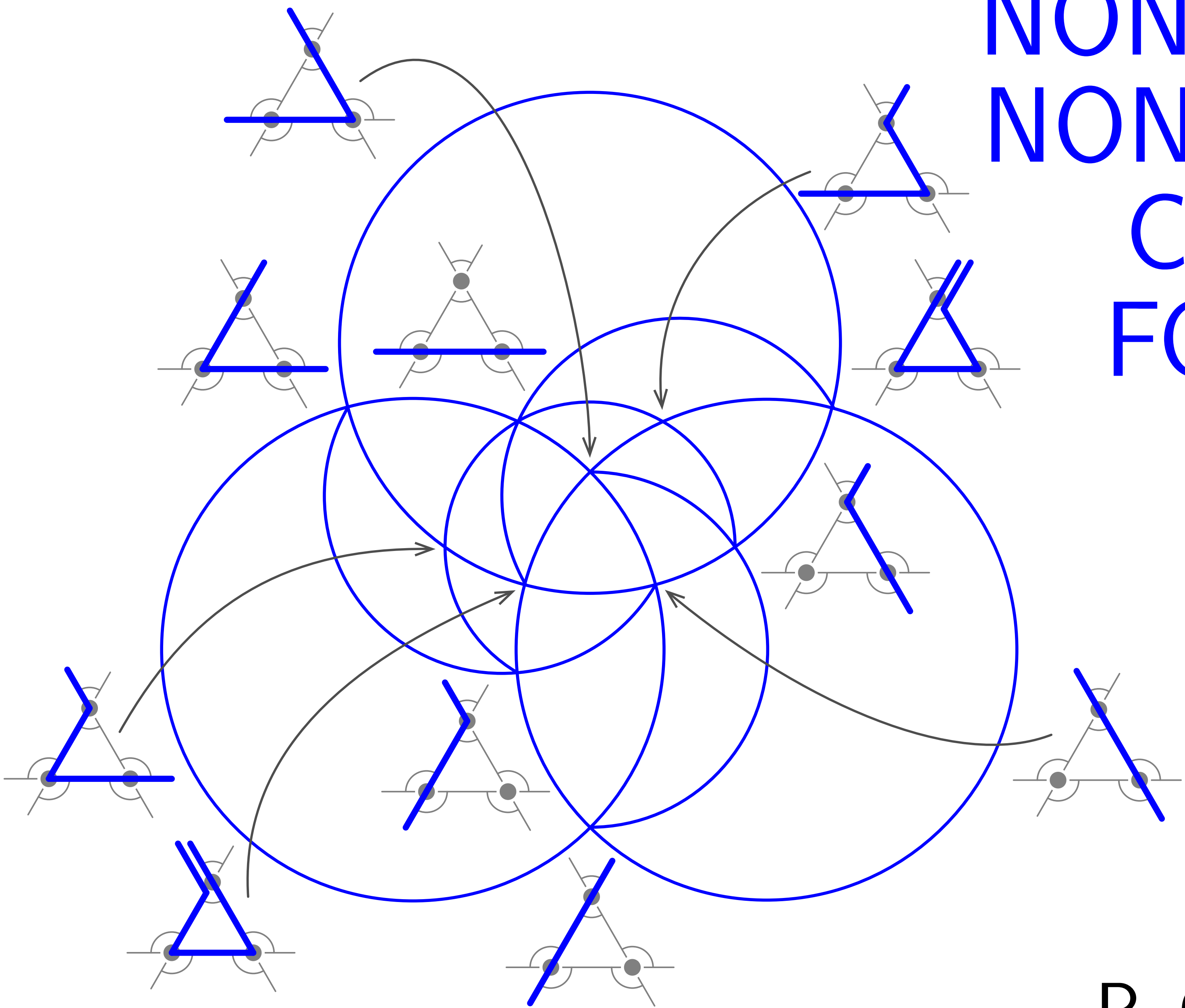


NON-KISSING & NON-CROSSING COMPLEXES FOR GENTLE ALGEBRAS

Y. PALU

Univ. Amiens

V. PILAUD

CNRS & LIX,

École Poytechnique

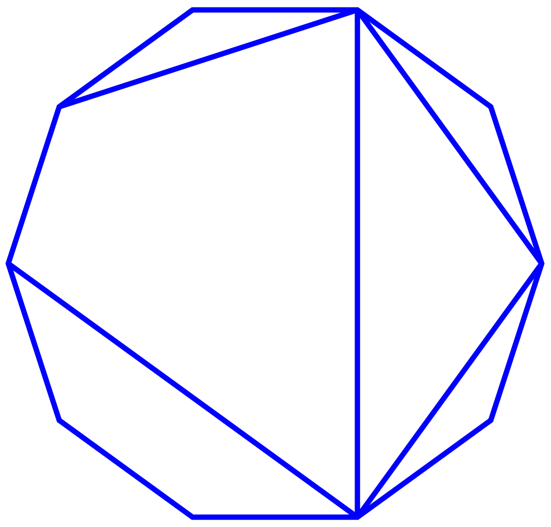
P.-G. PLAMONDON

Univ. Orsay

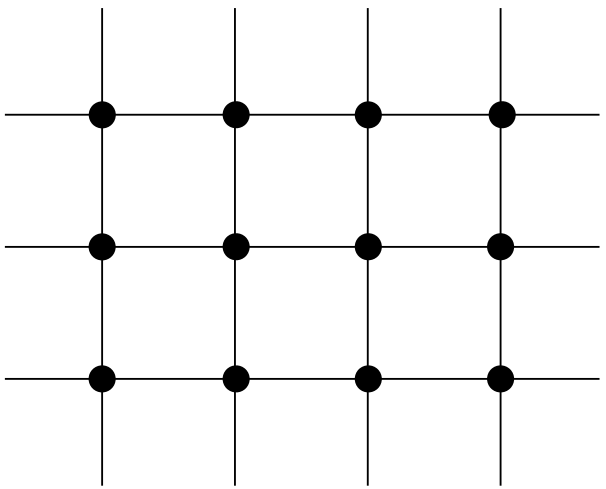
MOTIVATION

Baryshnikov, *On Stokes sets* ('01)
Chapoton, *Stokes posets and serpent nests* ('16)
Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('18)

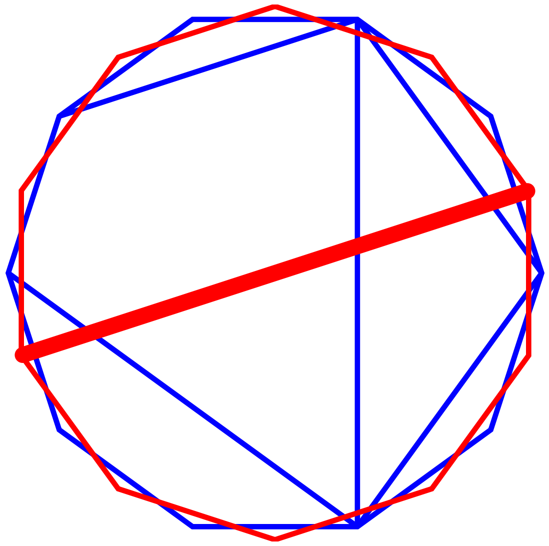
Petersen–Pylyavskyy–Speyer, *A non-crossing standard monomial theory* ('10)
Santos–Stump-Welker, *Non-crossing sets and the Grassmann-assoc.* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)



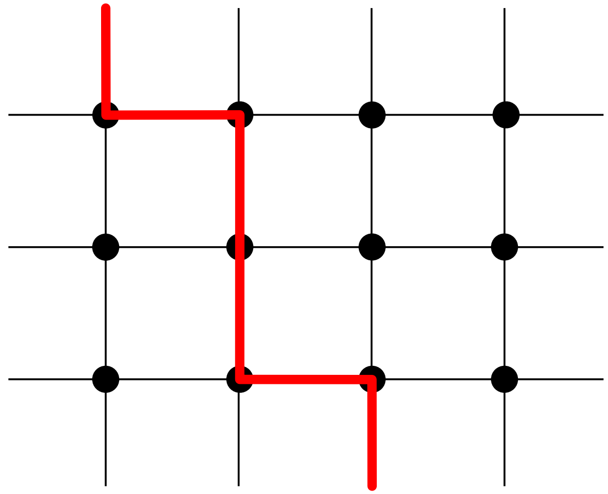
dissection



subset of $\mathbb{Z}^2$

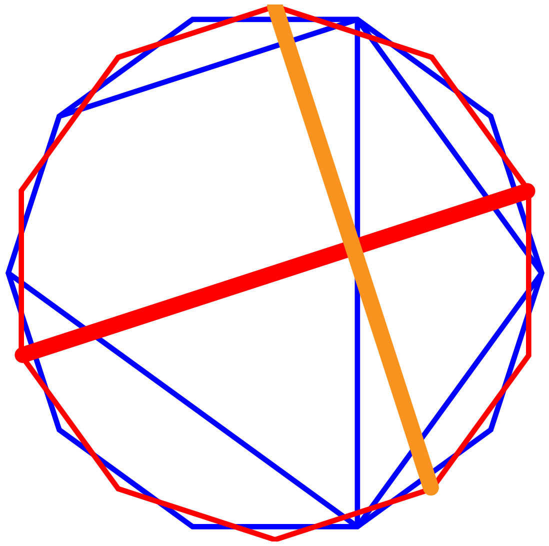


dissection
accordion

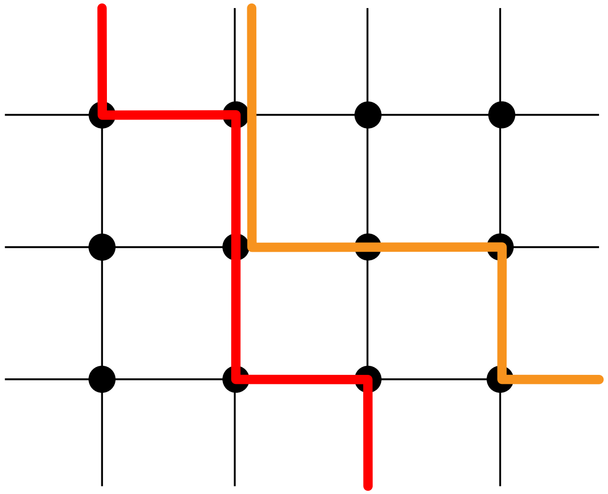


subset of $\mathbb{Z}^2$
monotone path

TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

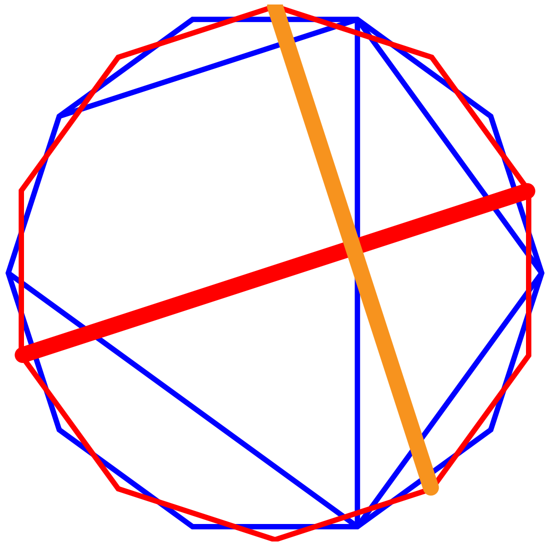


dissection
accordion
non-crossing complex

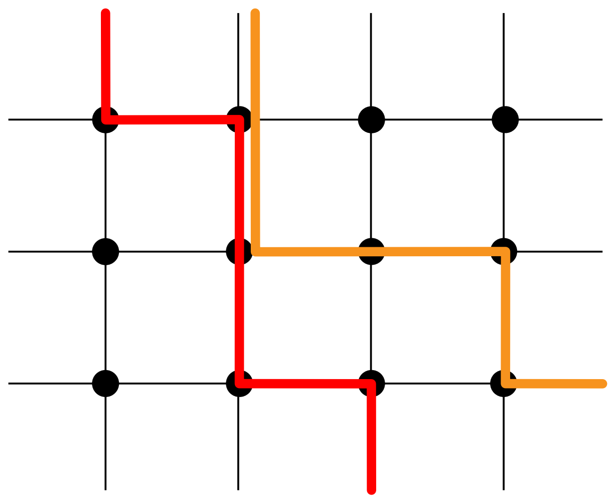


subset of $\mathbb{Z}^2$
monotone path
non-kissing complex

TWO GENERALIZATIONS OF THE ASSOCIAHEDRON



dissection
accordion
non-crossing complex



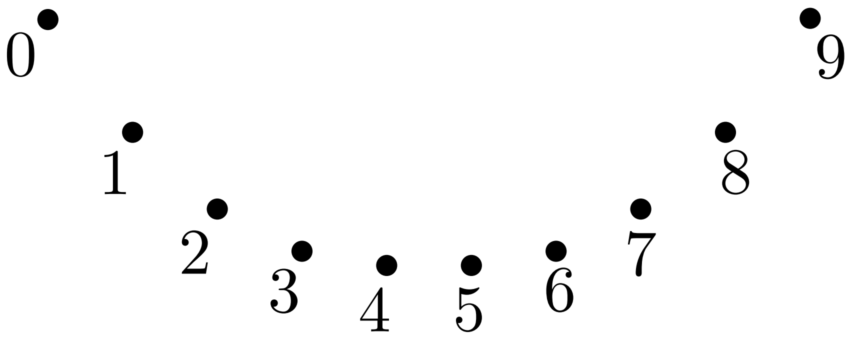
subset of $\mathbb{Z}^2$
monotone path
non-kissing complex

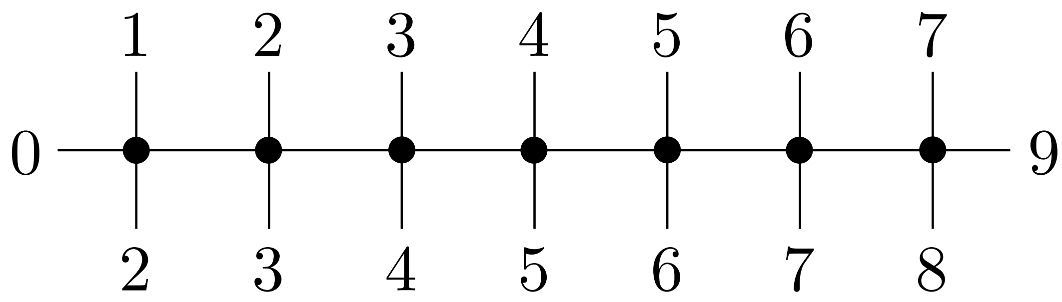
Baryshnikov, *On Stokes sets* ('01)
Chapoton, *Stokes posets and serpent nests* ('16)
Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('18)

Petersen–Pylyavskyy–Speyer, *A non-crossing standard monomial theory* ('10)
Santos–Stump–Welker, *Non-crossing sets and the Grassmann-assoc.* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)
Garver–McConville, *Enumerative properties of grid-associahedra* ('17$^+$)

simplicial associahedron $=$ simplicial complex with

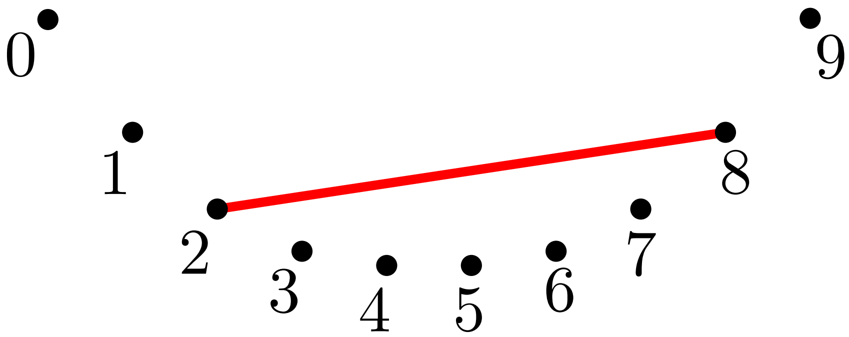
- vertices $=$ internal diagonals of an $(n+3)$-gon
- faces $=$ collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

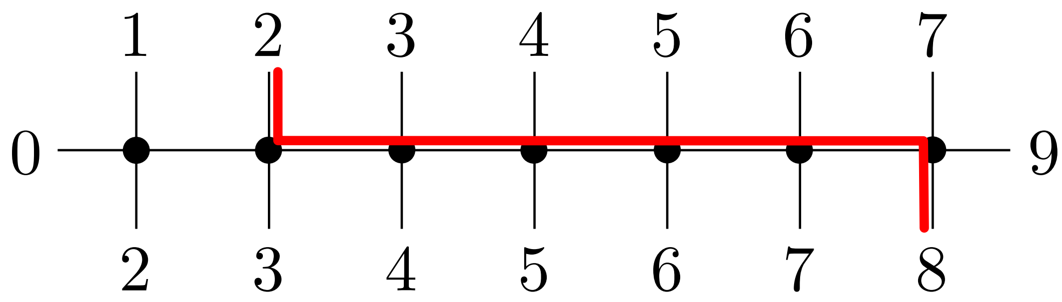




simplicial associahedron $=$ simplicial complex with

- vertices $=$ internal diagonals of an $(n+3)$-gon
- faces $=$ collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

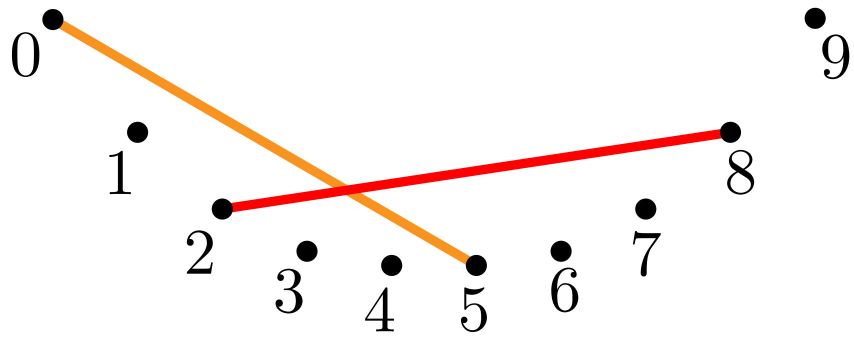


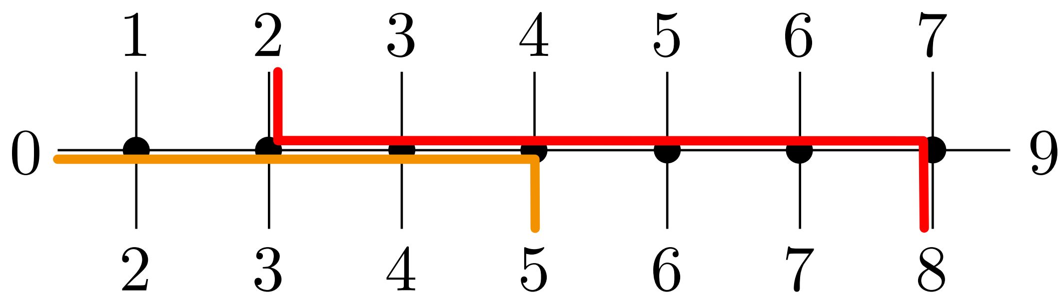


diagonal $\longleftrightarrow$ walk

simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

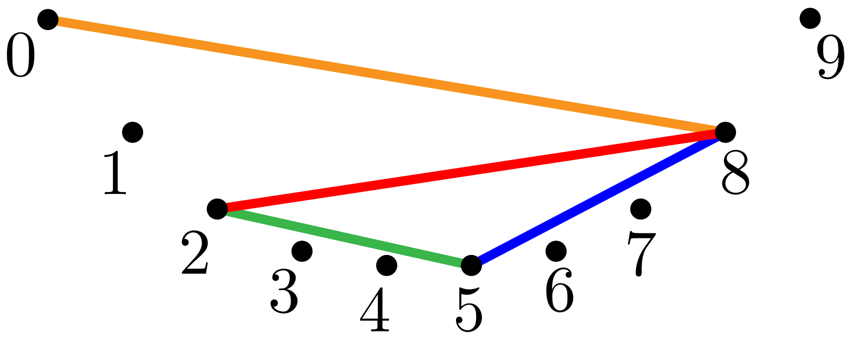


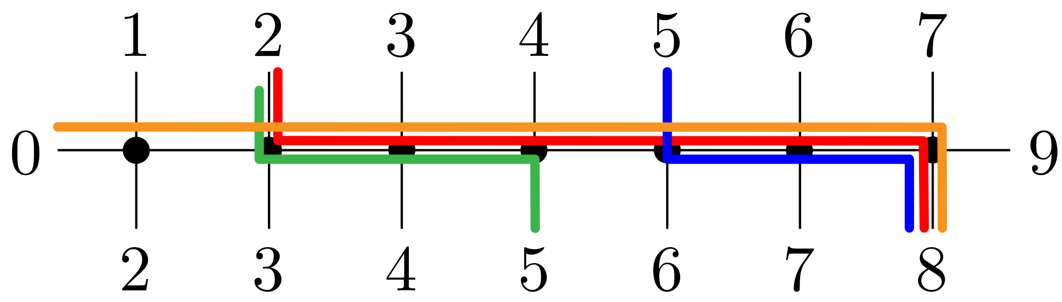


diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing

simplicial associahedron $=$ simplicial complex with

- vertices $=$ internal diagonals of an $(n+3)$-gon
- faces $=$ collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

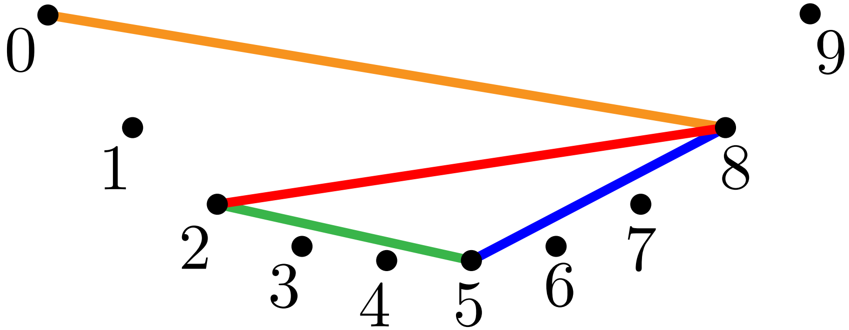


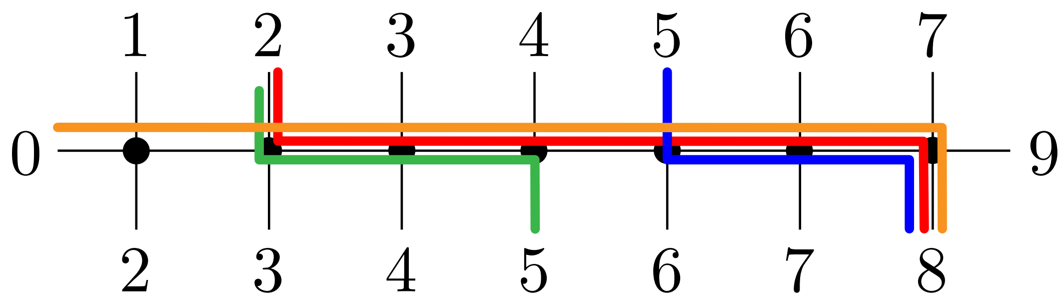


diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face

simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon



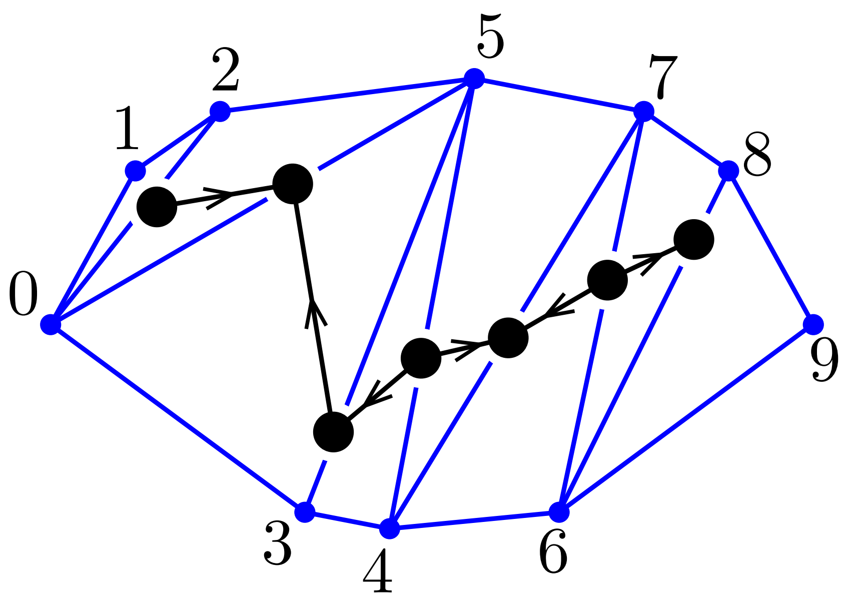


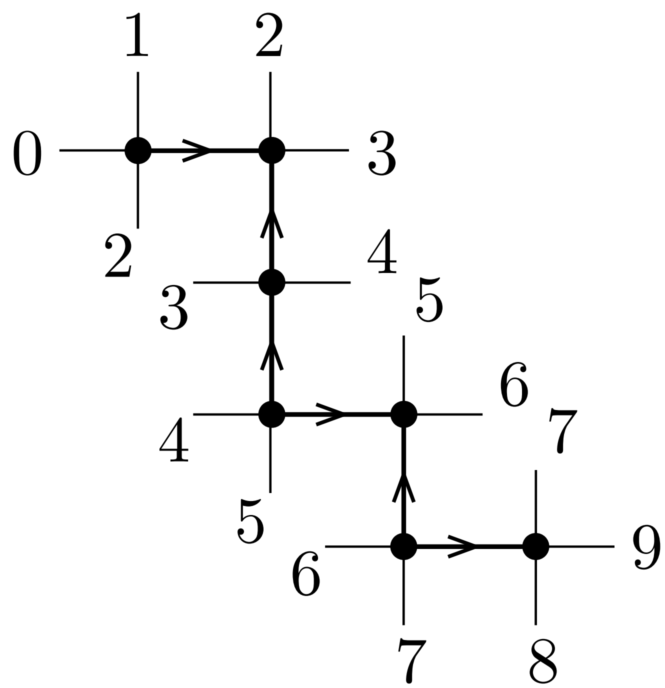
diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face
simplicial associahedron	$\longleftrightarrow$	non-kissing complex

McConville, *Lattice structures of grid Tamari orders* ('17)

simplicial associahedron $=$ simplicial complex with

- vertices $=$ internal diagonals of an $(n+3)$-gon
- faces $=$ collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

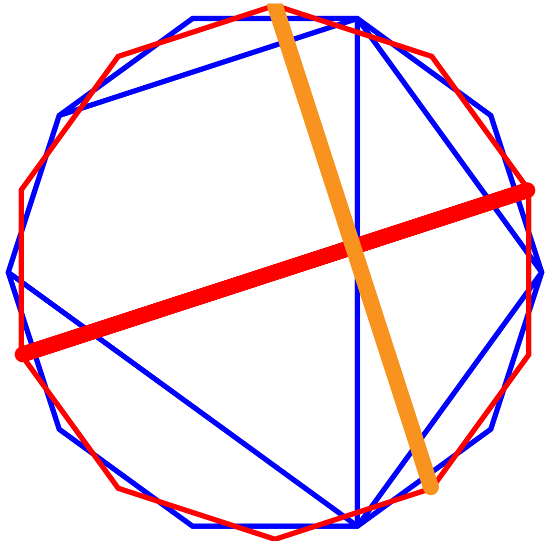




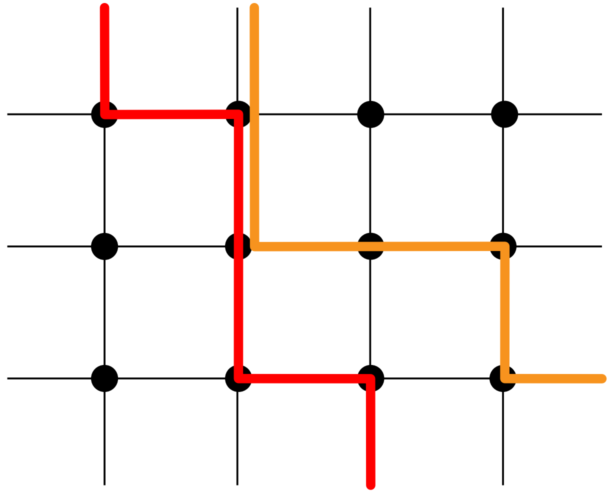
McConville, *Lattice structures of grid Tamari orders* ('17)

