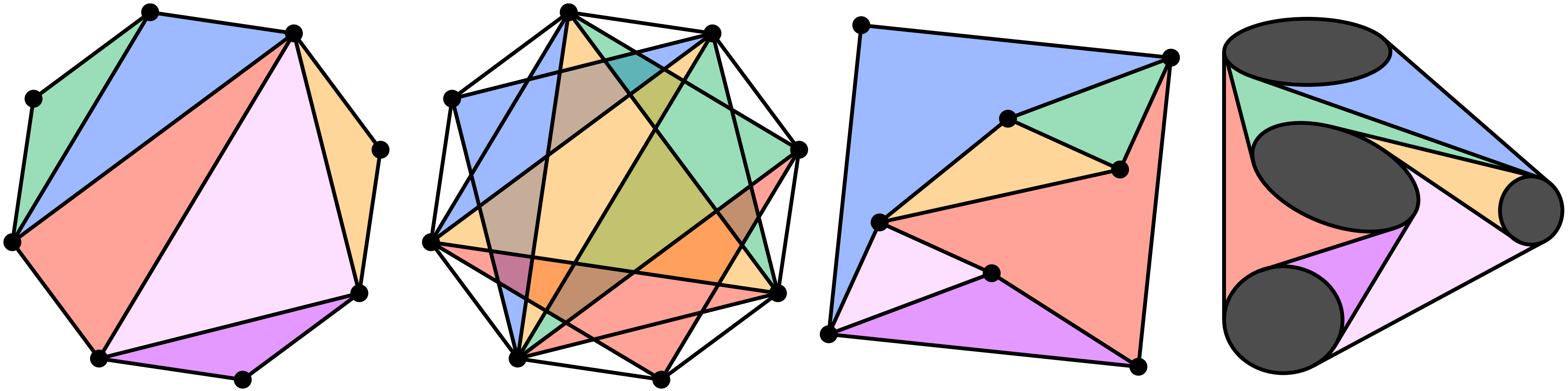
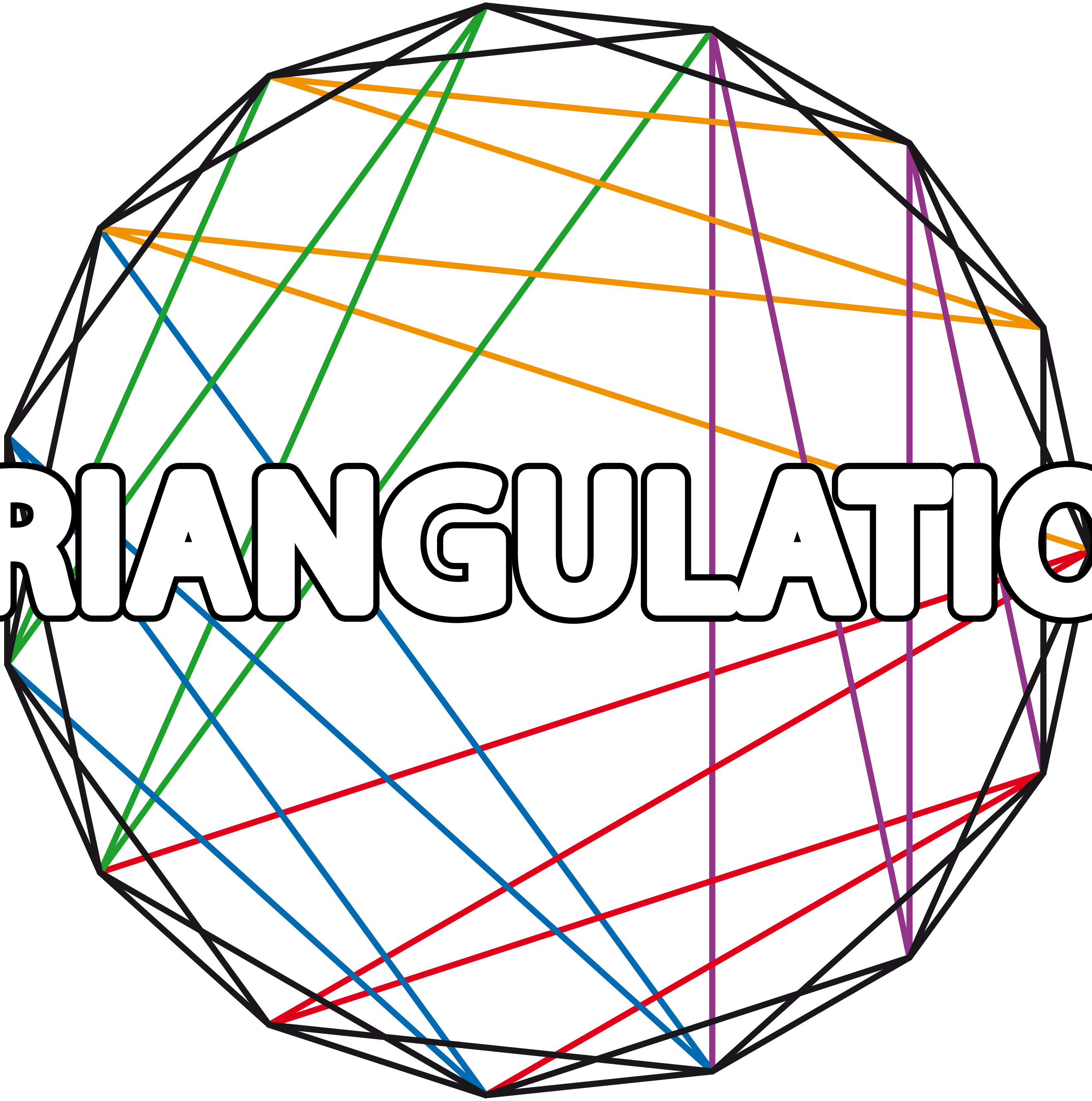
Oberwolfach workshop on Friezes, November 2015



Vincent PILAUD

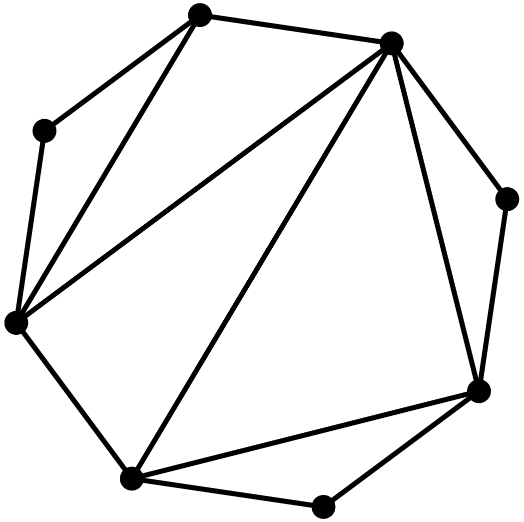
(CNRS & LIX, École Polytechnique)

...TRIANGULATIONS

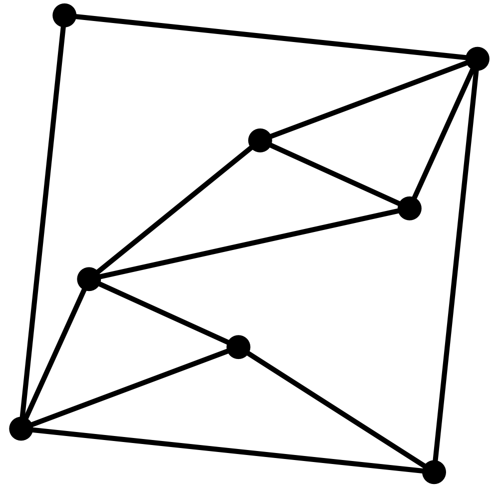


THREE GEOMETRIC FAMILIES

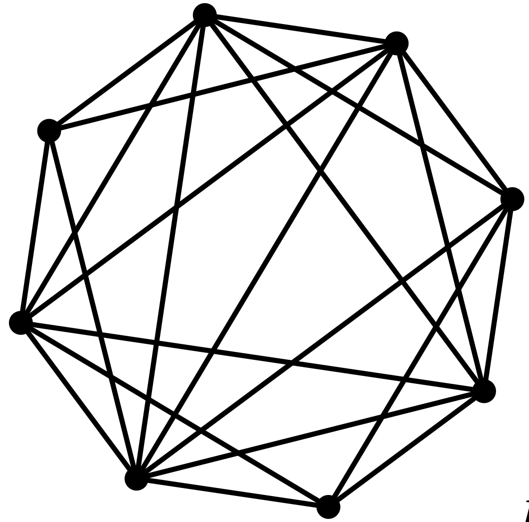
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

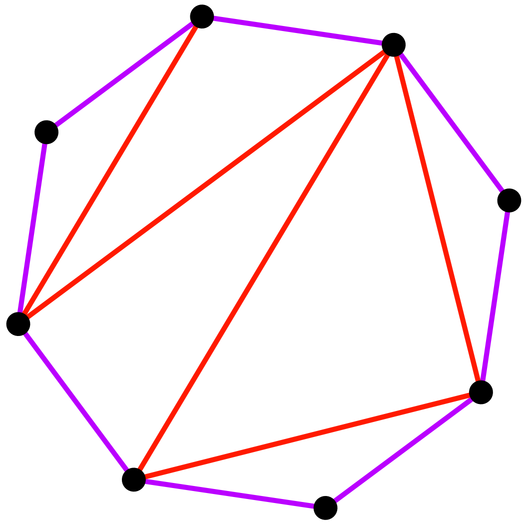
triangulation $=$ maximal crossing-free set of edges

pseudotriangulation $=$ maximal crossing-free pointed set of edges

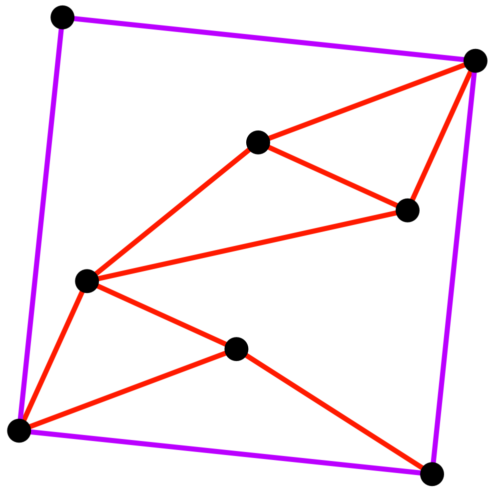
k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges

RELEVANT EDGES

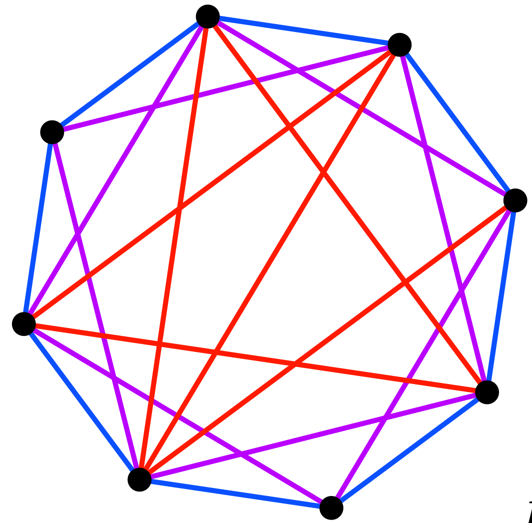
Triangulations



Pseudotriangulations



Multitriangulations



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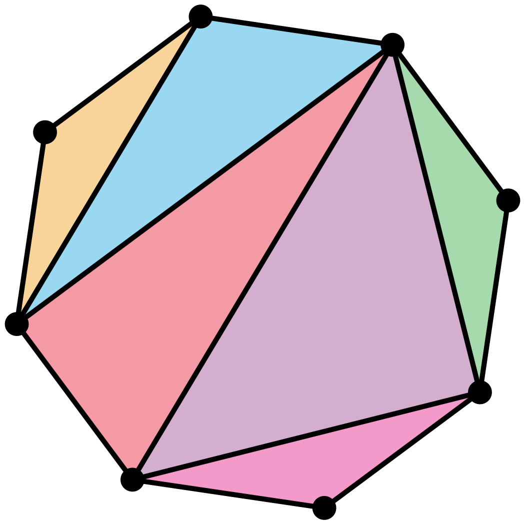
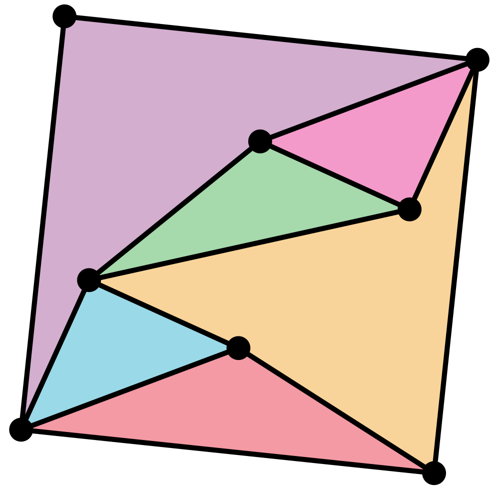
TRIANGLES – PSEUDOTRIANGLES – STARS

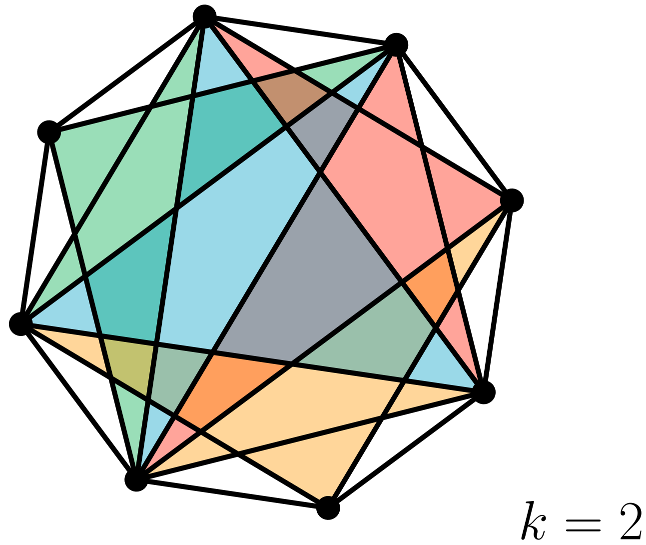
Triangulations



Pseudotriangulations



Multitriangulations

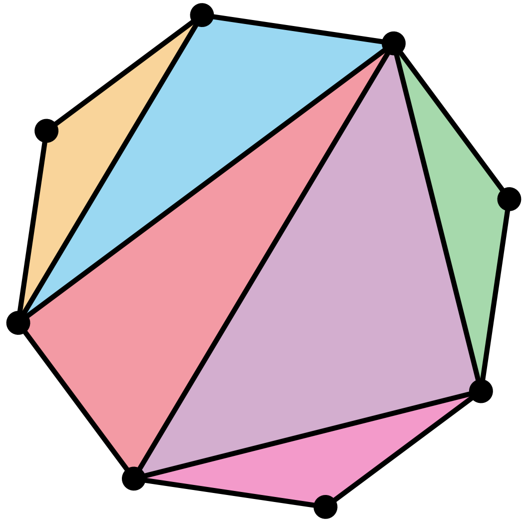


triangulation	= maximal crossing-free set of edges
	= decomposition into triangles
pseudotriangulation	= maximal crossing-free pointed set of edges
	= decomposition into pseudotriangles
k-triangulation	= maximal $(k+1)$-crossing-free set of edges
	= decomposition into k-stars

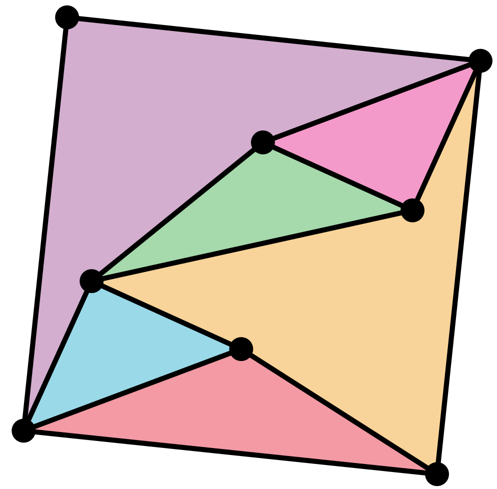
VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

TRIANGLES – PSEUDOTRIANGLES – STARS

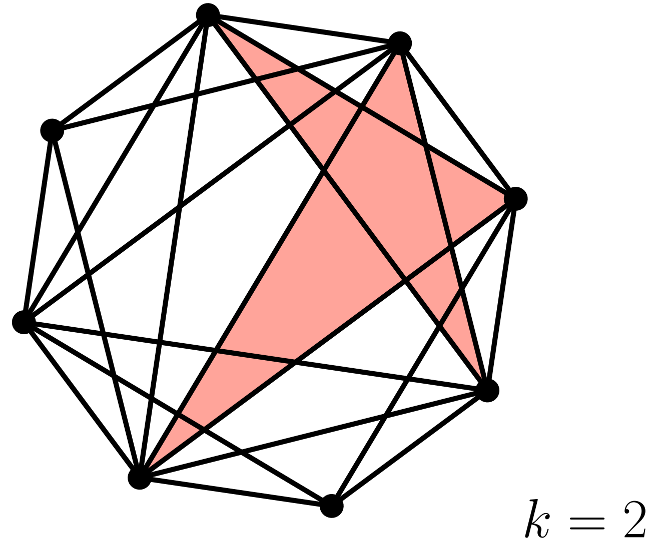
Triangulations



Pseudotriangulations



Multitriangulations



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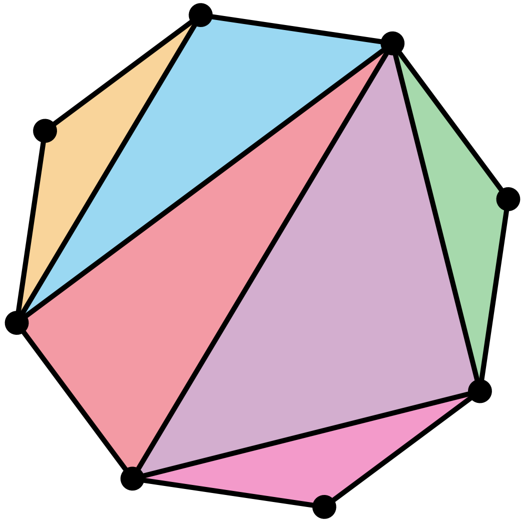
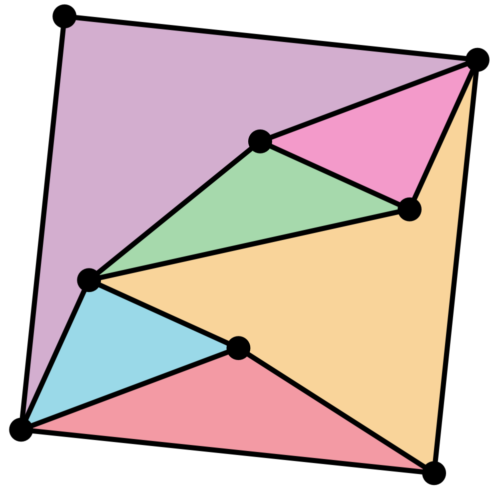
VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

TRIANGLES – PSEUDOTRIANGLES – STARS

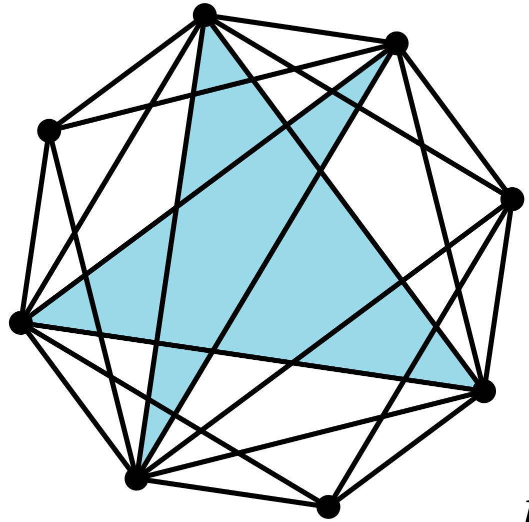
Triangulations



Pseudotriangulations



Multitriangulations



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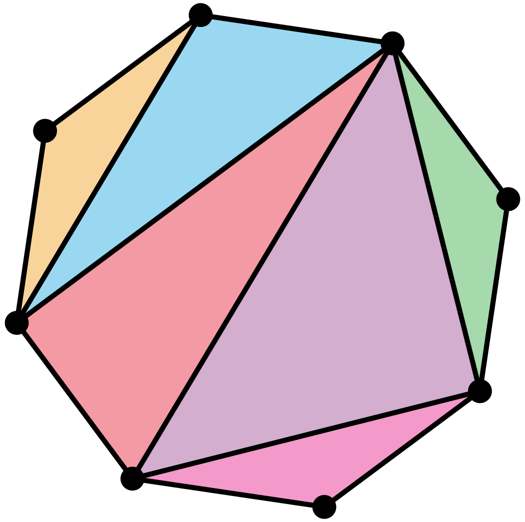
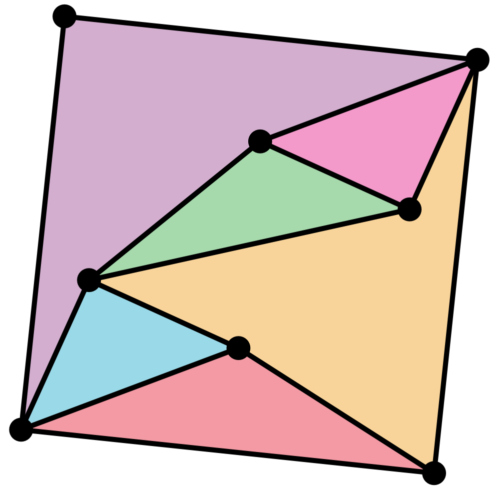
VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

