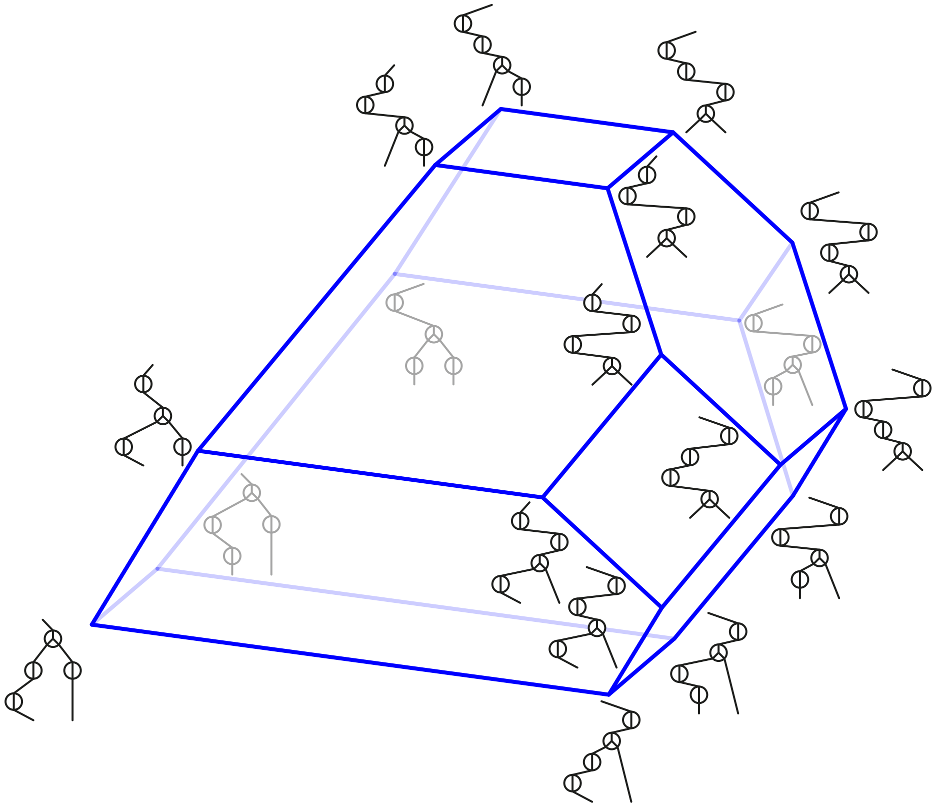
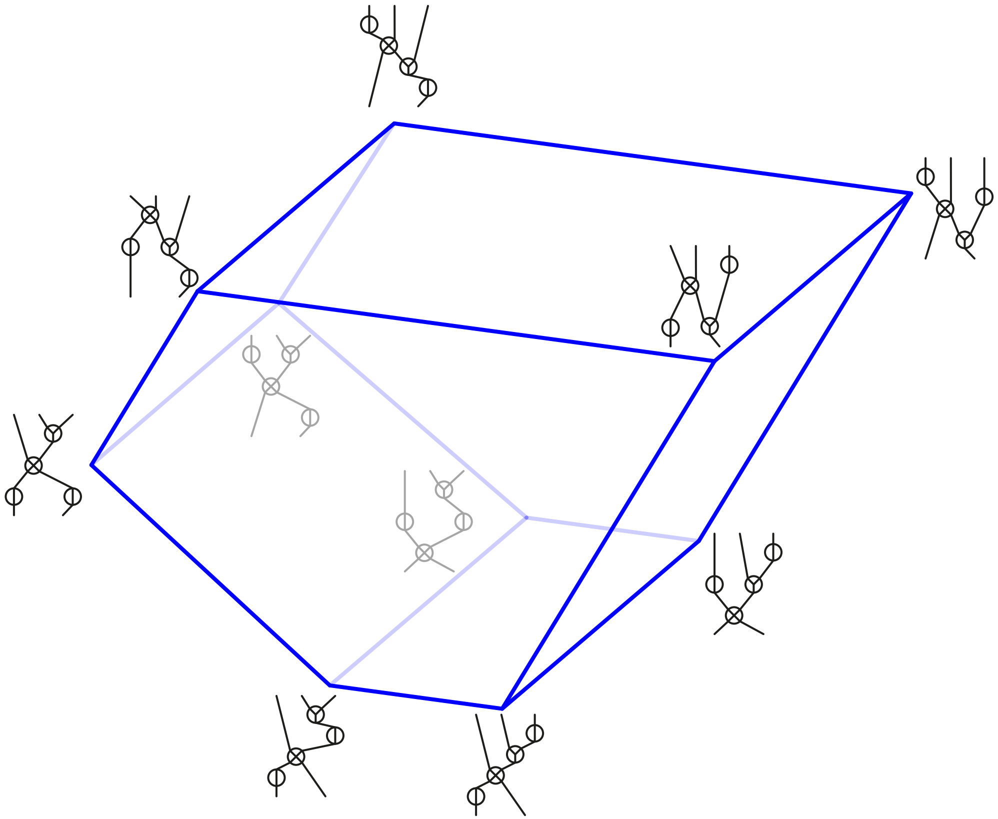
PERMUTREES

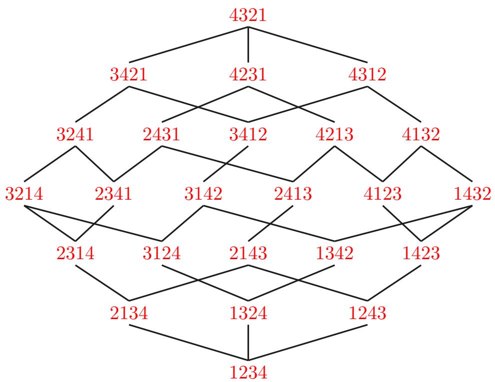
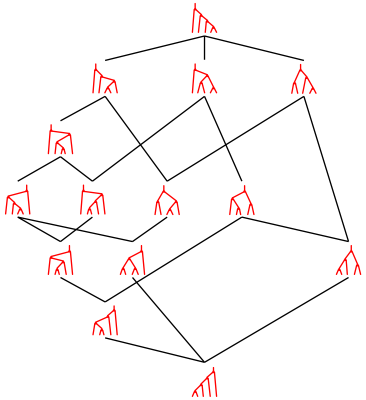
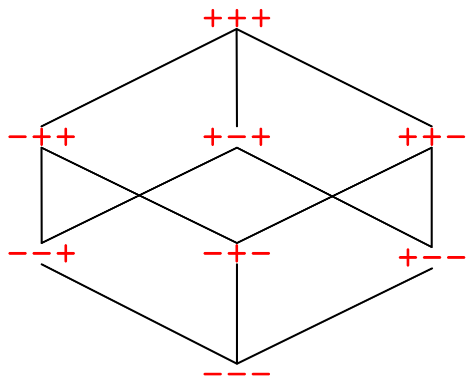
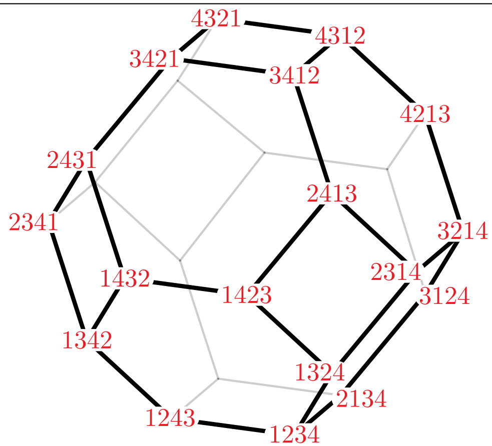
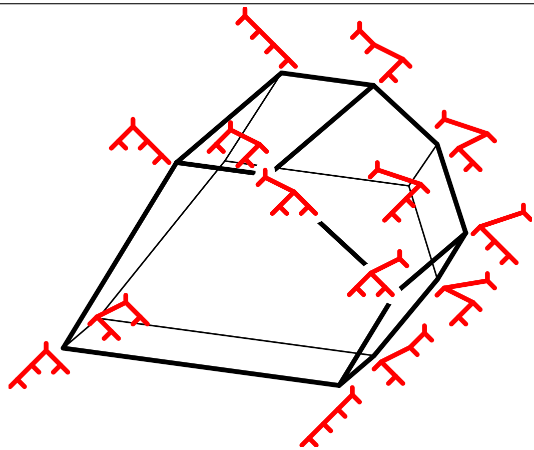
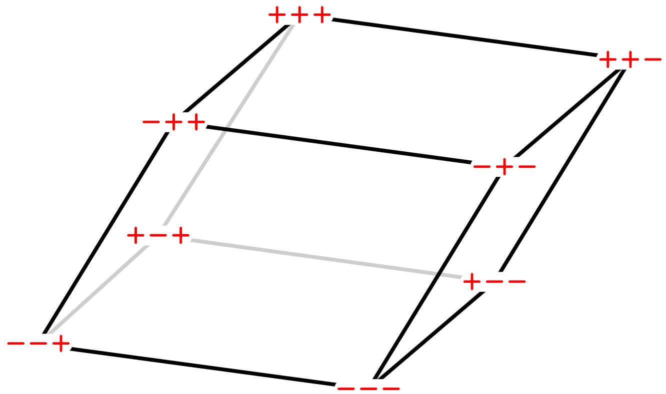




Vincent PILAUD
(CNRS & Ecole Polytechnique)

Viviane PONS
(LRI, Univ. Orsay)

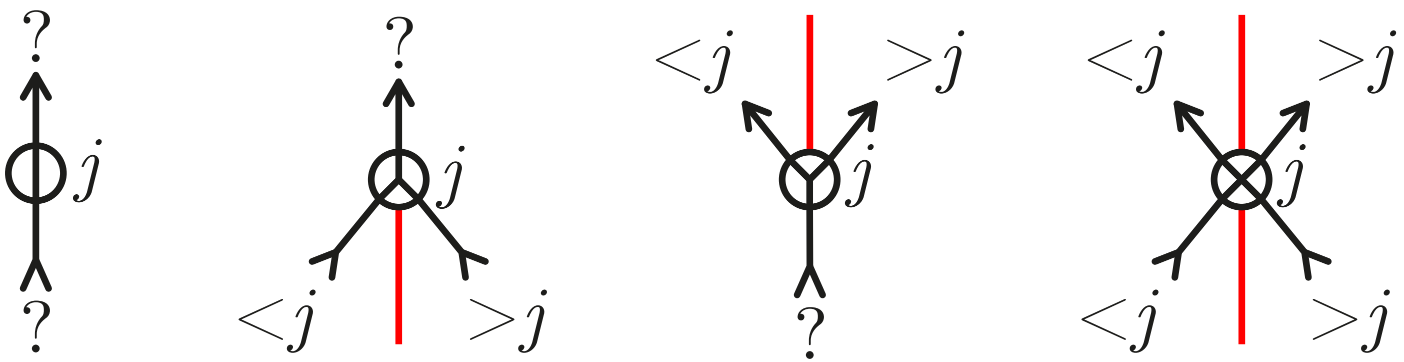
MOTIVATION

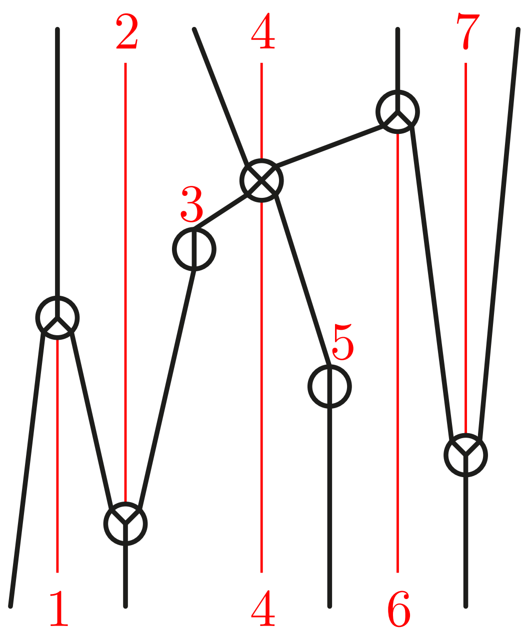
	permutations	binary trees	binary sequences
Combinatorics			
Geometry			
Algebra	Malvenuto-Reutenauer algebra $\mathsf{FQSym} = \mathrm{vect}\langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$ $\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma$ $\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$	Loday-Ronco algebra $\mathsf{PBT} = \mathrm{vect}\langle \mathbb{P}_{\mathrm{T}} \mid \mathrm{T} \in \mathcal{BT} \rangle$ $\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{T \nearrow^{T'} \leq T'' \leq {}^{T}\nwarrow_{T'}} \mathbb{P}_{T''}$ $\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$	Solomon algebra $\mathsf{Rec} = \mathrm{vect}\langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$ $\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$ $\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$

COMBINATORICS

PERMUTREES

permutree = directed (bottom to top) and labeled (bijectively by $[n]$) tree such that

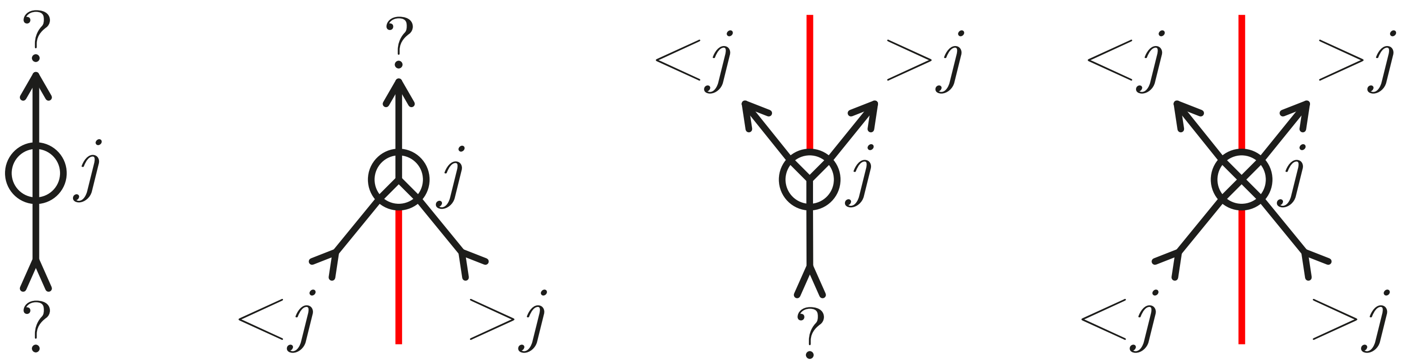




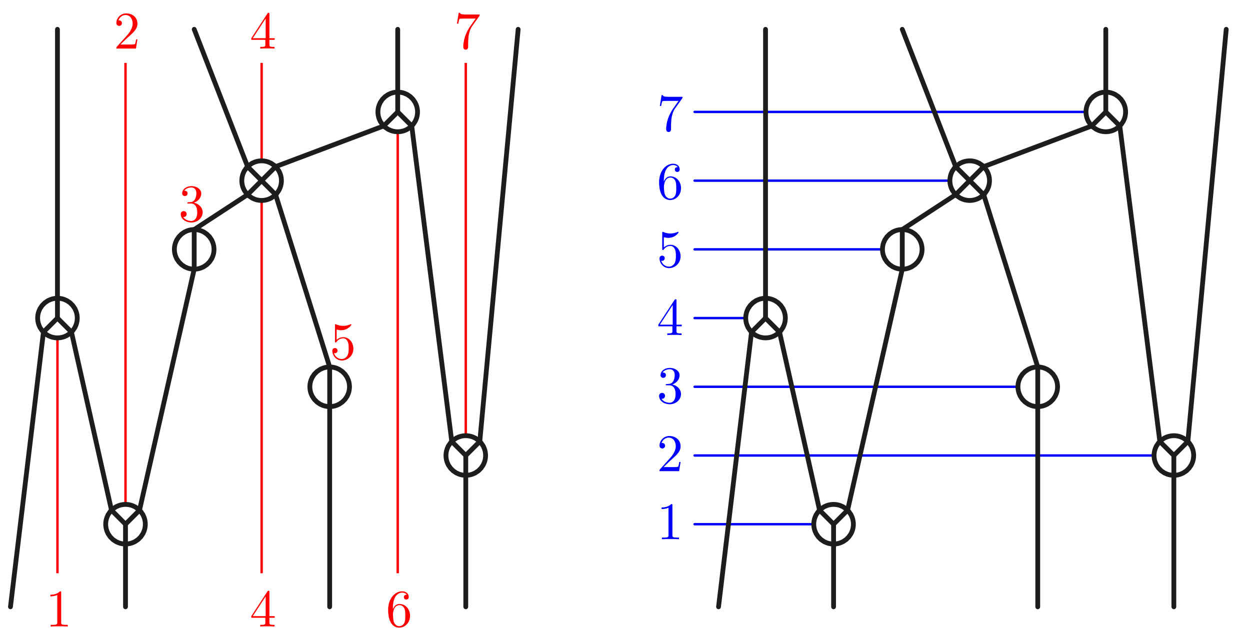
decoration = 

PERMUTREES

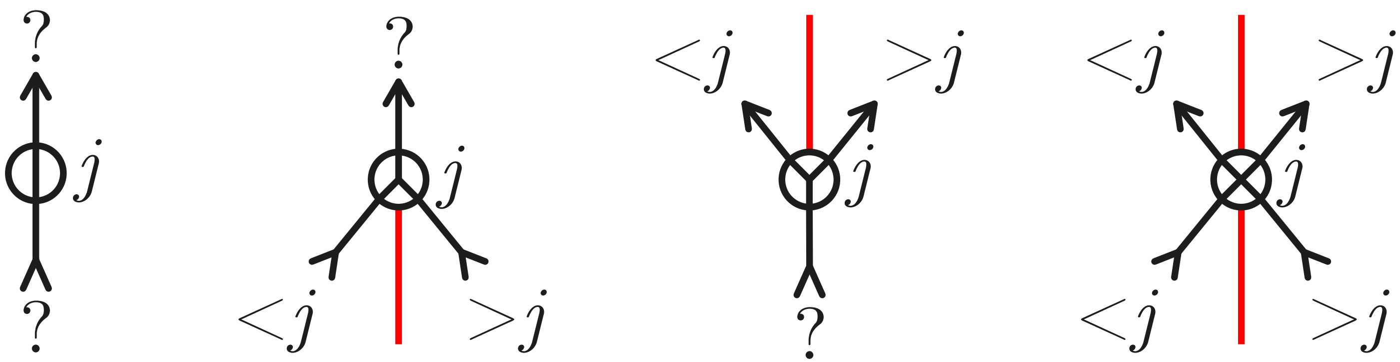
permutree = directed (bottom to top) and labeled (bijectively by $[n]$) tree such that



increasing tree = directed and labeled tree such that labels increase along arcs

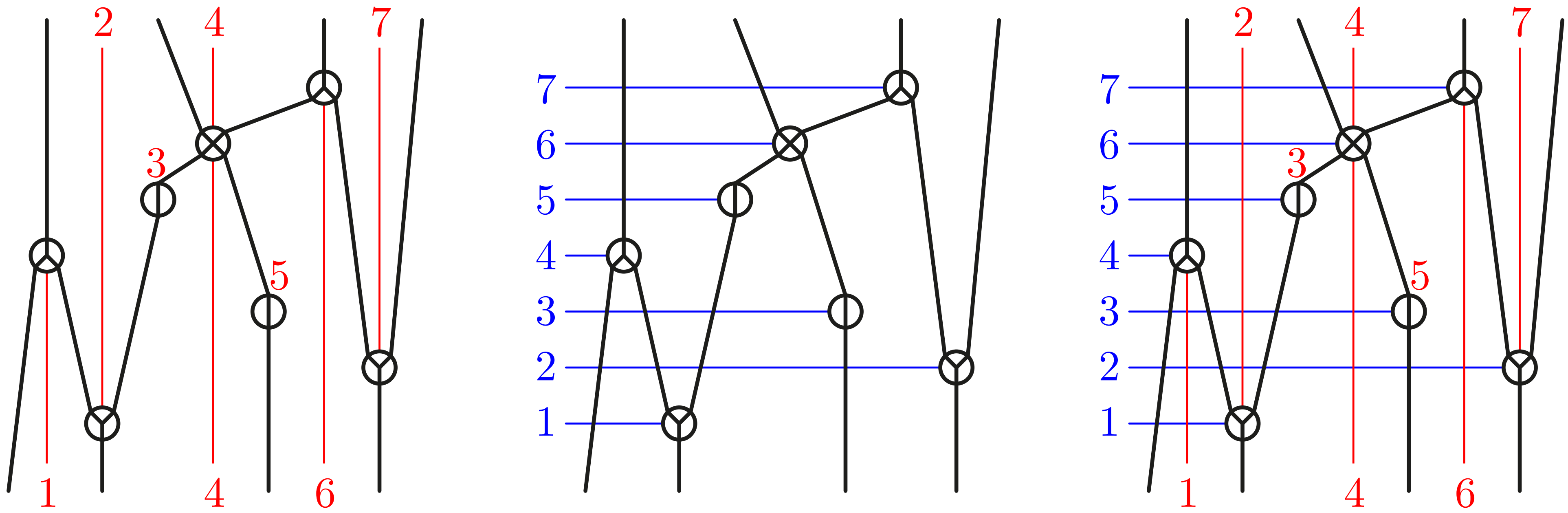


permutree = directed (bottom to top) and labeled (bijectively by $[n]$) tree such that



increasing tree = directed and labeled tree such that labels increase along arcs

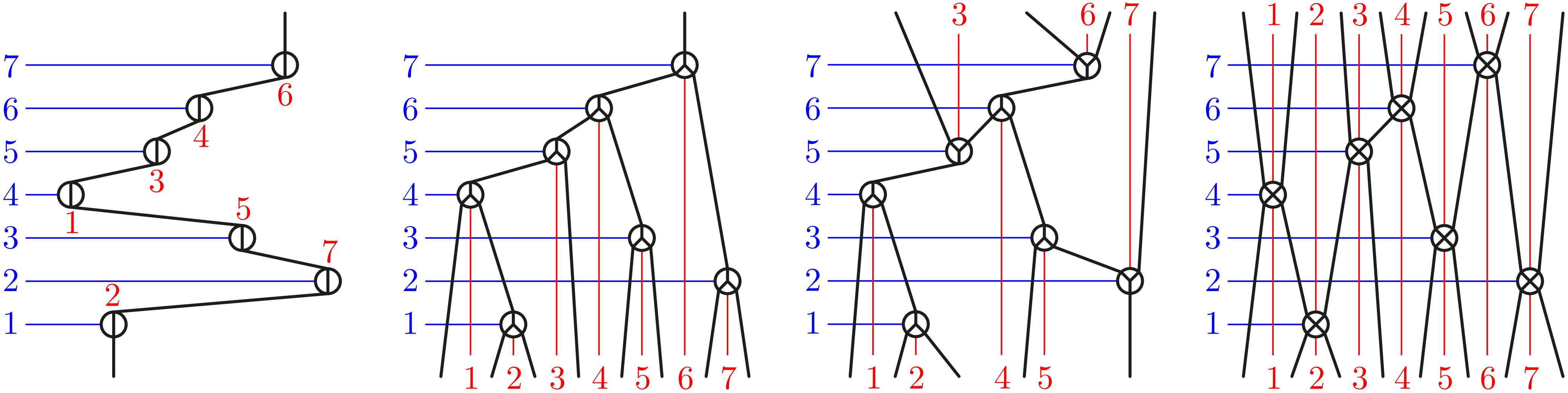
leveled permutree = directed tree with a permutree labeling and an increasing labeling



