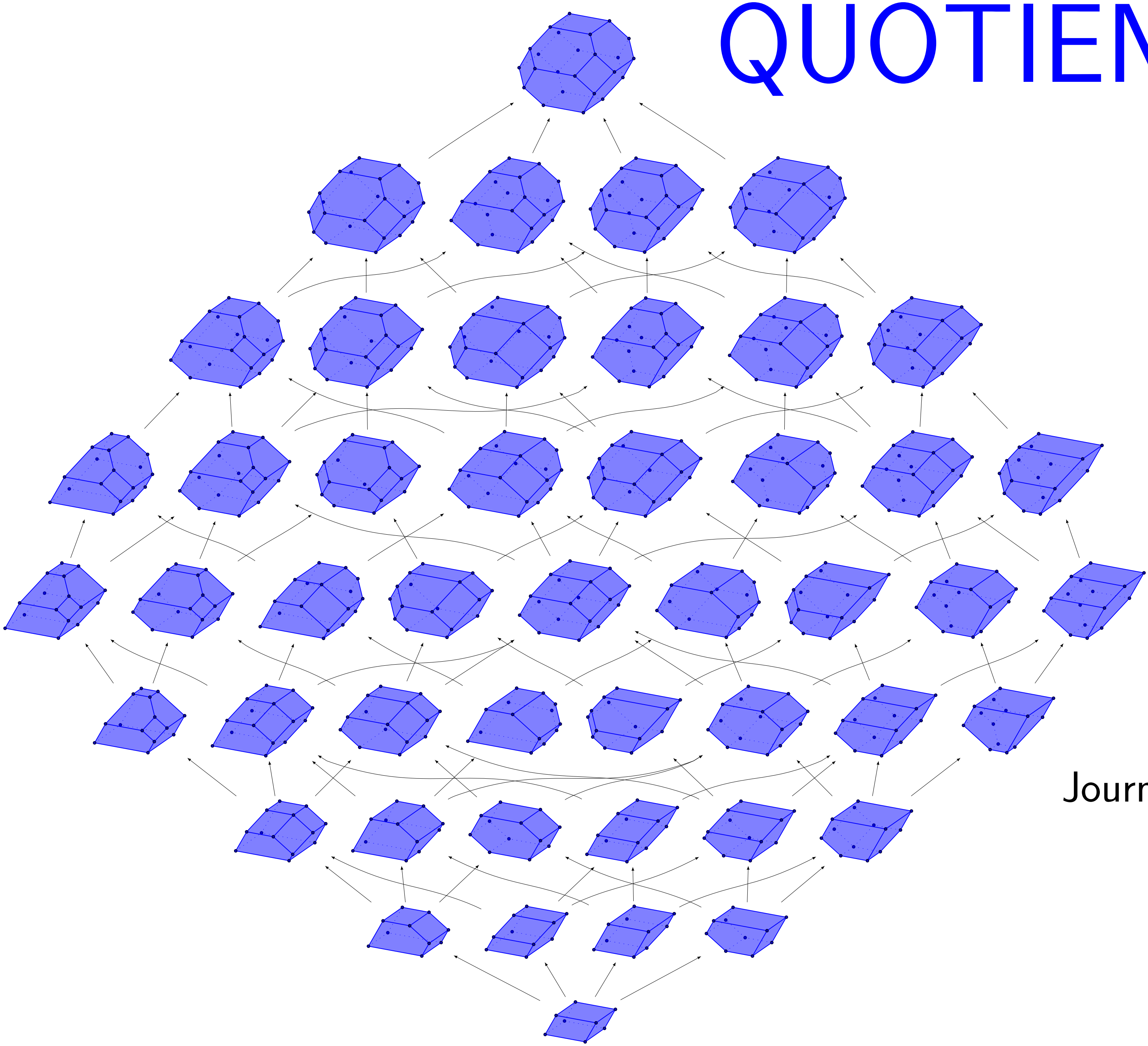


QUOTIENTOPES

V. PILAUD
(CNRS & LIX)

F. SANTOS
(Univ. Cantabira)

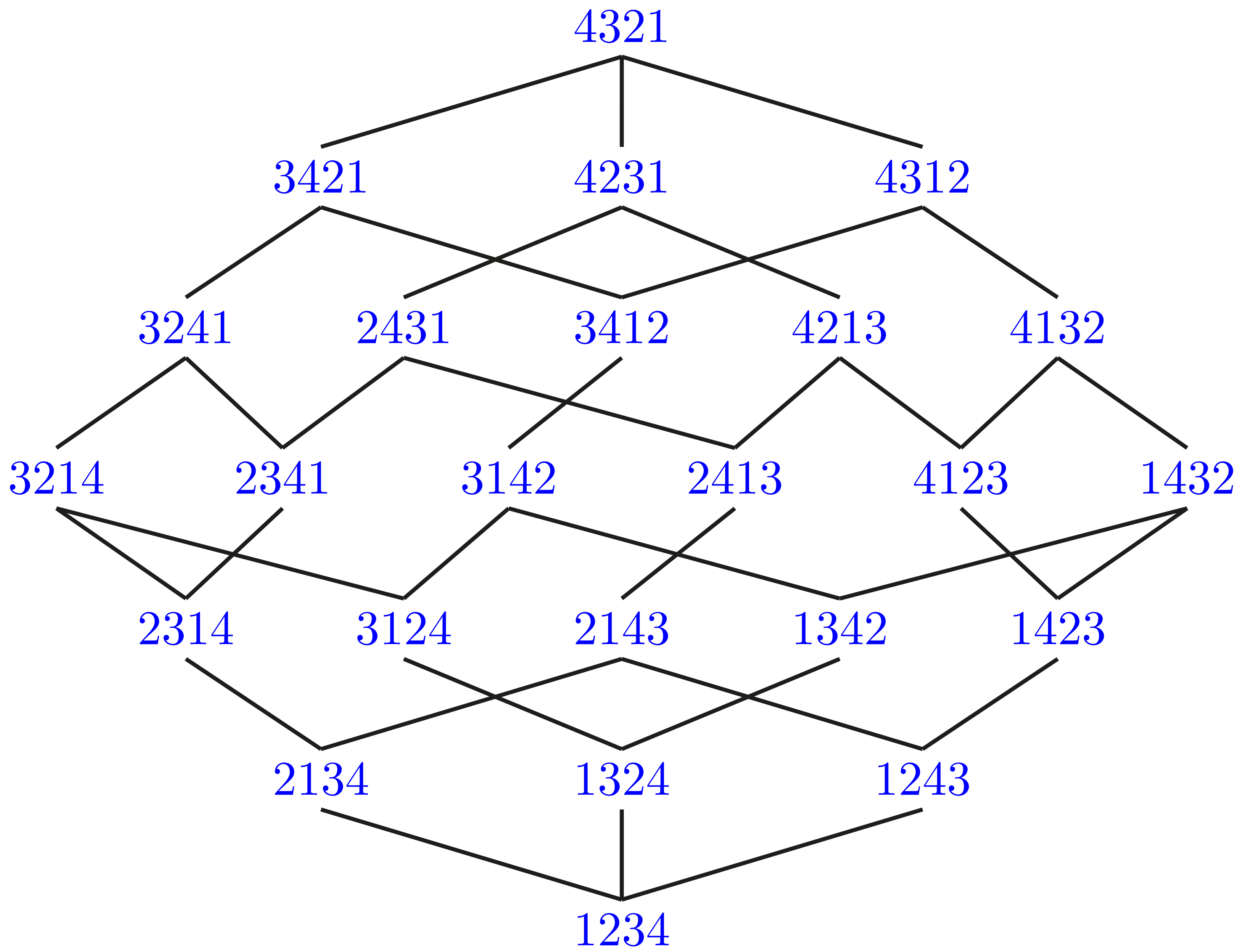
Polytopes à Paris
Journées de l'ANR CAPPS
23 – 24 janvier 2018

WEAK ORDER & PERMUTAHEDRON

WEAK ORDER

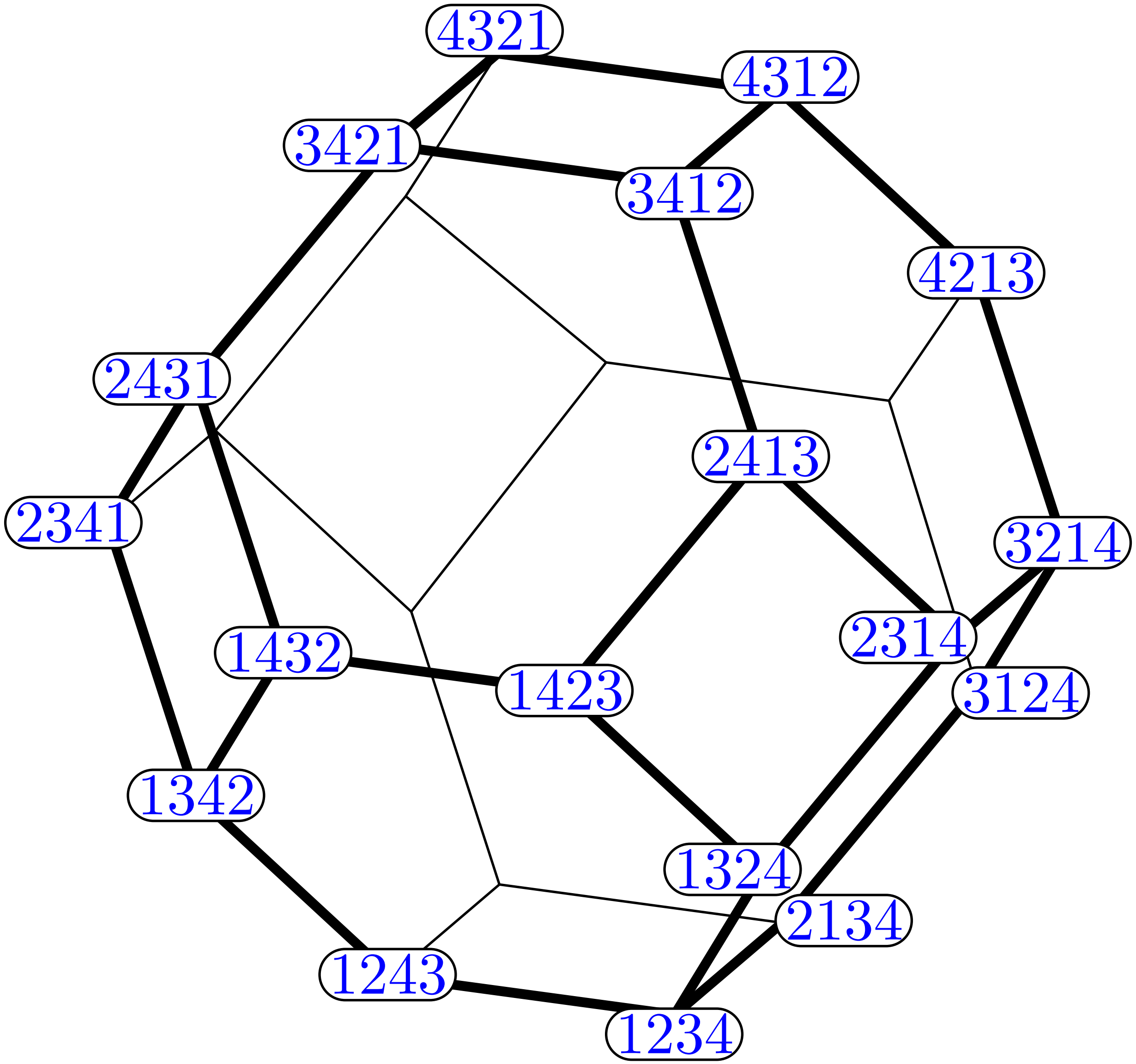
inversions of $\sigma \in \mathfrak{S}_n$ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered by inclusion of inversion sets



PERMUTAHEDRON

Permutohedron $\mathsf{Perm}(n) = \operatorname{conv}\{(\sigma(1), \dots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in \mathfrak{S}_n\}$



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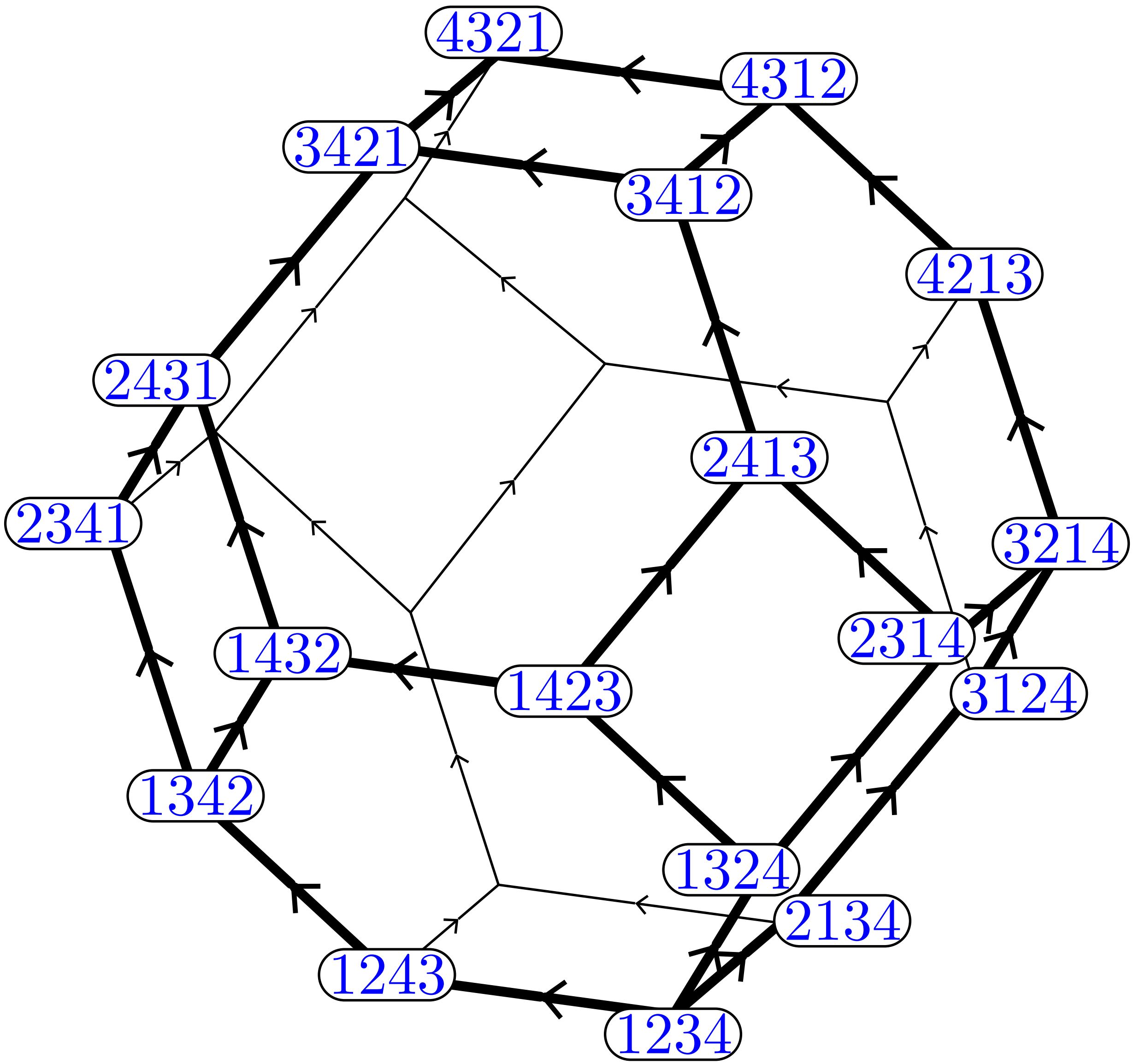
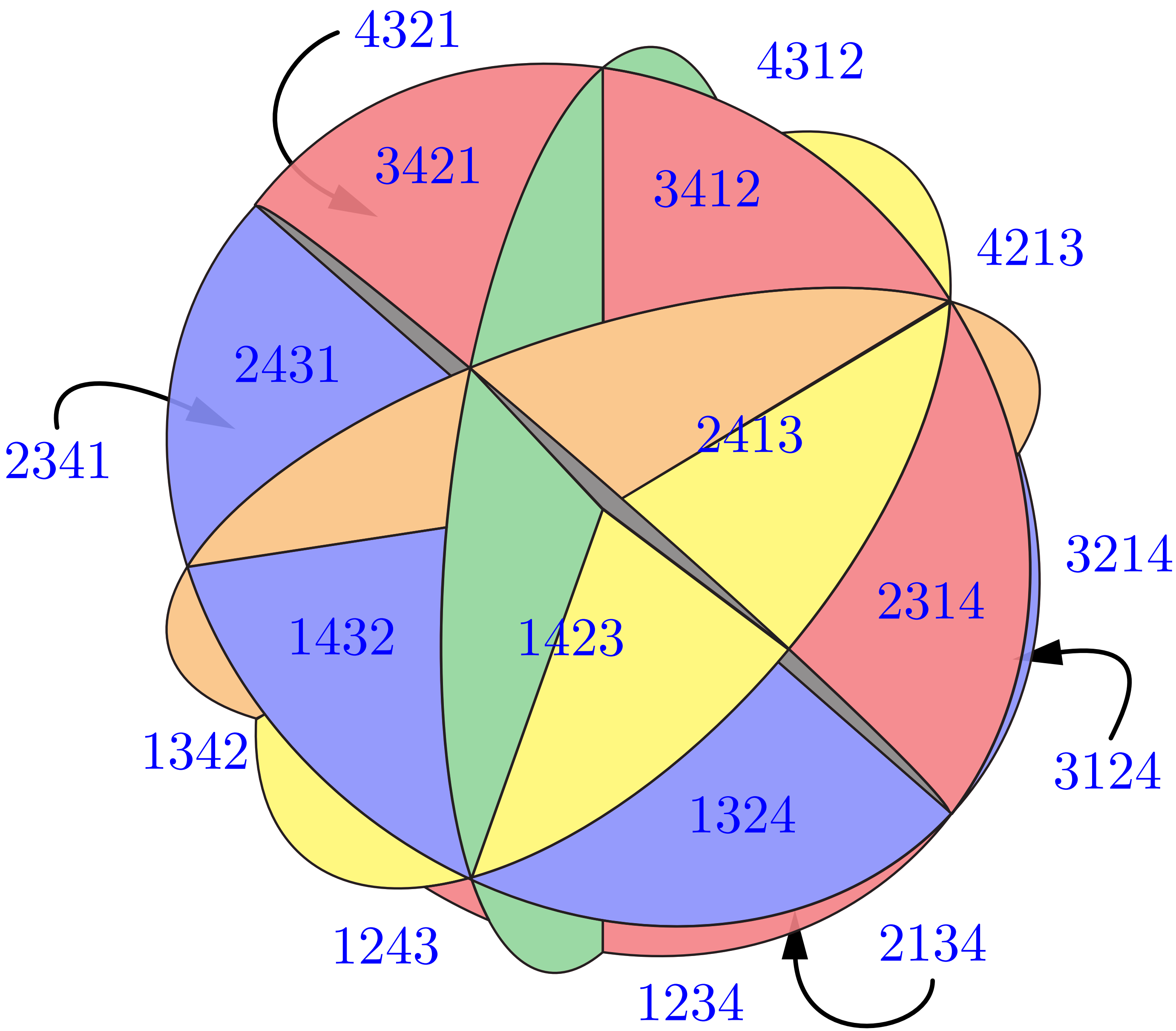


