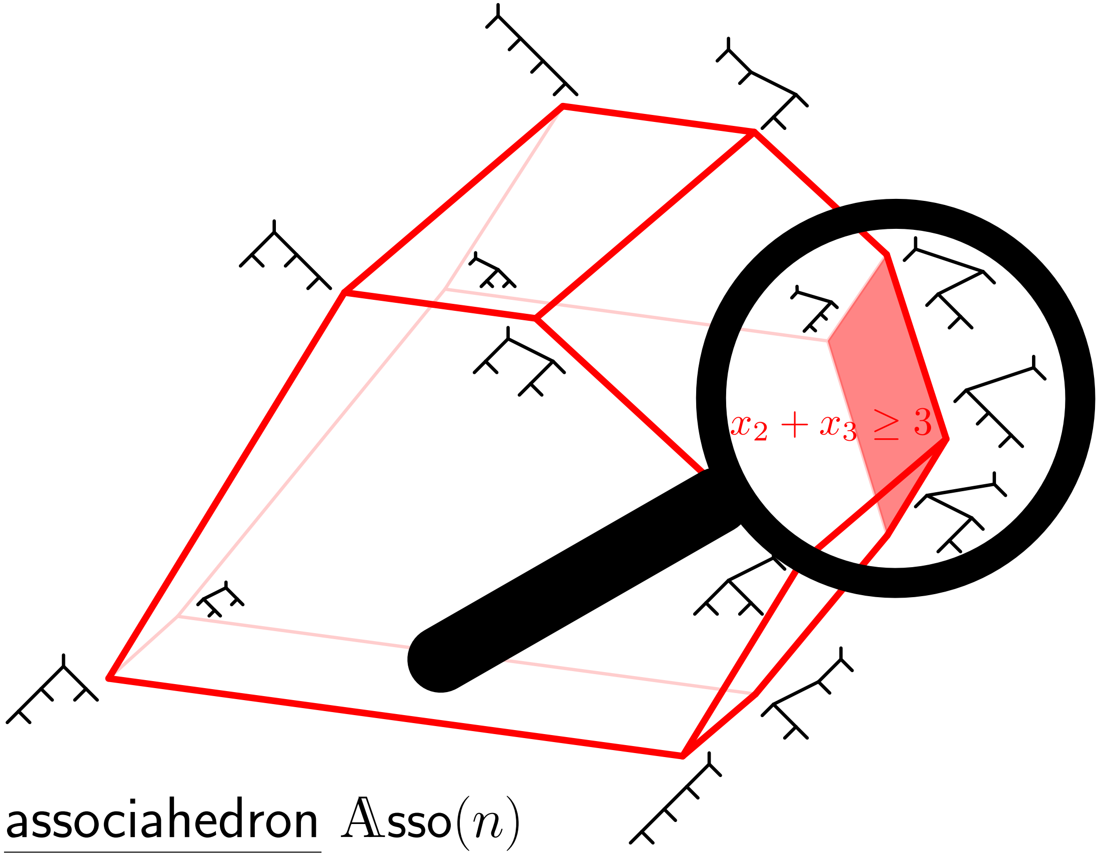

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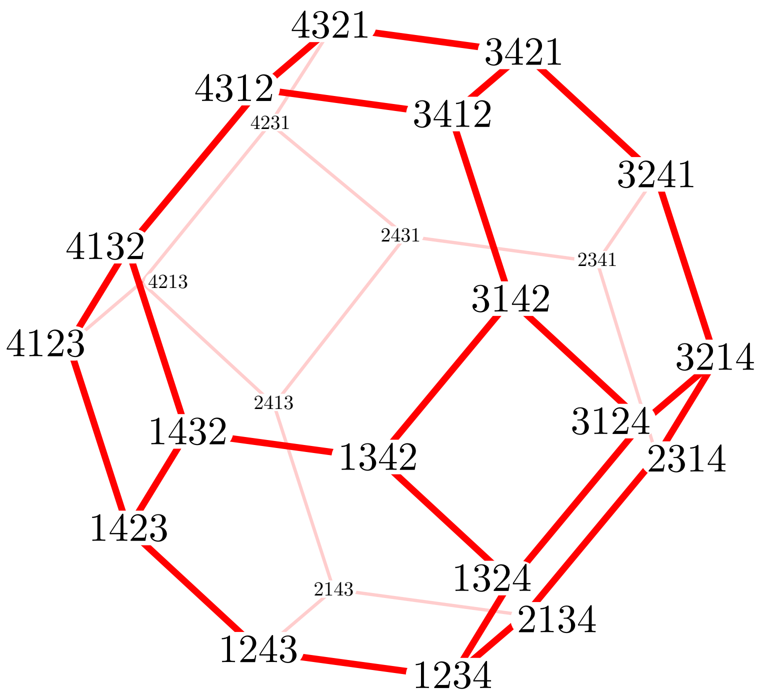

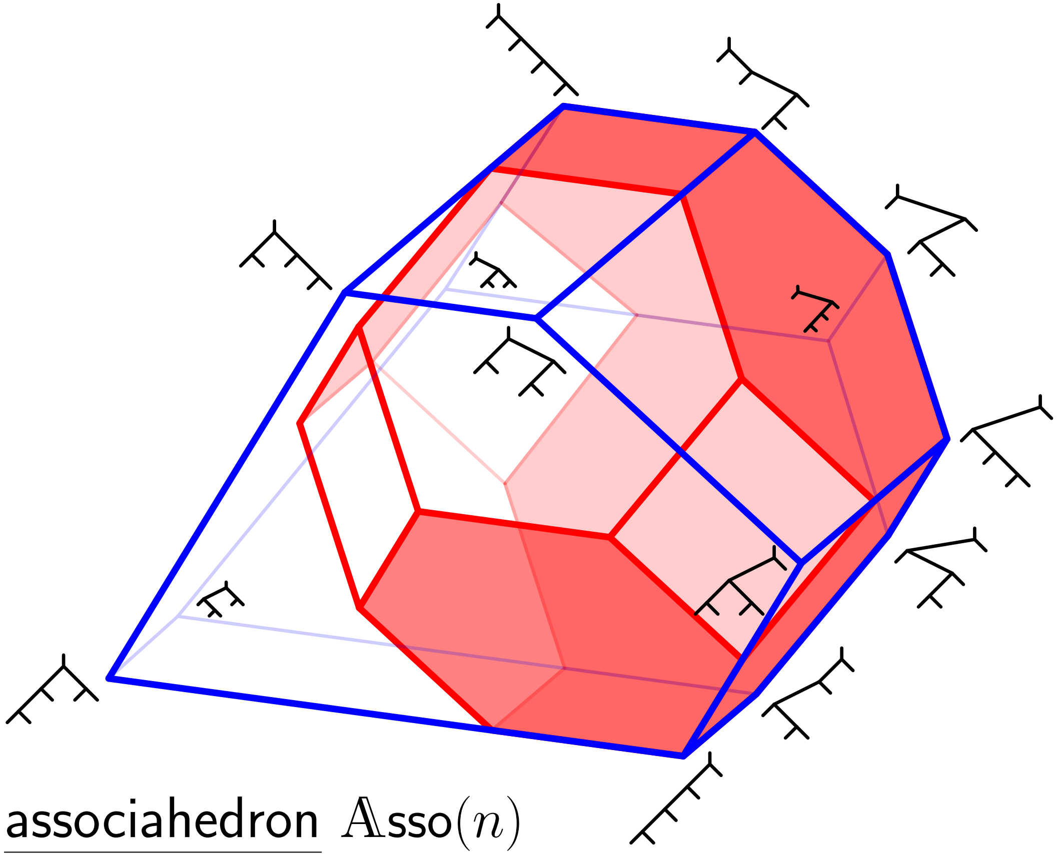

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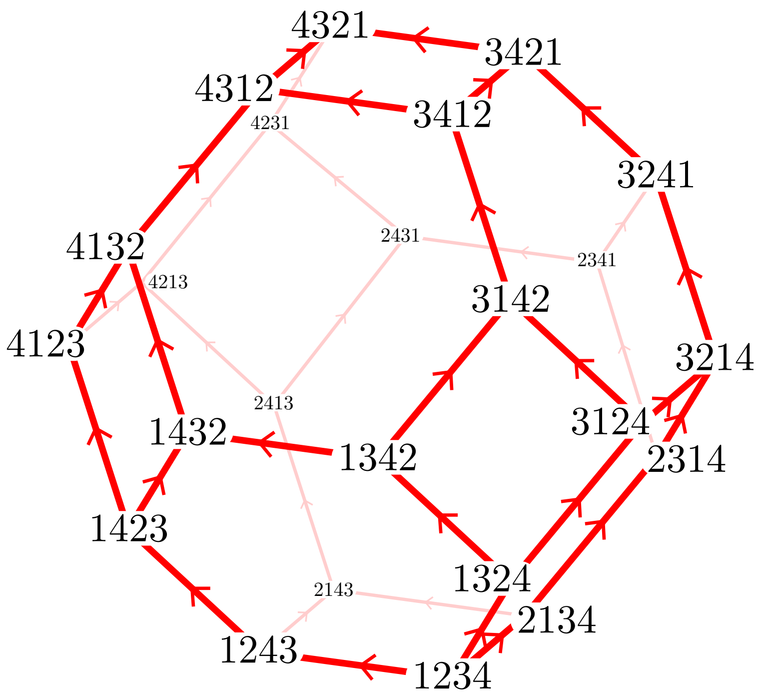

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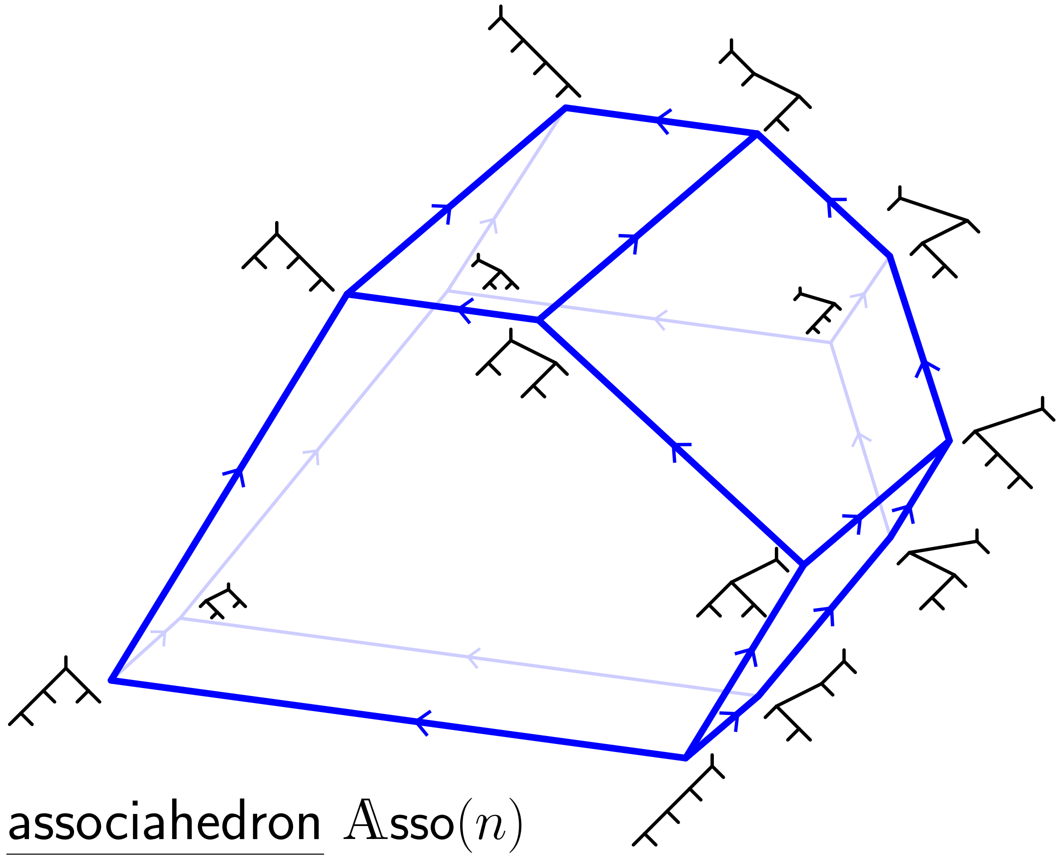
$\Longrightarrow$ weak order on permutations



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$\Longrightarrow$ Tamari lattice on binary trees

Hasse diagram of	weak order = graph of	permutahedron oriented	$12\ldots n \to n\ldots 21$
	Tamari lattice	associahedron	left $\to$ right comb

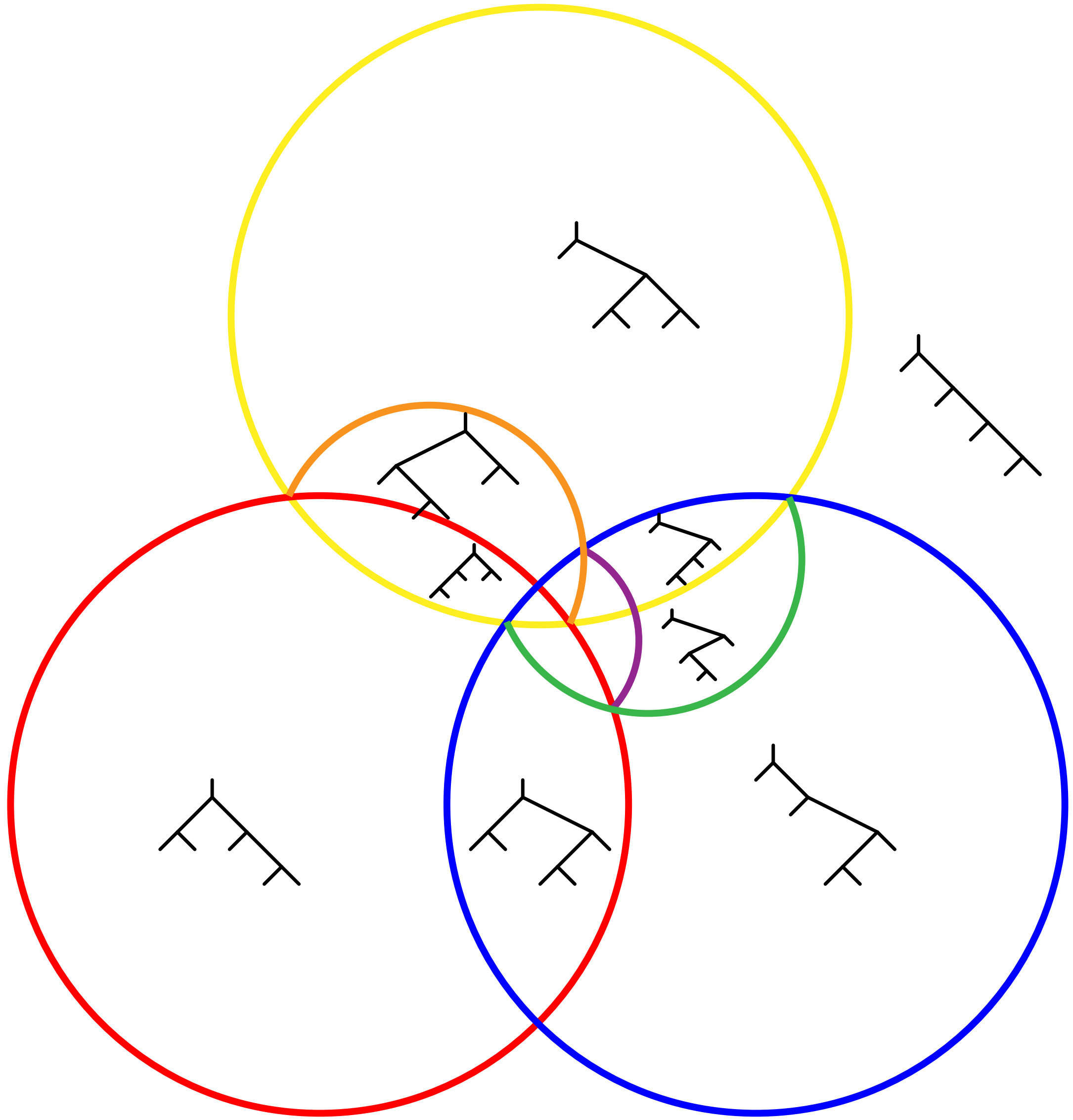
QUOTIENT FANS AND QUOTIENTOPES

QUOTIENT FAN

lattice congruence = equivalence relation on L compatible with meets and joins:

$$x \equiv x' \text{ and } y \equiv y' \text{ implies } x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

quotient fan $\mathcal{F}_{\equiv}$ = chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations σ in the same congruence class of $\equiv$ Reading ('05)



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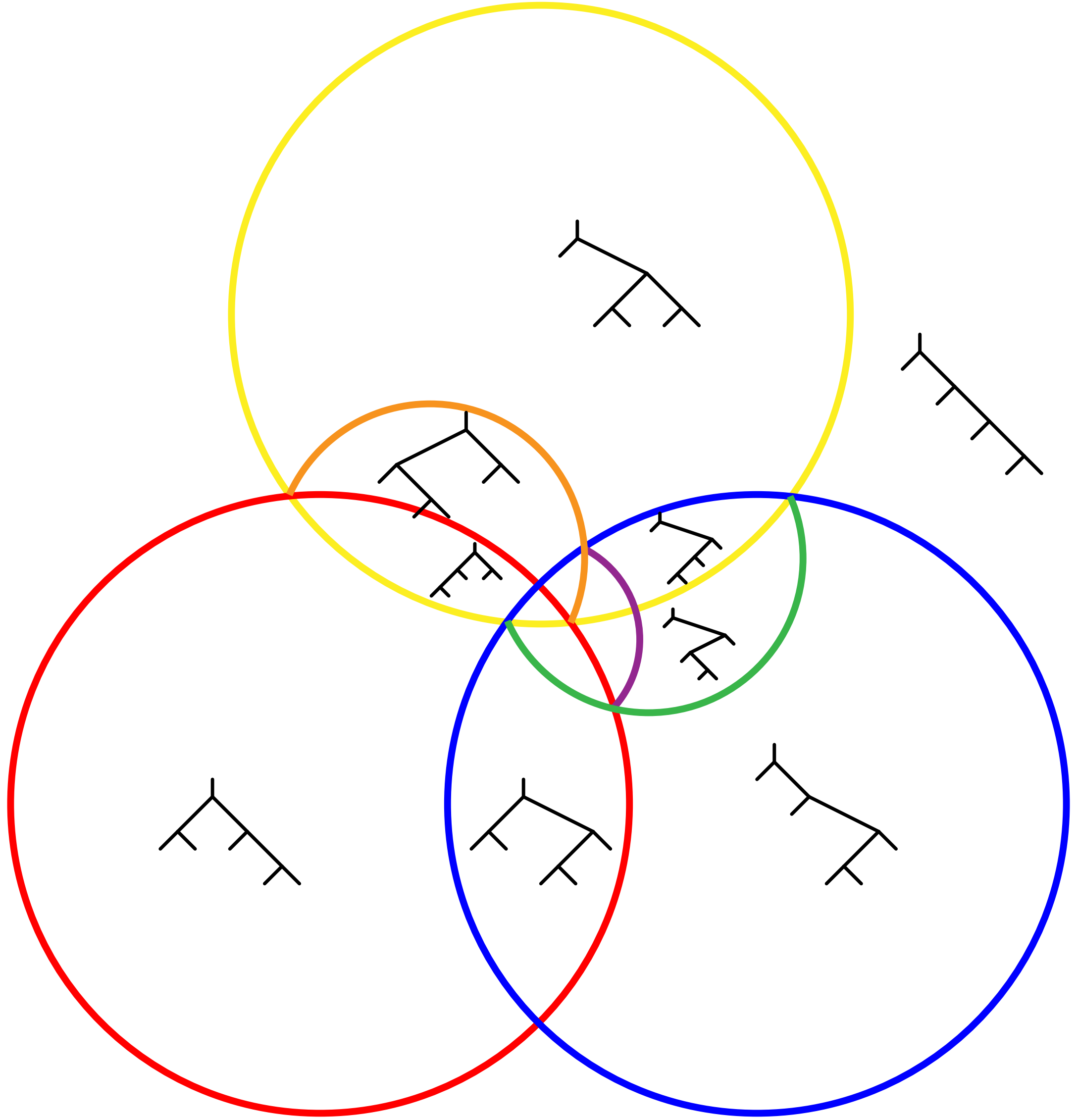
$\boldsymbol{W}_\equiv$ = walls of the quotient fan $\mathcal{F}_\equiv$

Describe the possible sets of walls $\boldsymbol{W}_\equiv$



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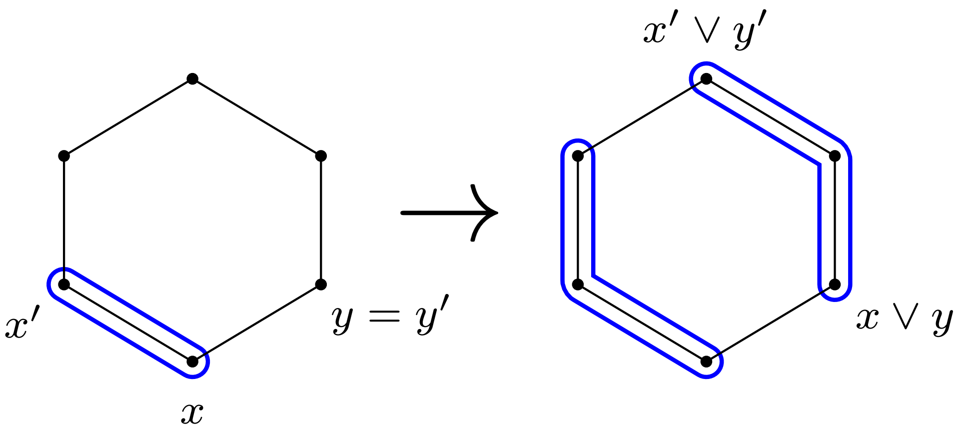
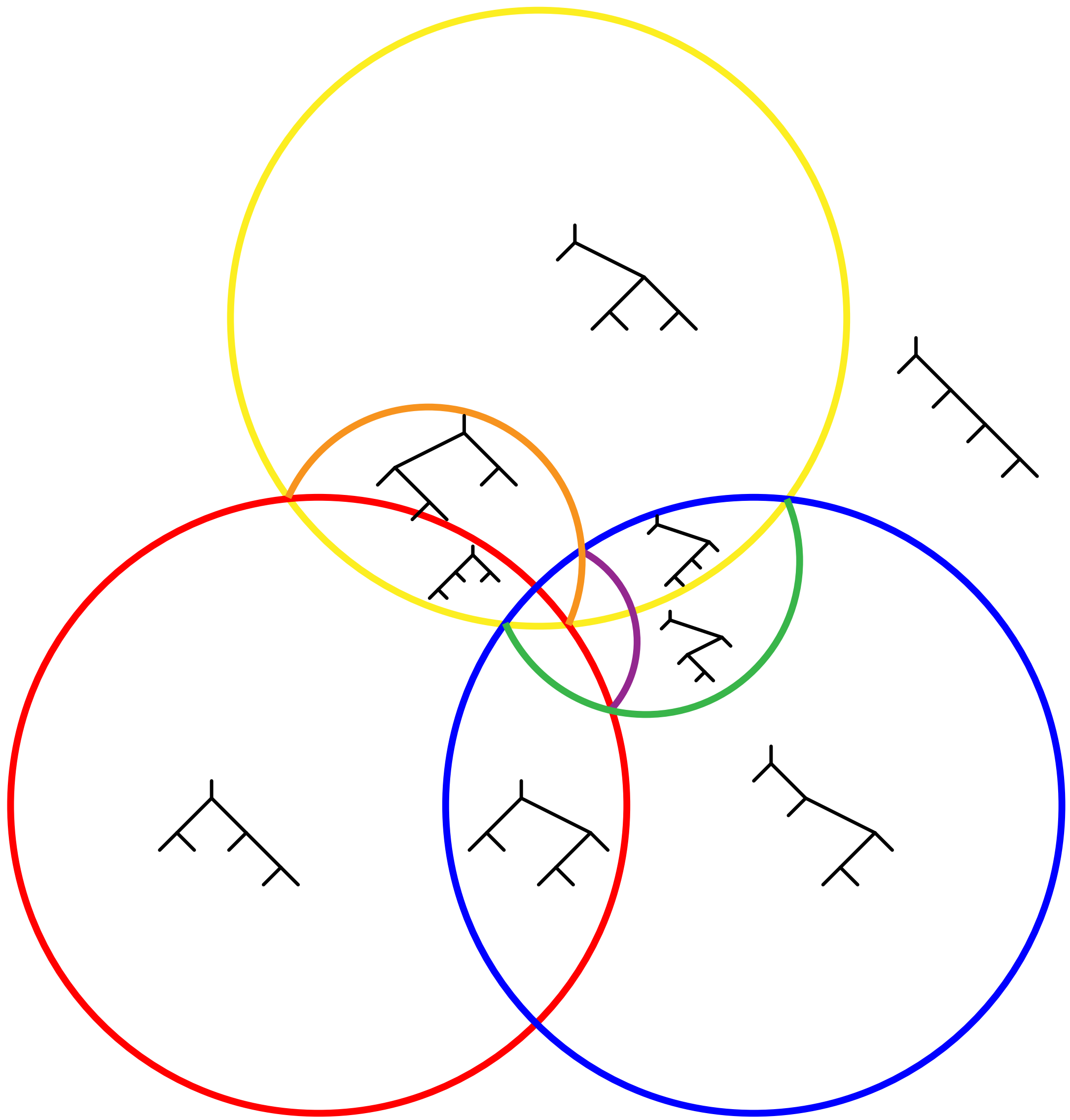
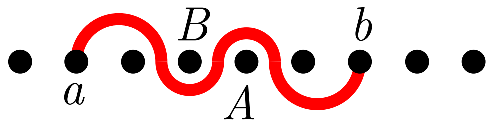