Show that non-crossing and non-kissing complexes coincide

To this end, generalize both:



non-crossing complex
to dissections of surfaces

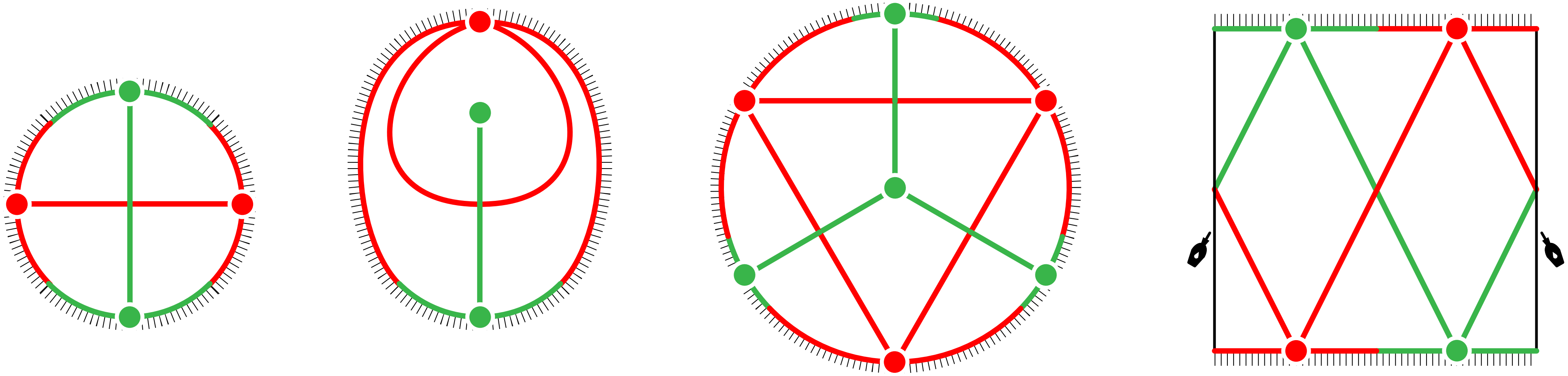


non-kissing complex
to gentle quivers

Palu–P.–Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras* ('18$^+$)

NON-CROSSING COMPLEX

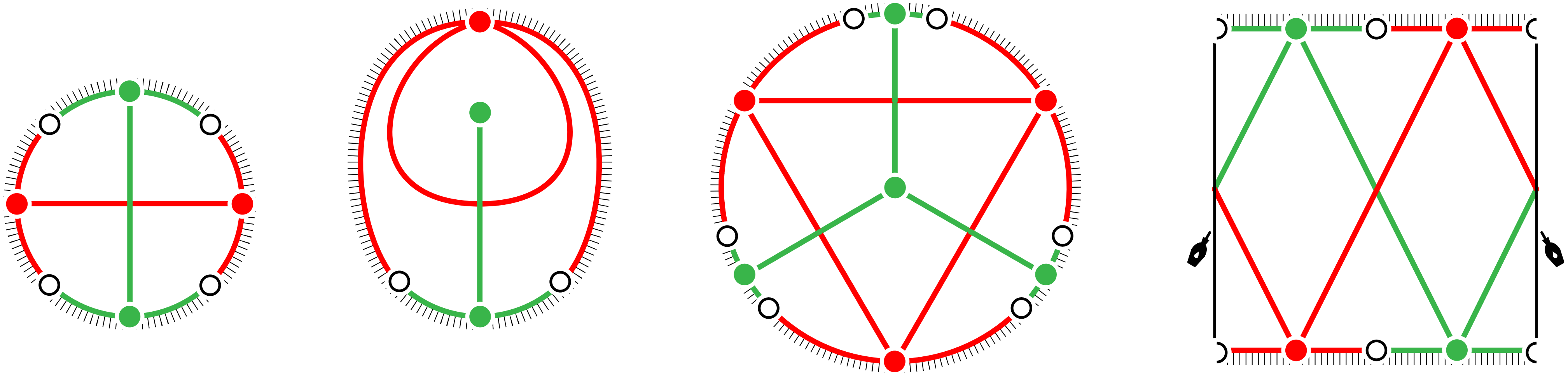
Palu–P.–Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras* ('18$^+$)



$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$



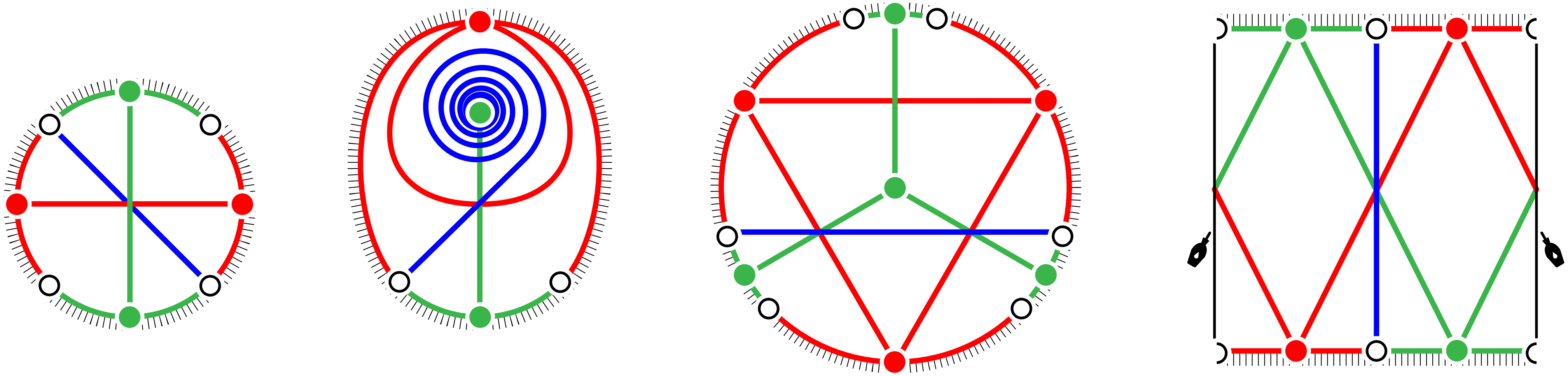
$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

blossom vertices = white vertices, alternating with $\mathrm{V} \cup \mathrm{V}^*$ along the boundary of $\mathcal{S}$

DUAL DISSECTIONS



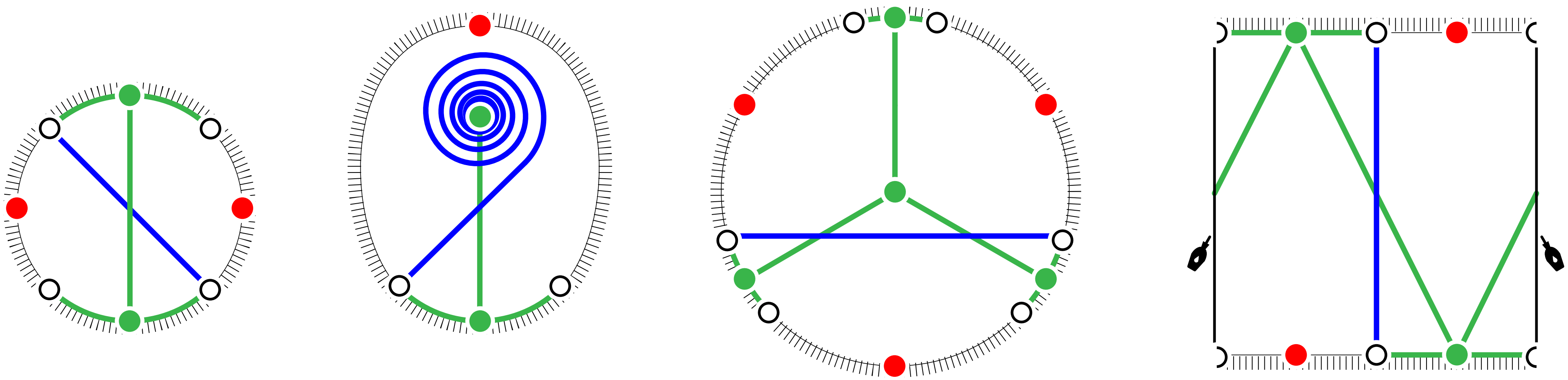
$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

blossom vertices = white vertices, alternating with $\mathrm{V} \cup \mathrm{V}^*$ along the boundary of $\mathcal{S}$

B-curve = curve which at each endpoint either reaches a blossom point or infinitely circles around a puncture of $\mathcal{S}$



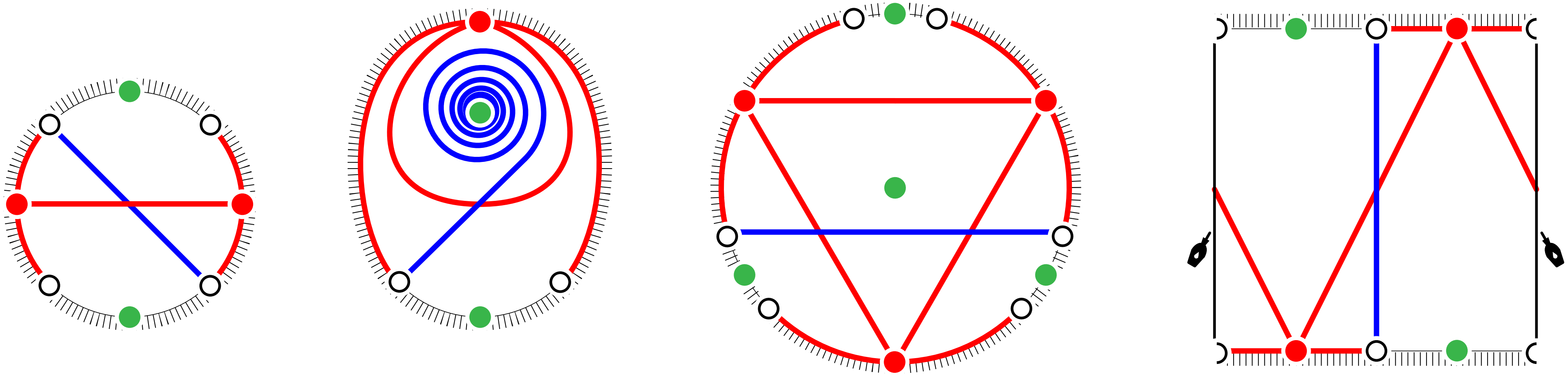
D-accordion $=$ B-curve α such that whenever α meets a face f of D,

(i) it enters crossing an edge a of f and leaves crossing an edge b of f

(ii) the two edges a and b of f crossed by α are consecutive along the boundary of f,

(iii) α, a and b bound a disk inside f that does not contain f^*.

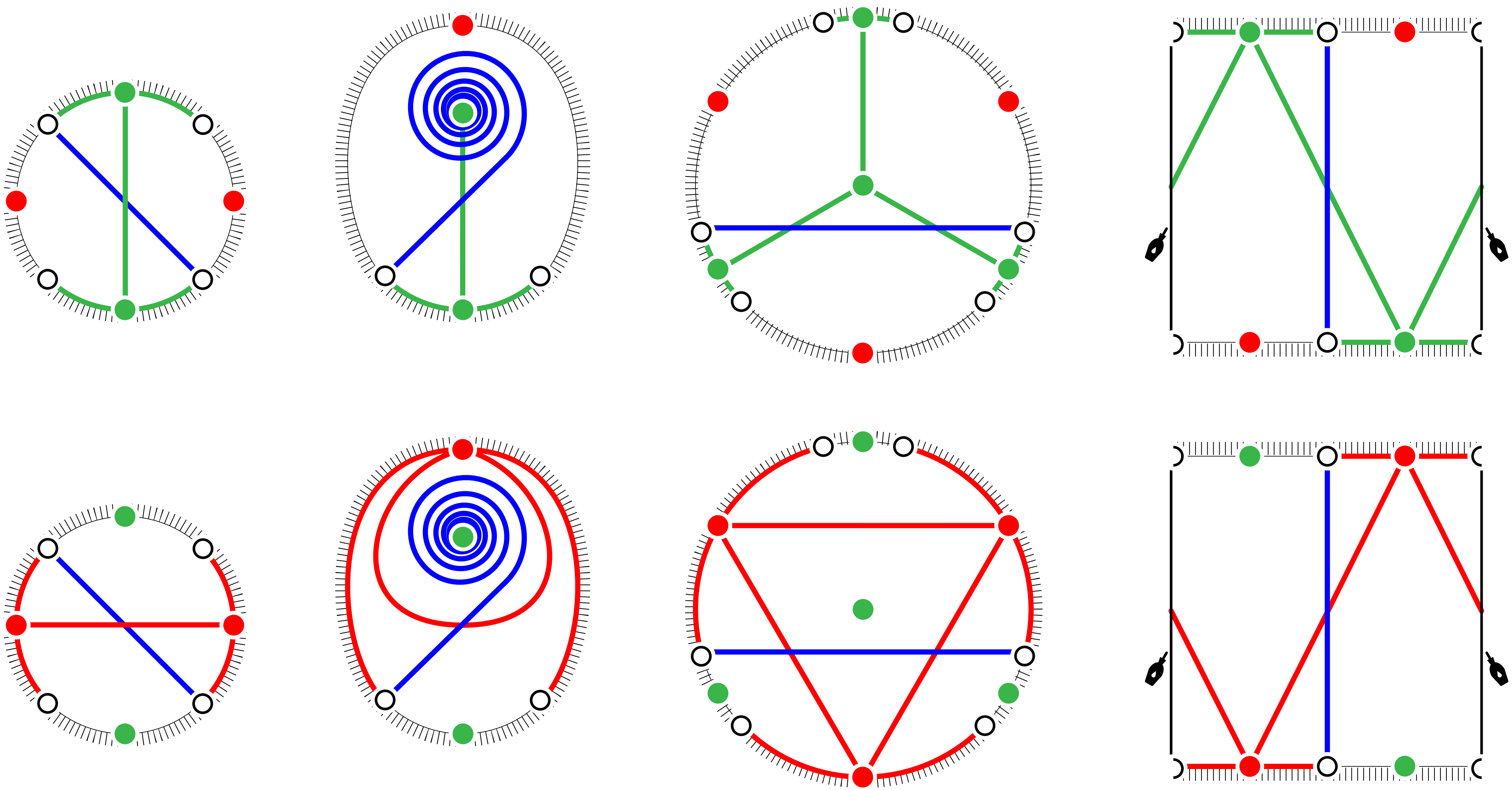
D-accordion complex $=$ simplicial complex of pairwise non-crossing sets of D-accordions



D*-slalom = B-curve α of $\bar{\mathcal{S}}$ such that, whenever α crosses an edge a^* of D* contained in two faces f^*, g^* of D*, the marked points f and g lie on opposite sides of α in the union of f^* and g^* glued along a^*.

D*-slalom complex = simplicial complex of pairwise non-crossing sets of D*-slaloms

$\color{green}{\mathrm{D}}$-ACCORDIONS $=$ $\color{red}{\mathrm{D}^*}$-SLALOMS

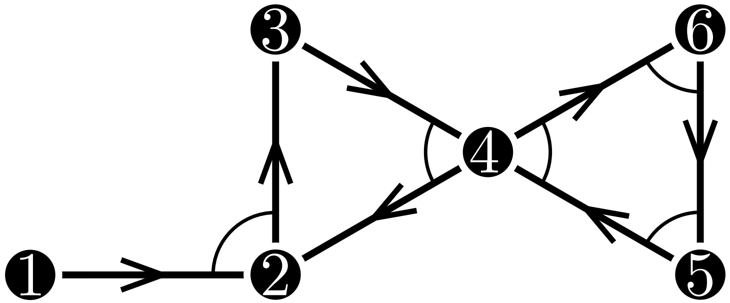


$(\color{green}{\mathrm{D}}, \color{red}{\mathrm{D}^*})$-non-crossing complex $=$ $\color{green}{\mathrm{D}}$-accordion complex $=$ $\color{red}{\mathrm{D}^*}$-slalom complex

NON-KISSING COMPLEX

Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17^{+})
Brüstle–Douville–Mousavand–Thomas–Yıldırım, *On the combinatorics of gentle algebras* ('17^{+})

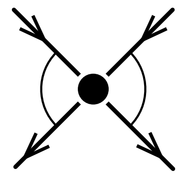
GENTLE QUIVERS AND STRINGS



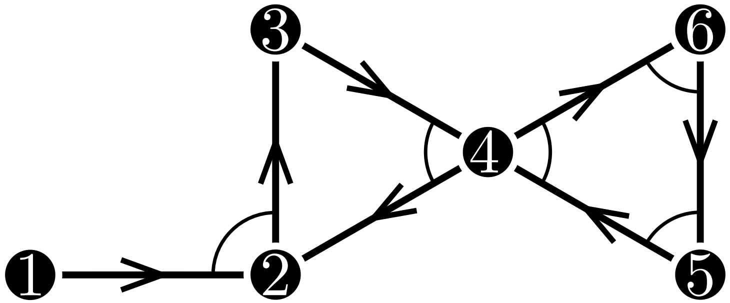
gentle quiver $\bar{Q} =$

- quiver $Q =$ oriented graph (Q_0, Q_1, s, t)
- relations $I =$ forbid certain paths

where

- forbidden paths all of length 2
- locally at each vertex, subgraph of 

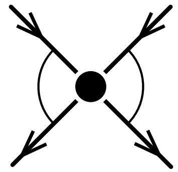
GENTLE QUIVERS AND STRINGS

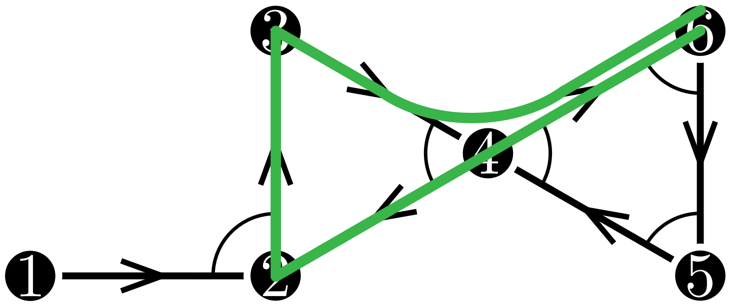


gentle quiver $\bar{Q}$ =

- quiver Q = oriented graph (Q_0, Q_1, s, t)
- relations I = forbid certain paths

where

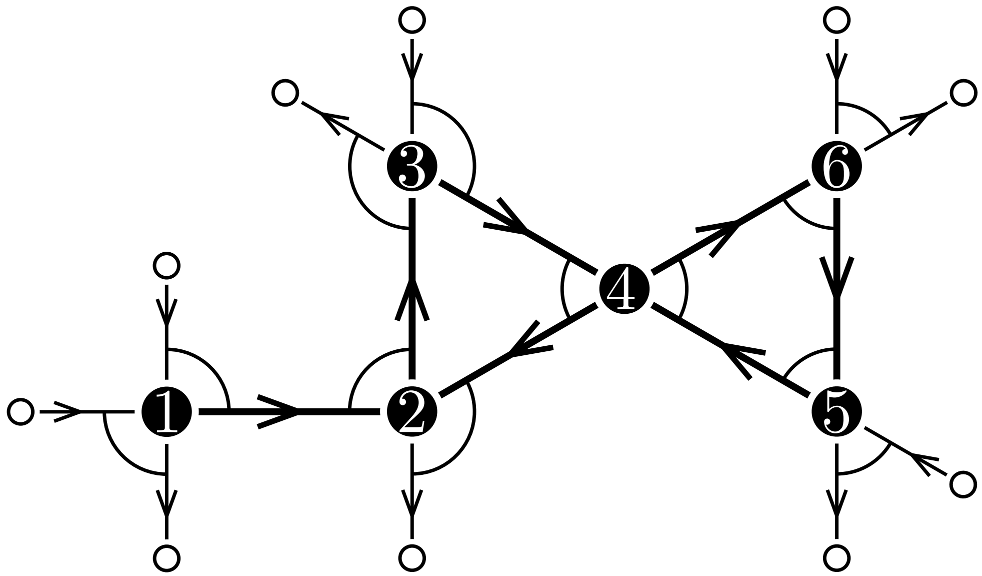
- forbidden paths all of length 2
- locally at each vertex, subgraph of 



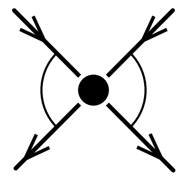
string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$ with $\alpha_k \in Q_1$, $\varepsilon_k \in \{-1, 1\}$

such that

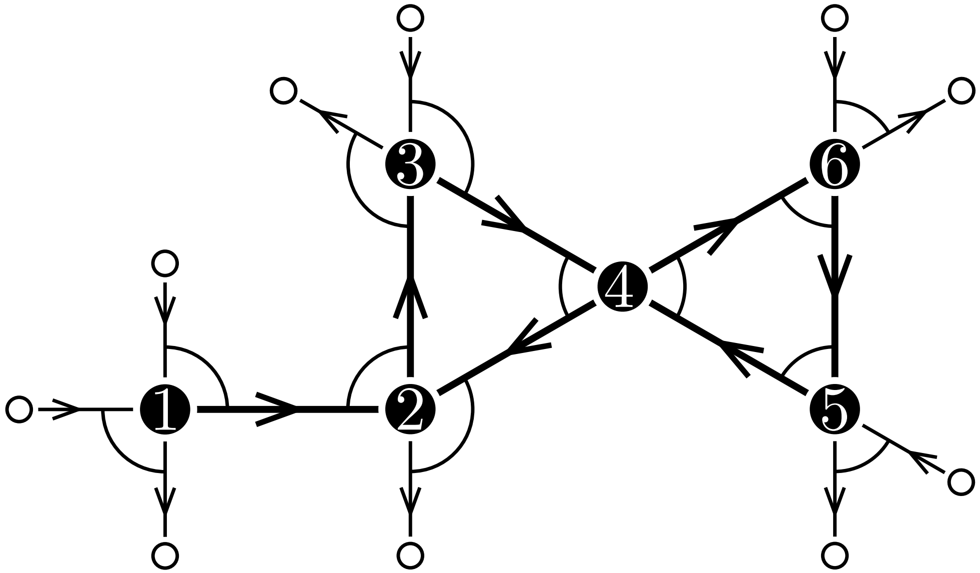
- $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$
- contains no factor π or π^{-1} for any path $\pi \in I$
- contains no $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for any arrow $\alpha \in Q_1$