SPECIAL PERMUTREES

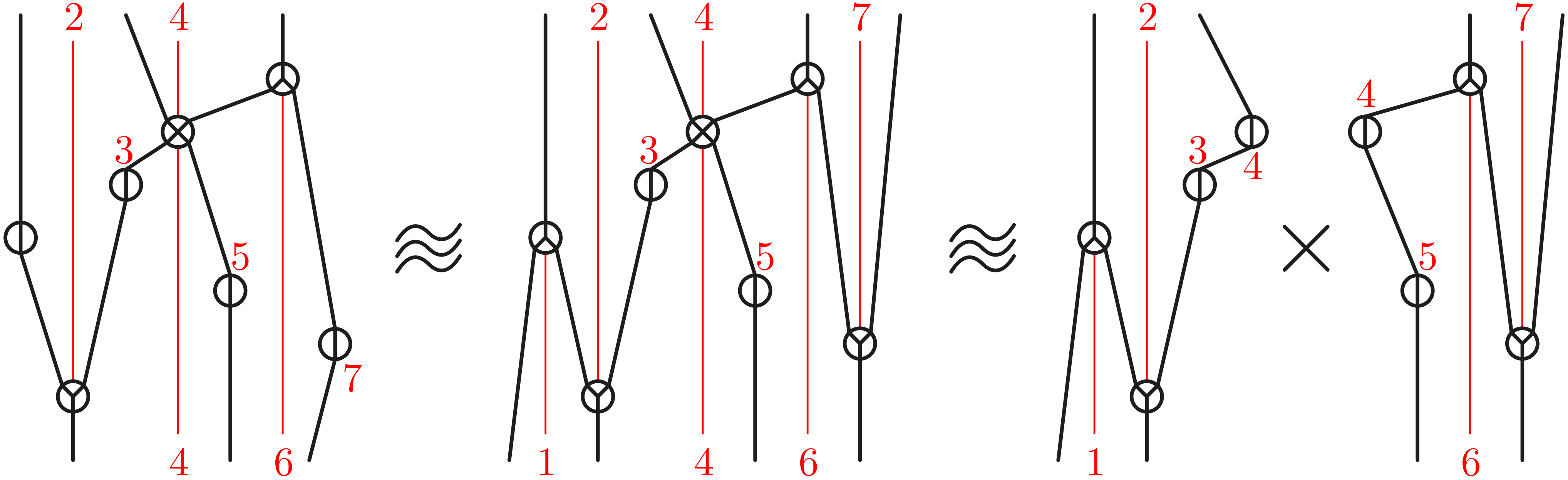
Examples.

decoration δ		permutrees
⦶^{n}	$\longleftrightarrow$	permutations of $[n]$
☮^{n}	$\longleftrightarrow$	standard binary search trees
$\{$☮, Ⓨ$\}^n$	$\longleftrightarrow$	Cambrian trees
$\otimes^n$	$\longleftrightarrow$	binary sequences



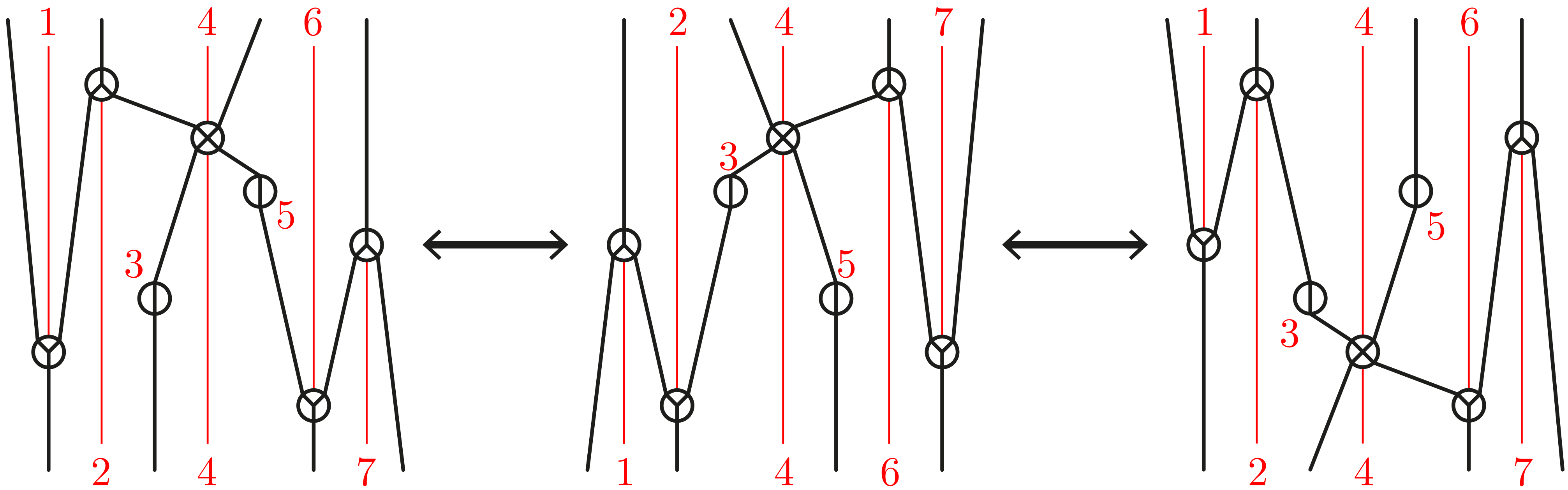
REMARKS

- the first and last decorations do not really matter
- $\otimes$ vertices $\longleftrightarrow$ product



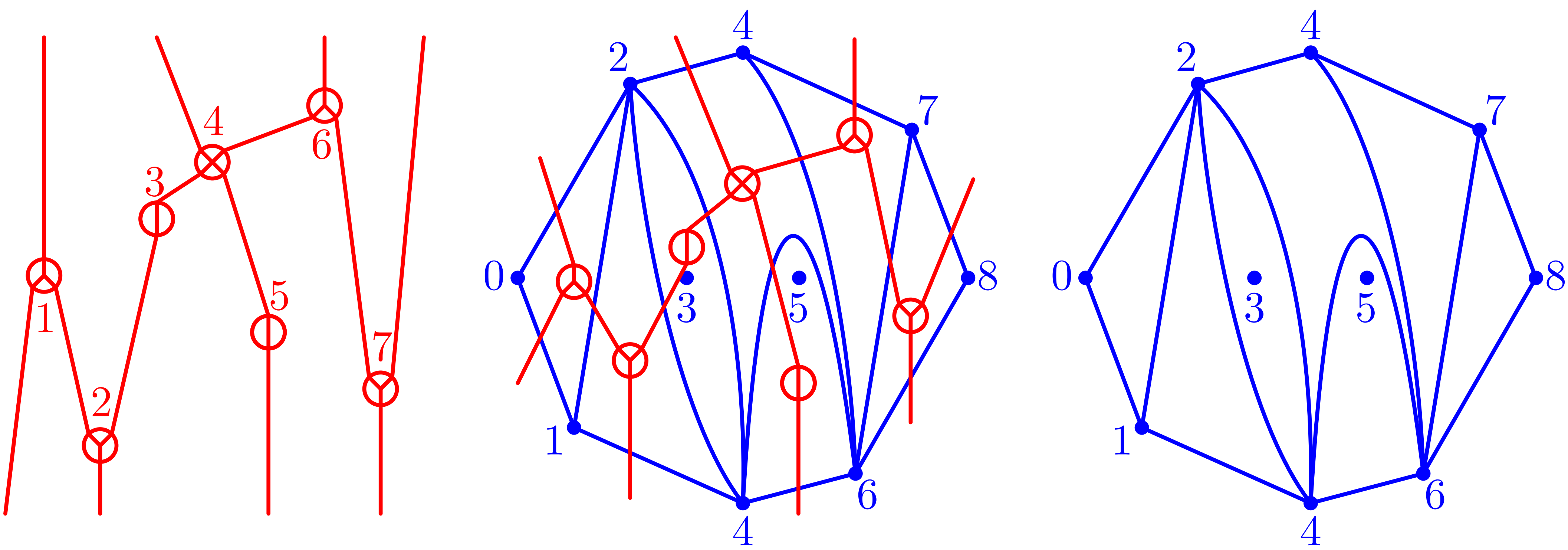
REMARKS

- the first and last decorations do not really matter
- $\otimes$ vertices $\longleftrightarrow$ product
- horizontal and vertical symmetrees



PERMUTREES AND TREEANGULATIONS

permutrees are dual to $\{2, 3, 4\}$-angulations of polygons

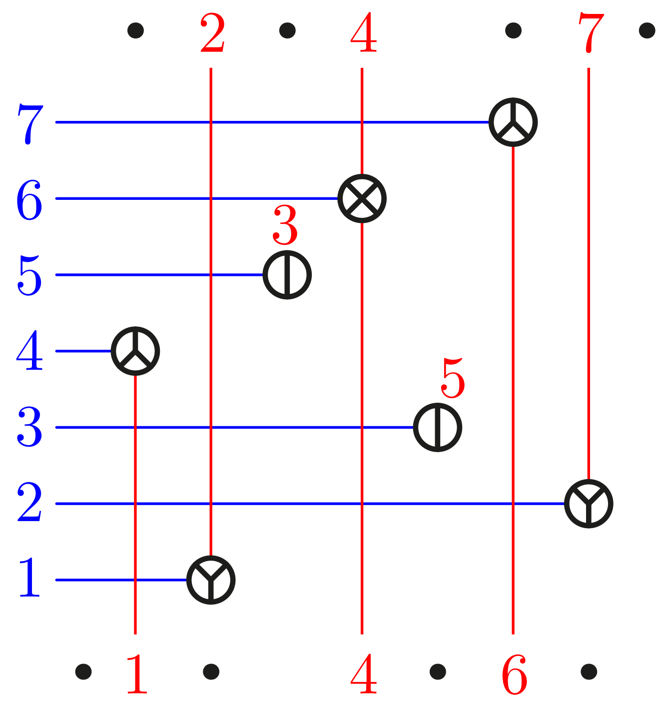


vertices above/below/inside $[0, 8]$	$\longleftrightarrow$	decoration
diangle enclosing j	$\longleftrightarrow$	node ⦶ labeled j
triangle $i < j < k$ with j below	$\longleftrightarrow$	node ☮ labeled j
triangle $i < j < k$ with j above	$\longleftrightarrow$	node Ⓨ labeled j
quadrangle $i < j^-, j^+ < k$	$\longleftrightarrow$	node ⊗ labeled j

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{27}51\underline{1}3\overline{\underline{4}}\underline{6}$



Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13$^+$)
Chatel-P., *Cambrian Hopf algebras* ('14$^+$)
P.-Pons, *Permutrees* ('16$^+$)

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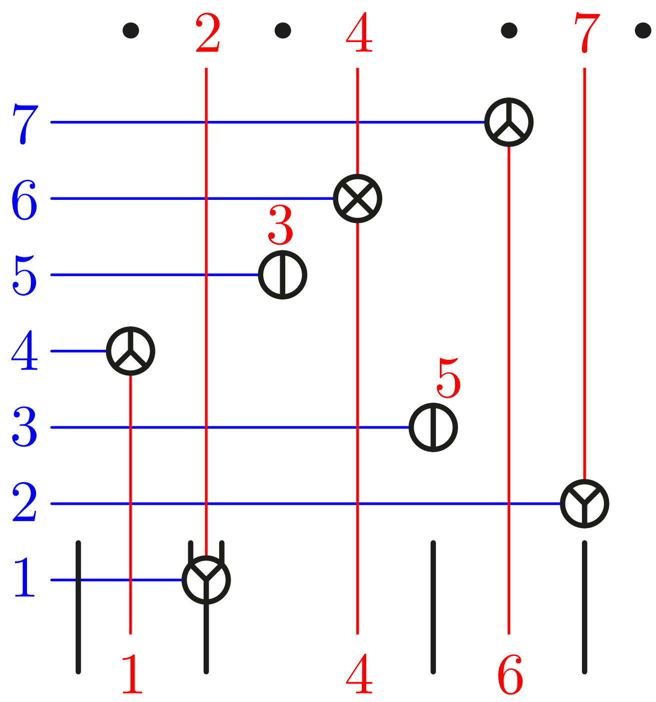

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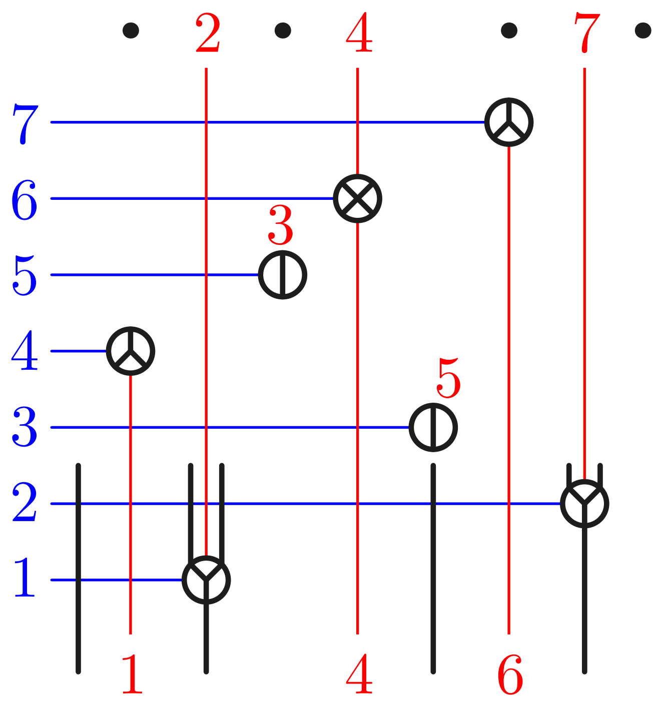
PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}51\underline{1}3\underline{\overline{4}6}$



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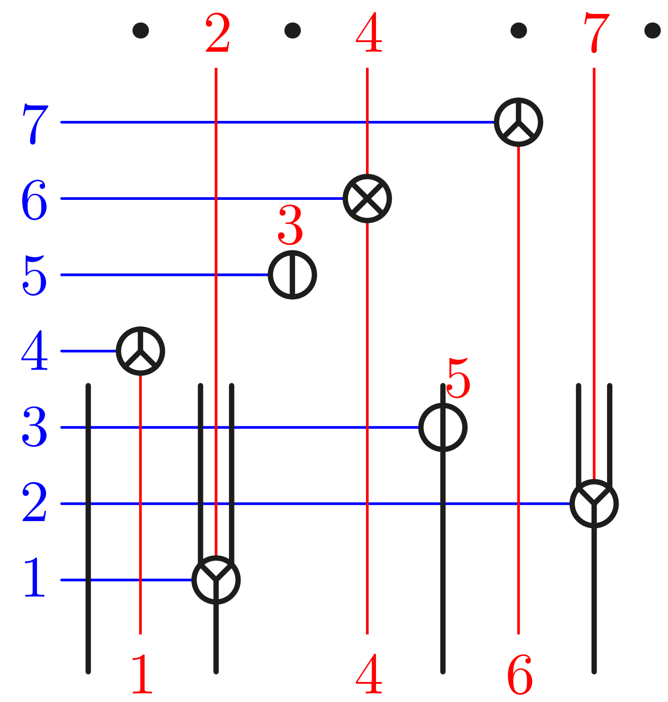

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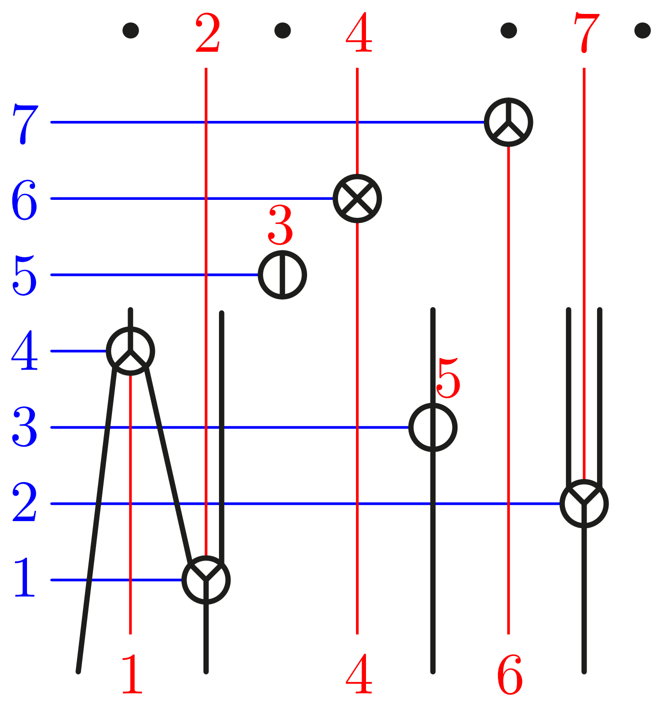

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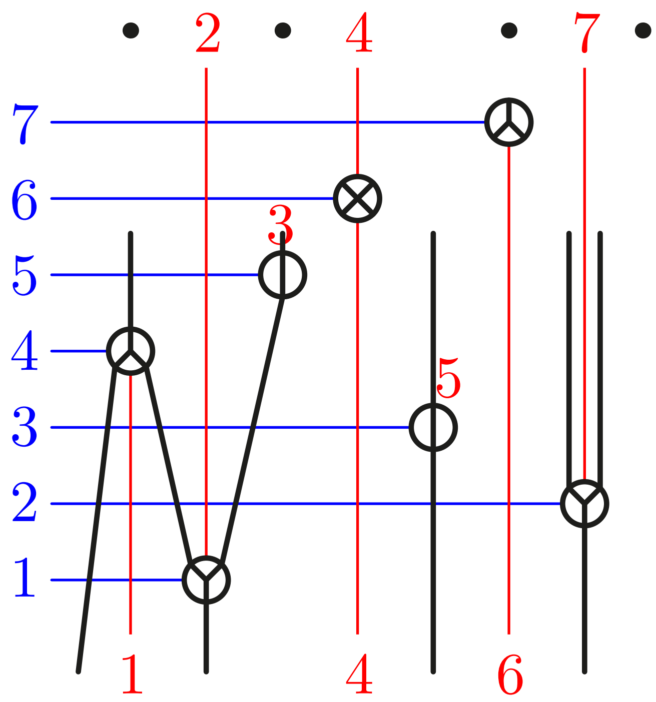