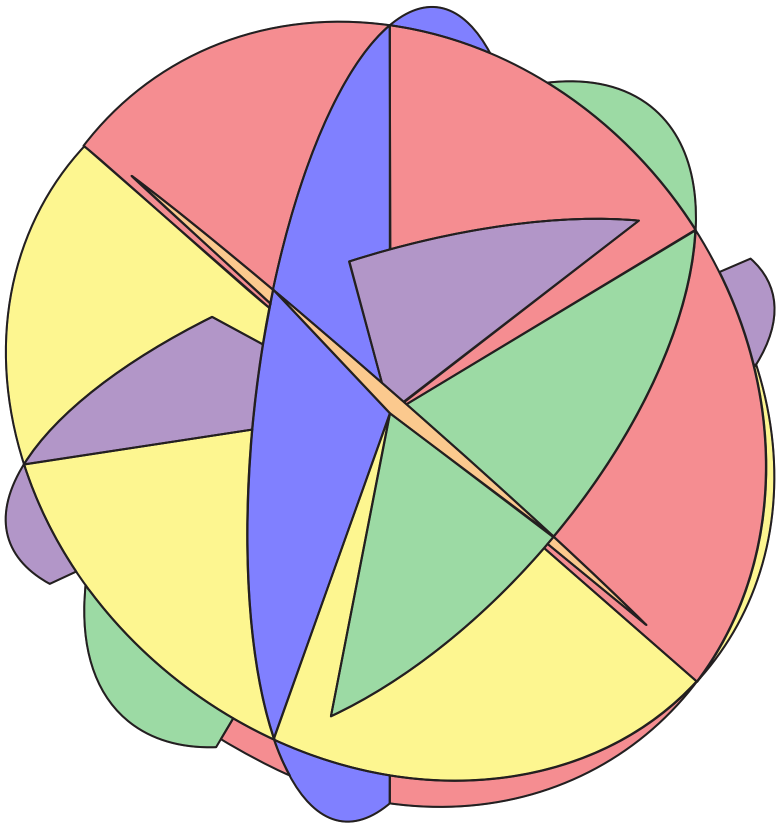


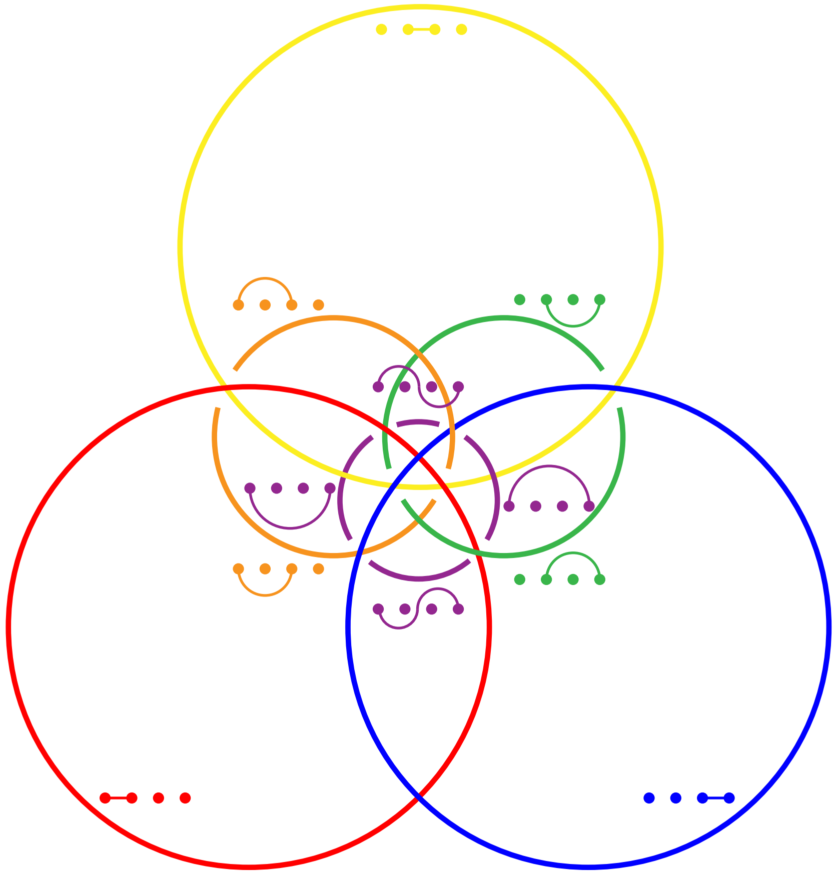
ARCS AND SHARDS

arc (a, b, A, B) with $1 \leq a < b \leq n$ and $A \sqcup B = \,]a, b[$



shard $\Sigma(a, b, A, B) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{a'} \leq x_a = x_b \leq x_{b'} \text{ for all } a' \in A \text{ and } b' \in B \right\}$





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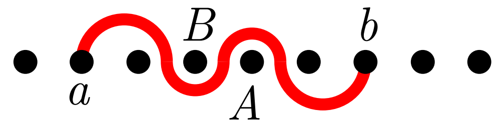
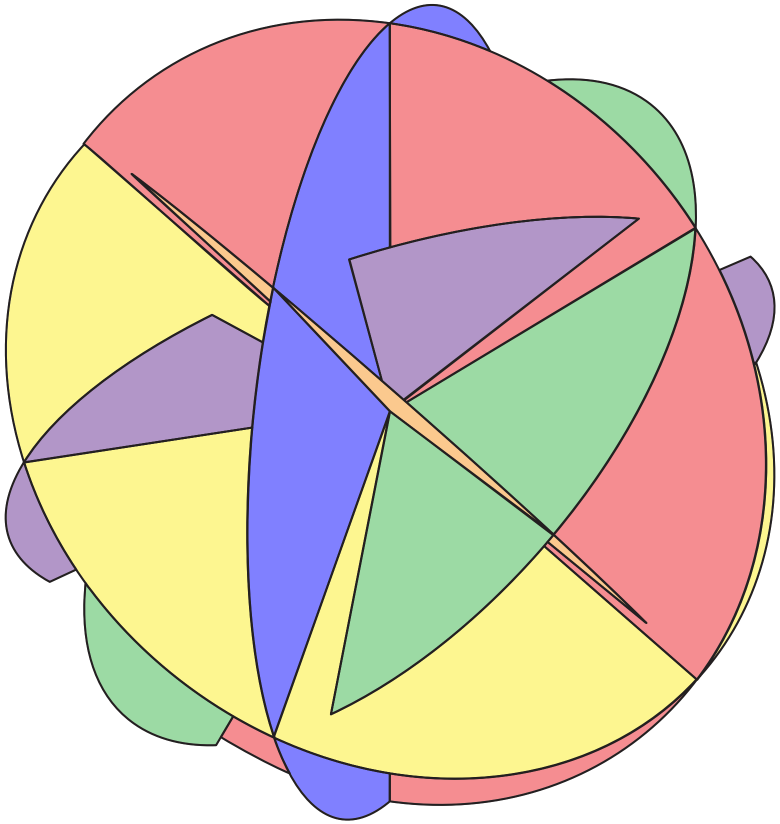
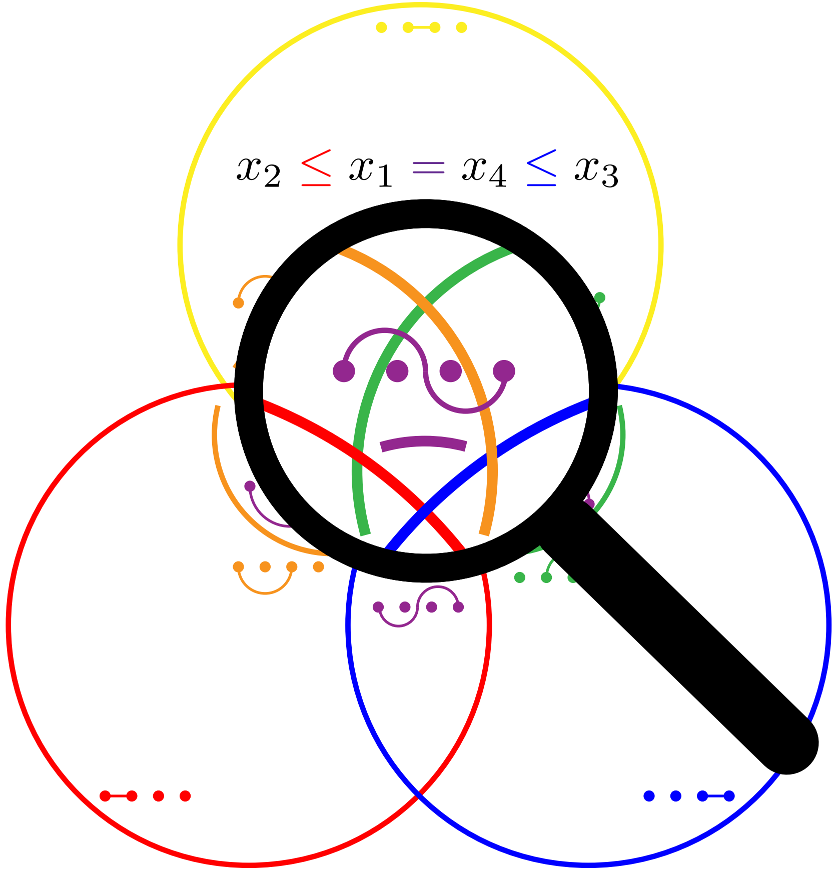


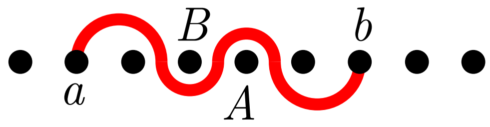
shard $\Sigma(a, b, A, B) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_{a'} \leq x_a = x_b \leq x_{b'} \text{ for all } a' \in A \text{ and } b' \in B\}$





ARCS AND SHARDS

<u>arc</u> (a, b, A, B) with $1 \leq a < b \leq n$ and $A \sqcup B = \,]a, b[$



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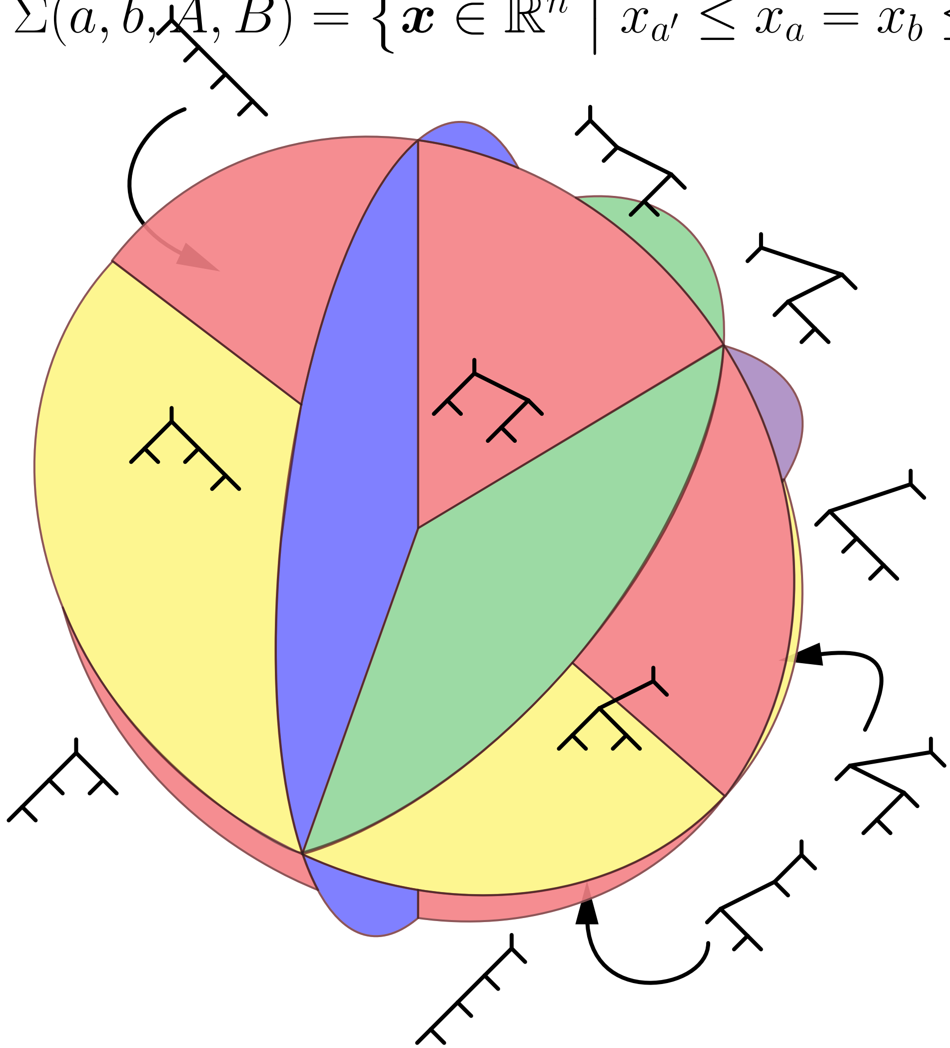
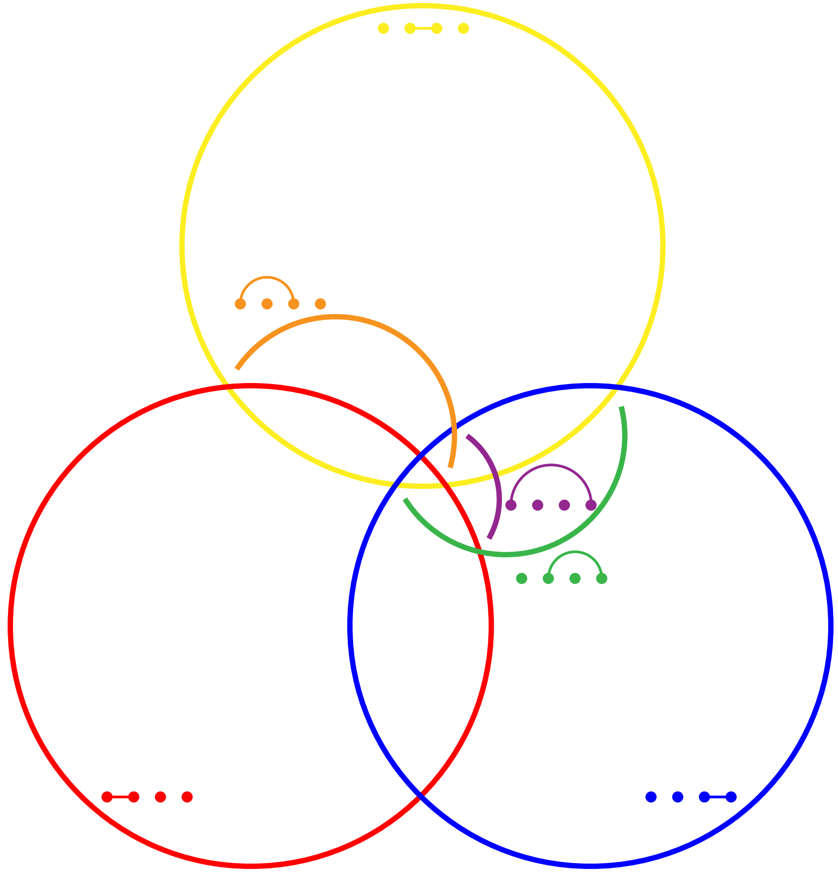
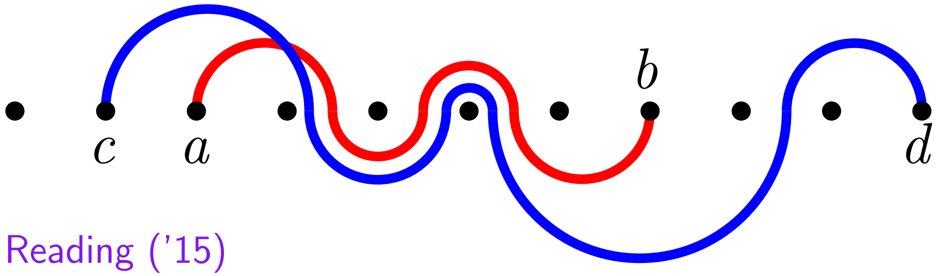




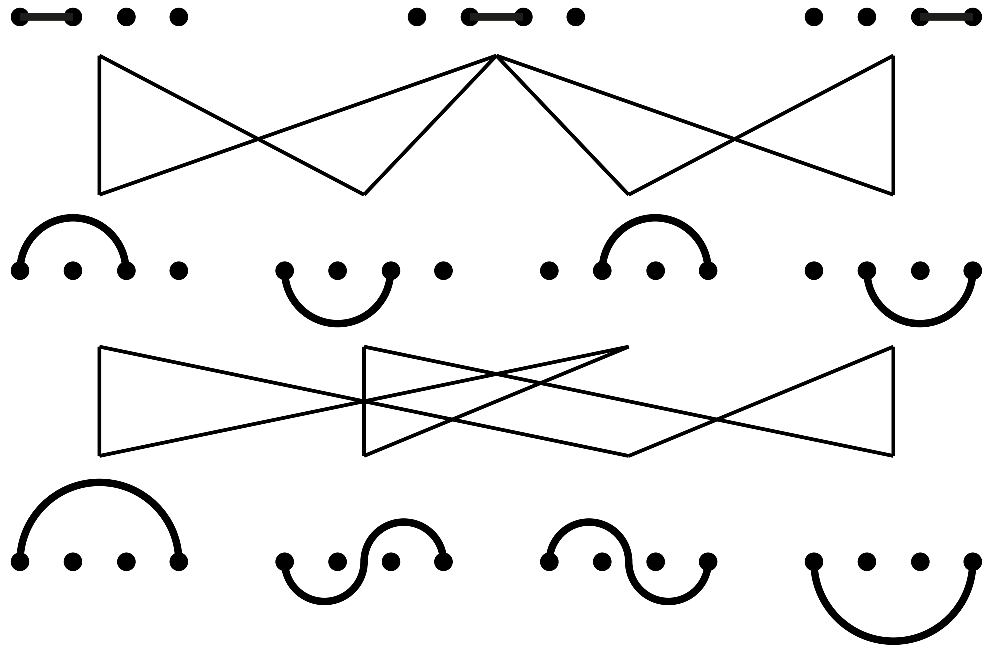
The set of walls $\boldsymbol{W}_{\equiv}$ of the quotient fan $\mathcal{F}_{\equiv}$ is a union of shards $\boldsymbol{\Sigma}_{\equiv}$ Reading ('05)

FORCING

$\Sigma(a,b,A,B)$ <u>forces</u> $\Sigma(c,d,C,D)$ =
$c \leq a < b \leq d$ and $A \subseteq C$ and $B \subseteq D$



Reading ('15)



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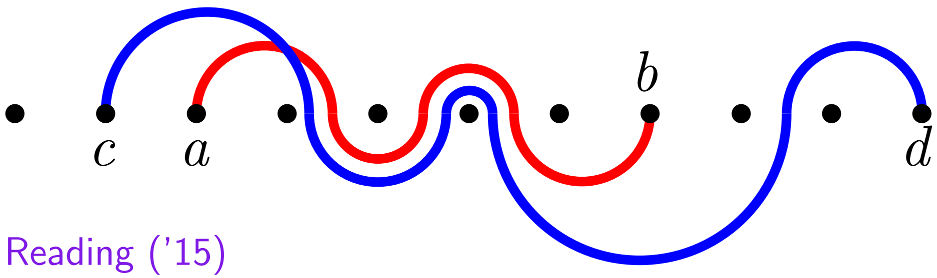

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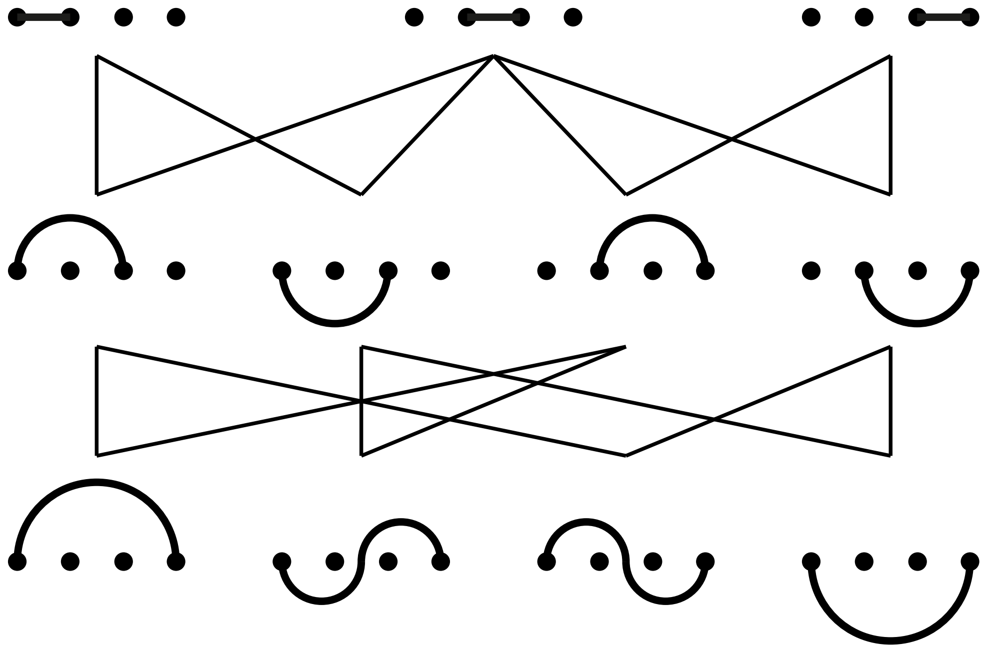
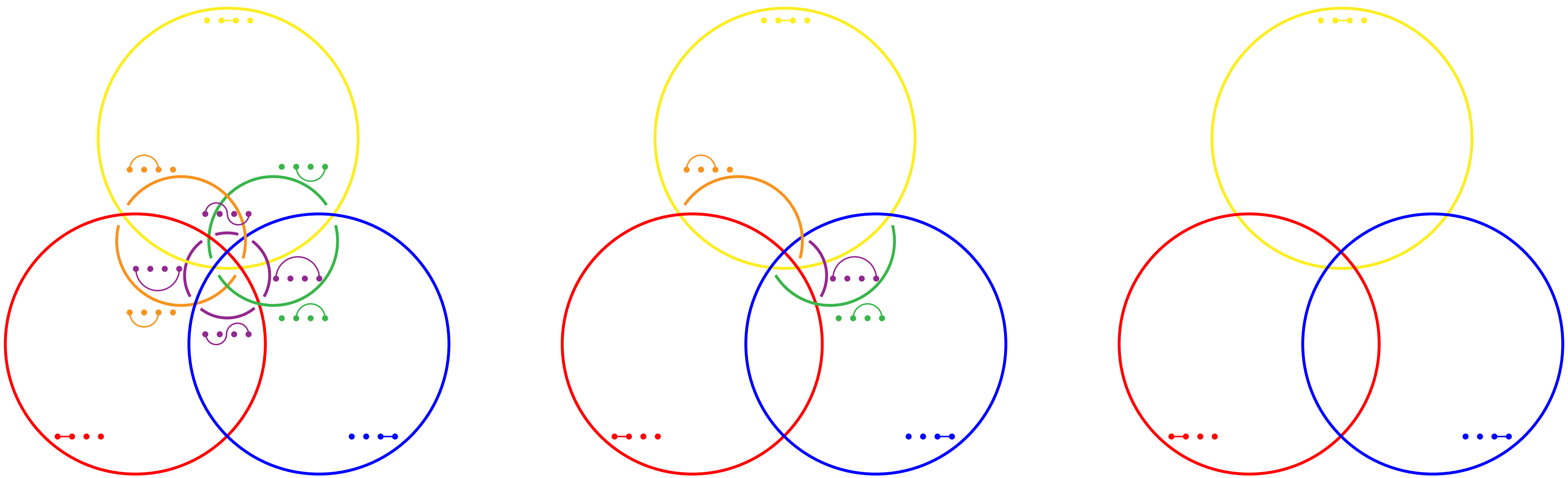