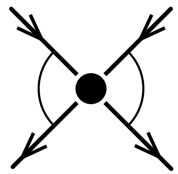


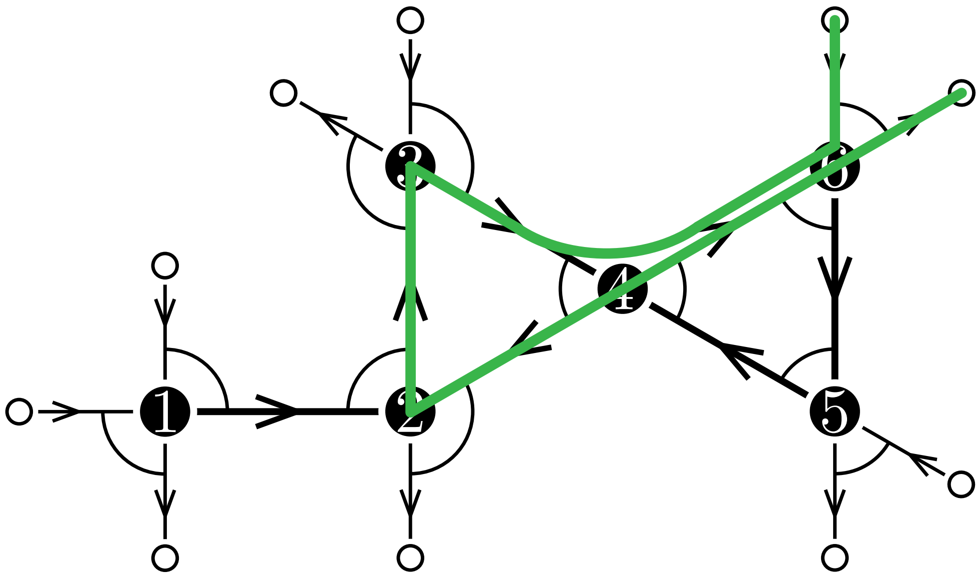
blossoming quiver $\bar{Q}^{❀} =$

add blossoms to complete each vertex to 

BLOSSOMING QUIVERS AND WALKS

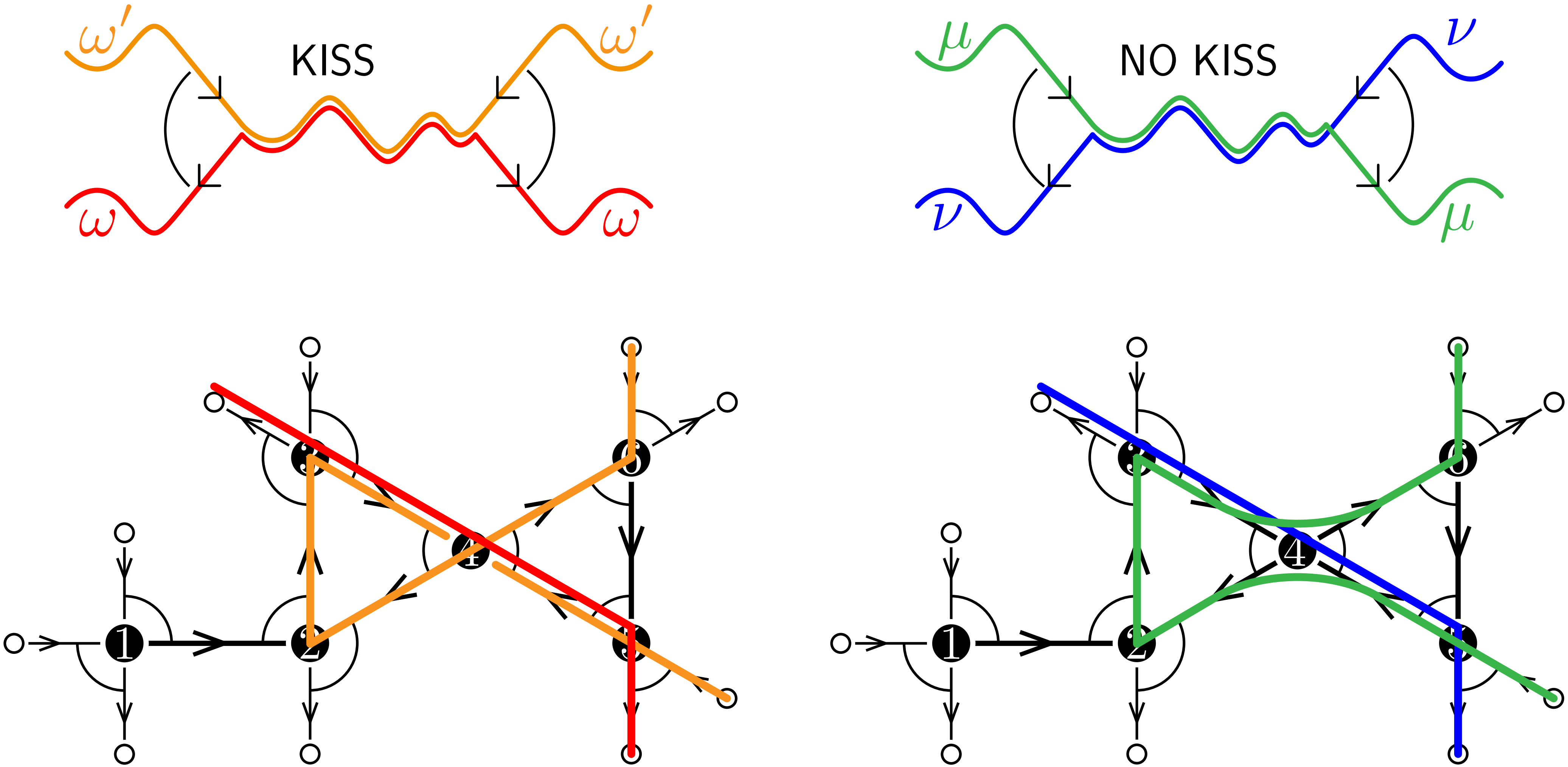


blossoming quiver $\bar{Q}^{\text{❀}}$ =
add blossoms to complete each vertex to 

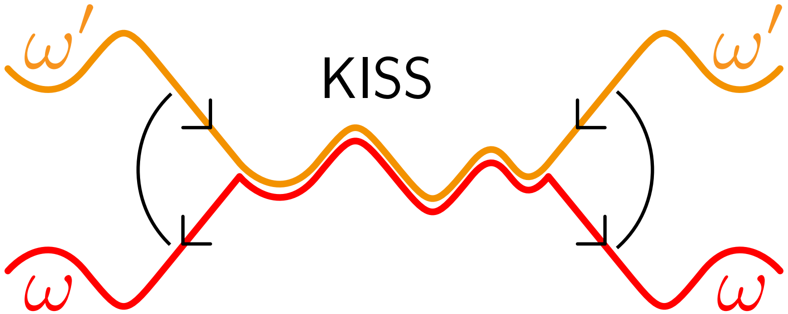


walk ω = maximal string in $\bar{Q}^{\text{❀}}$
from blossoms to blossoms

KISSING

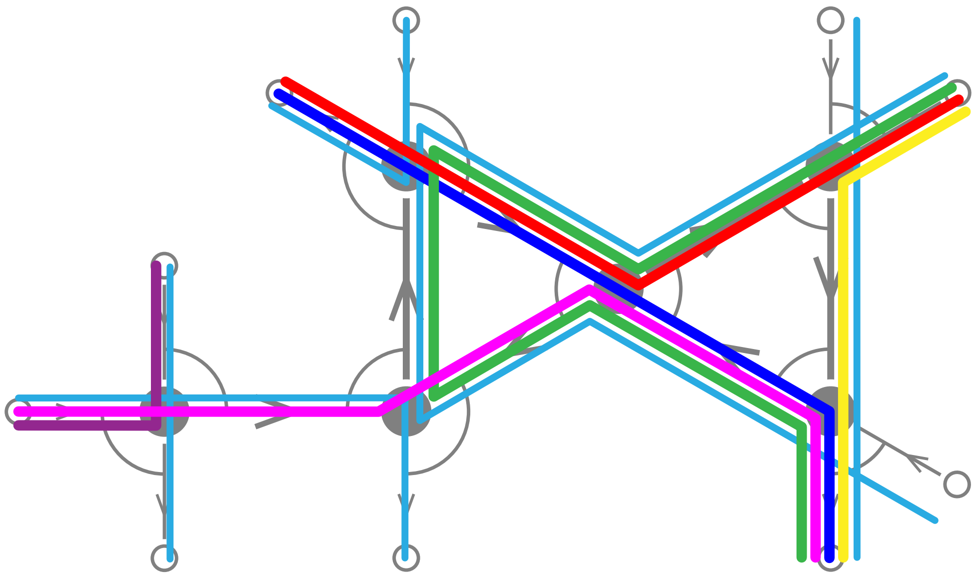


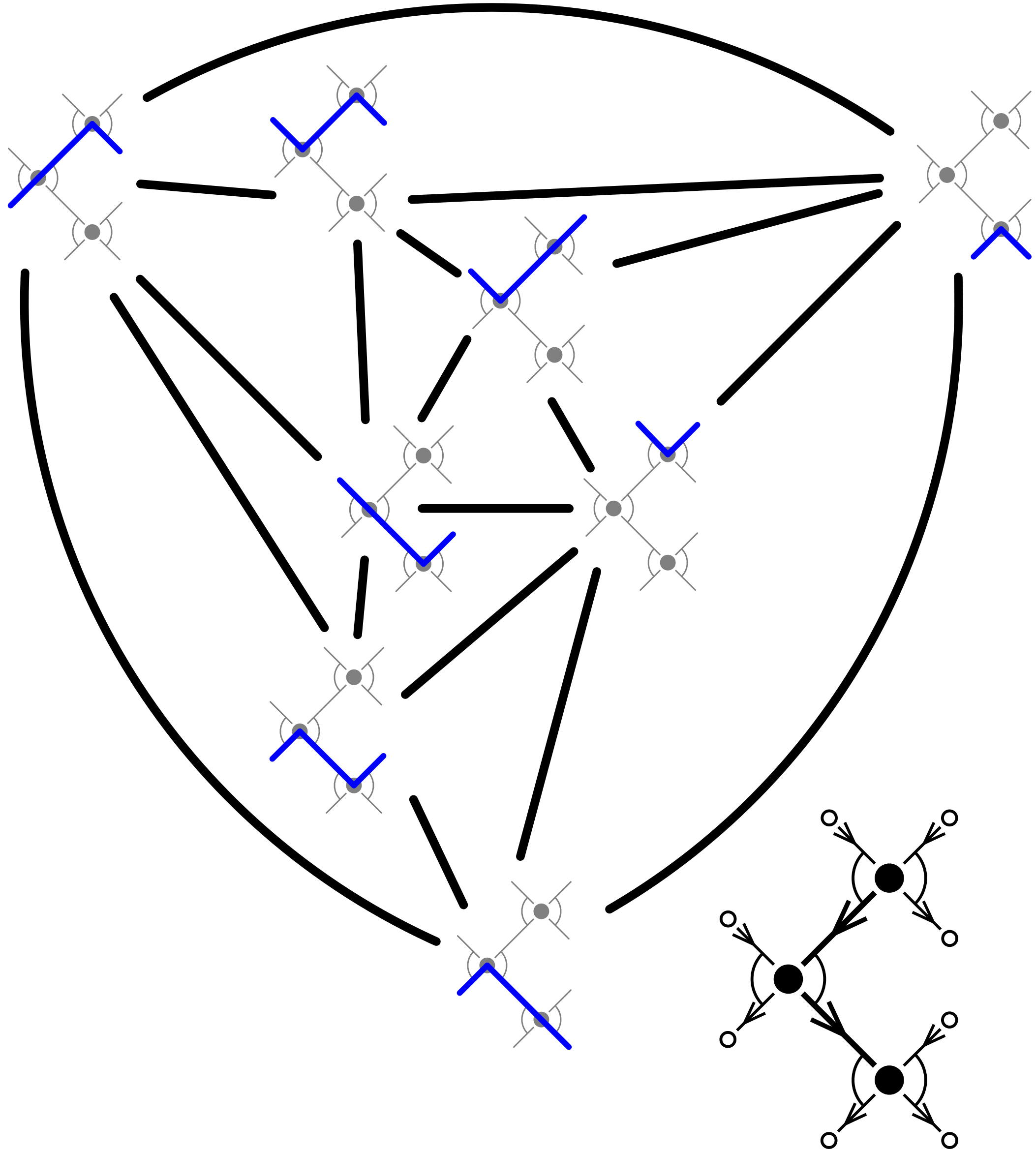
NON-KISSING COMPLEX



[reduced] <u>non-kissing complex</u> $\mathcal{NK}(\bar{Q}) =$

- vertices = [bending] walks in $\bar{Q}^{\circledast}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing [bending] walks in $\bar{Q}^{\circledast}$



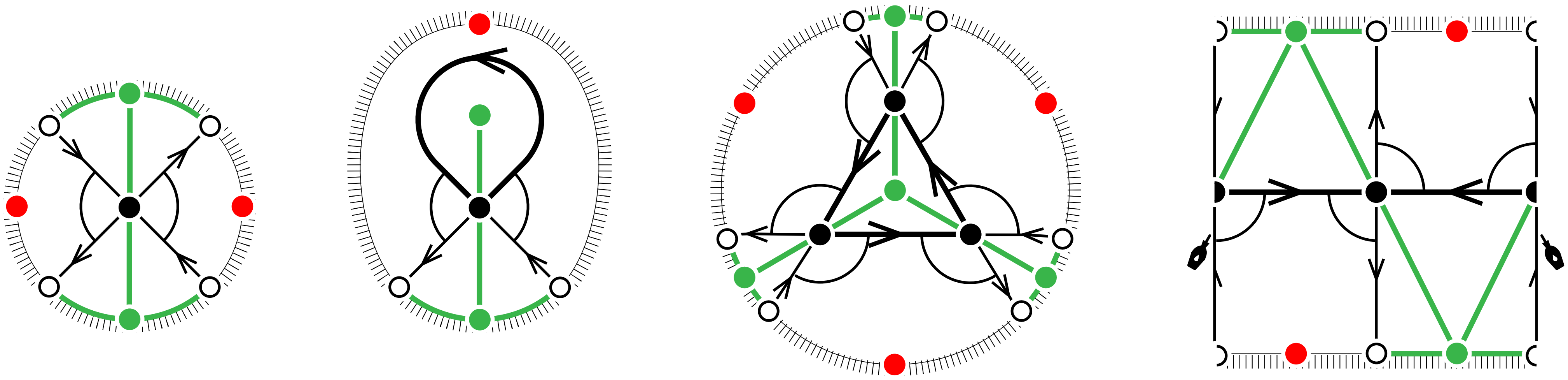


NON-CROSSING VS NON-KISSING

Palu–P.–Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras* ('18^{+})

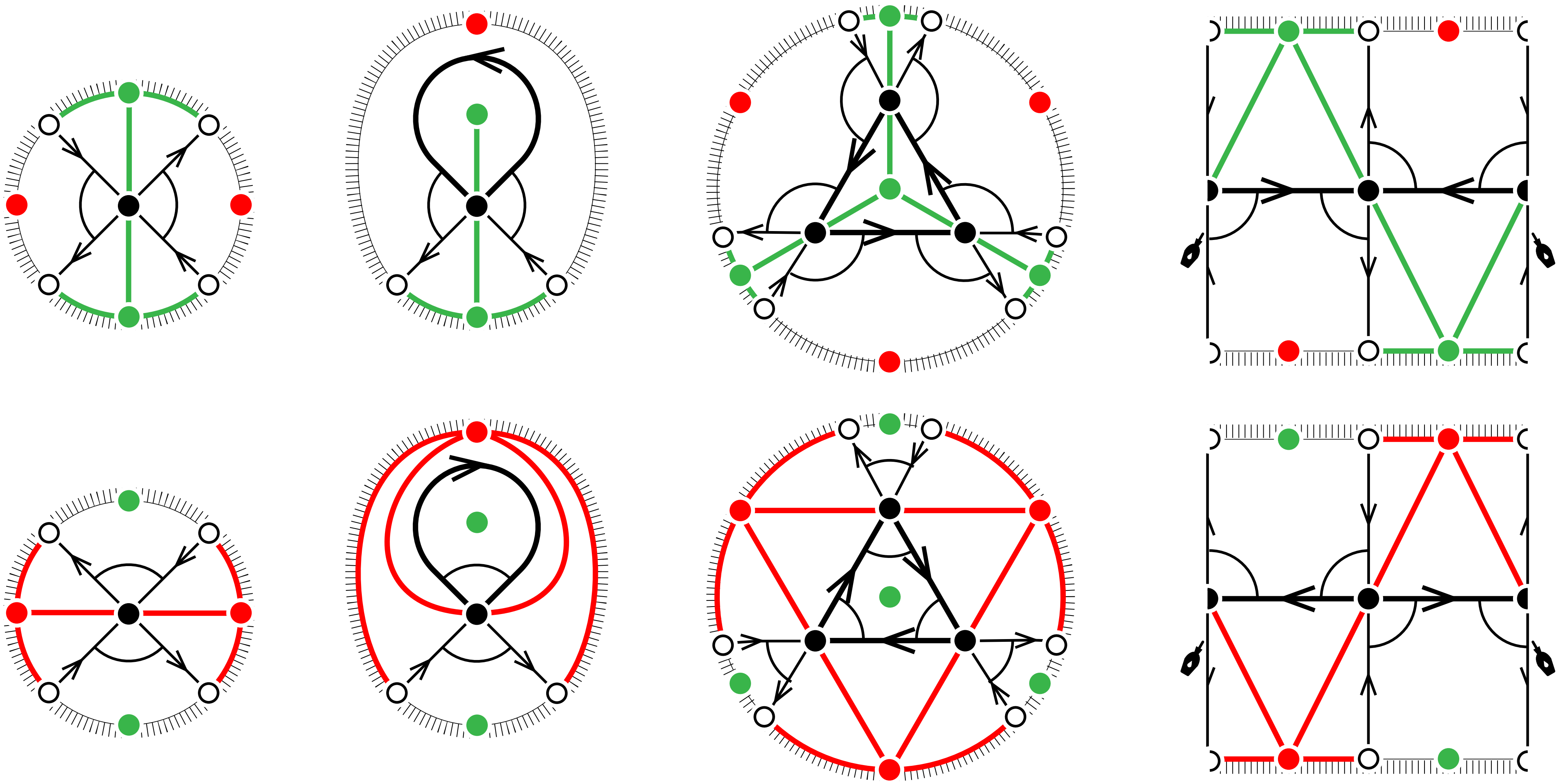
quiver $\bar{Q}_{\mathrm{D}}$ of a dissection $=$

- vertices $=$ edges of D (boundary edges are blossom vertices)
- arrows $=$ two consecutive edges around a face of D
- relations $=$ three consecutive edges around a face of D



quiver $\bar{Q}_{\mathrm{D}}$ of a dissection =

- vertices = edges of D (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of D
- relations = three consecutive edges around a face of D

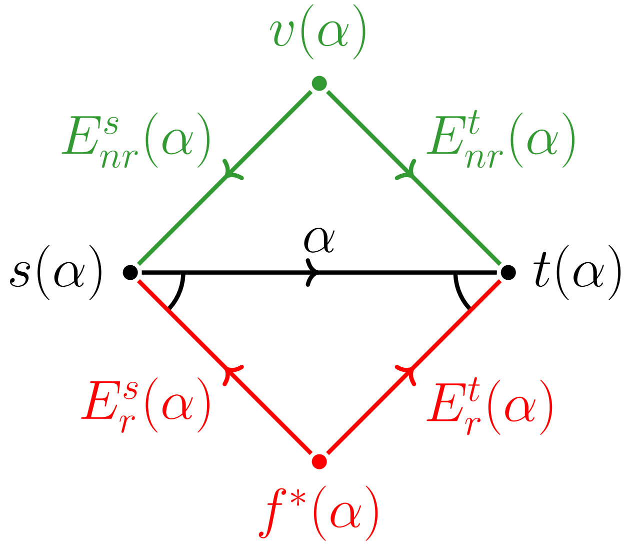


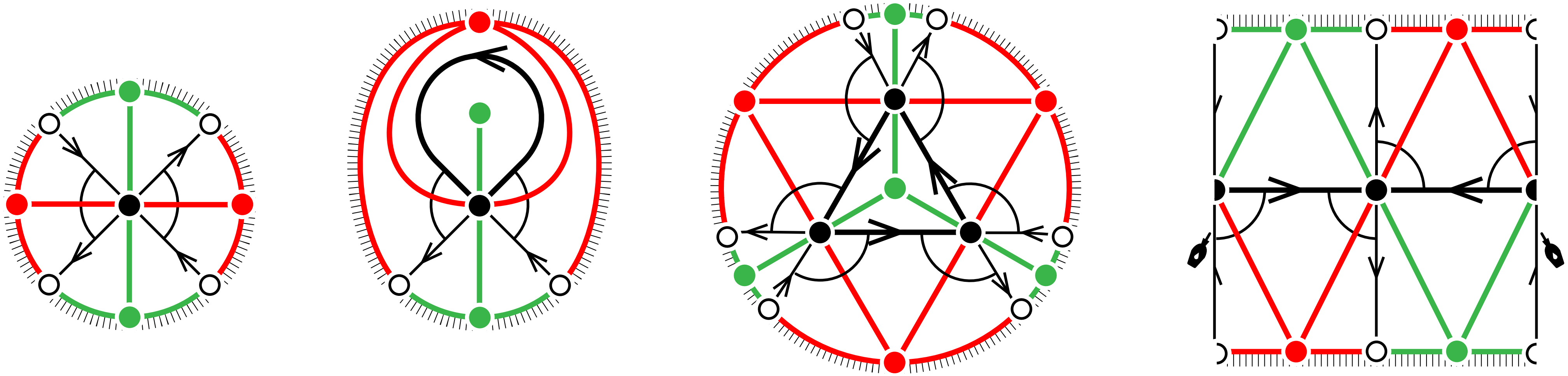
surface $\mathcal{S}_{\bar{Q}}$ of quiver $\bar{Q}$ = surface obtained from the blossoming quiver $\bar{Q}^{❀}$ as follows:

(i) for each arrow $\alpha \in Q_1^{❀}$, consider a lozenge

(ii) for any $\alpha, \beta \in Q_1^{❀}$ with $t(\alpha) = s(\beta)$, proceed to the following identifications:

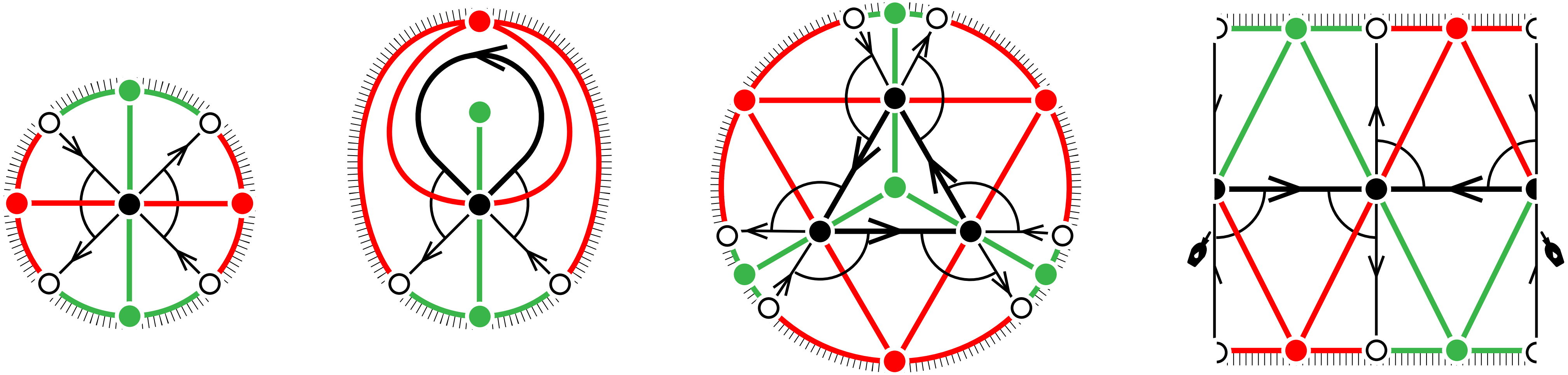
- if $\alpha\beta \in I$, then glue $E_r^t(\alpha)$ with $E_r^s(\beta)$,
- if $\alpha\beta \notin I$, then glue $E_{nr}^t(\alpha)$ with $E_{nr}^s(\beta)$.





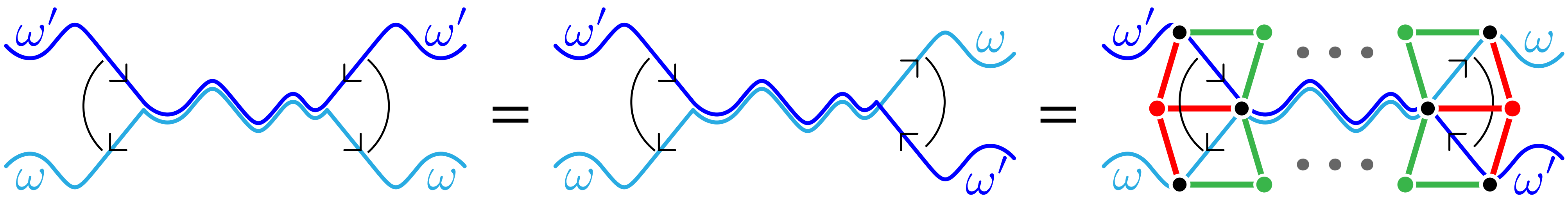
PROP. The two previous constructions are inverse to each other and define bijections:

pairs of dual dissections on a surface $\longleftrightarrow$ gentle quivers



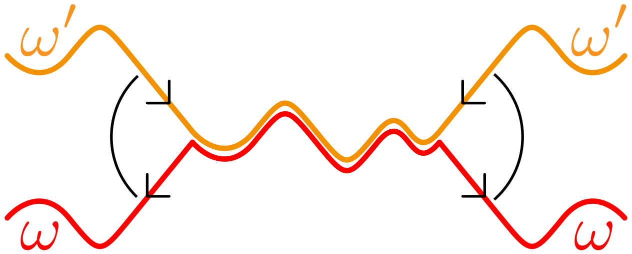
PROP. It defines isomorphisms between:

non-crossing complex of dissections $\longleftrightarrow$ non-kissing complex of gentle quiver



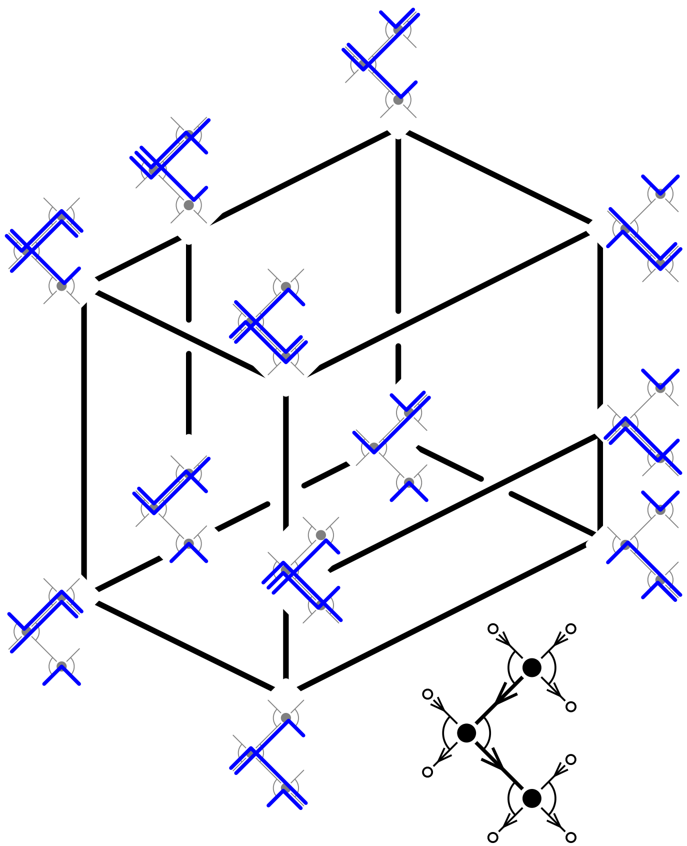
non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices $=$ walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces $=$ collections of pairwise non-kissing walks in $\bar{Q}^{❀}$

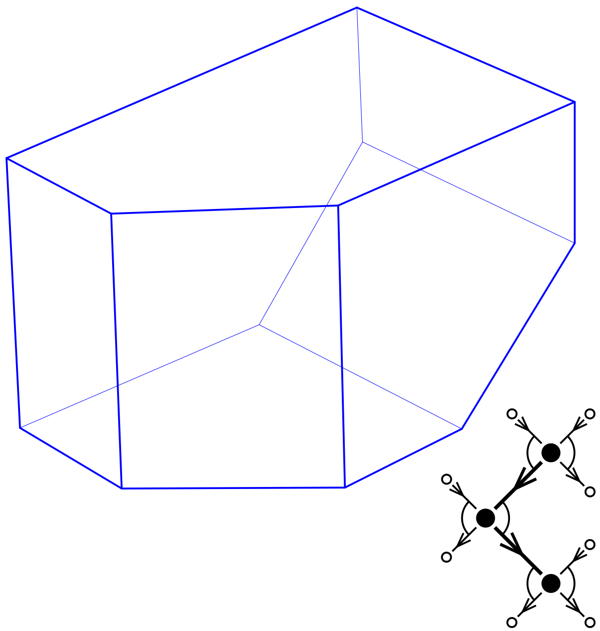


... generalizing the associahedron

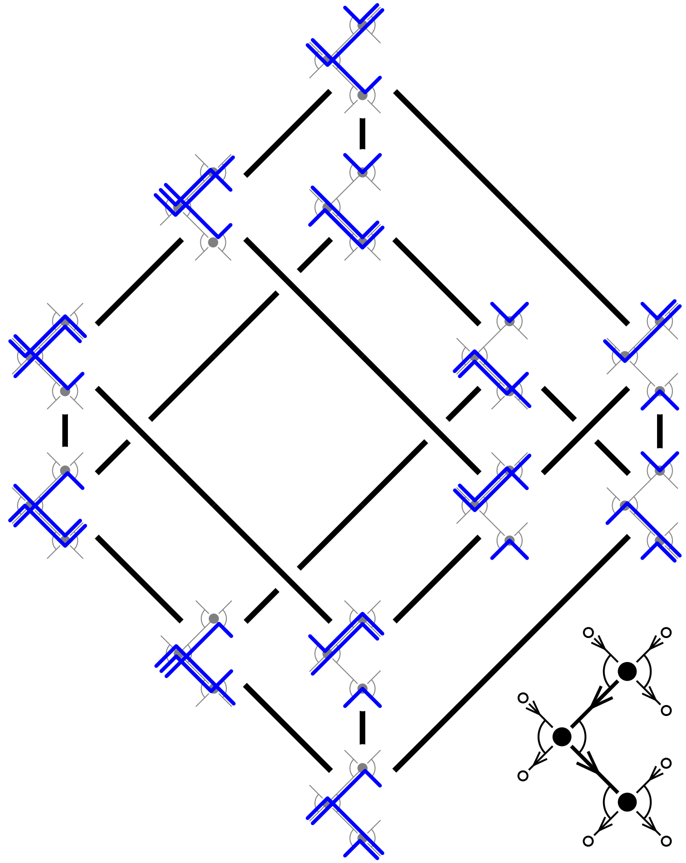
Flip graph



Associahedron

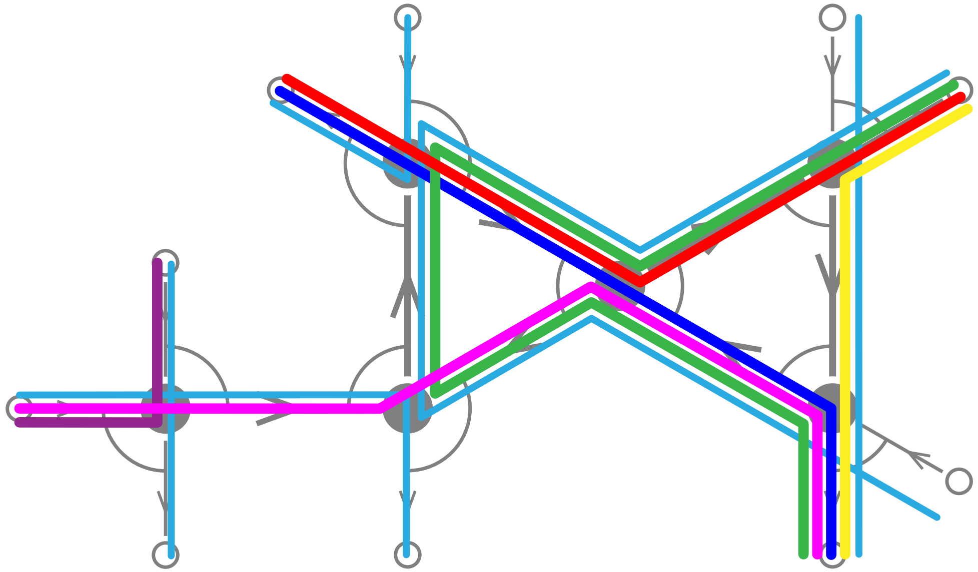


Tamari lattice

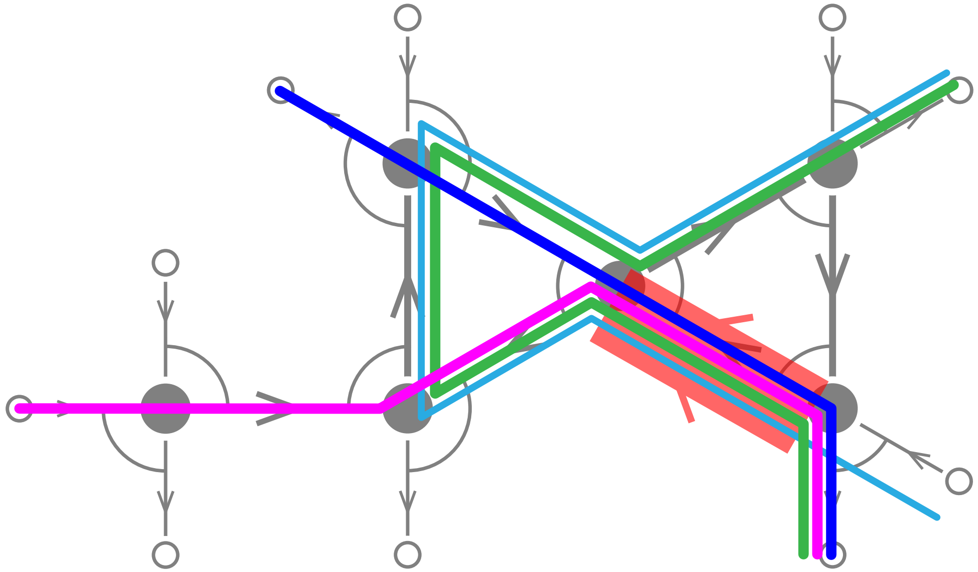


DISTINGUISHED ARROWS AND FLIPS

McConville, *Lattice structures of grid Tamari orders* ('17)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17$^+$)