TRIANGLES – PSEUDOTRIANGLES – STARS

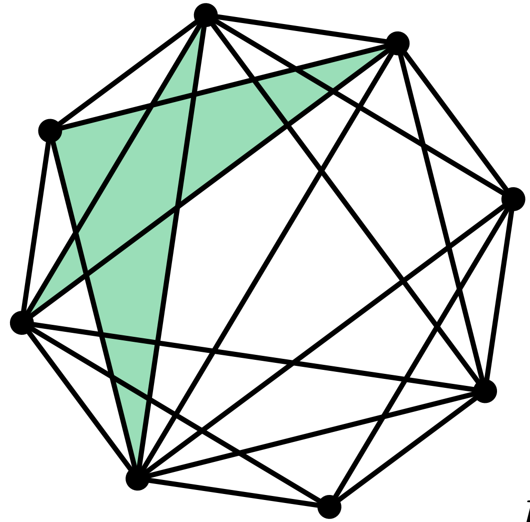
Triangulations



Pseudotriangulations



Multitriangulations



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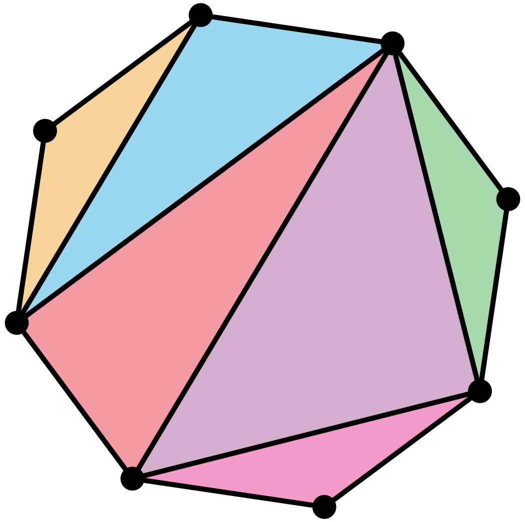
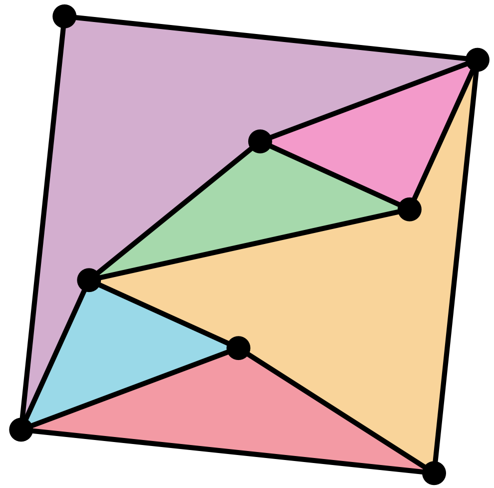
TRIANGLES – PSEUDOTRIANGLES – STARS

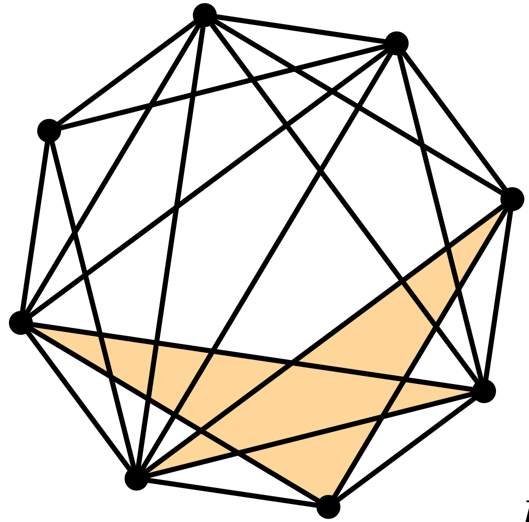
Triangulations



Pseudotriangulations



Multitriangulations



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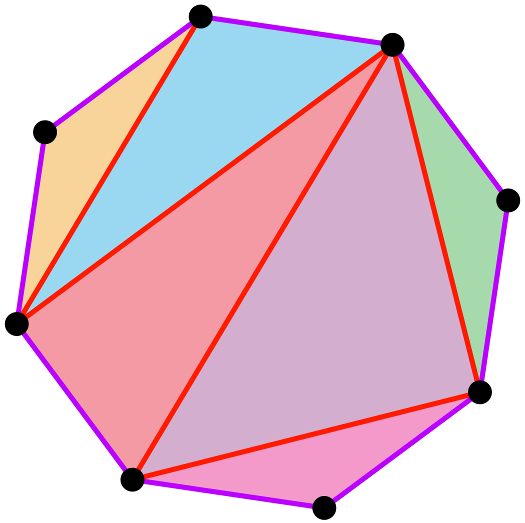
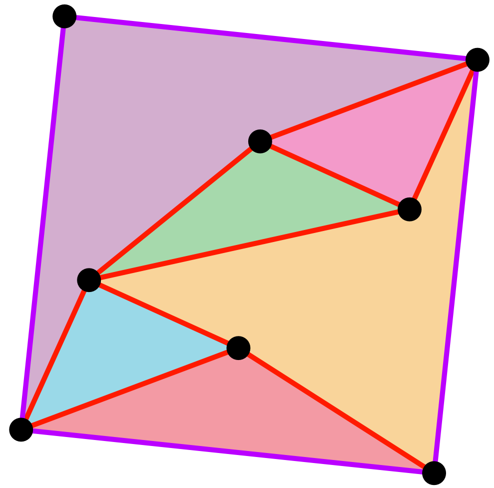
TRIANGLES – PSEUDOTRIANGLES – STARS

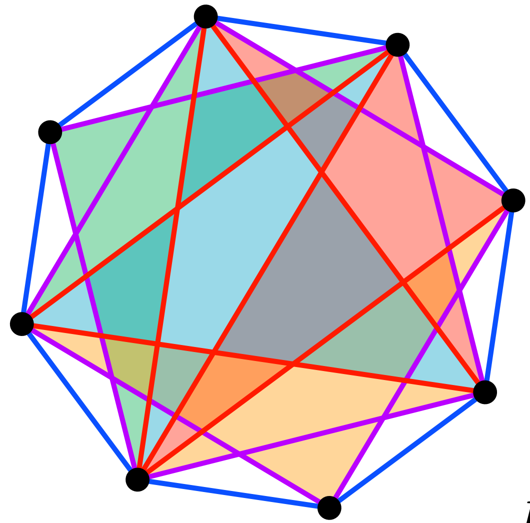
Triangulations



Pseudotriangulations



Multitriangulations



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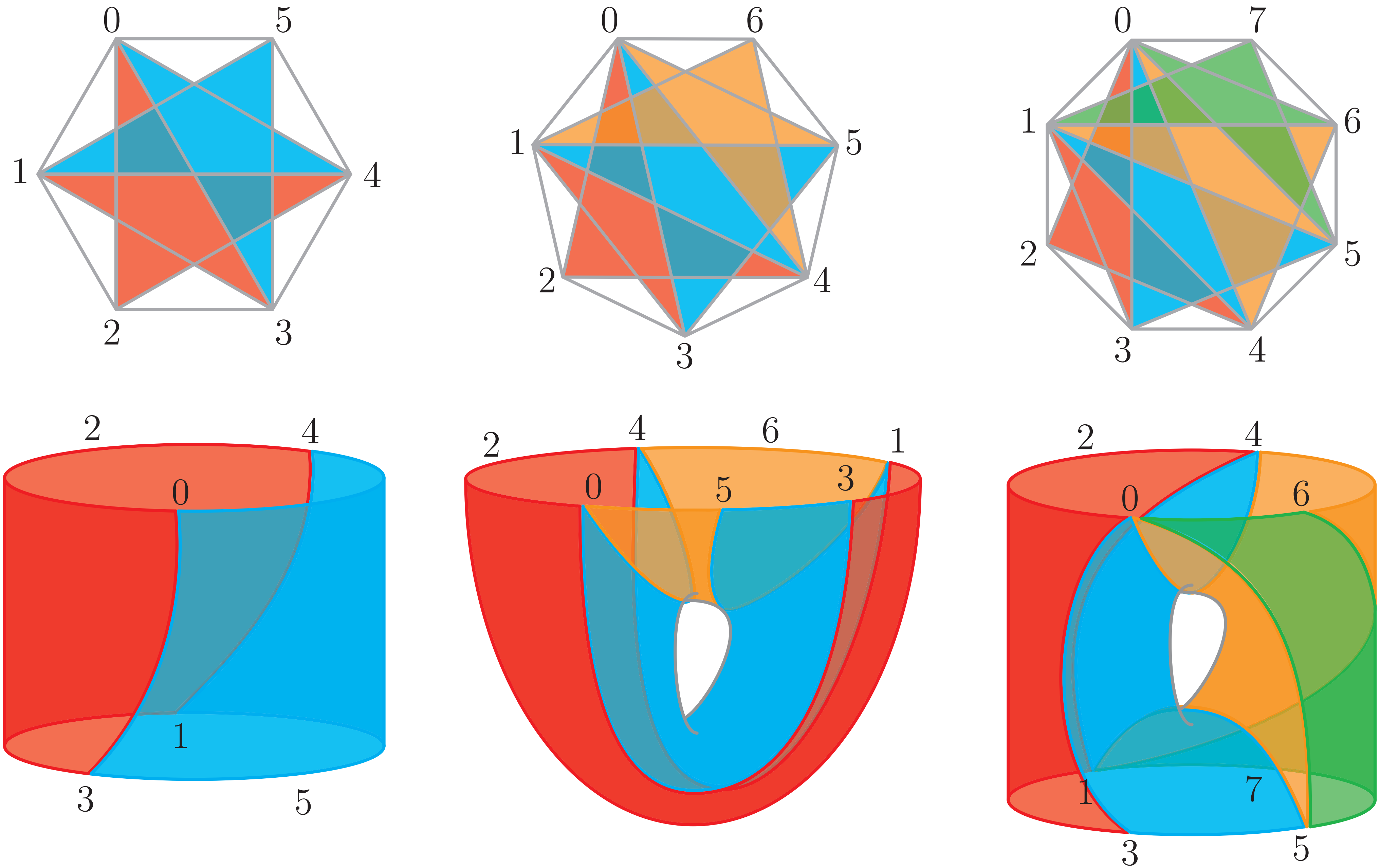
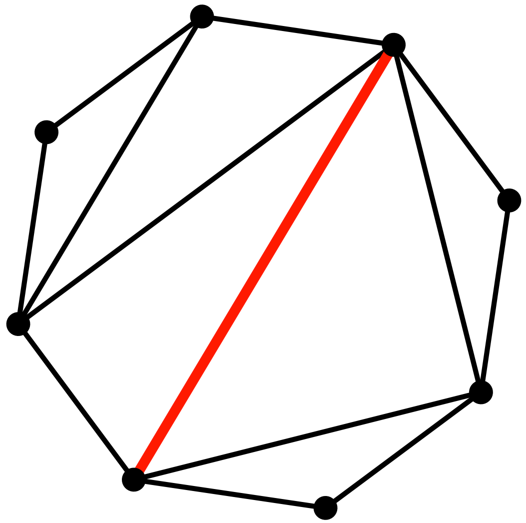
VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

DECOMPOSITIONS OF SURFACES

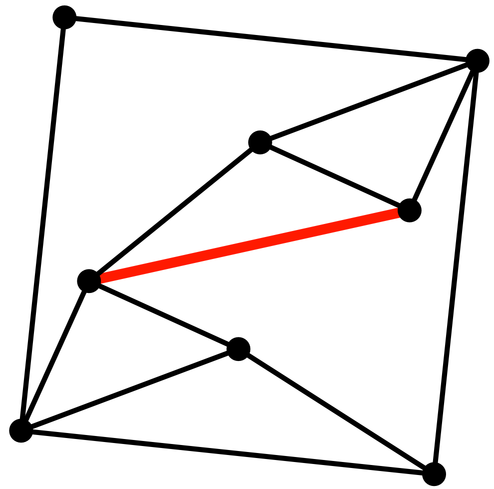


FLIPS

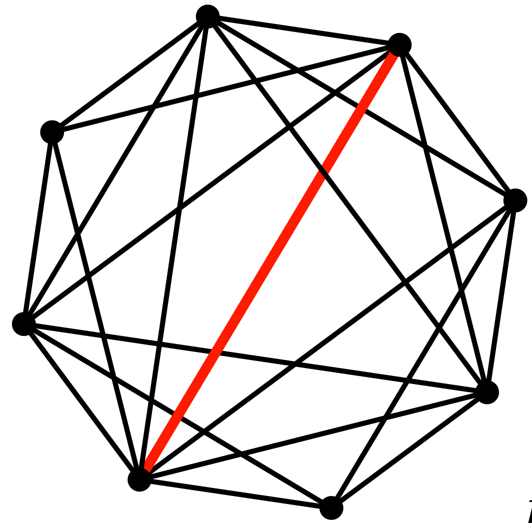
Triangulations



Pseudotriangulations



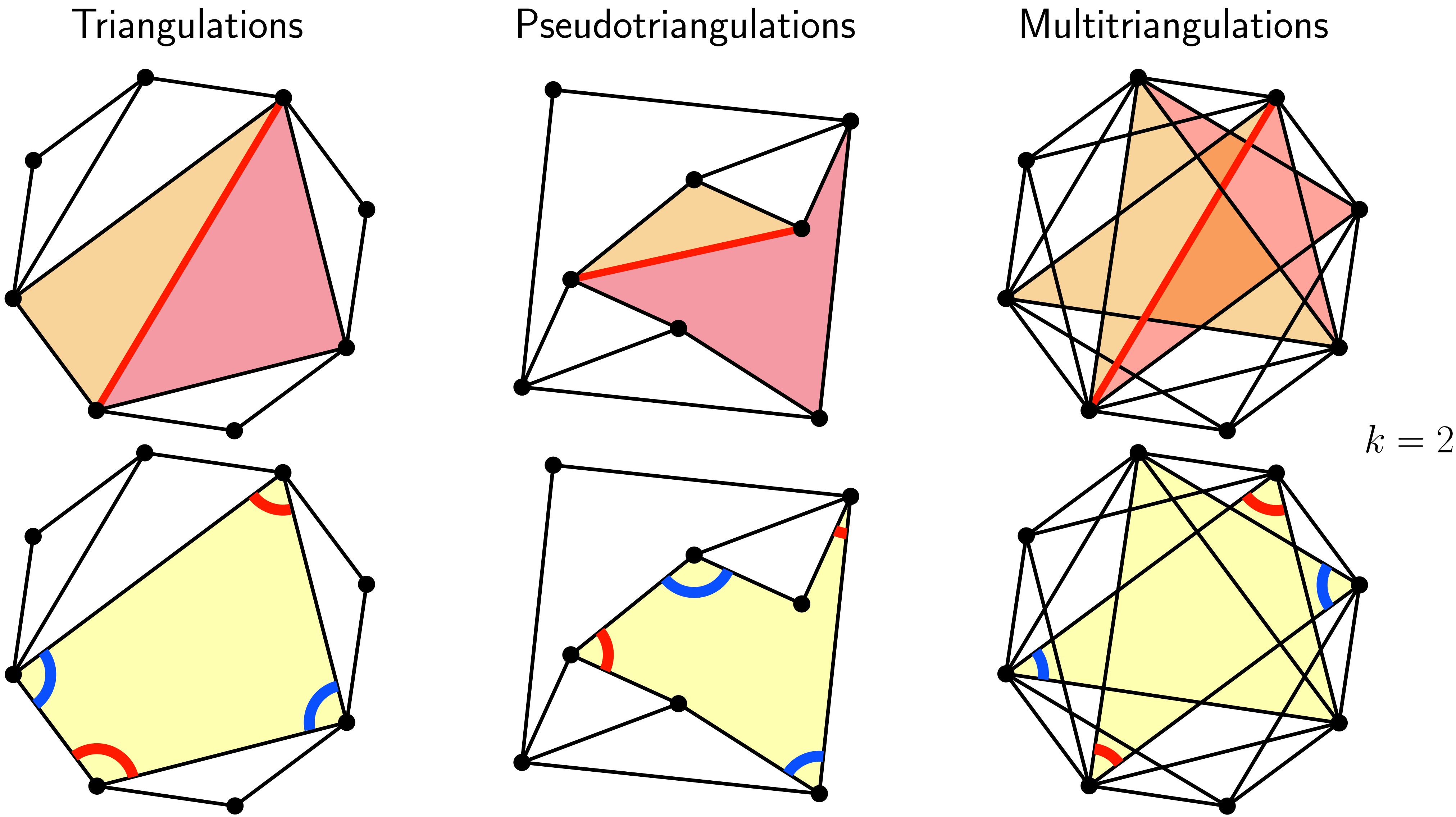
Multitriangulations



$k = 2$

flip = exchange an internal edge with the common bisector of the two adjacent cells

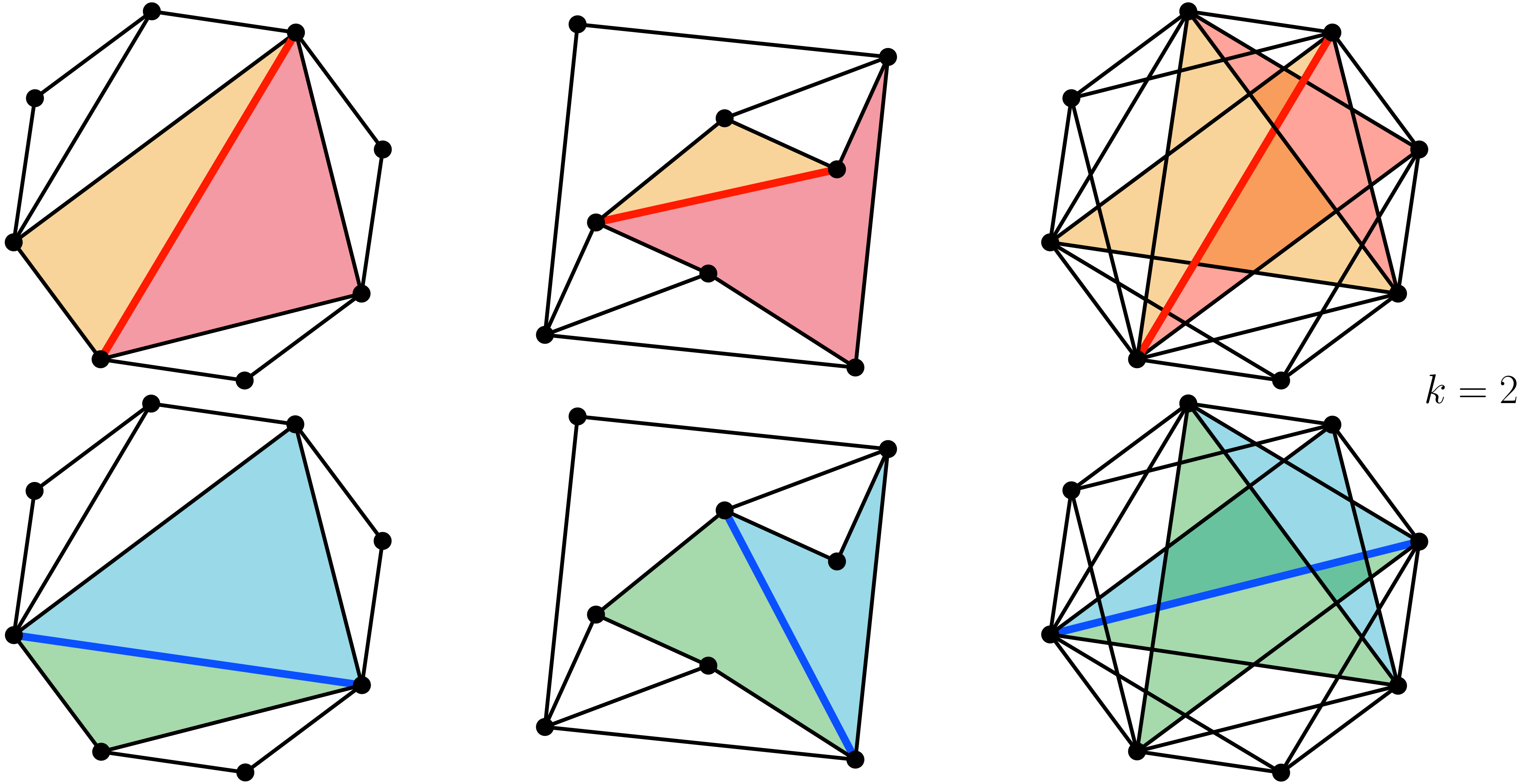
FLIPS



flip = exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

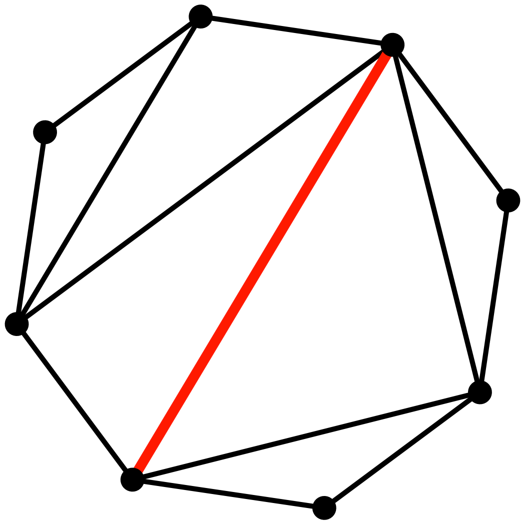
Triangulations — Pseudotriangulations — Multitriangulations