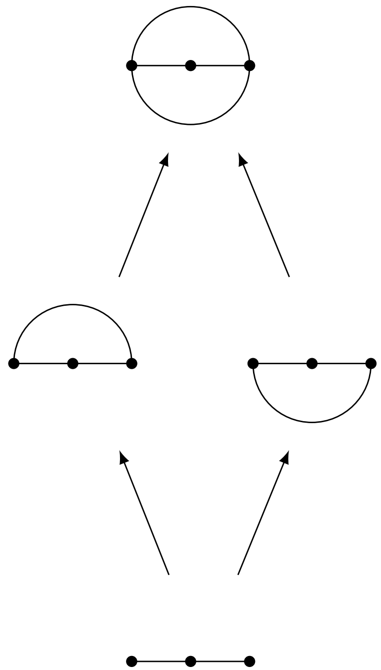
TFAE for a set of shards $\boldsymbol{\Sigma}$:

- there is a congruence $\equiv$ with $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\equiv}$
- $\boldsymbol{\Sigma}$ is an upper ideal in forcing order



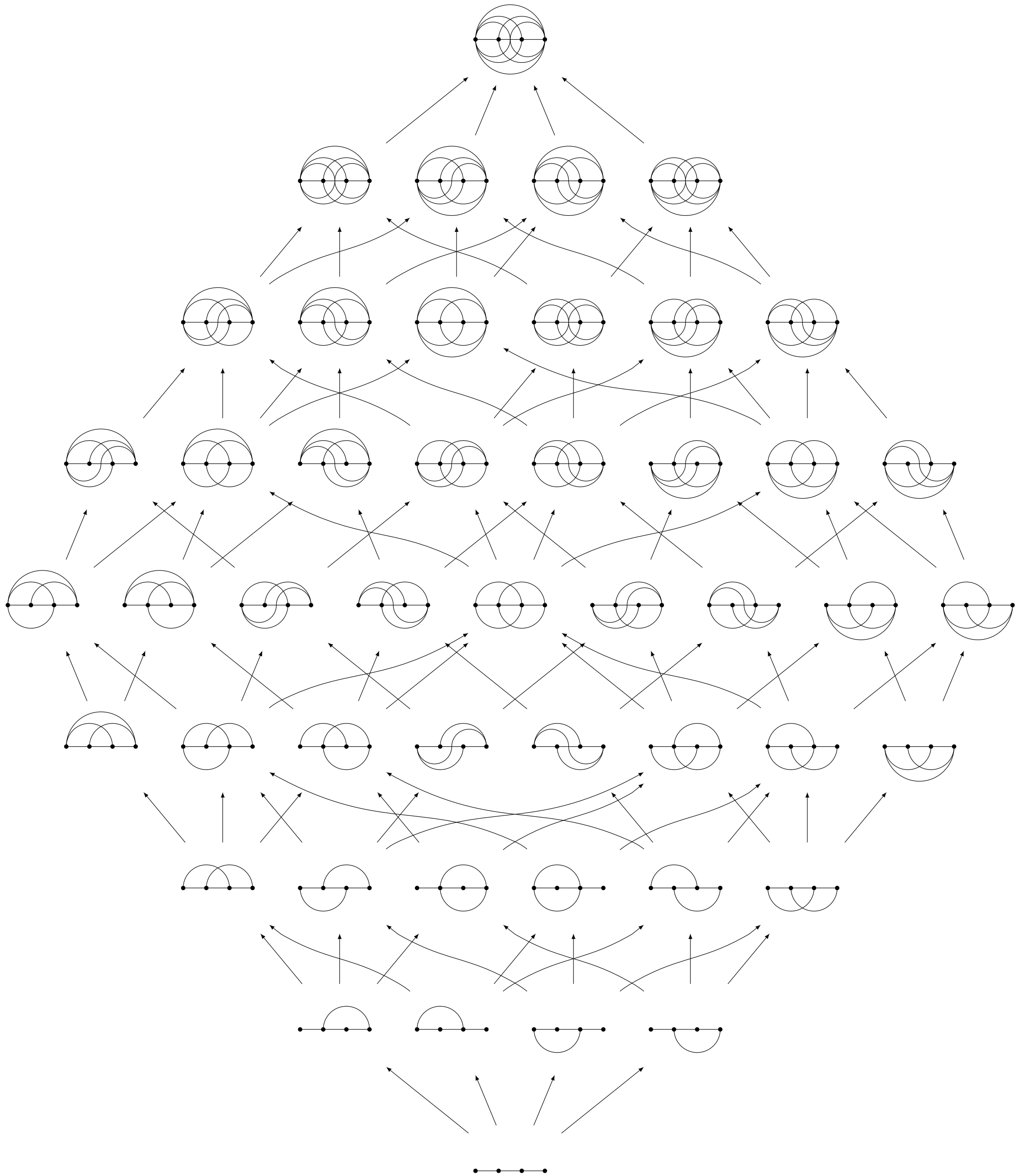
SHARD IDEALS

shard ideal = upper ideal in forcing order



essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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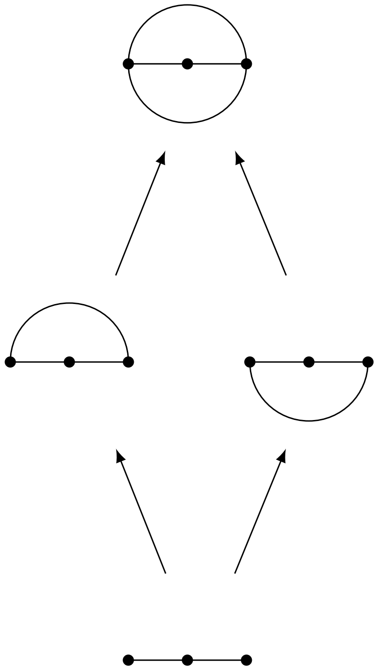

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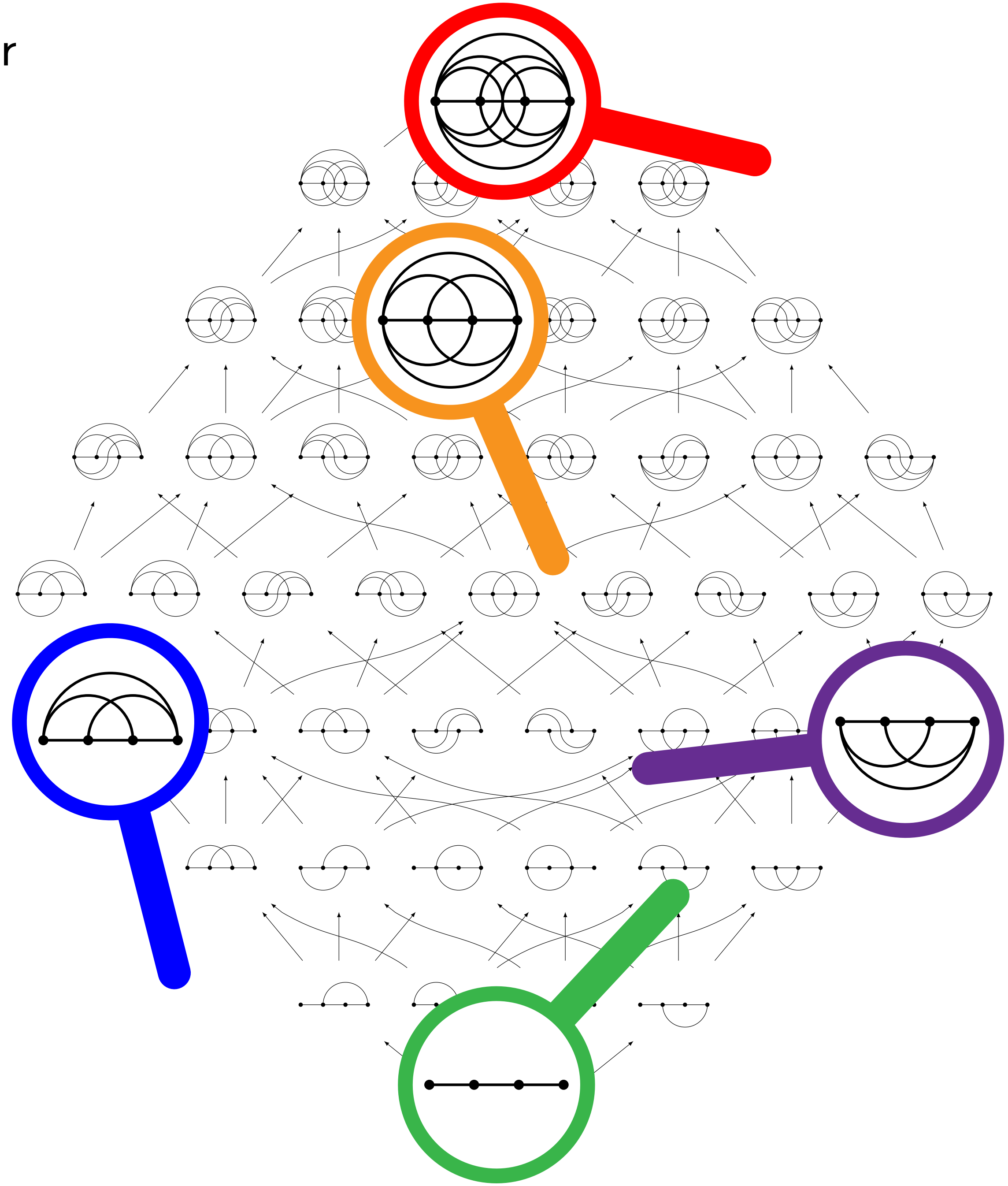
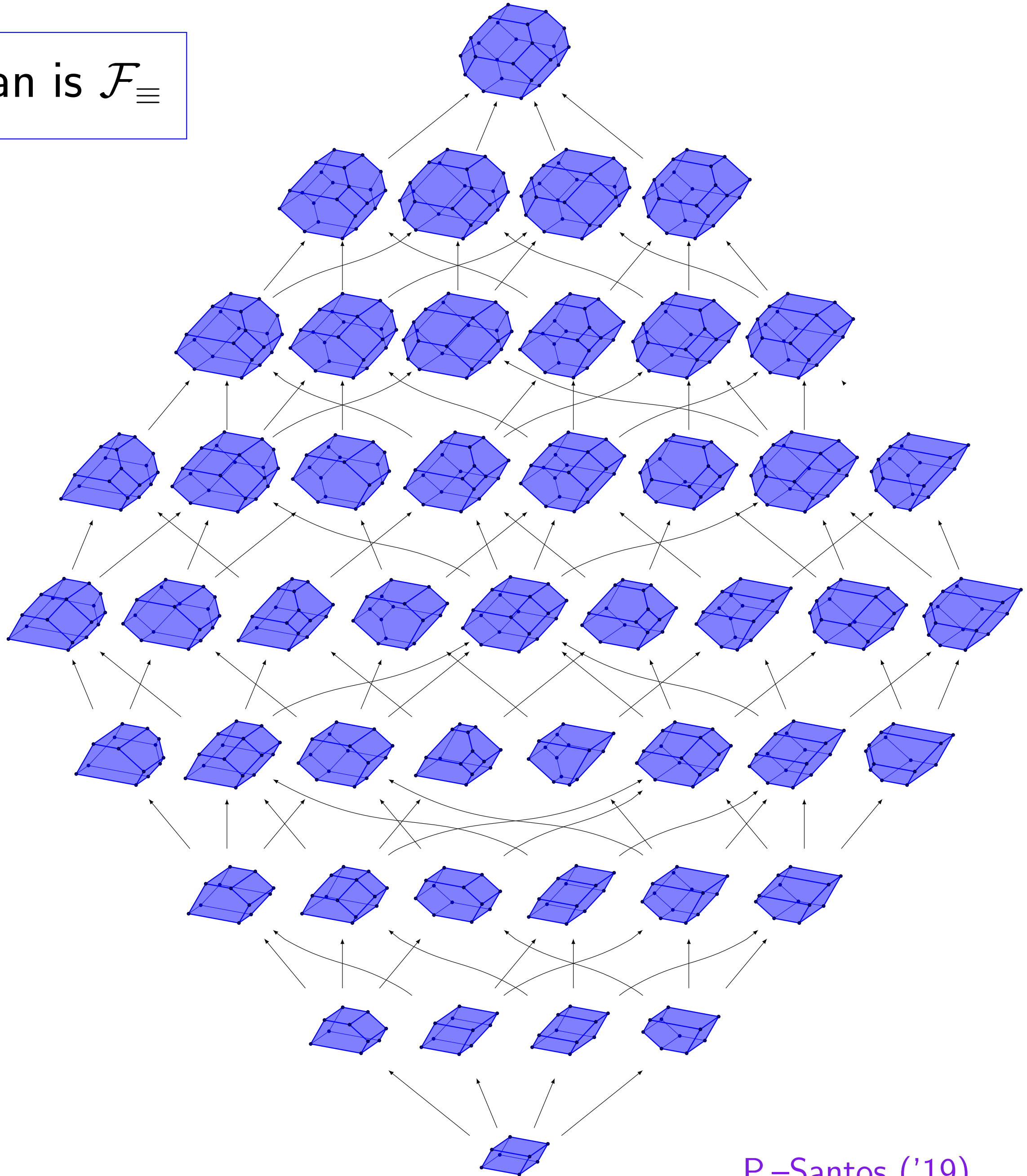


QUOTIENTOPES

quotientope = polytope whose normal fan is $\mathcal{F}_{\equiv}$



P.–Santos ('19)

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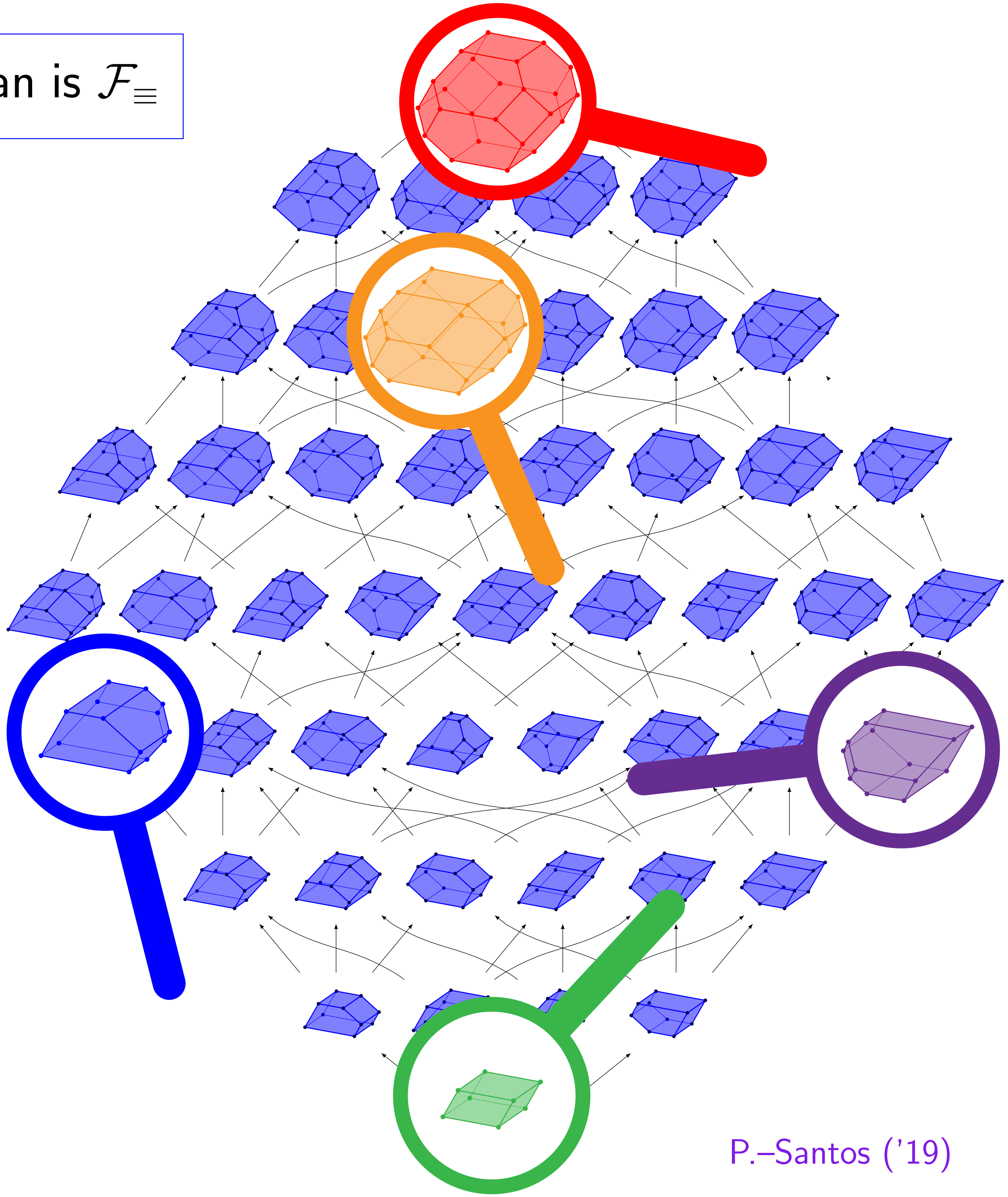

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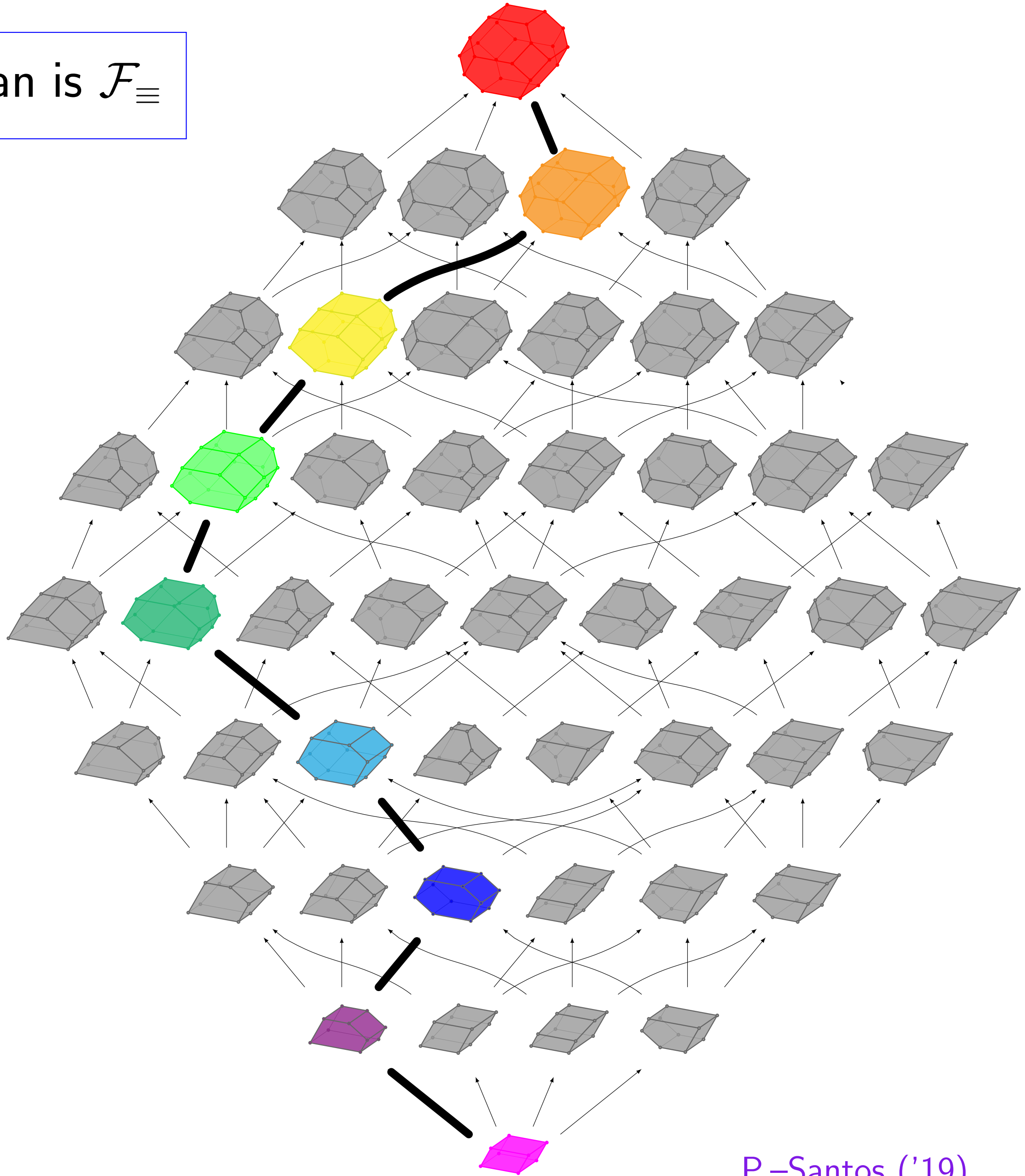
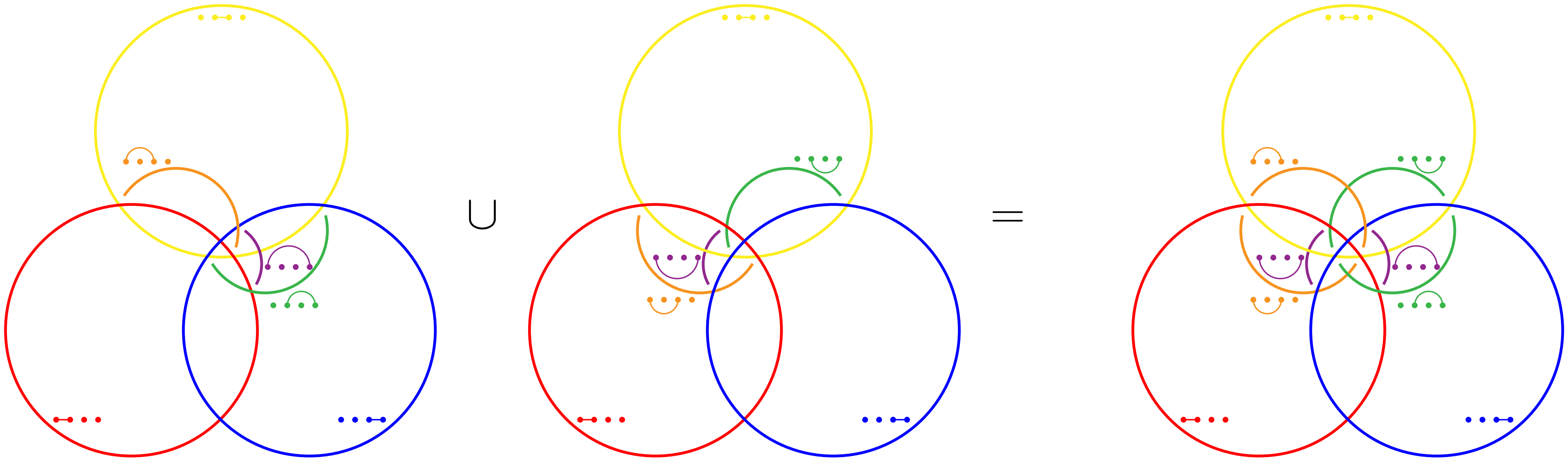