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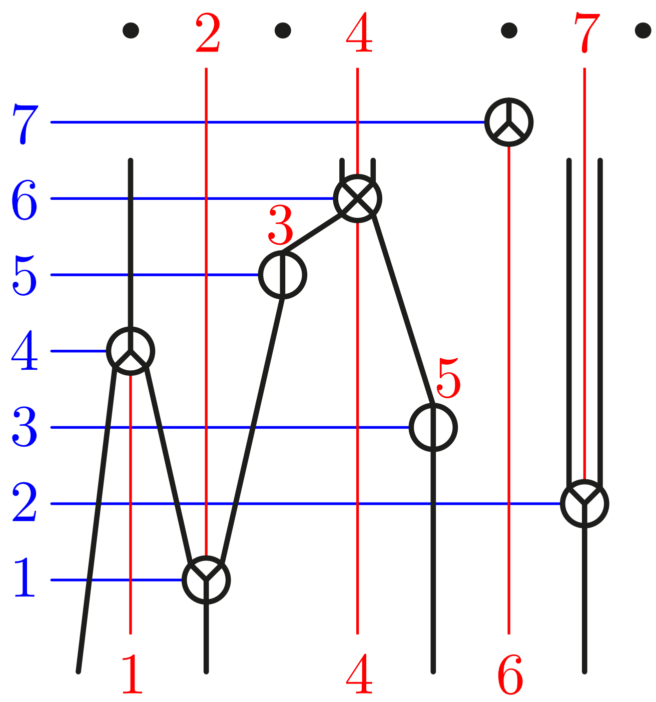

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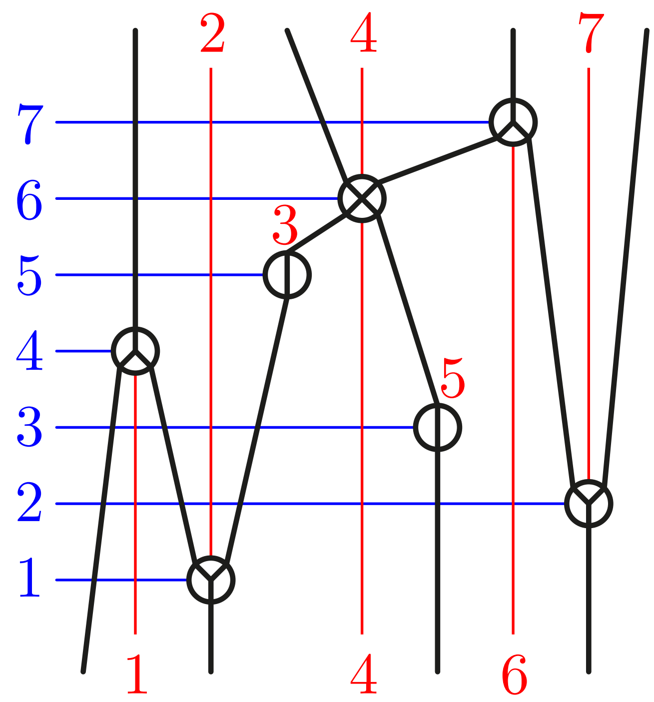

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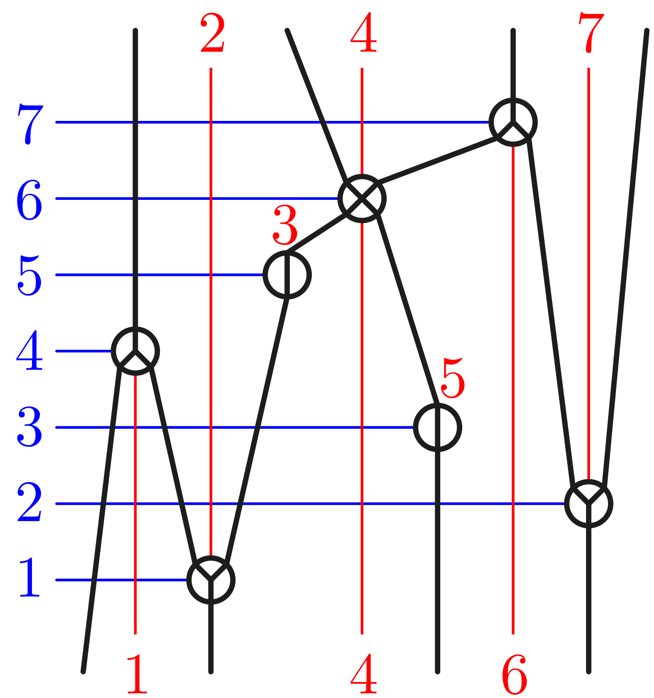
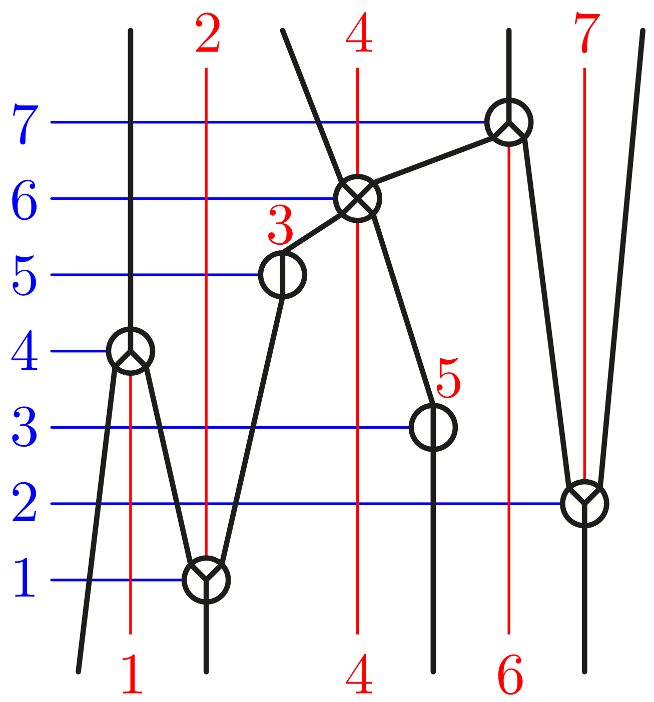


PROP. bijection decorated permutations $\longleftrightarrow$ leveled permutrees

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}51\underline{1}3\overline{\underline{4}}\underline{6}$



$\mathbf{P}(\tau) = \mathbf{P}$-symbol of τ = permutree produced by permutree correspondence

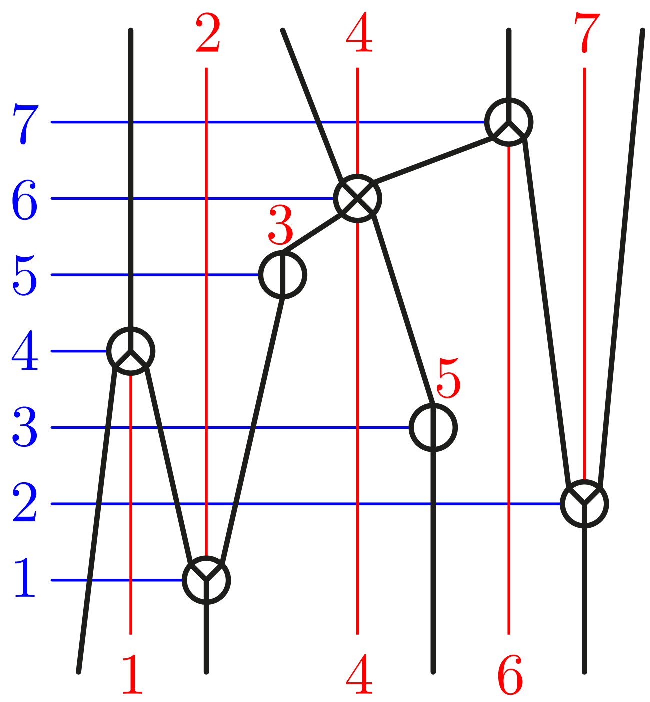
$\mathbf{Q}(\tau) = \mathbf{Q}$-symbol of τ = increasing tree produced by permutree correspondence

(analogy to Robinson-Schensted algorithm)

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$



$\mathbf{P}(\tau) = \mathbf{P}$-symbol of τ = permutree produced by permutree correspondence
$\mathbf{Q}(\tau) = \mathbf{Q}$-symbol of τ = increasing tree produced by permutree correspondence

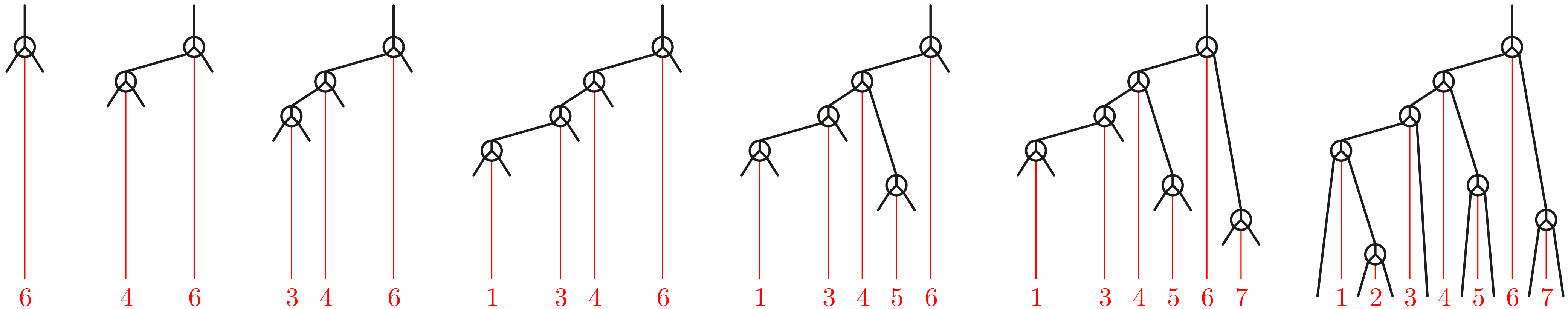
PROP. permutree congruence class labeled by permutree T

$$\{\tau \in \mathfrak{S}^\delta \mid \mathbf{P}(\tau) = \mathrm{T}\} = \{\text{linear extensions of } \mathrm{T}\}$$

CORRESPONDENCE FOR SPECIAL DECORATIONS

Examples.

decoration δ		permutree insertion map
⦶^{n}	$\longleftrightarrow$	identity
☮^{n}	$\longleftrightarrow$	binary search tree insertion
$\{$☮, ⍦$\}^n$	$\longleftrightarrow$	Cambrian trees insertion
⊗^{n}	$\longleftrightarrow$	recoil map



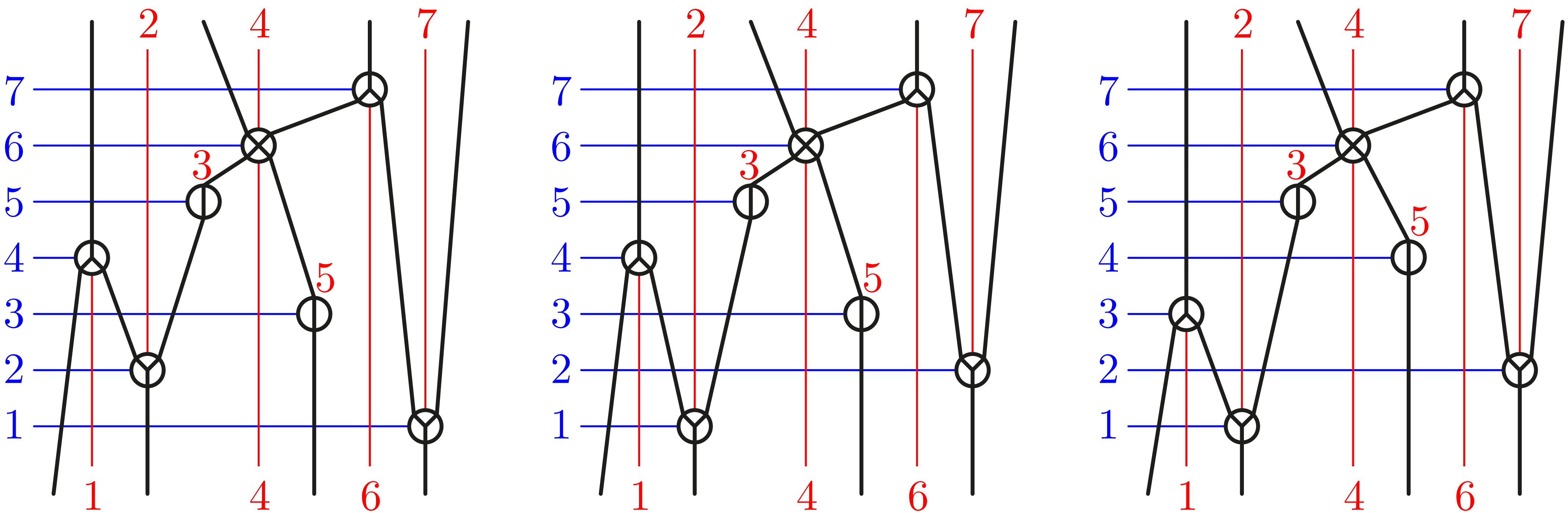
PERMUTREE CONGRUENCE

δ-permutree congruence = transitive closure of the rewriting rules

$$UacVbW \equiv_\delta UcaVbW \quad \text{if } a < b < c \text{ and } \delta_b \in \{\text{⅄⃝}, \otimes\}$$
$$UbVacW \equiv_\delta UbVcaW \quad \text{if } a < b < c \text{ and } \delta_b \in \{\text{Y⃝}, \otimes\}$$

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

PROP. $\tau \equiv_\delta \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$



$\overline{7}\,\overline{2}\,5\,\underline{1}\,3\,\underline{\overline{4}}\,\underline{6} \equiv \overline{2}\,\overline{7}\,5\,\underline{1}\,3\,\underline{\overline{4}}\,\underline{6} \equiv \overline{2}\,\overline{7}\,\underline{1}\,5\,3\,\underline{\overline{4}}\,\underline{6}$

PERMUTREE CONGRUENCE

δ-permutree congruence $=$ transitive closure of the rewriting rules

$$UacVbW \equiv_\delta UcaVbW \quad \text{if } a < b < c \text{ and } \delta_b \in \{\text{☮}, \otimes\}$$
$$UbVacW \equiv_\delta UbVcaW \quad \text{if } a < b < c \text{ and } \delta_b \in \{\text{Ⓨ}, \otimes\}$$

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

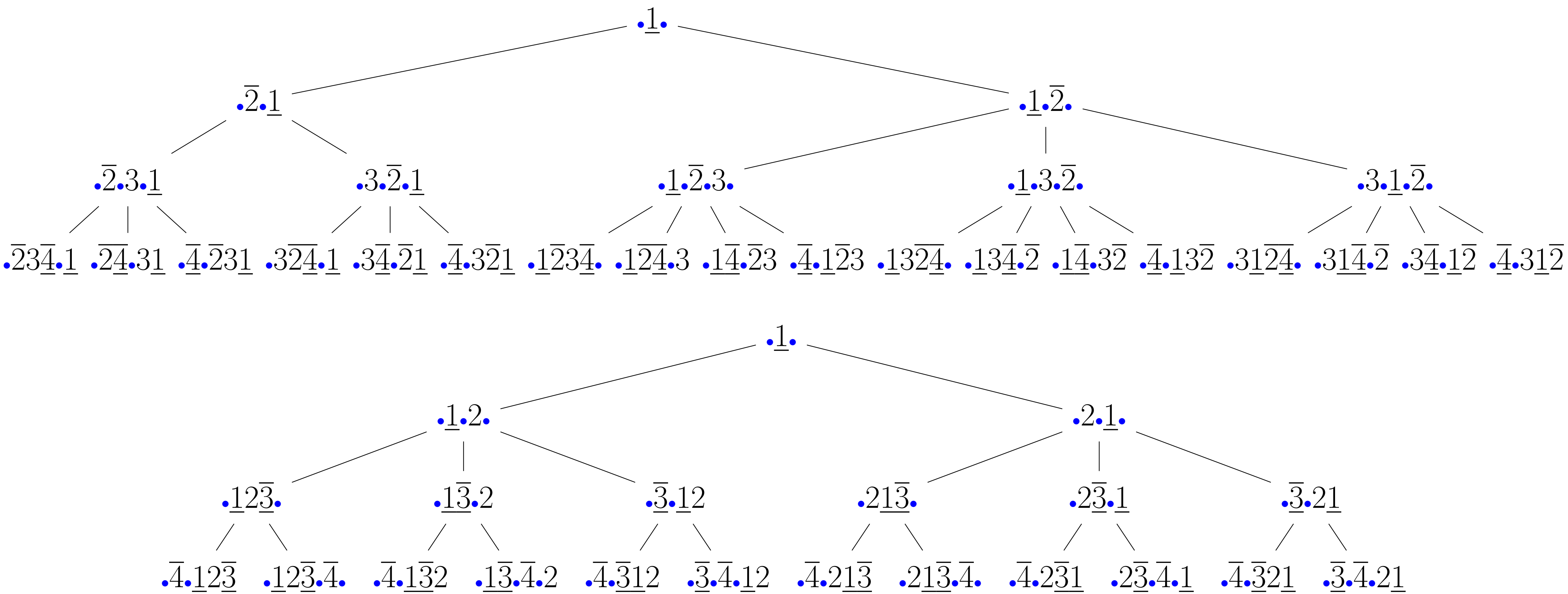
PROP. $\tau \equiv_\delta \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$

PROP. Permutree classes are intervals of the weak order
minimums avoid $b - ca$ with $\delta_b \in \{\text{Ⓨ}, \otimes\}$ and $ca - b$ with $\delta_b \in \{\text{☮}, \otimes\}$
maximums avoid $b - ac$ with $\delta_b \in \{\text{Ⓨ}, \otimes\}$ and $ac - b$ with $\delta_b \in \{\text{☮}, \otimes\}$.