weak order = orientation of the graph of Perm(n)

COXETER ARRANGEMENT

Coxeter fan = fan defined by the hyperplane arrangement $\{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}_{1 \le i < j \le n}$



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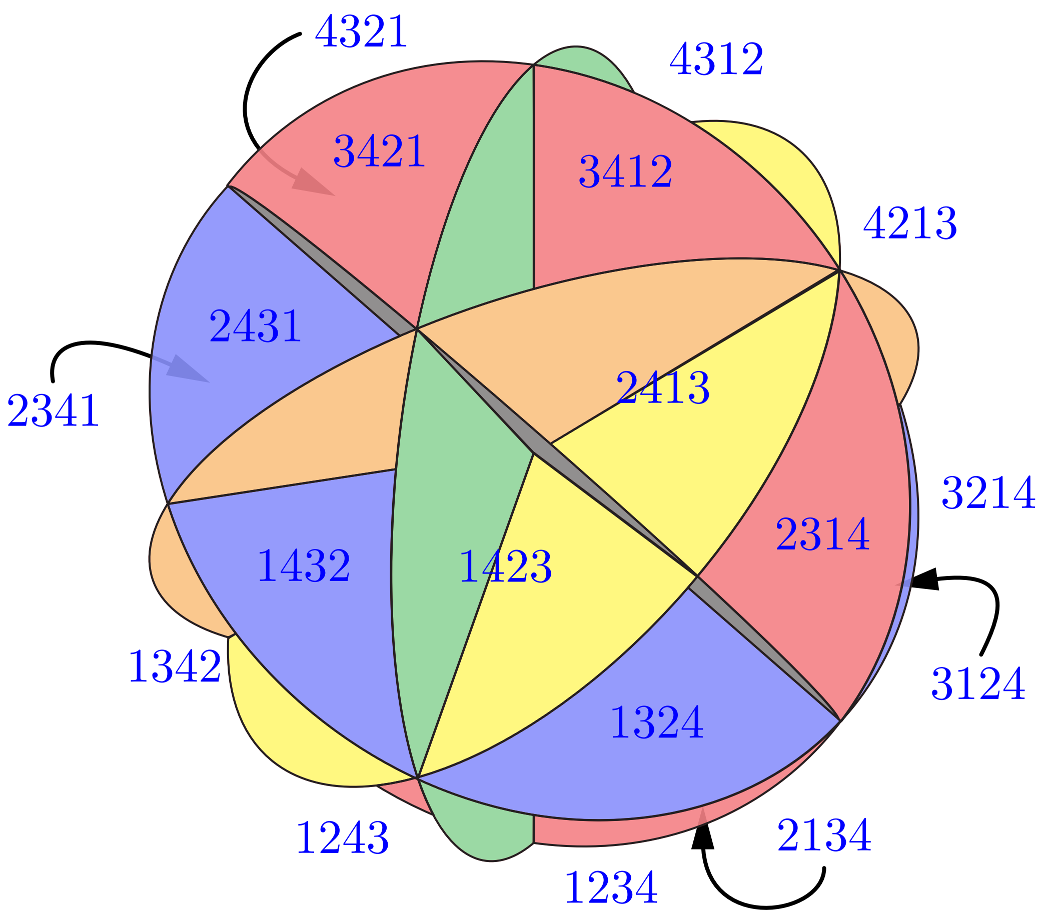
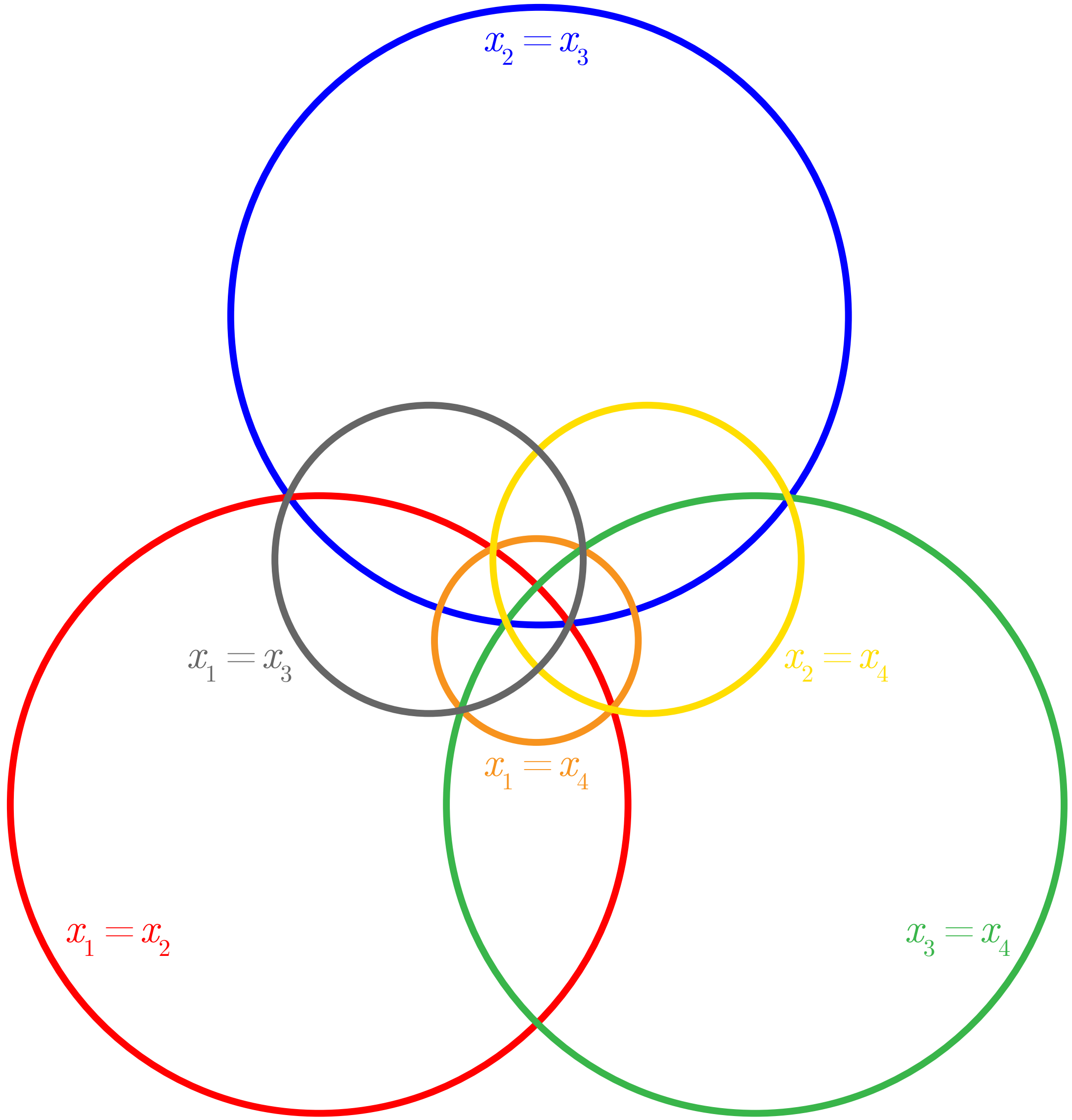




LATTICE SETUP

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

CANONICAL JOIN REPRESENTATIONS

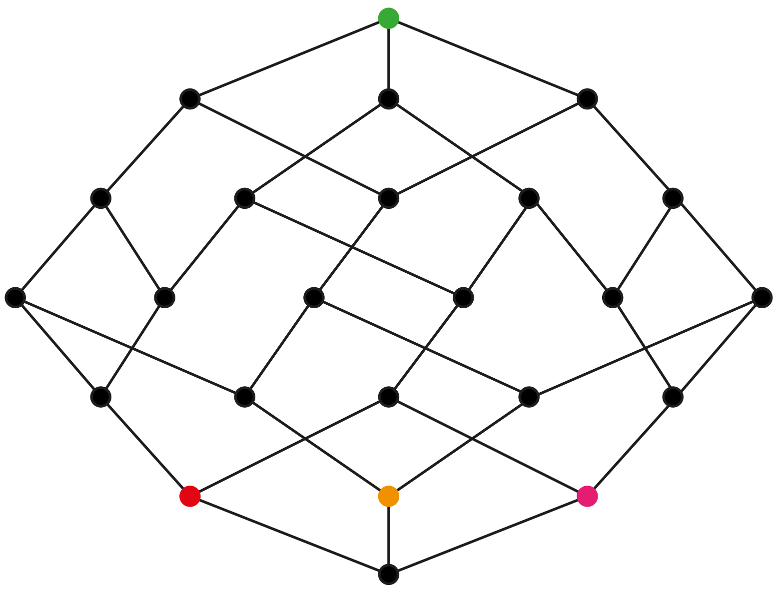
lattice = poset $(L, \leq)$ with a meet $\wedge$ and a join $\vee$

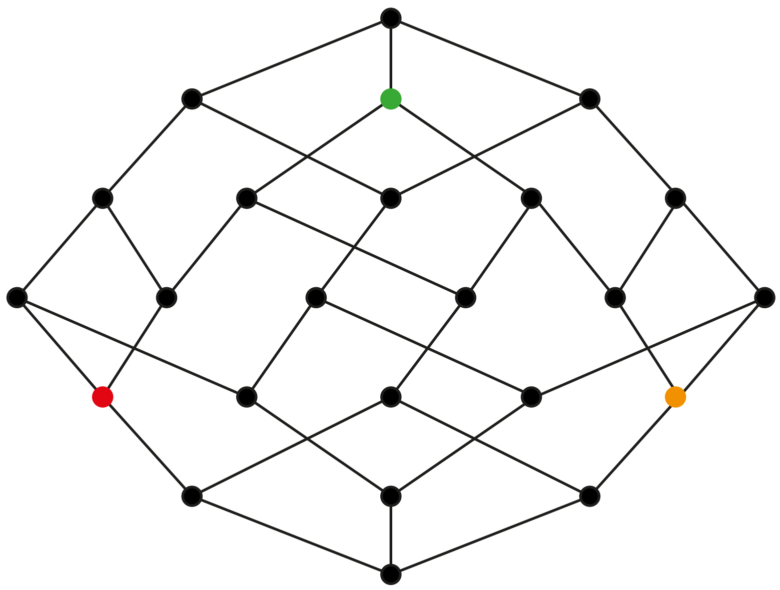
join representation of $x \in L$ = subset $J \subseteq L$ such that $x = \bigvee J$.