P.–Santos ('19)

MINKOWSKI SUMS OF ASSOCIAHEDRA

INTERSECTIONS OF CONGRUENCES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
then the quotient fan $\mathcal{F}_{\equiv}$ is the common refinement of the quotient fans $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$



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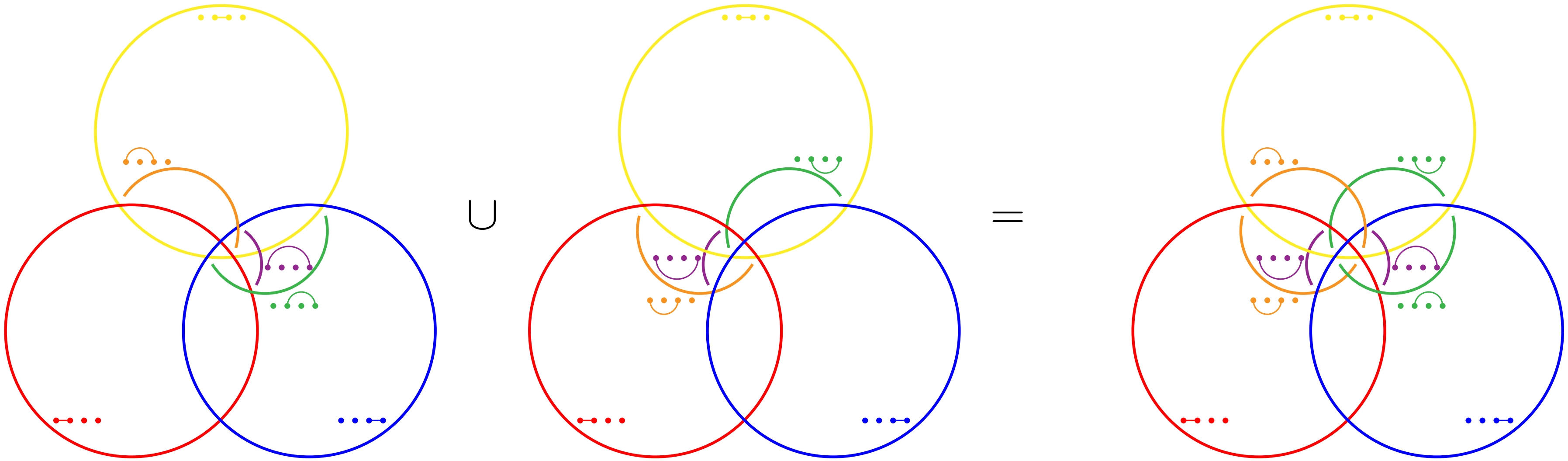
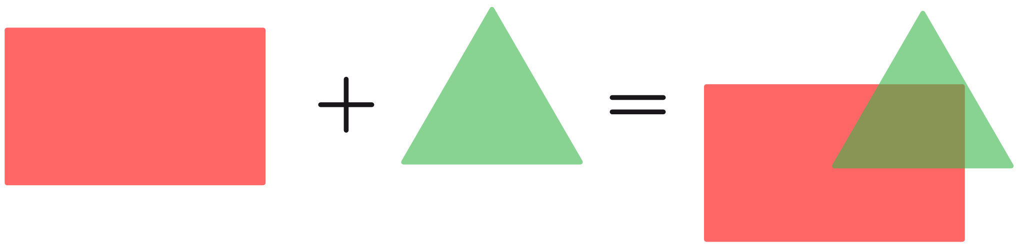


Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$



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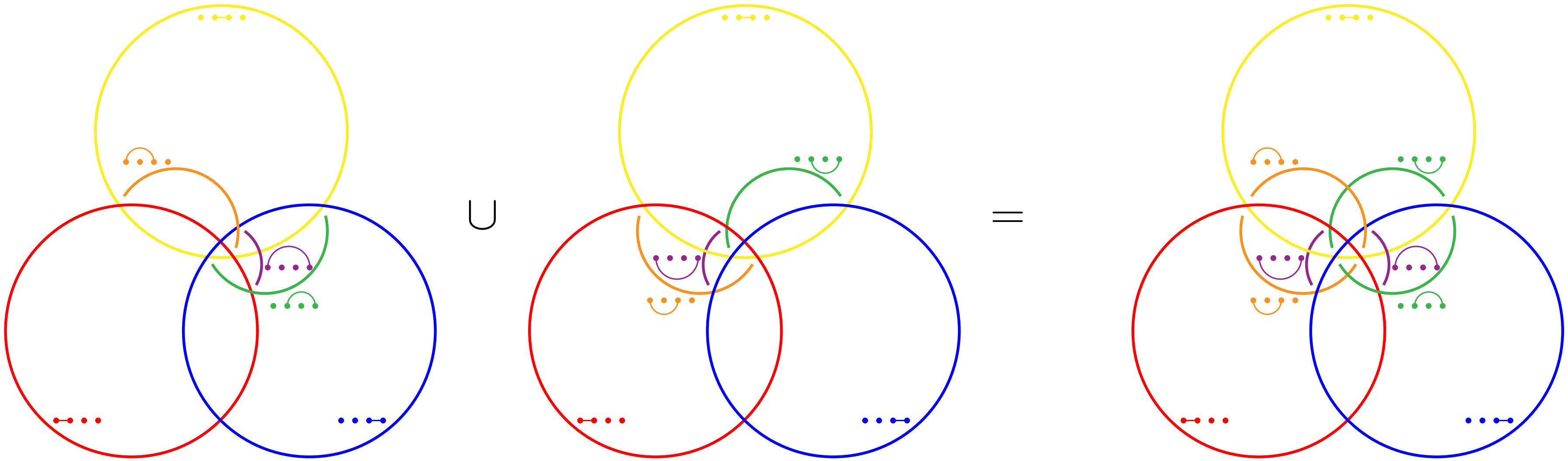

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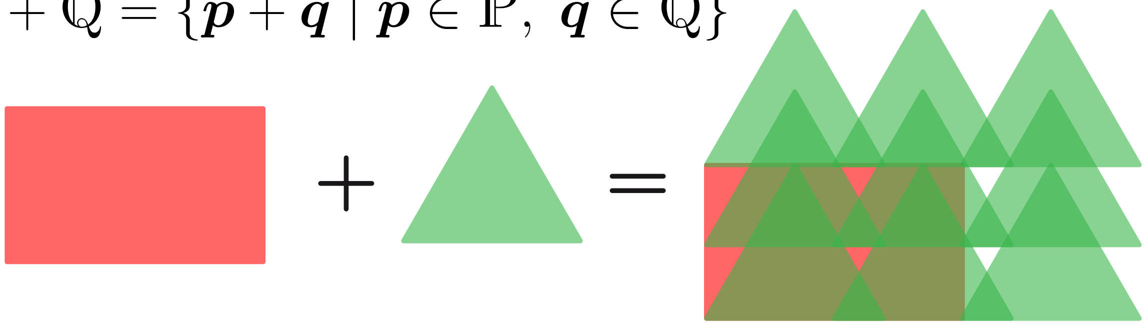

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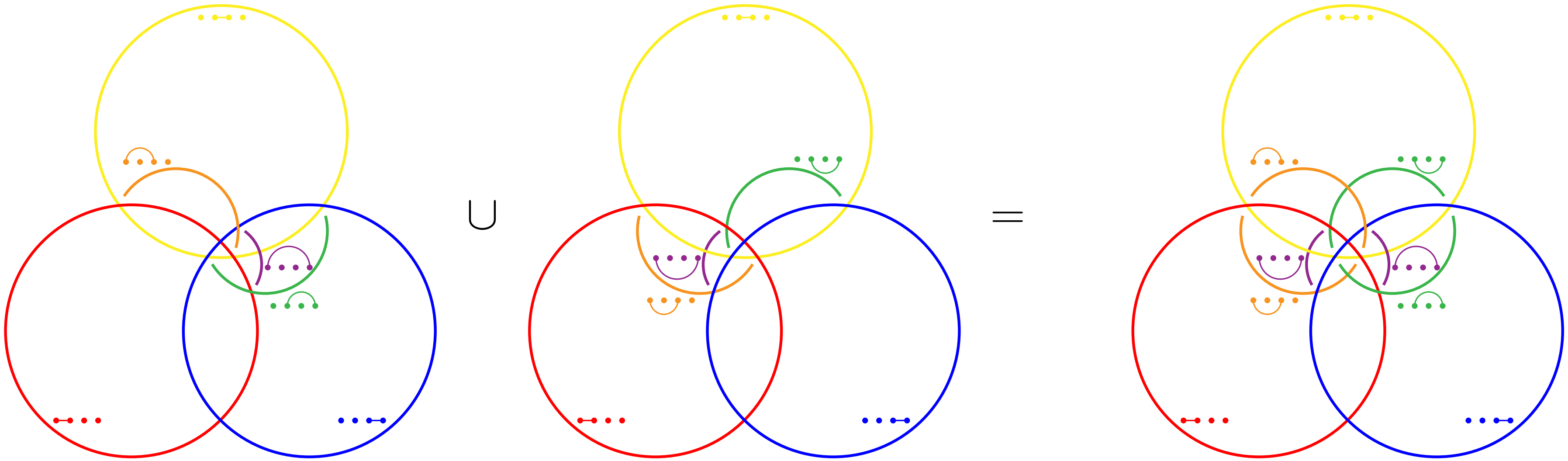

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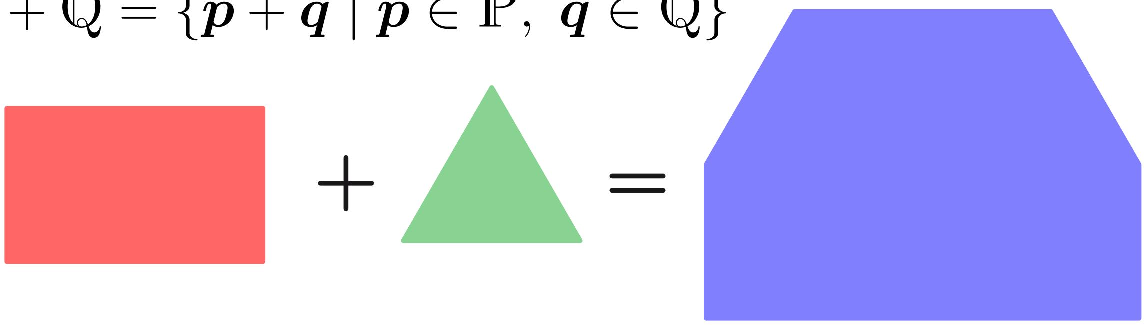

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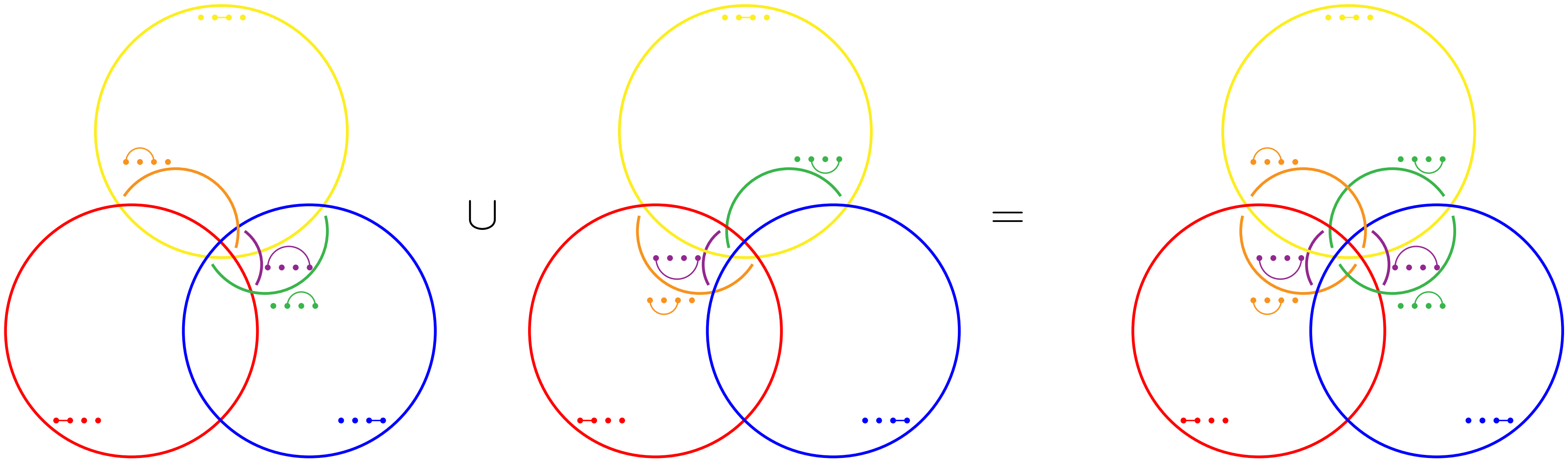

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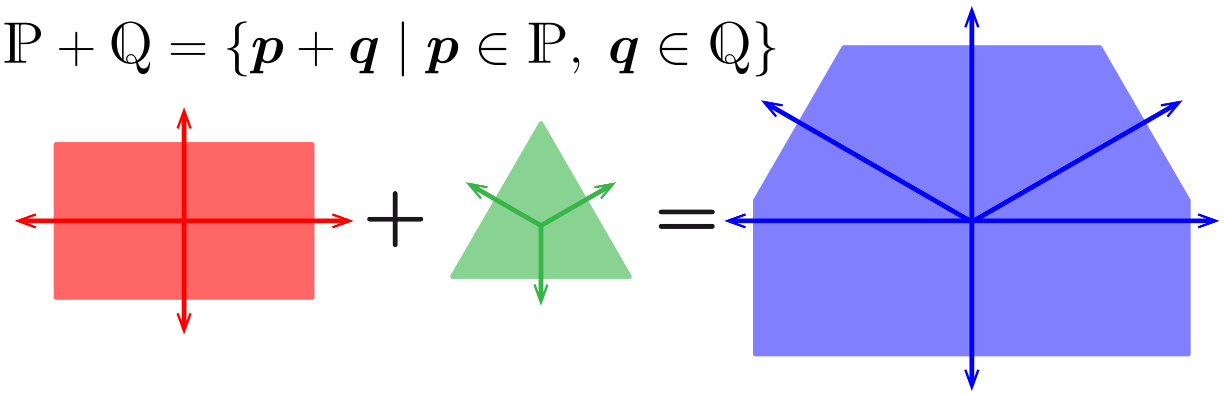

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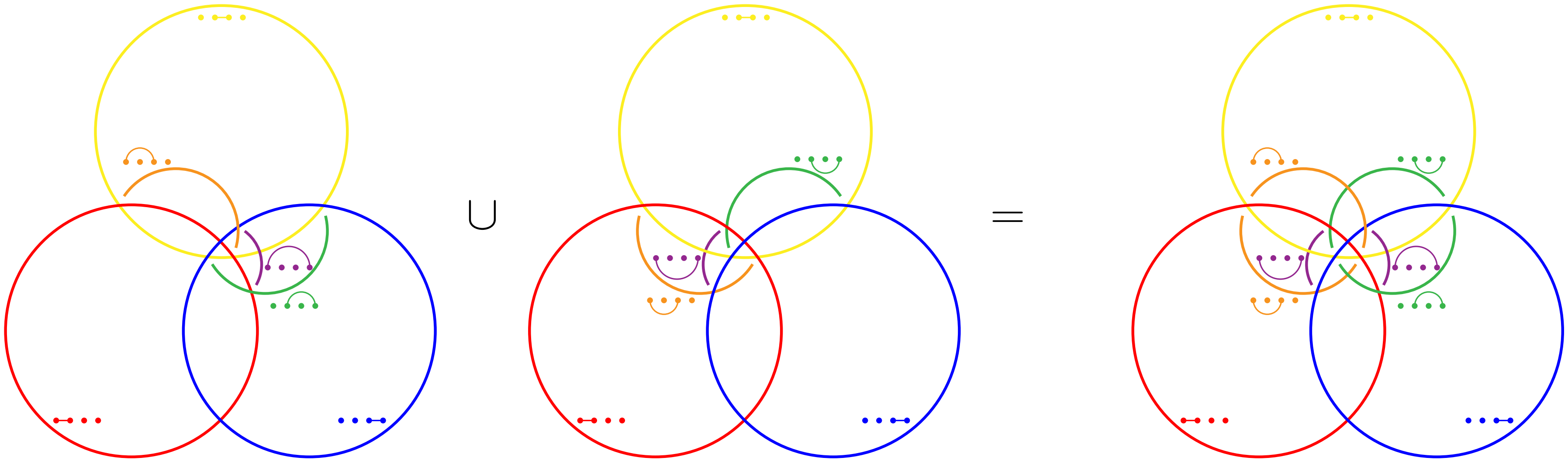

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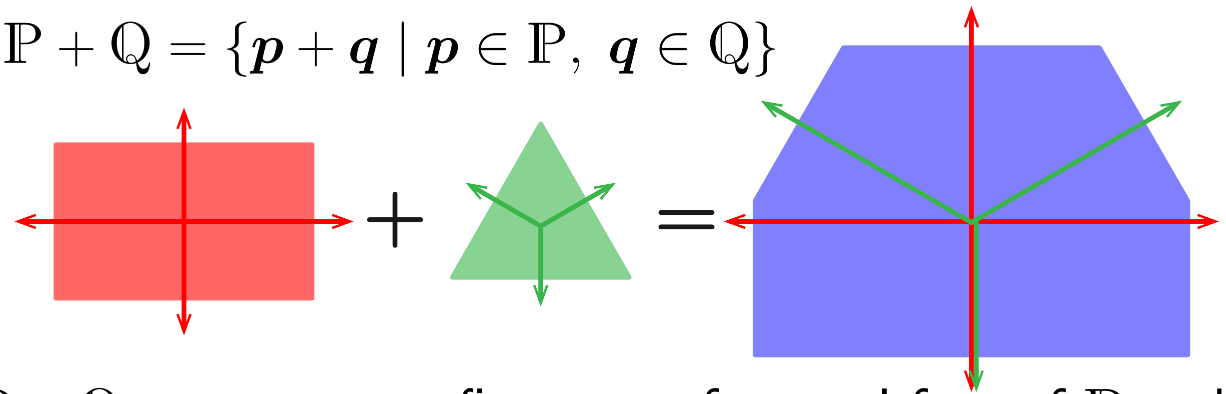

Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$

MINKOWSKI SUMS OF QUOTIENTOPES

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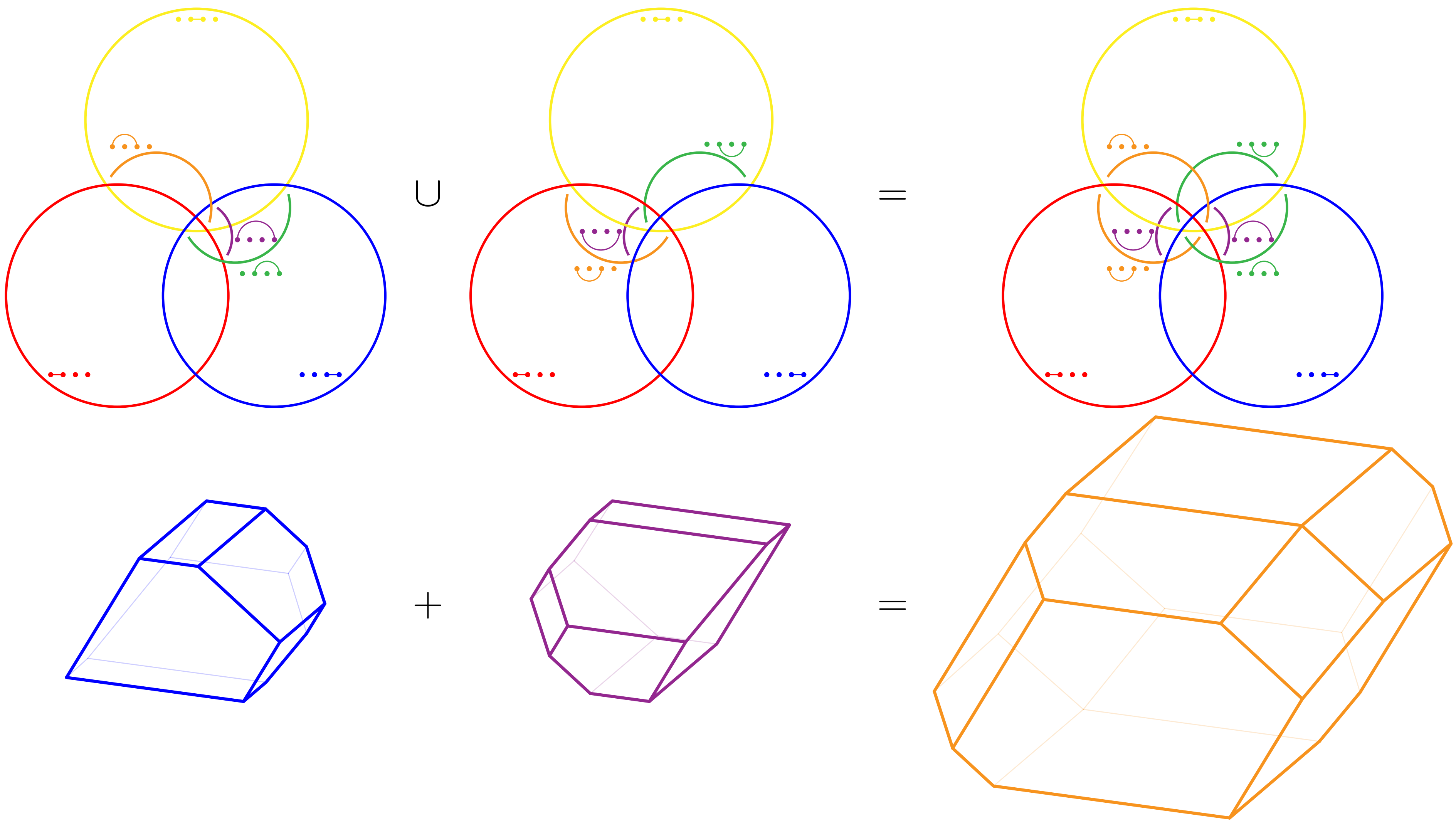

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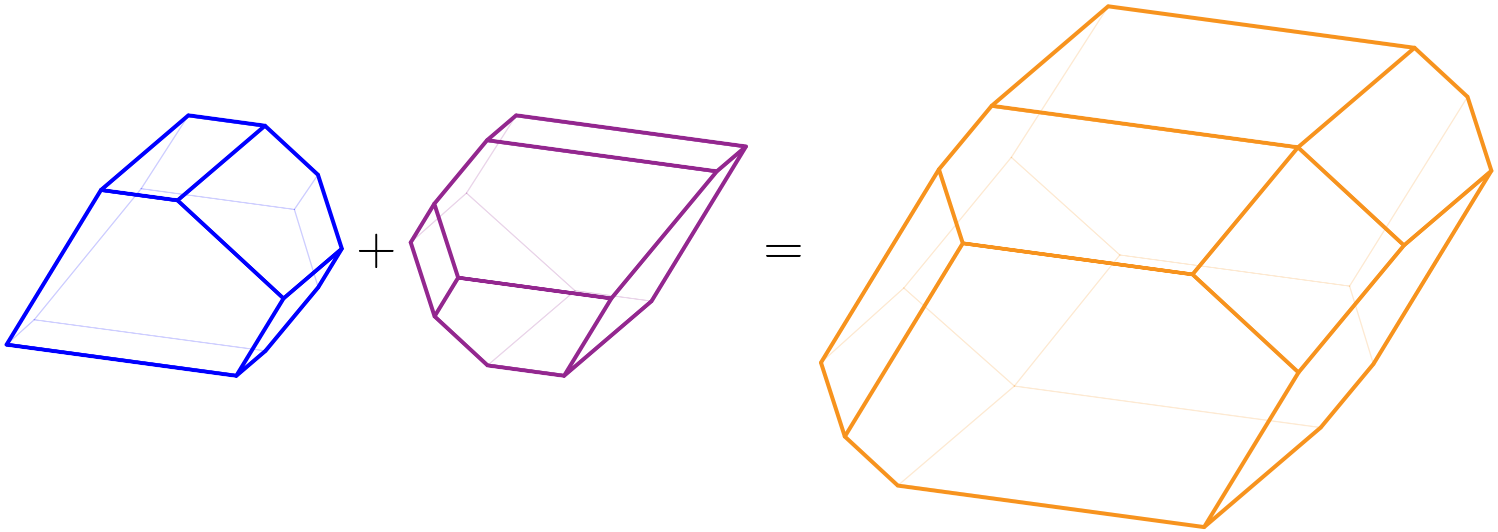
Principal arc ideals are Cambrian congruences