Reading, *Cambrian lattices* ('06)
P.-Pons, *Permutrees* ('16$^+$)

NUMEROLOGY

$\mathcal{T}_\delta$ = generating tree of the weak order maximal permutations of δ-permutree classes (permutations avoiding $b - ac$ with $\delta_b \in \{⊕, ⊗\}$ and $ac - b$ with $\delta_b \in \{⊕, ⊗\}$)



PROP. The generating tree $\mathcal{T}_\delta$ only depends on the positions of ⦶ and ⊗

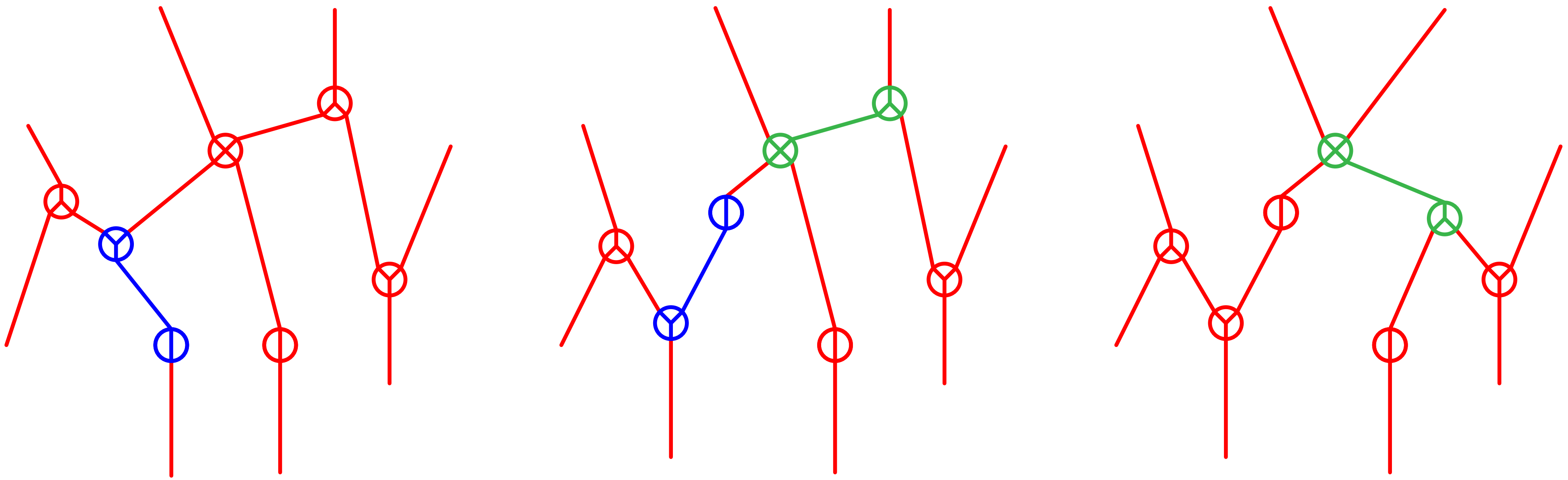
PROP. The number $\mathbf{C}(\delta, g)$ of δ-trees with g free gaps $\bullet$ is given by

$$\mathbf{C}(\delta, g) = \begin{cases} \mathbb{1}_{g>2} \cdot (g-1) \cdot \mathbf{C}(\delta', g-1) & \text{if } \delta_n = ⦶ \\ \mathbb{1}_{g\geq 2} \cdot \sum_{g' \geq g-1} \mathbf{C}(\delta', g') & \text{if } \delta_n = Ⓨ \text{ or } ☮ \\ \mathbb{1}_{g=2} \cdot \sum_{g' \geq 2} g' \cdot \mathbf{C}(\delta', g') & \text{if } \delta_n = \otimes \end{cases}$$

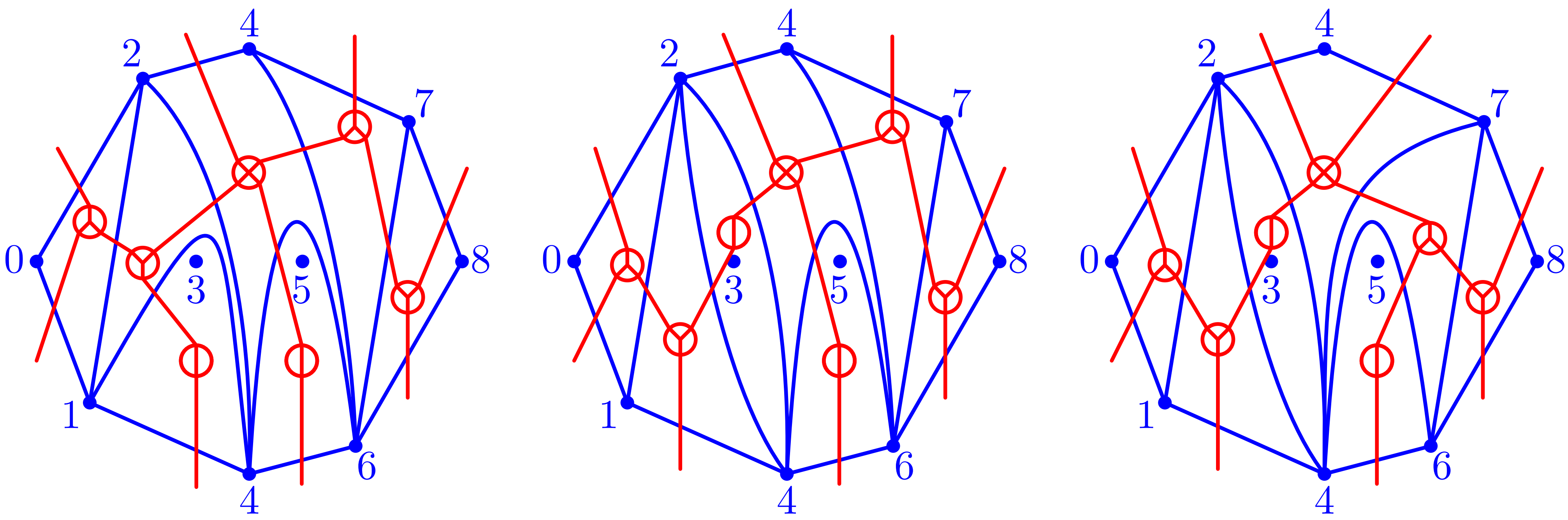
Examples.

decoration δ		$\mathbf{C}(\delta)$
$⦶^n$	$\longleftrightarrow$	factorial $n!$
$☮^n$	$\longleftrightarrow$	Catalan $C_n = \frac{1}{n+1}\binom{2n}{n}$
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	$2^n C_n$
$\otimes^n$	$\longleftrightarrow$	2^{n-1}

Rotation operation preserves permutrees:

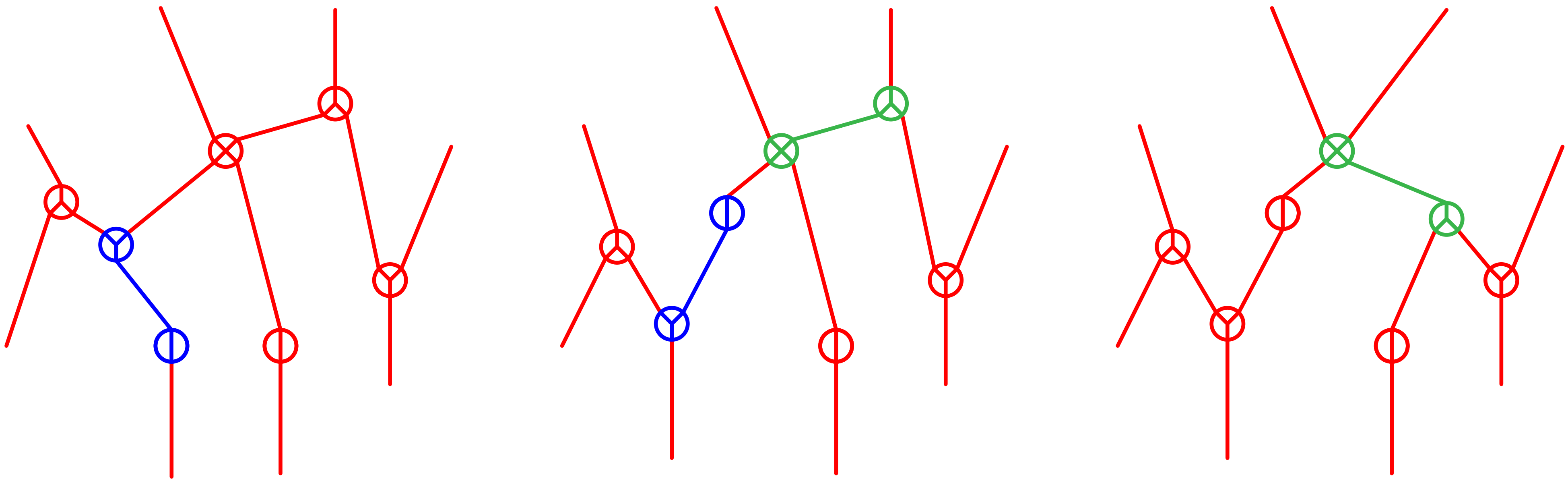


Rotation operation preserves permutrees:



ROTATIONS AND PERMUTREE LATTICES

Rotation operation preserves permutrees:

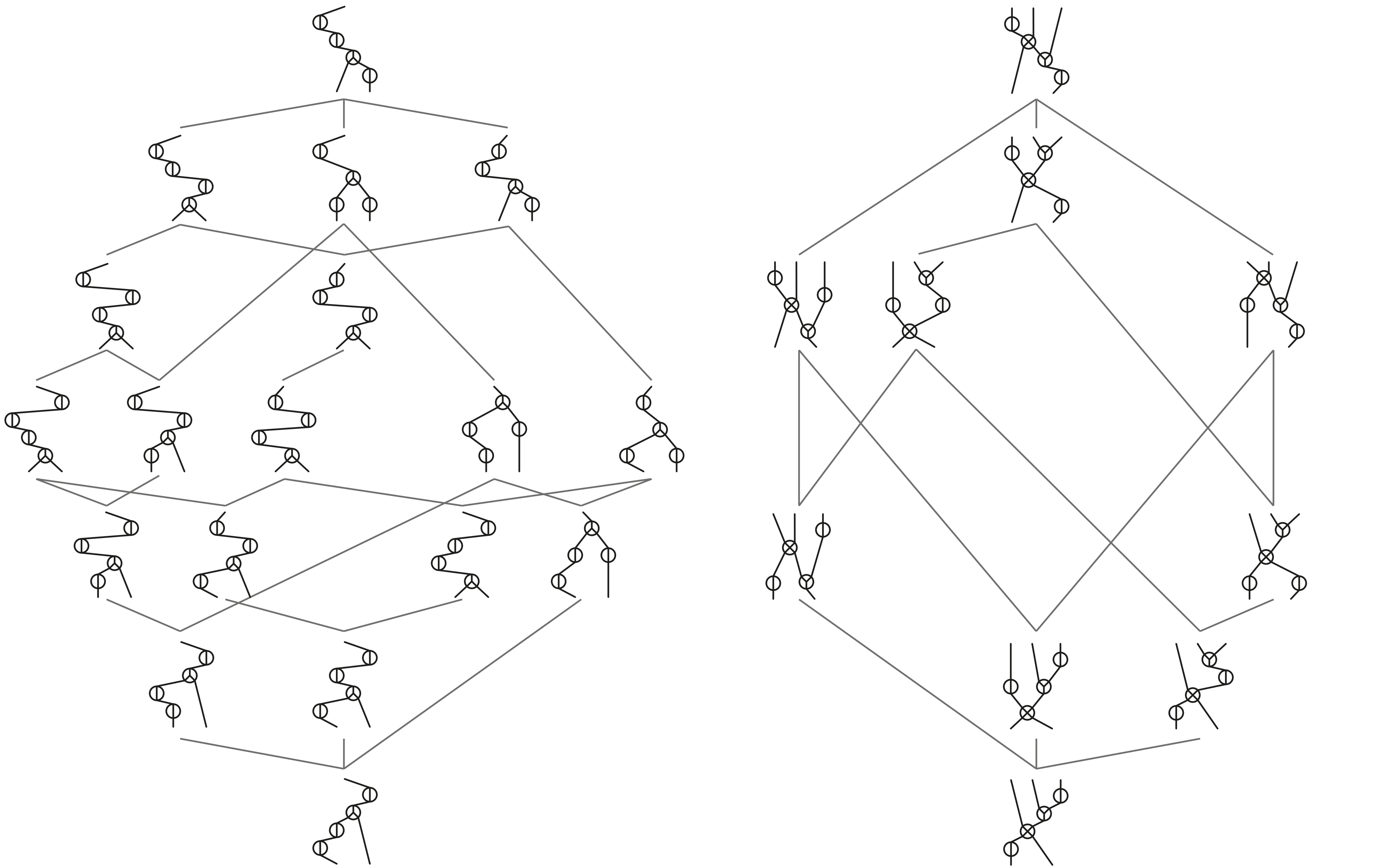


increasing rotation $=$ rotation of edge $i \to j$ where $i < j$

PROP. The transitive closure of the increasing rotation graph is the permutree lattice
$\mathbf{P}$ defines a lattice homomorphism from weak order to permutree lattice

Reading, *Cambrian lattices* ('06)
P.-Pons, *Permutrees* ('16^{+})

ROTATIONS AND CAMBRIAN LATTICES



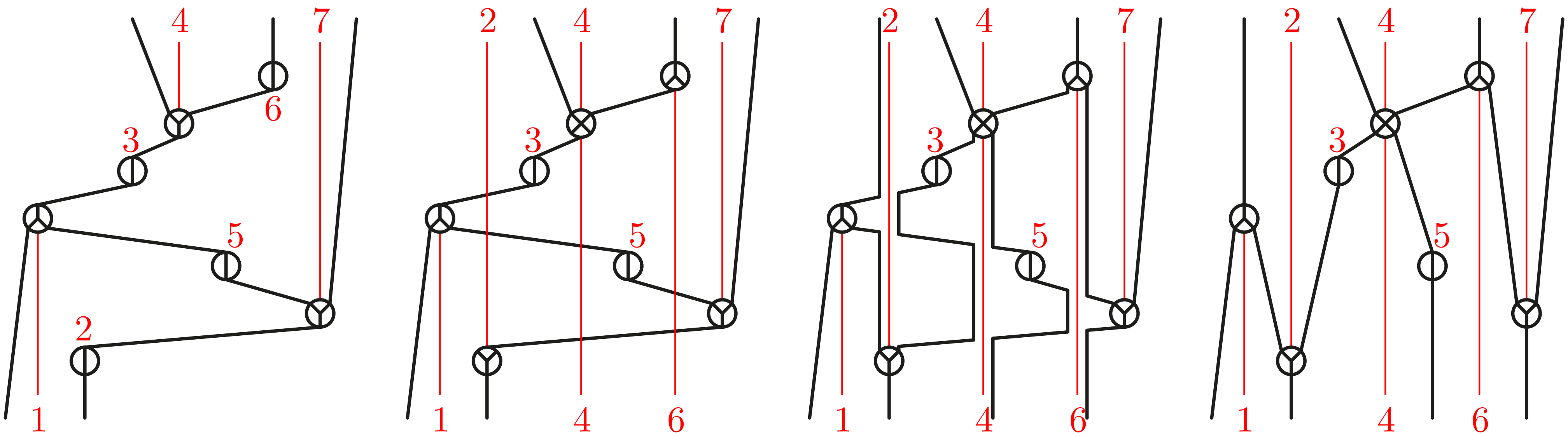
Reading, *Cambrian lattices* ('06)
P.-Pons, *Permutrees* ('16^{+})

DECORATION REFINEMENTS

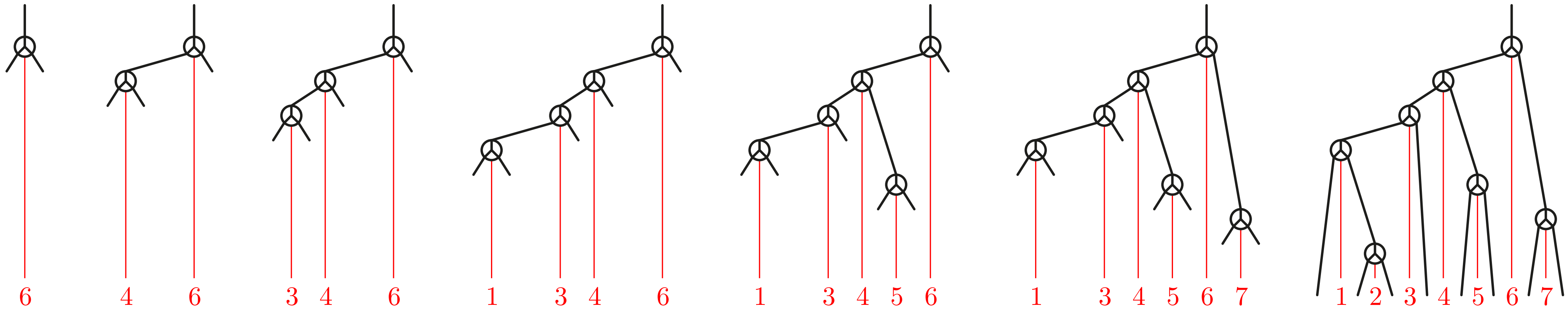
δ **refines** δ' when $\delta_i \preccurlyeq \delta'_i$ for all $i \in [n]$ for the order ⦶ $\preccurlyeq$ Ⓨ, ⅄⃝ $\preccurlyeq$ ⊗

PROP. When δ refines δ', the δ-permutree congruence classes refine the δ-permutree congruence classes refine: $\sigma \equiv_{\delta} \tau \Longrightarrow \sigma \equiv_{\delta'} \tau$

It defines a surjection $\Psi_{\delta}^{\delta'}$ from the δ-permutrees to δ'-permutrees

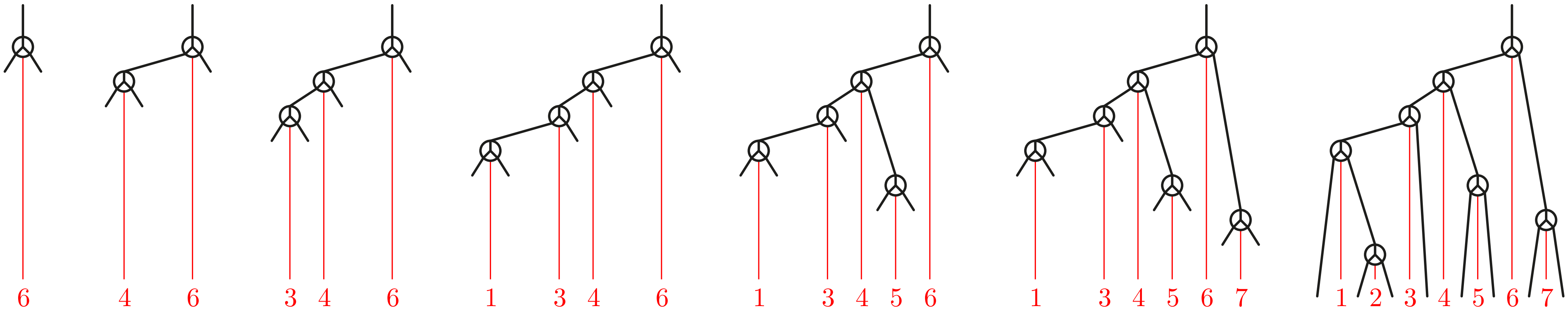


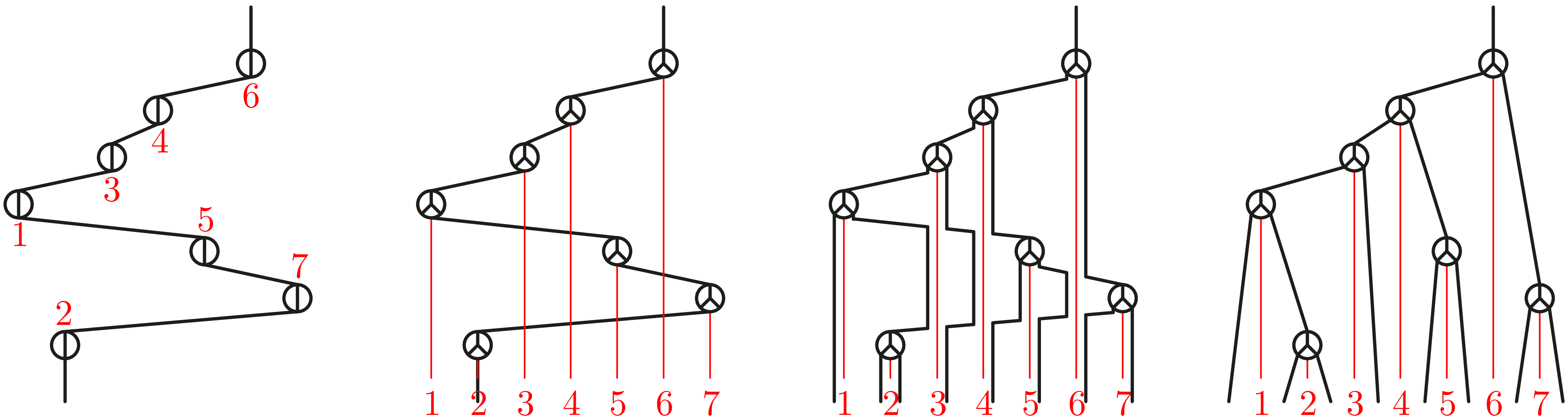
Example: Binary search tree insertion



DECORATION REFINEMENTS

Example: Binary search tree insertion with cisors and elastics





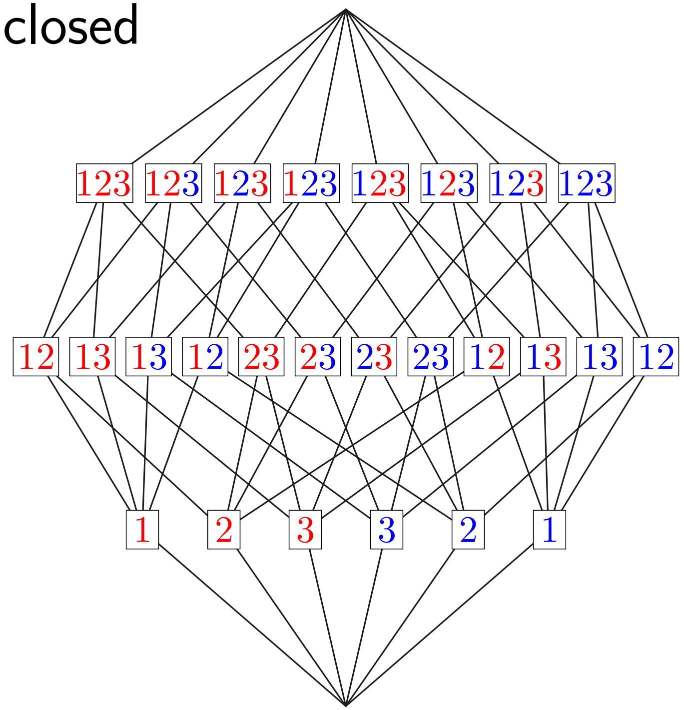
GEOMETRY

SIMPLICIAL COMPLEX

simplicial complex $=$ collection of subsets of X downward closed

exm:

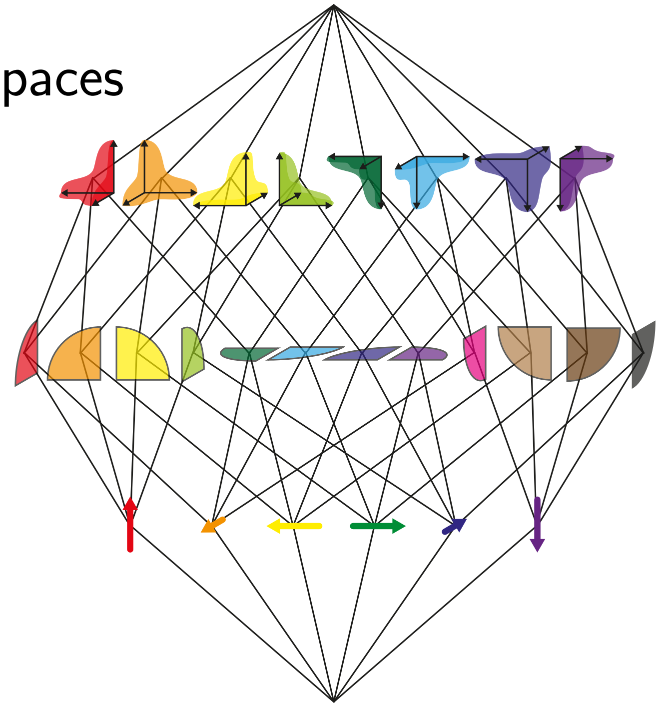
$$X = [n] \cup [n]$$
$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I\}$$

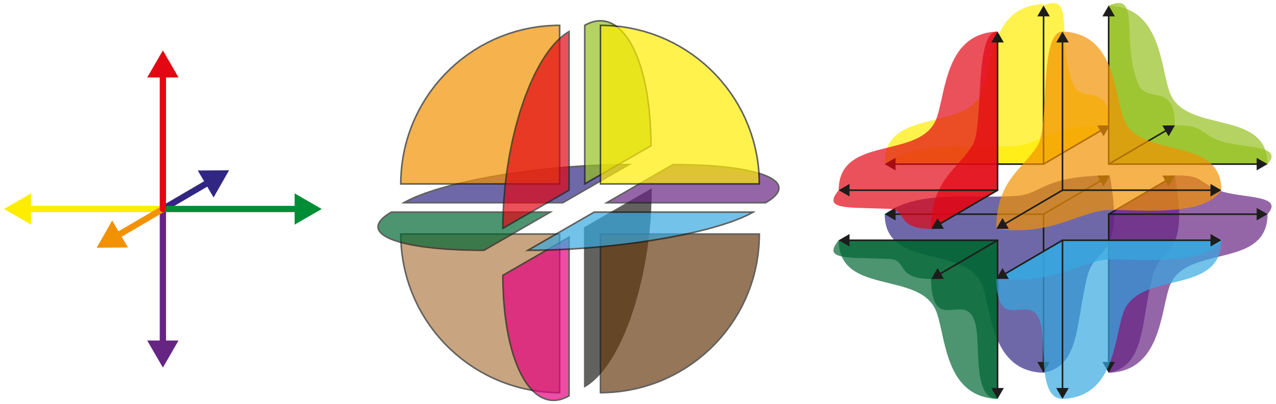


FANS

polyhedral cone $=$ positive span of a finite set of $\mathbb{R}^d$
$=$ intersection of finitely many linear half-spaces

fan $=$ collection of polyhedral cones closed by faces
and where any two cones intersect along a face





simplicial fan $=$ maximal cones generated by d rays

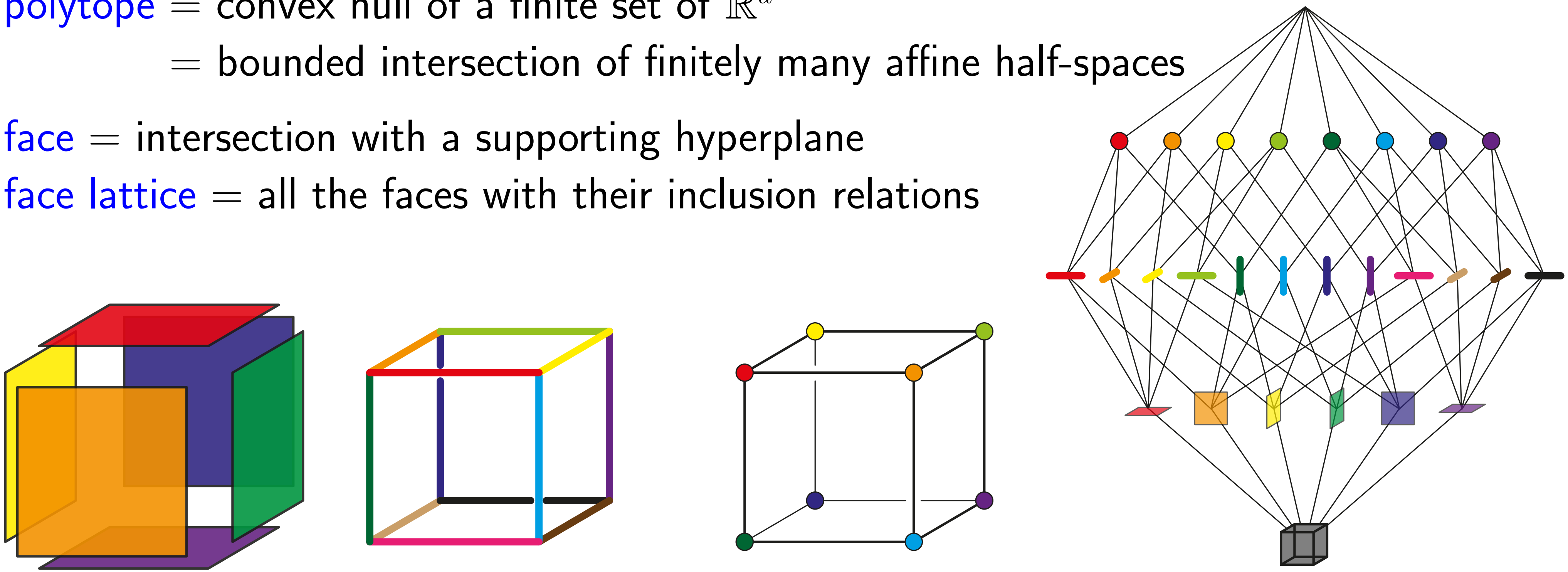
POLYTOPES

polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many affine half-spaces

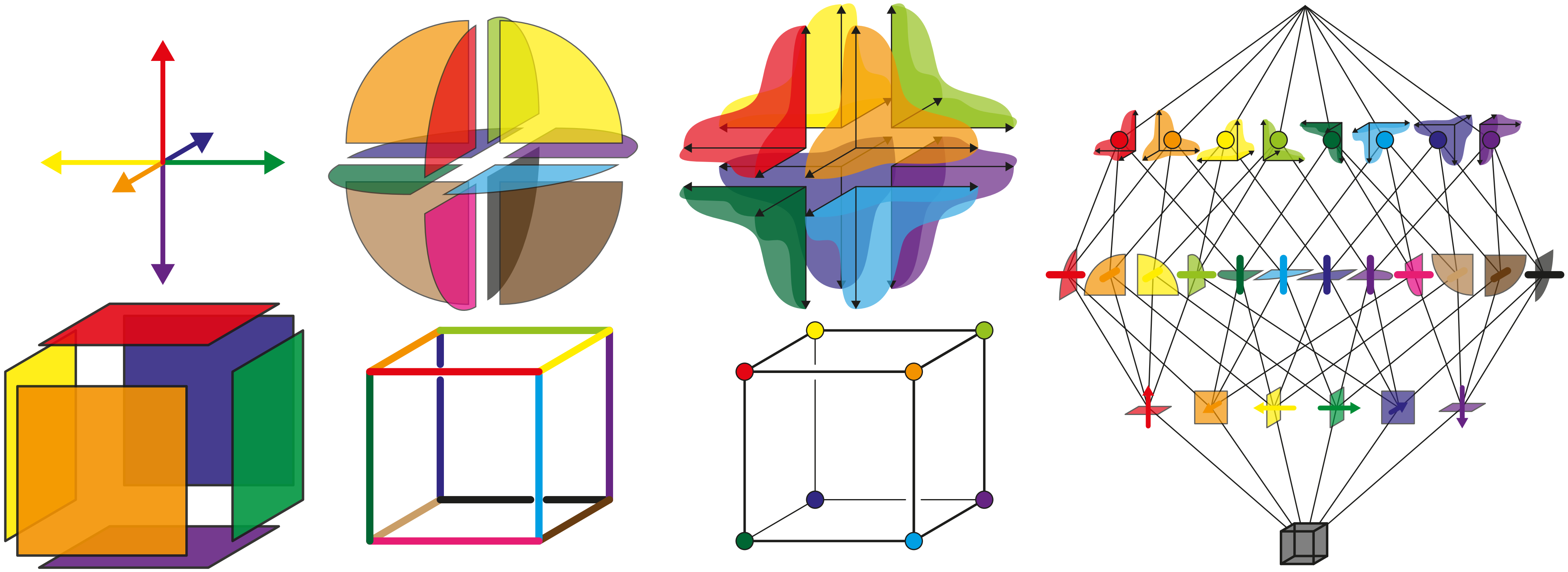
face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations



simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



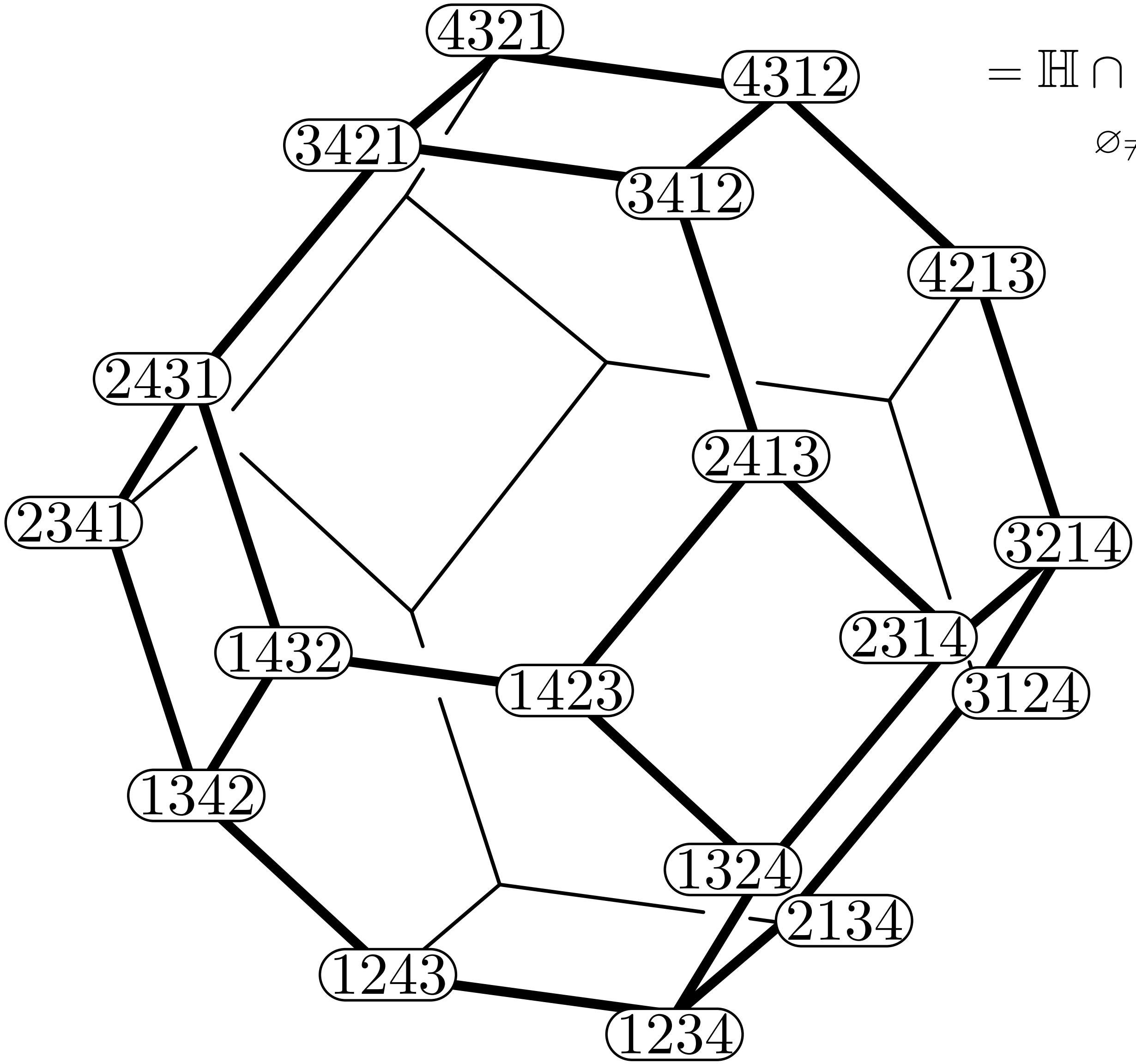
P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = { normal cone of F | F face of P }

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

PERMUTAHEDRON

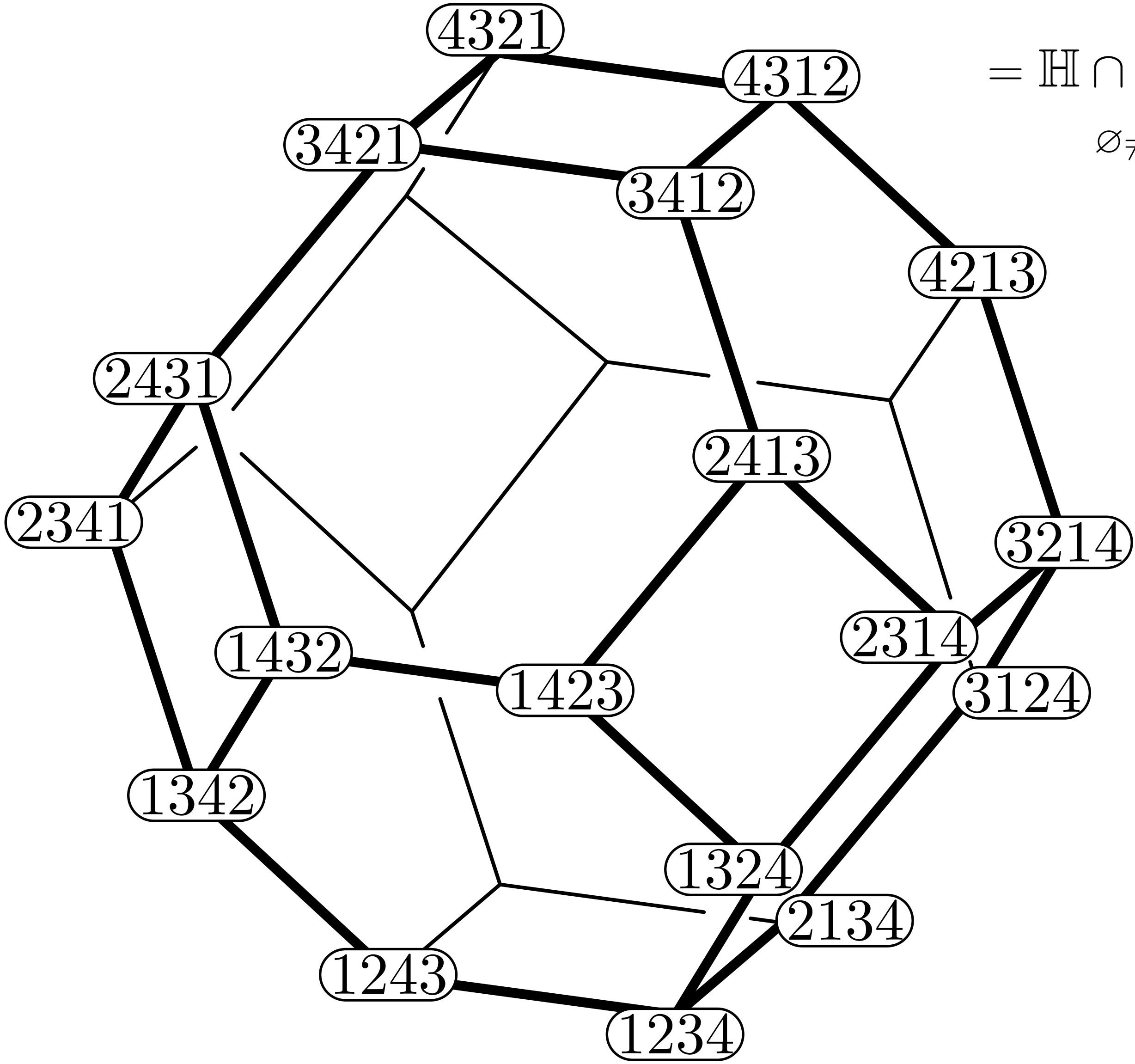


Permutahedron $\mathbb{P}(n)$

$$= \operatorname{conv}\{(\sigma(1), \ldots, \sigma(n)) \mid \sigma \in \Sigma_n\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

PERMUTAHEDRON



Permutahedron $\mathbb{P}(n)$

$$= \mathrm{conv}\left\{(\sigma(1),\ldots,\sigma(n)) \mid \sigma \in \Sigma_n\right\}$$

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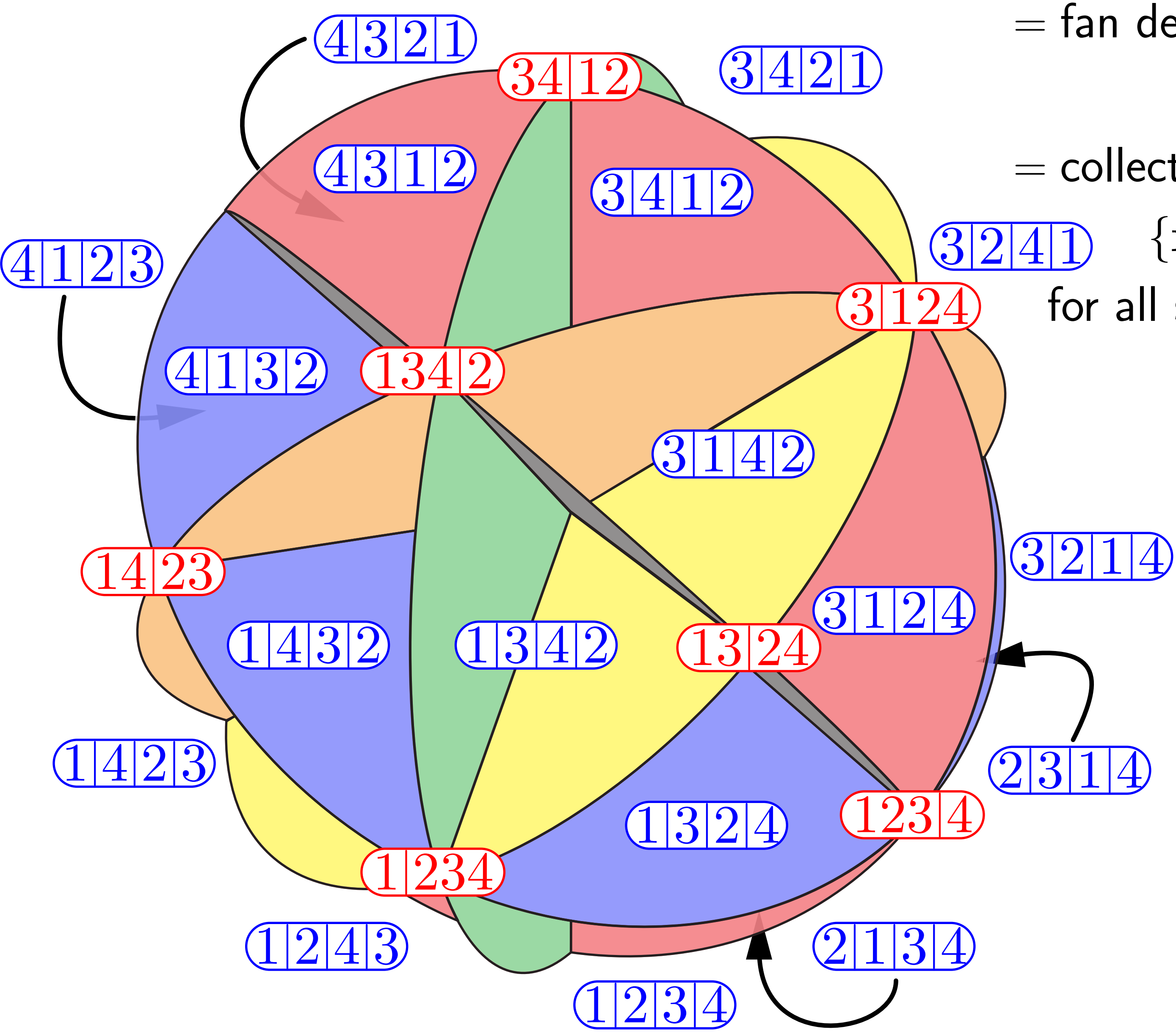
connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group