$x = \bigvee J$ irredundant if $\nexists\, J' \subsetneq J$ with $x = \bigvee J'$

JR are ordered by containement of order ideals: $J \leq J' \iff \forall\, y \in J,\ \exists\, y' \in J',\ y \leq y'$

canonical join representation of x = minimal irred. join representation of x (if it exists)





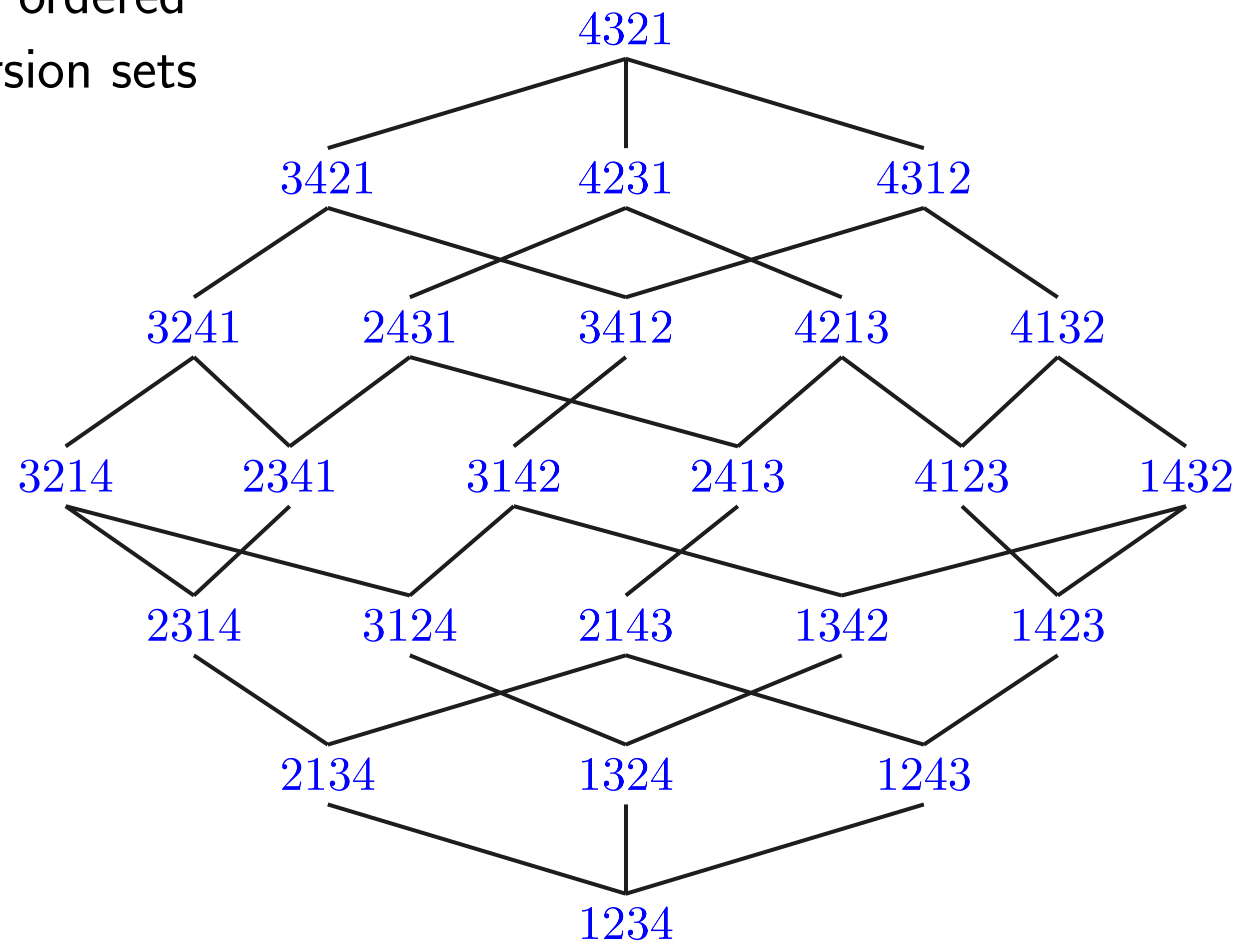
$\Longrightarrow$ "lowest way to write x as a join"

CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

σ permutation

inversions of σ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered by inclusion of inversion sets



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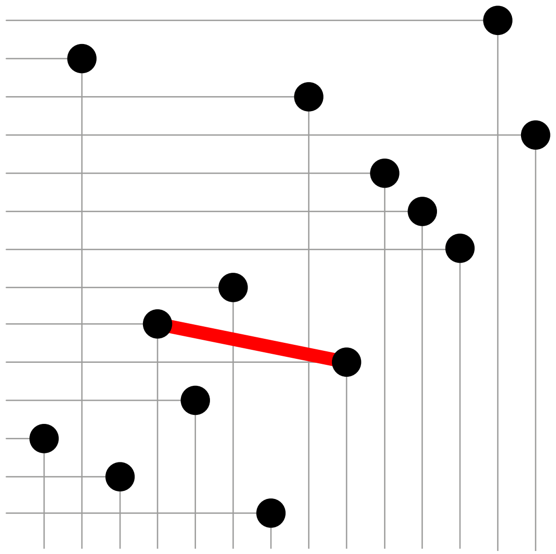
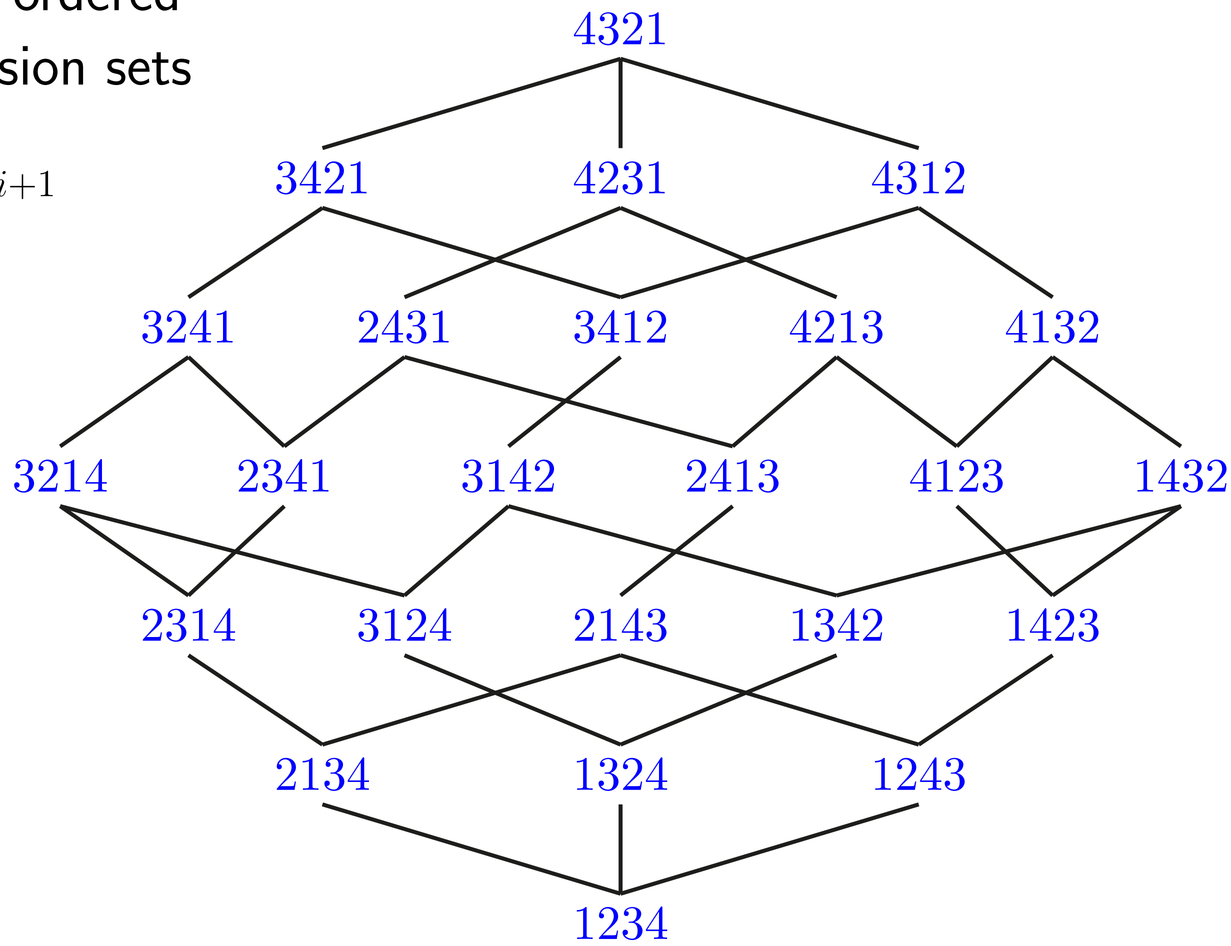
descent of σ = i such that $\sigma_i > \sigma_{i+1}$

$\sigma =$





CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

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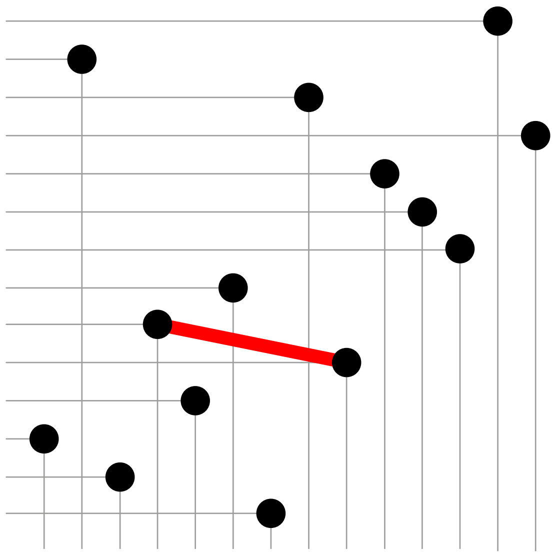
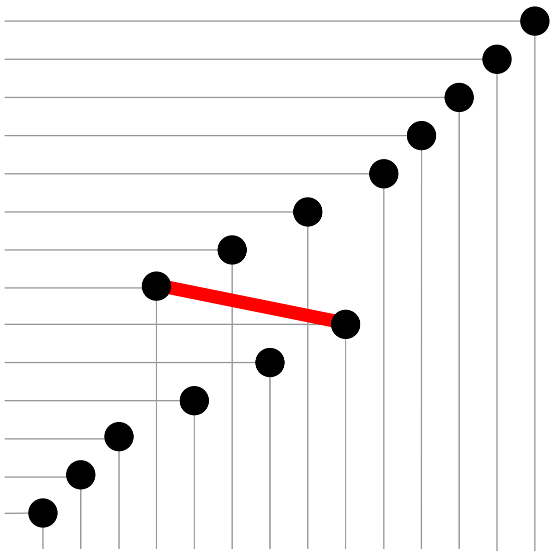
inversions of σ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered
by inclusion of inversion sets

descent of σ = i such that $\sigma_i > \sigma_{i+1}$

join-irreducible $\lambda(\sigma, i)$

$\sigma =$  $\longrightarrow$ $\lambda(\sigma, i) =$ 

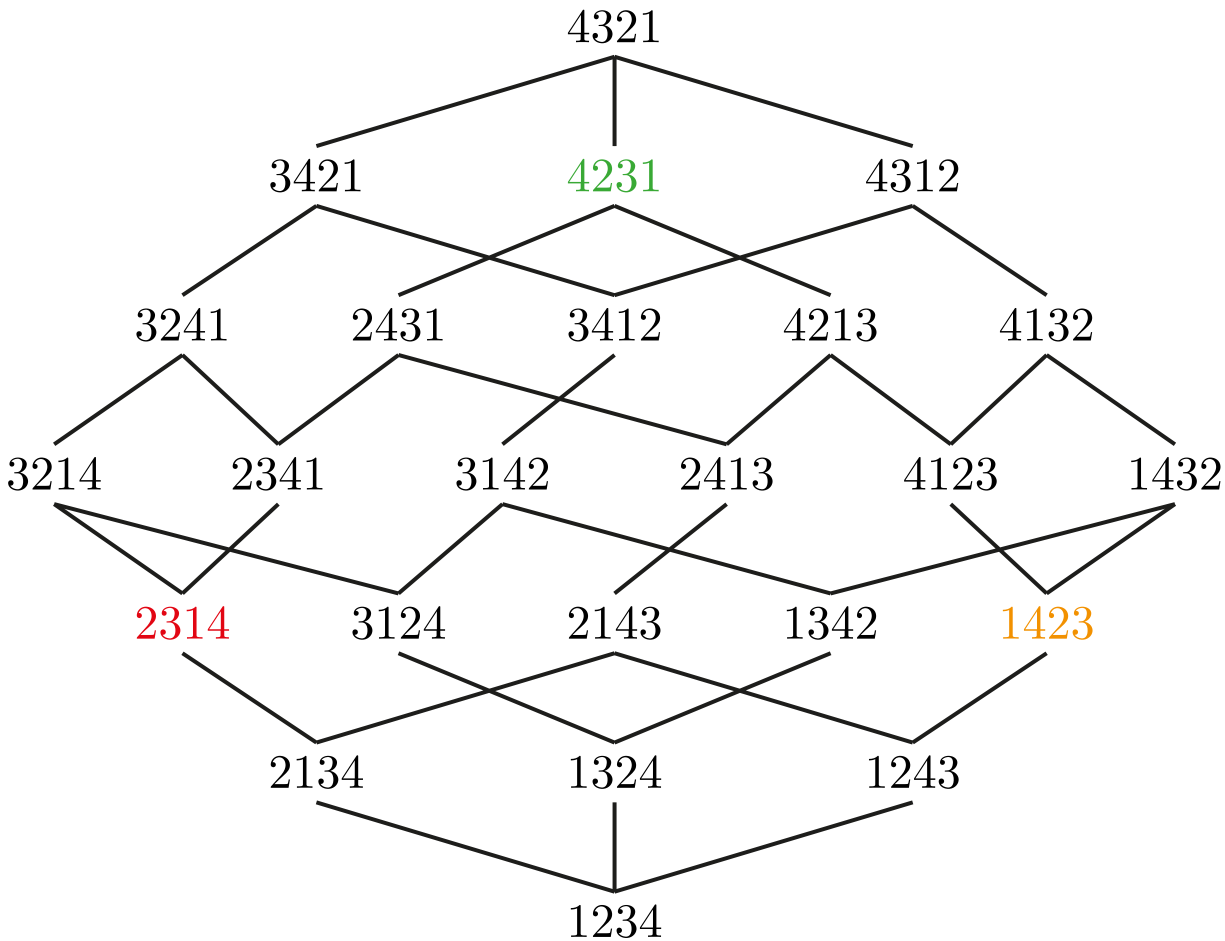
THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$.

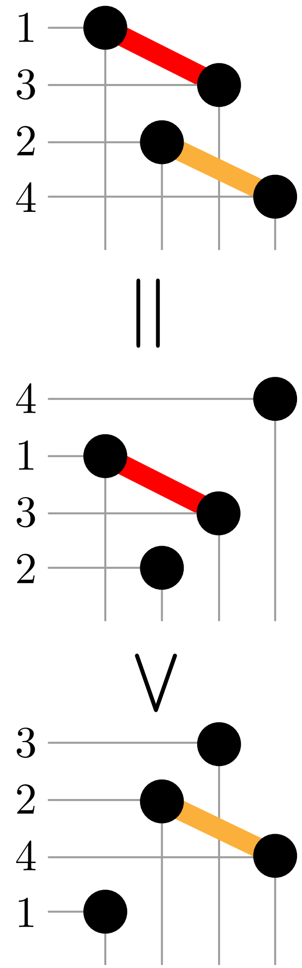
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)

CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

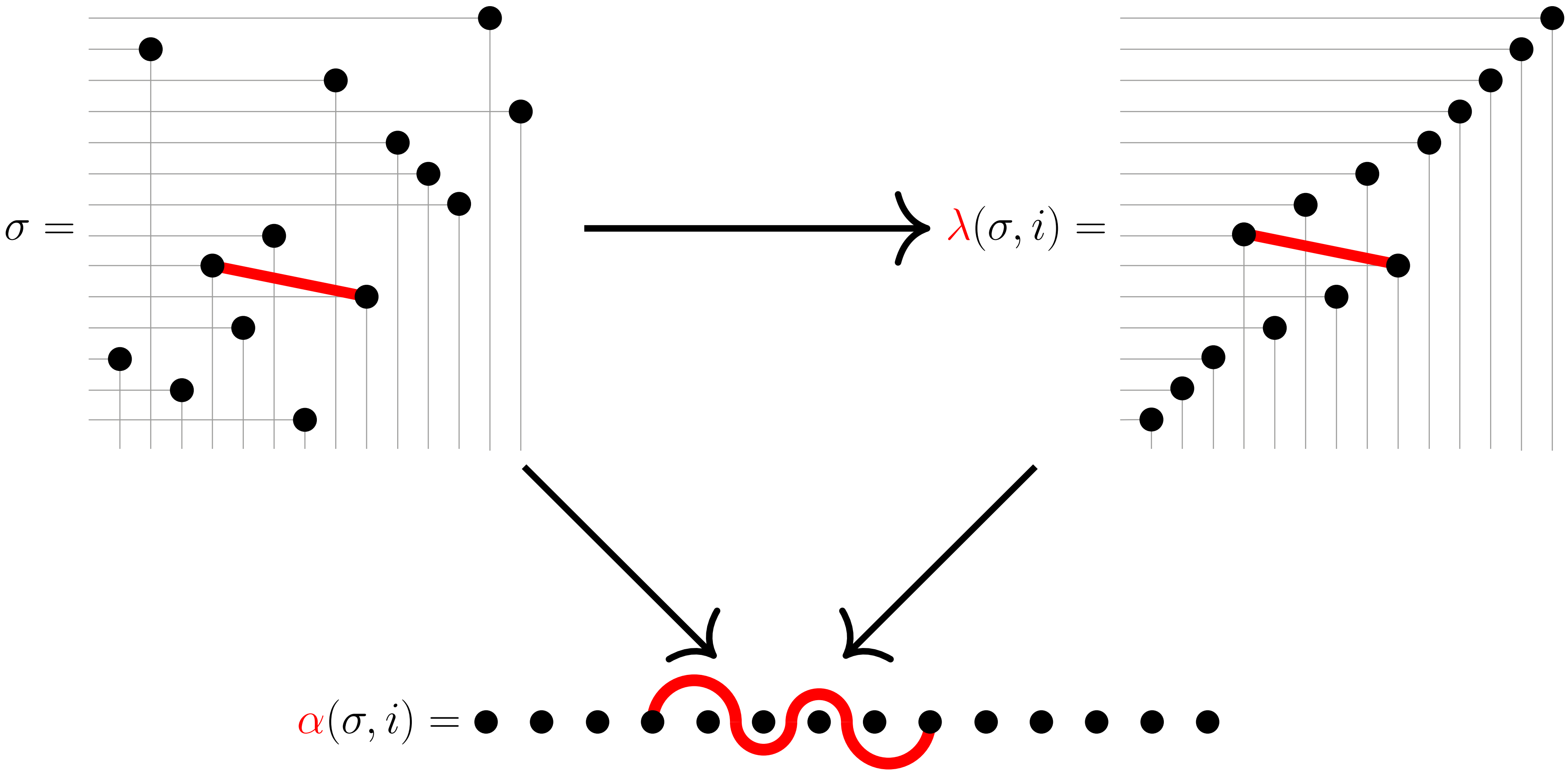
THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$.