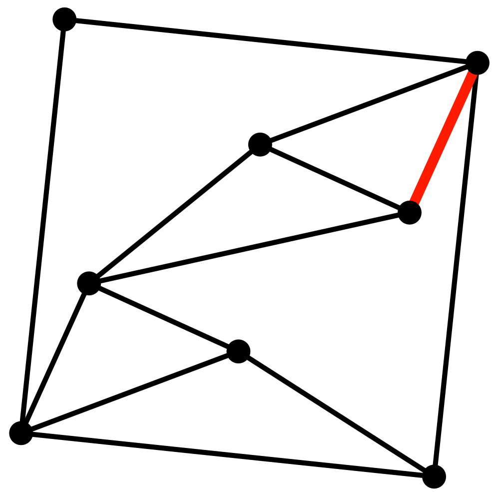
flip = exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

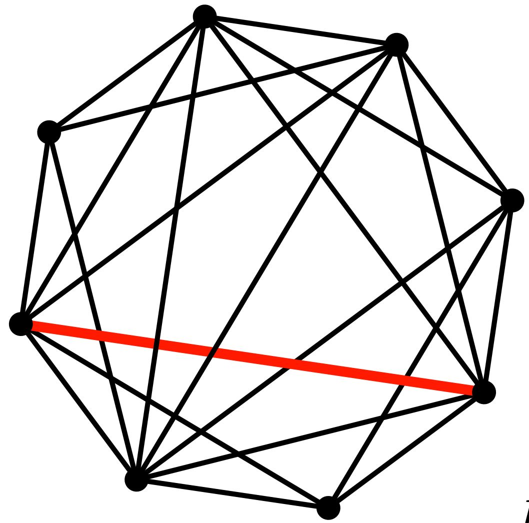
Triangulations



Pseudotriangulations



Multitriangulations

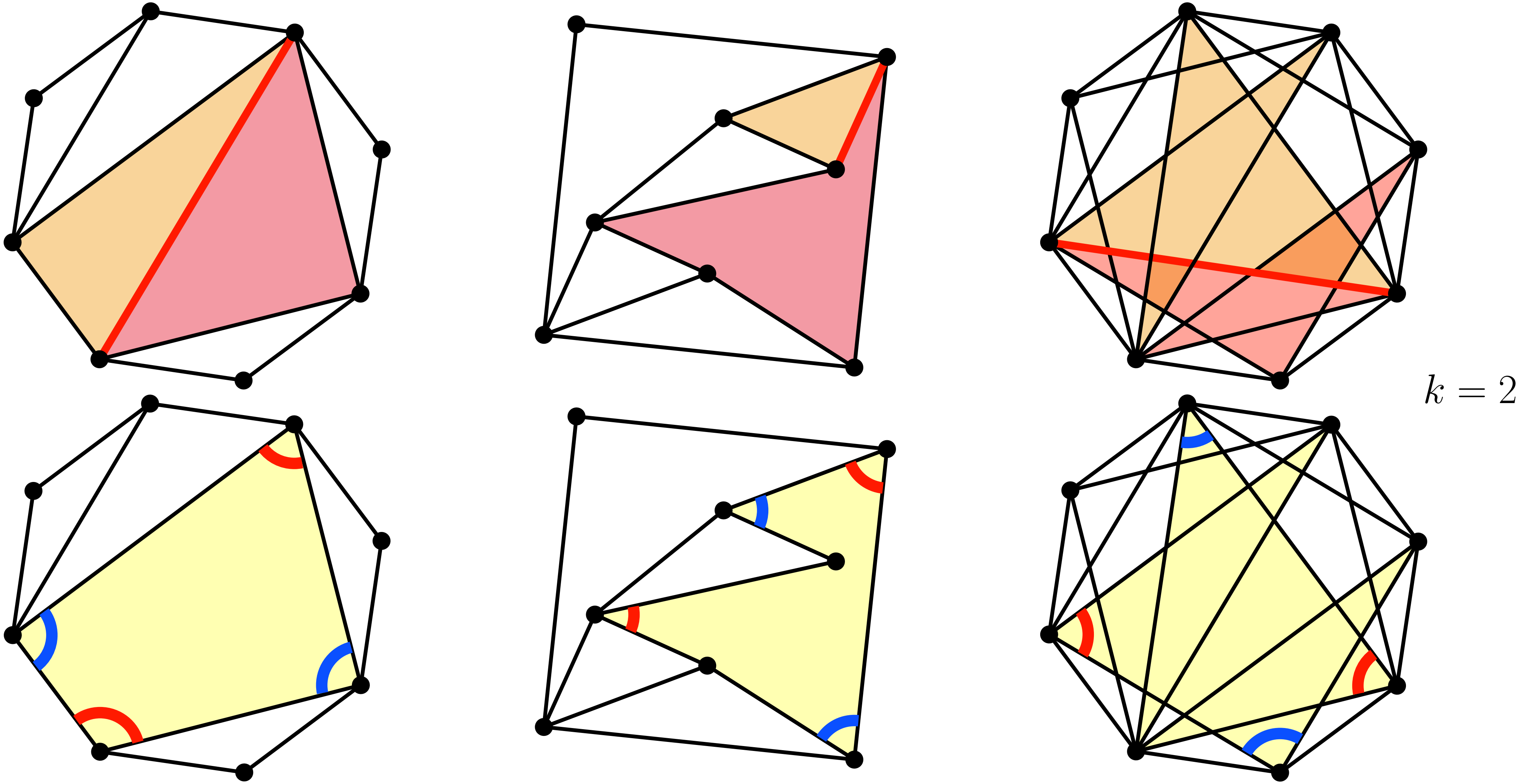


$k = 2$

flip = exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

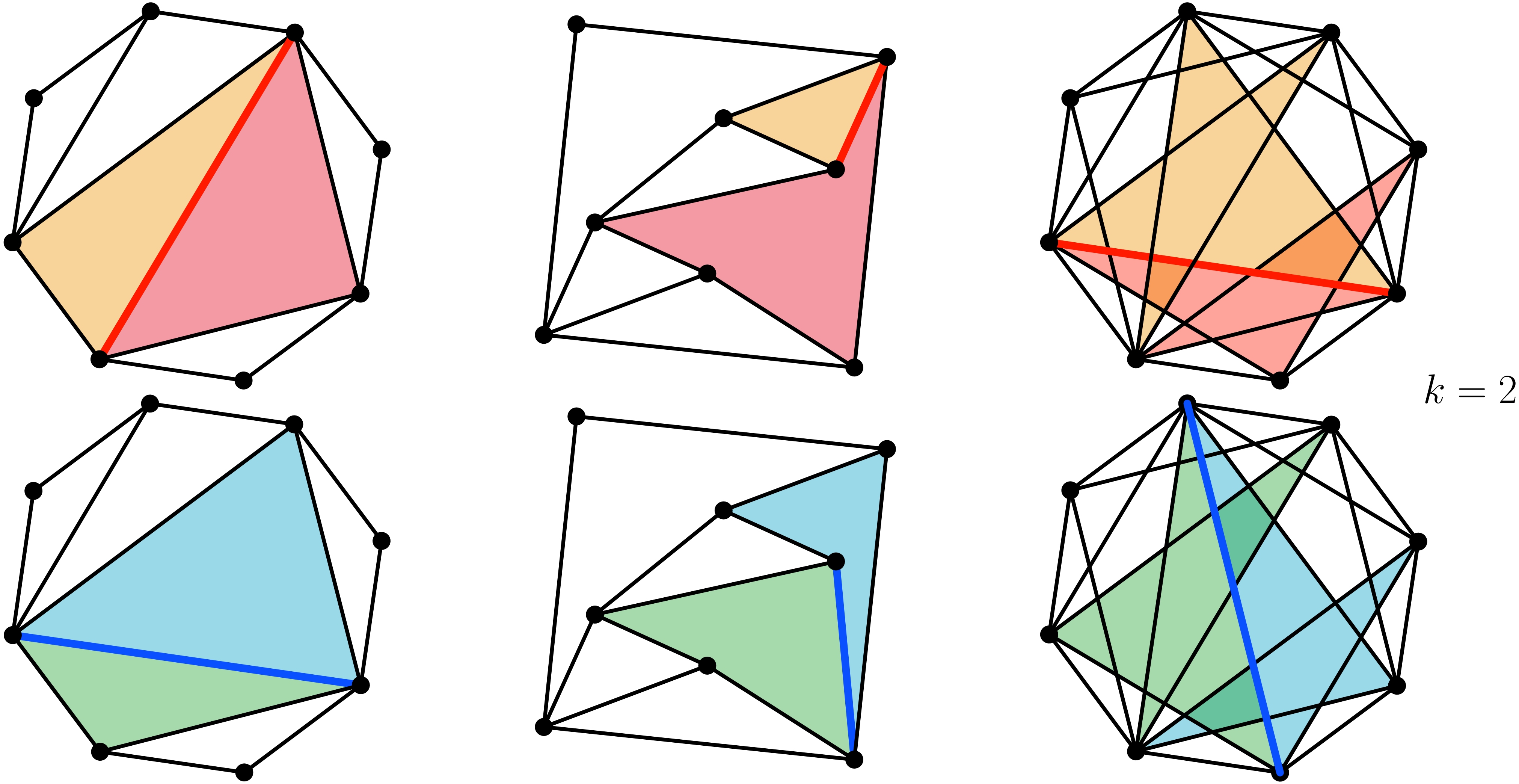
Triangulations | Pseudotriangulations | Multitriangulations



flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

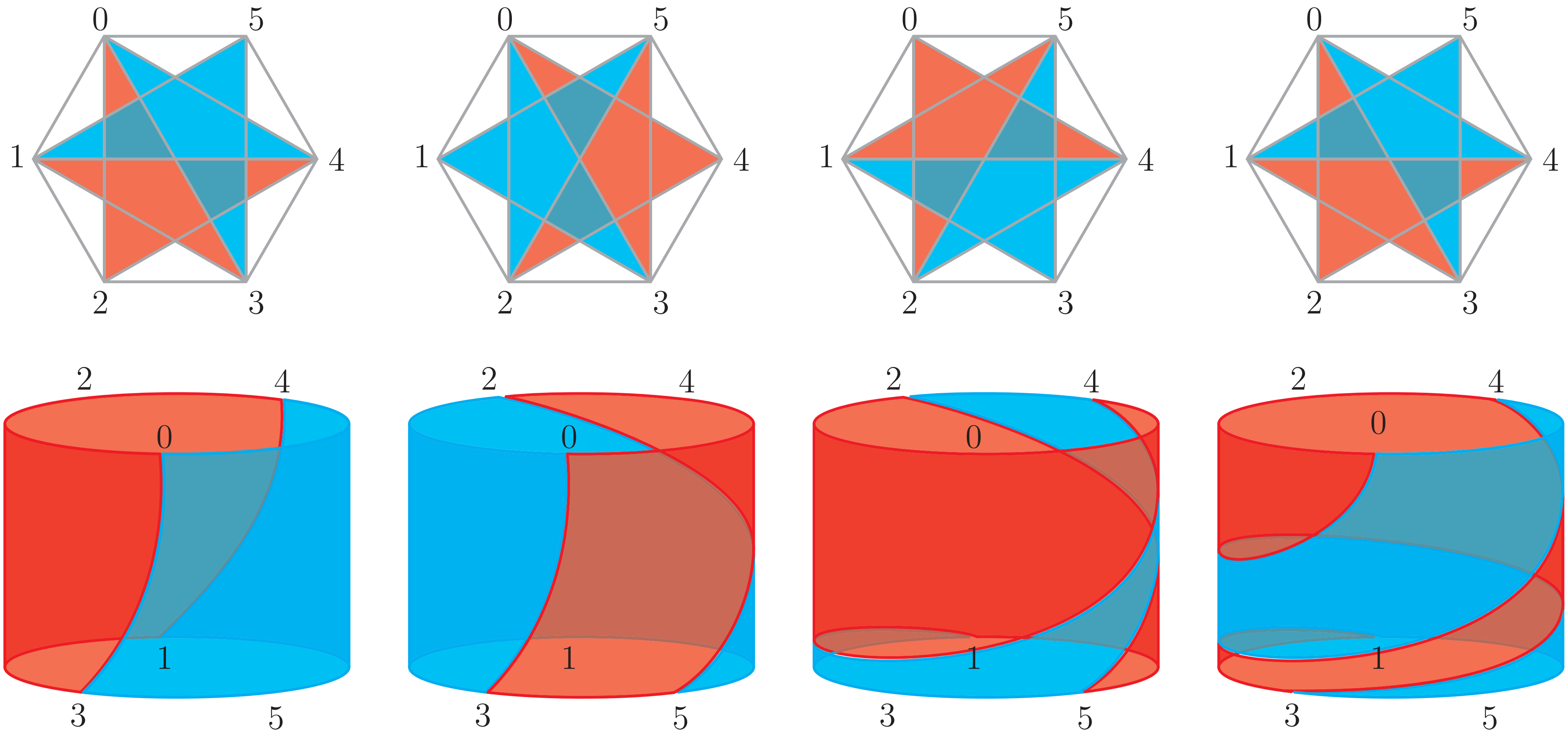
FLIPS

Triangulations | Pseudotriangulations | Multitriangulations



flip = exchange an internal edge with the common bisector of the two adjacent cells

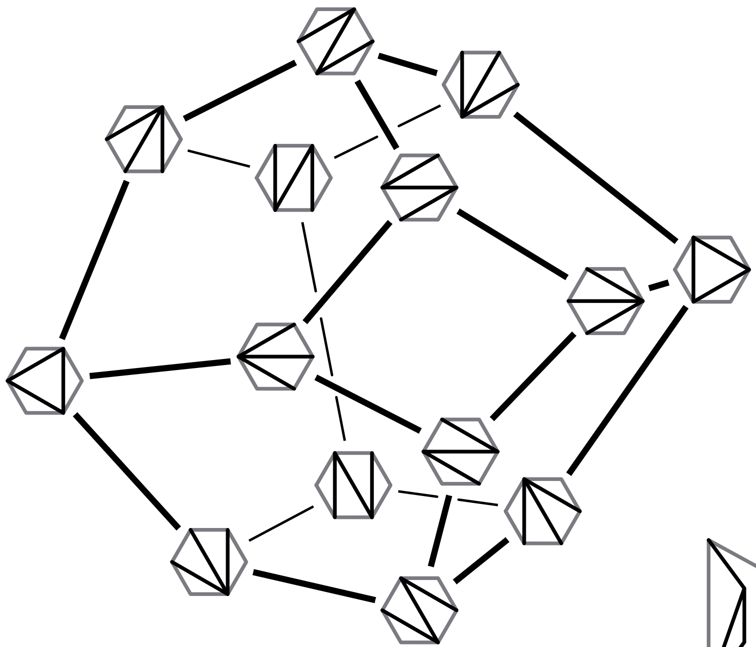
FLIPS ON SURFACES



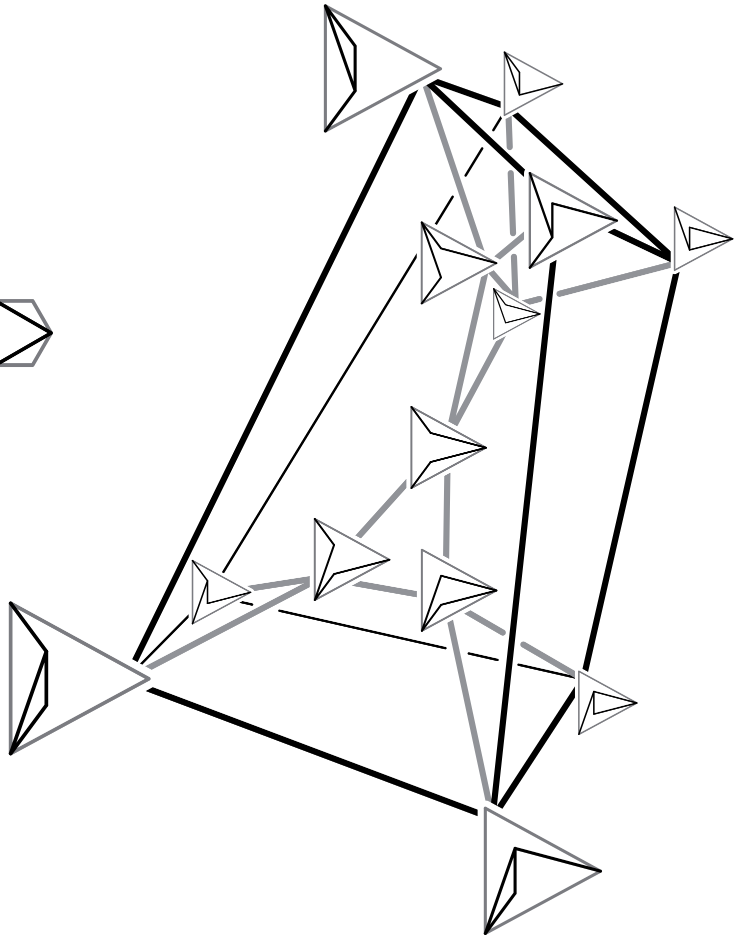
fundamental group of the flip graph $G_{n,k} \longmapsto$ mapping class group of the surface $\mathcal{S}_{n,k}$

THREE GEOMETRIC STRUCTURES

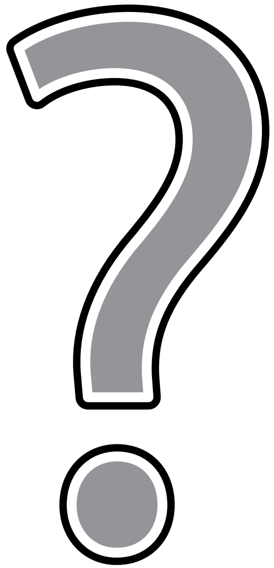
Triangulations



Pseudotriangulations



Multitriangulations

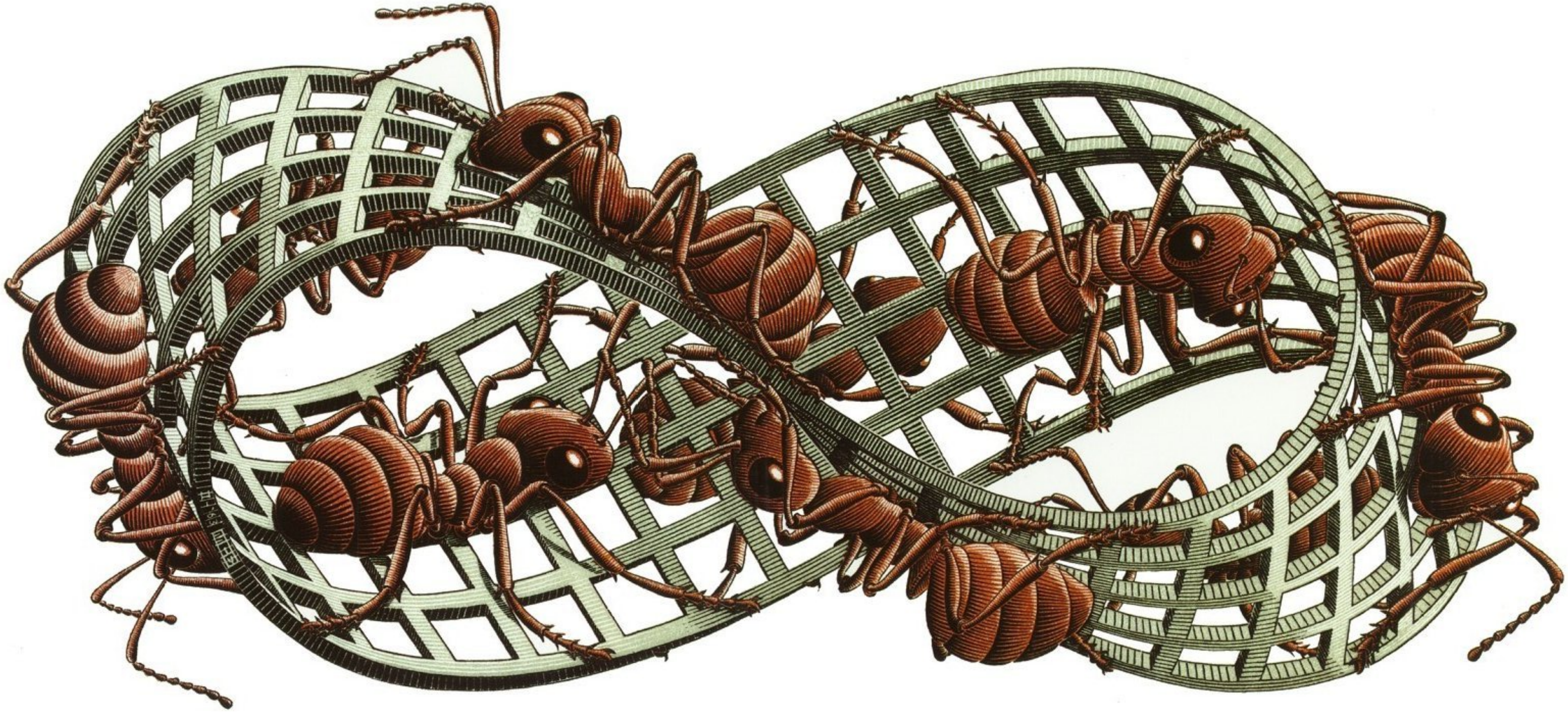


associahedron	$\longleftrightarrow$	crossing-free sets of internal edges
pseudotriangulations polytope	$\longleftrightarrow$	pointed crossing-free sets of internal edges
multiassociahedron	$\longleftrightarrow$	$(k+1)$-crossing-free sets of k-internal edges