Any quotient fan is realized by a Minkowski sum of (low dim.) associahedra



Padrol-P.-Ritter ('20$^+$)

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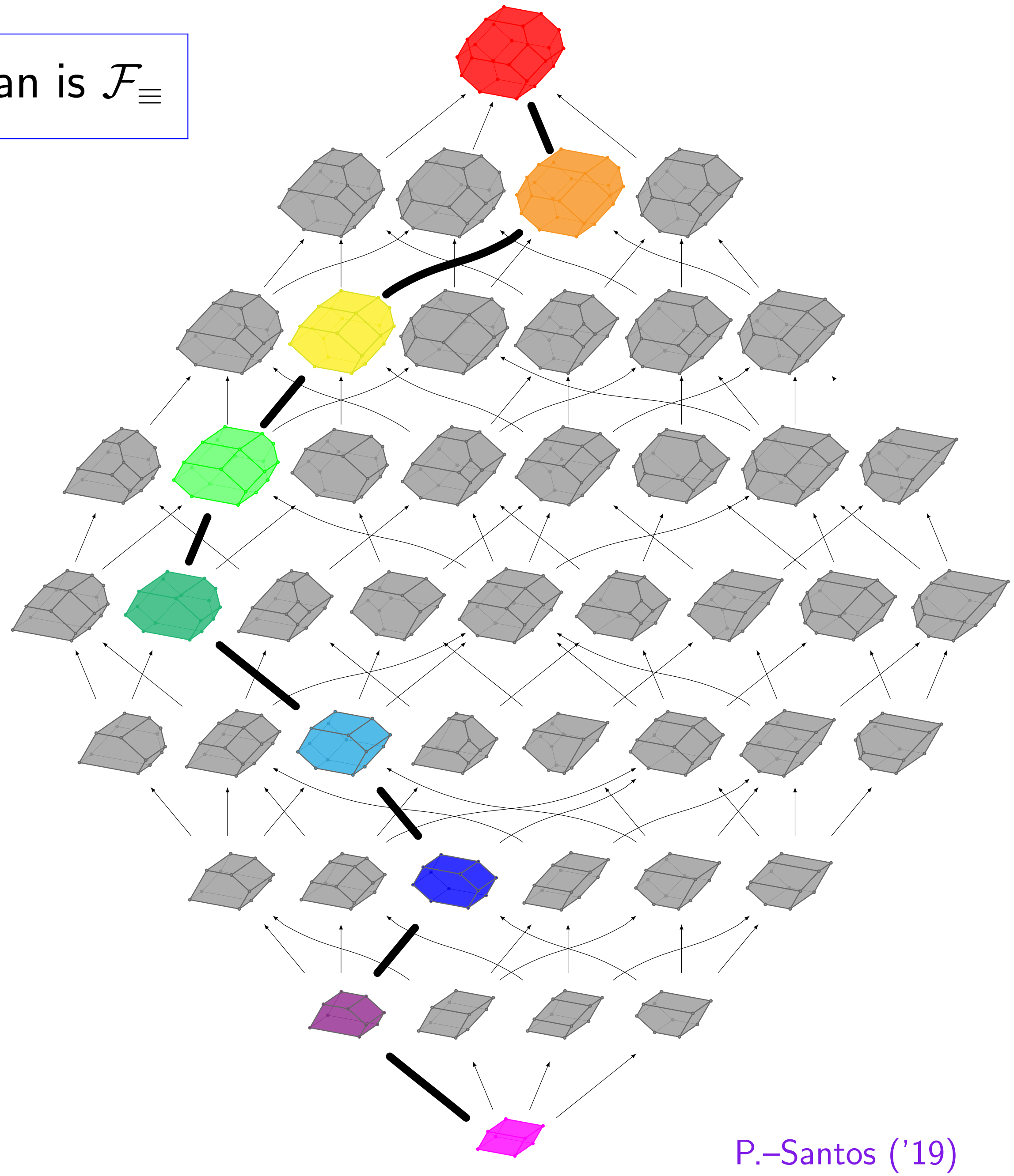
quotientope = polytope whose normal fan is $\mathcal{F}_{\equiv}$



P.–Santos ('19)

SHARD POLYTOPES

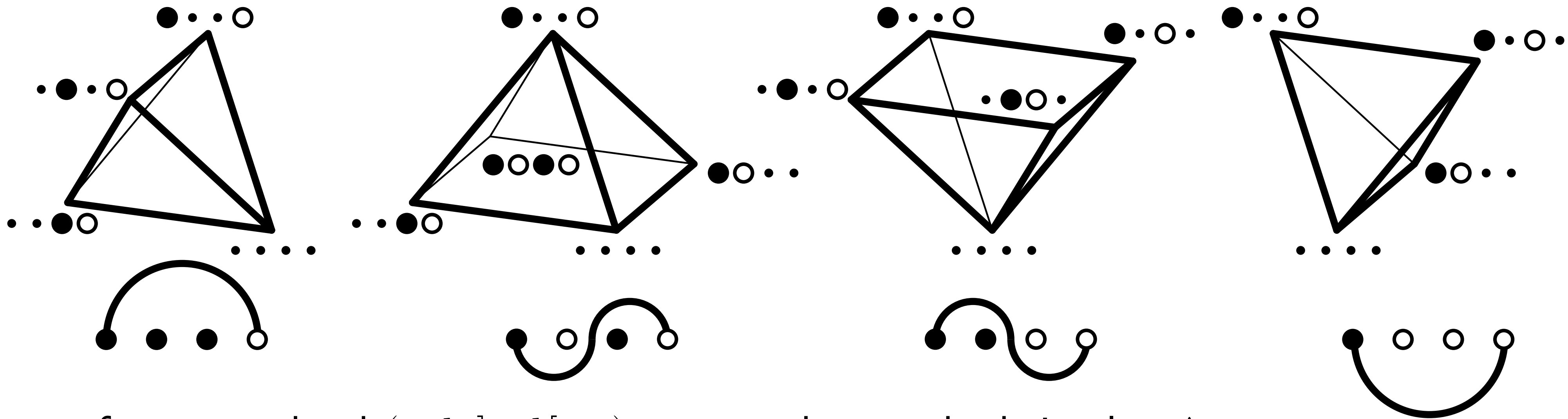
SHARD POLYTOPES

for a shard $\Sigma = \Sigma(a, b, A, B)$, define

- Σ-matching = sequence $a \le a_1 < b_1 < \cdots < a_k < b_k \le b$ where $\begin{cases} a_i \in \{a\} \cup A \\ b_i \in B \cup \{b\} \end{cases}$
- characteristic vector $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$

shard polytope $\mathbb{SP}(\Sigma) = \operatorname{conv}\left\{\chi(M) \;\middle|\; M\ \Sigma\text{-matching}\right\}$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{cl} x_j = 0 & \text{for all } j \in [n] \smallsetminus [a,b] \\ 0 \le x_{a'} \le 1 & \text{for all } a' \in \{a\} \cup A \\ -1 \le x_{b'} \le 0 & \text{for all } b' \in B \cup \{b\} \\ 0 \le \sum_{i \le j} x_i \le 1 & \text{for all } j \in [n] \end{array} \right\}$$

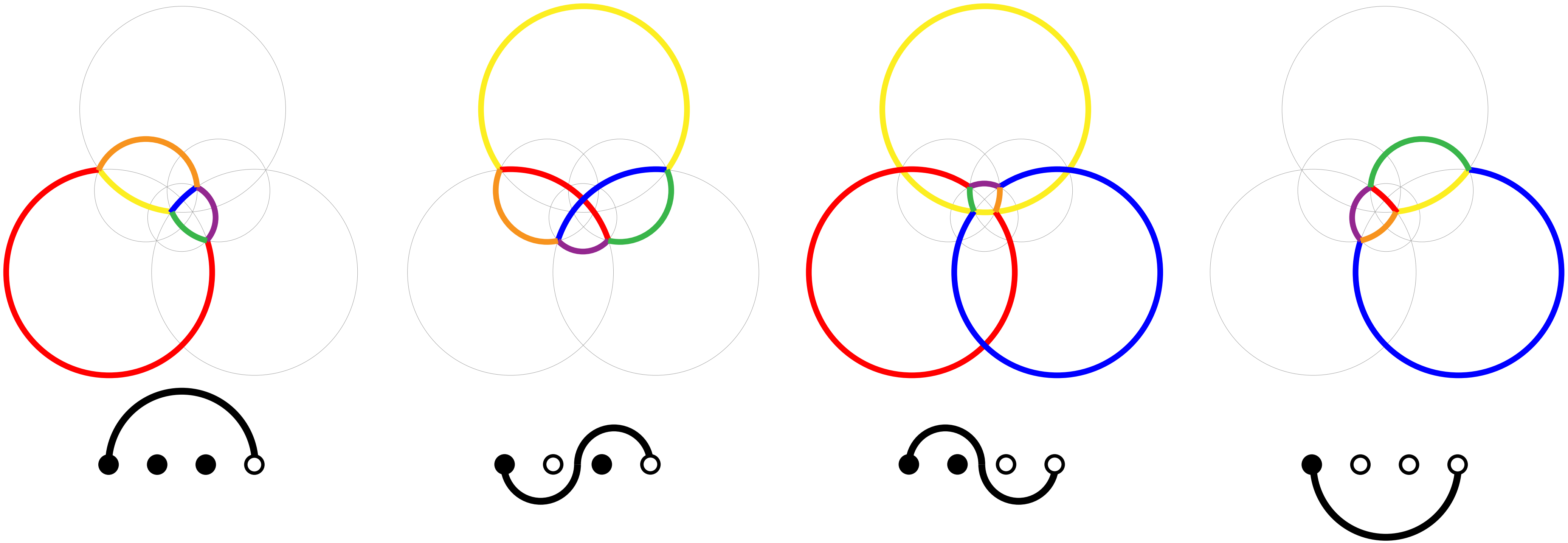


exm: for an up shard $(a, b,]a, b[, \varnothing)$, we get the standard simplex $\triangle_{[a,b]} - \boldsymbol{e}_b$

shard polytope $\mathbb{SP}(\Sigma) = \operatorname{conv}\left\{\chi(M) \;\middle|\; M\ \Sigma\text{-matching}\right\}$

The normal fan of the shard polytope $\mathbb{SP}(\Sigma)$

- contains the shard Σ,
- is contained in the union of the shards forcing Σ



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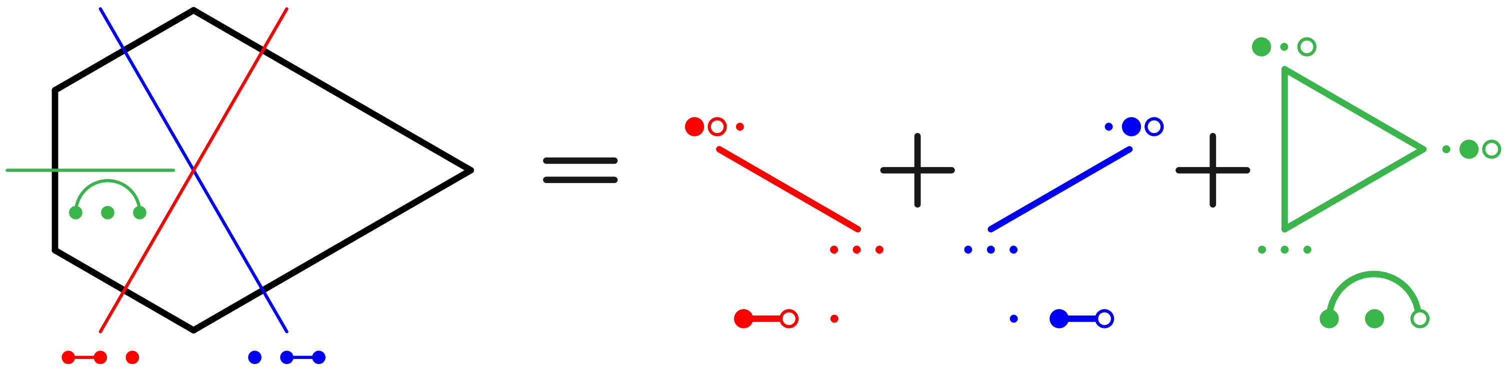
For any lattice congruence $\equiv$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the Minkowski sum of the shard polytopes $\mathbb{SP}(\Sigma)$ for $\Sigma \in \boldsymbol{\Sigma}_\equiv$ Padrol-P.-Ritter (20^+)



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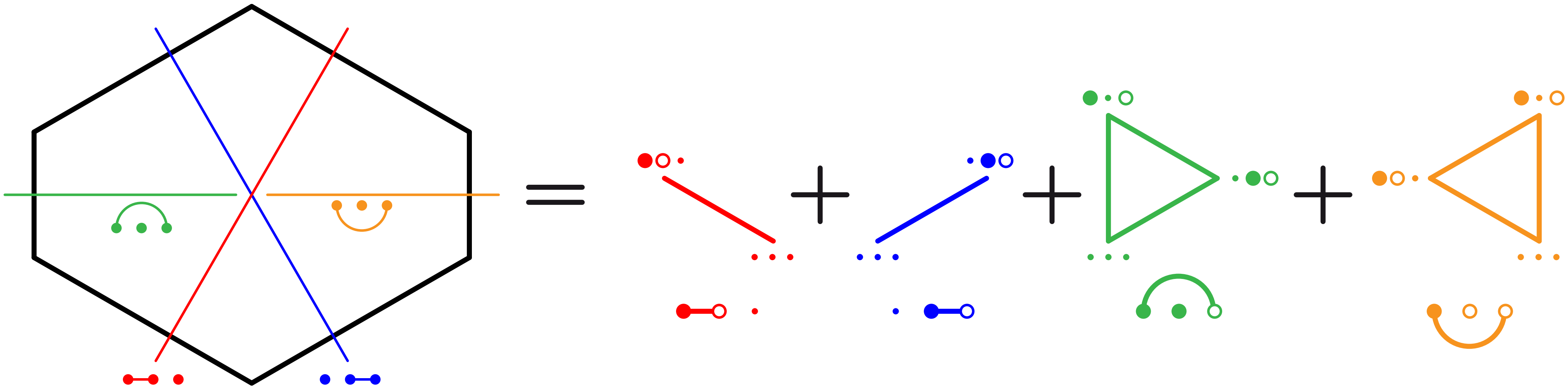


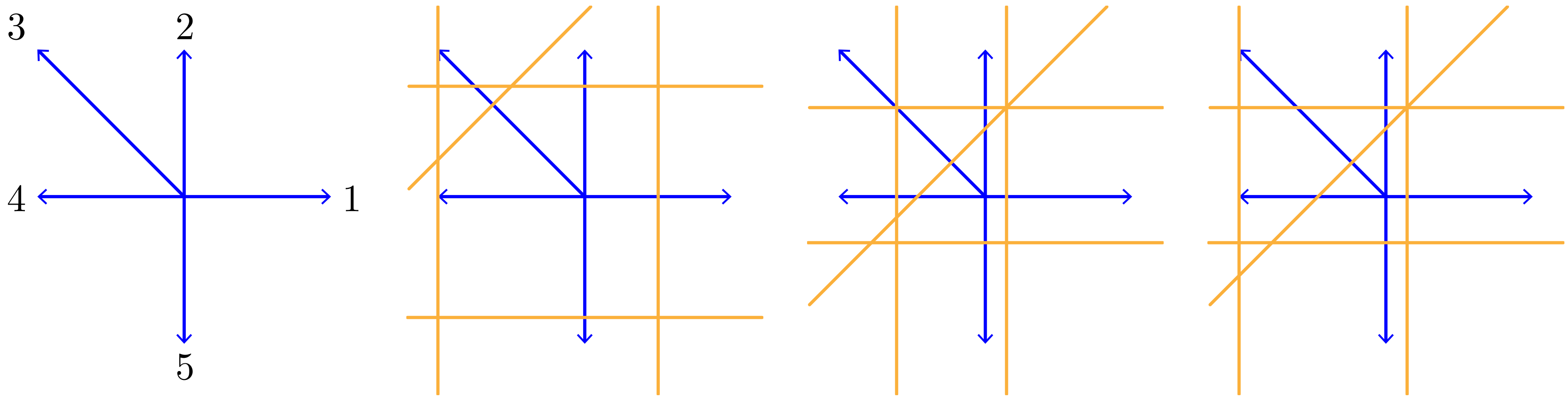
SHARD POLYTOPES AND TYPE CONES

CHOOSING RIGHT-HAND-SIDES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

$\boldsymbol{G} = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$



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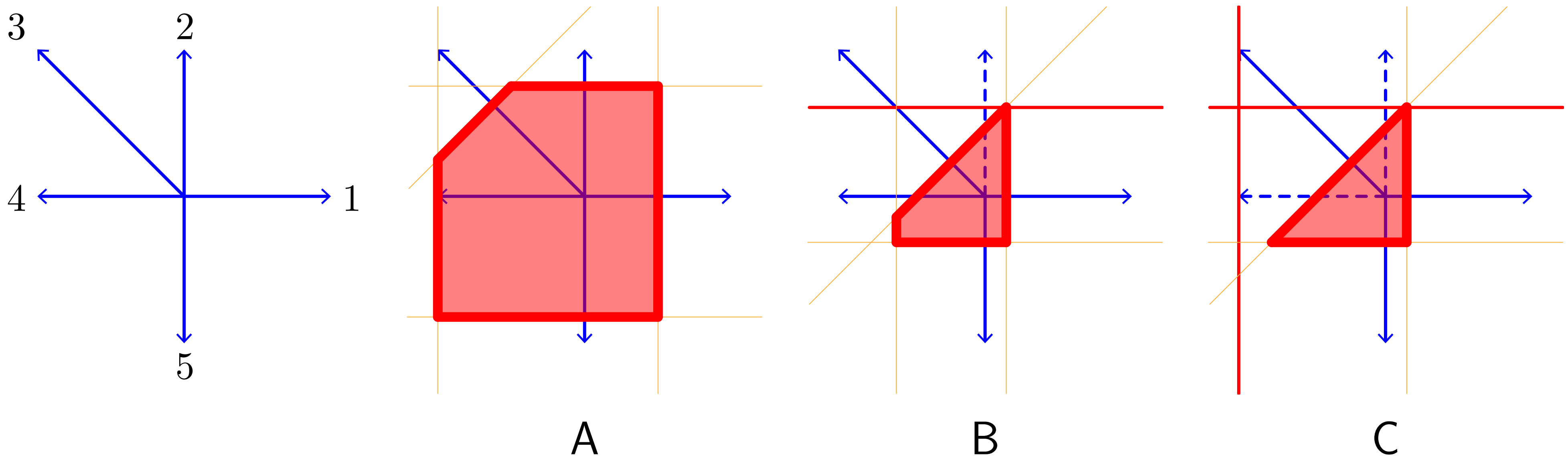

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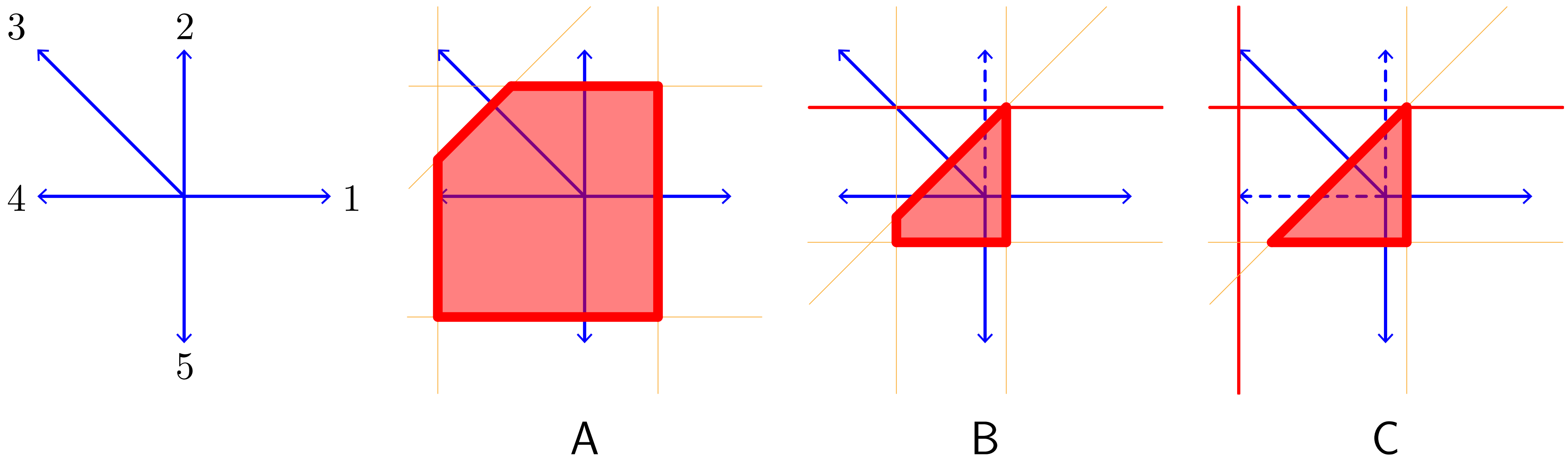

When is $\mathcal{F}$ the normal fan of $\mathbb{P}_{\boldsymbol{h}}$?