F face of $\mathcal{NK}(\bar{Q})$

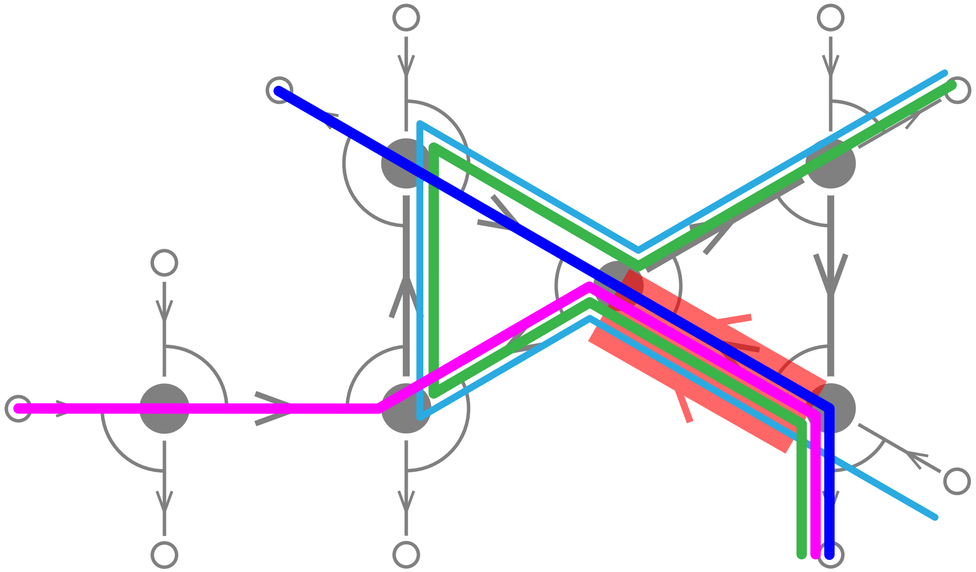


F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$

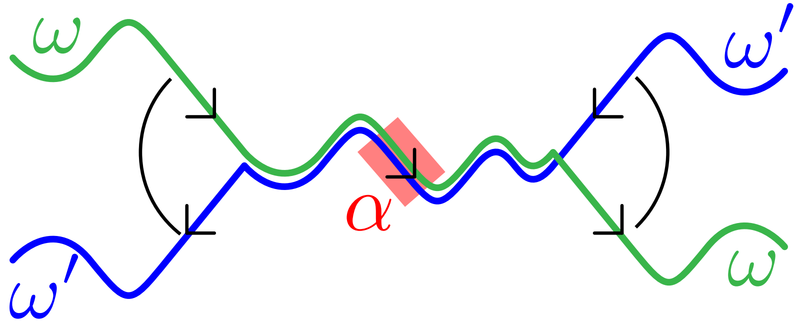
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

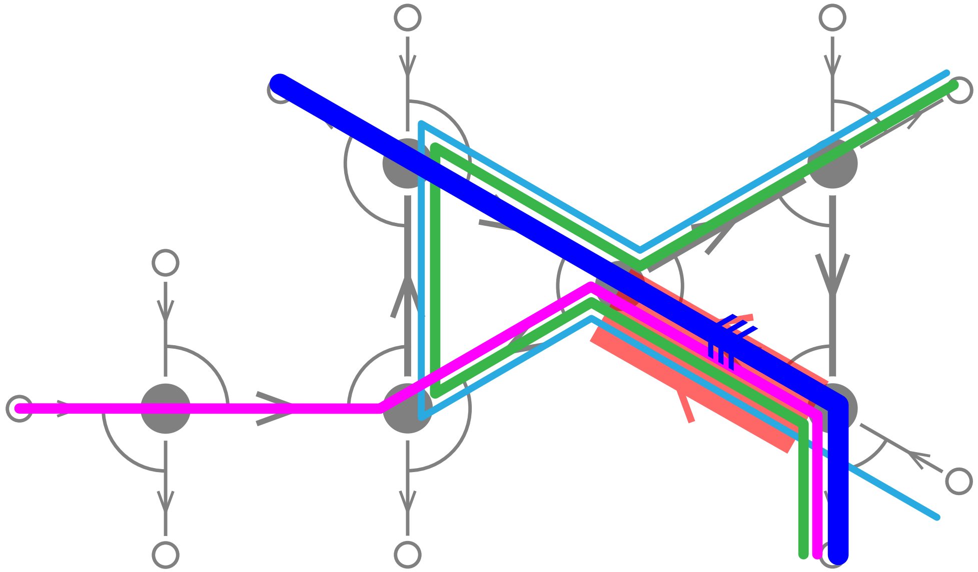
$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ <u>countercurrent order</u> at α

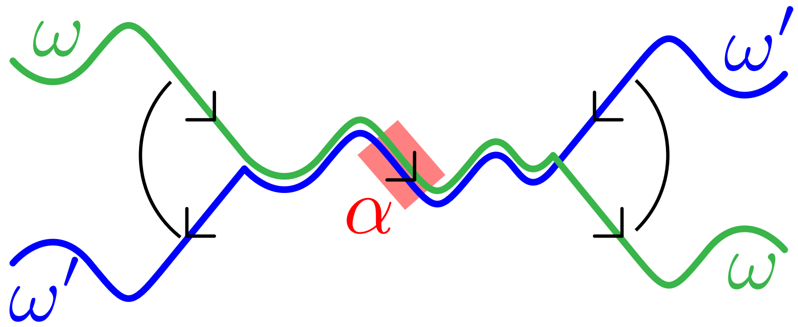
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

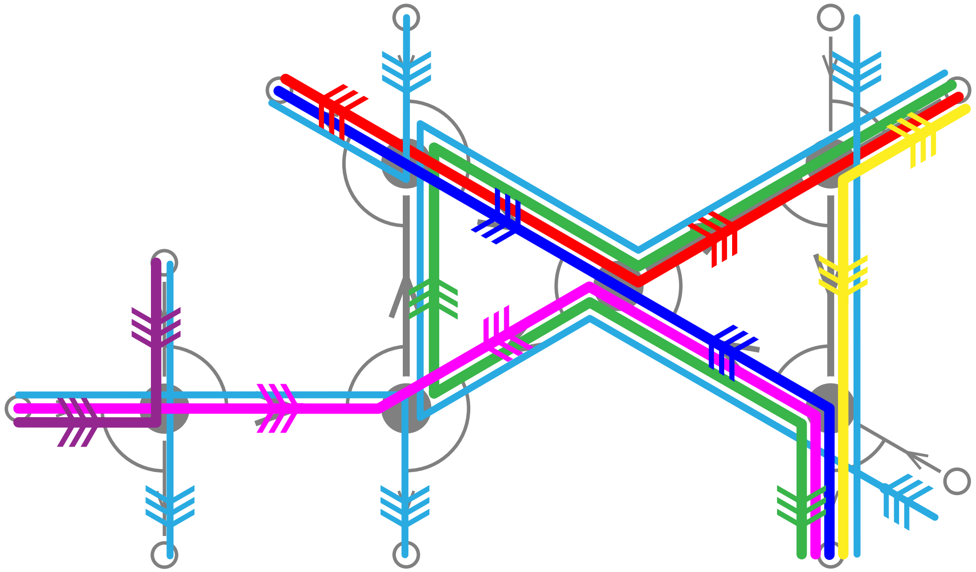
$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

distinguished walk at α in F = dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

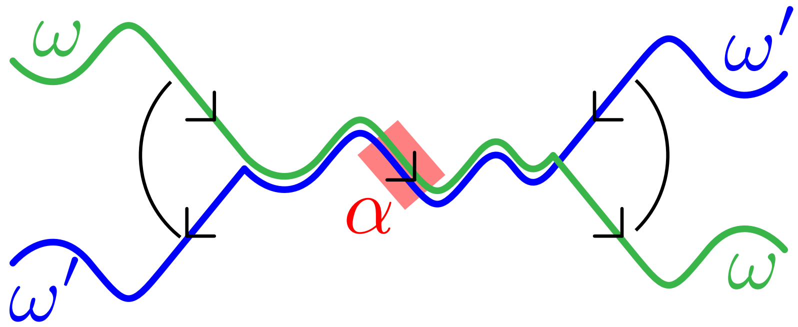
distinguished arrows of ω in F = da$(\omega, F) = \{\alpha \in Q_1 \mid \omega = \text{dw}(\alpha, F)\}$



F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

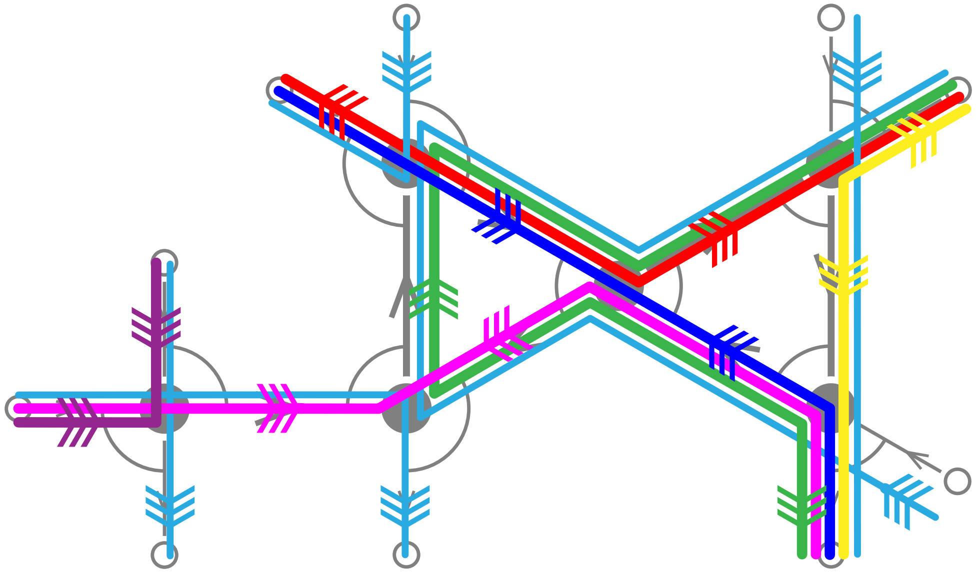
distinguished walk at α in F = dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in F = da$(\omega, F) = \{\alpha \in Q_1 \mid \omega = \text{dw}(\alpha, F)\}$

PROP. For any facet $F \in \mathcal{NK}(\bar{Q})$,

- each bending walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

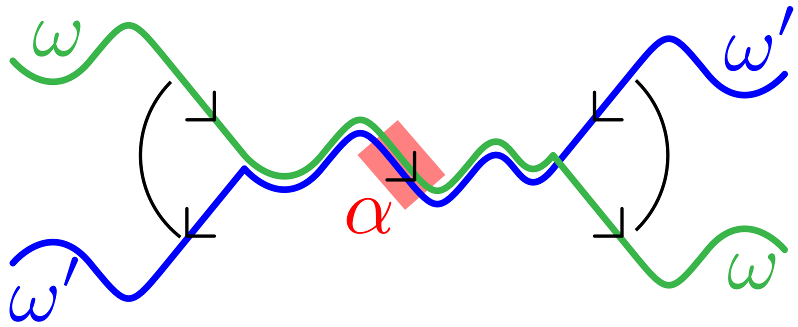
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ <u>countercurrent order</u> at α

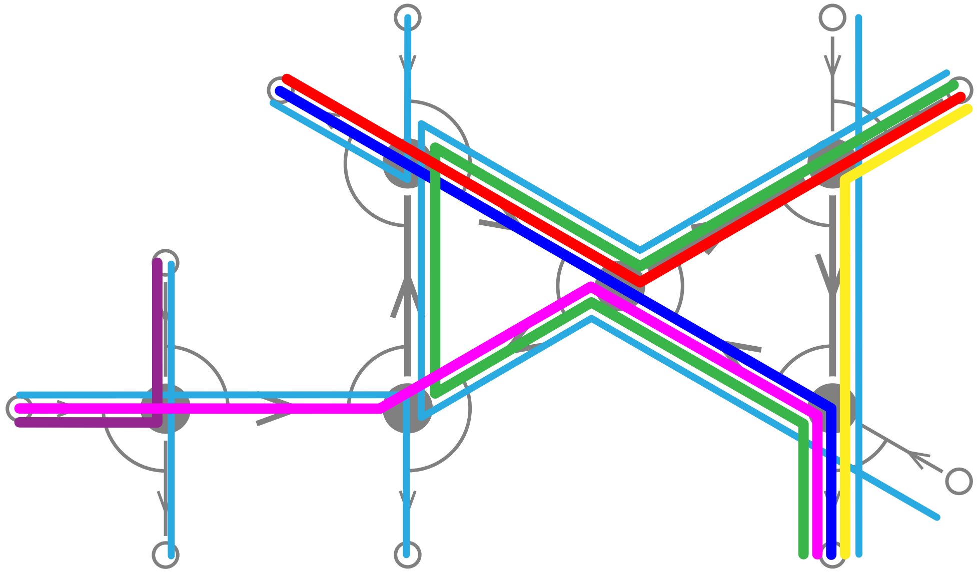
<u>distinguished walk</u> at α in F = dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

<u>distinguished arrows</u> of ω in F = da$(\omega, F) = \{\alpha \in Q_1 \mid \omega = \mathrm{dw}(\alpha, F)\}$

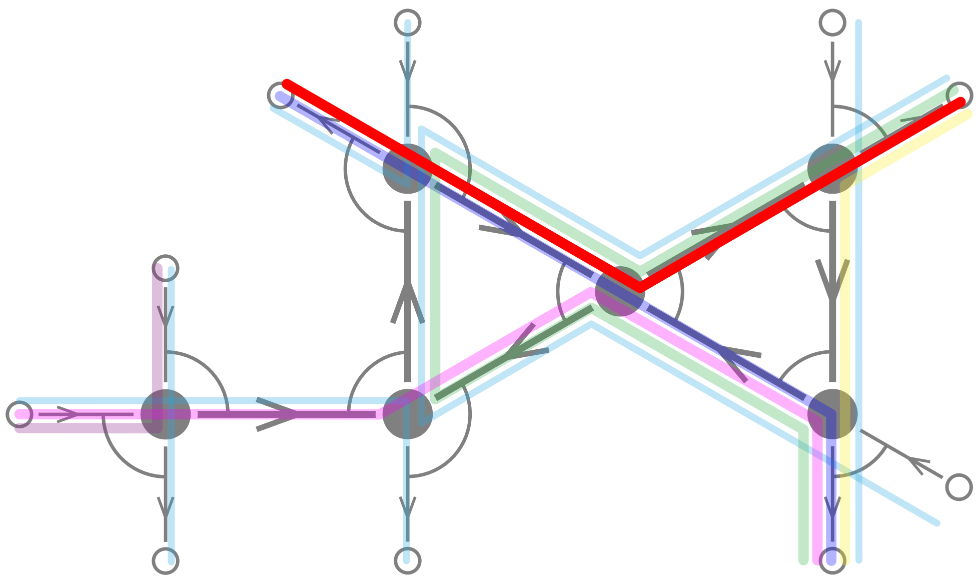
PROP. For any facet $F \in \mathcal{NK}(\bar{Q})$,

- each bending walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

CORO. $\mathcal{NK}(\bar{Q})$ is pure of dimension $|Q_0|$.

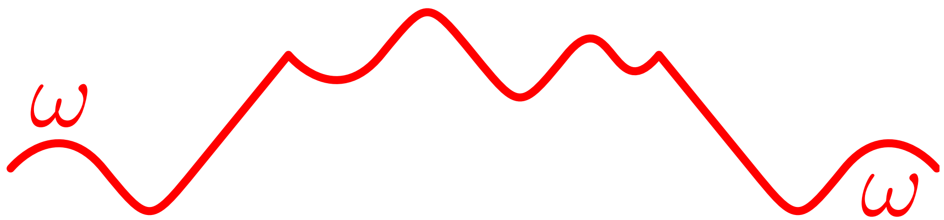


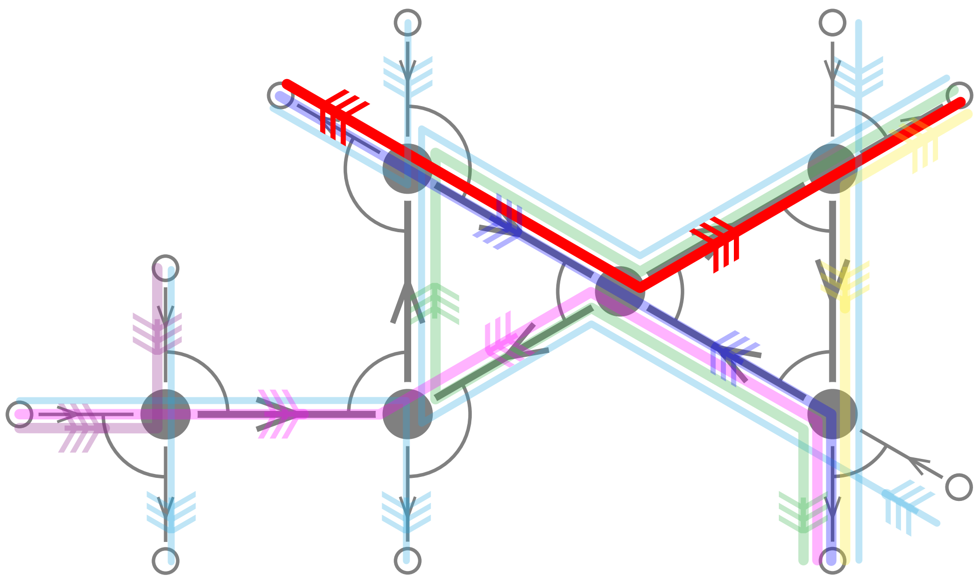
F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)



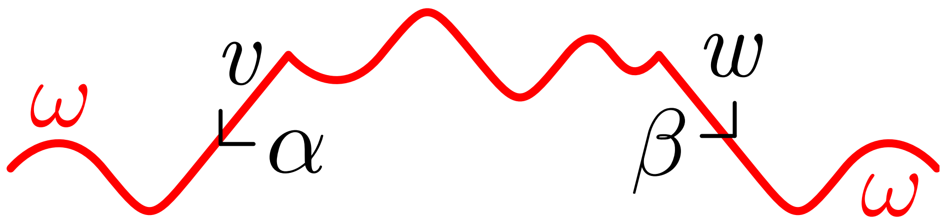
F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

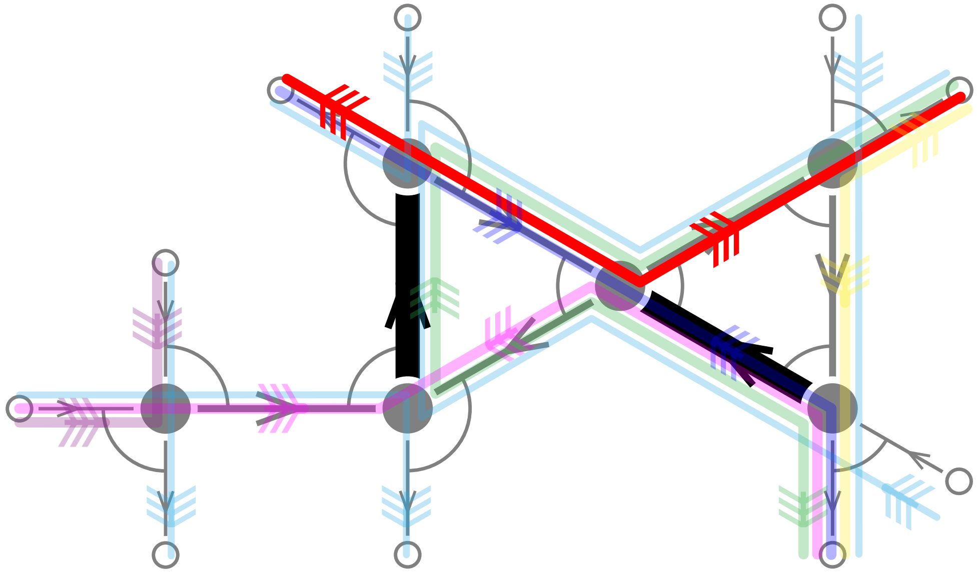
$\omega \in F$ we want to "flip"





F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$\omega \in F$ we want to "flip"
$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$



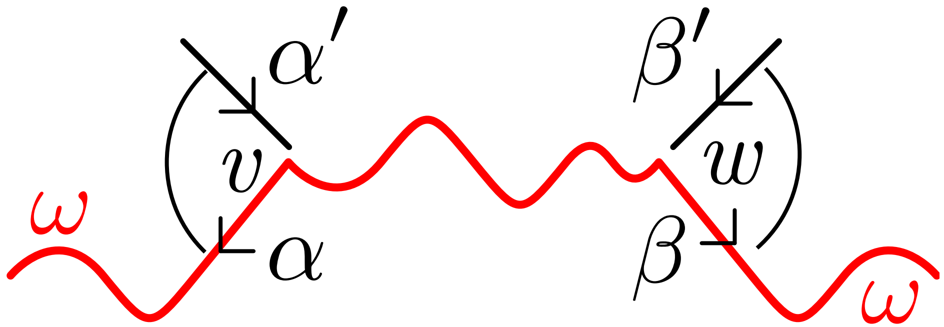


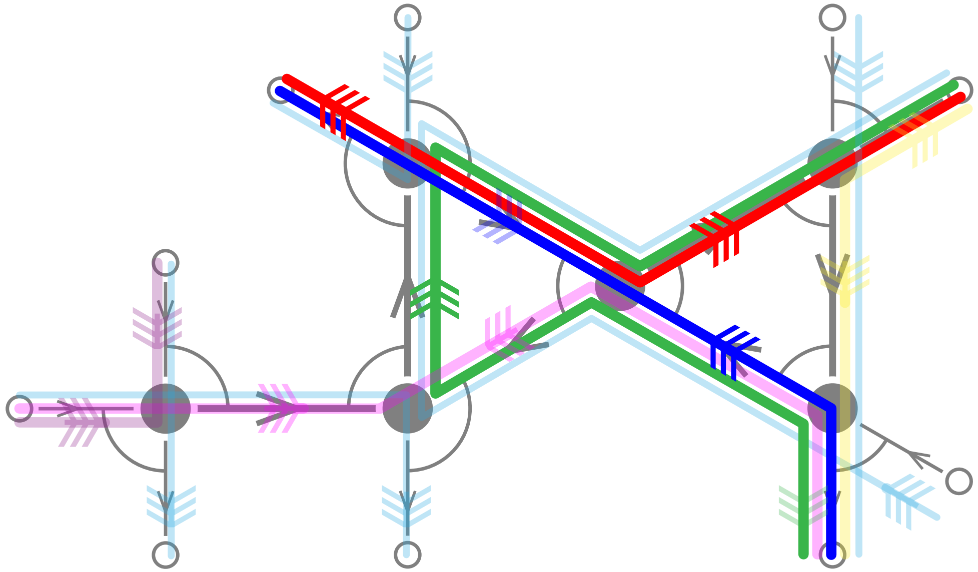
F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathrm{da}(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$





F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

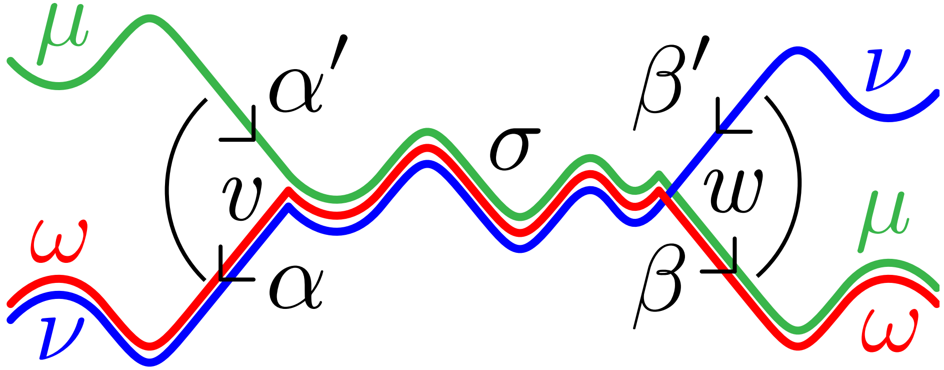
$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

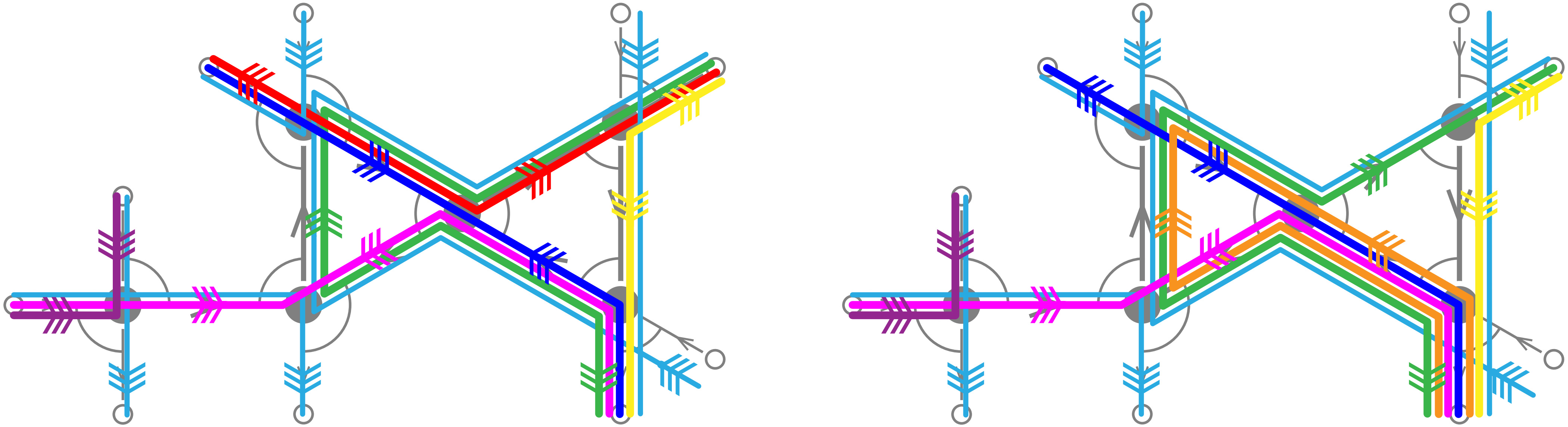
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = \mathsf{dw}(\alpha', F)$ and $\nu = \mathsf{dw}(\beta', F)$

$\omega = \nu[\cdot, v]\, \sigma\, \mu[w, \cdot]$



FLIPS



F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

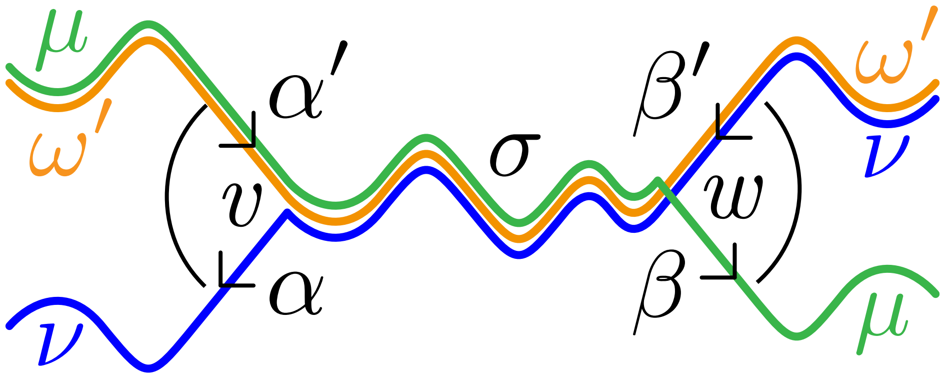
$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

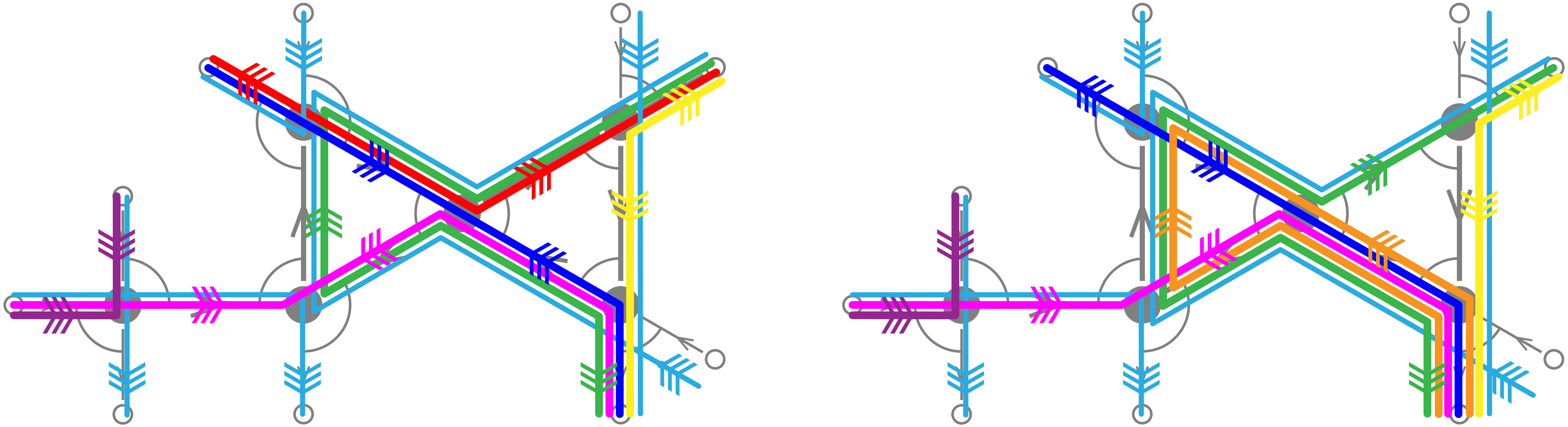
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = \mathsf{dw}(\alpha', F)$ and $\nu = \mathsf{dw}(\beta', F)$