DUALITY

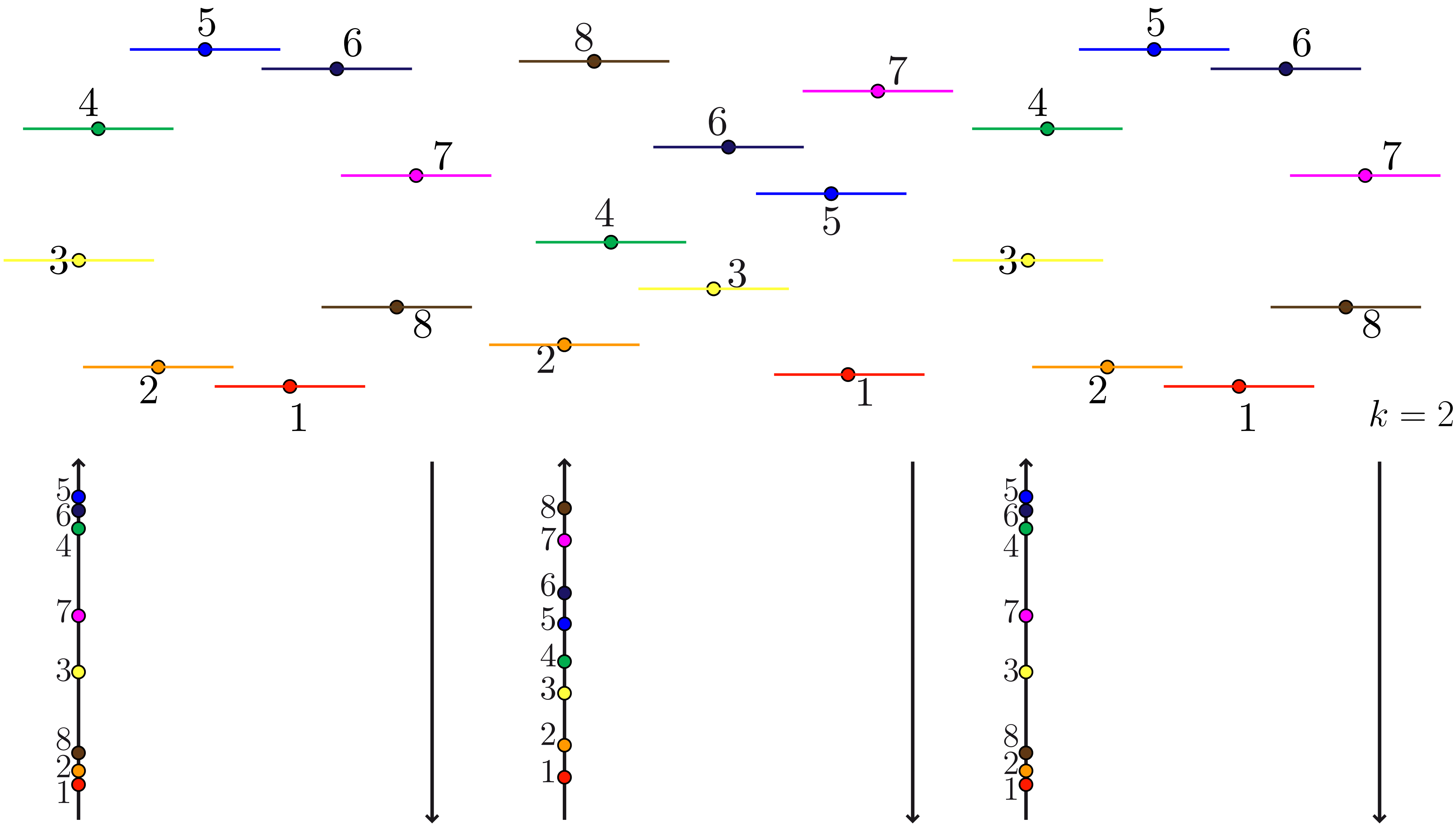
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

LINE SPACE OF THE PLANE = MÖBIUS STRIP



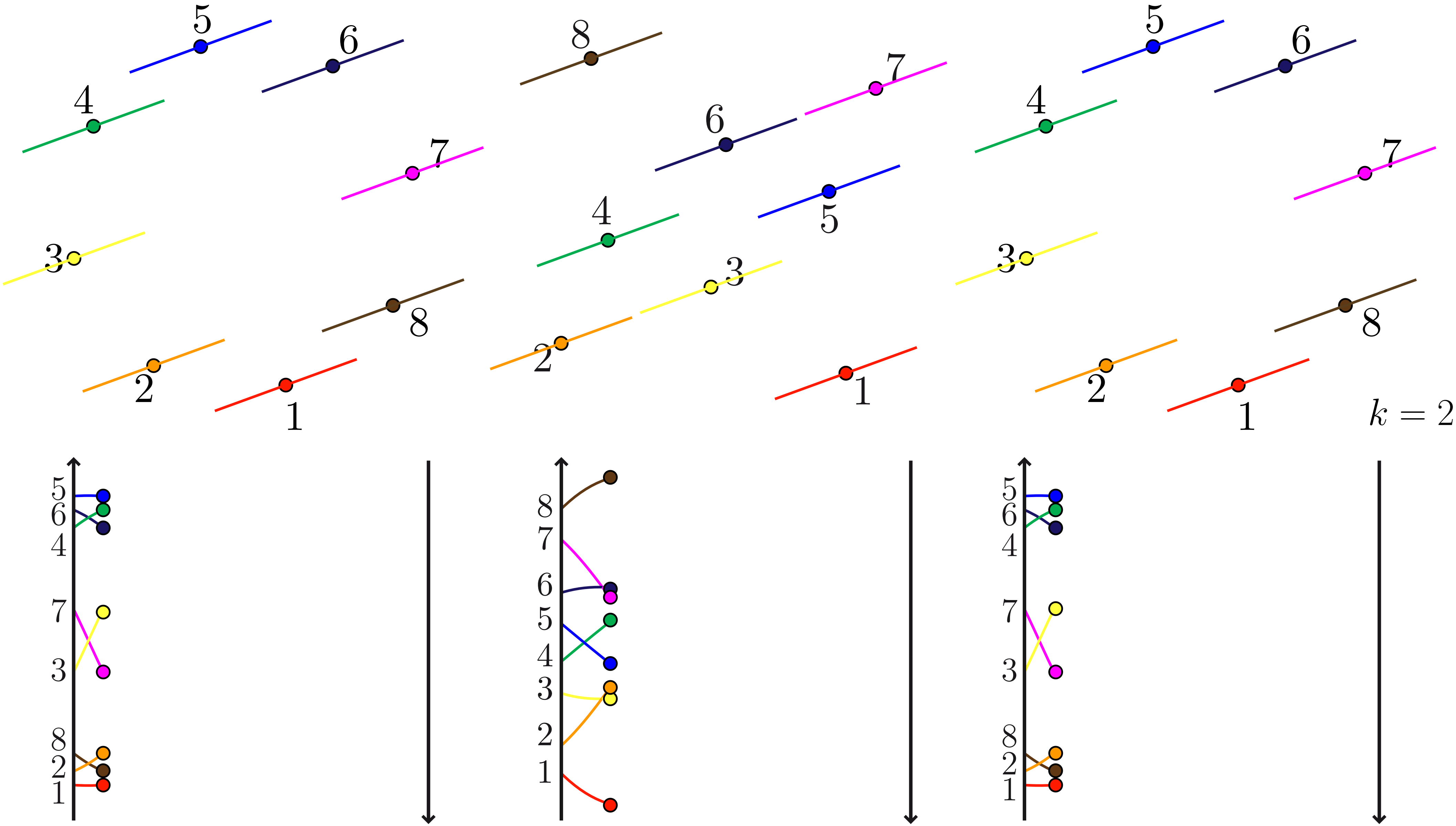
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



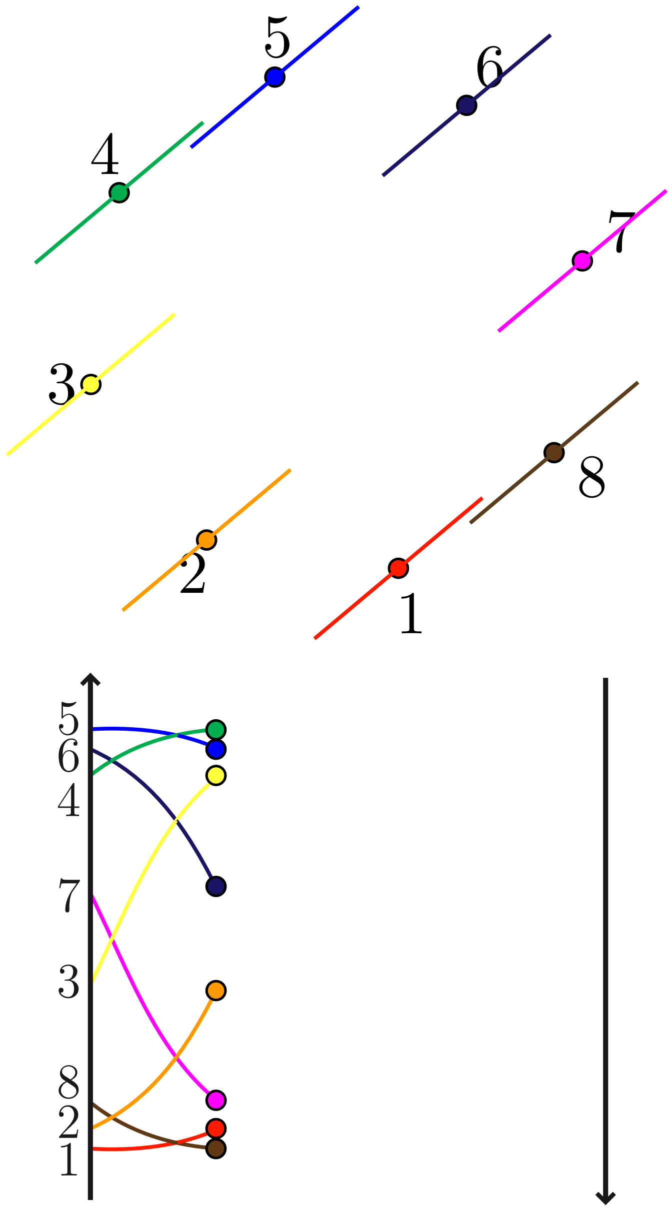
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations

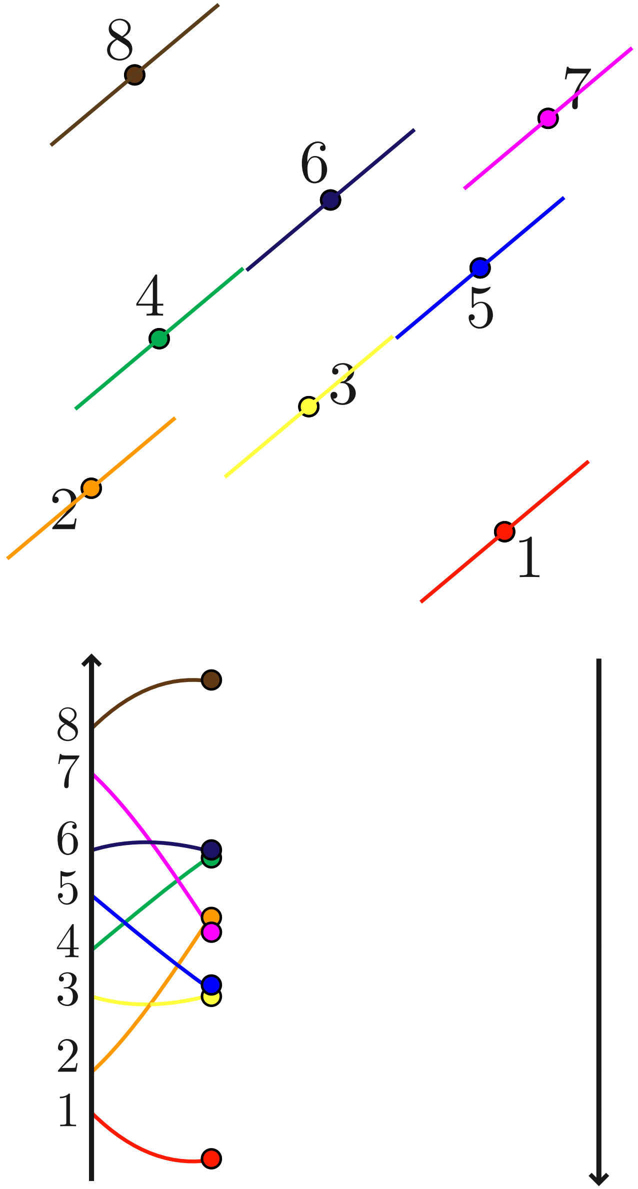


DUALITY

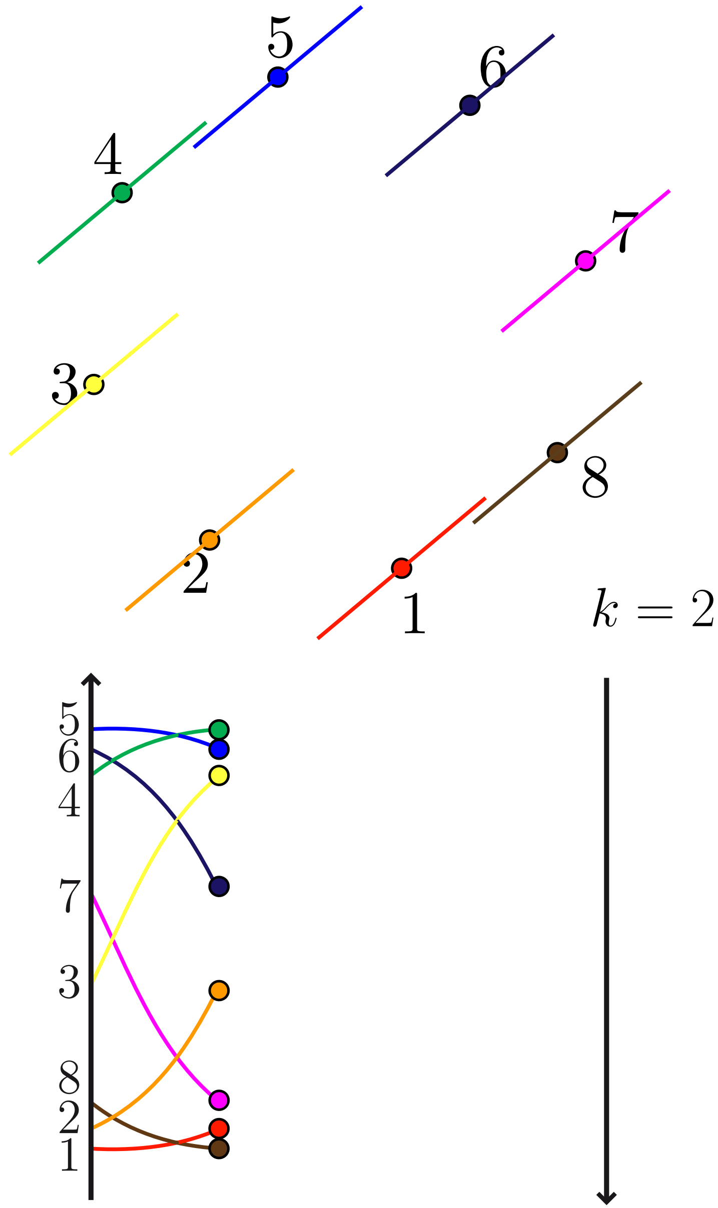
Triangulations



Pseudotriangulations

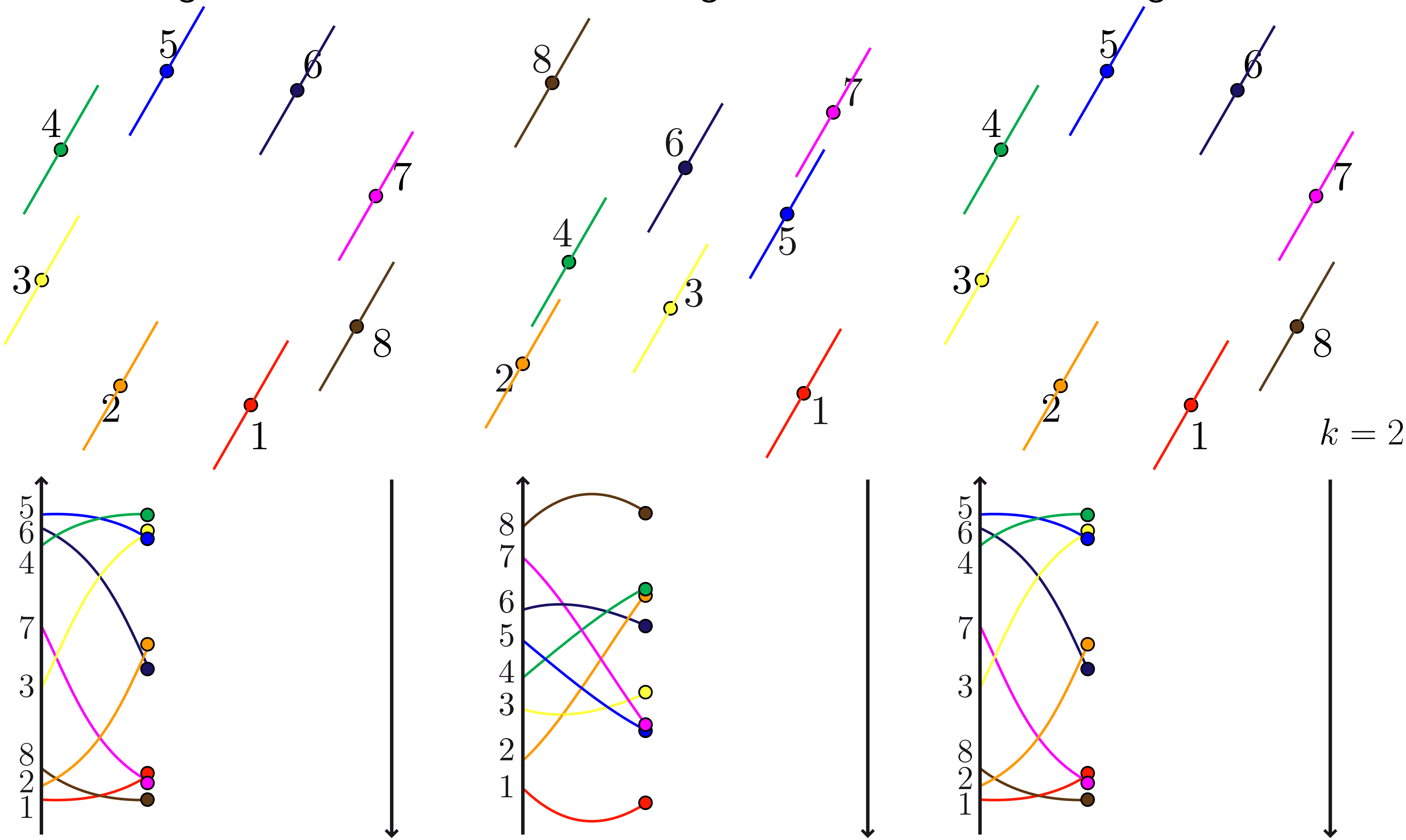


Multitriangulations



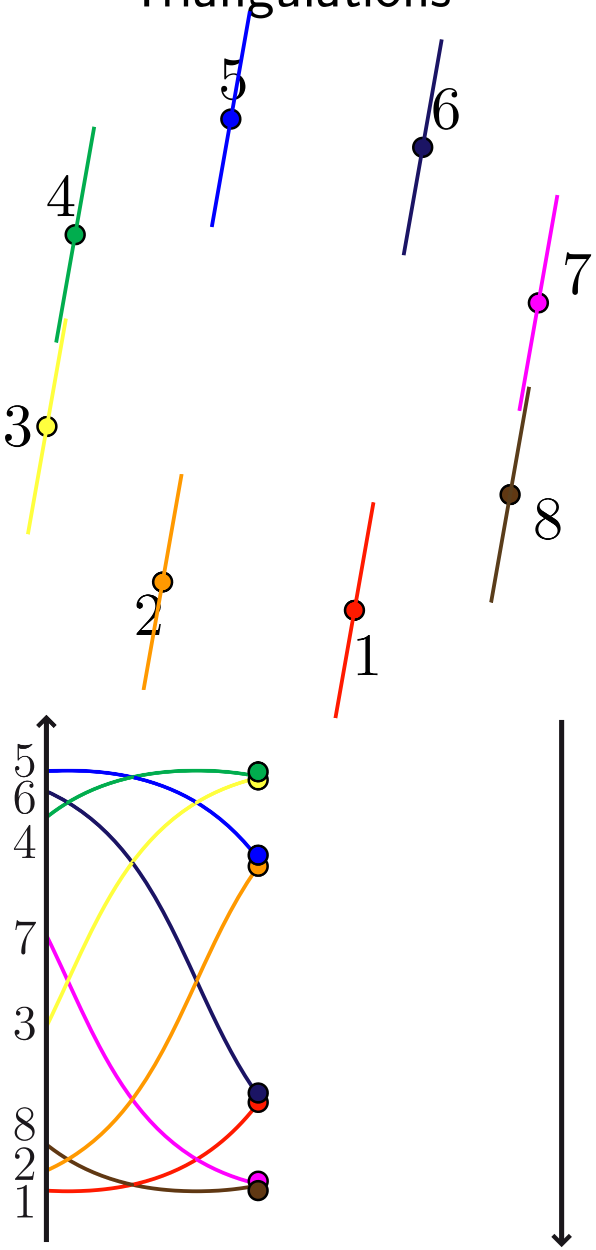
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations

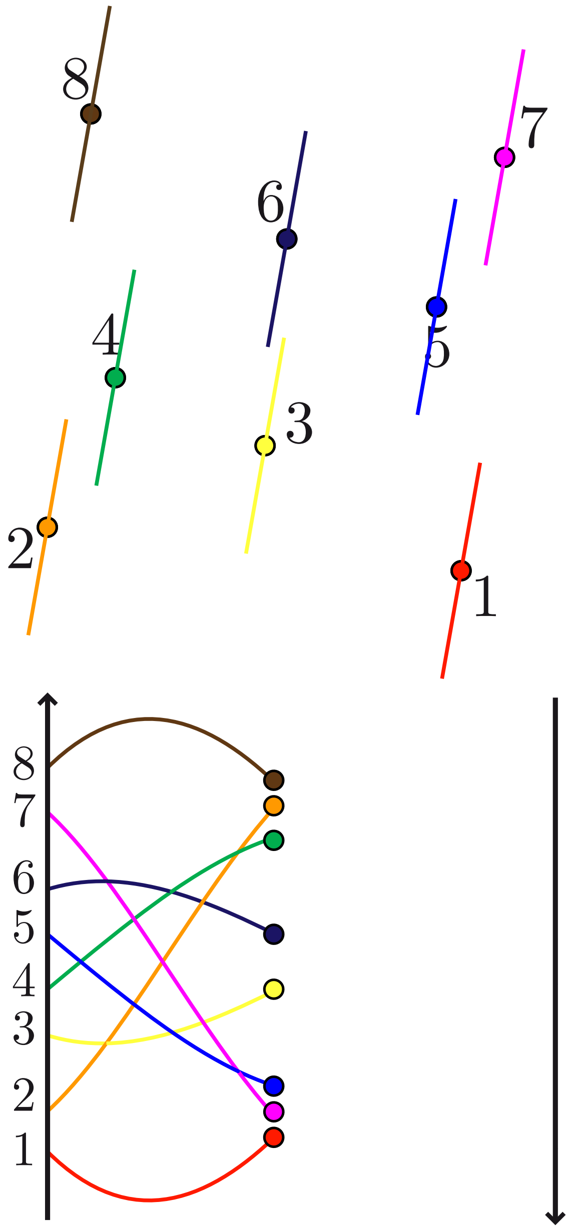


DUALITY

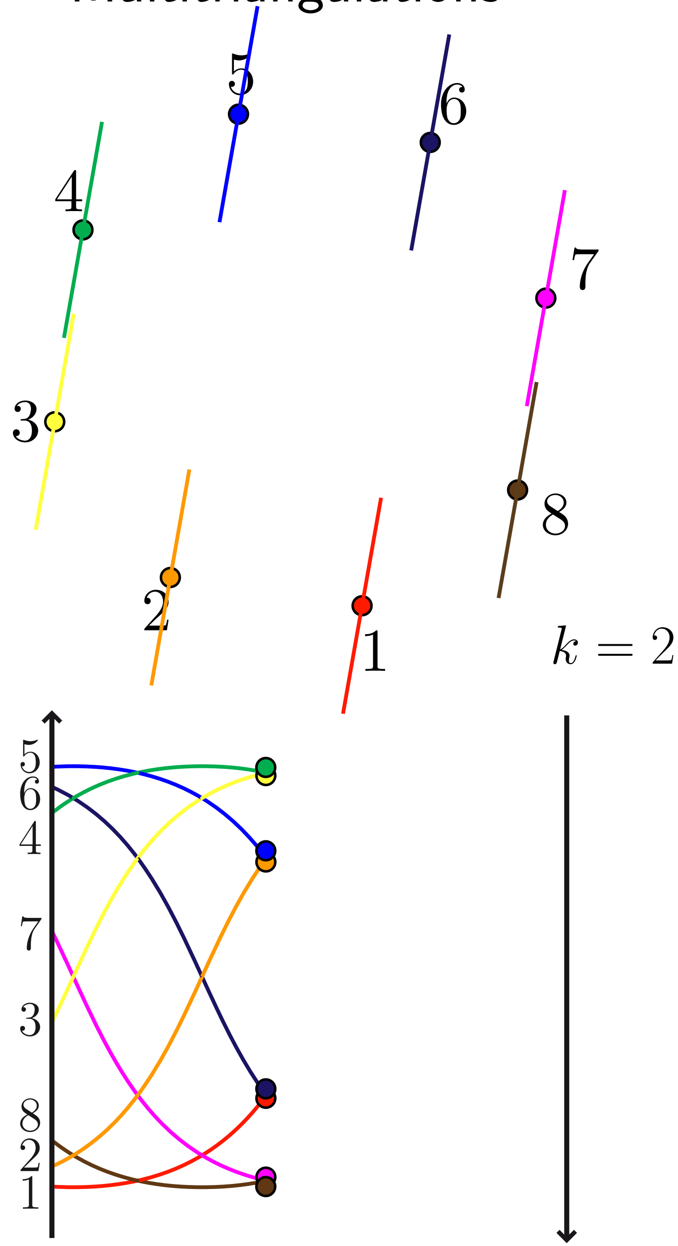
Triangulations



Pseudotriangulations

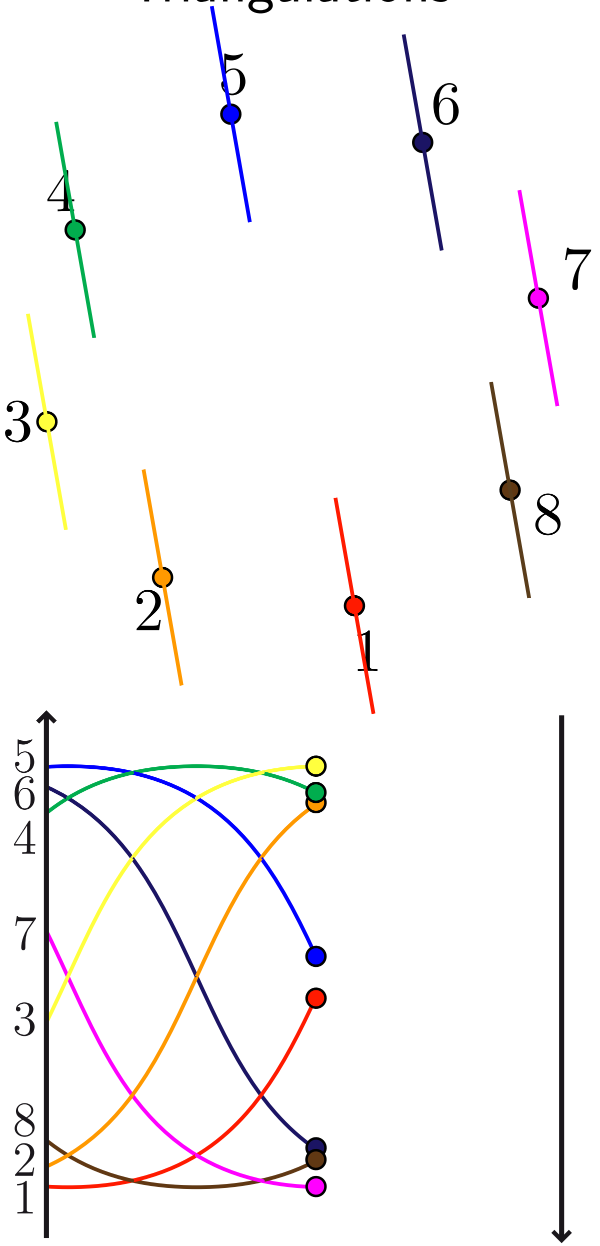


Multitriangulations

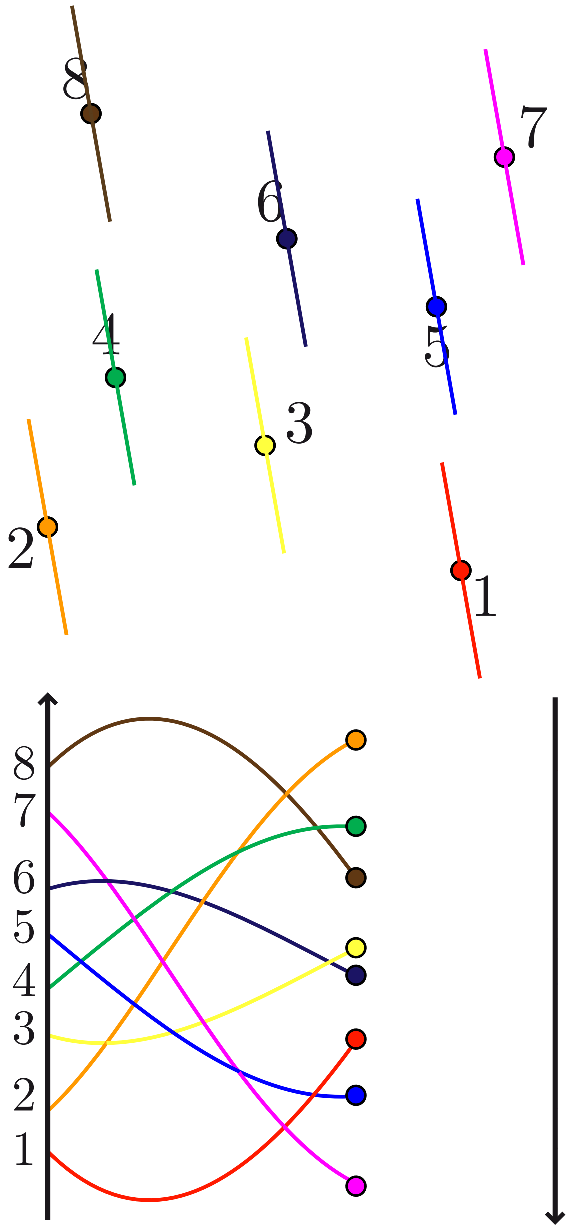


DUALITY

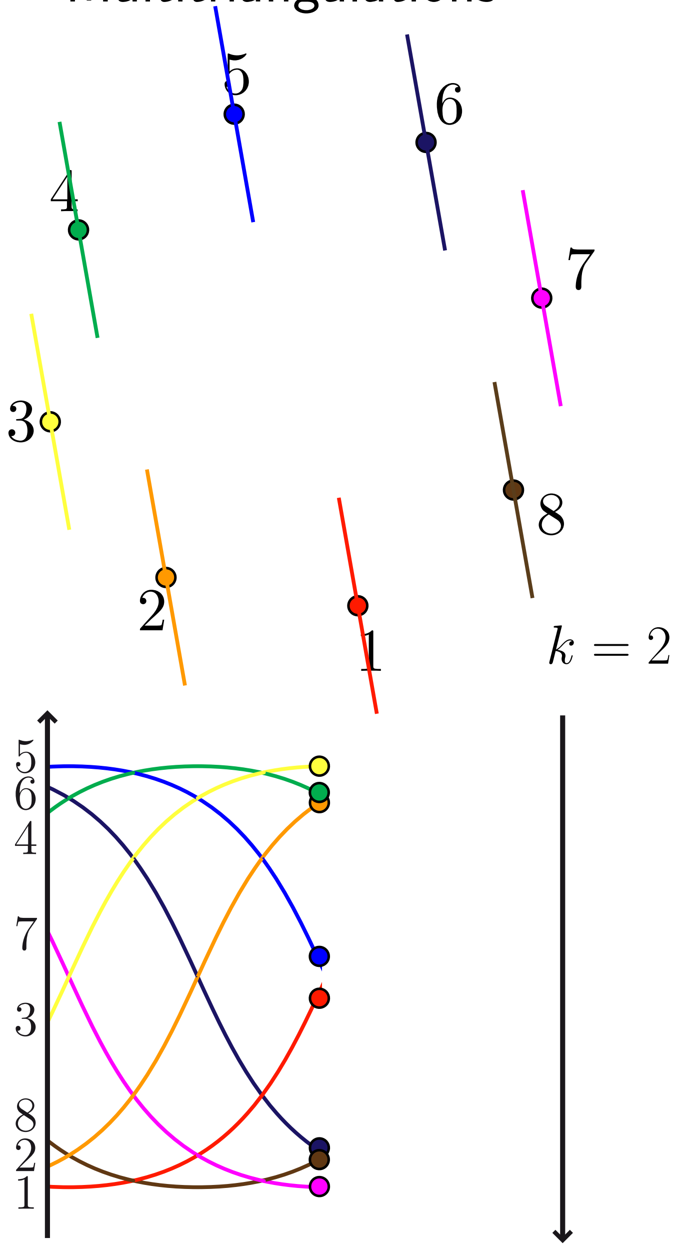
Triangulations



Pseudotriangulations

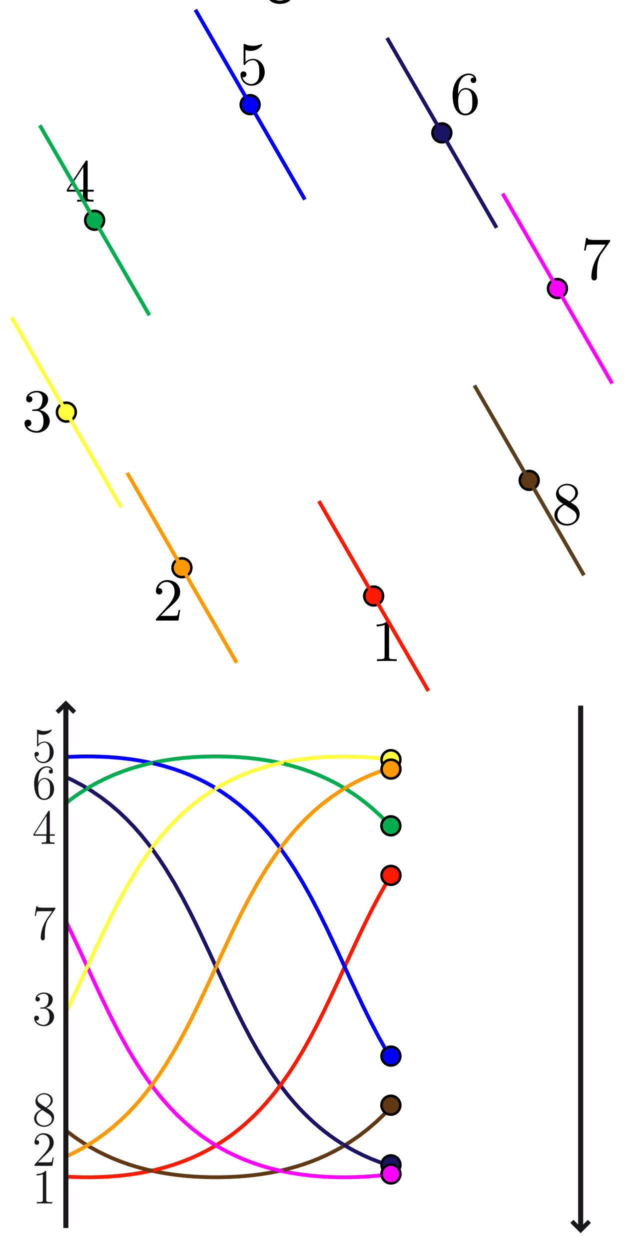


Multitriangulations

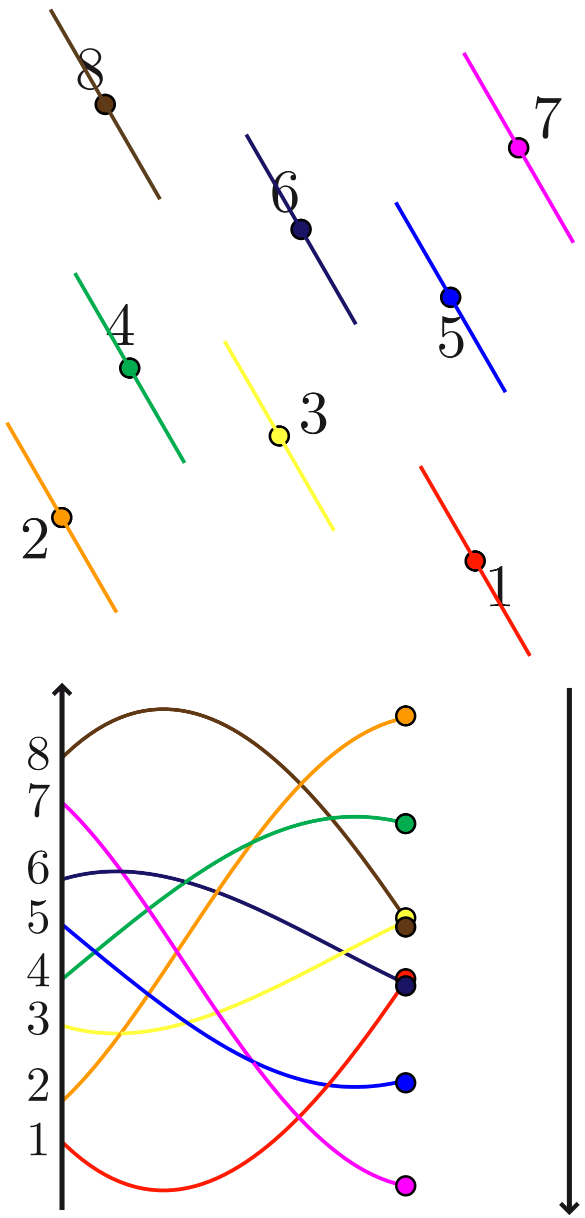


DUALITY

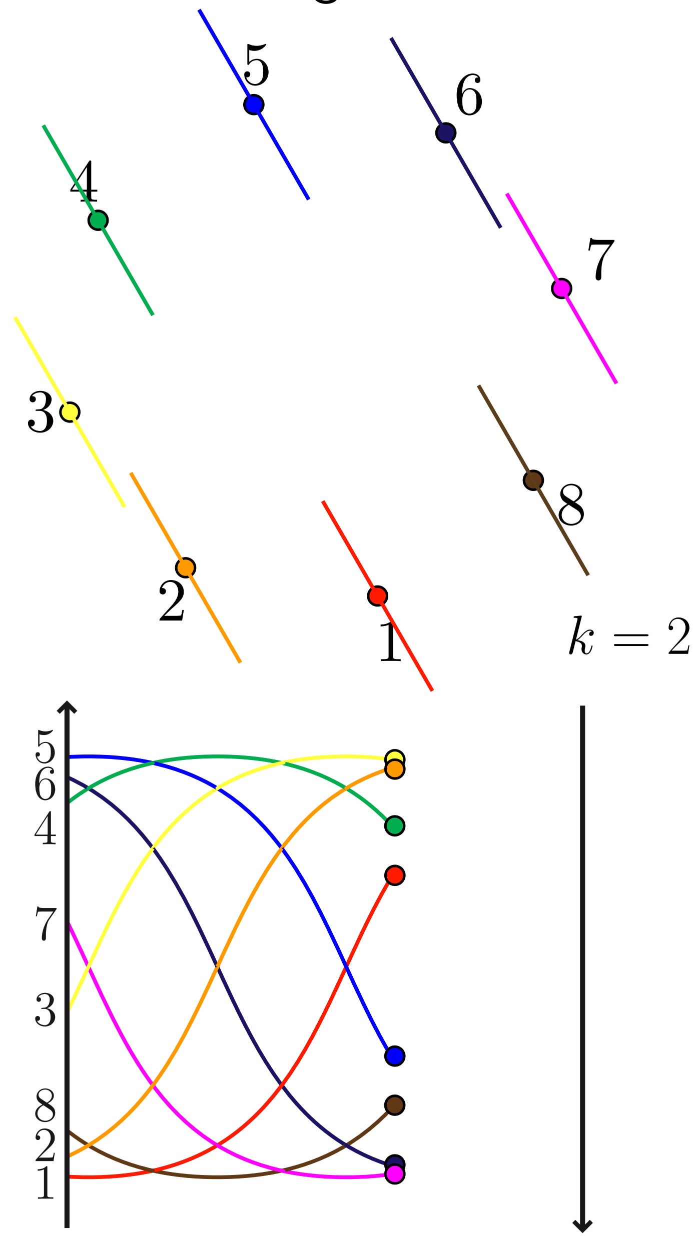
Triangulations



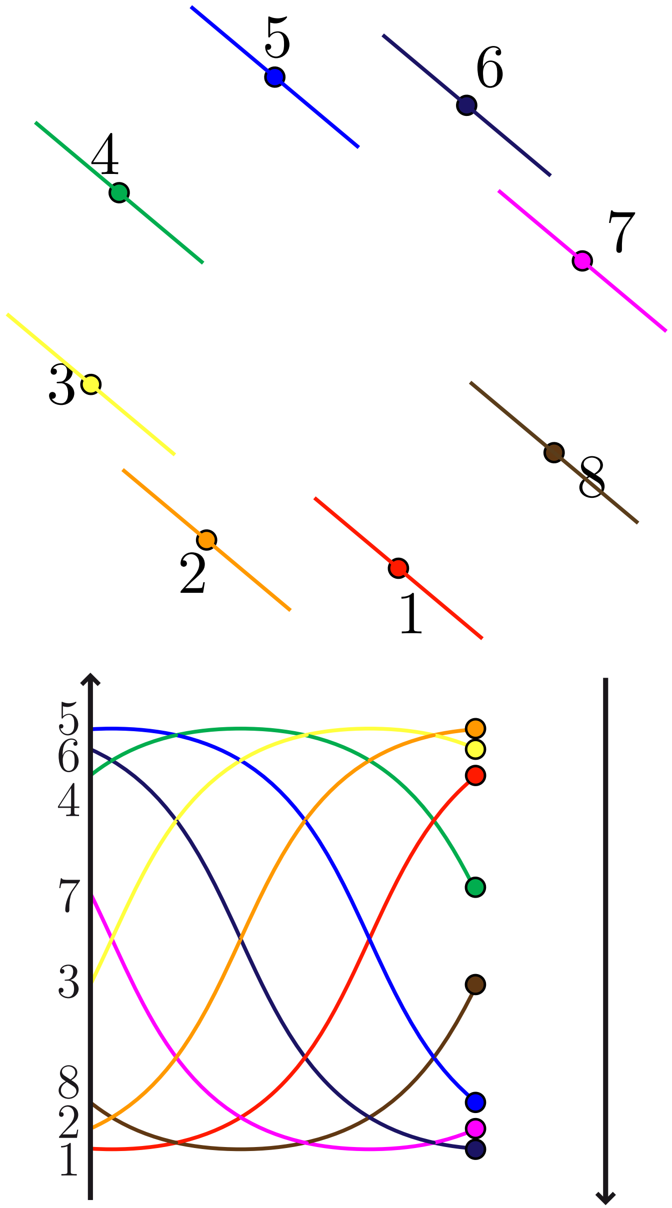