Coxeter fan

= fan defined by the hyperplane arrangement

$$\{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}_{1 \le i < j \le n}$$

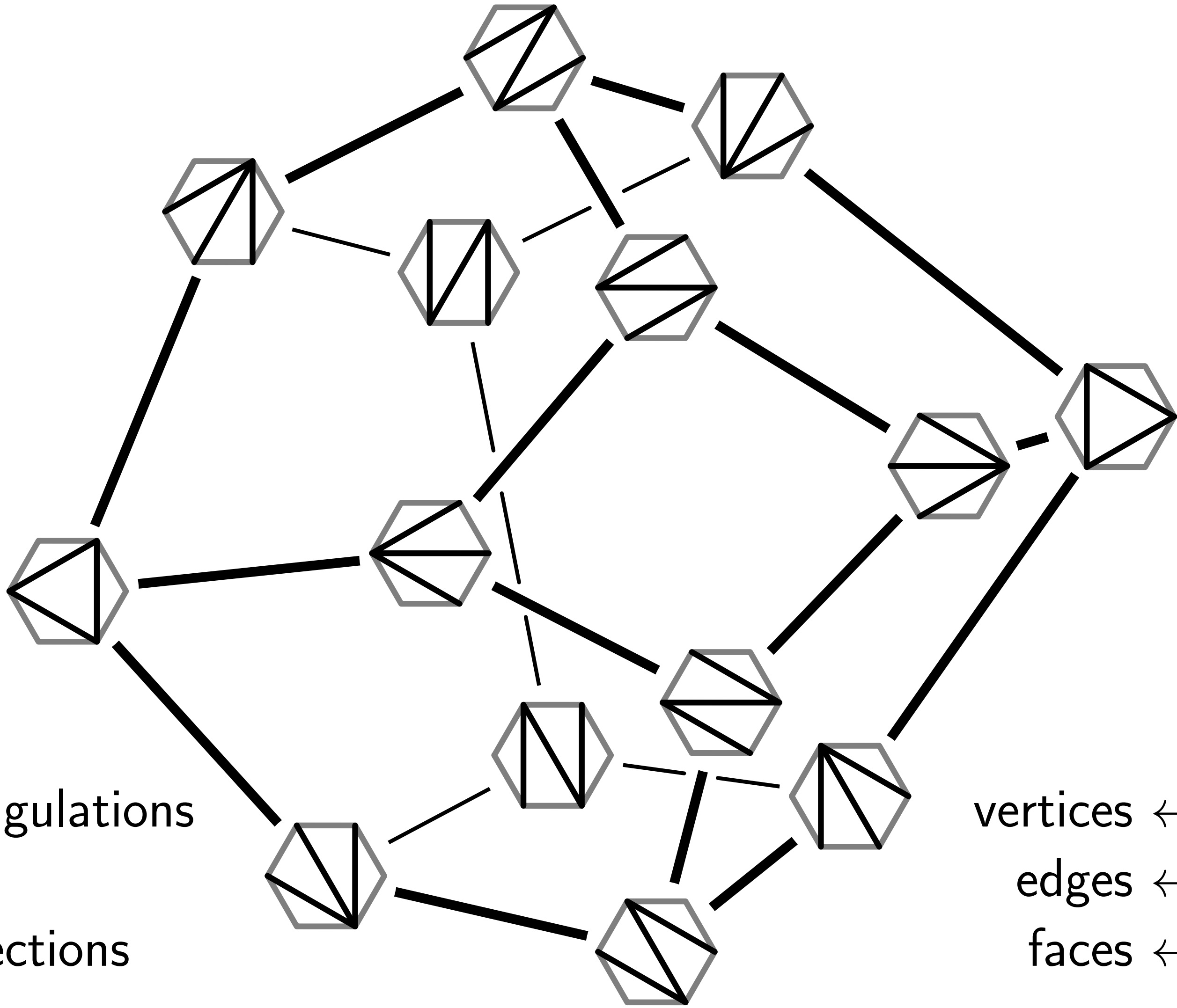
= collection of all cones

$$\{\mathbf{x} \in \mathbb{R}^n \mid x_i < x_j \text{ if } \pi(i) < \pi(j)\}$$

for all surjections $\pi : [n] \to [n-k]$

ASSOCIAHEDRON

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+2)$-gon, ordered by reverse inclusion

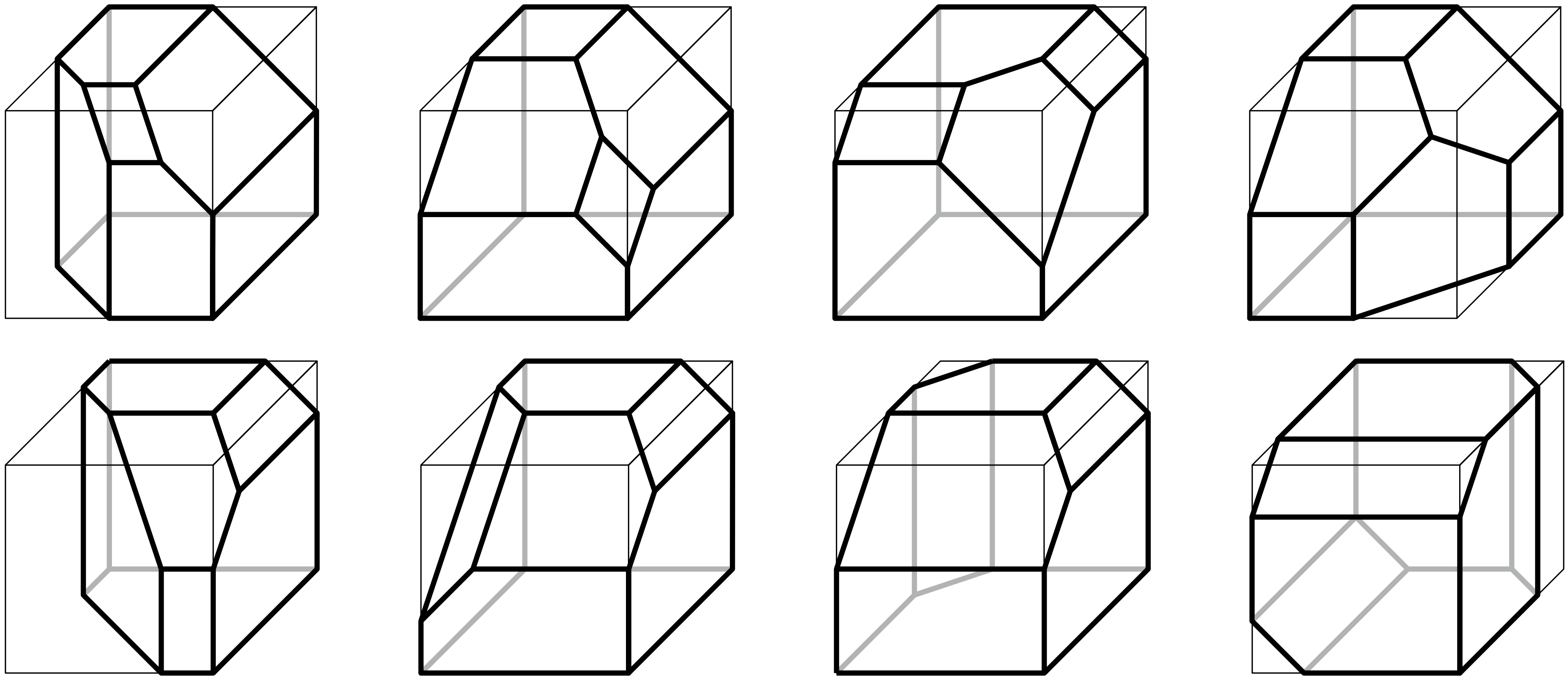


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+2)$-gon, ordered by reverse inclusion



(Pictures by Ceballos-Santos-Ziegler)

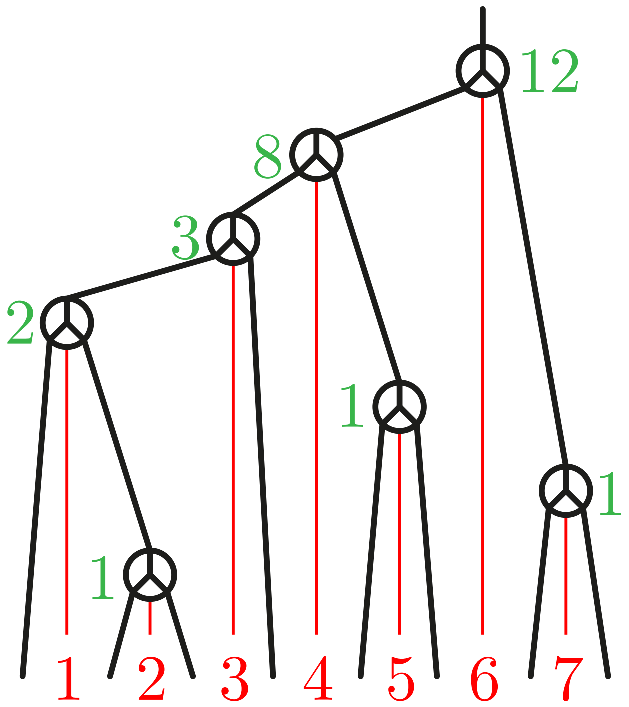
Lee ('89), Gel'fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), . . . , Ceballos-Santos-Ziegler ('11)
Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12$^+$), Lange-P. ('13$^+$)

LODAY'S ASSOCIAHEDRON

$$\mathbb{A}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n} \mathbf{H}^{\geq}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\big]_{i\in[n]} \qquad \mathbf{H}^{\geq}(i,j) := \left\{\mathbf{x}\in\mathbb{R}^n \;\middle|\; \sum_{i\le k\le j} x_i \ge \binom{j-i+2}{2}\right\}$$

Loday, *Realization of the Stasheff polytope* ('04)



LODAY'S ASSOCIAHEDRON

$$\mathbb{A}(n) := \operatorname{conv}\left\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\right\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \left[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\right]_{i\in[n]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, \sum_{i\le k\le j} x_i \ge \binom{j-i+2}{2}\right\}$$

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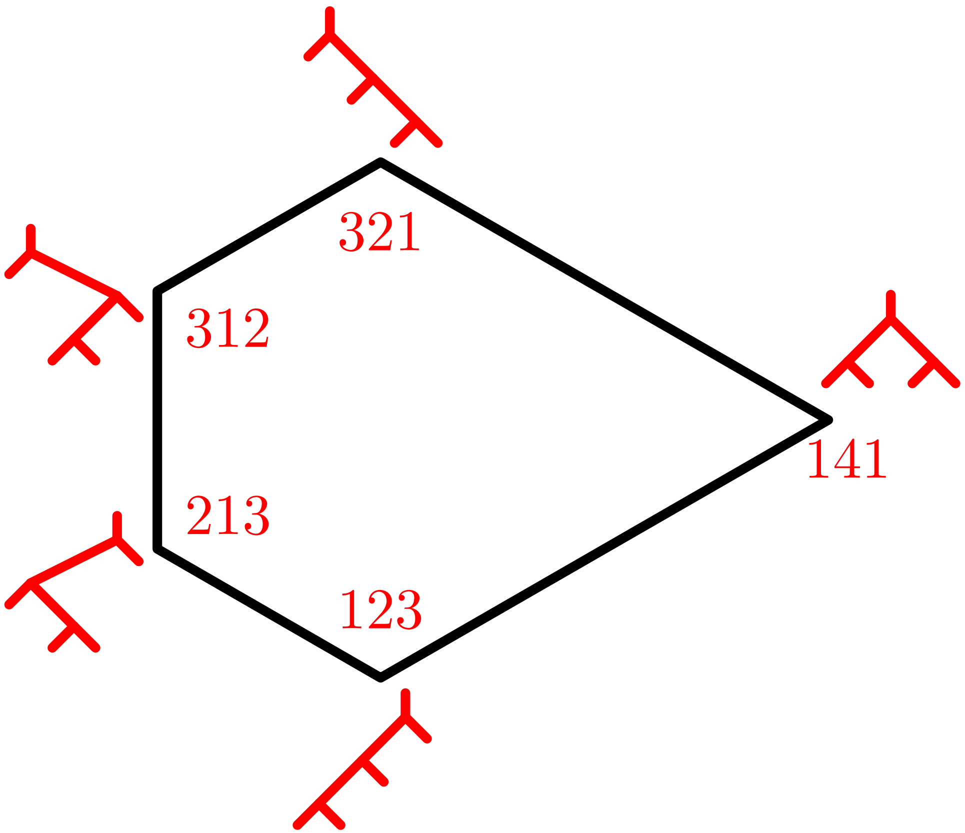
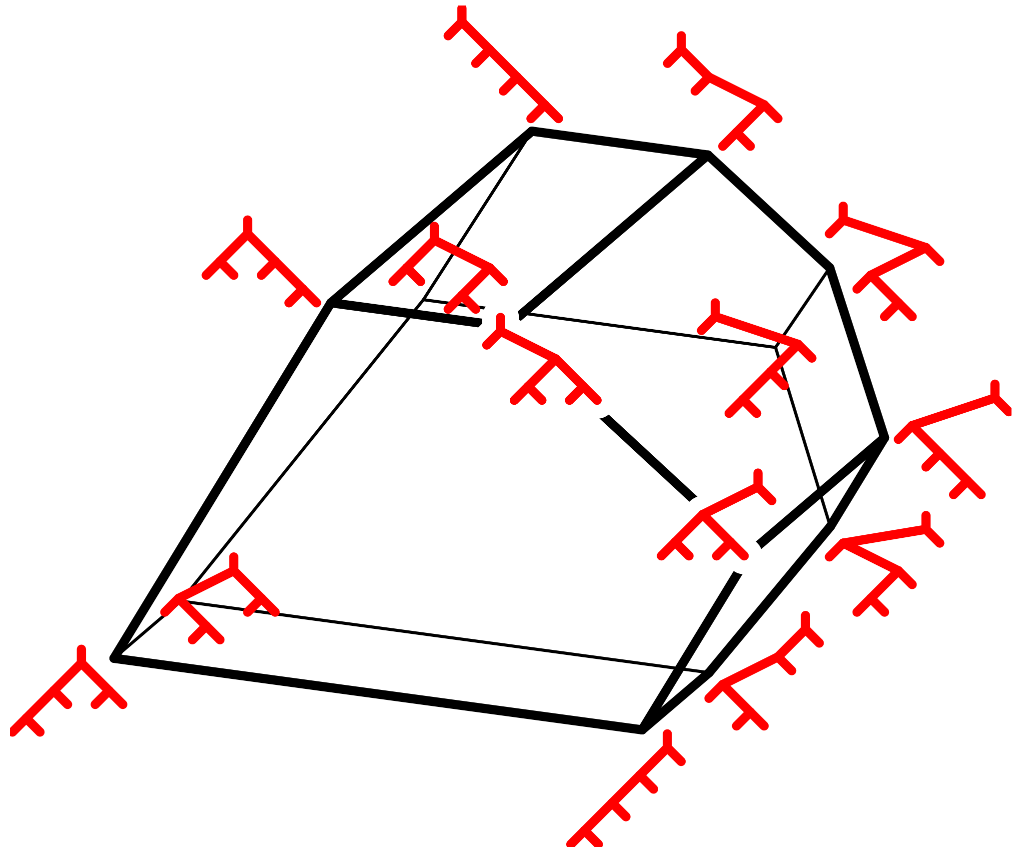
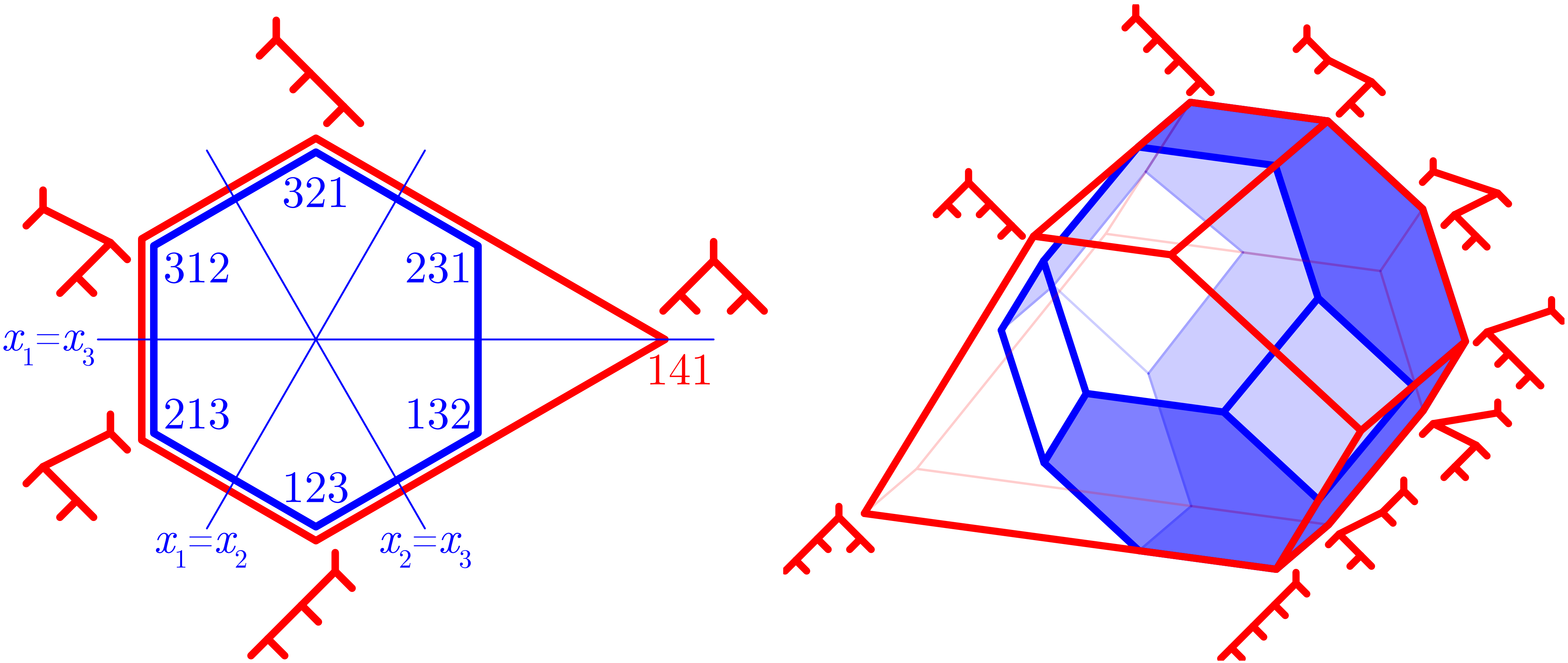




LODAY'S ASSOCIAHEDRON

$\mathbb{A}(n)$ obtained by deleting inequalities in facet description of the permutahedron

$$\text{normal cone of } \mathbf{L}(\mathrm{T}) \text{ in } \mathbb{A}(n) = \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}$$
$$= \bigcup_{\sigma^{-1} \in \mathcal{L}(\mathrm{T})} \text{normal cone of } \sigma \text{ in } \mathbb{P}(n)$$



POLYWOOD

Loday's Associahedron

PERMUTREE FAN

For a permutree T, define

$$\begin{aligned} \mathrm{C}^{\diamond}(\mathrm{T}) &:= \{\mathbf{x} \in \mathbb{H} \mid x_i \le x_j \text{ for any } i \to j \text{ in } \mathrm{T}\} \\ &= \mathbb{1} + \mathrm{cone}\left\{ \sum_{j \in J} |I| \mathbf{e}_j - \sum_{i \in I} |J| \mathbf{e}_i \;\middle|\; \text{for all edge cuts } (I \parallel J) \text{ in } \mathrm{T}) \right\} \end{aligned}$$

THM. For any $\delta \in \{⦶, ☮, Ⓨ, ⊗\}^n$, the collection of cones $\{\mathrm{C}^{\diamond}(\mathrm{T}) \mid \mathrm{T}\ \delta\text{-permutree}\}$ together with all their faces define a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$

P.-Pons, *Permutrees* ('16$^+$)

Examples.

decoration δ		permutree fan $\mathcal{F}(\delta)$
$⦶^n$	$\longleftrightarrow$	braid fan
$☮^n$	$\longleftrightarrow$	binary tree fan
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian fan
$⊗^n$	$\longleftrightarrow$	fan of the arrangement $\{x_i = x_{i+1} \mid i \in [n-1]\}$

PERMUTREEHEDRA

THM. The permutree fan $\mathcal{F}(\delta)$ is the normal fan of the permutreehedron $\mathbb{PT}(\delta)$, defined equivalently as