WALL-CROSSING INEQUALITIES

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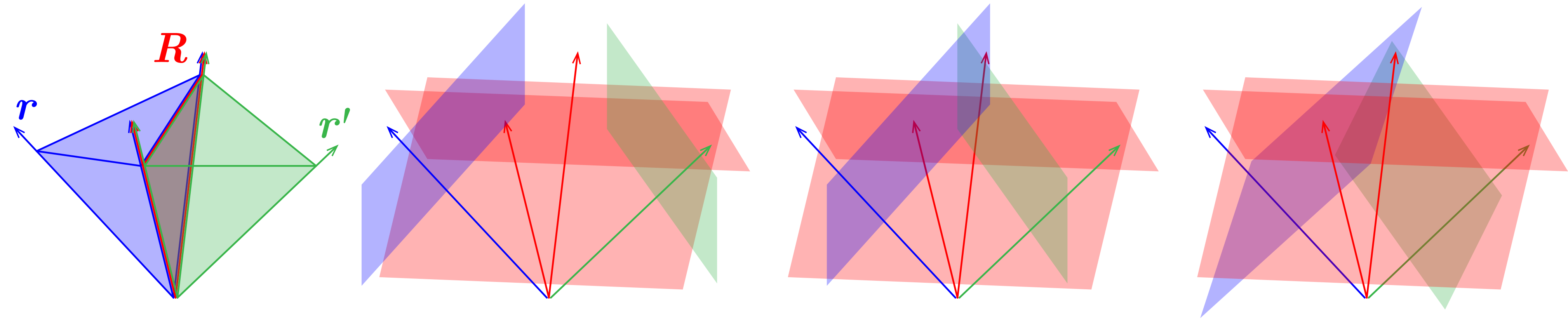

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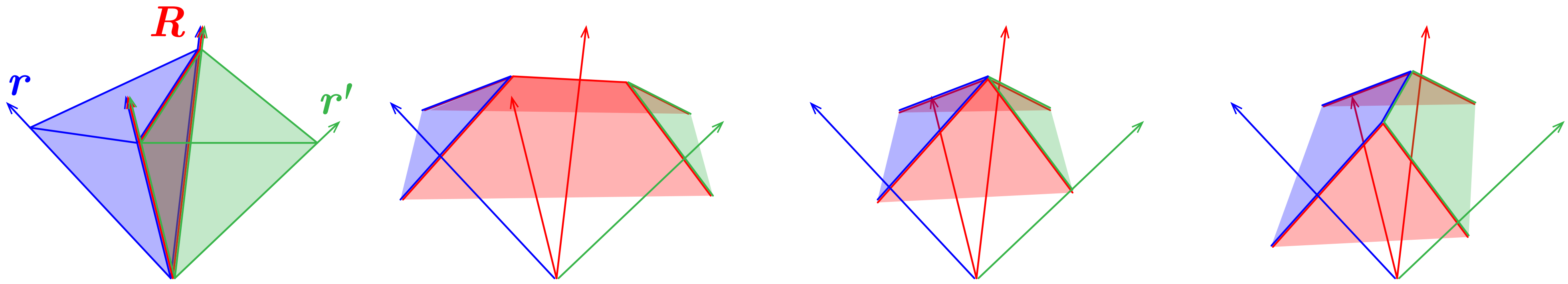

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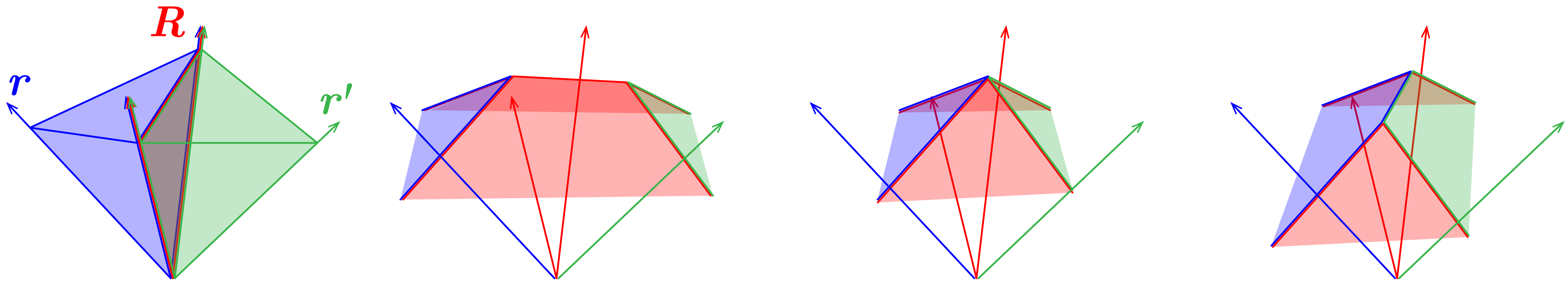

wall-crossing inequality for a wall $\boldsymbol{R}$ = $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, h_{\boldsymbol{s}} > 0$ where

- $\boldsymbol{r}, \boldsymbol{r'}$ = rays such that $\boldsymbol{R} \cup \{\boldsymbol{r}\}$ and $\boldsymbol{R} \cup \{\boldsymbol{r'}\}$ are chambers of $\mathcal{F}$
- $\alpha_{\boldsymbol{R},\boldsymbol{s}}$ = coeff. of unique linear dependence $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, \boldsymbol{s} = 0$ with $\alpha_{\boldsymbol{R},\boldsymbol{r}} + \alpha_{\boldsymbol{R},\boldsymbol{r'}} = 2$

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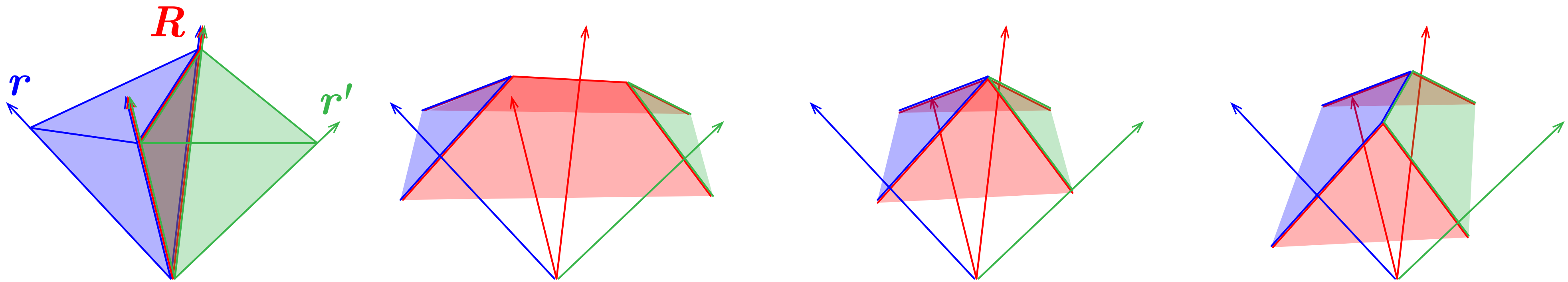
for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$



wall-crossing inequality for a wall $\boldsymbol{R}$ = $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r}'\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, h_{\boldsymbol{s}} > 0$ where

- $\boldsymbol{r}, \boldsymbol{r}'$ = rays such that $\boldsymbol{R} \cup \{\boldsymbol{r}\}$ and $\boldsymbol{R} \cup \{\boldsymbol{r}'\}$ are chambers of $\mathcal{F}$
- $\alpha_{\boldsymbol{R},\boldsymbol{s}}$ = coeff. of unique linear dependence $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r}'\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, \boldsymbol{s} = 0$ with $\alpha_{\boldsymbol{R},\boldsymbol{r}} + \alpha_{\boldsymbol{R},\boldsymbol{r}'} = 2$

$\mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}$ $\iff$ $\boldsymbol{h}$ satisfies all wall-crossing inequalities of $\mathcal{F}$

WALL-CROSSING INEQUALITIES

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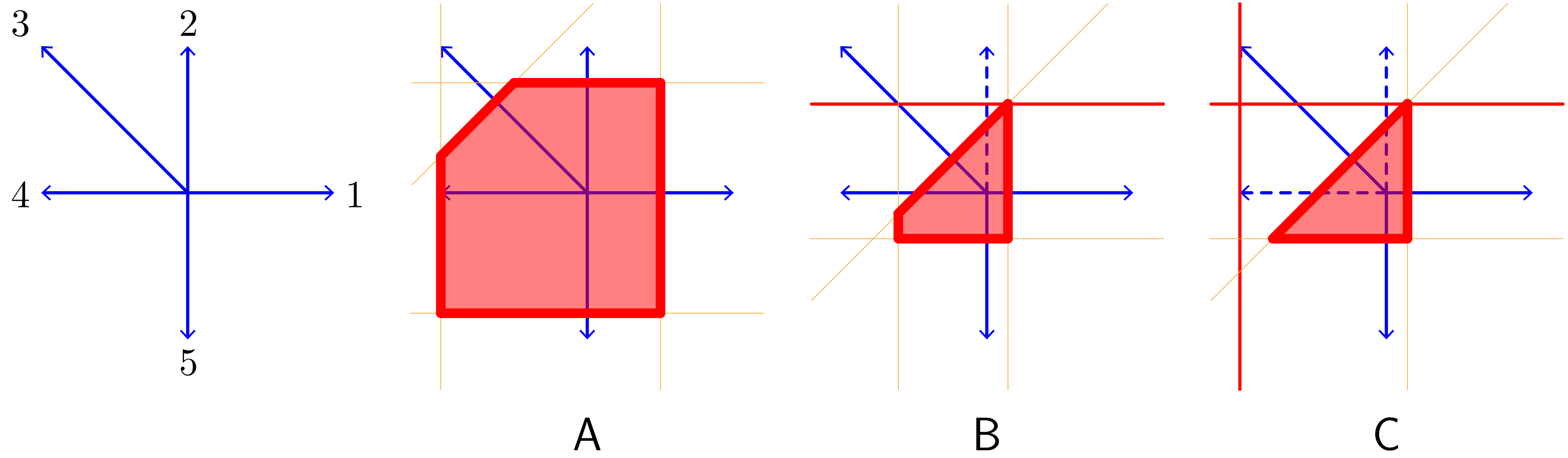


wall-crossing inequalities:

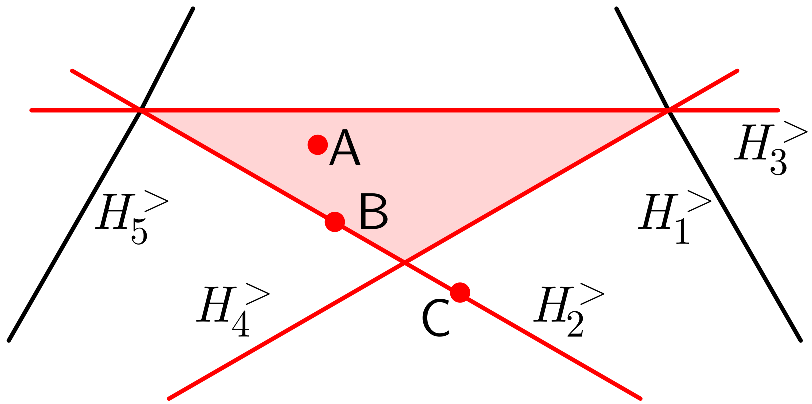
wall $1: \quad h_2 + h_5 > 0$

wall $2: \quad h_1 + h_3 > h_2$

wall $3: \quad h_2 + h_4 > h_3$

wall $4: \quad h_3 + h_5 > h_4$

wall $5: \quad h_1 + h_4 > 0$



TYPE CONE

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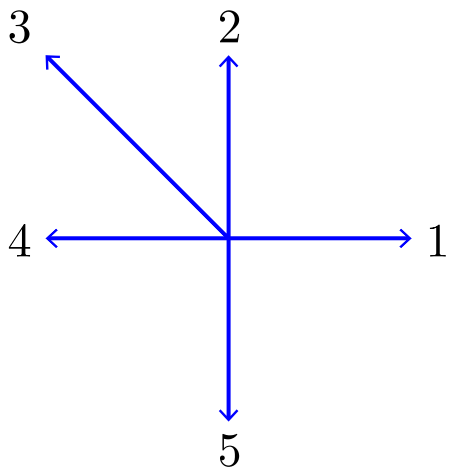
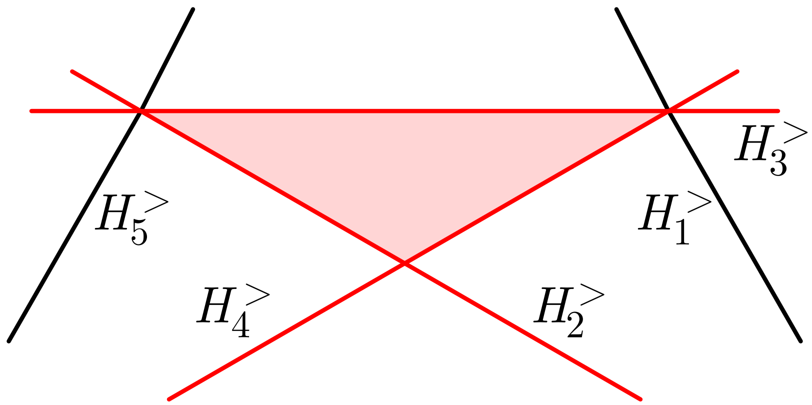
<u>type cone</u> $\mathbb{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ McMullen ('73)

$= \big\{ \boldsymbol{h} \in \mathbb{R}^N \;\big|\; \mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}} \big\}$

$= \big\{ \boldsymbol{h} \in \mathbb{R}^N \;\big|\; \boldsymbol{h}$ satisfies all wall-crossing inequalities of $\mathcal{F} \big\}$





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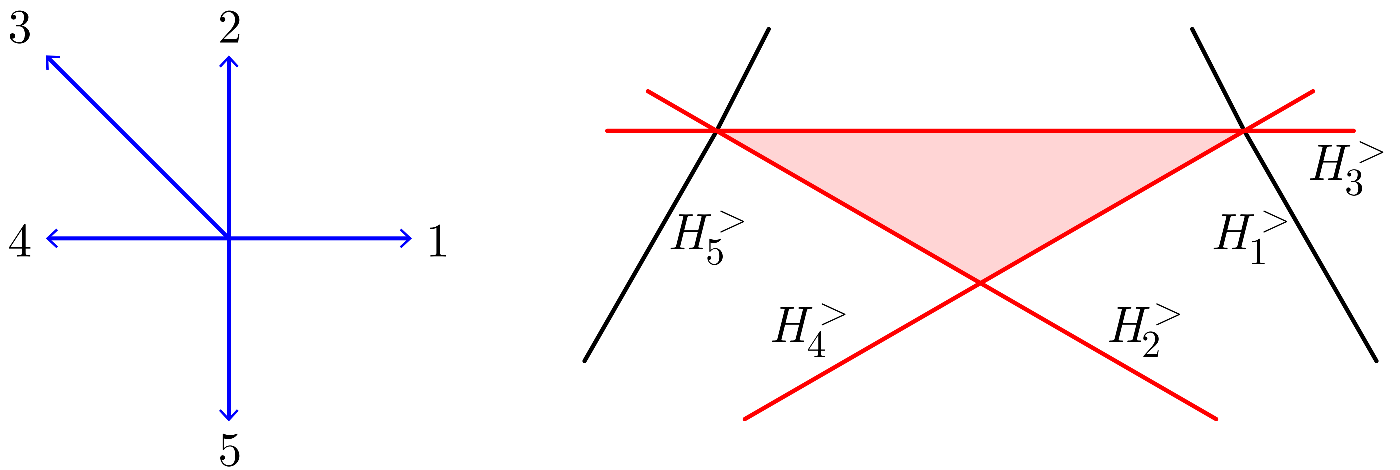

some properties of $\mathbb{TC}(\mathcal{F})$:

- $\mathbb{TC}(\mathcal{F})$ is an open cone (dilations preserve normal fans)
- $\mathbb{TC}(\mathcal{F})$ has lineality space $\boldsymbol{G}\,\mathbb{R}^n$ (translations preserve normal fans)
- dimension of $\mathbb{TC}(\mathcal{F})/\boldsymbol{G}\,\mathbb{R}^n = N - n$

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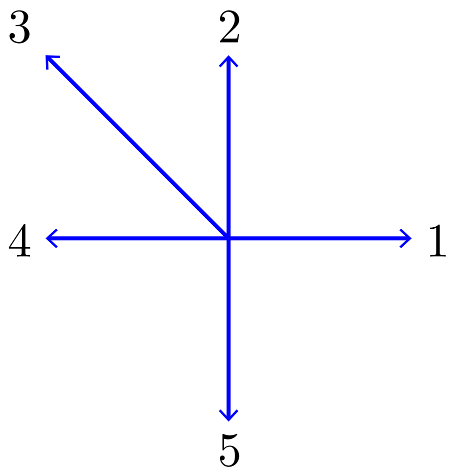
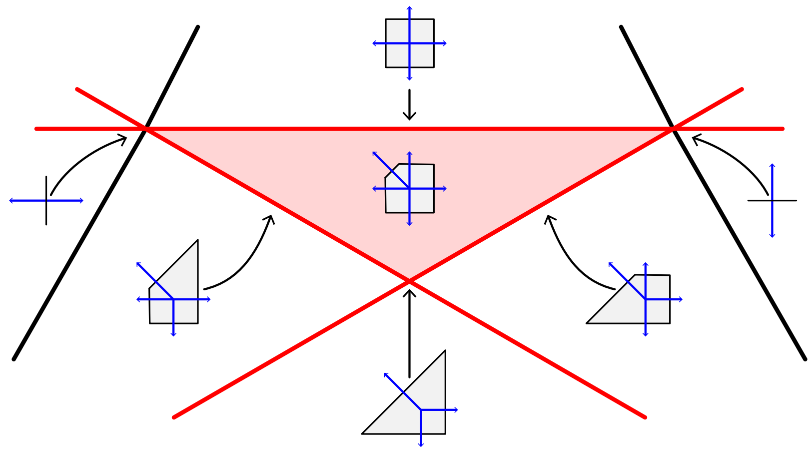




some properties of $\mathbb{TC}(\mathcal{F})$:

- closure of $\mathbb{TC}(\mathcal{F})$ = polytopes whose normal fan coarsens $\mathcal{F}$ = deformation cone
- Minkowski sums $\longleftrightarrow$ positive linear combinations

SIMPLICIAL TYPE CONE

Assume that the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial

$\boldsymbol{K} = (N-n) \times N$-matrix whose rows are inner normal vectors of the facets of $\mathbb{TC}(\mathcal{F}(\delta))$

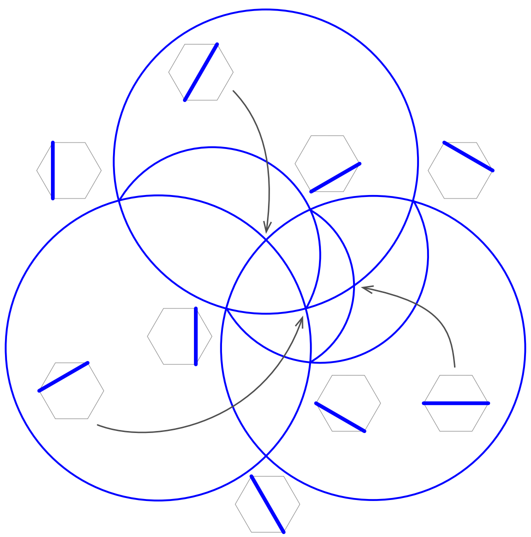
All polytopal realizations of $\mathcal{F}$ are affinely equivalent to

$$\mathbb{R}_{\boldsymbol{\ell}} = \left\{ \boldsymbol{z} \in \mathbb{R}^N \;\middle|\; \boldsymbol{K}\boldsymbol{z} = \boldsymbol{\ell} \text{ and } \boldsymbol{z} \geq 0 \right\}$$

for any positive vector $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$

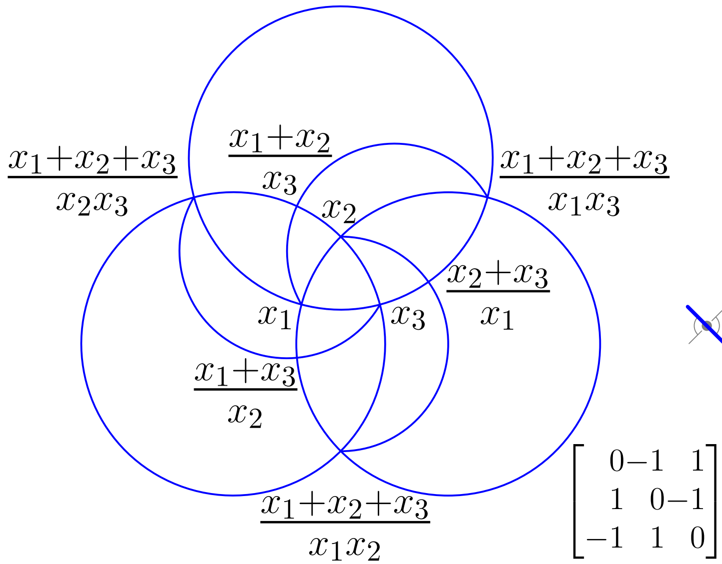
Padrol–Palu–P.–Plamondon ('19$^+$)

Fundamental exms: $\boldsymbol{g}$-vector fans of cluster-like complexes



sylvester fans

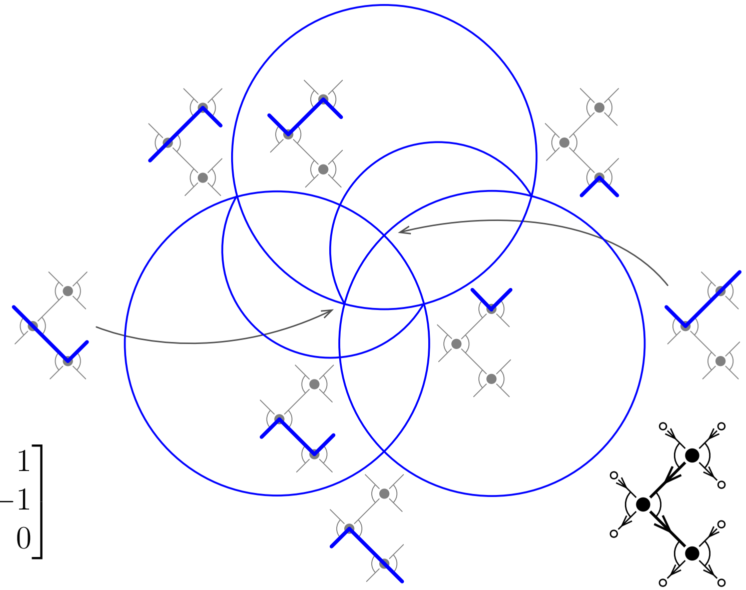
Arkani-Hamed–Bai–He–Yan ('18)



finite type $\boldsymbol{g}$-vector fans

wrt any seed (acyclic or not)

BMDMTY ('18$^+$)

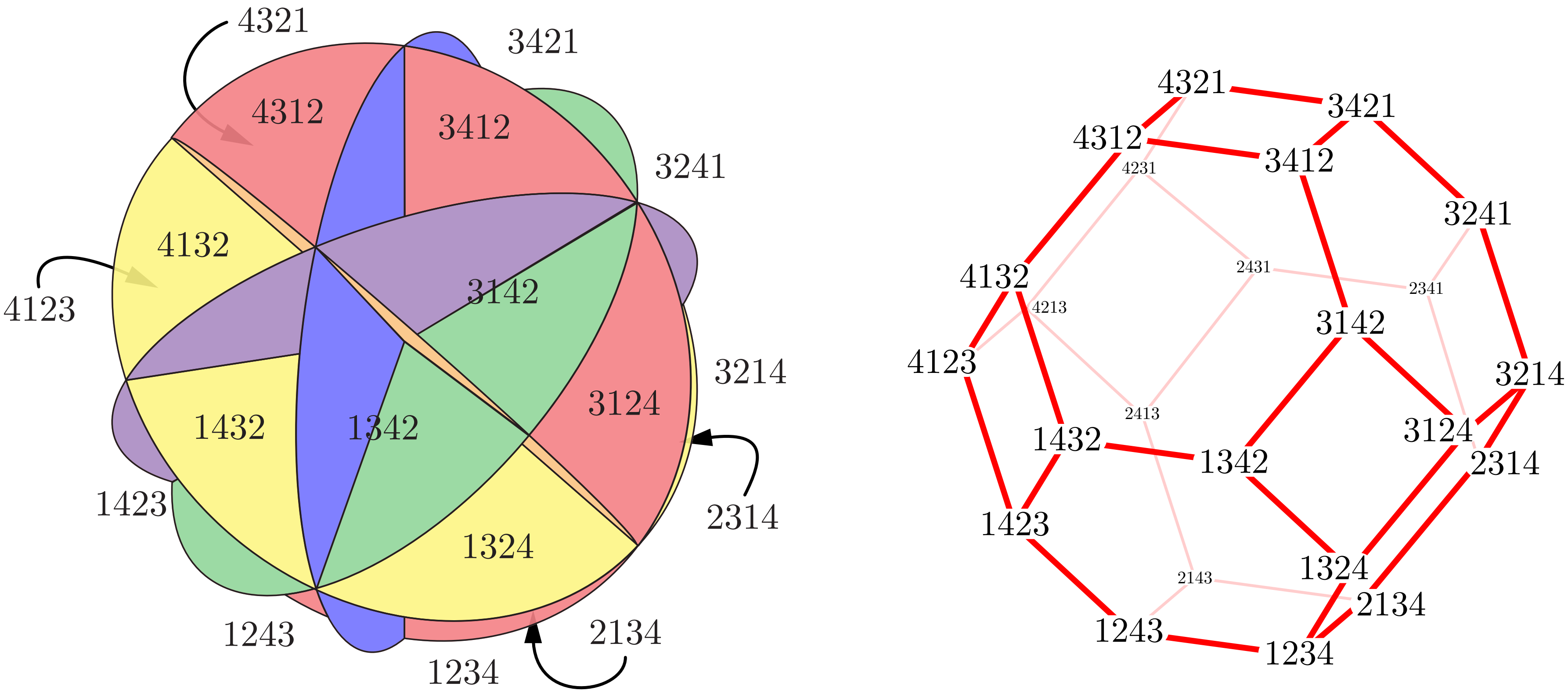


finite gentle fans

for brick and 2-acyclic quivers

Palu–P.–Plamondon ('18)