Reading, *Noncrossing arc diagrams and canonical join representations* ('15)





ARCS



arc $= (a, b, n, S)$ with $1 \leq a < b \leq n$ and $S \subseteq \,]a, b[$

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

$\sigma = 2537146$

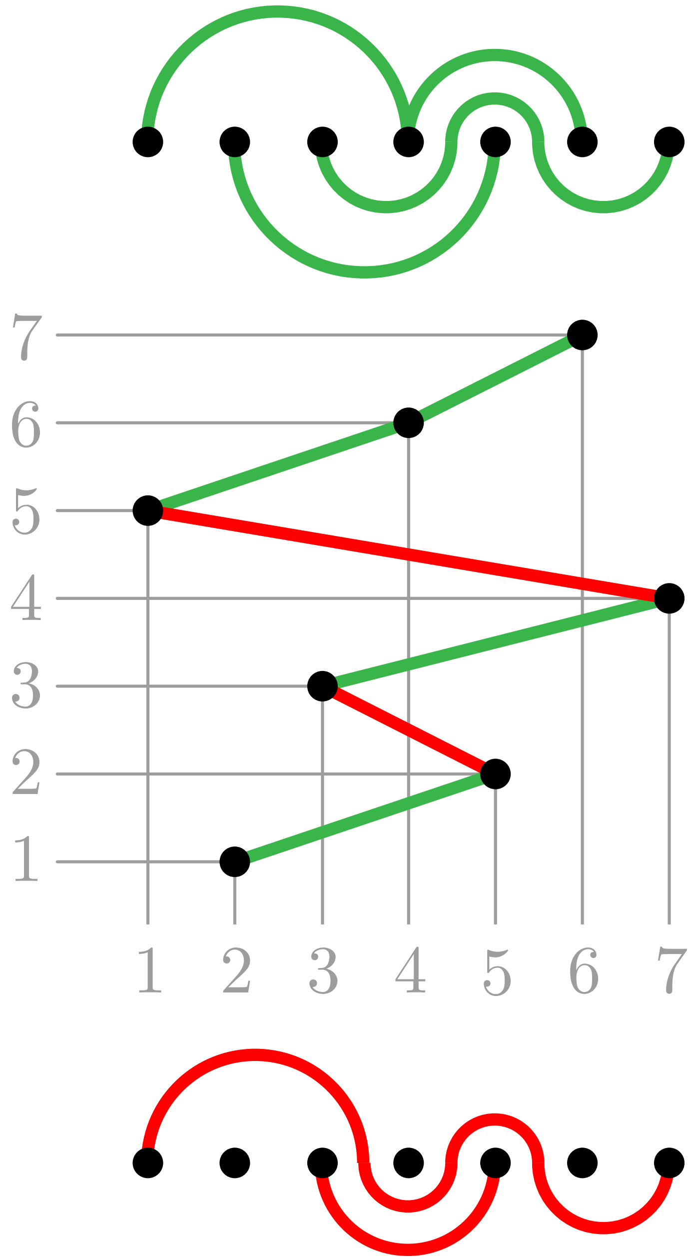
draw the table of points (σ_i, i)
draw all arcs (σ_i, i) — $(\sigma_{i+1}, i+1)$ with
descents in red and ascent in green

project down the red arcs and up the green arcs
allowing arcs to bend but not to cross or pass points

$\delta(\sigma)$ = projected red arcs
$\delta(\sigma)$ = projected green arcs

noncrossing arc diagrams = set $\mathcal{D}$ of arcs st. $\forall\, \alpha, \beta \in \mathcal{D}$:

- left$(\alpha) \neq$ left(β) and right$(\alpha) \neq$ right(β),
- α and β are not crossing.

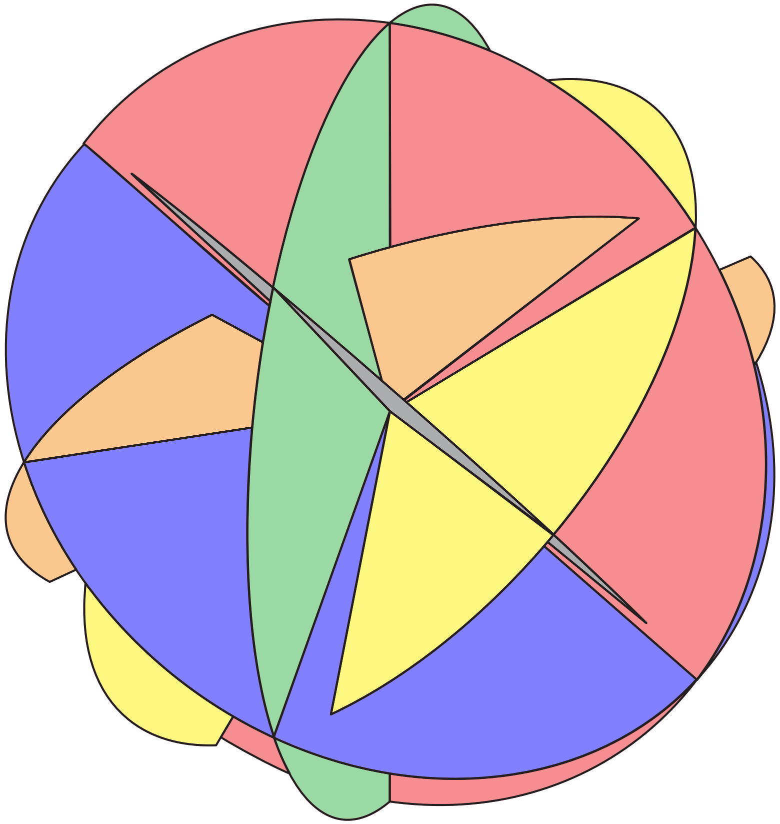


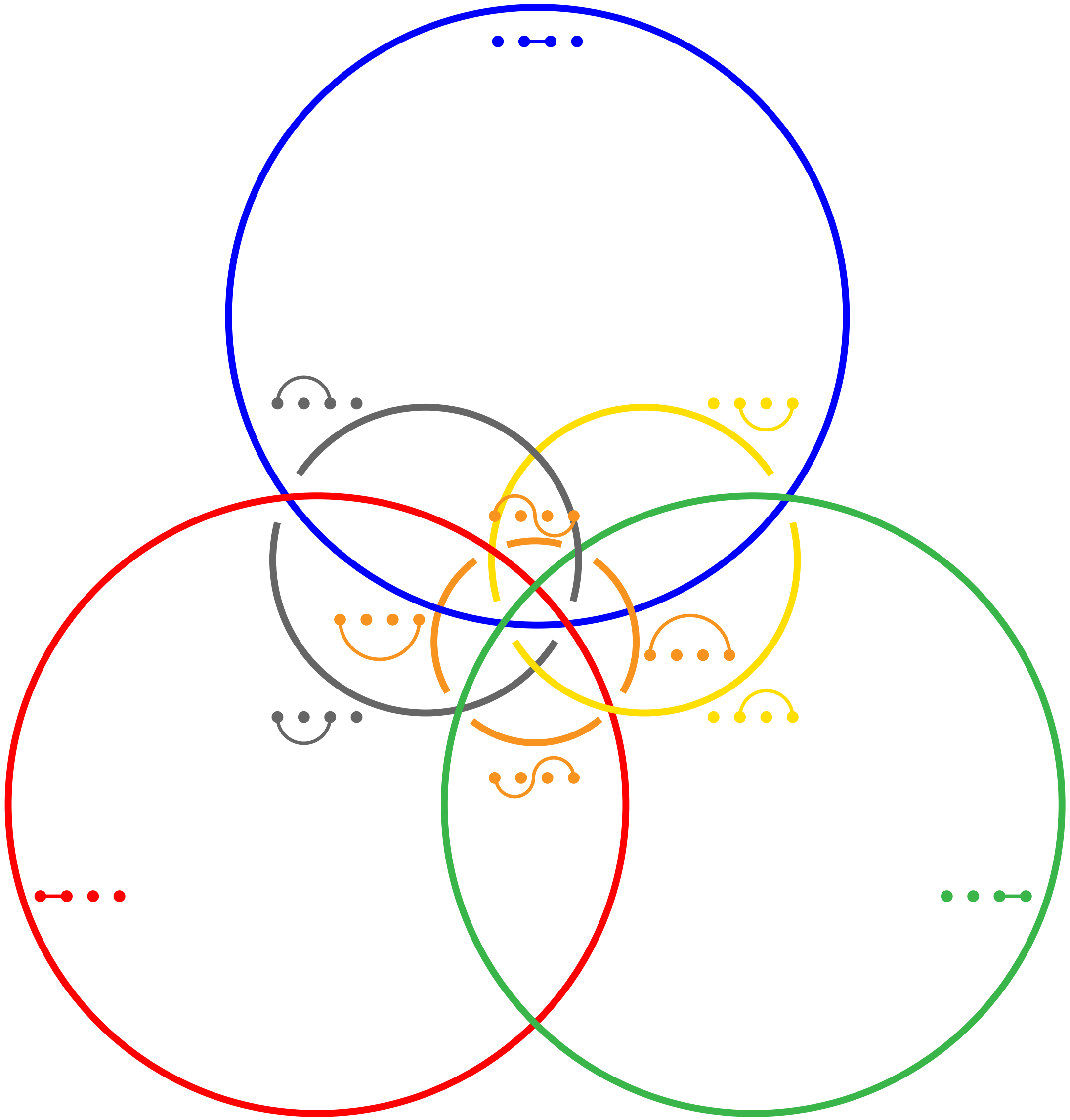
THM. $\sigma \to \delta(\sigma)$ and $\sigma \to \delta(\sigma)$ are bijections from permutations to noncrossing arc diagrams.

Reading, *Noncrossing arc diagrams and can. join representations* ('15)

SHARDS

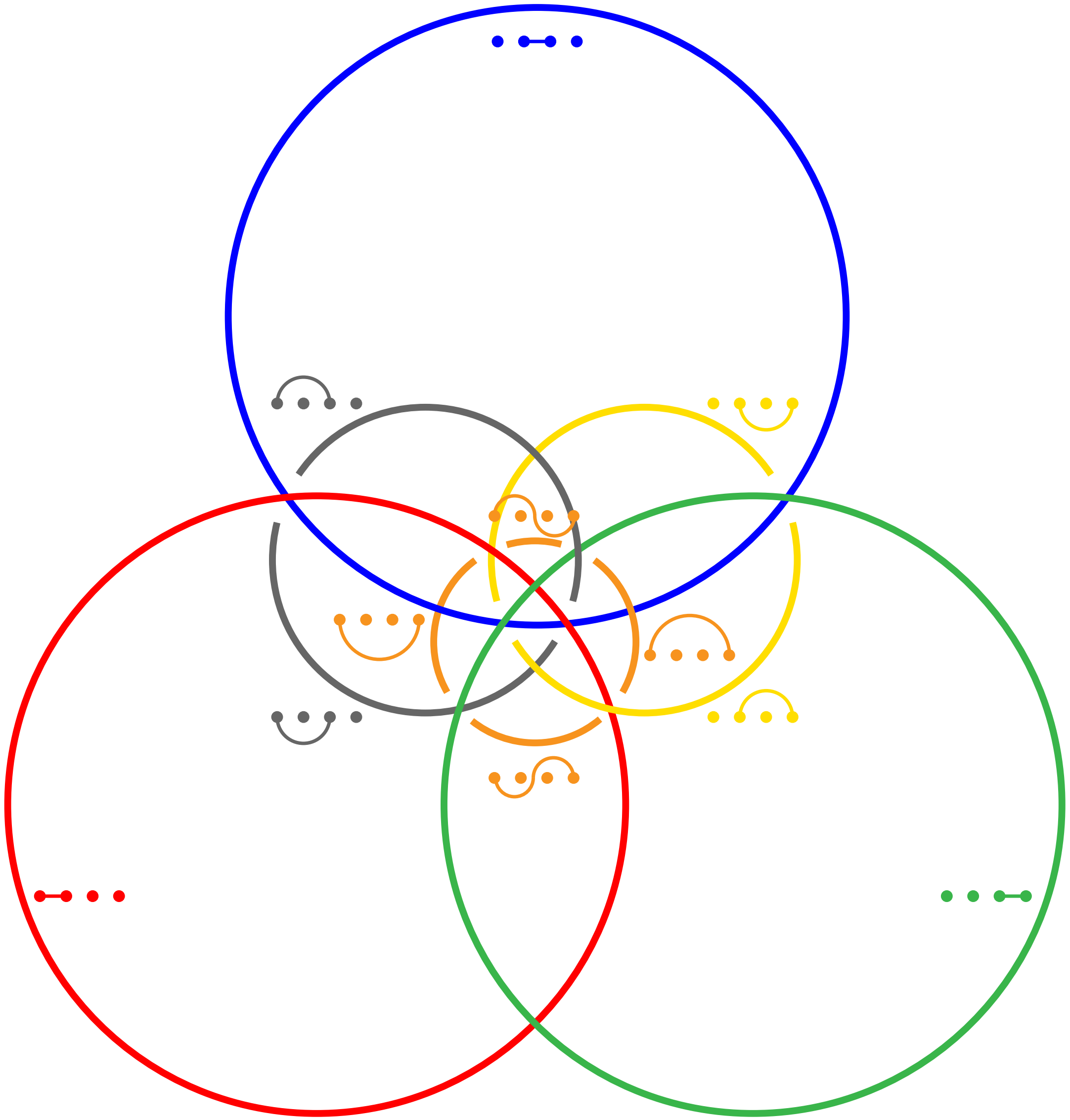
<u>shard</u> $\Sigma(i, j, n, S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \leq x_k \text{ for all } k \in S \text{ while} \\ x_i \geq x_k \text{ for all } k \in]i, j[\smallsetminus S \end{array} \right. \right\}$





SHARDS

shard $\Sigma(i, j, n, S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i, j[\smallsetminus S \end{array} \right. \right\}$



REM. The shards $\Sigma(i, j, n, S)$ for all subsets $S \subseteq \,]i, j[$ decompose the hyperplane $x_i = x_j$ into 2^{j-i-1} pieces.

REM. A chamber of the Coxeter fan is characterized by the shards below it.