(i) the convex hull of the points

$$\mathbf{a}(\mathrm{T})_i = \begin{cases} d+1 & \text{if } \delta_i = \text{⦶}, \\ (\underline{\ell}+1)(\underline{r}+1) & \text{if } \delta_i = \text{☮}, \\ (d+1) - \overline{\ell}\overline{r} & \text{if } \delta_i = \text{Ⓨ}, \\ (\underline{\ell}+1)(\underline{r}+1) - \overline{\ell}\overline{r} & \text{if } \delta_i = \otimes, \end{cases}$$

for all δ-permutrees T

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

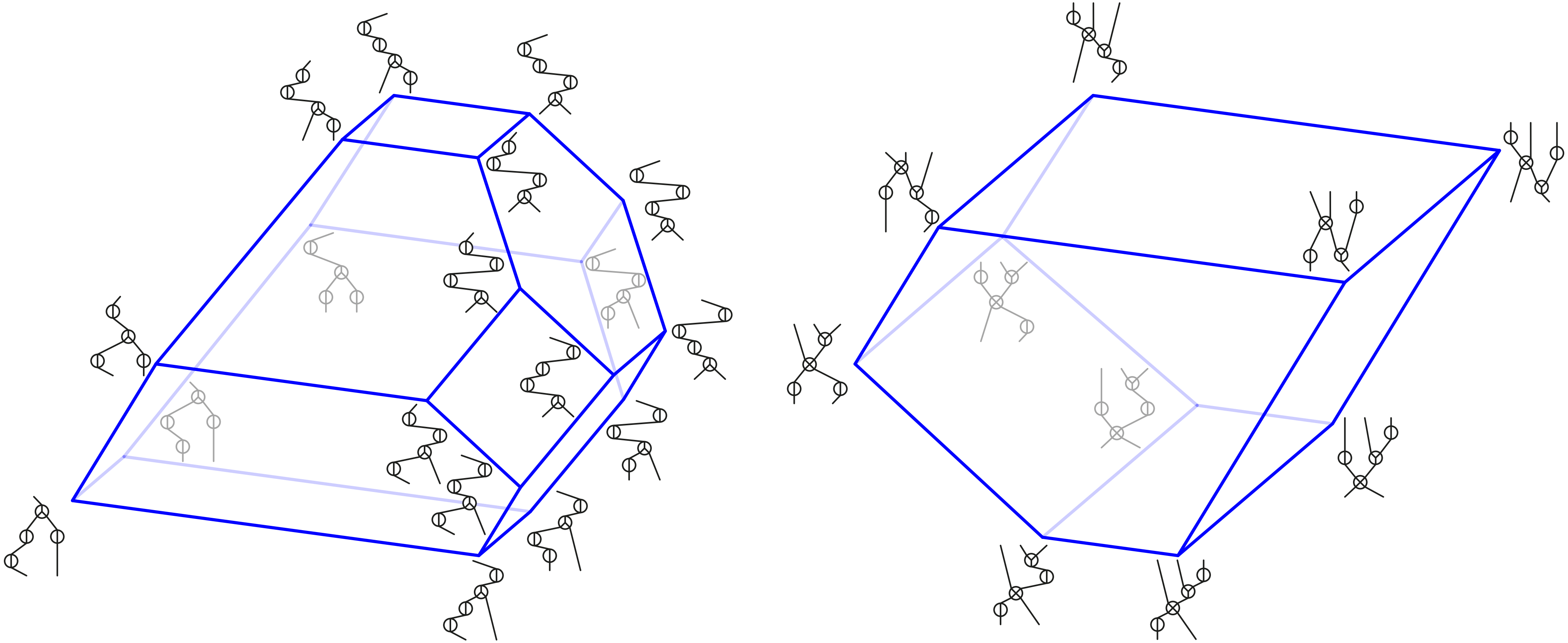
$$\mathbf{H}^{\geq}(I) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \right\}$$

for all edge cuts $(I \parallel J)$ of all δ-permutrees

P.-Pons, *Permutrees* ('16$^+$)

PERMUTREEHEDRA

THM. The permutree fan $\mathcal{F}(\delta)$ is the normal fan of the permutreehedron $\mathbb{PT}(\delta)$



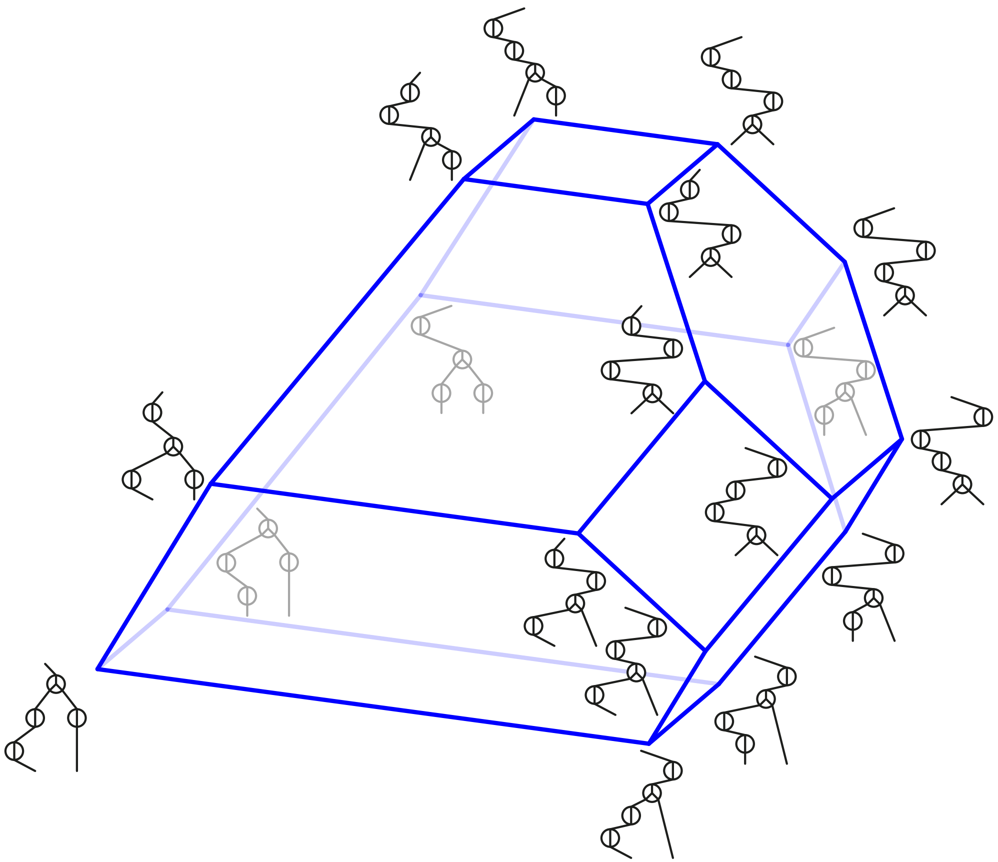
Examples.

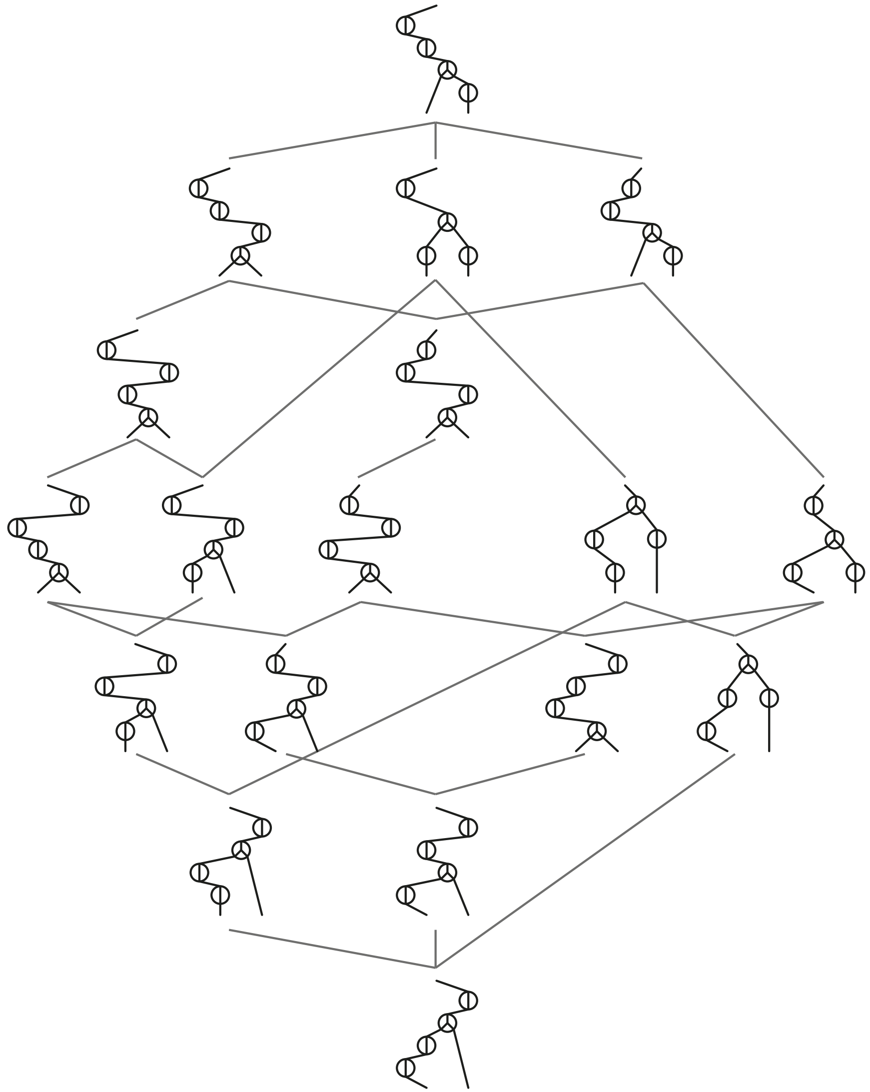
decoration δ		permutreehedron $\mathbb{PT}(\delta)$
$⦶^n$	$\longleftrightarrow$	permutahedron
$☮^n$	$\longleftrightarrow$	Loday's associahedron
$\{☮, \rotatebox{180}{☮}\}^n$	$\longleftrightarrow$	Hohlweg-Lange's associahedra
$\otimes^n$	$\longleftrightarrow$	cube

FURTHER GEOMETRIC TOPICS

PROP. $U := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n+1-2i)\,\mathbf{e}_i$

graph of $\mathbb{PT}(\delta)$ oriented by U = Hasse diagram of the δ-permutree lattice





FURTHER GEOMETRIC TOPICS

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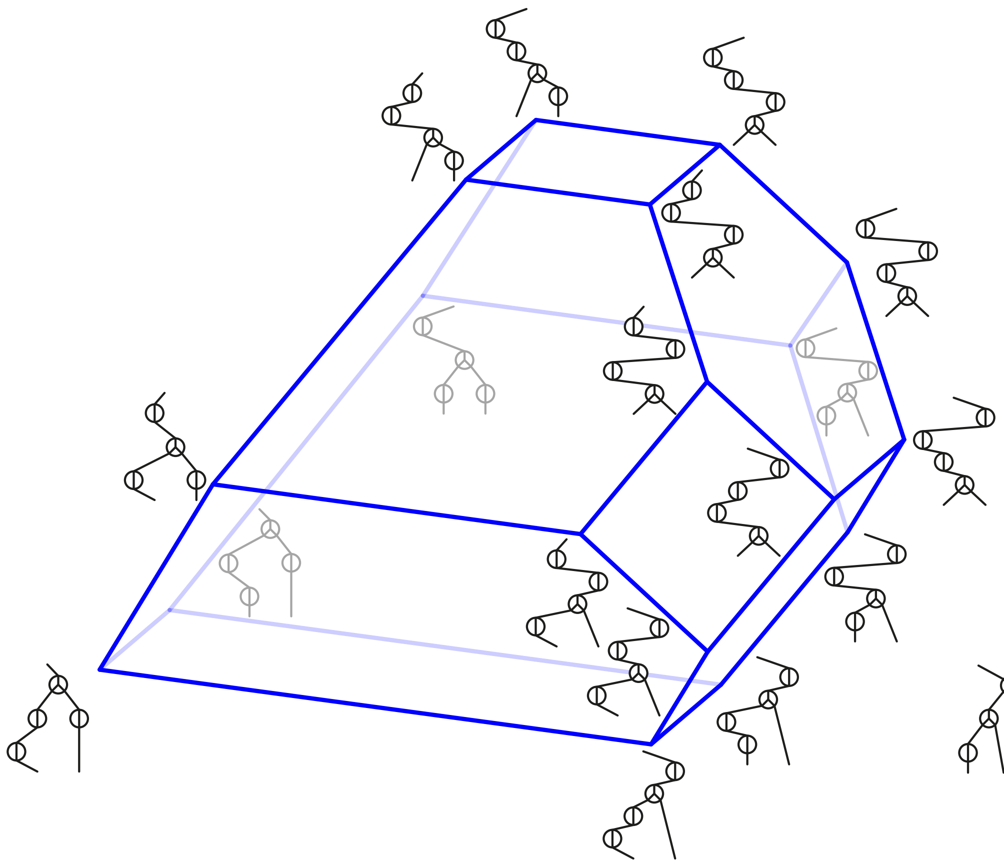
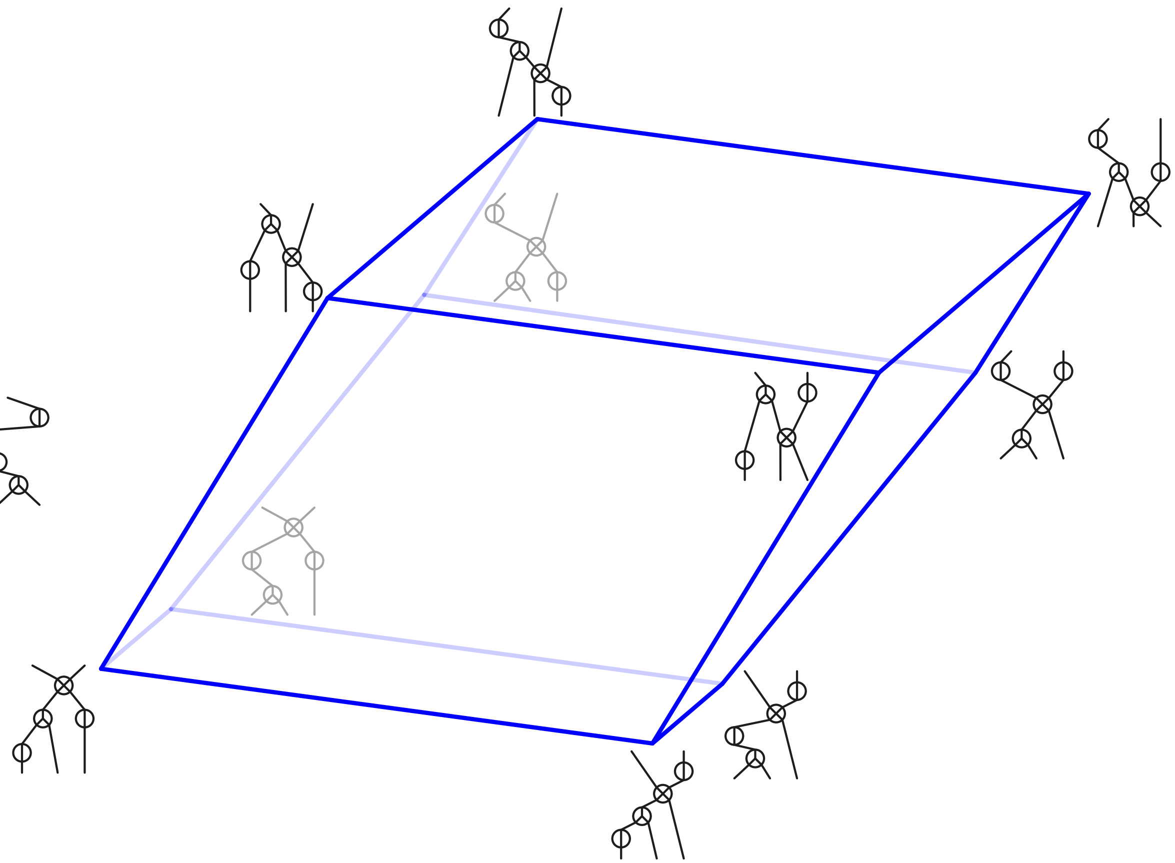
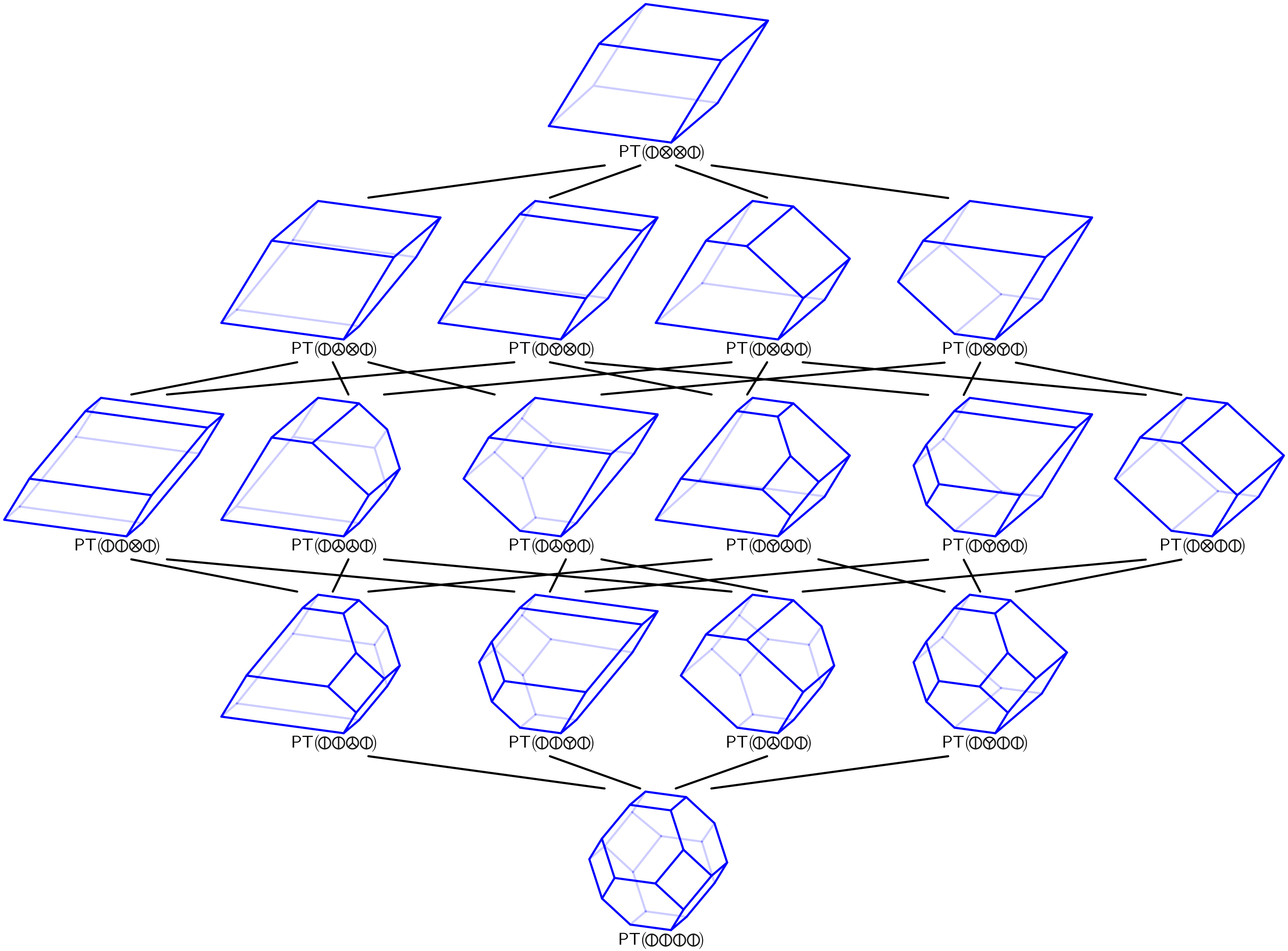
PROP. refinement $\delta \preccurlyeq \delta' \quad \Longrightarrow \quad$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$