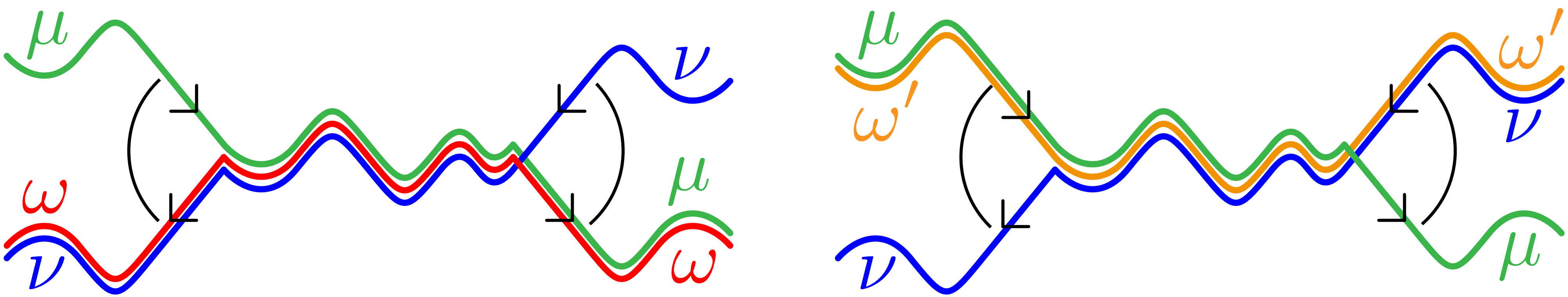
$\omega = \nu[\cdot, v]\, \sigma\, \mu[w, \cdot]$

$\omega' = \mu[\cdot, v]\, \sigma\, \nu[w, \cdot]$



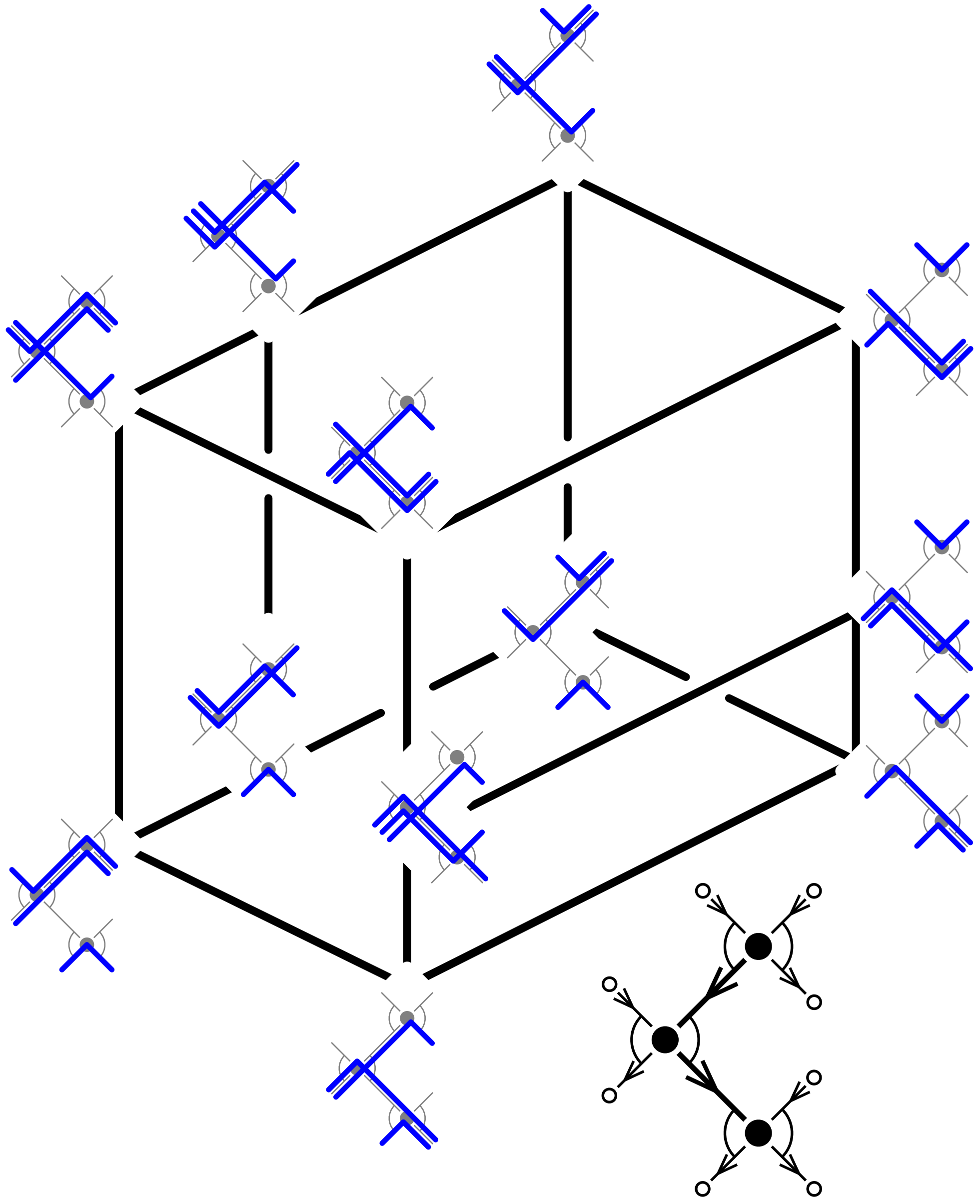


PROP. ω' kisses ω but no other walk of F. Moreover, ω' is the only such walk.



flip graph =

- vertices = non-kissing facets
- edges = flips



GENTLE ASSOCIAHEDRA

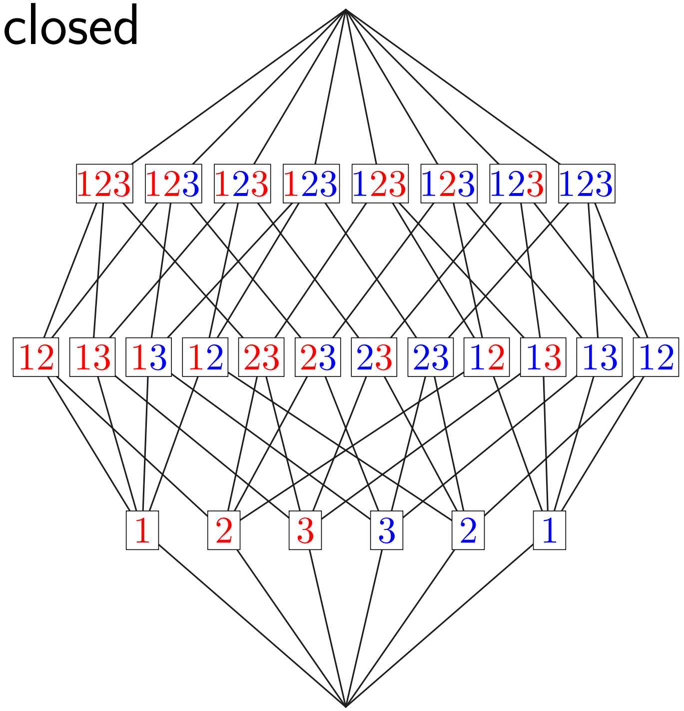
Manneville–P., *Geometric realizations of the accordion complex* ('17$^+$)
Hohlweg–P.–Stella, Polytopal realizations of finite type g-vector fans ('17$^+$)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17$^+$)

SIMPLICIAL COMPLEX

<u>simplicial complex</u> = collection of subsets of X downward closed

exm:

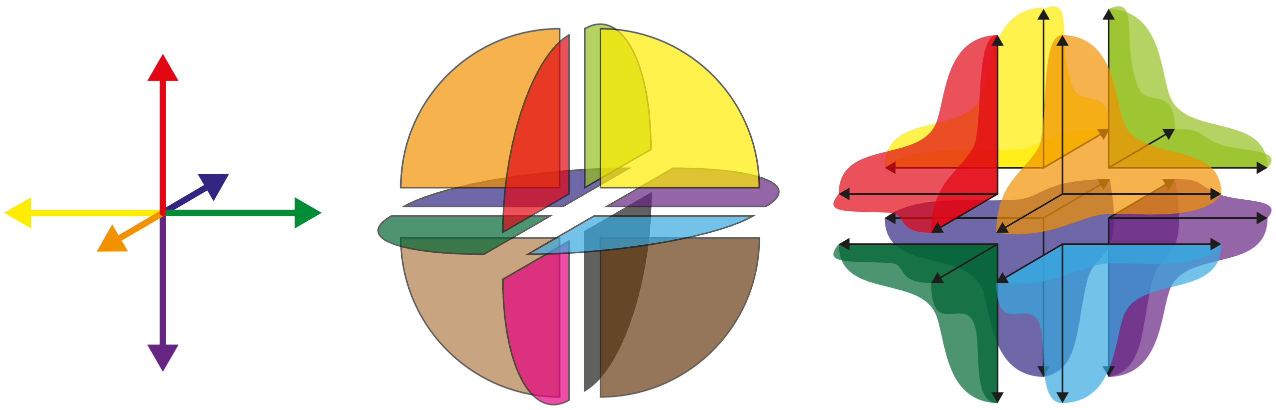
$$X = [n] \cup \underline{[n]}$$
$$\Delta = \{ I \subseteq X \mid \forall i \in [n],\ \{i, \underline{i}\} \not\subseteq I \}$$

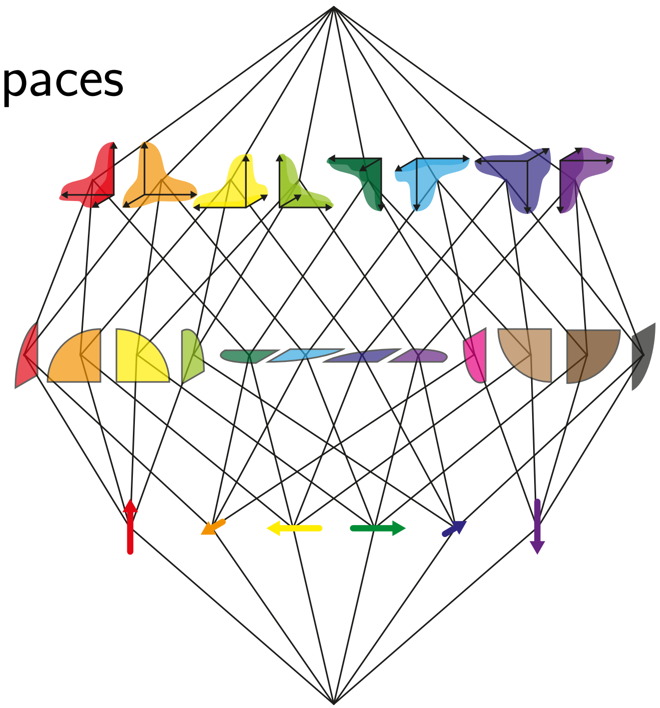


FANS

<u>polyhedral cone</u> $=$ positive span of a finite set of $\mathbb{R}^d$
$=$ intersection of finitely many linear half-spaces

<u>fan</u> $=$ collection of polyhedral cones closed by faces
and where any two cones intersect along a face

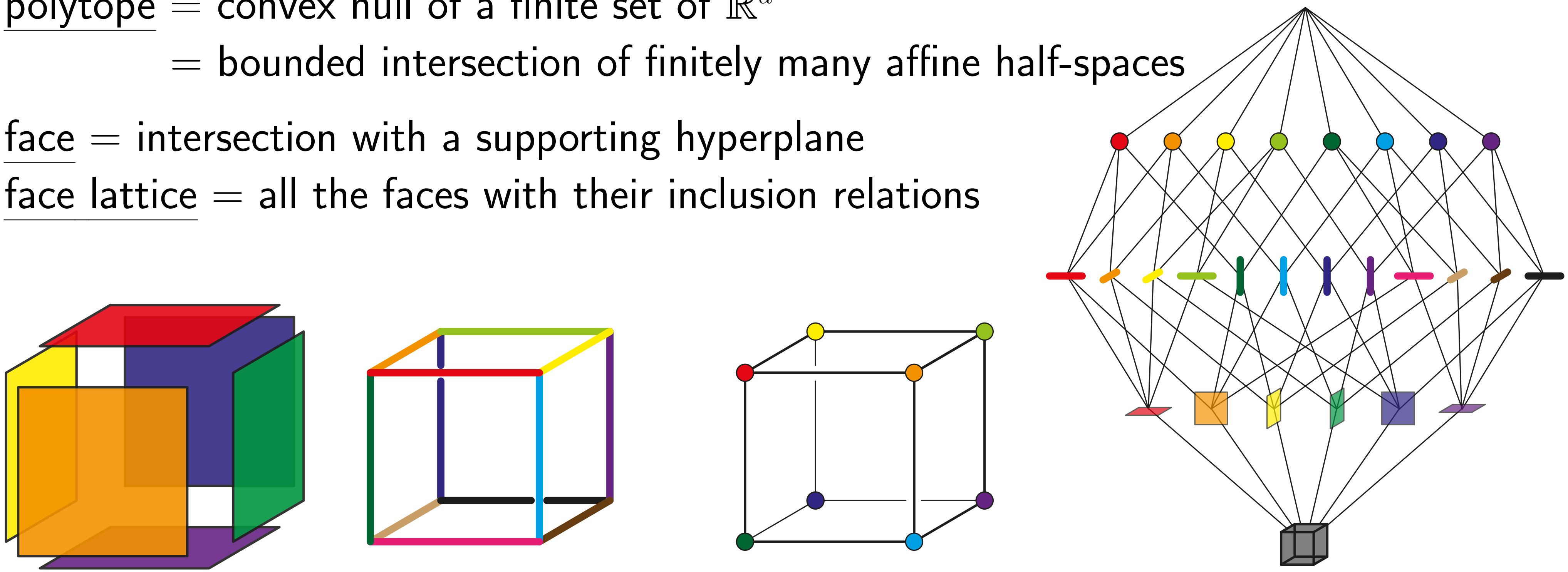




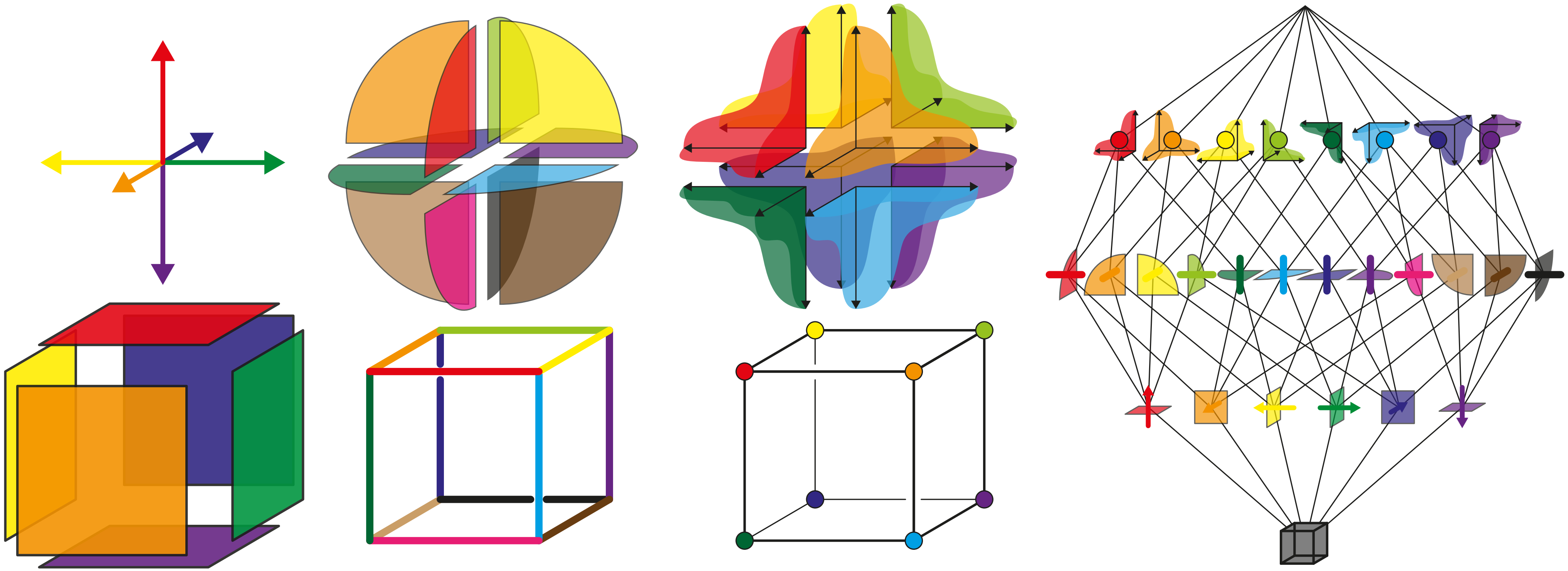
<u>simplicial</u> fan $=$ maximal cones generated by d rays

polytope $=$ convex hull of a finite set of $\mathbb{R}^d$
$=$ bounded intersection of finitely many affine half-spaces

face $=$ intersection with a supporting hyperplane
face lattice $=$ all the faces with their inclusion relations



simple polytope $=$ facets in general position $=$ each vertex incident to d facets



P polytope, F face of P

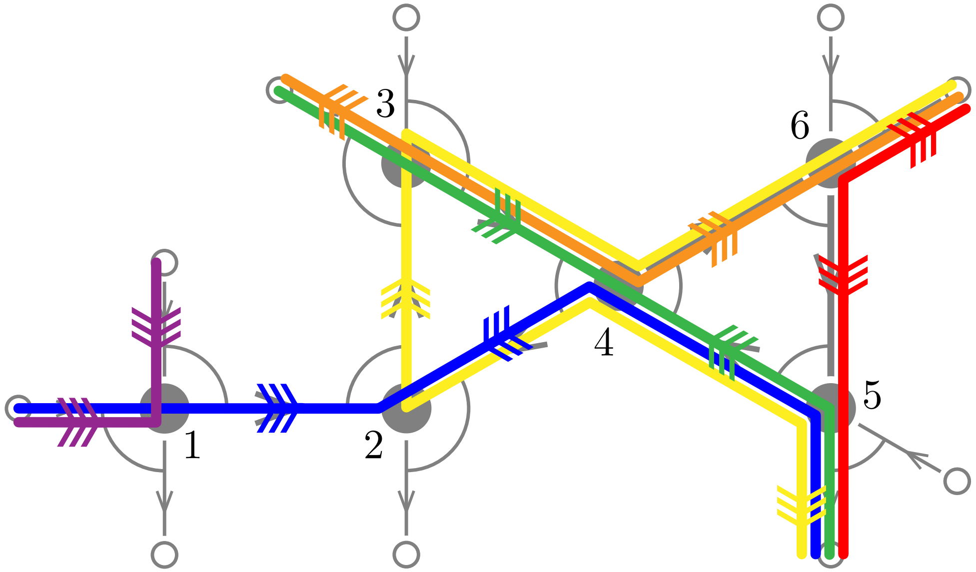
<u>normal cone</u> of F = positive span of the outer normal vectors of the facets containing F

<u>normal fan</u> of $P = \{$ normal cone of $F \mid F$ face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex multiplicity vector $\mathbf{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of Q_0 $\quad = \quad \sum_{i \in [m]} \mathbf{e}_{v_i} \in \mathbb{R}^{Q_0}$

g-vector $\mathbf{g}(\omega)$ of a walk ω $\quad = \quad \mathbf{m}_{\text{peaks}(\omega)} - \mathbf{m}_{\text{deeps}(\omega)}$

c-vector $\mathbf{c}(\omega \in F)$ of a walk ω in a non-kissing facet F $\quad = \quad \varepsilon(\omega, F)\, \mathbf{m}_{\text{ds}(\omega,F)}$



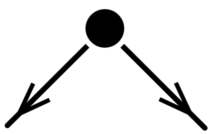
$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

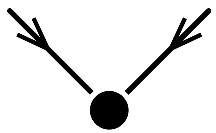
$\mathbf{g}F$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{c}F$

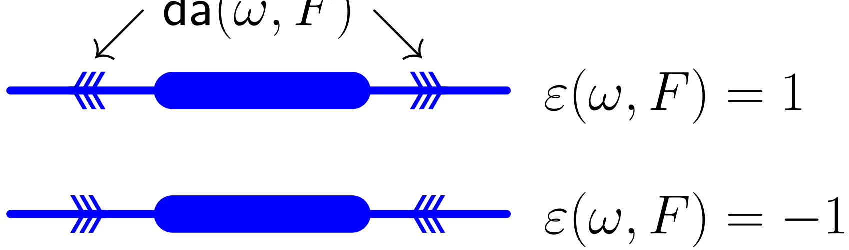
peak





deep

$\text{da}(\omega, F)$



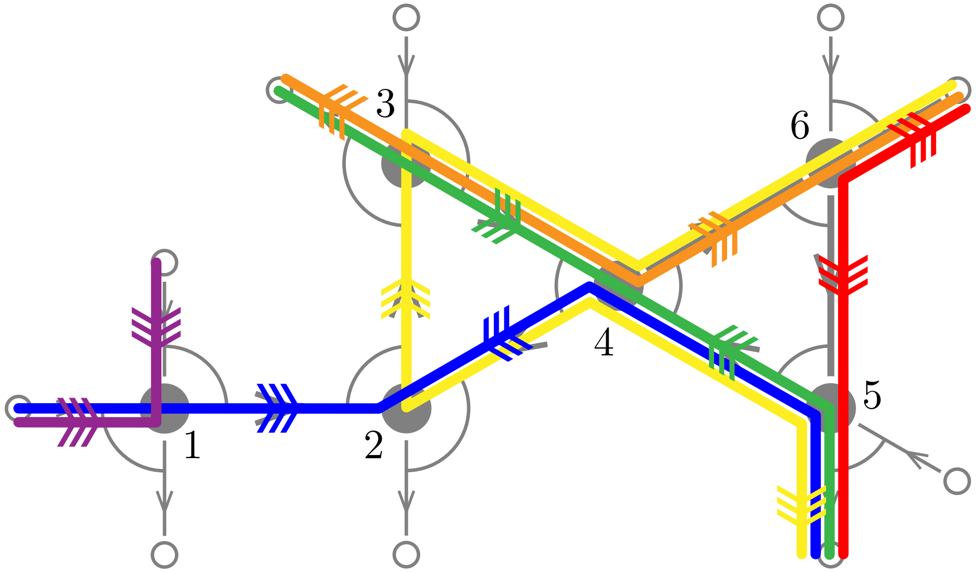
$\text{ds}(\omega, F)$

G-VECTORS & C-VECTORS

multiplicity vector $\mathbf{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of $Q_0 \quad = \quad \sum_{i \in [m]} \mathbf{e}_{v_i} \; \in \; \mathbb{R}^{Q_0}$

g-vector $\mathbf{g}(\omega)$ of a walk $\omega \quad = \quad \mathbf{m}_{\text{peaks}(\omega)} - \mathbf{m}_{\text{deeps}(\omega)}$

c-vector $\mathbf{c}(\omega \in F)$ of a walk ω in a non-kissing facet $F \quad = \quad \varepsilon(\omega, F)\, \mathbf{m}_{\text{ds}(\omega, F)}$



$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{g}F \qquad\qquad\qquad \mathbf{c}F$$

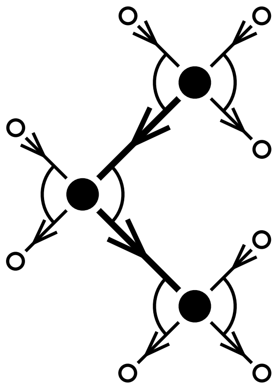
PROP. For any non-kissing facet F, the sets of vectors

$$\mathbf{g}(F) := \{\mathbf{g}(\omega) \mid \omega \in F\} \qquad \text{and} \qquad \mathbf{c}(F) := \{\mathbf{c}(\omega \in F) \mid \omega \in F\}$$

form dual bases.

Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17+)

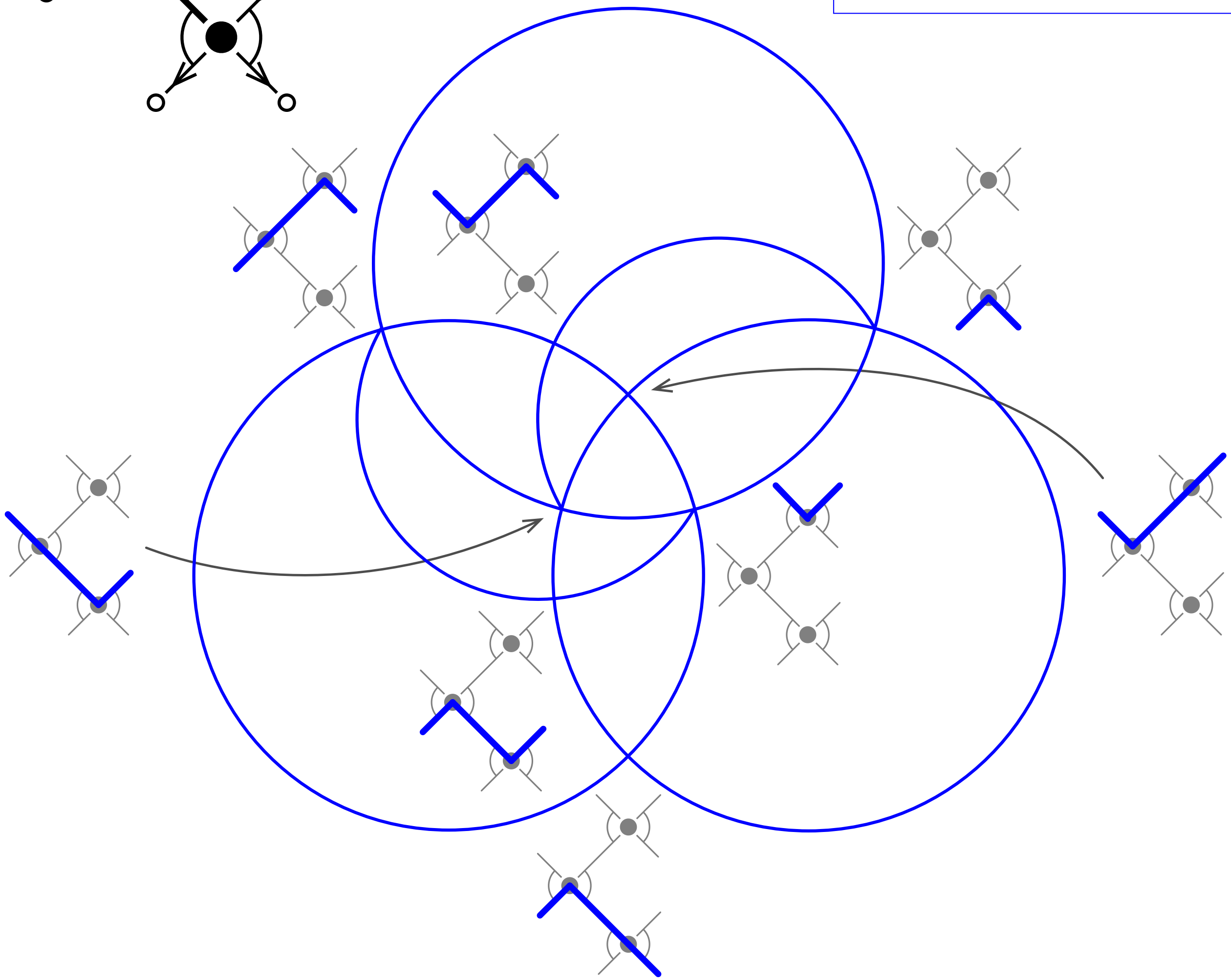
$\mathbf{G}$-VECTOR FAN



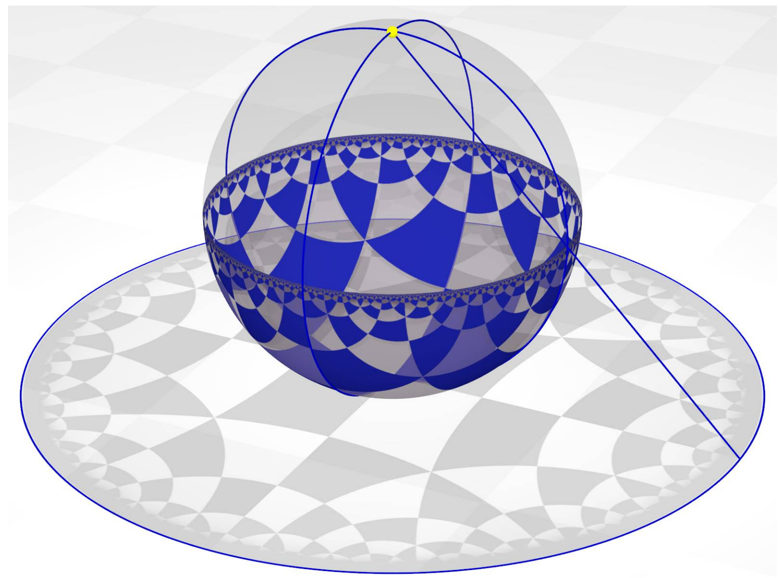
THM. For any gentle quiver $\bar{Q}$, the collection of cones

$$\mathcal{F}^{\mathbf{g}}(\bar{Q}) := \left\{ \mathbb{R}_{\geq 0}\mathbf{g}(F) \;\middle|\; F \in \mathcal{NK}(\bar{Q}) \right\}$$

forms a compl. simpl. fan, called <u>g-vector fan</u> of $\bar{Q}$.