LATTICE QUOTIENTS

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

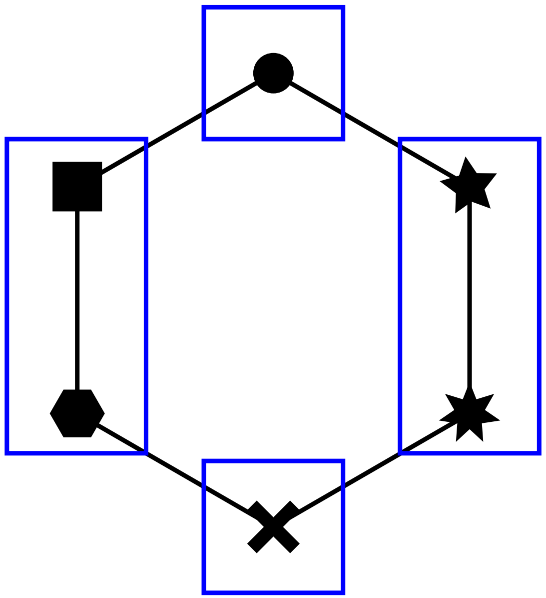
LATTICE CONGRUENCES

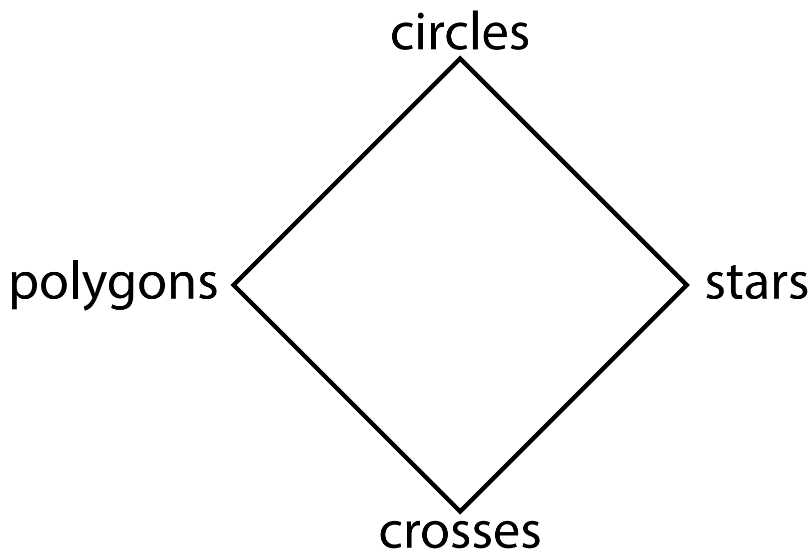
lattice congruence $=$ equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \qquad \Longrightarrow \qquad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

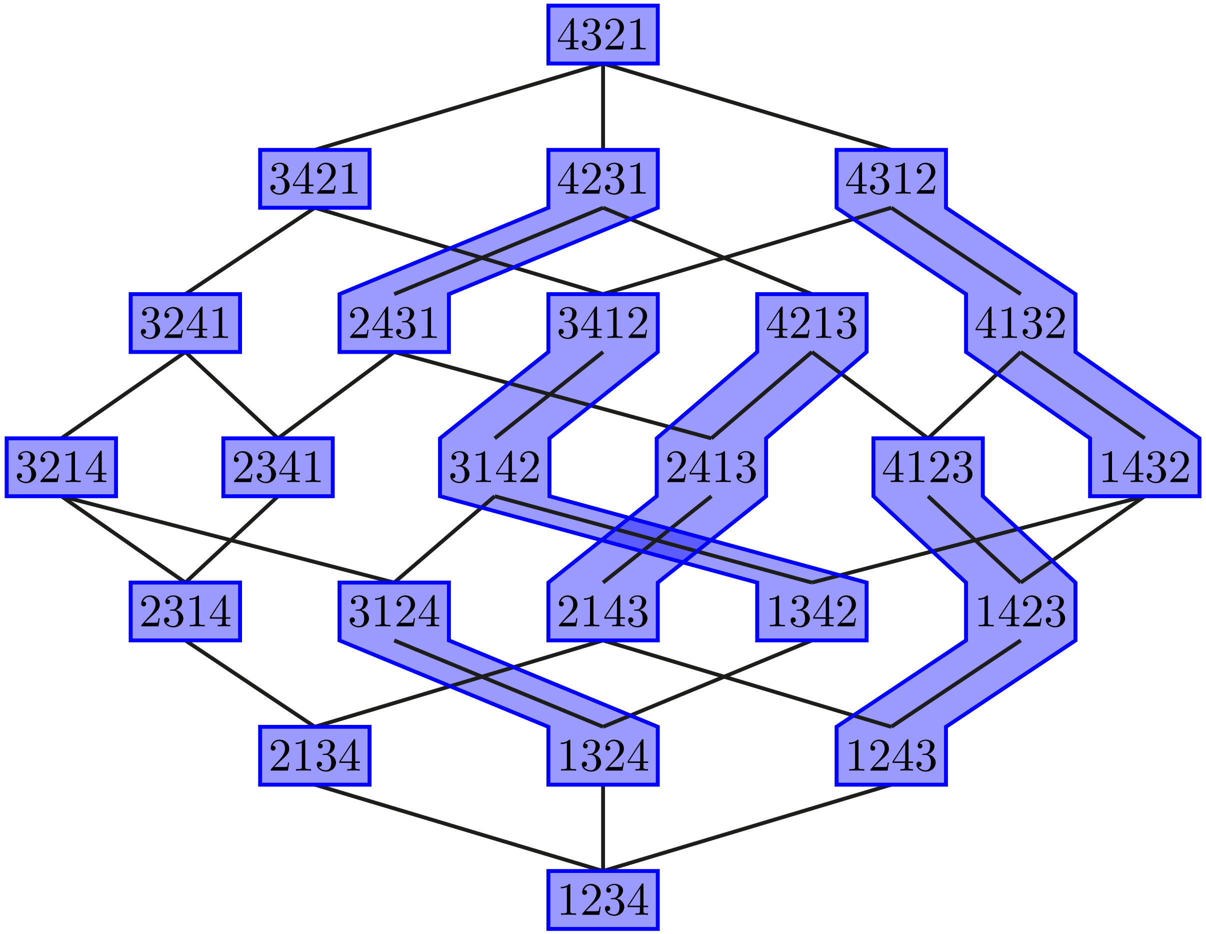
lattice quotient of $L/\equiv$ $=$ lattice on equiv. classes of L under $\equiv$ where

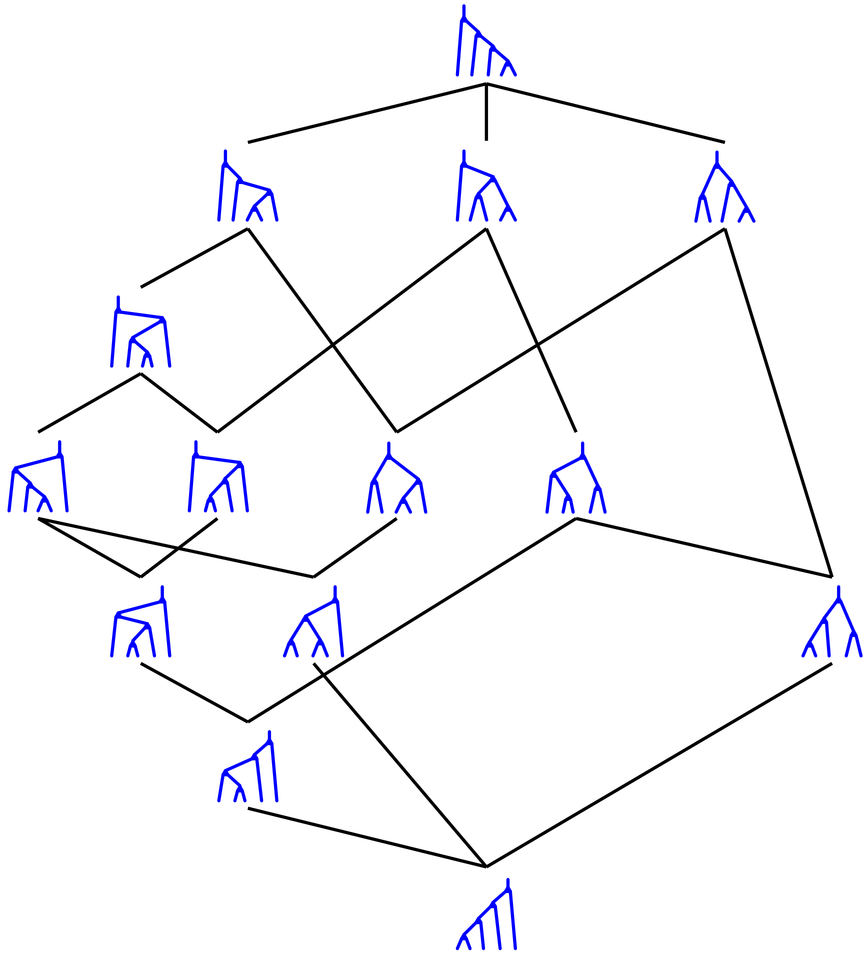
- $X \le Y \iff \exists x \in X,\ y \in Y,\quad x \le y$
- $X \wedge Y$ $=$ equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ $=$ equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$





EXM: TAMARI LATTICE





Tamari lattice = lattice quotient of the weak order by the relation "same binary tree"

Catalan combinatorics — Associahedron — Non-crossing partitions — ...

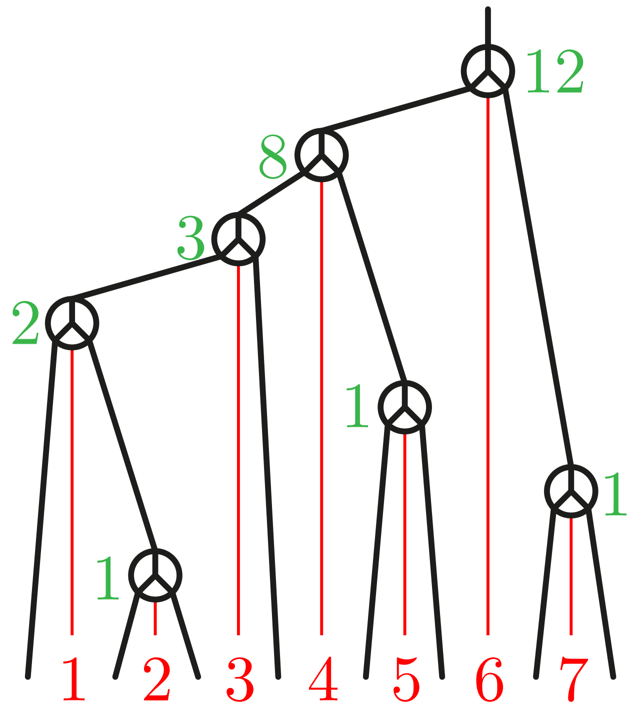
LODAY'S ASSOCIAHEDRON

$$\mathrm{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T}\ \text{binary tree}\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n+1} \mathbf{H}^{\geq}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\big]_{i\in[n+1]} \qquad \mathbf{H}^{\geq}(i,j) := \left\{\mathbf{x}\in\mathbb{R}^{n+1} \;\middle|\; \sum_{i\le k\le j} x_i \geq \binom{j-i+2}{2}\right\}$$

Shnider-Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)

Loday, *Realization of the Stasheff polytope* ('04)