MATRIOCHKA PERMUTREEHEDRA

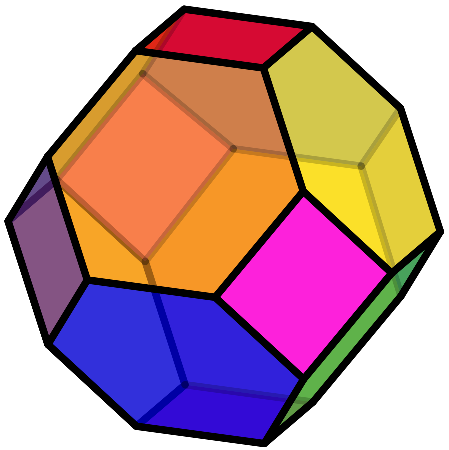
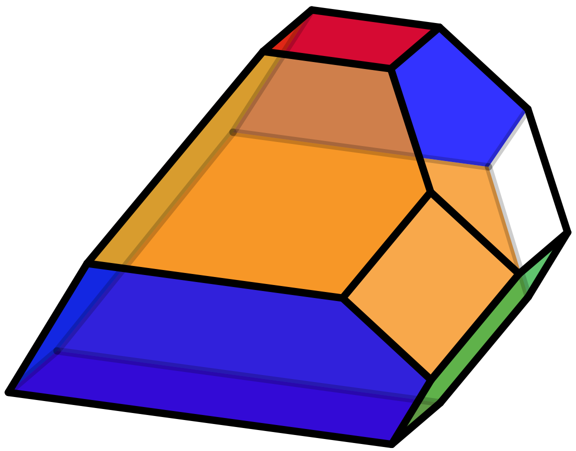
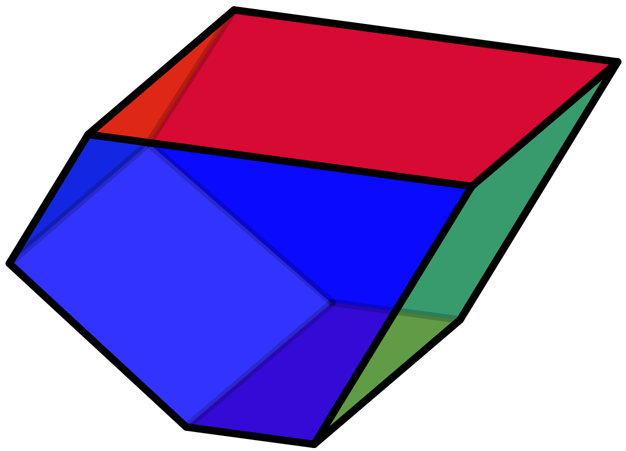


FURTHER GEOMETRIC TOPICS

PROP. $U := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n+1-2i)\, \mathbf{e}_i$
graph of $\mathbb{PT}(\delta)$ oriented by U = Hasse diagram of the δ-permutree lattice

PROP. refinement $\delta \preccurlyeq \delta'$ $\implies$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$

PROP. Assume $\delta_1 = \delta_n \neq$ ⦶ and let $n_1, \dots, n_k$ = sizes of the blocks of consecutive ⦶
$\mathbb{PT}(\delta)$ has $\sum_i (2^{n_i+1} - 1)$ pairs of parallel facets

FURTHER GEOMETRIC TOPICS

PROP. $U := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n + 1 - 2i)\, \mathbf{e}_i$
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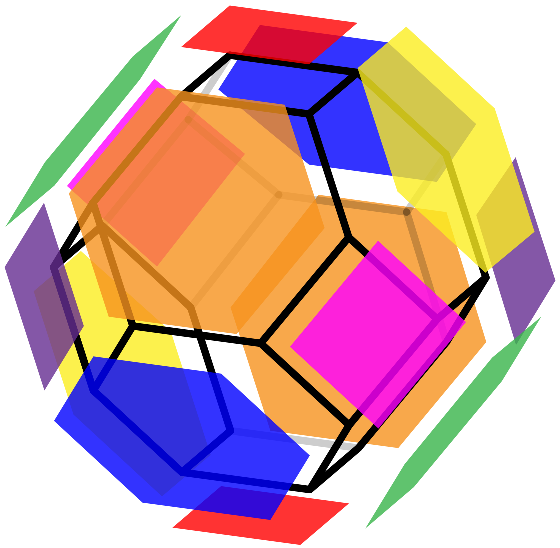
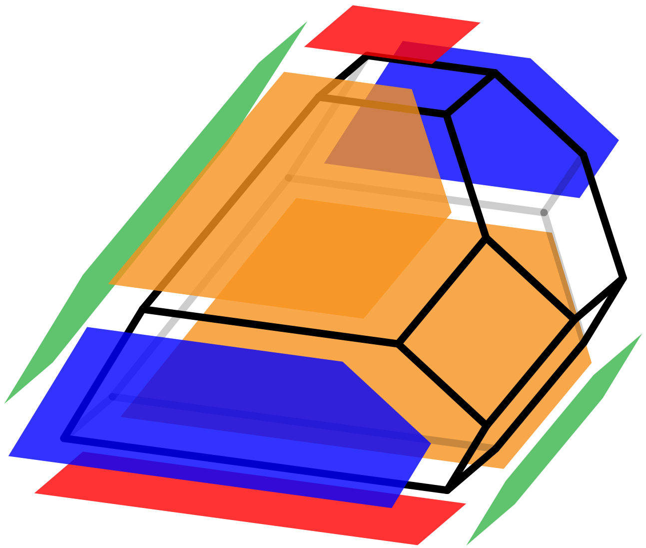
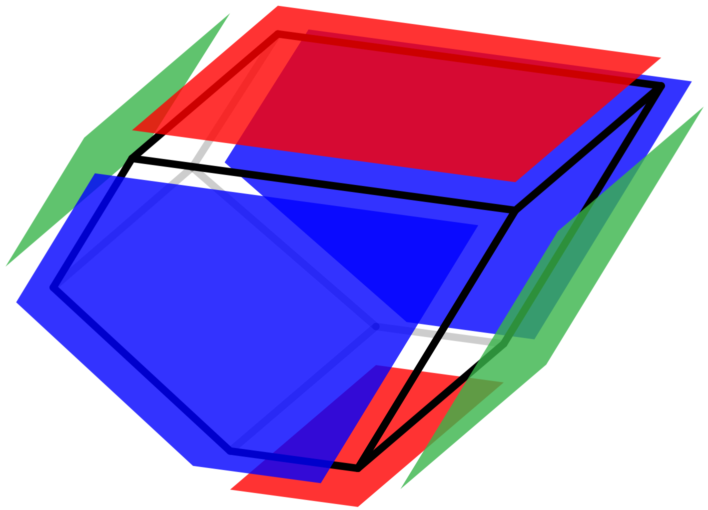

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PROP. refinement $\delta \preccurlyeq \delta' \quad \Longrightarrow \quad$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$

PROP. Assume $\delta_1 = \delta_n \ne ⦶$ and let $n_1, \dots, n_k$ = sizes of the blocks of consecutive ⦶
$\mathbb{PT}(\delta)$ has $\sum_i (2^{n_i+1} - 1)$ pairs of parallel facets

PROP. For $\delta \preccurlyeq \delta'$, a δ-permutree T and a δ'-permutree T', the following are equivalent:

- the vertex $\mathbf{a}(\mathrm{T})$ of $\mathbb{PT}(\delta)$ coincides with the vertex $\mathbf{a}(\mathrm{T}')$ of $\mathbb{PT}(\delta')$
- the normal cone $\mathrm{C}^\diamond(\mathrm{T})$ of $\mathbb{PT}(\delta)$ coincides with the normal cone $\mathrm{C}^\diamond(\mathrm{T}')$ of $\mathbb{PT}(\delta')$
- the fiber of T' under the surjection $\Psi_\delta^{\delta'}$ is the singleton $(\Psi_\delta^{\delta'})^{-1}(\mathrm{T}') = \{\mathrm{T}\}$
- T and T' have the same linear extensions
- T and T' coincide up to empty ancestors or descendants

FURTHER GEOMETRIC TOPICS

PROP. $U := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n+1-2i)\, \mathbf{e}_i$
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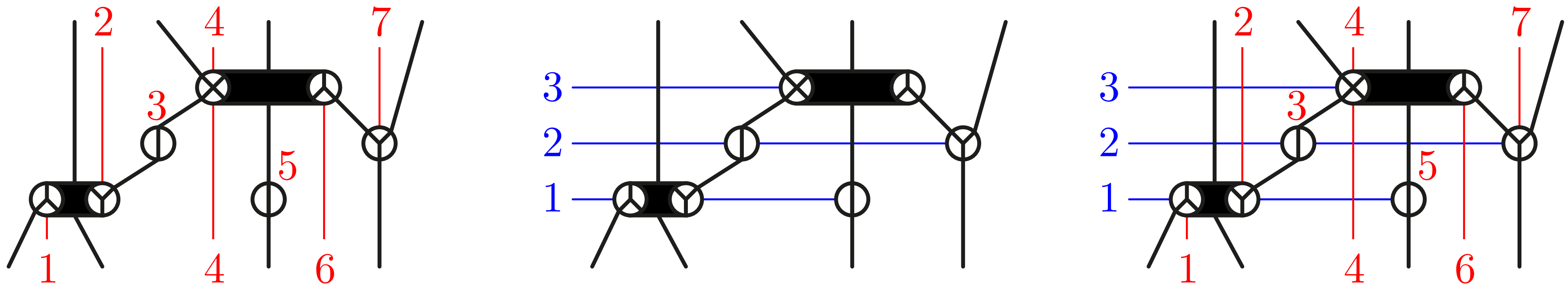
PROP. For $\delta \preccurlyeq \delta'$, a δ-permutree T and a δ'-permutree T', the following are equivalent:

- the vertex $\mathbf{a}(\mathrm{T})$ of $\mathbb{PT}(\delta)$ coincides with the vertex $\mathbf{a}(\mathrm{T}')$ of $\mathbb{PT}(\delta')$
- T and T' have the same linear extensions

PROP. $\delta, \delta' \in$ ⦶ $\cdot$ {⦶, ☮, Ⓨ, ⊗}$^{n-2}$ $\cdot$ ⦶
let $I_1, \dots, I_p$ and $I'_1, \dots, I'_{p'}$ be the blocks of consecutive ⦶ in δ and δ'
The isometries between the two permutreehedra $\mathbb{PT}(\delta)$ and $\mathbb{PT}(\delta')$ are given by
$(\mathfrak{S}_{I_1} \times \dots \times \mathfrak{S}_{I_p}) \circ \langle$vertical and horizontal symmetry$\rangle \circ (\mathfrak{S}_{I'_1} \times \dots \times \mathfrak{S}_{I'_{p'}})$

EXTENSIONS

- Schröder permutrees



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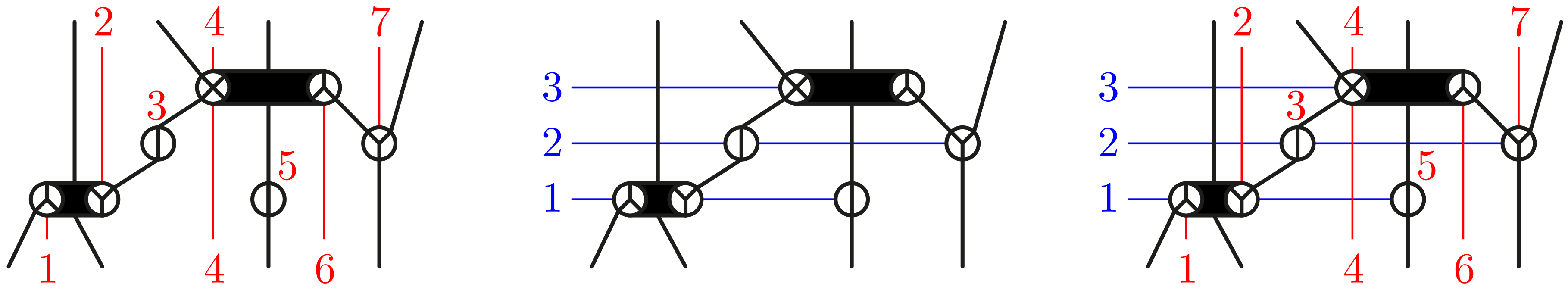
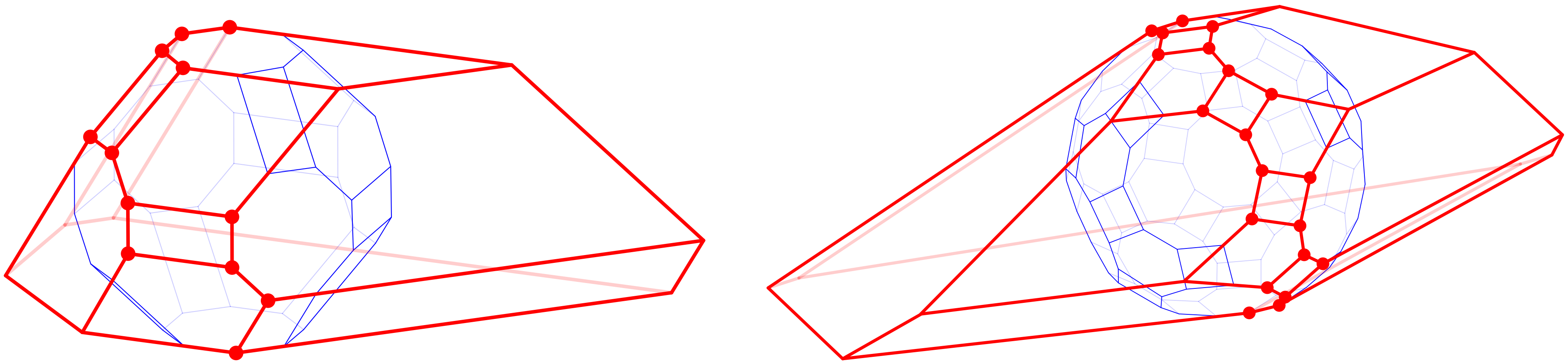


- arbitrary finite Coxeter groups
  somewhere between the W-permutahedron and the W-associahedron



EXTENSIONS

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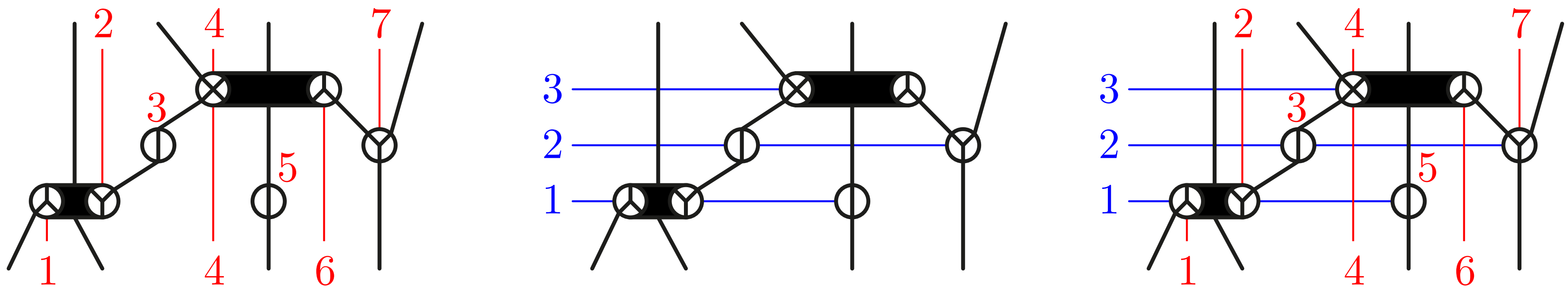

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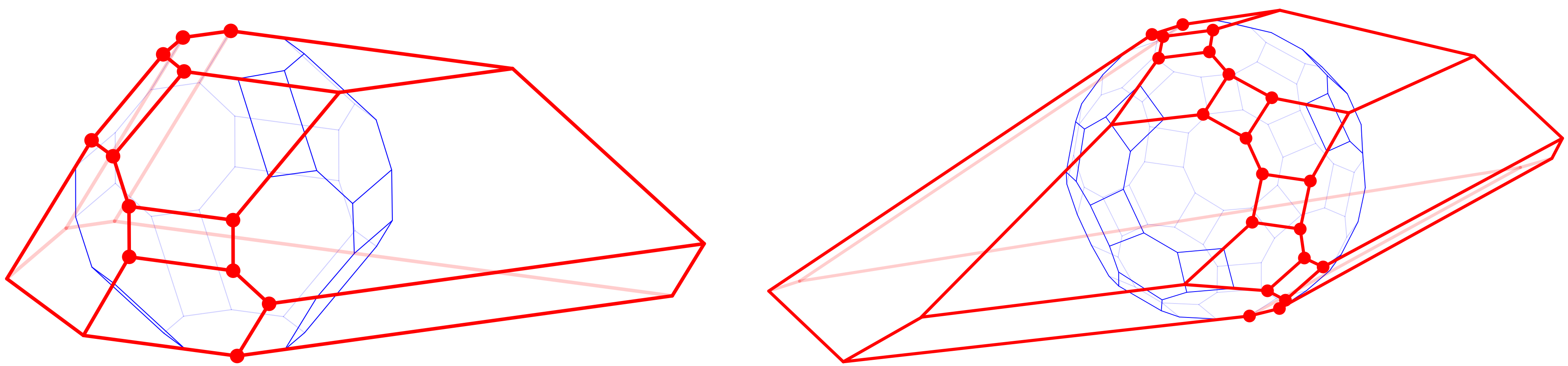
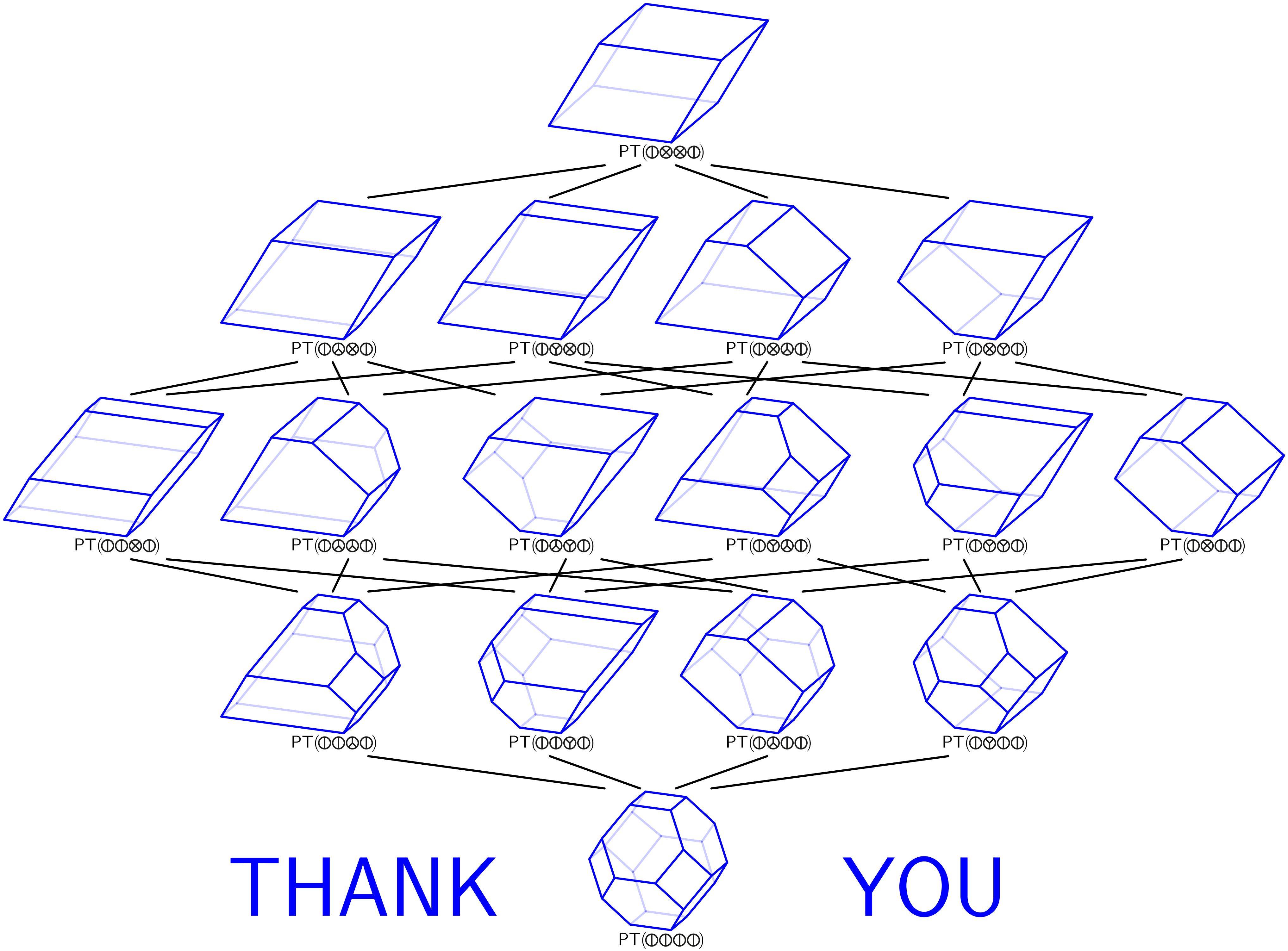


- non-crossing arc diagrams passing a limited number of walls



THANK YOU