SUBMODULAR FUNCTIONS



closed type cone of braid fan = {deformed permutahedra} = {submodular functions}

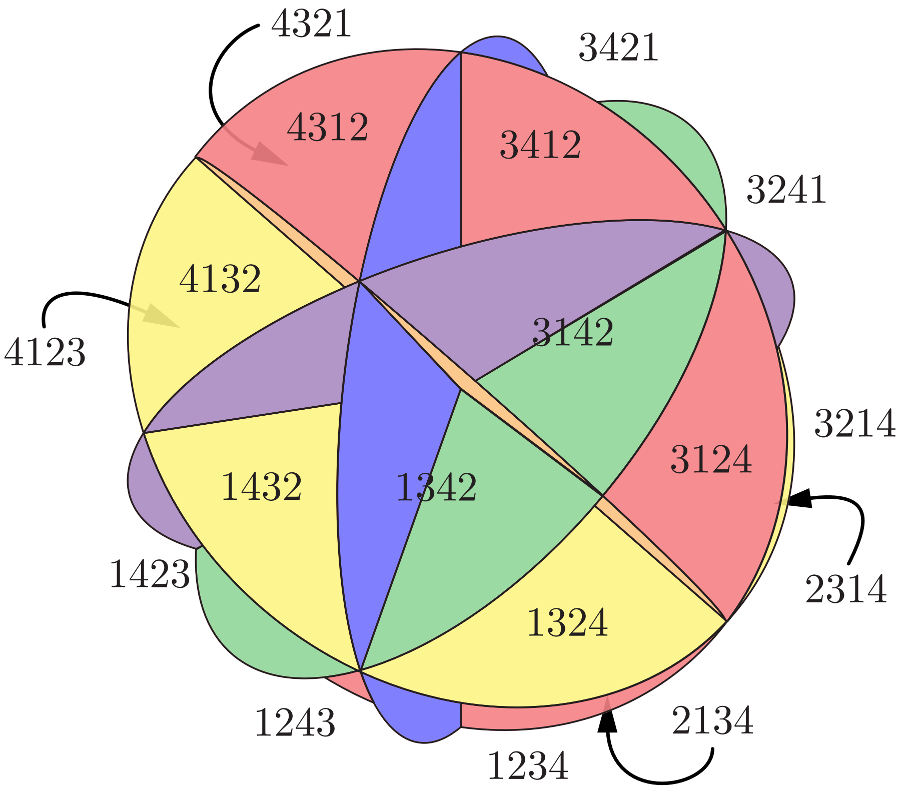
deformed permutahedron = polytope whose normal fan coarsens the braid fan

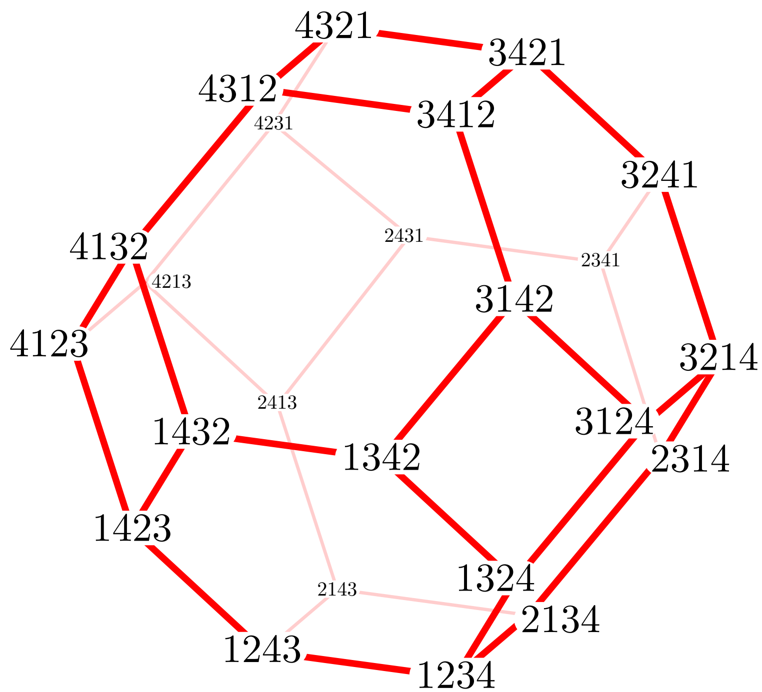
$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \langle \mathbb{1} \mid \boldsymbol{x} \rangle = z_{[n]} \text{ and } \langle \mathbb{1}_R \mid \boldsymbol{x} \rangle \ge z_R \text{ for all } R \subseteq [n] \right\}$$

for some vector $\boldsymbol{z} \in \mathbb{R}^{2^{[n]}}$ such that $z_R + z_S \le z_{R \cup S} + z_{R \cap S}$ and $z_\varnothing = 0$

Postnikov ('09) Postnikov–Reiner–Williams ('08)

SUBMODULAR FUNCTIONS





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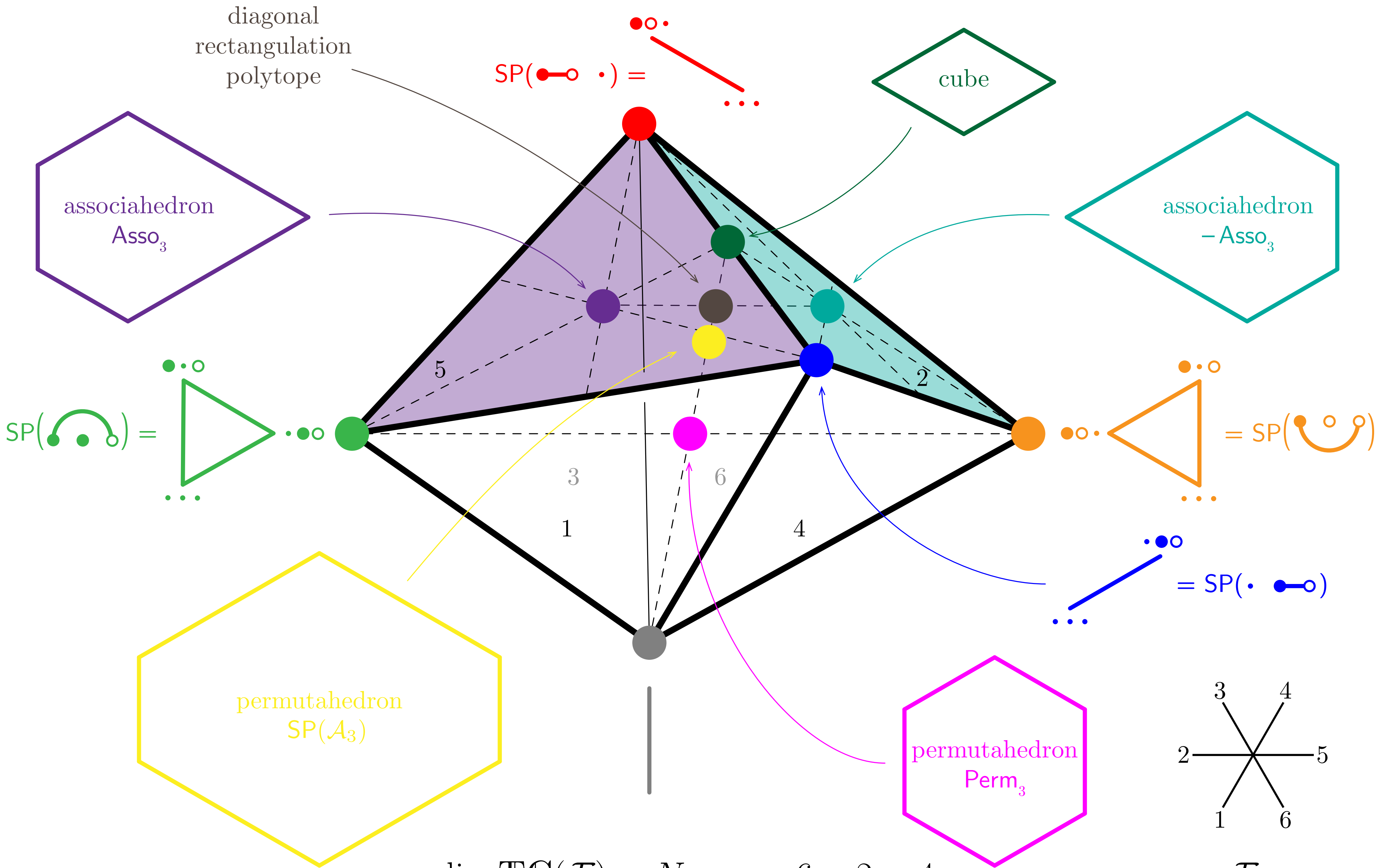
for some vector $\boldsymbol{z} \in \mathbb{R}^{2^{[n]}}$ such that $z_R + z_S \leq z_{R \cup S} + z_{R \cap S}$ and $z_\varnothing = z_{\{i\}} = 0$, where $\mathcal{J} = \{J \subseteq [n] \mid |J| \geq 2\}$

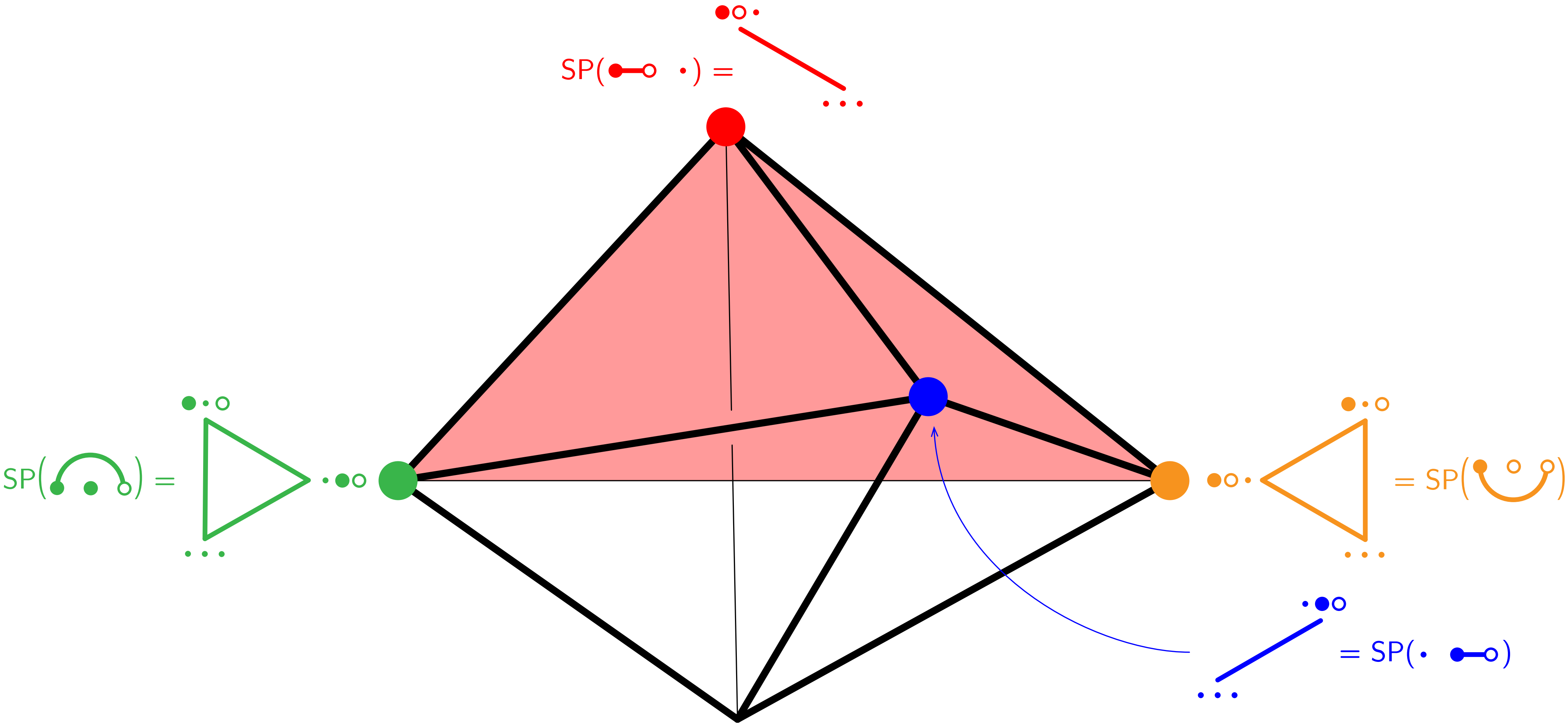
Postnikov ('09) Postnikov–Reiner–Williams ('08)

SUBMODULAR FUNCTIONS



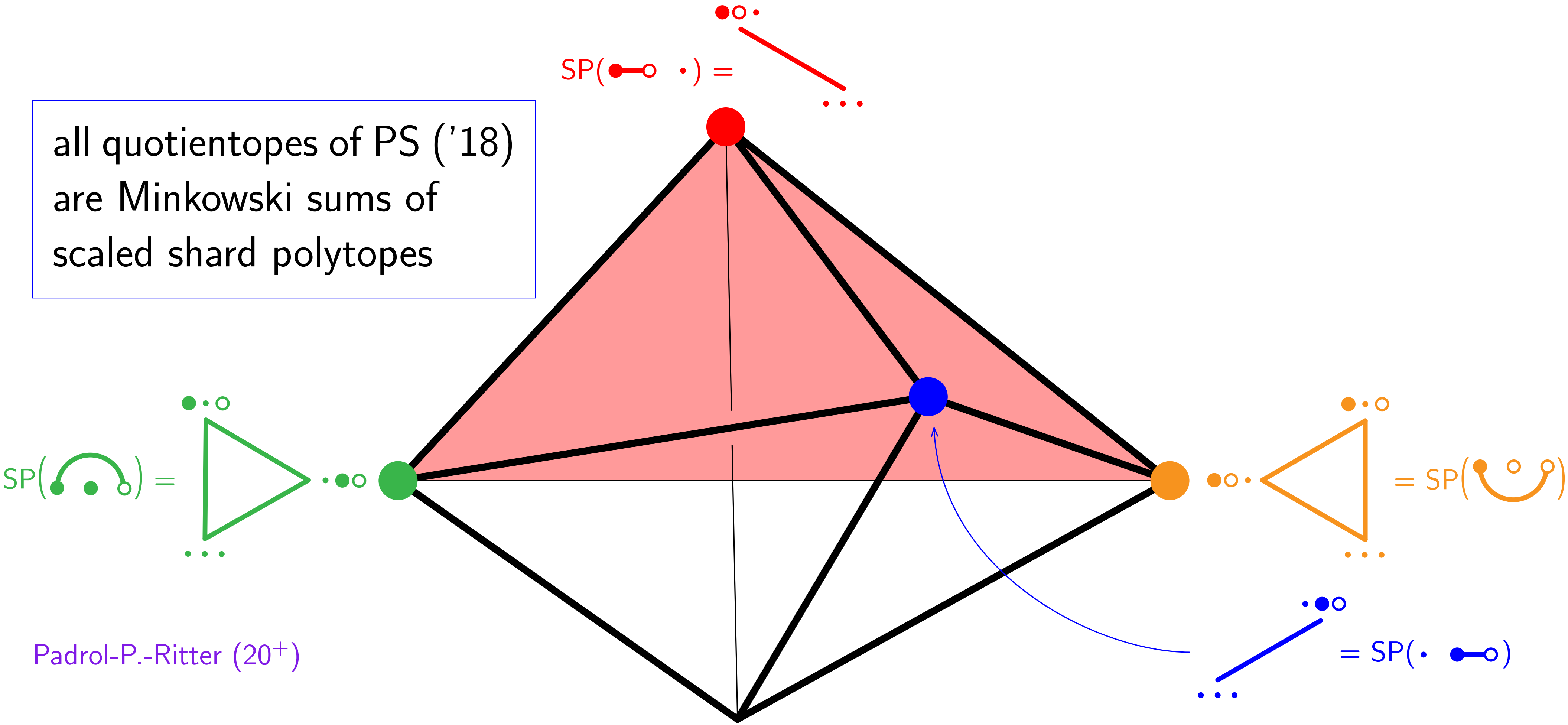
$$\dim \mathbb{TC}(\mathcal{F}) = N - n = 6 - 2 = 4$$

SUBMODULAR FUNCTIONS



SUBMODULAR FUNCTIONS

all quotientopes of PS ('18) are Minkowski sums of scaled shard polytopes



Padrol-P.-Ritter (20^{+})

SUBMODULAR FUNCTIONS

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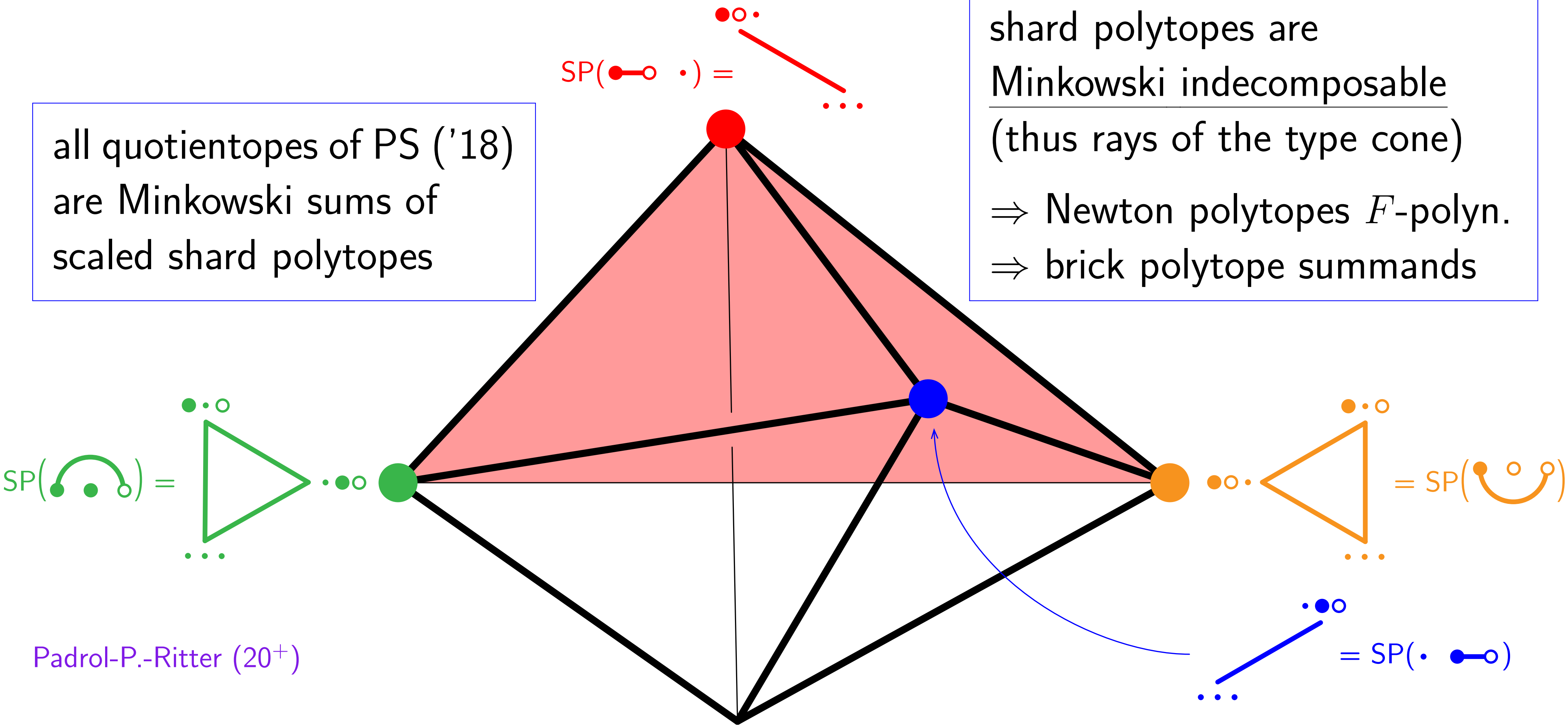
shard polytopes are Minkowski indecomposable (thus rays of the type cone)

⇒ Newton polytopes F-polyn.

⇒ brick polytope summands



Padrol-P.-Ritter (20^{+})

SUBMODULAR FUNCTIONS

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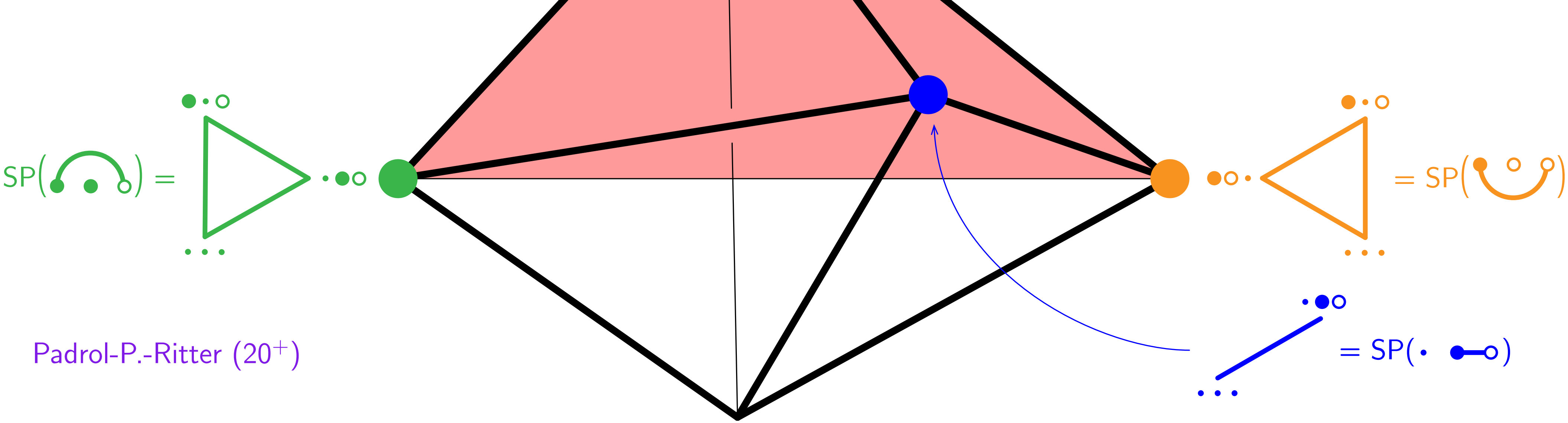


Padrol-P.-Ritter (20$^+$)

Any deformed permutahedron is a Minkowski sum and difference of shard polytopes

$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \sum_{J \in \mathcal{J}} y_J \, \triangle_J = \sum_{I \in \mathcal{J}} s_I \, \mathbb{SP}(\Sigma_I)$$

with explicit (combinatorial) exchange matrices between the parameters $\boldsymbol{s}$, $\boldsymbol{y}$ and $\boldsymbol{z}$

OPEN QUESTIONS

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
base region B = distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$
inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C
poset of regions $\mathrm{PR}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

The poset of regions $\mathrm{PR}(\mathcal{H}, B)$ — Björner-Edelman-Ziegler ('90)

- is never a lattice when B is not a simplicial region
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement

If $\mathrm{PR}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a congruence of $\mathrm{PR}(\mathcal{H}, B)$, the cones obtained by glueing the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan $\mathcal{F}_\equiv$

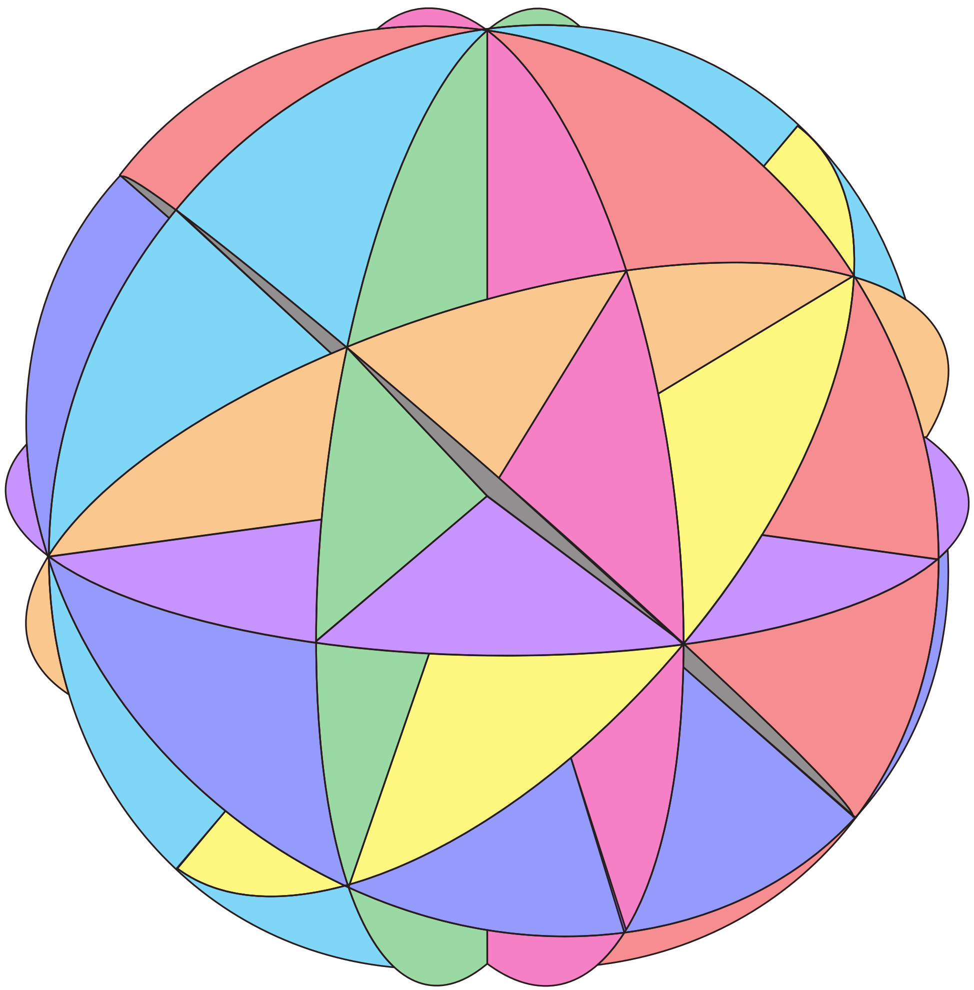
Reading ('05)