LODAY'S ASSOCIAHEDRON

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Shnider-Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)
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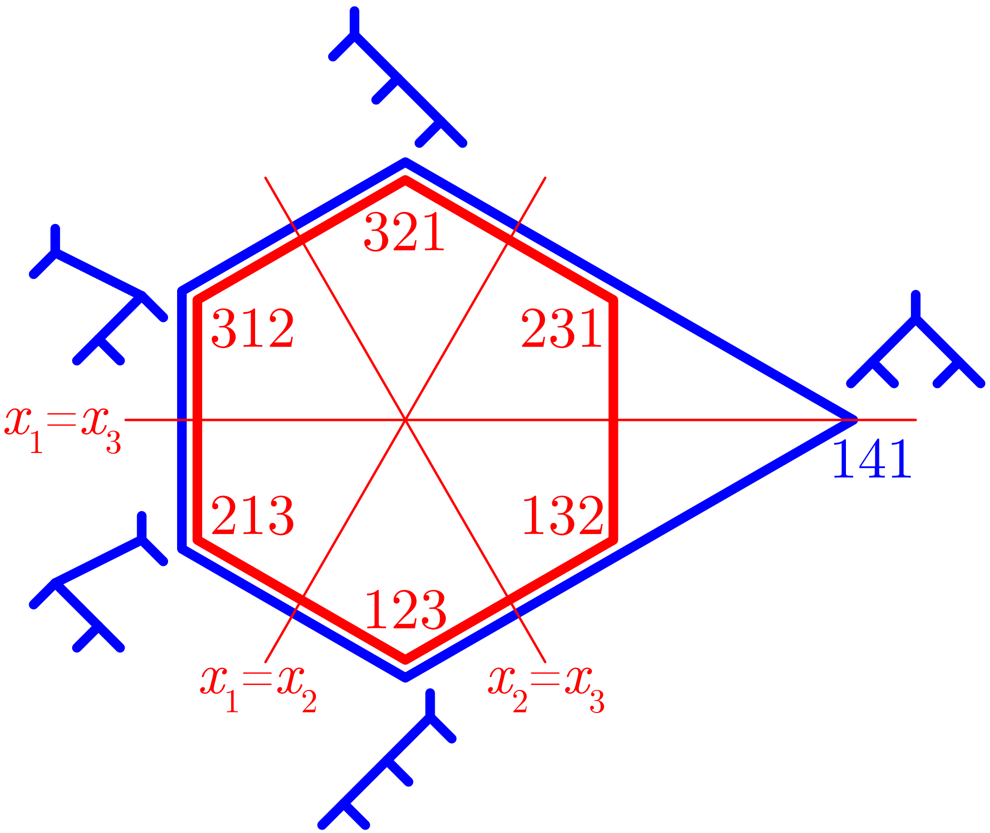
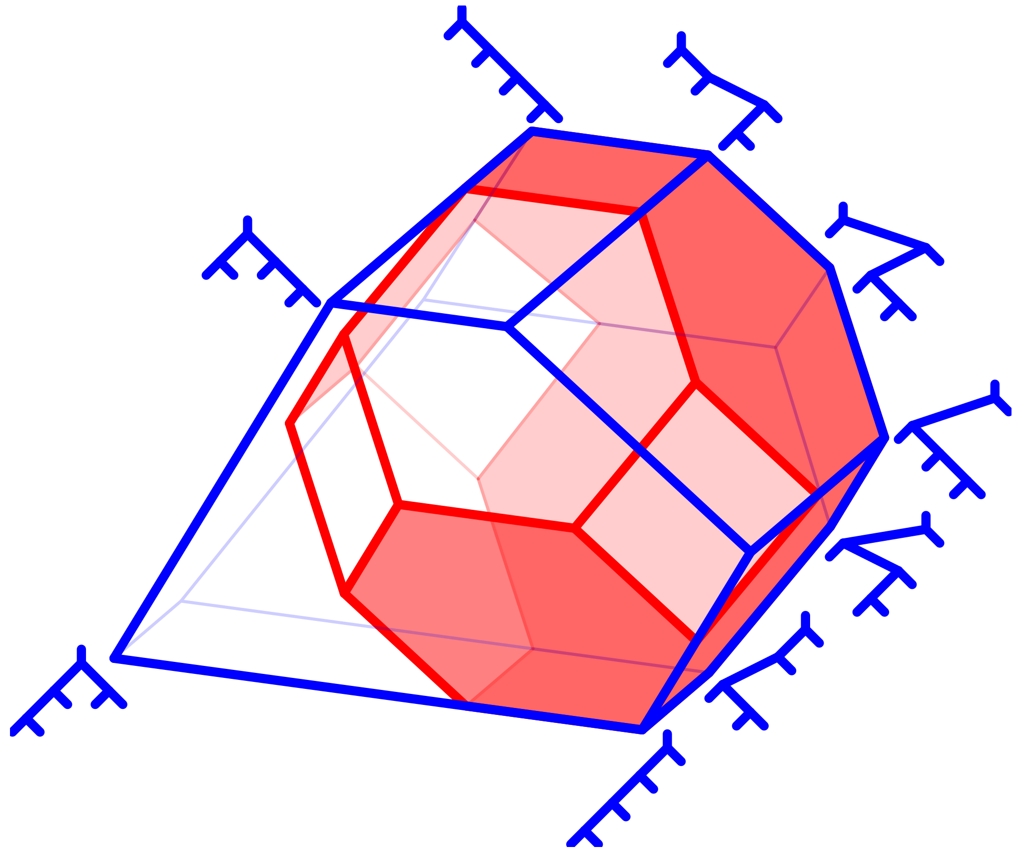
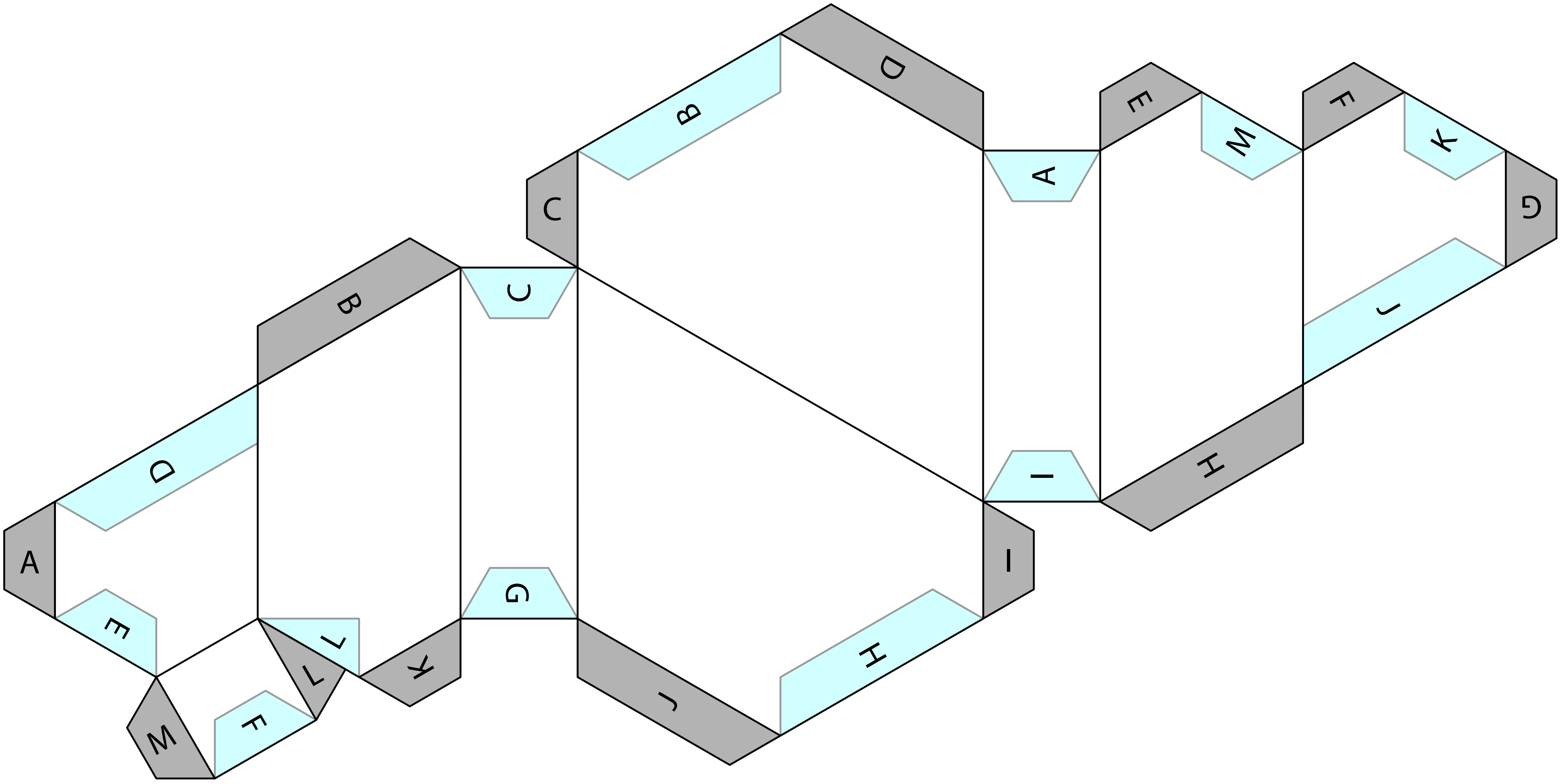




POLYWOOD

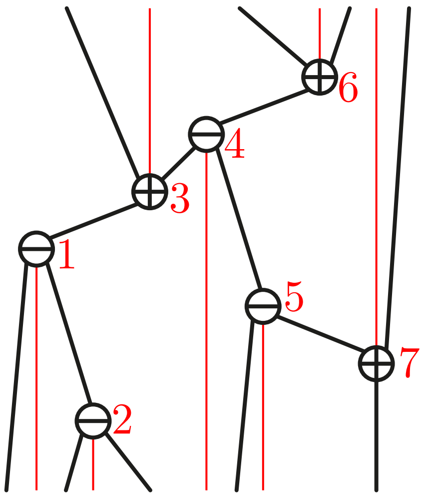
POLYWOOD



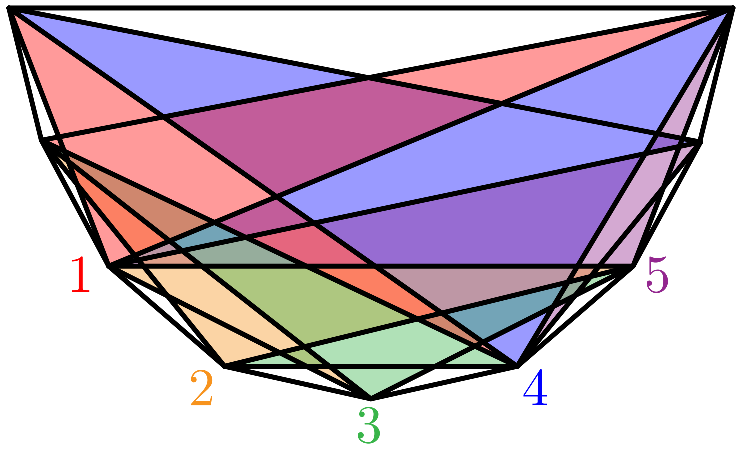
LODAY'S ASSOCIAHEDRON

RELEVANT LATTICE QUOTIENTS OF THE WEAK ORDER

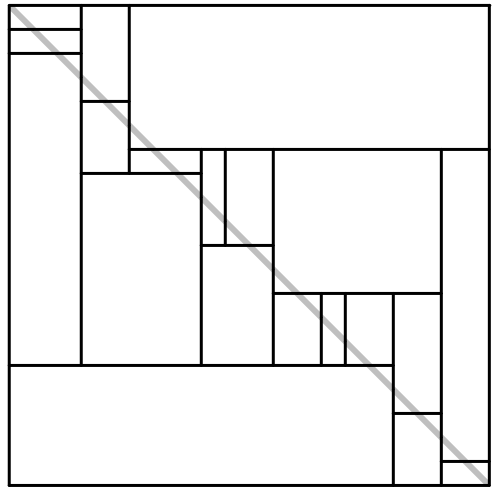
Cambrian trees



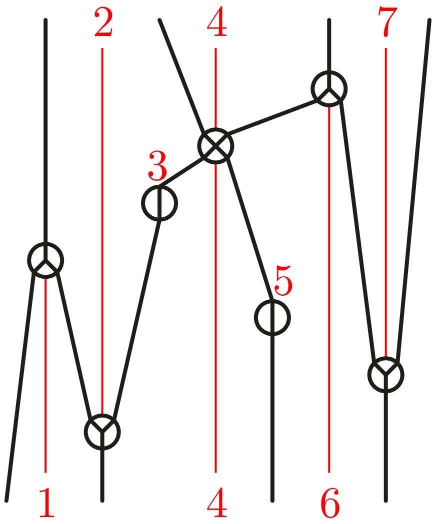
Acyclic k-triangulations



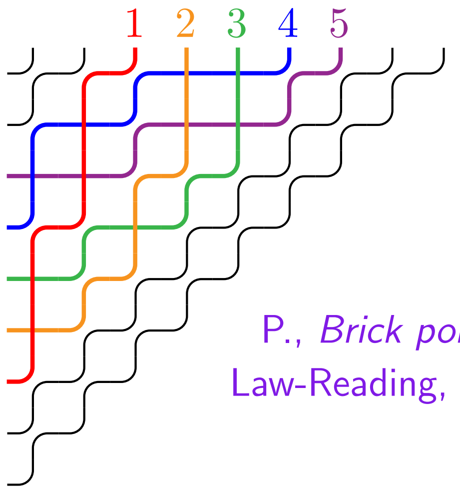
diagonal rectangulations



Permutrees



Pipe dreams

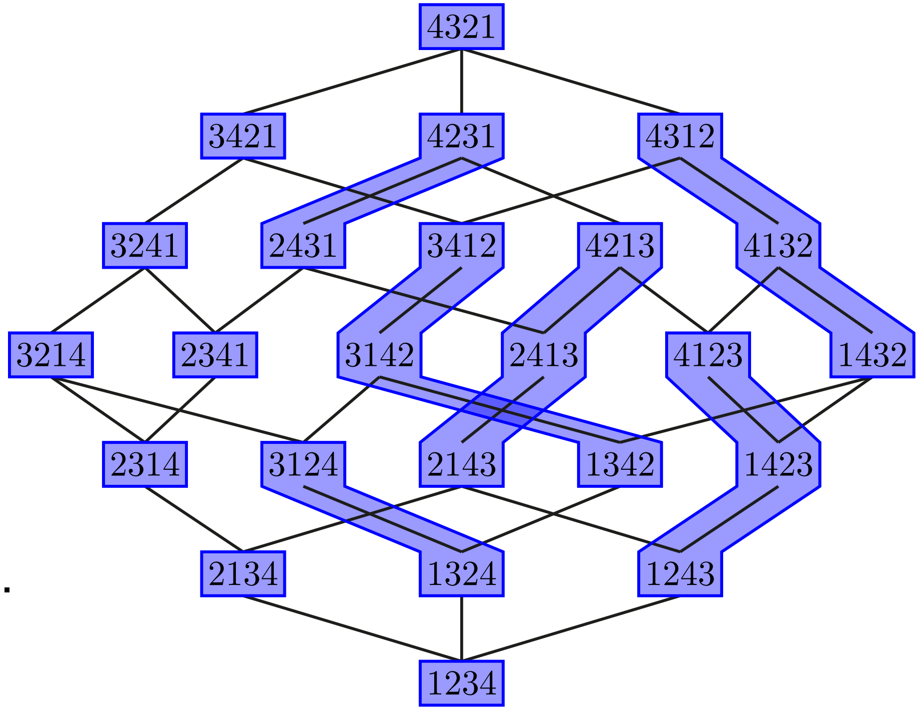


Reading, *Cambrian lattices* ('06)
Chatel-P., *Cambrian algebra* ('17)
P., *Brick polytopes, lattice quotients, and Hopf algebras* ('15$^+$)
Law-Reading, *The Hopf algebra of diagonal rectangulations* ('12)
P.-Pons, *Permutrees* ('17)

LATTICE QUOTIENTS AND CANONICAL JOIN REPRESENTATIONS

$\equiv$ lattice congruence on L, then

- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/\equiv$ is isomorphic to $\pi_\downarrow(L)$ (as poset)
- canonical join representations in $L/\equiv$ are canonical join representations in L that do not involve join irreducibles x with $\pi_\downarrow(x) \neq x$.

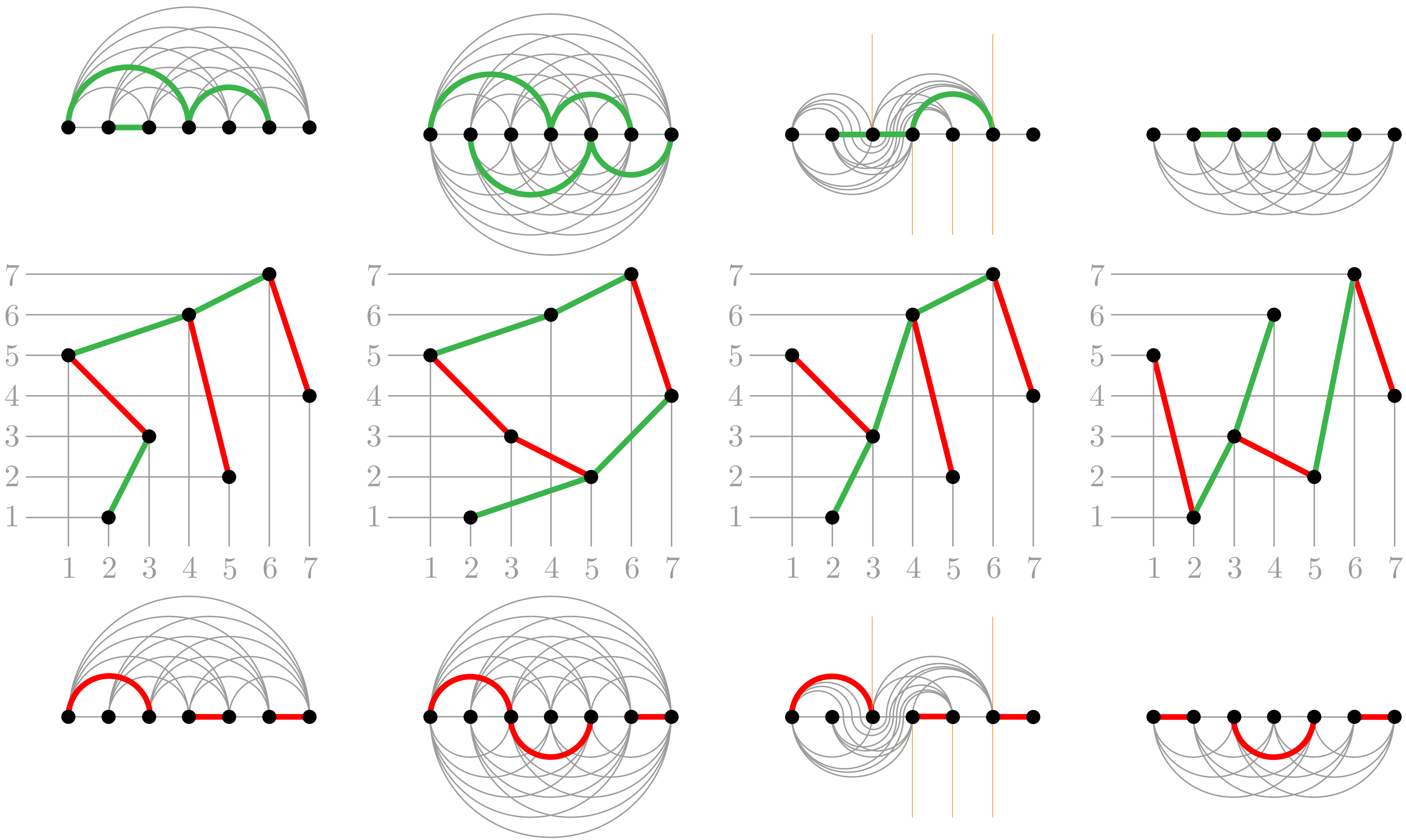


THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
Let $\mathcal{I}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$
Then

- $\pi_\downarrow(\sigma) = \sigma \iff \delta(\sigma) \subseteq \mathcal{I}_\equiv$.
- the map $\mathfrak{S}_n/\equiv \longrightarrow$ {nc arc diagrams in $\mathcal{I}_\equiv$} is a bijection.
  $X \longmapsto \delta(\pi_\downarrow(X))$

Reading, *Noncrossing arc diagrams and can. join representations* ('15)

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN



binary trees | diagonal quadrangulations | permutrees | k-sashes

FORCING AND ARC IDEALS

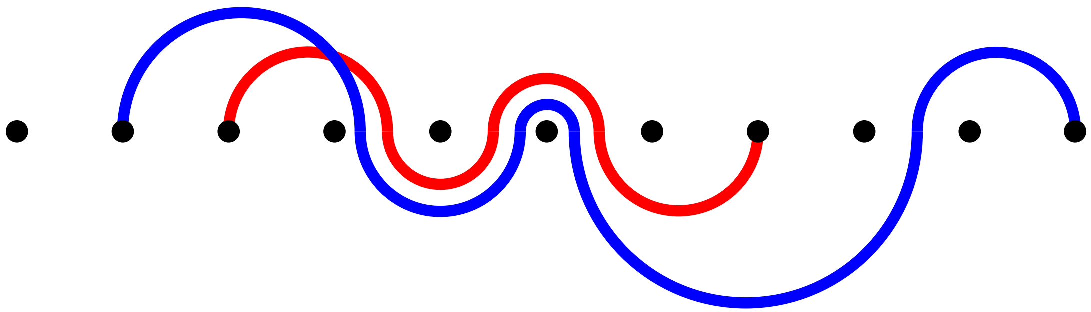
THM. $\mathcal{I}_{\equiv}$ = arcs corresponding to join irreducibles σ with $\pi_{\downarrow}(\sigma) = \sigma$.
Bijection $\mathfrak{S}_n/{\equiv} \longleftrightarrow$ {nc arc diagrams in $\mathcal{I}_{\equiv}$}.