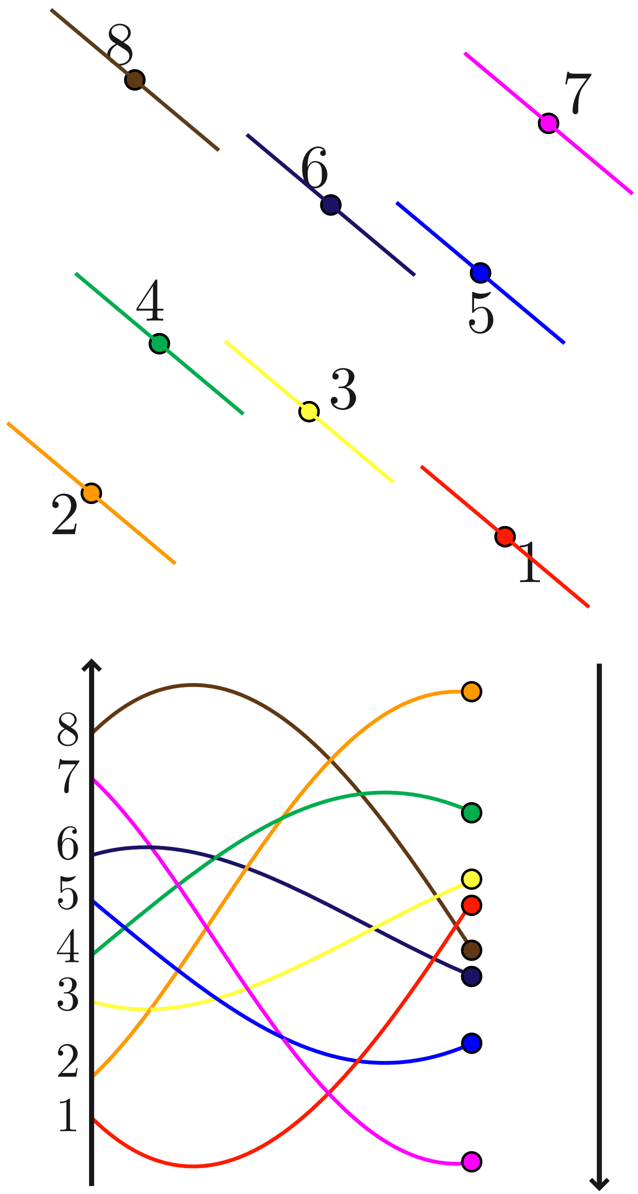
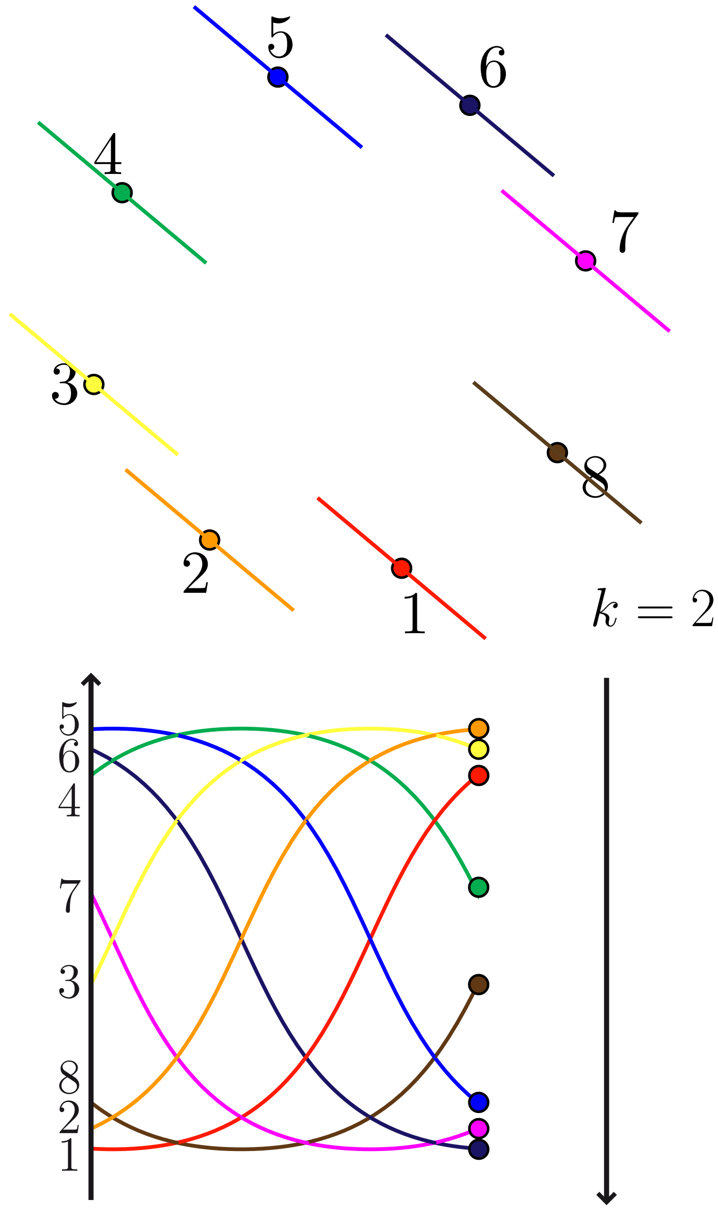
Pseudotriangulations



Multitriangulations

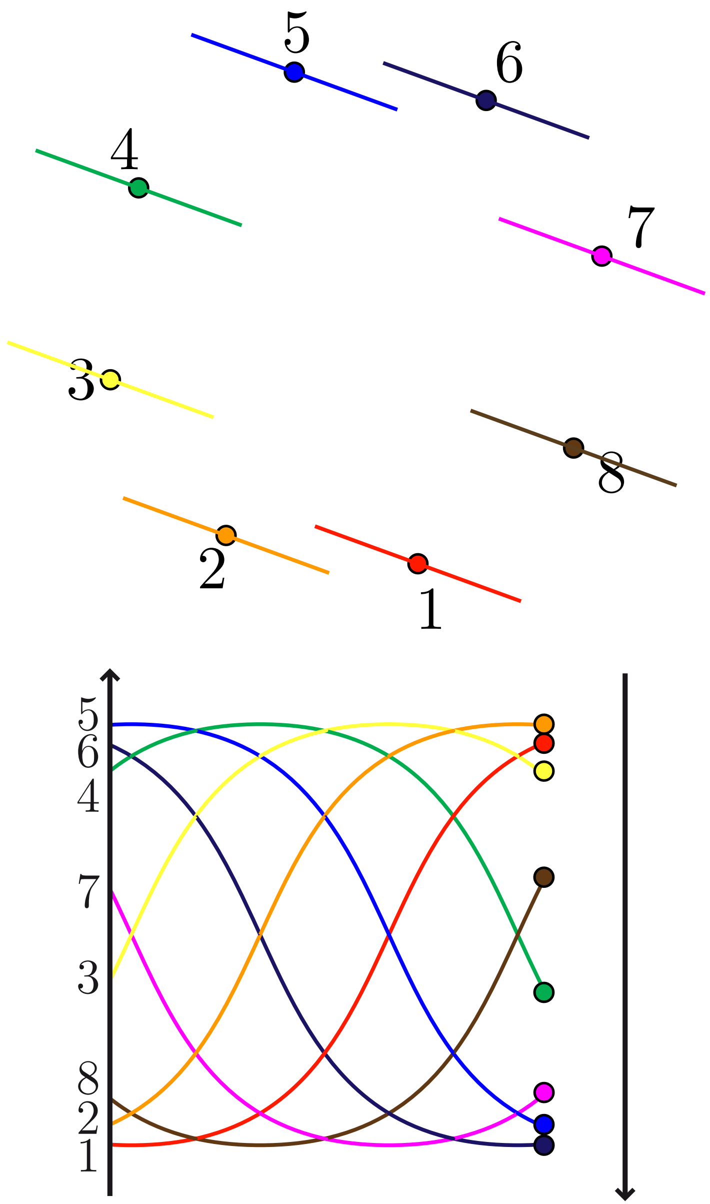


DUALITY

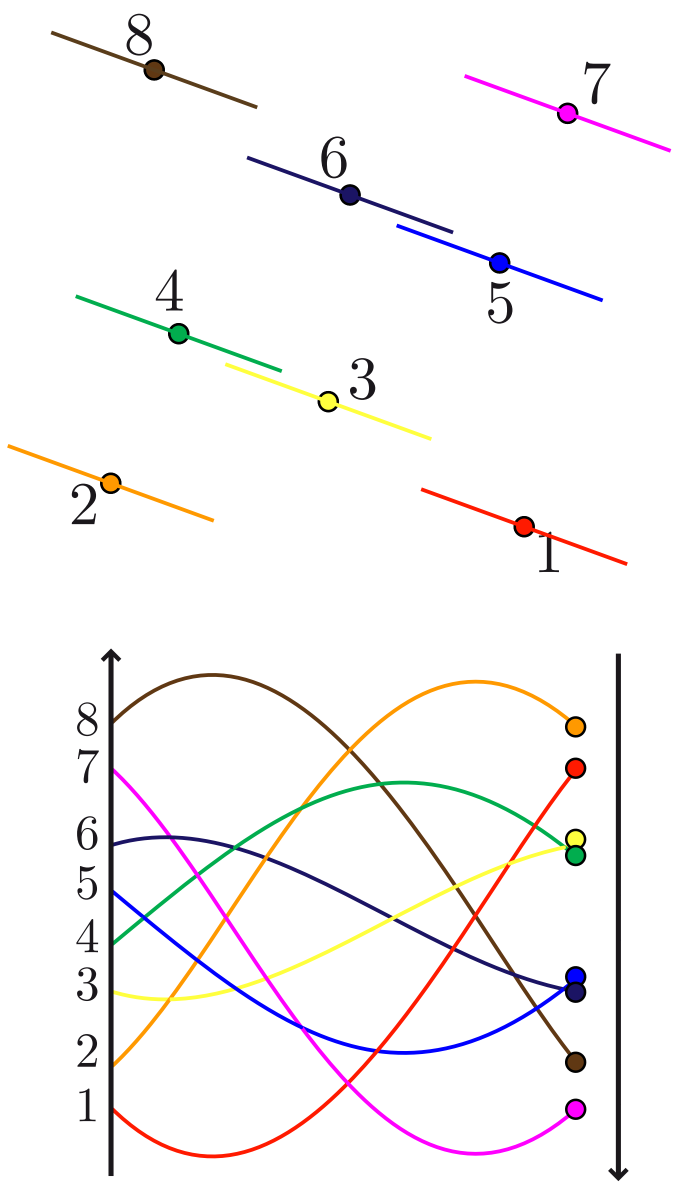
Triangulations	Pseudotriangulations	Multitriangulations
		

DUALITY

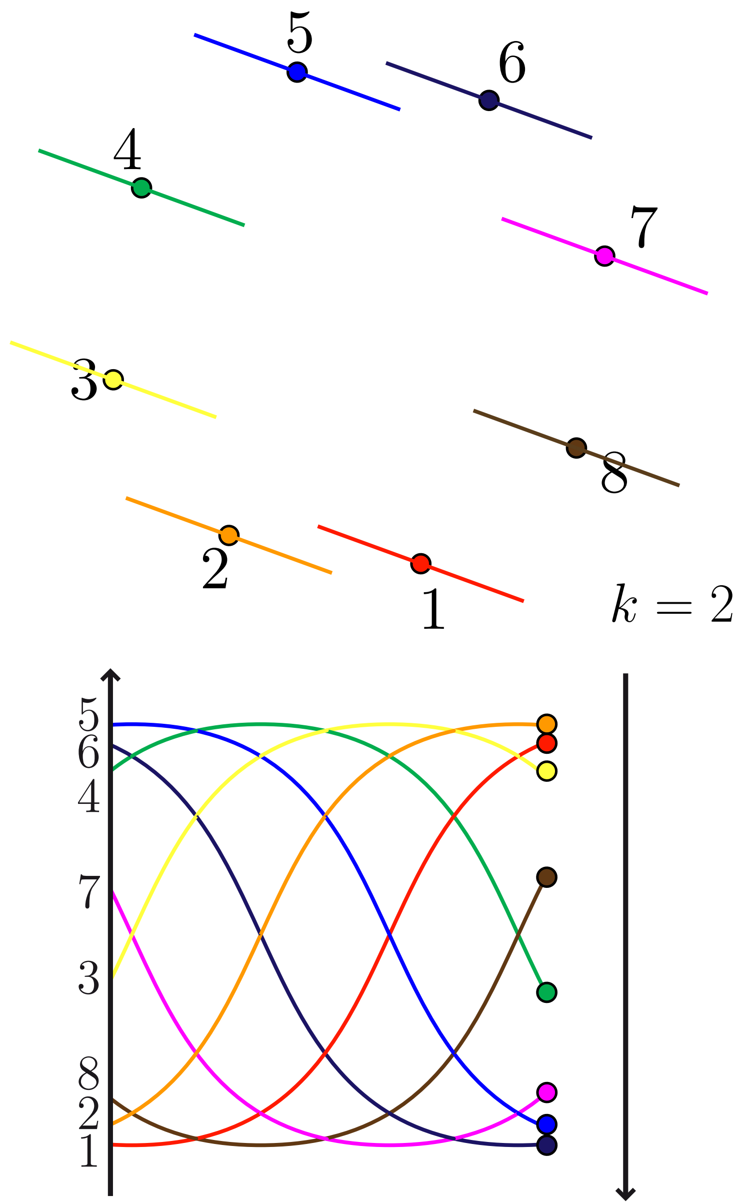
Triangulations



Pseudotriangulations

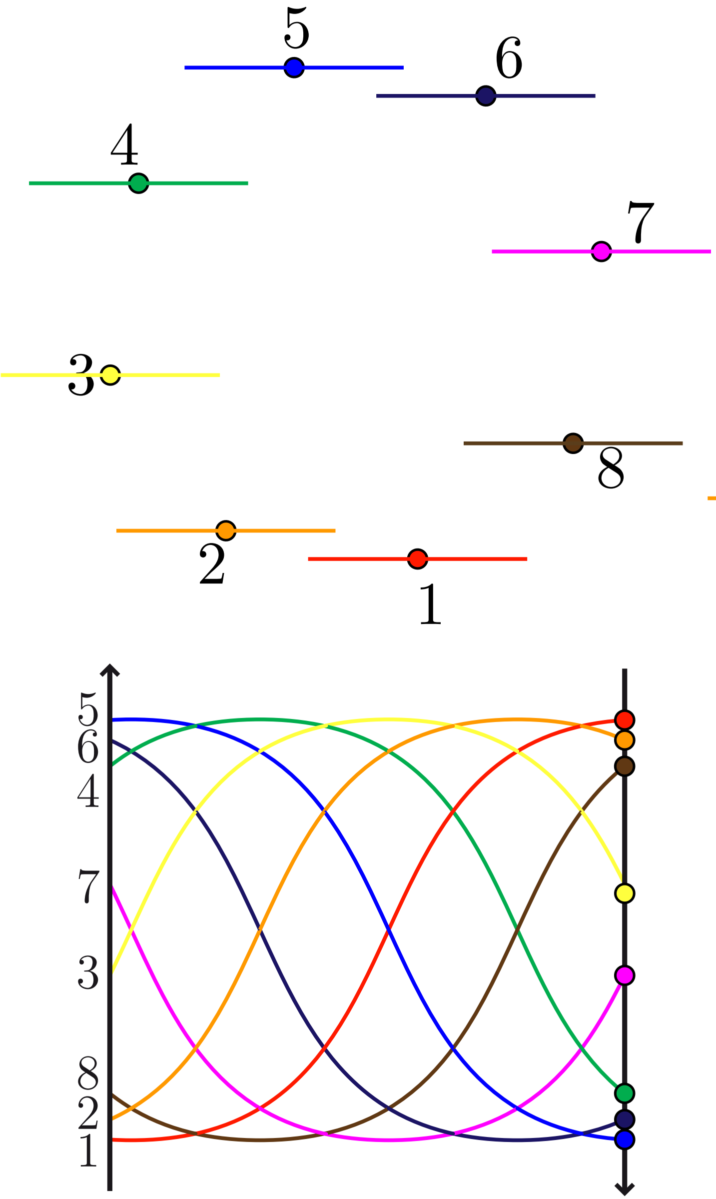


Multitriangulations



DUALITY

Triangulations



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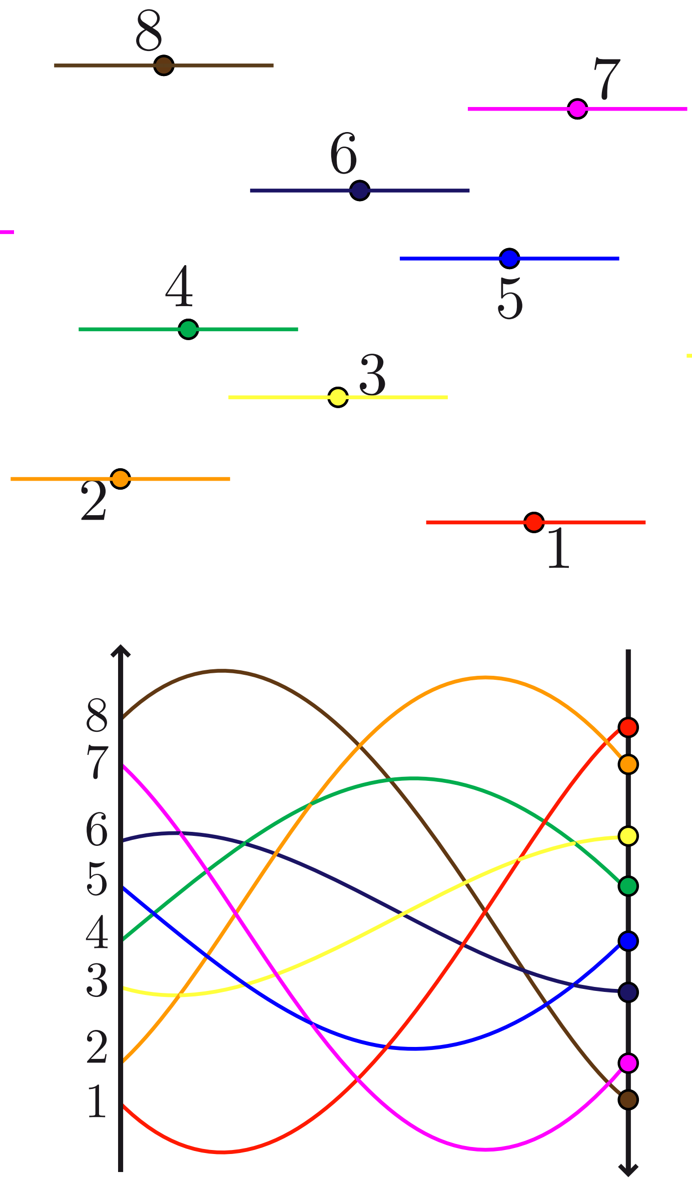