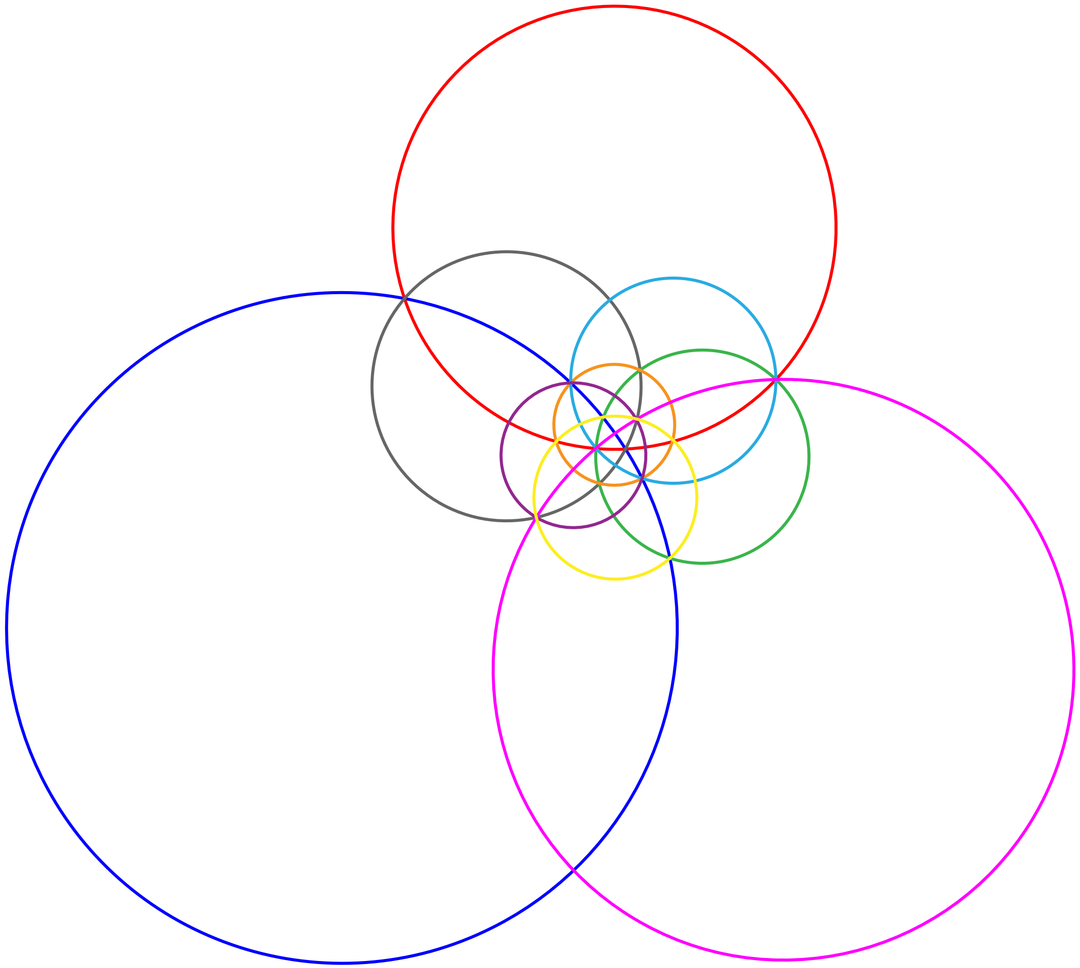
Is the quotient fan $\mathcal{F}_\equiv$ always polytopal?

SHARDS FOR HYPERPLANE ARRANGEMENTS

shard = piece of hyperplane obtained after cutting all rank 2 subgroups

shard poset = (pre)poset of forcing relations among shards





Reading ('03)

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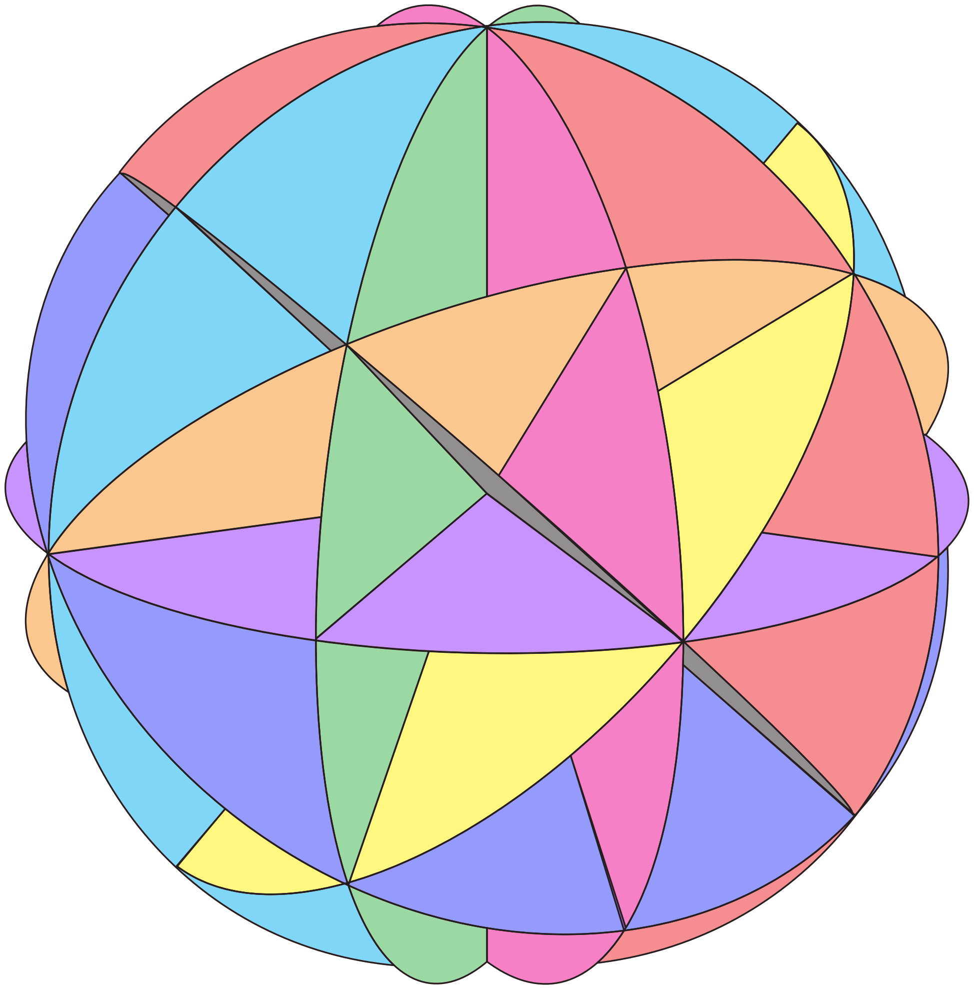

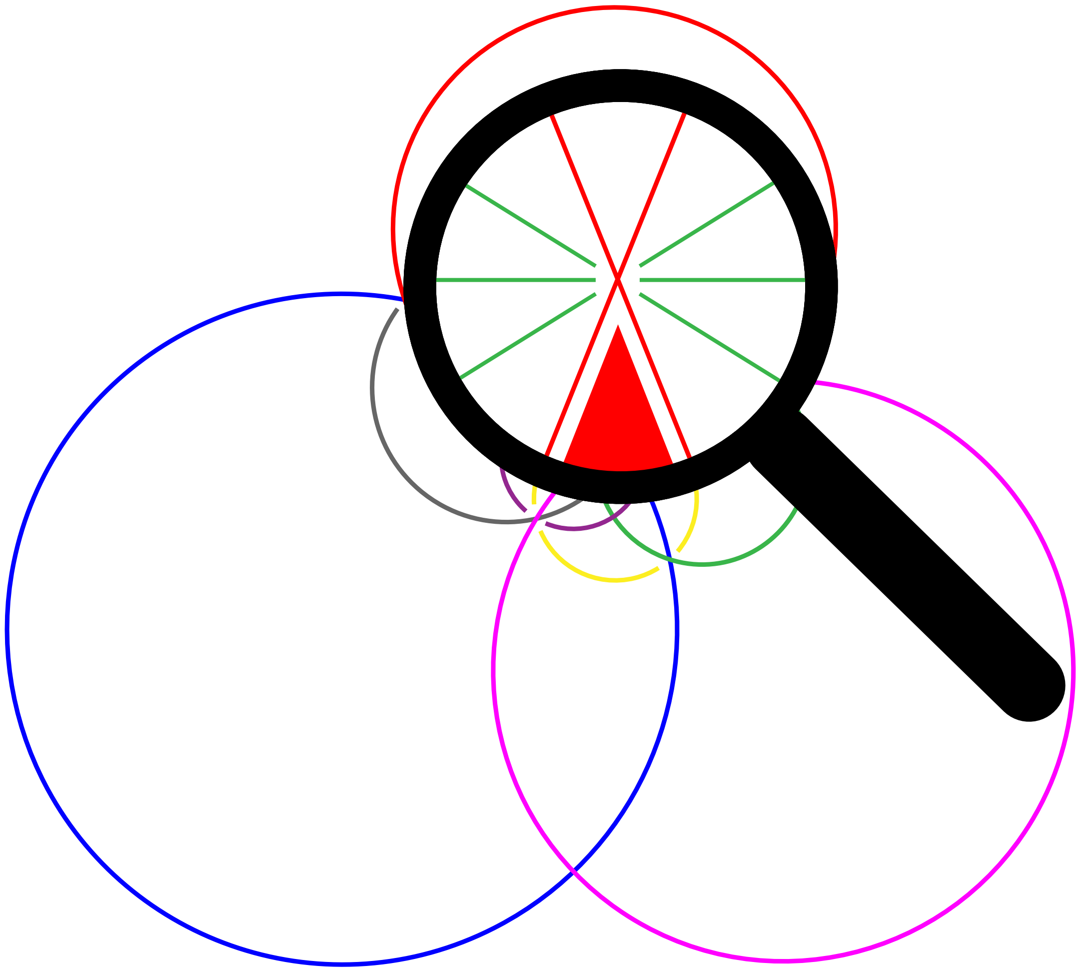

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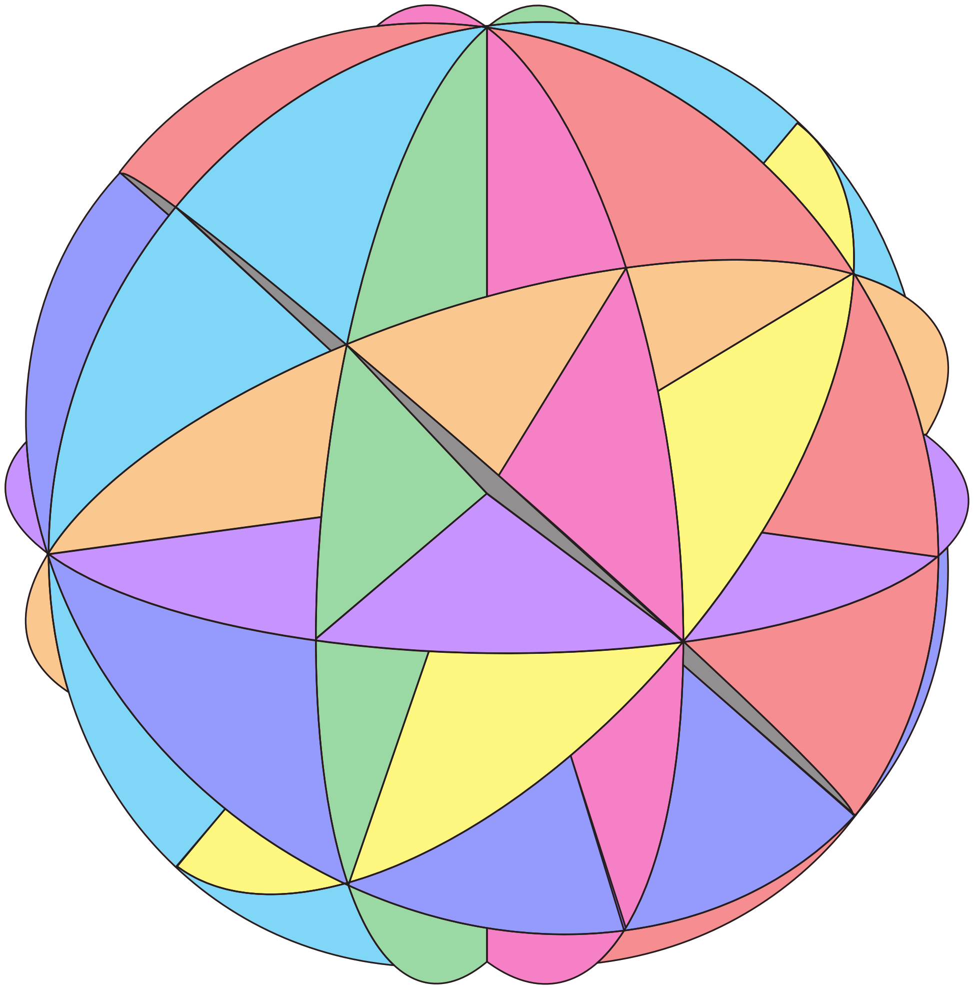

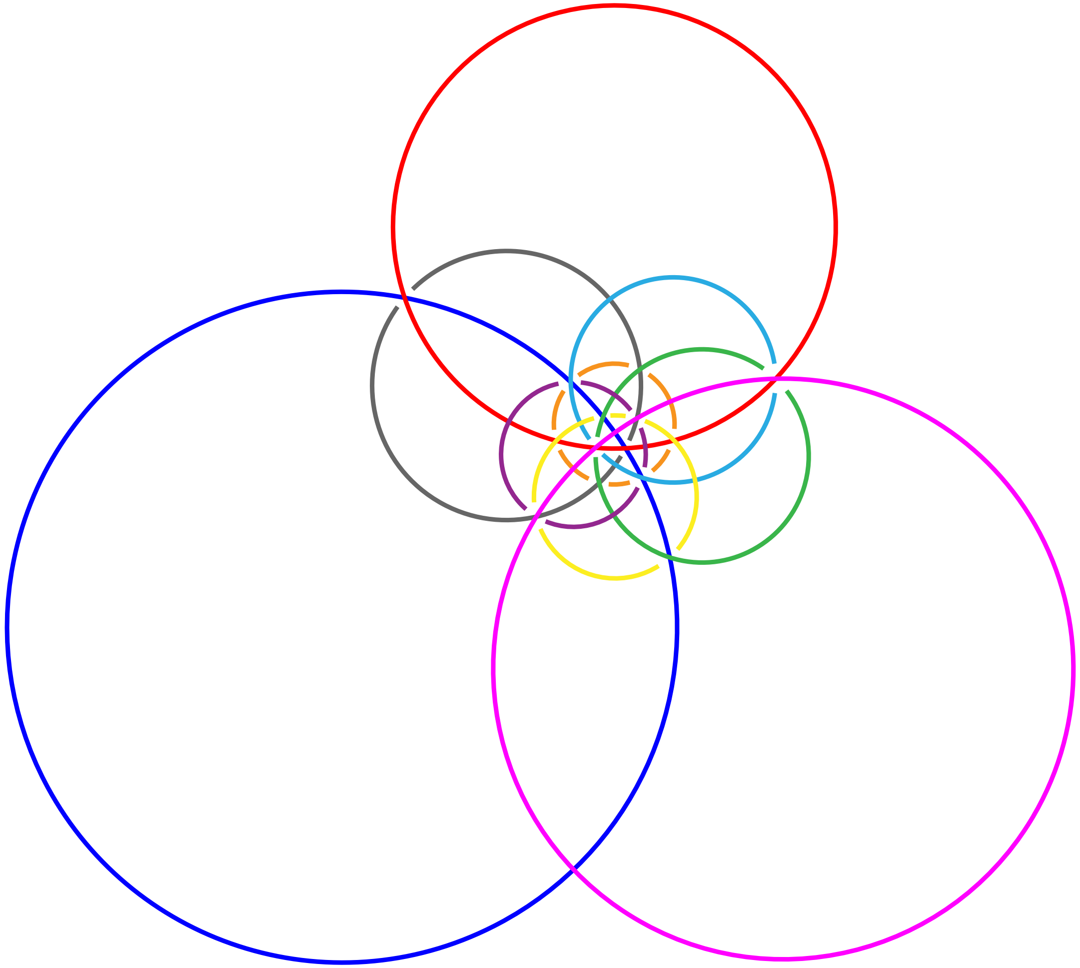

Reading ('03)

SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

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shard polytope for a shard Σ = polytope whose normal fan

- contains the shard Σ,
- is contained in the union of the shards forcing Σ

Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

If any shard Σ admits a shard polytope $\mathbb{SP}(\Sigma)$, then

- for any lattice congruence $\equiv$ of PR$(\mathcal{H}, B)$, the quotient fan $\mathcal{F}_{\equiv}$ is the normal of the Minkowski sum of the shard polytopes $\mathbb{SP}(\Sigma)$ for Σ in the shard ideal $\boldsymbol{\Sigma}_{\equiv}$
- if the arrangement $\mathcal{H}$ is simplicial, then the shard polytopes $\mathbb{SP}(\Sigma)$ form a basis for the type cone of the fan defined by $\mathcal{H}$ (up to translation)

Padrol-P.-Ritter (20^{+})

SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

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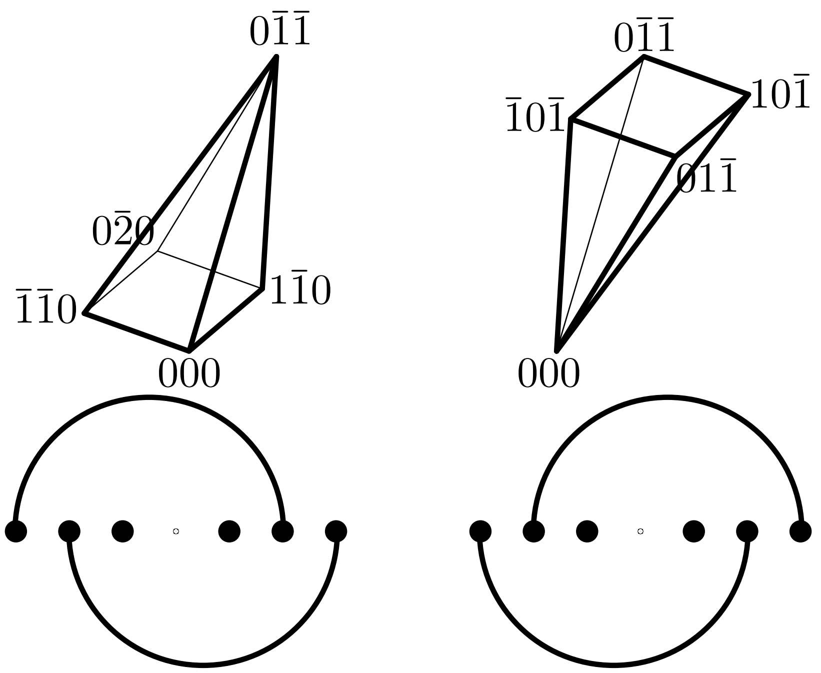
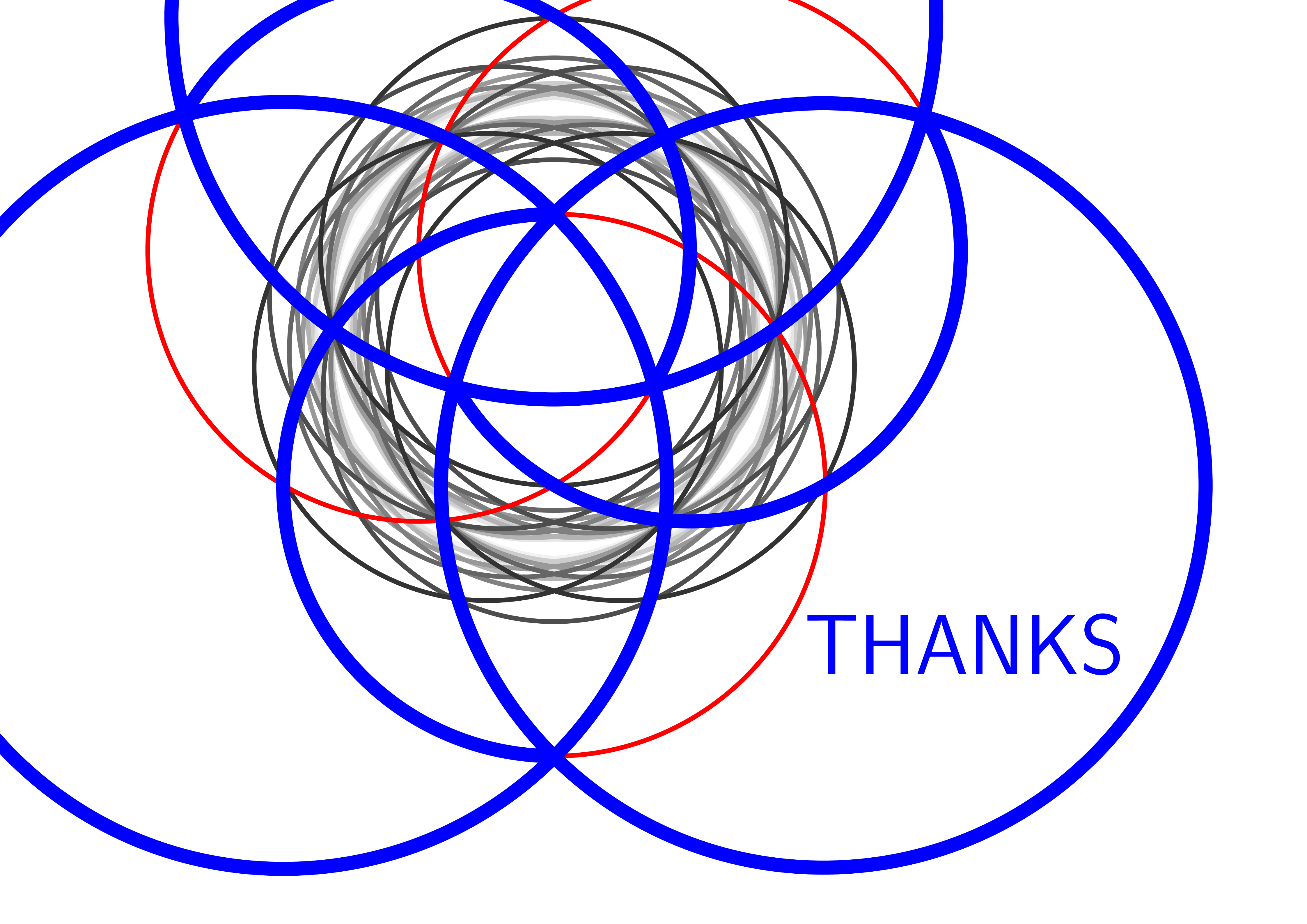
Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

For crystallographic arrangements,
Newton polytopes of F-polynomials
all seem to be shard polytopes,
but some shards are missing...




THANKS