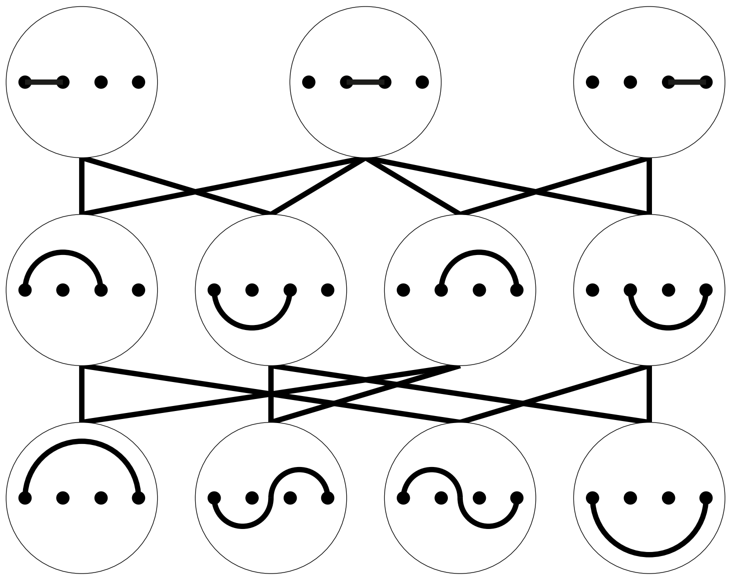
What sets of arcs can be $\mathcal{I}_{\equiv}$?

THM. The following are equivalent for a set of arcs $\mathcal{I}$:

- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{I} = \mathcal{I}_{\equiv}$,
- $\mathcal{I}$ is an upper ideal for the order $(a, d, n, S) \prec (b, c, n, T) \iff a \le b < c \le d$ and $T = S \cap \,]b, c[$.

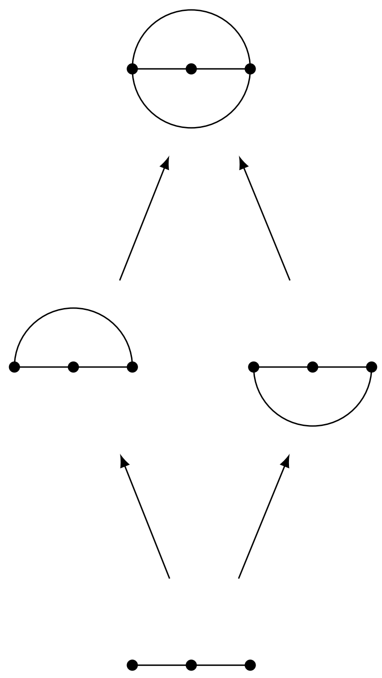
Reading, *Noncrossing arc diagrams and can. join representations* ('15)



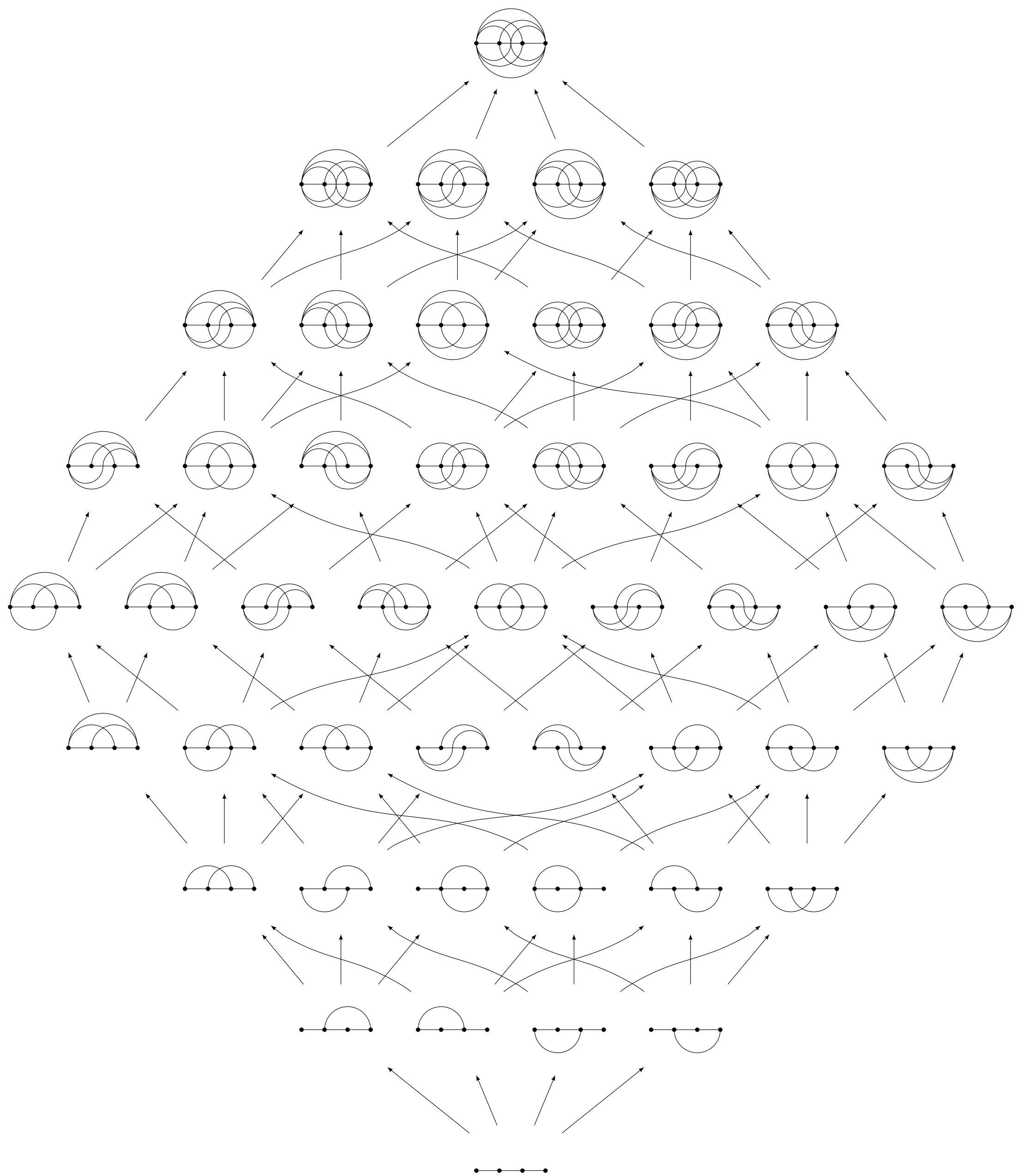


ARC IDEALS

arc ideal = ideal of the forcing poset on arcs = subsets of arcs closed by forcing



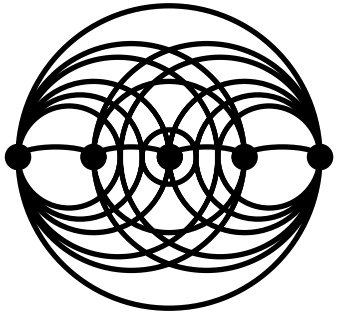
1, 1, 4, 47, 3322, ...
1, 2, 7, 60, 3444, ...
OEIS A091687



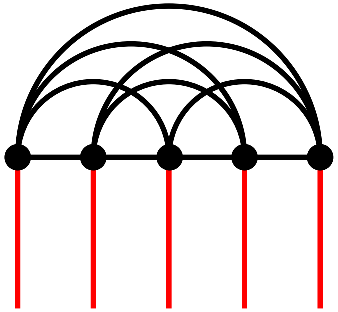
BOUNDED CROSSINGS ARC IDEALS

arc ideal = ideal of the forcing poset on arcs = subsets of arcs closed by forcing

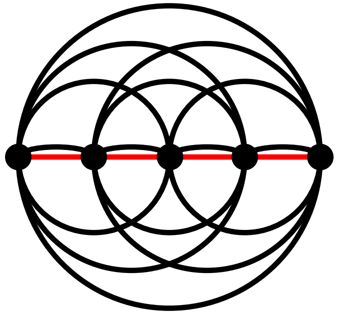
fix $k \geq 0$ and some red walls above, below and in between the points
allow arcs that cross at most k walls



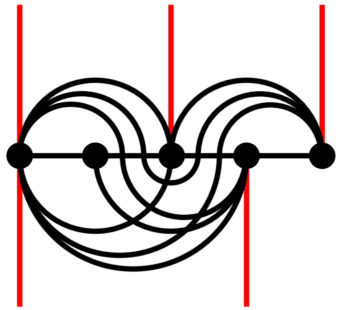
weak order



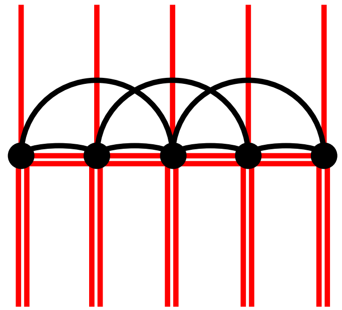
Tamari lattice



diagonal rectangulations



Cambrian lattices



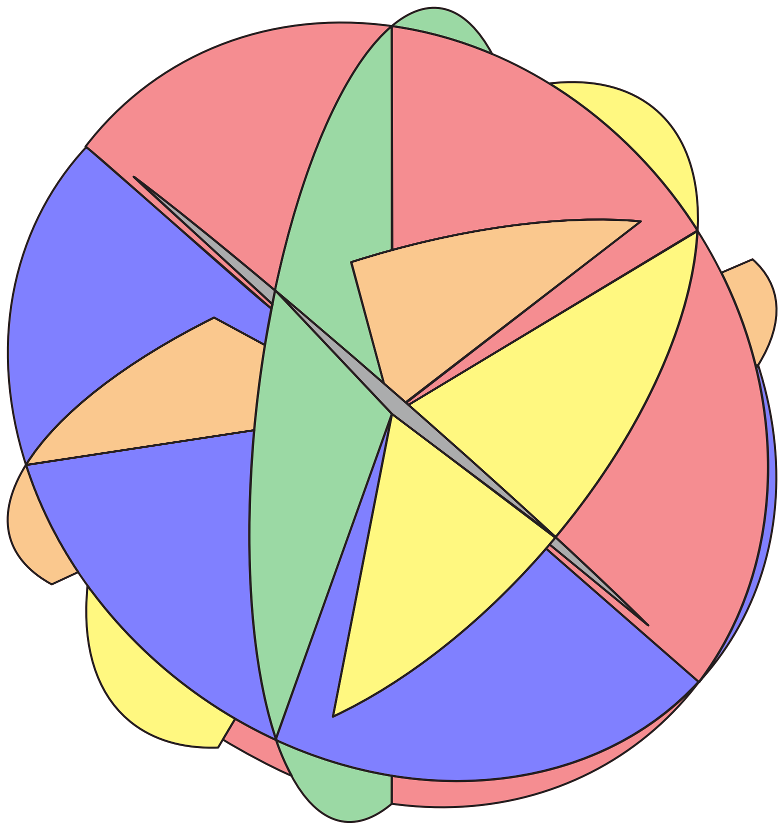
k-sashes lattices

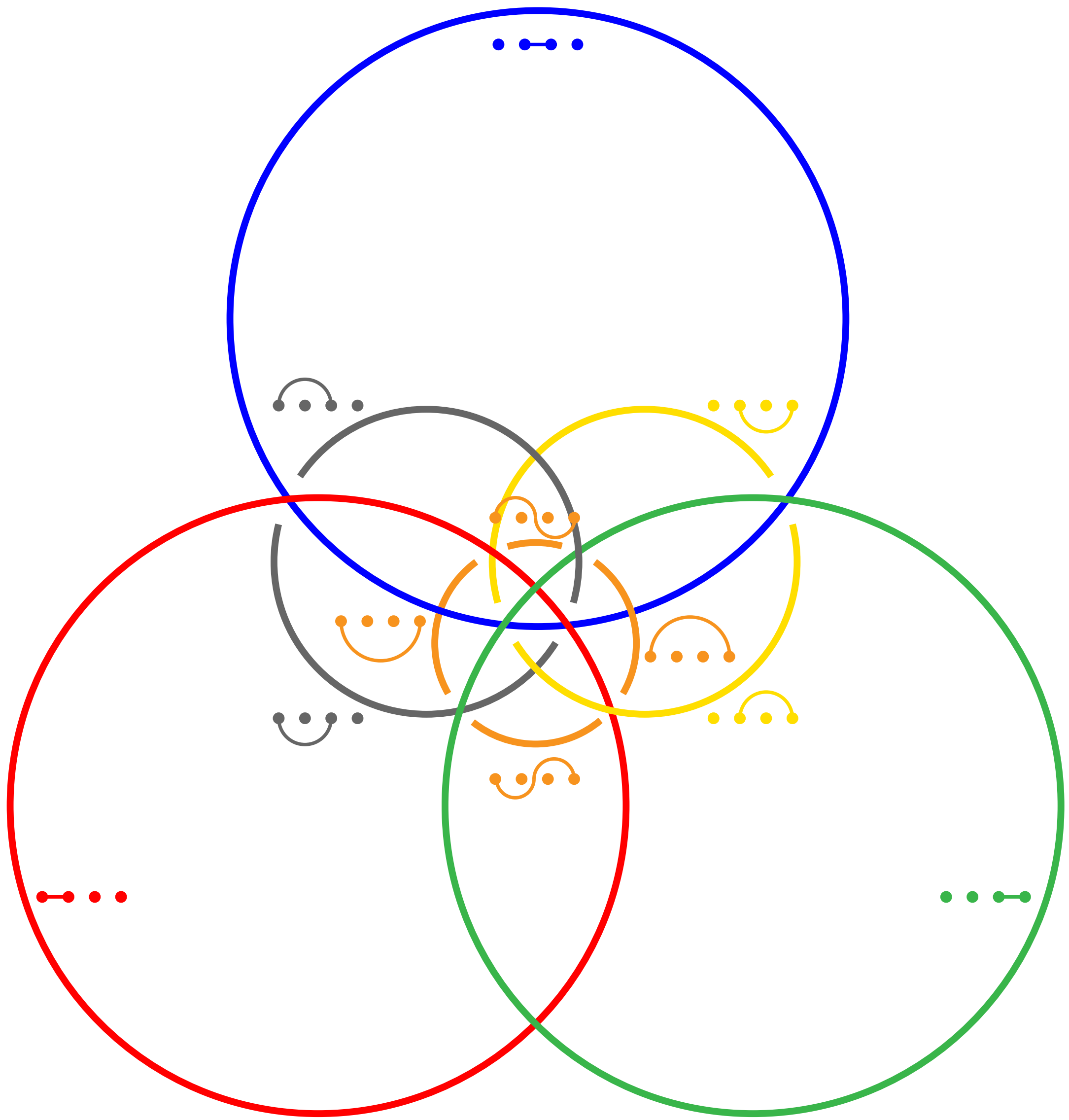
QUOTIENTOPES

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)
Pilaud-Santos, *Quotientopes* ('17$^+$)