stereographic projection from $(1, 1, 1)$



kissing number $\mathsf{kn}(\omega) = \sum_{\omega'}$ number of times ω and ω' kiss

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{NK}(\bar{Q})$, the two sets of $\mathbb{R}^{Q_0}$ given by

(i) the convex hull of the points

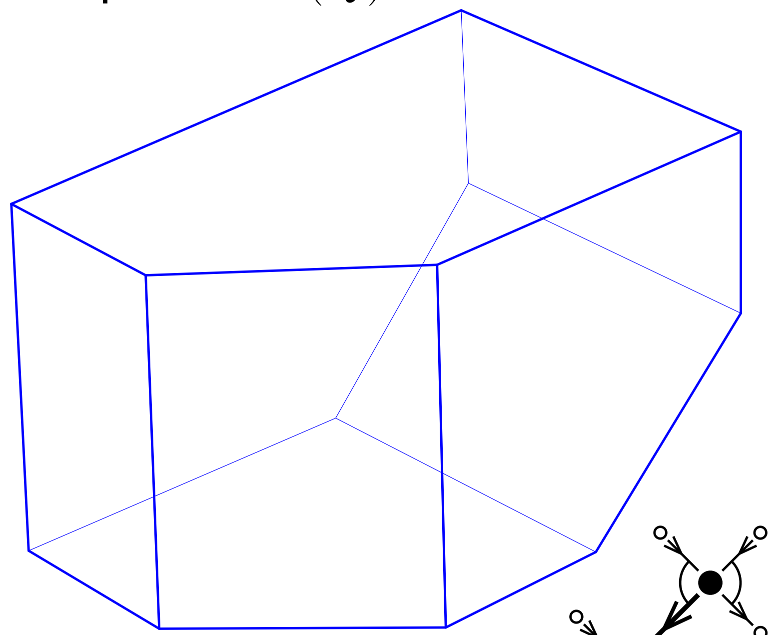
$$\mathbf{p}(F) := \sum_{\omega \in F} \mathsf{kn}(\omega)\, \mathbf{c}(\omega \in F),$$

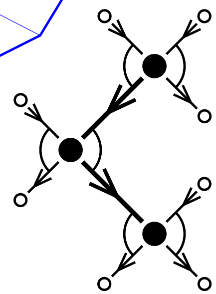
for all non-kissing facets $F \in \mathcal{NK}(\bar{Q})$,

(ii) the intersection of the halfspaces

$$\mathbf{H}^{\geq}(\omega) := \left\{ \mathbf{x} \in \mathbb{R}^{Q_0} \;\middle|\; \langle\, \mathbf{g}(\omega) \mid \mathbf{x} \,\rangle \leq \mathsf{kn}(\omega) \right\}.$$

for all walks ω of $\bar{Q}$,





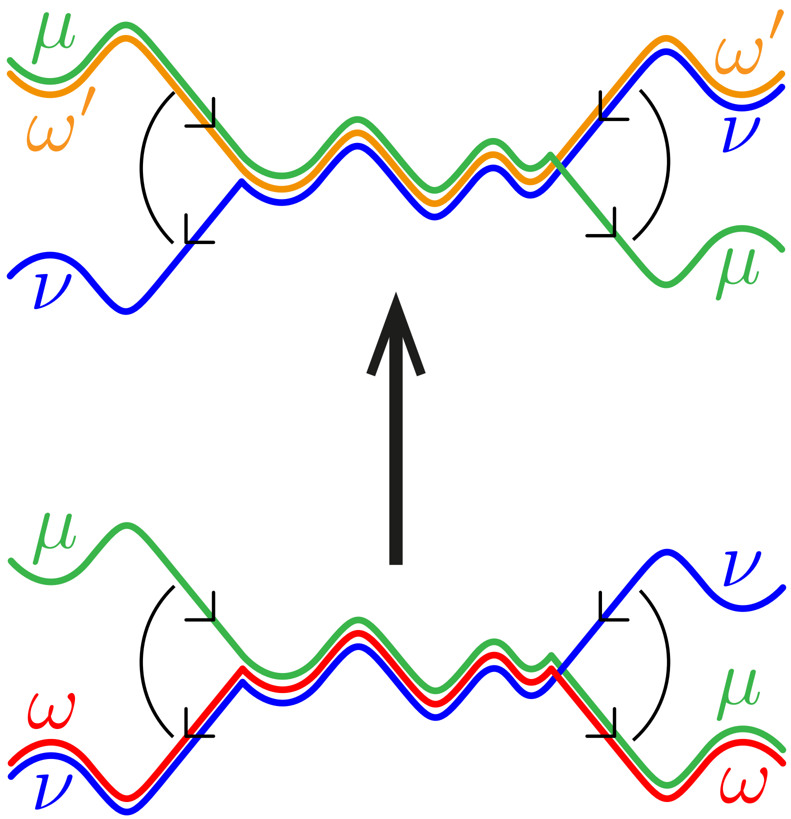
define the same polytope, whose normal fan is the g-vector fan $\mathcal{F}^{\mathbf{g}}$. We call it the $\bar{Q}$-associahedron and denote it by Asso.

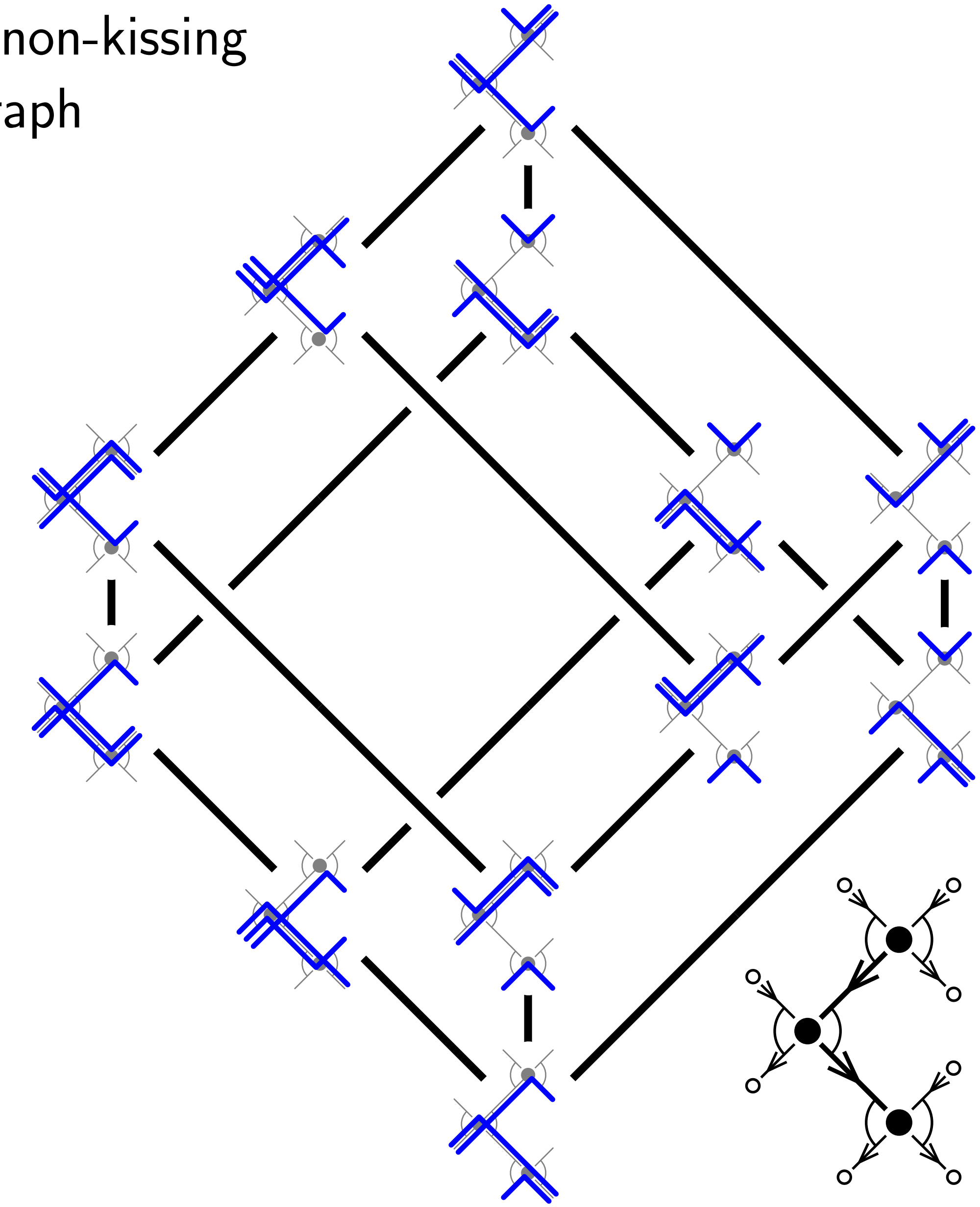
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17$^+$)

NON-KISSING LATTICE

McConville, *Lattice structures of grid Tamari orders* ('17)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17$^+$)

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{NK}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.





Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17$^+$)

LATTICE QUOTIENTS

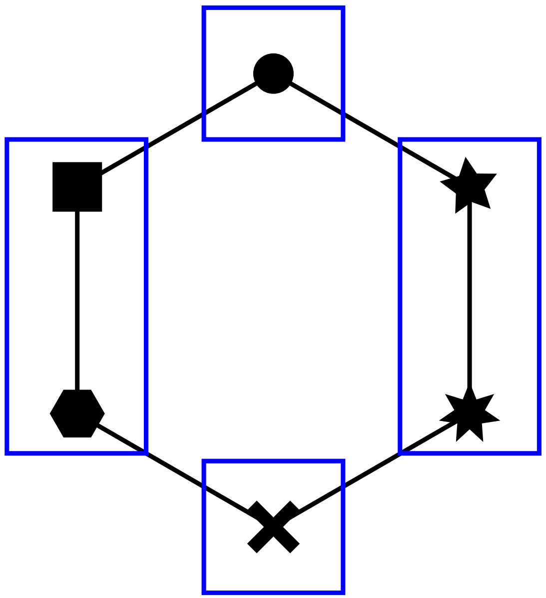
lattice $=$ poset $(L, \leq)$ with a meet $\wedge$ and a join $\vee$

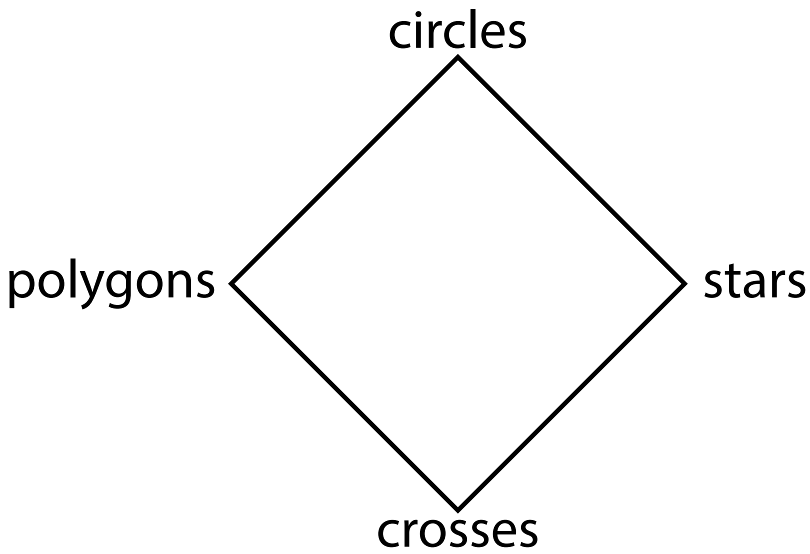
lattice congruence $=$ equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \qquad \Longrightarrow \qquad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

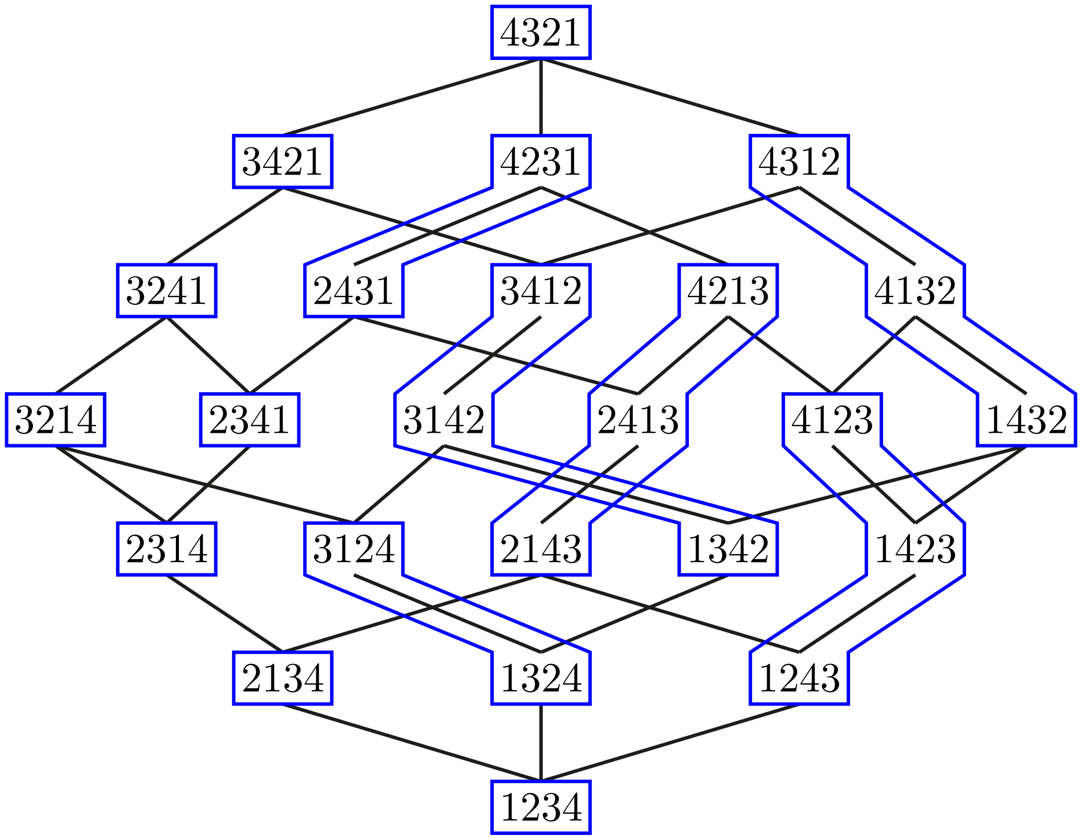
lattice quotient of $L/\equiv$ $=$ lattice on equiv. classes of L under $\equiv$ where

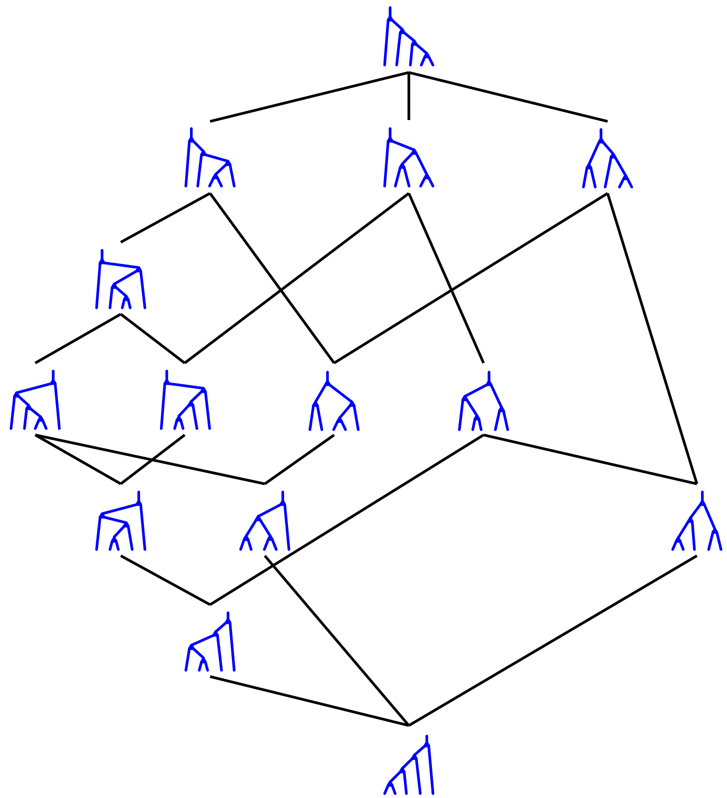
- $X \leq Y \iff \exists x \in X,\ y \in Y,\quad x \leq y$
- $X \wedge Y$ $=$ equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ $=$ equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$



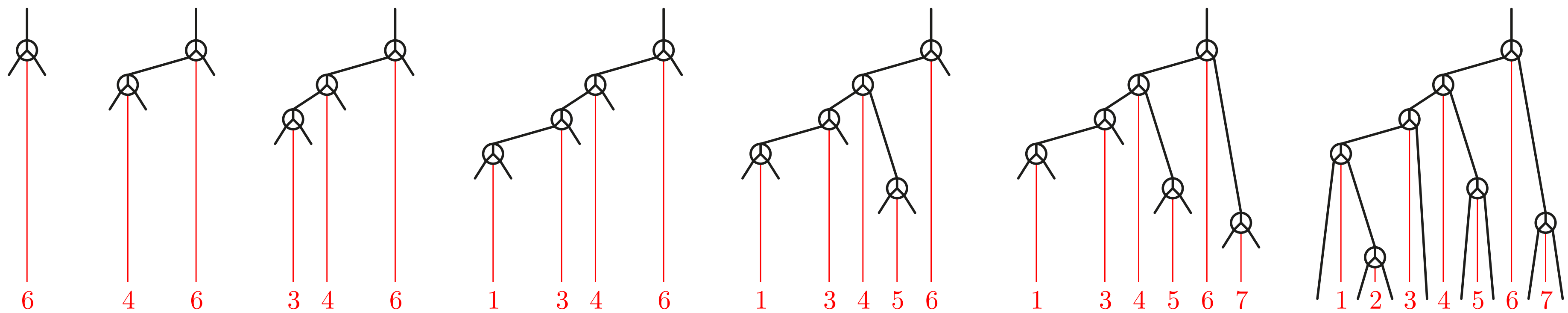


EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

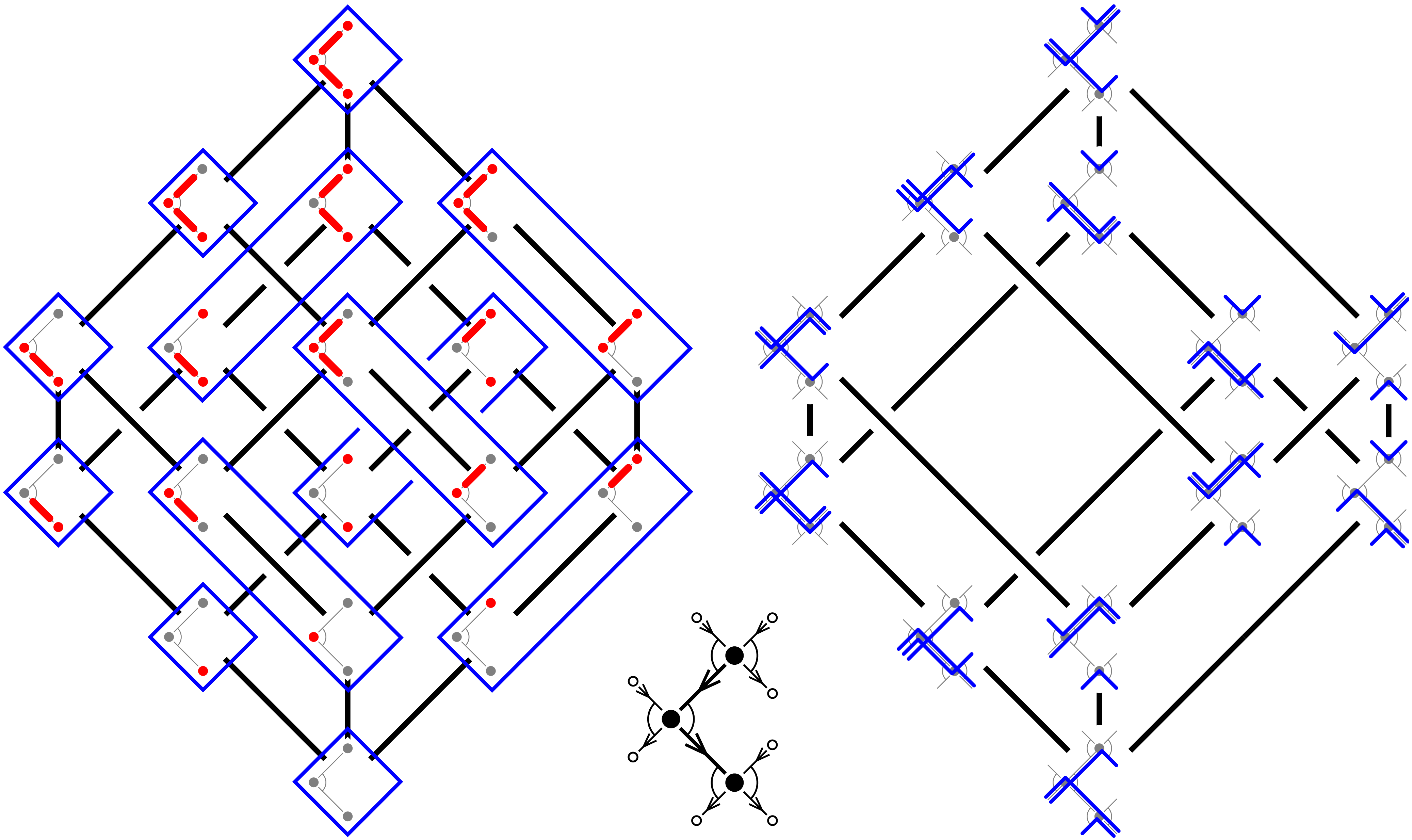




binary search tree insertion of 2751346



NON-KISSING LATTICE



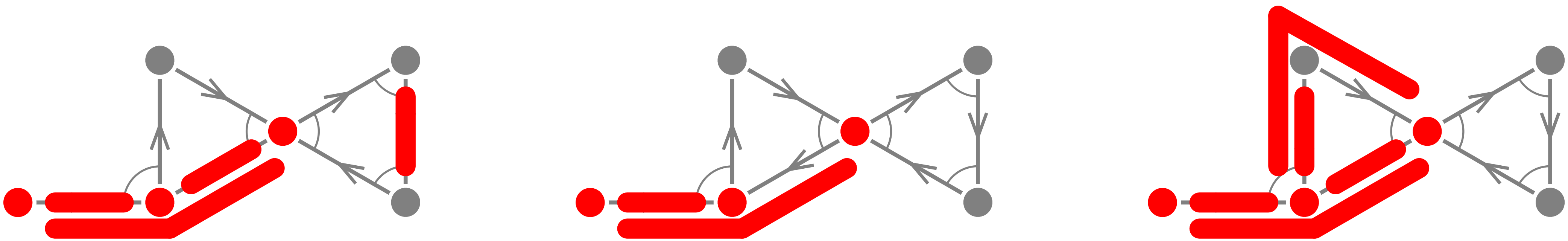
BICLOSED SETS OF STRINGS

σ, τ oriented strings

concatenation $\sigma \circ \tau = \big\{ \sigma\alpha\tau \;\big|\; \alpha \in Q_1 \text{ and } \sigma\alpha\tau \text{ string of } \bar{Q} \big\}$

closure $S^{\mathrm{cl}} = \displaystyle\bigcup_{\substack{\ell \in \mathbb{N} \\ \sigma_1, \dots, \sigma_\ell \in S}} \sigma_1 \circ \dots \circ \sigma_\ell =$ all strings obtained by concatenation of some strings of S

closed $\iff S^{\mathrm{cl}} = S$ coclosed $\iff \bar{S}^{\mathrm{cl}} = \bar{S}$ biclosed $=$ closed and coclosed



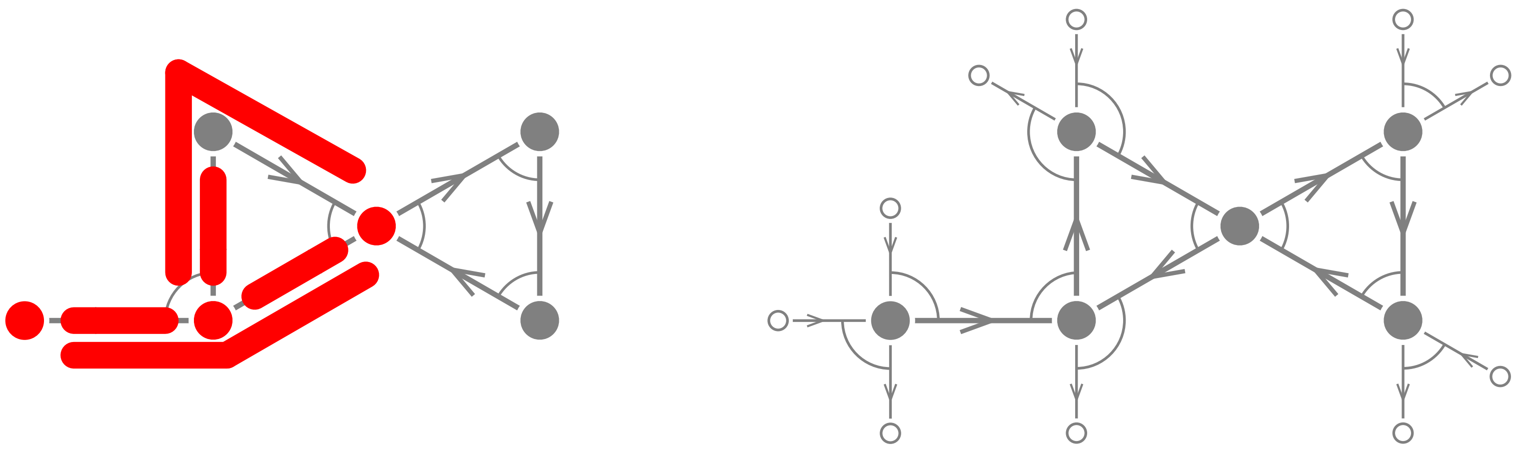
THM. For any gentle quiver $\bar{Q}$ such that $\mathcal{NK}(\bar{Q})$ is finite, the inclusion poset on biclosed sets of strings of $\bar{Q}$ is a congruence-uniform lattice.

McConville, *Lattice structures of grid Tamari orders* ('17)

Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('17$^+$)

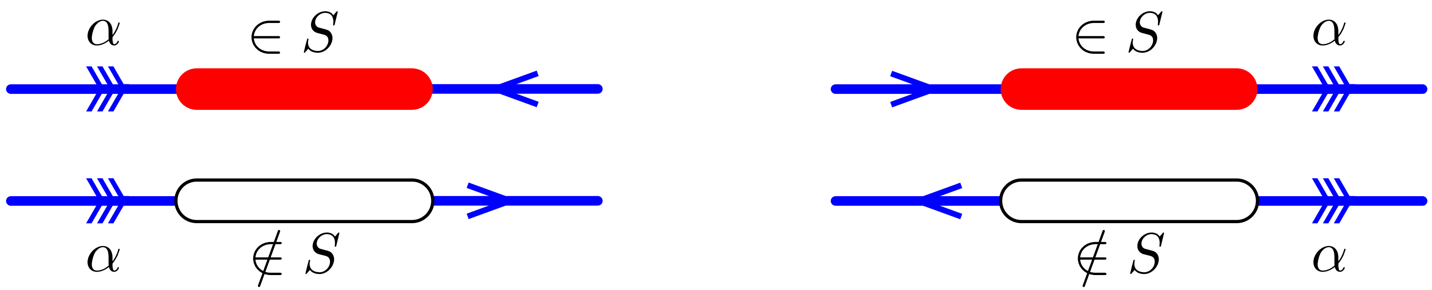
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17$^+$)

Surjection from biclosed sets of strings to non-kissing facets



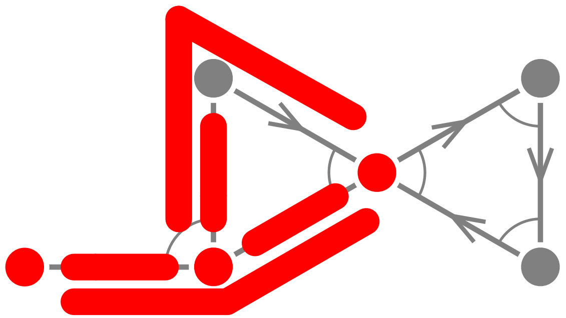
S biclosed, $\alpha \in Q_1$

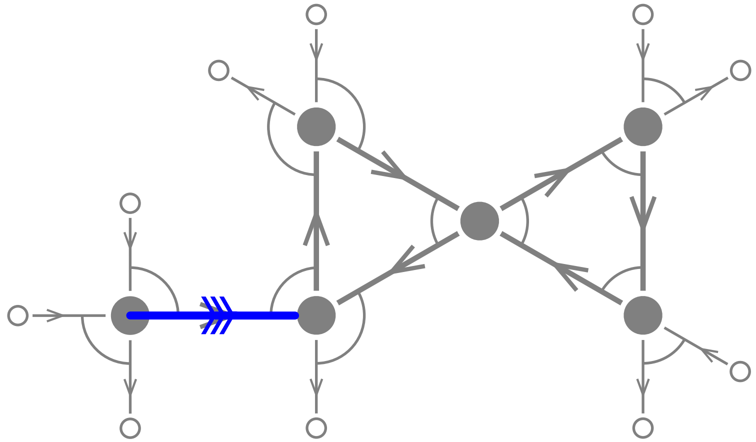
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

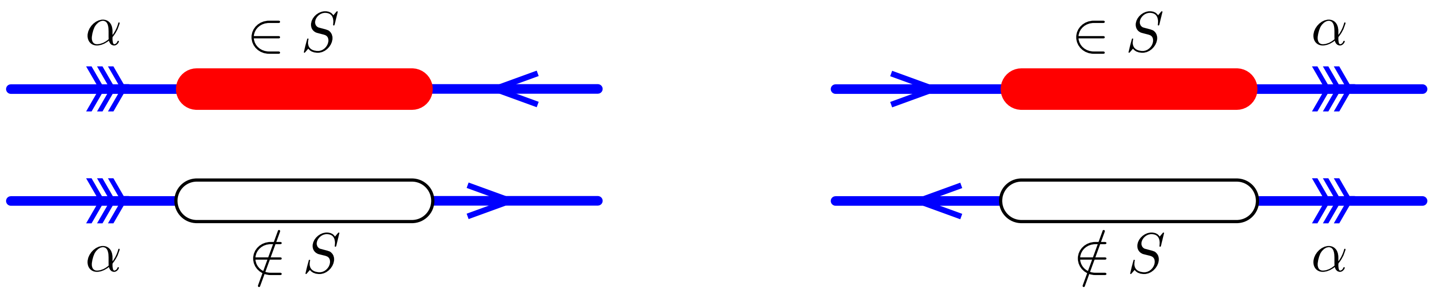
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

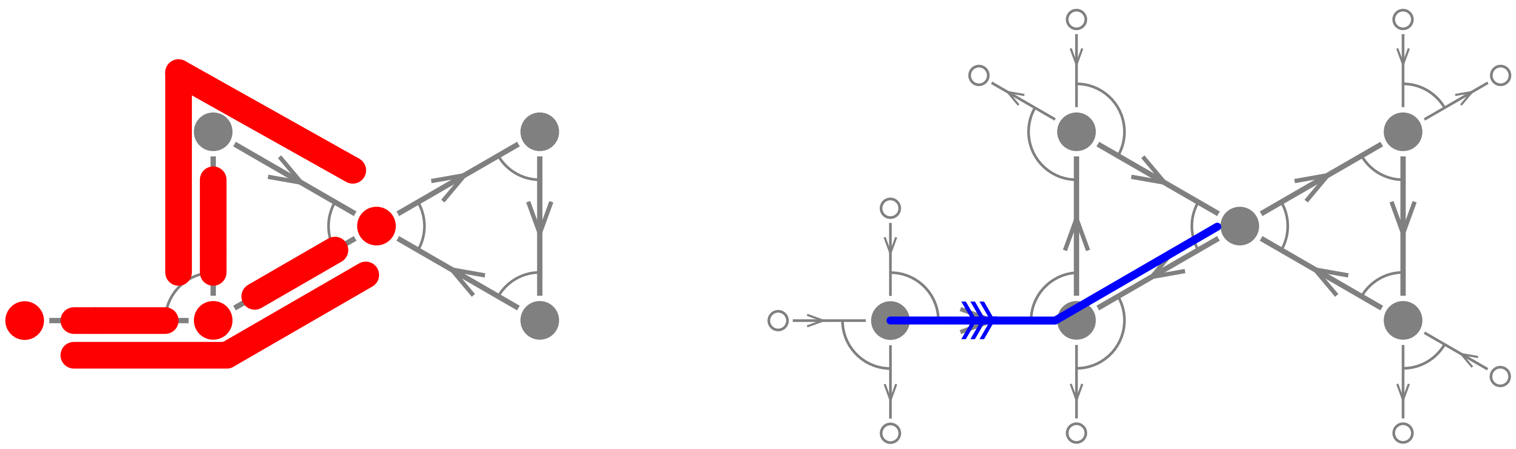
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

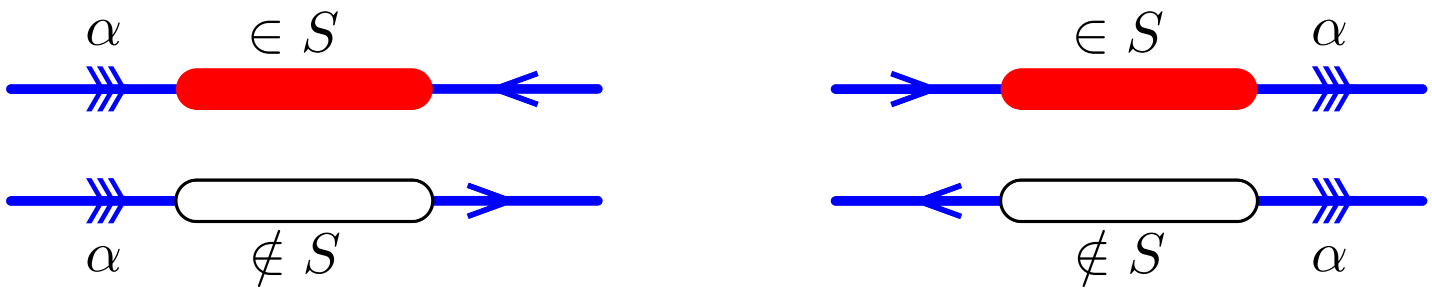
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

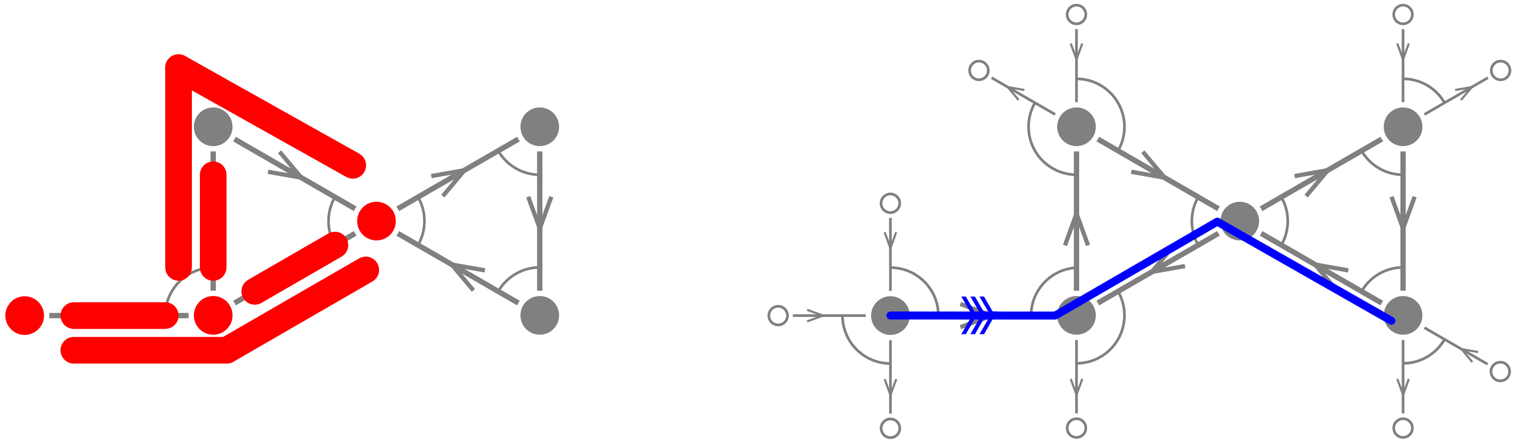
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

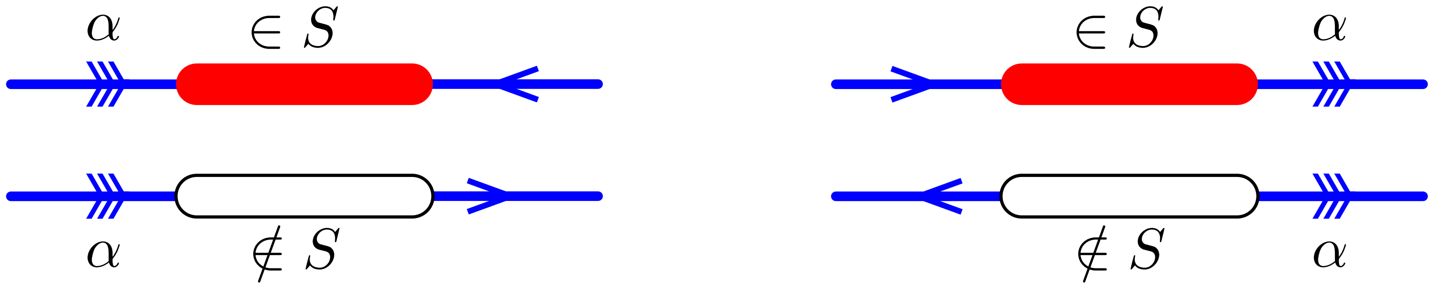
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

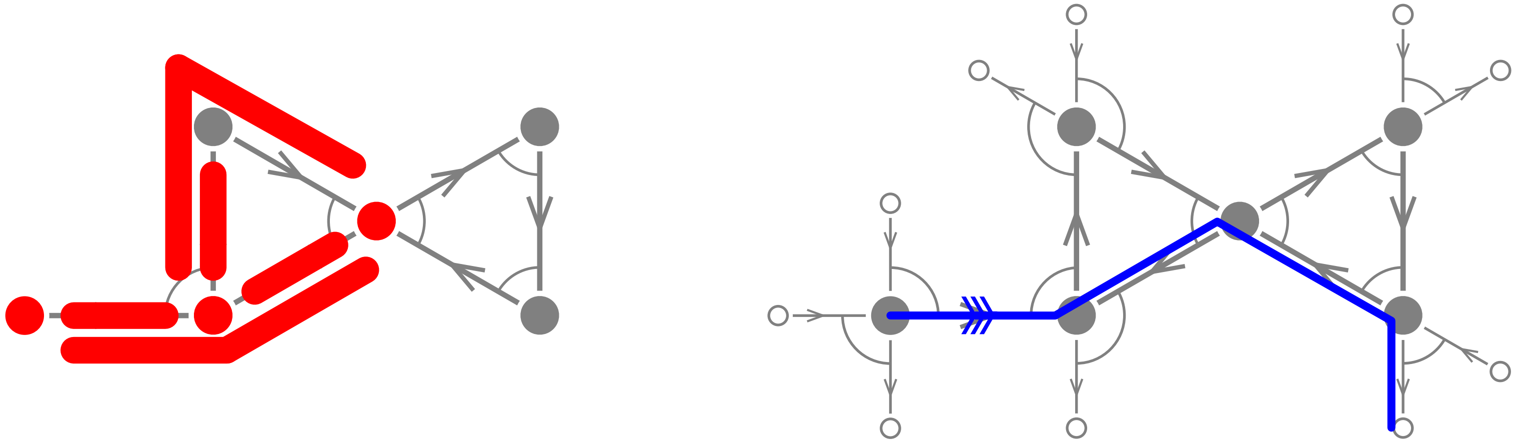
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

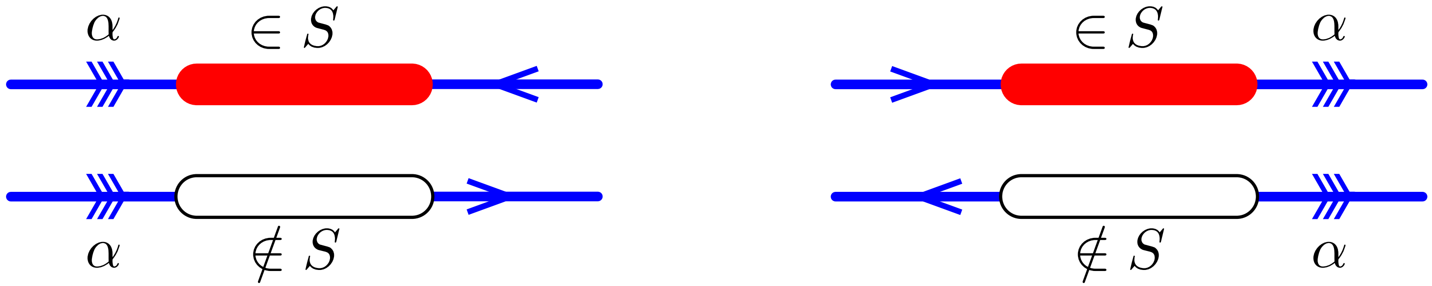
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



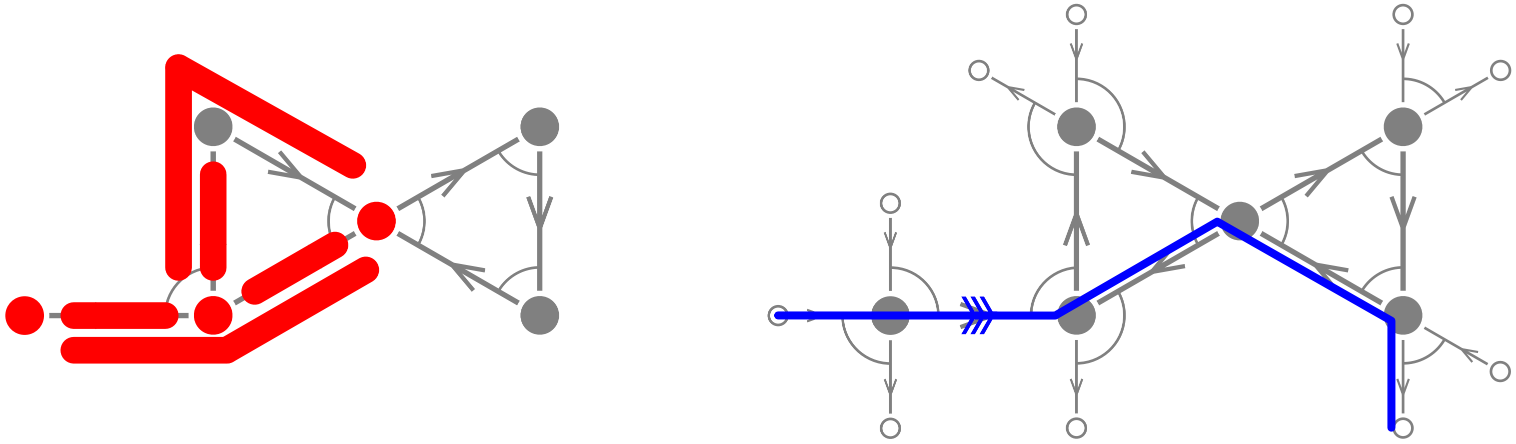
S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:



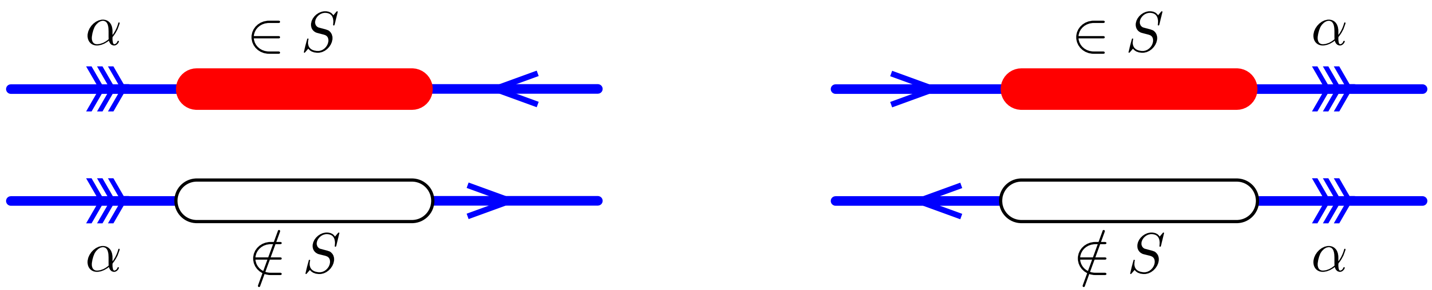
McConville, *Lattice structures of grid Tamari orders* ('17)

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

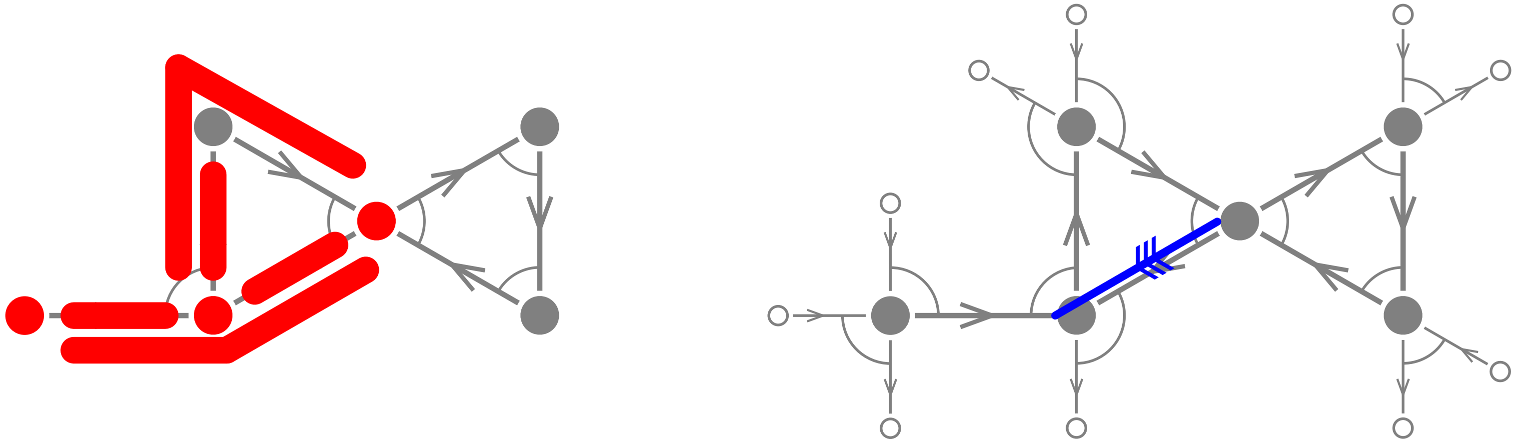
$\omega(\alpha, S)$ = walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

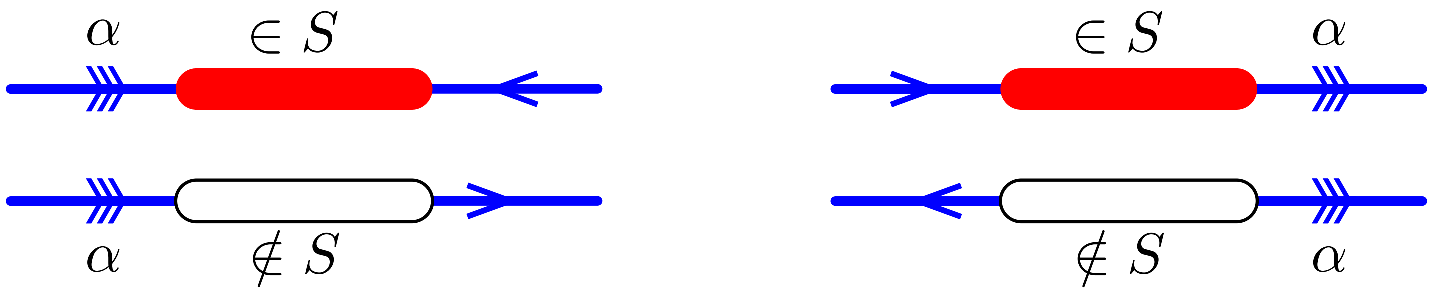
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

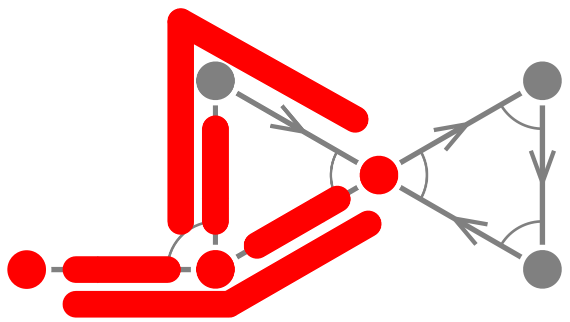
$\omega(\alpha, S)$ = walk constructed with the local rules:

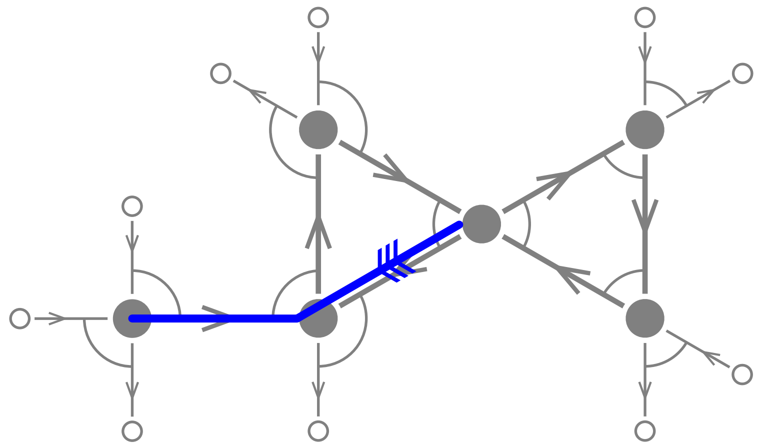


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

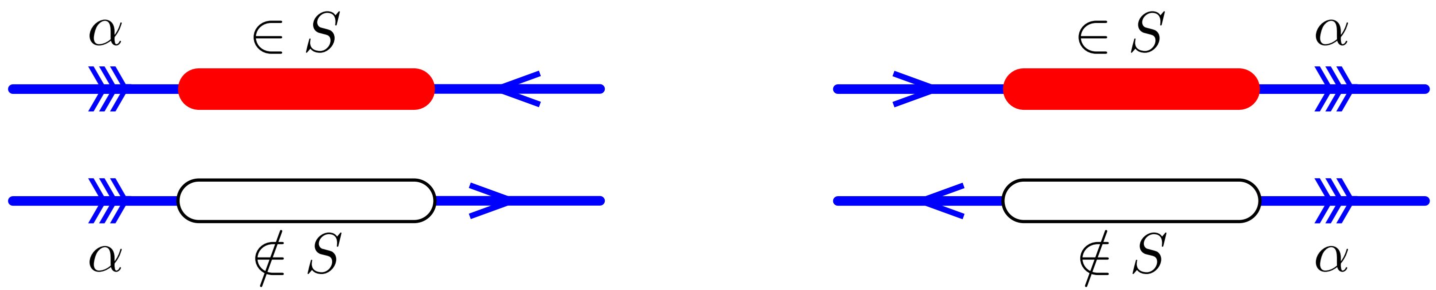
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

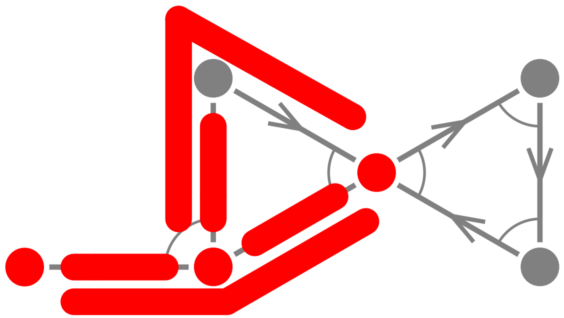
$\omega(\alpha, S) =$ walk constructed with the local rules:

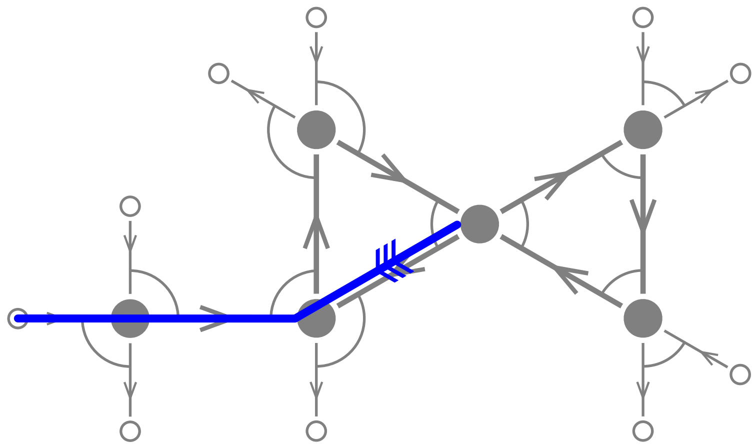


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

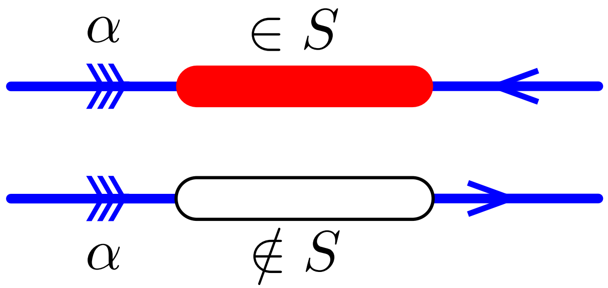
Surjection from biclosed sets of strings to non-kissing facets

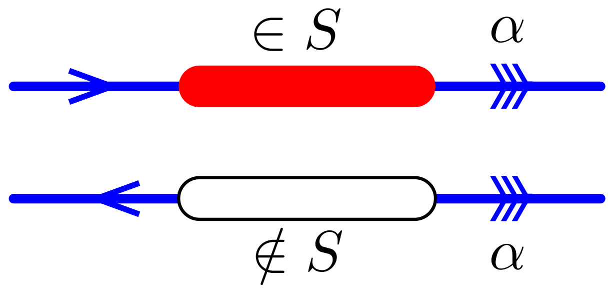




S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:

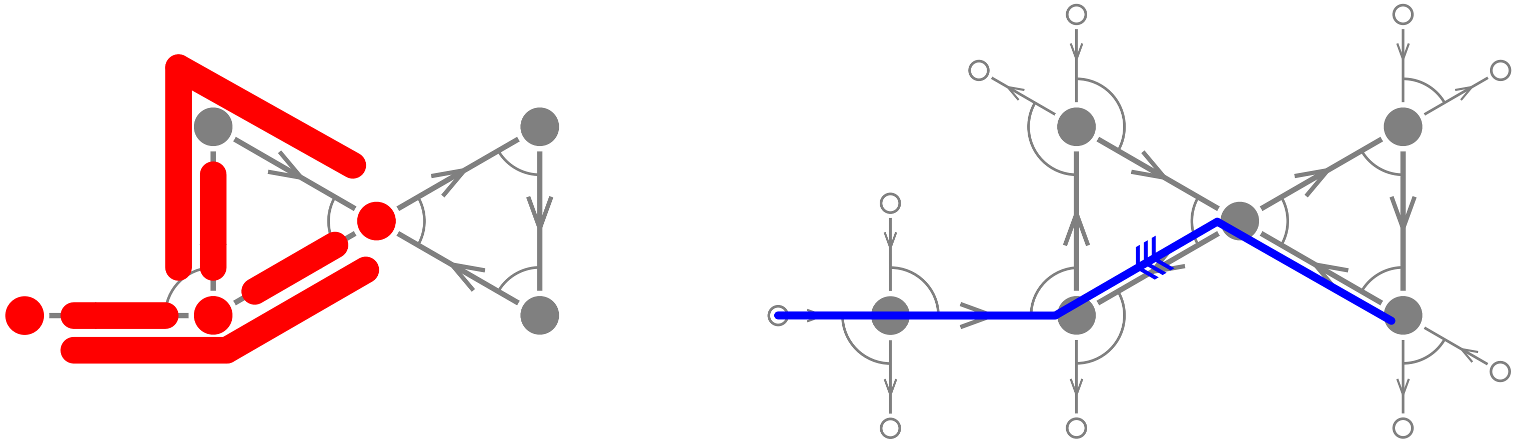




McConville, *Lattice structures of grid Tamari orders* ('17)

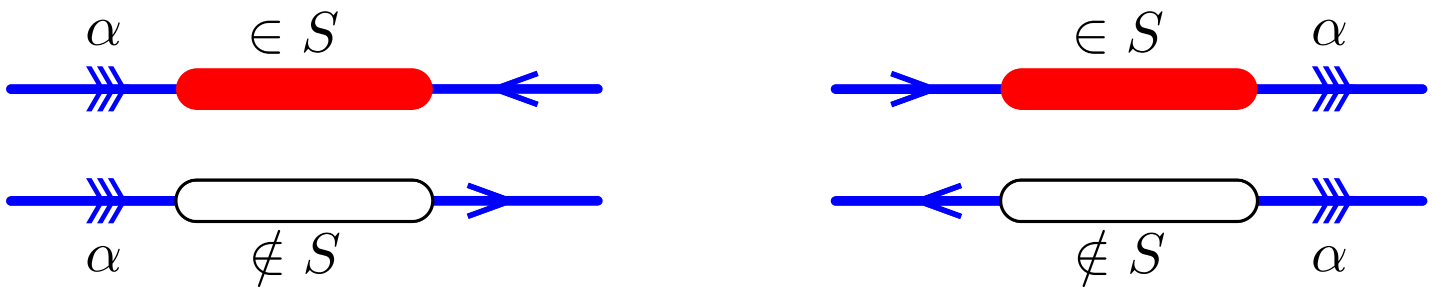
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