SHARDS

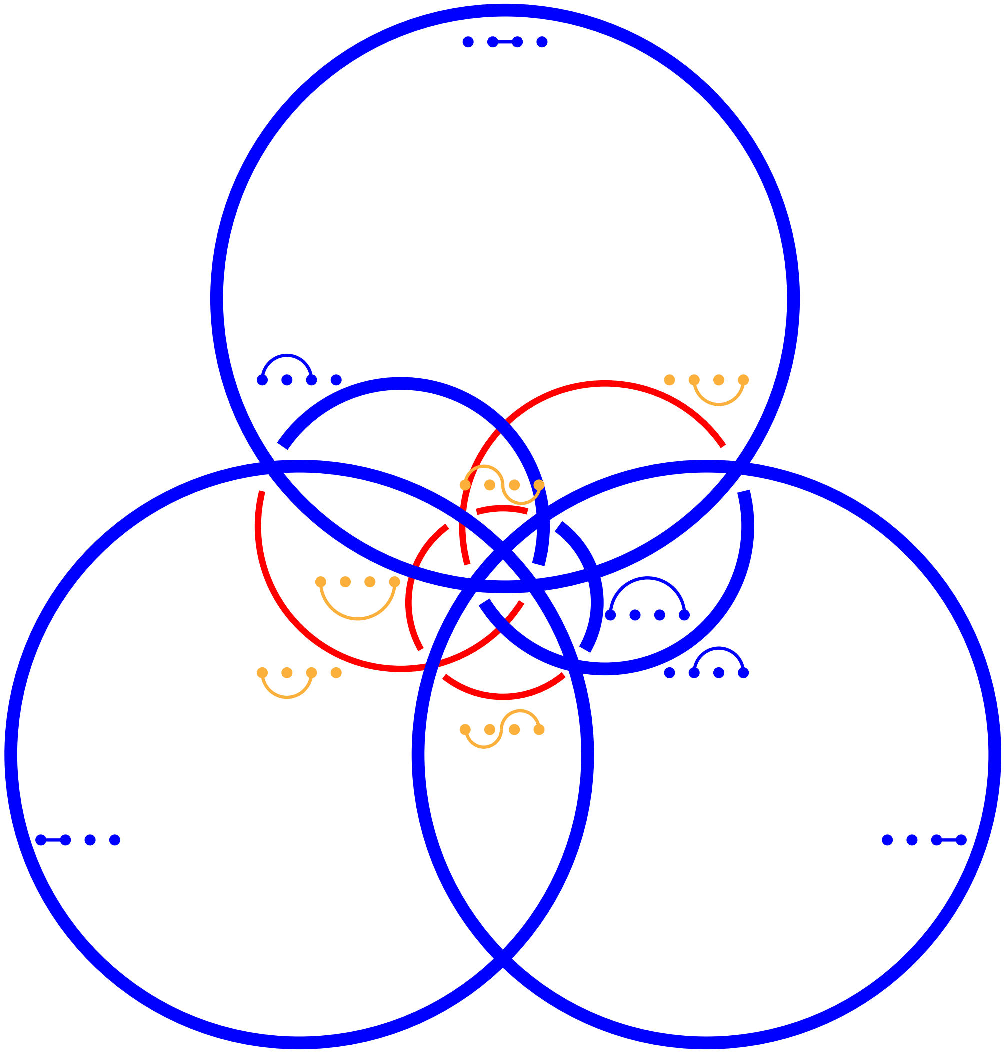
shard $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \begin{bmatrix} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{bmatrix} \right\}$





SHARDS AND QUOTIENT FAN

shard $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \begin{bmatrix} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{bmatrix} \right\}$



THM. For a lattice congruence $\equiv$ on $\mathfrak{S}_n$, the cones obtained by glueing the Coxeter regions of the permutations in the same congruence class of $\equiv$ form a fan $\mathcal{F}_\equiv$ of $\mathbb{R}^n$ whose dual graph realizes the lattice quotient $\mathfrak{S}_n/\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

THM. Each lattice congruence $\equiv$ on $\mathfrak{S}_n$ corresponds to a set of shards $\boldsymbol{\Sigma}_\equiv$ such that the cones of $\mathcal{F}_\equiv$ are the connected components of the complement of the union of the shards in $\boldsymbol{\Sigma}_\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

QUOTIENTOPE

fix a forcing dominant function $f : \sigma \to \mathbb{R}_{>\mathbf{0}}$ ie. st. $f(\Sigma) > \sum_{\Sigma' \succ \Sigma} f(\Sigma')$ for any shard Σ.

for a shard $\Sigma = (i, j, n, S)$ and a subset $\varnothing \neq R \subsetneq [n]$ define the contribution

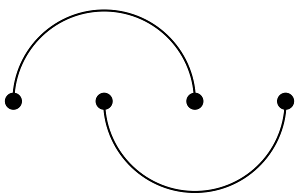
$$\gamma(\Sigma, R) := \begin{cases} 1 & \text{if } |R \cap \{i, j\}| = 1 \text{ and } S = R \cap \,]i, j[, \\ 0 & \text{otherwise} \end{cases}$$

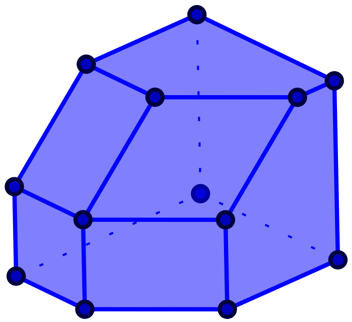
define height function h for $\varnothing \neq R \subsetneq [n]$ by $h^f_{\equiv}(R) := \sum_{\Sigma \in \mathbf{\Sigma}_{\equiv}} f(\Sigma)\, \gamma(\Sigma, R)$.

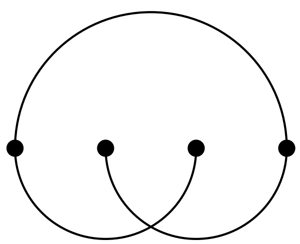
THM. For a lattice congruence $\equiv$ on $\mathfrak{S}_n$ and a forcing dominant function $f : \mathbf{\Sigma} \to \mathbb{R}_{>\mathbf{0}}$, the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of the polytope

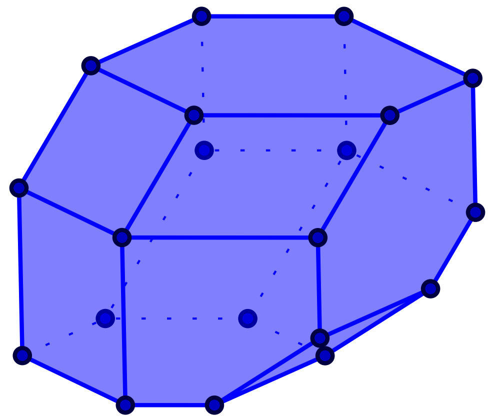
$$P^f_{\equiv} := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \langle\, \mathbf{r}(R) \mid \mathbf{x} \,\rangle \leq h^f_{\equiv}(R) \text{ for all } \varnothing \neq R \subsetneq [n] \right\}.$$

P.-Santos, *Quotientopes* ('17$^+$)

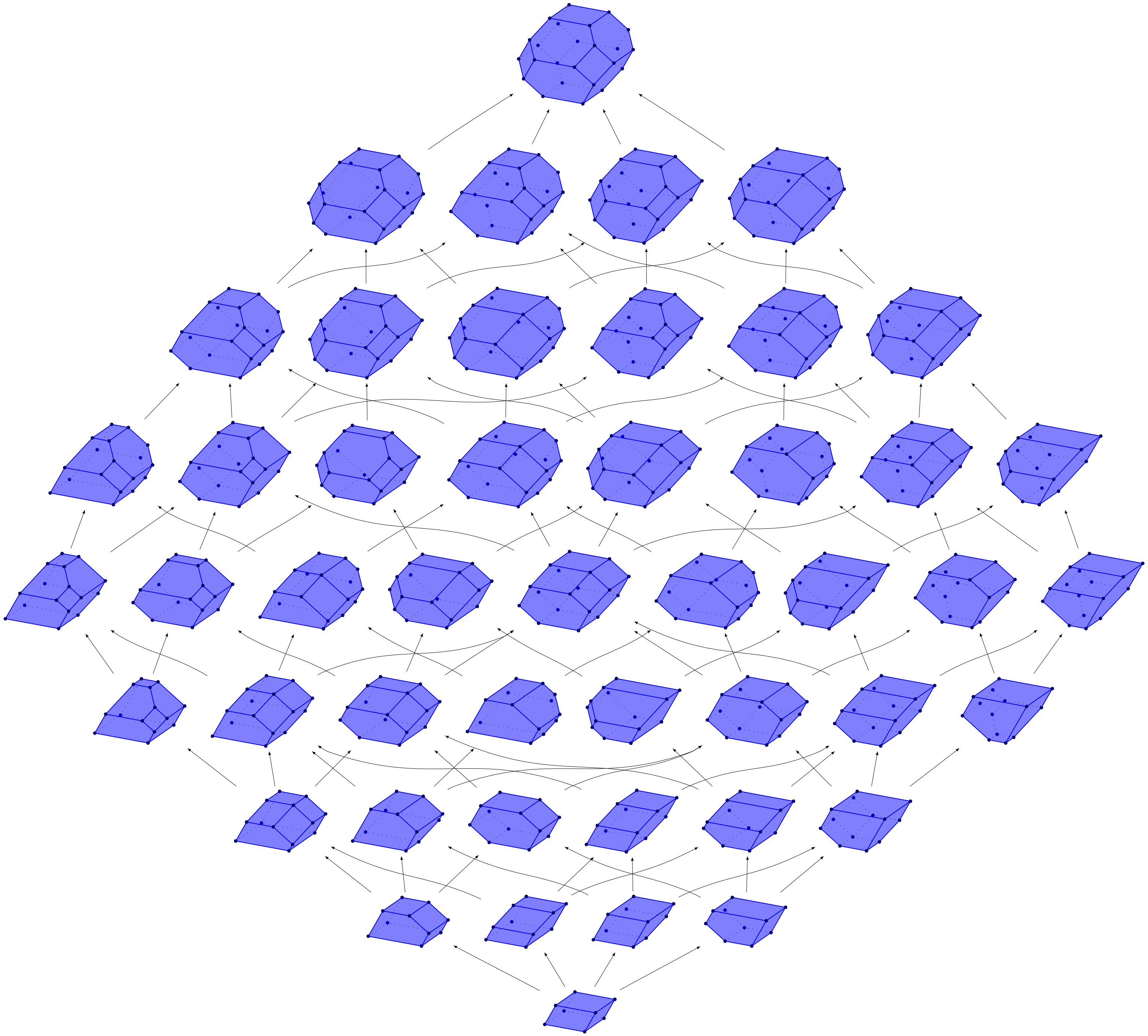




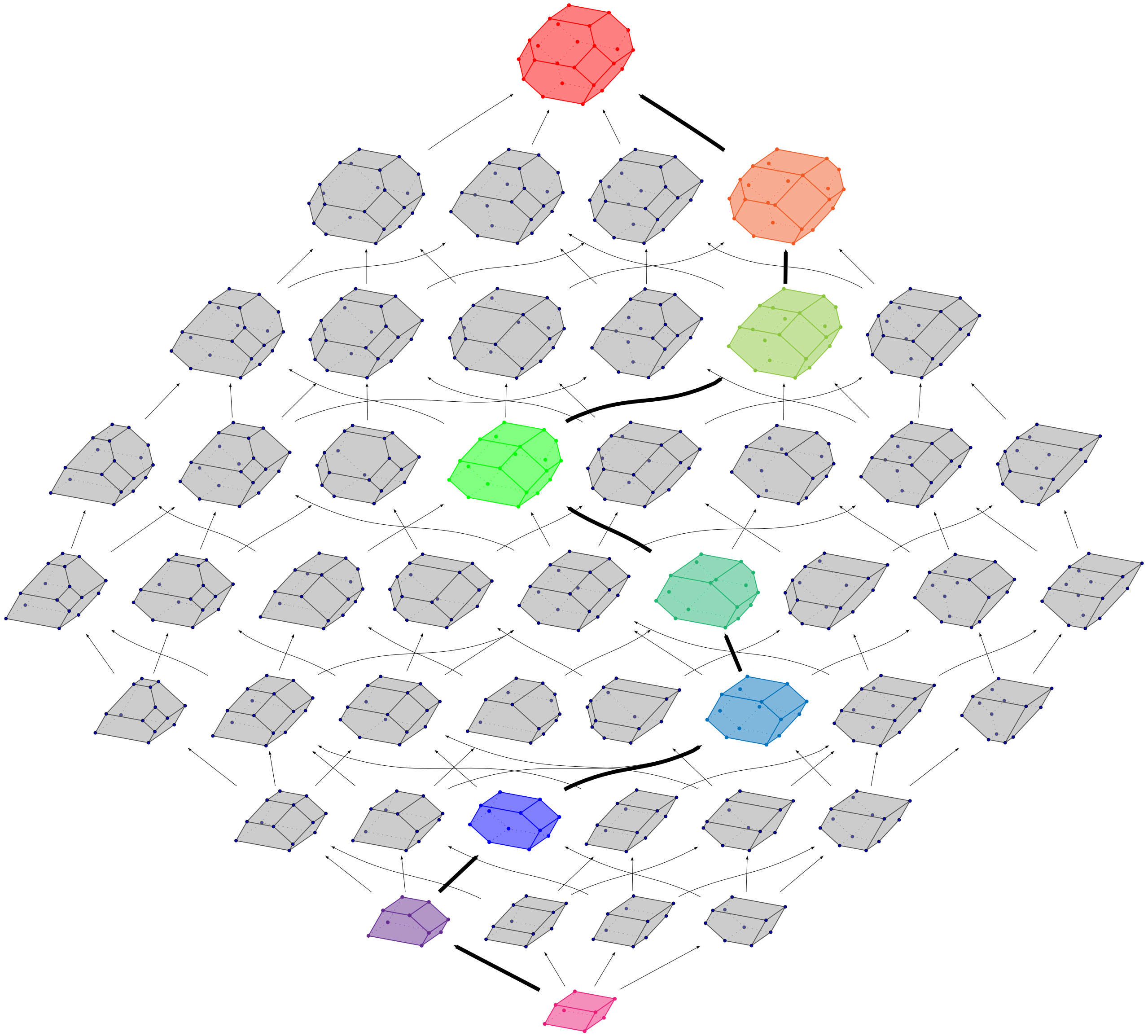




QUOTIENTOPE LATTICE



QUOTIENTOPE LATTICE



POLYWOOD

outsidahedra	insidahedra
permutrees	quotientopes

TOWARDS QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
B distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$
inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C
poset of regions $\mathrm{Pos}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

THM. The poset of regions $\mathrm{Pos}(\mathcal{H}, B)$

- is never a lattice when B is not a simple region,
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement.

Björner-Edelman-Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)

THM. If $\mathrm{Pos}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a lattice congruence of $\mathrm{Pos}(\mathcal{H}, B)$, the cones obtained by glueing together the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?

THANKS