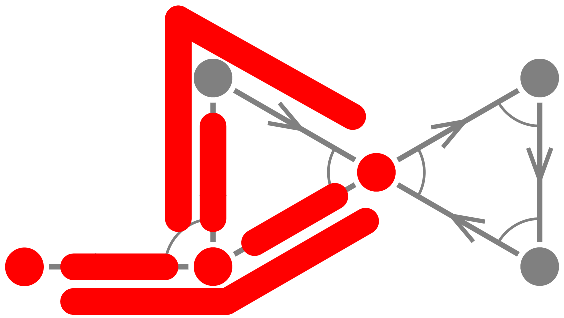
$\omega(\alpha, S) =$ walk constructed with the local rules:

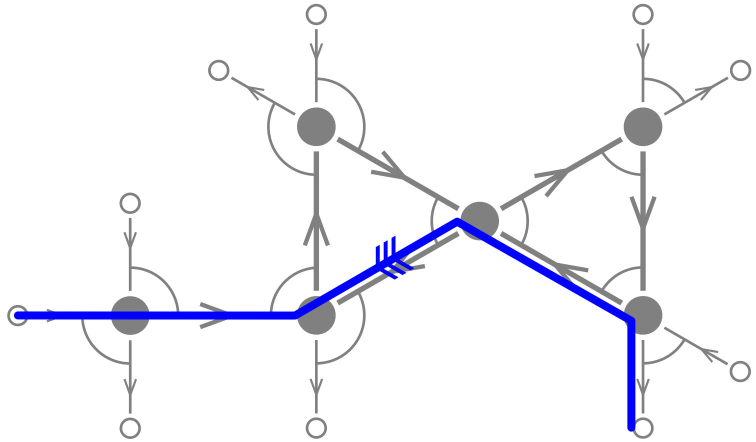


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

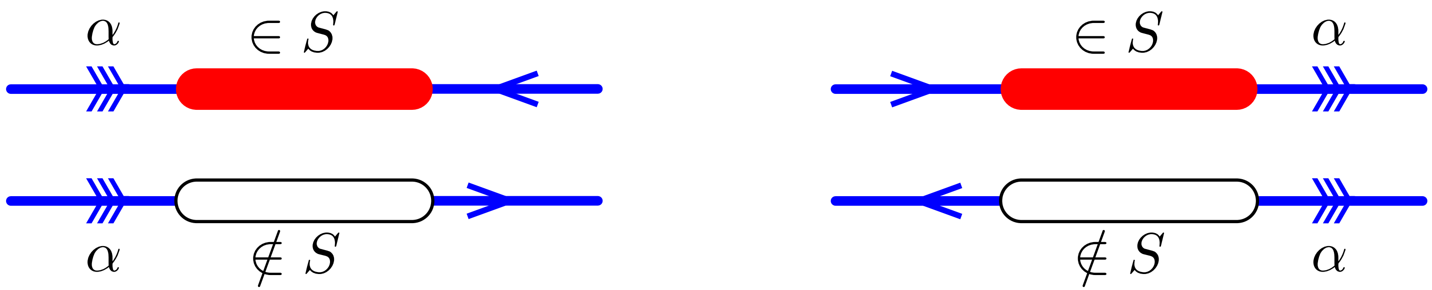
Surjection from biclosed sets of strings to non-kissing facets





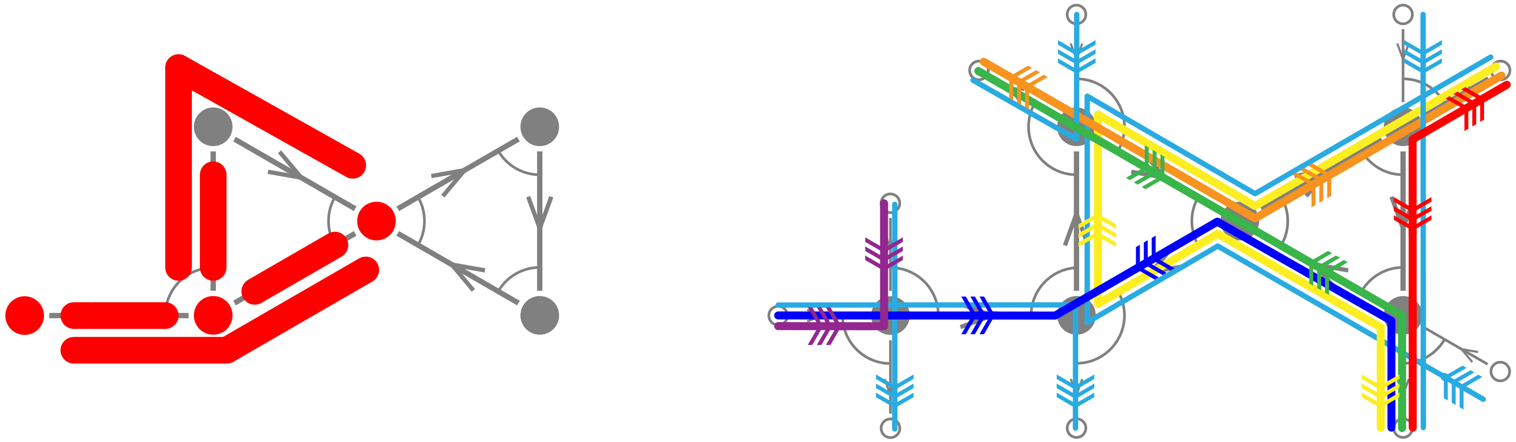
S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

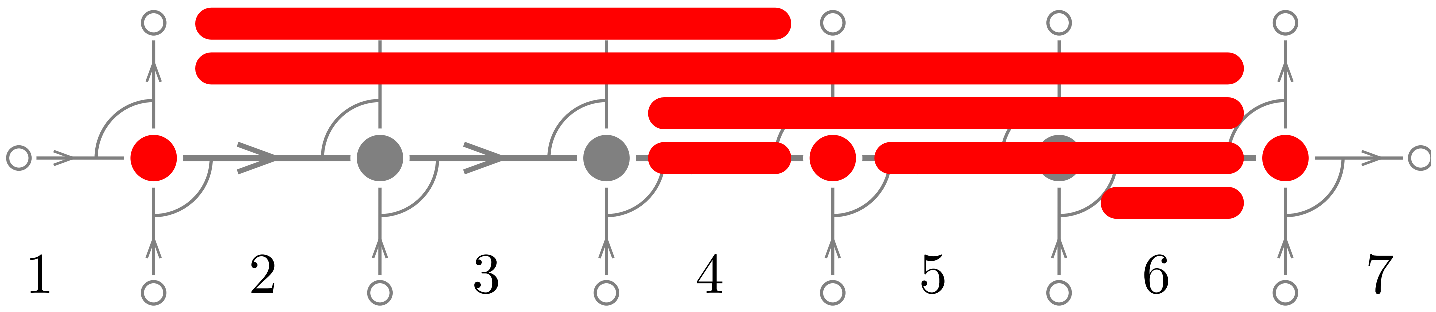
Surjection from biclosed sets of strings to non-kissing facets

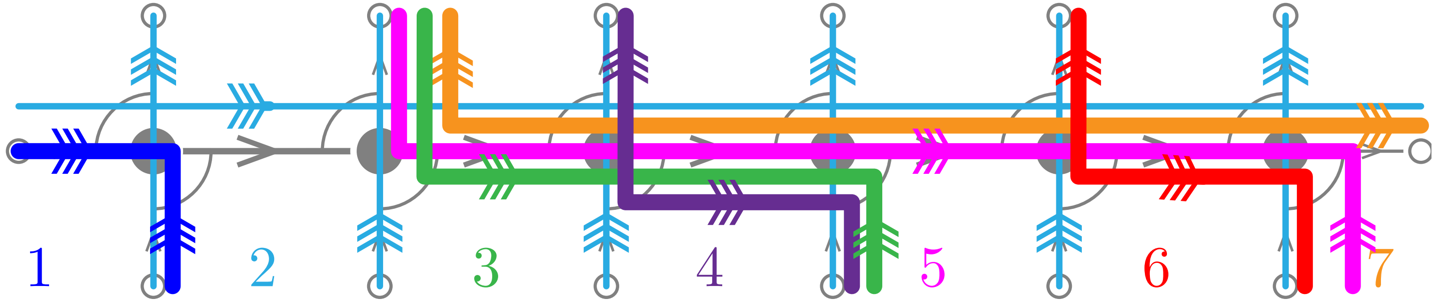


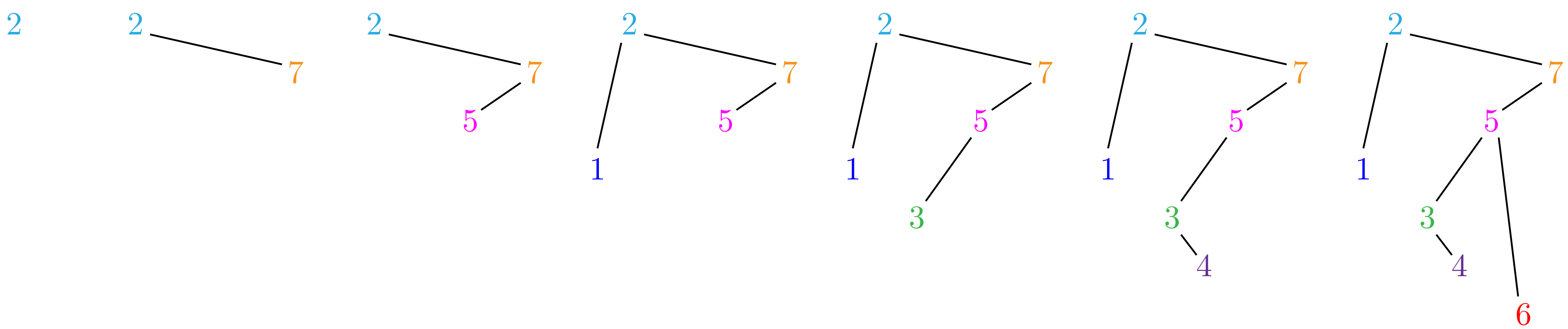
PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

McConville, *Lattice structures of grid Tamari orders* ('17)

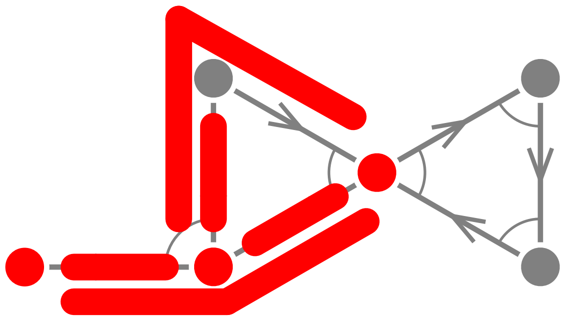
inversion set of 2751346

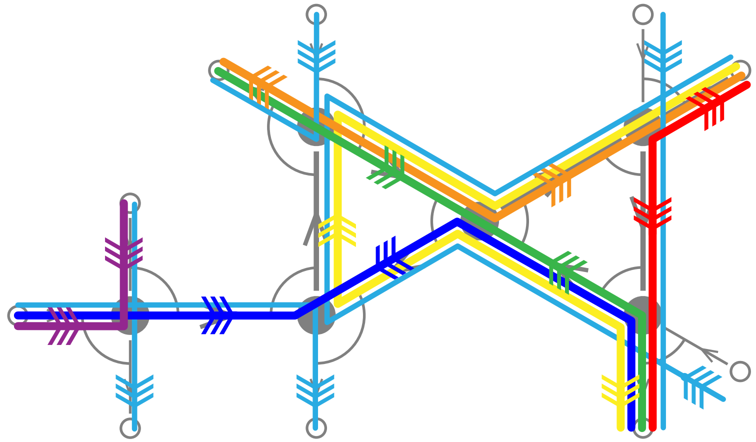






Surjection from biclosed sets of strings to non-kissing facets



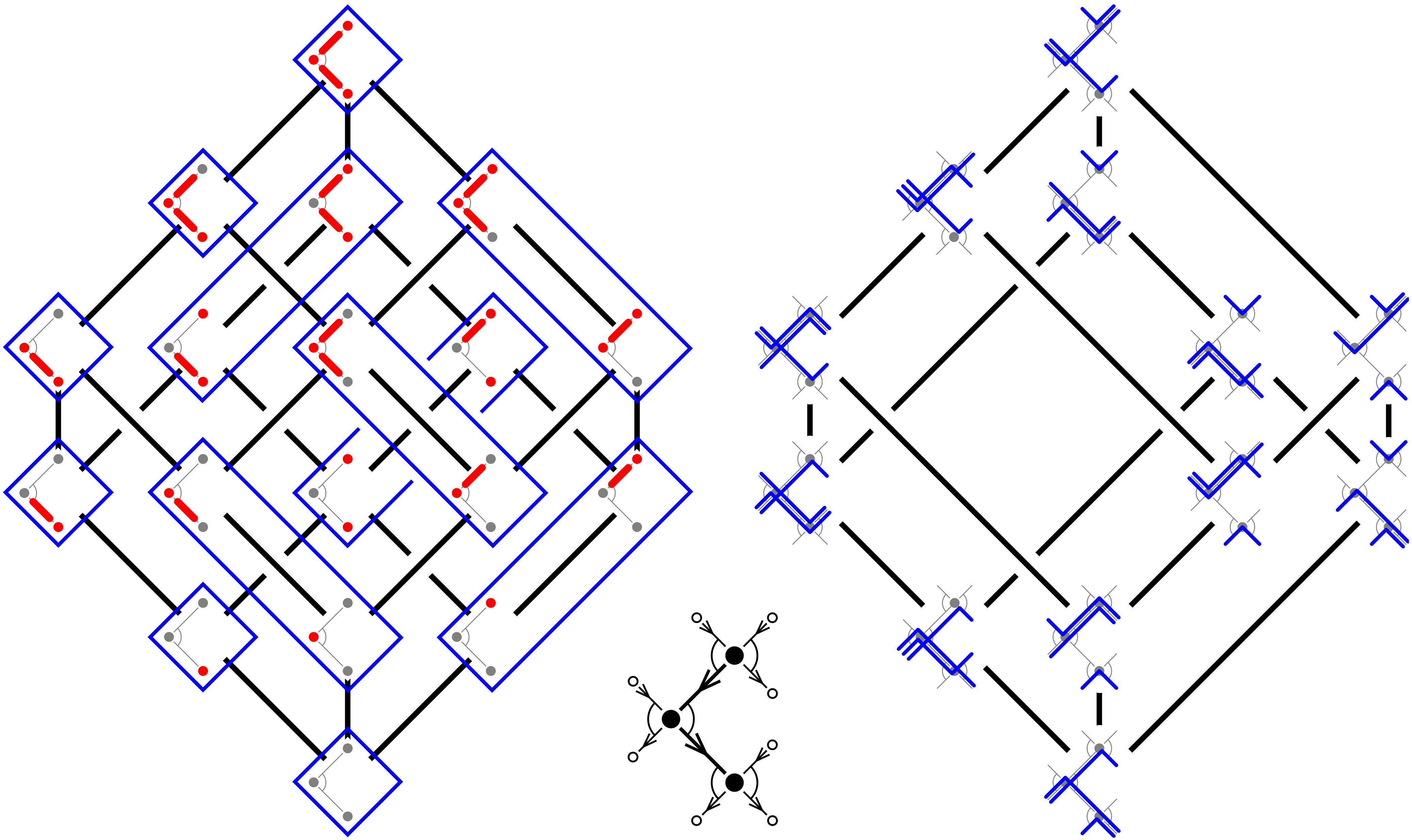


PROP. $\eta(S) := \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

THM. The map η defines a lattice morphism from biclosed sets to non-kissing facets.

McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING LATTICE



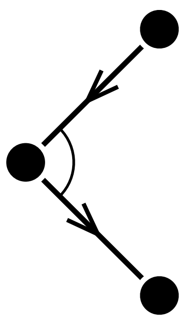
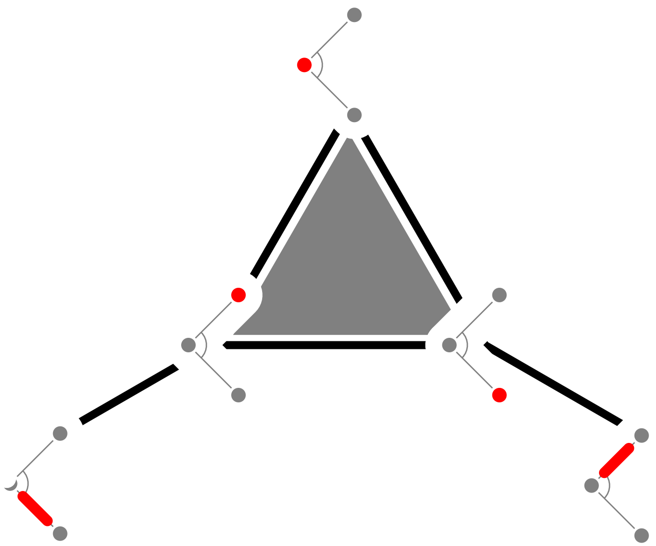
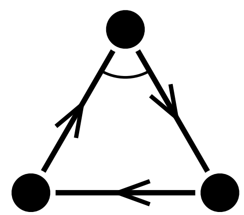
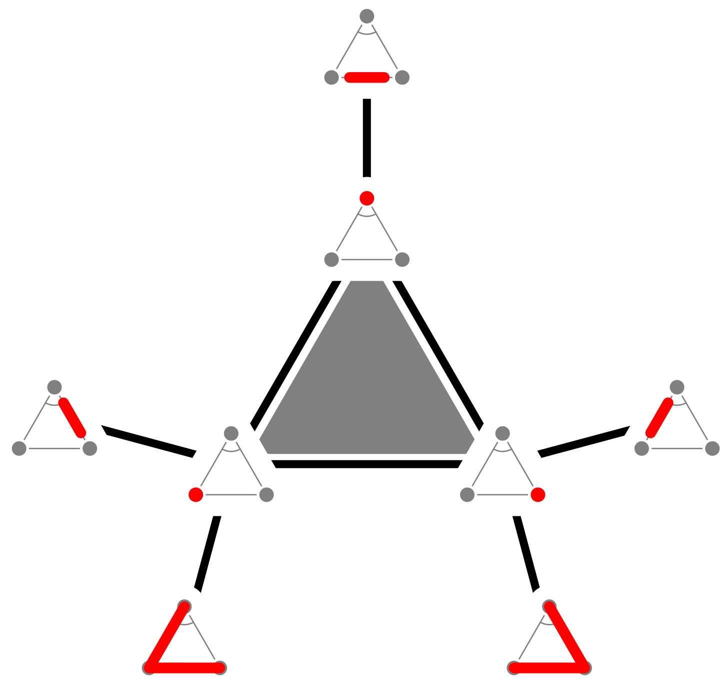
NON-KISSING LATTICE

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{NK}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17^+)

Much more nice combinatorics:

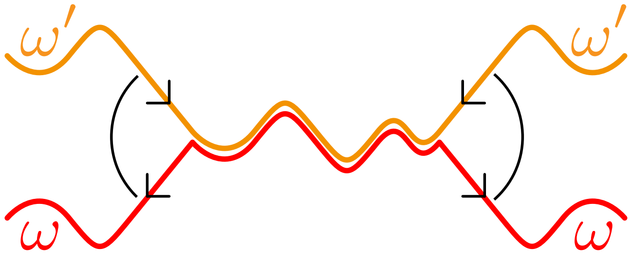
- join-irreducible elements of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ are in bijection with distinguishable strings
- canonical join complex of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is a generalization of non-crossing partitions

SUMMARY

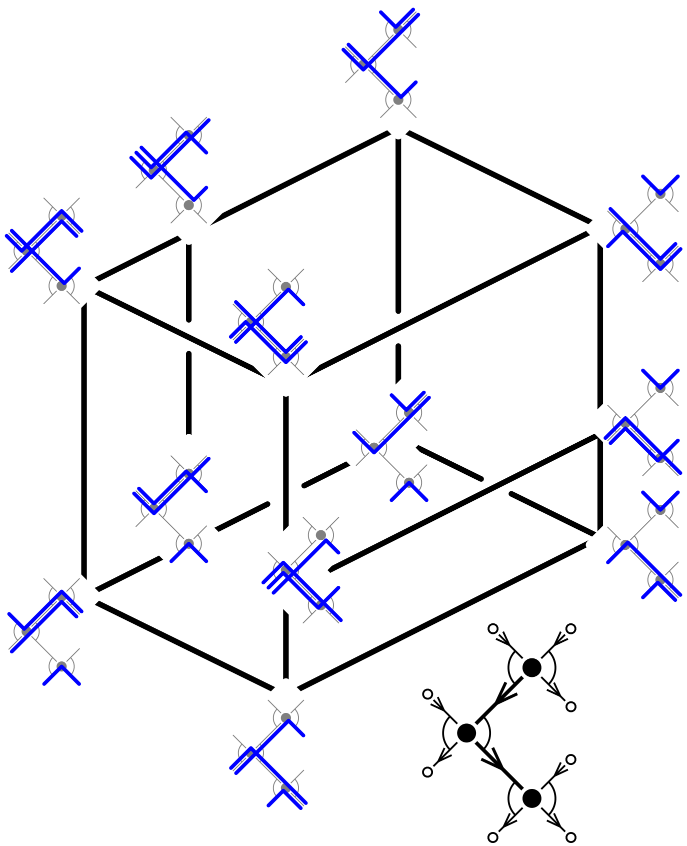
non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices $=$ walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces $=$ collections of pairwise non-kissing walks in $\bar{Q}^{❀}$

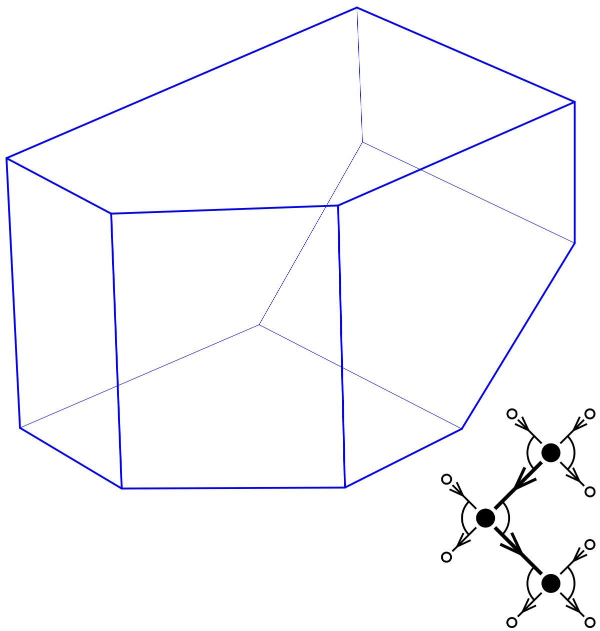


... generalizing the associahedron

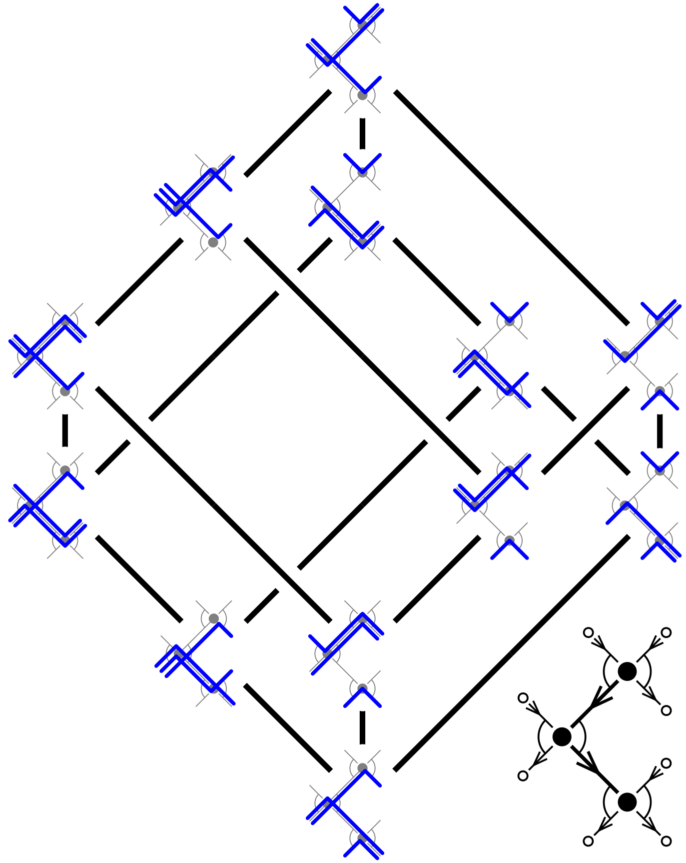
Flip graph

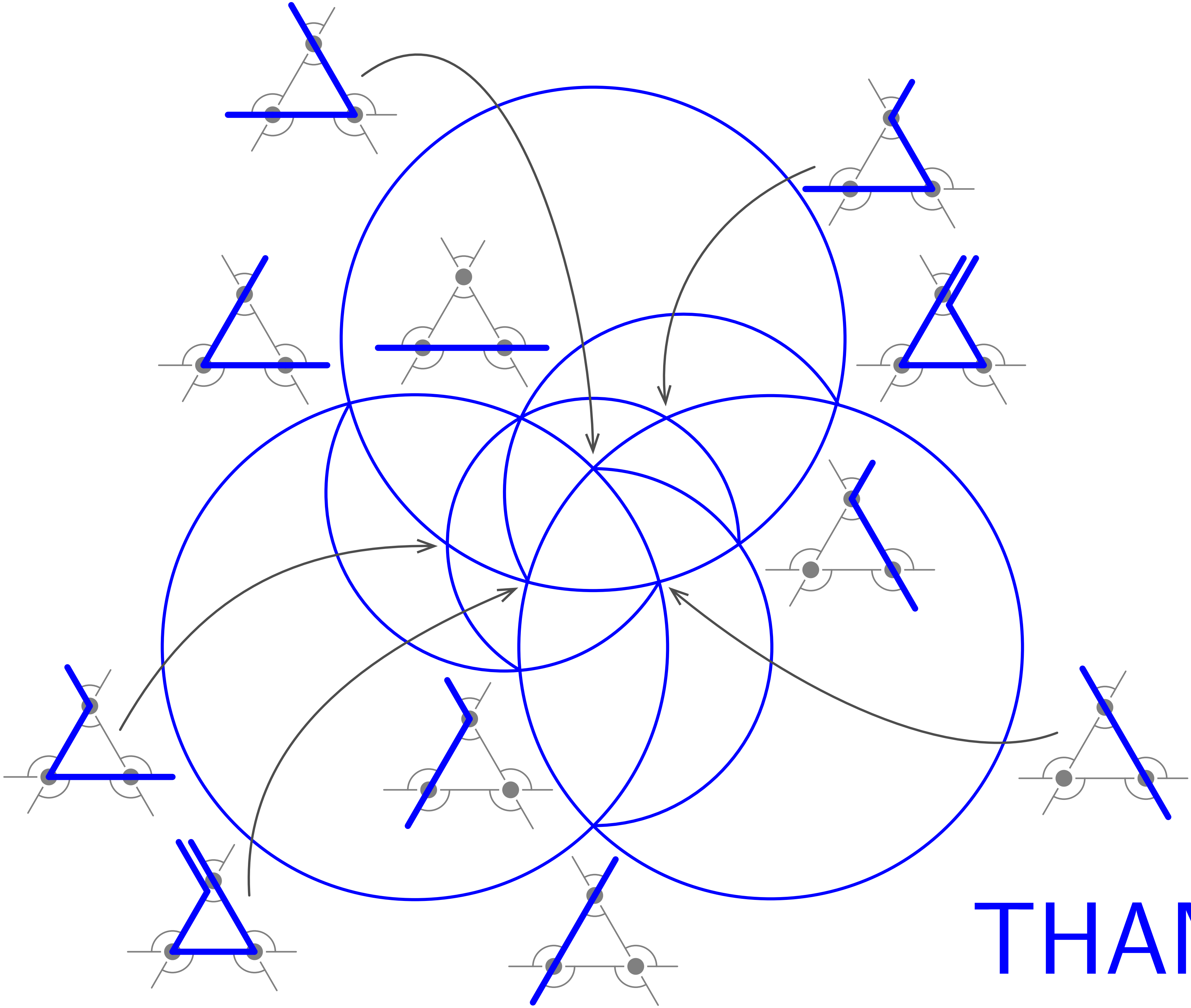


Associahedron



Tamari lattice





THANKS