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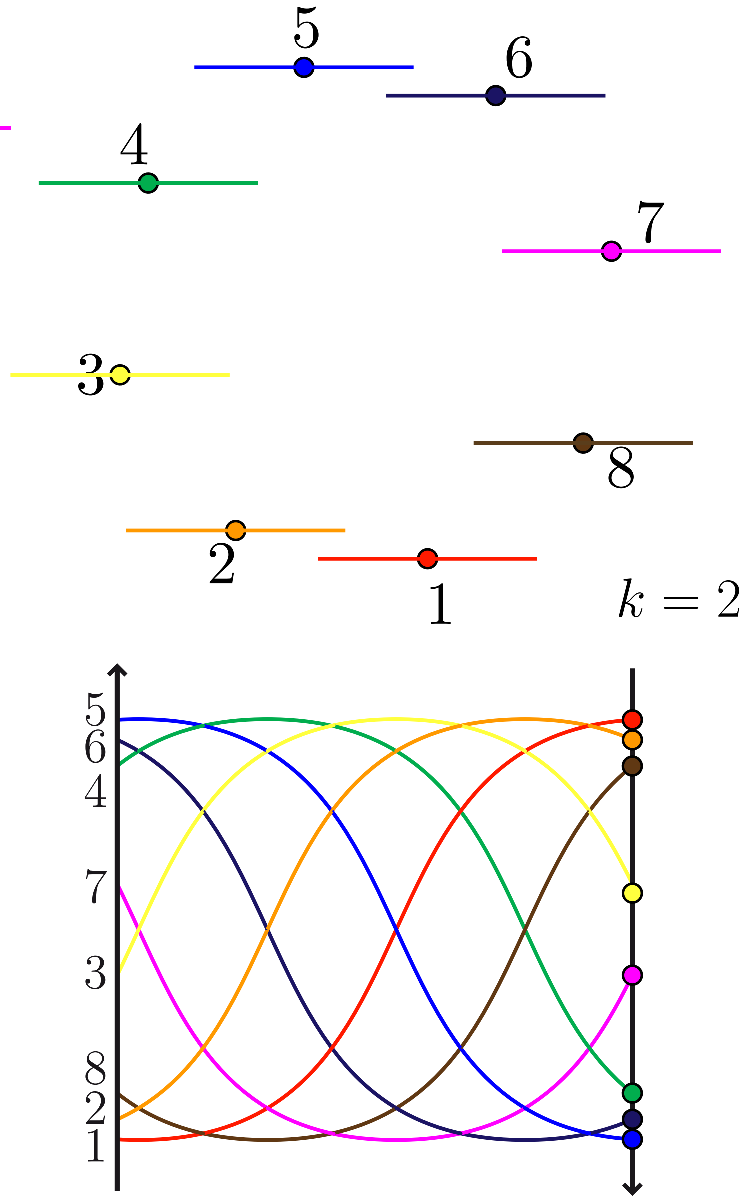

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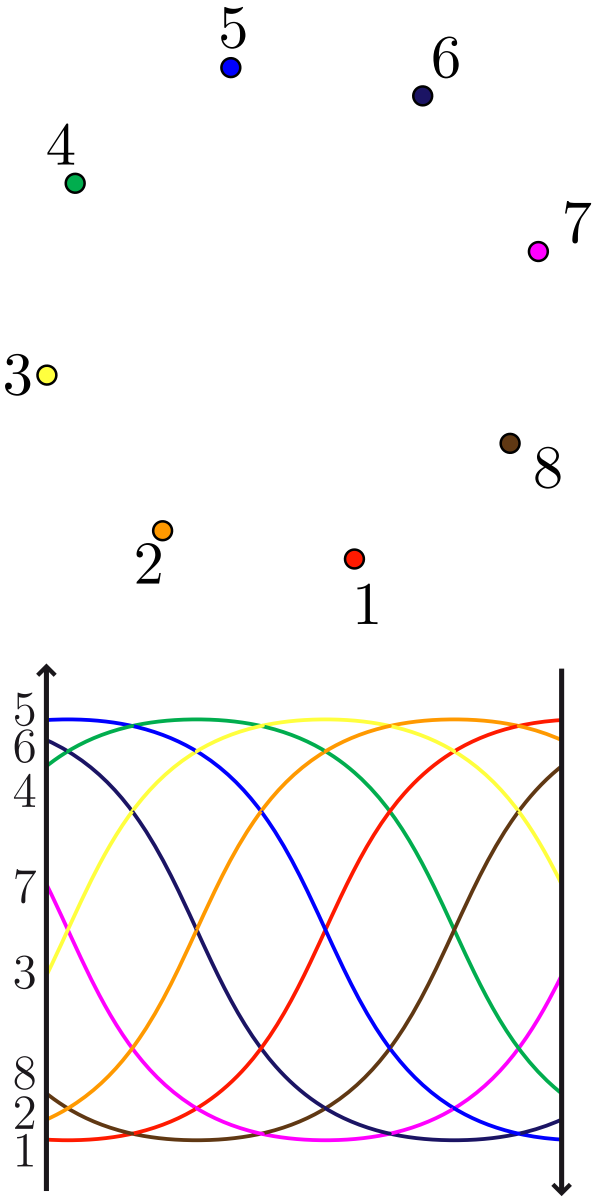

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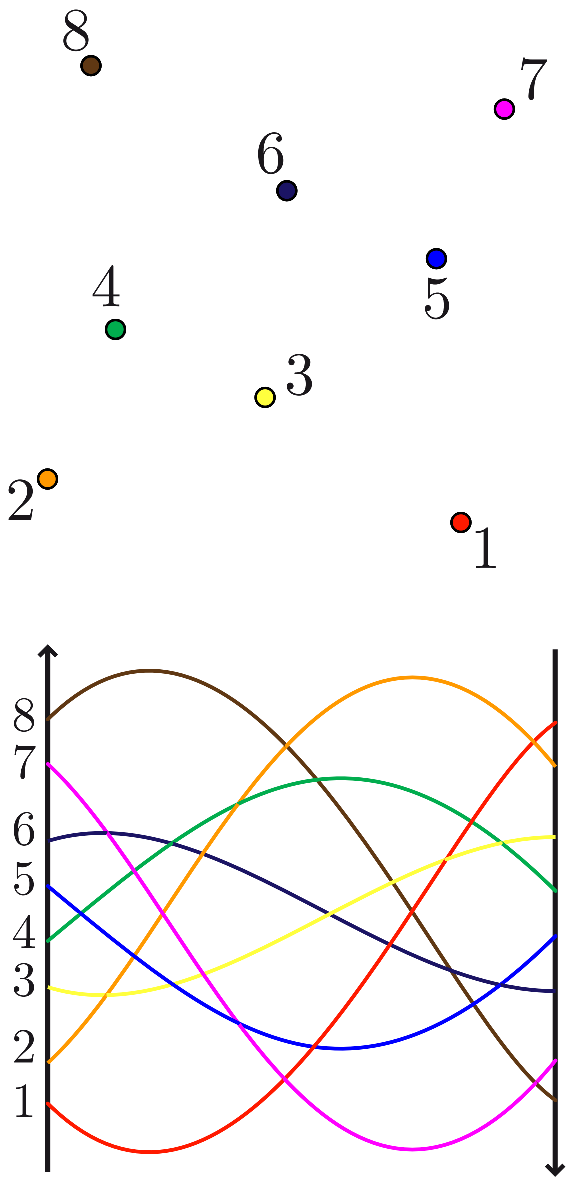

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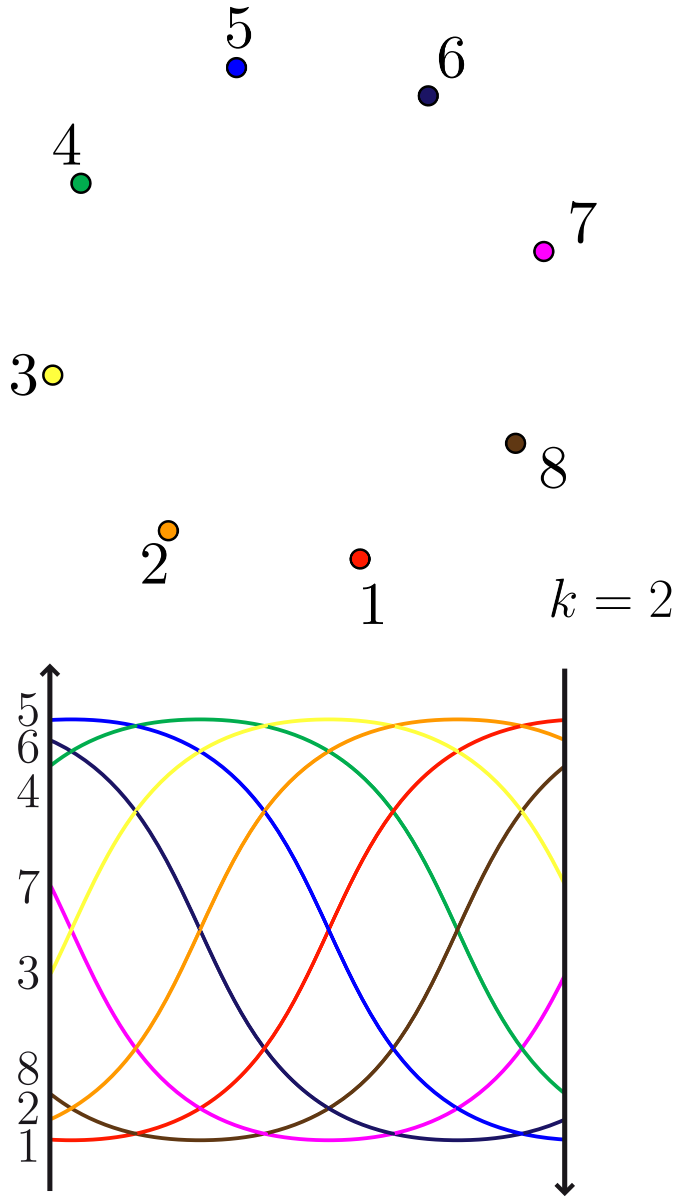

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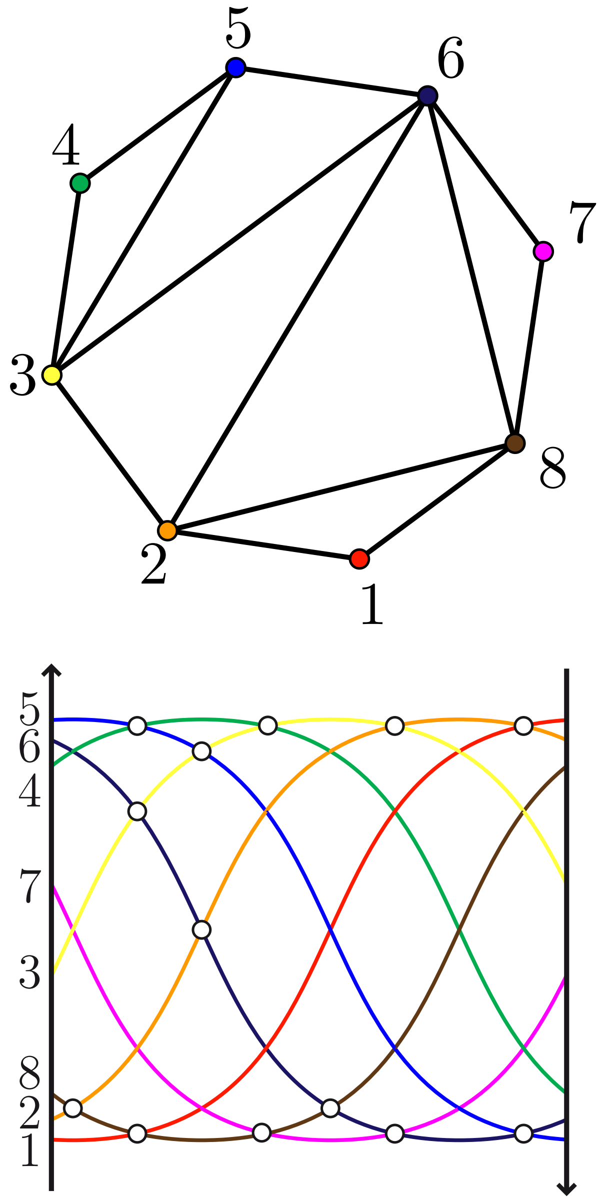

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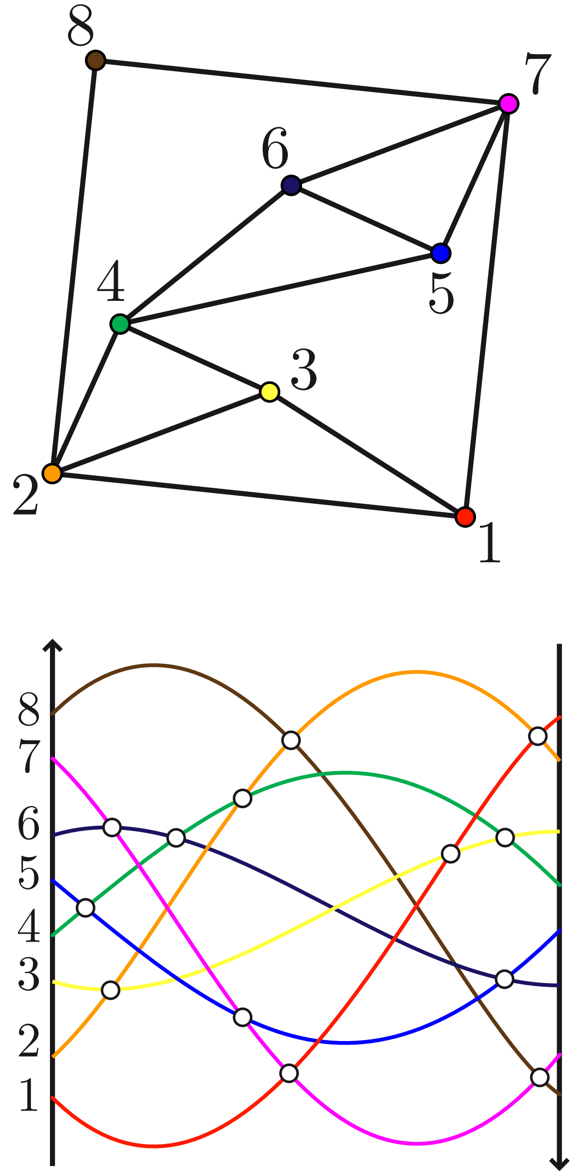

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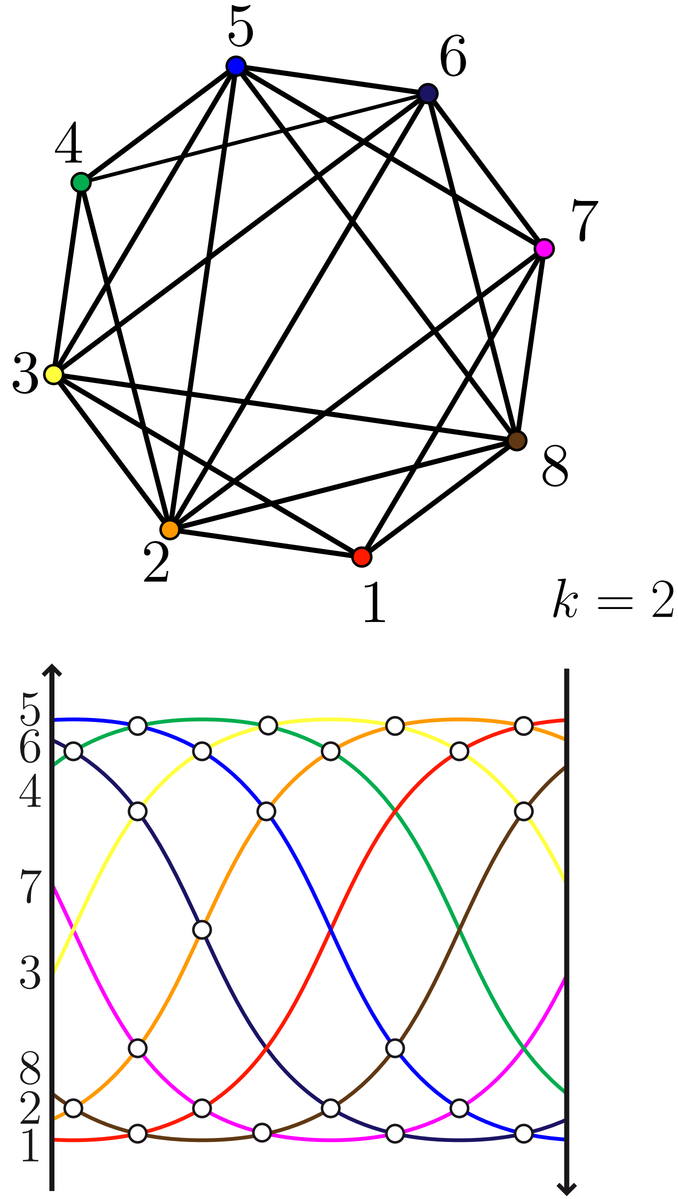
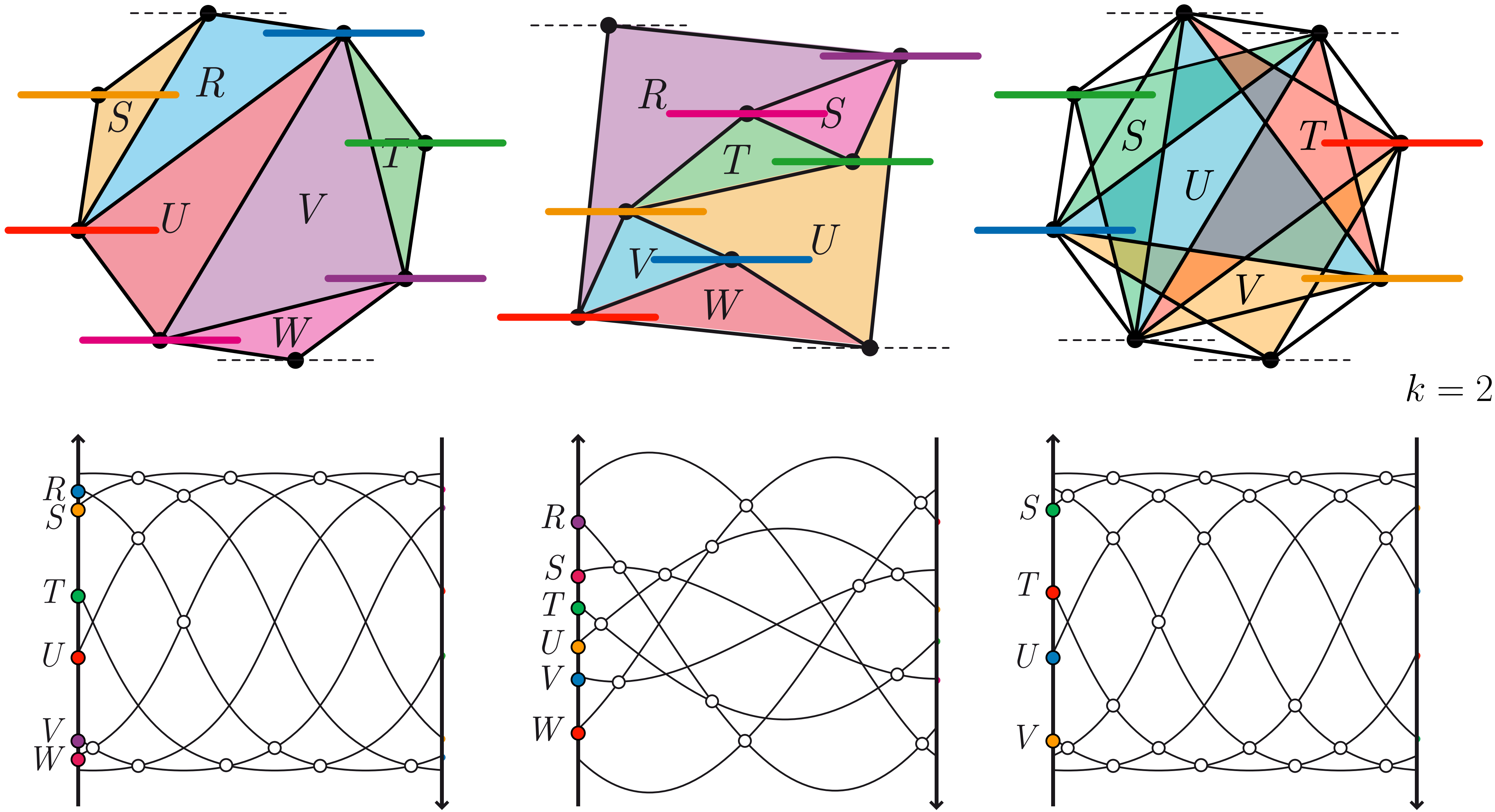


DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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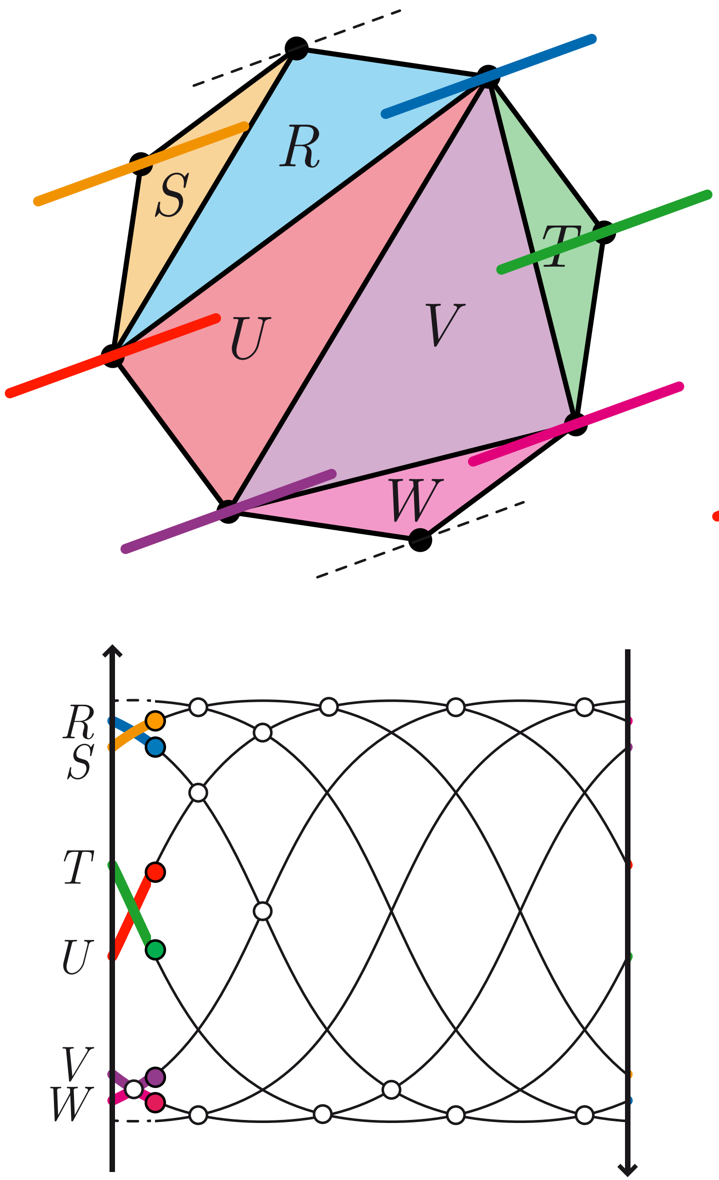

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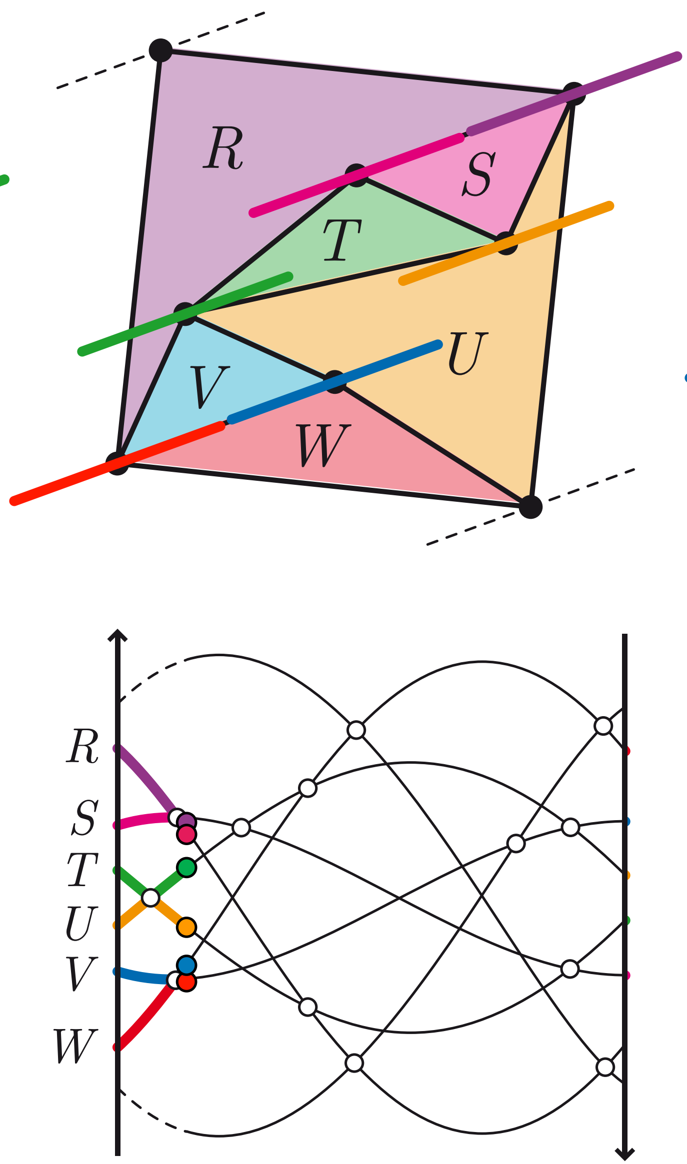

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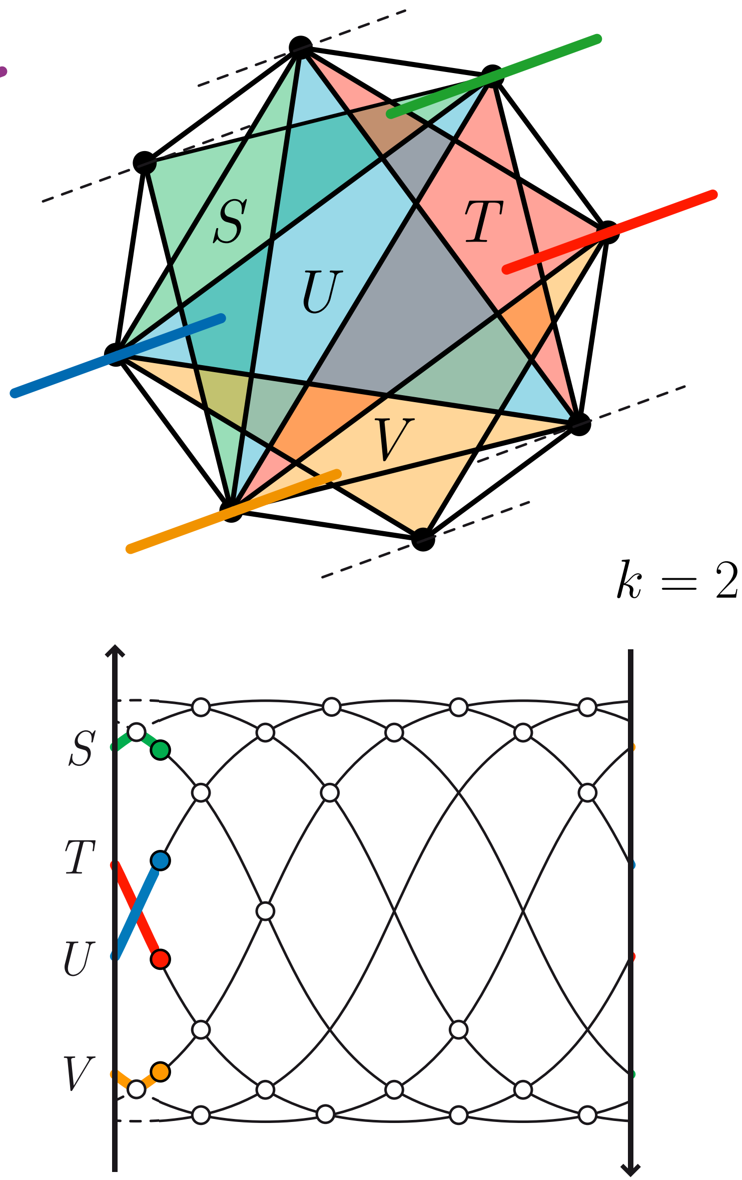

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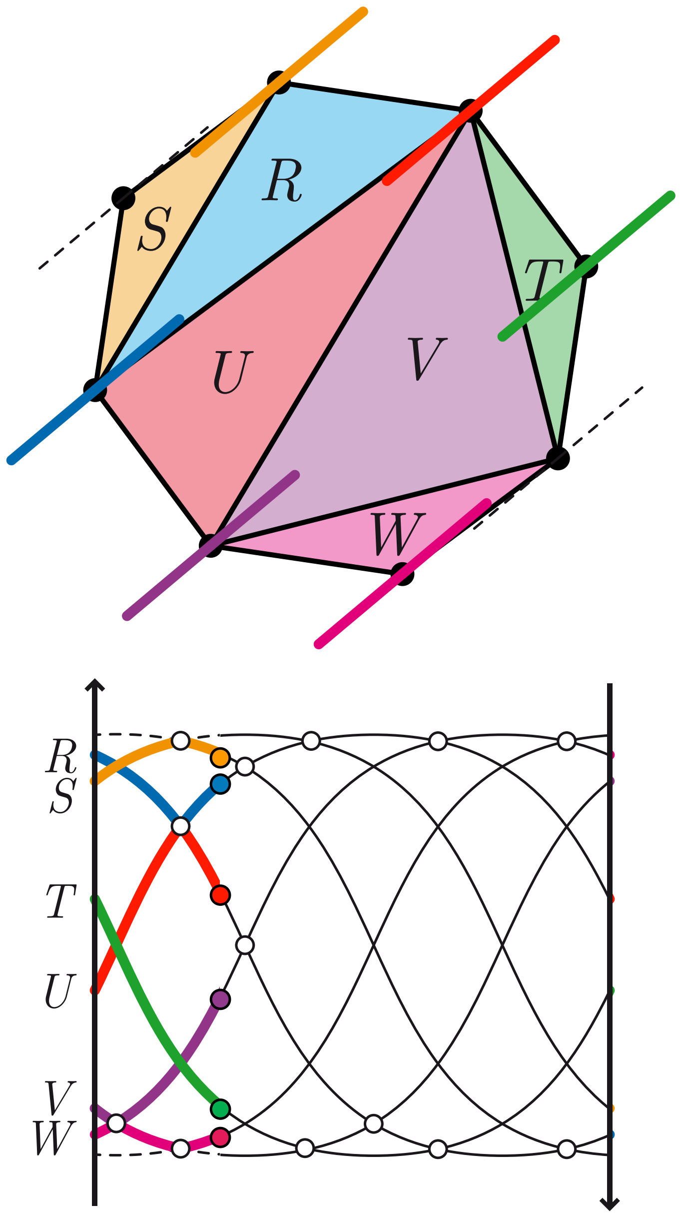
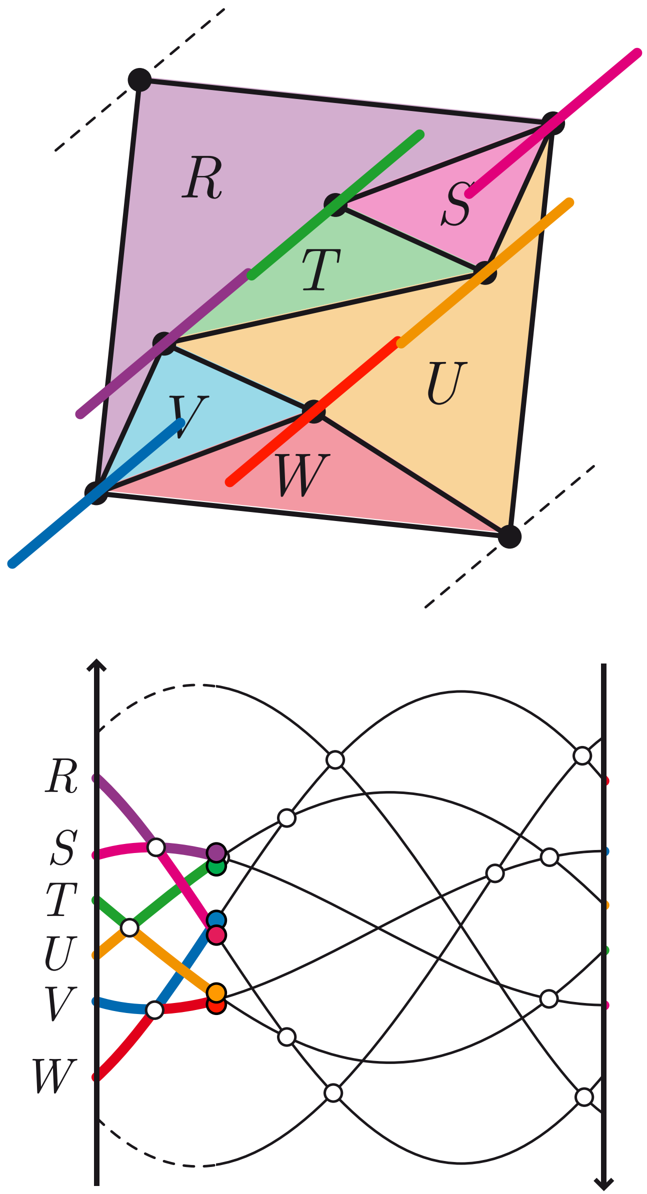
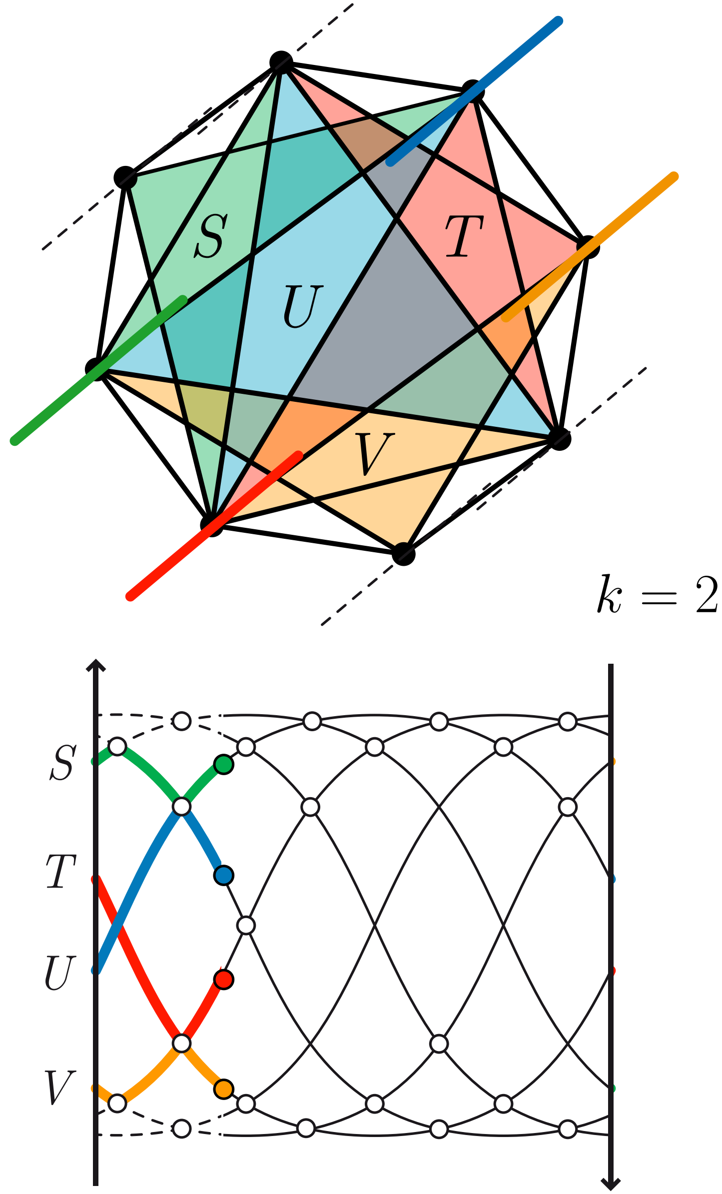
Triangulations



Pseudotriangulations



Multitriangulations



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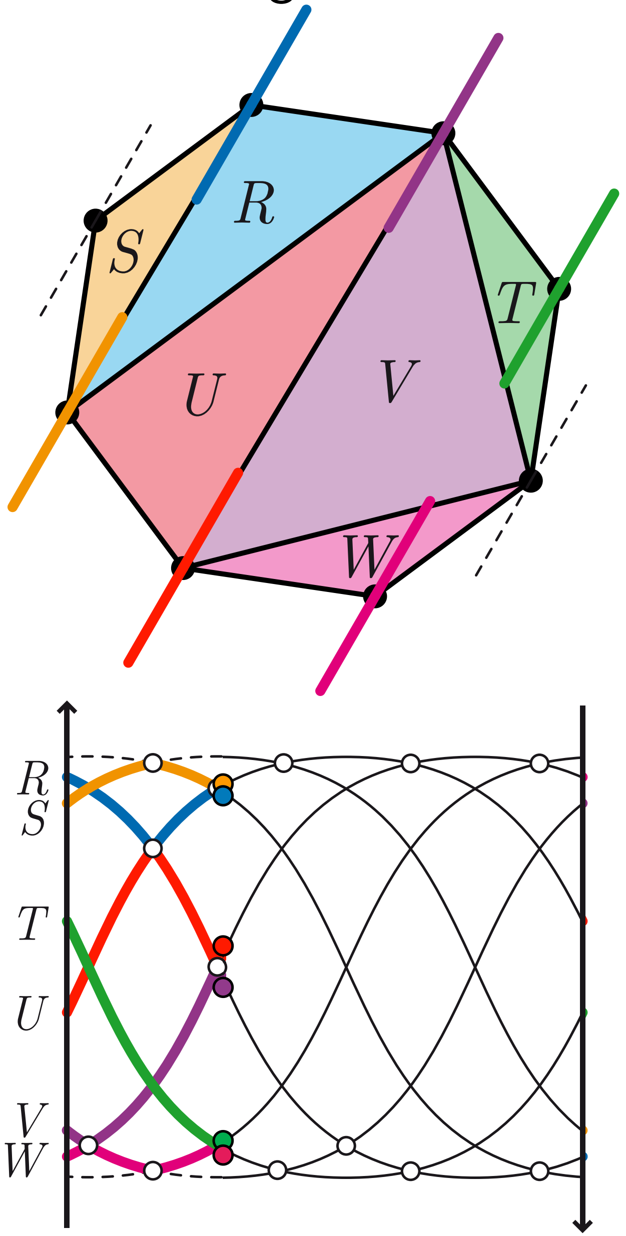
Triangulations



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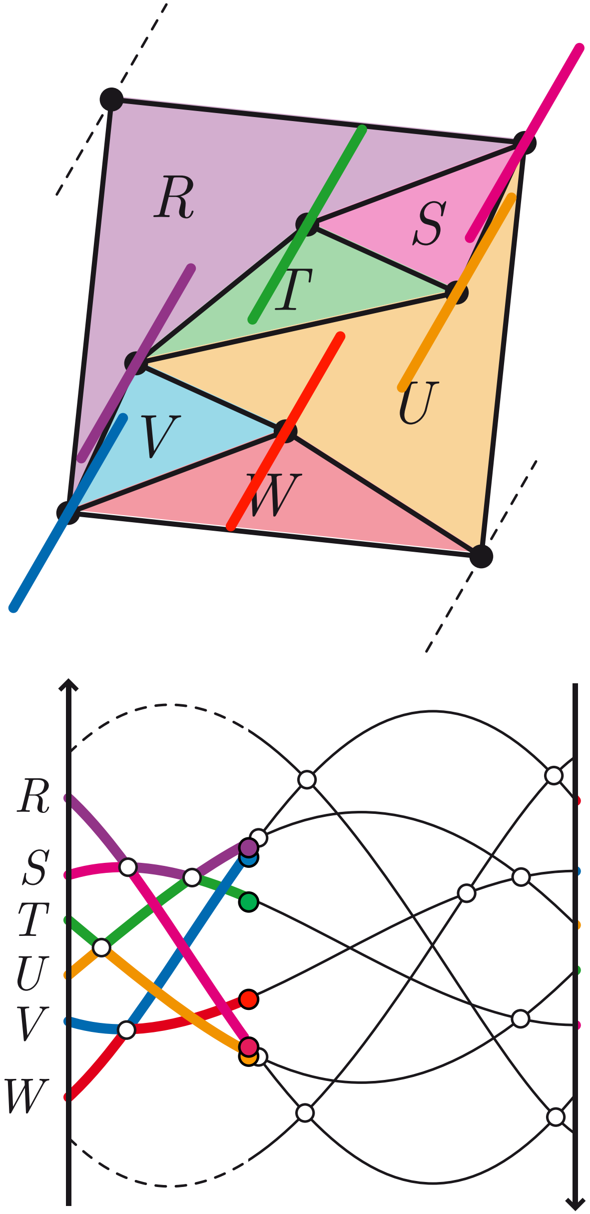

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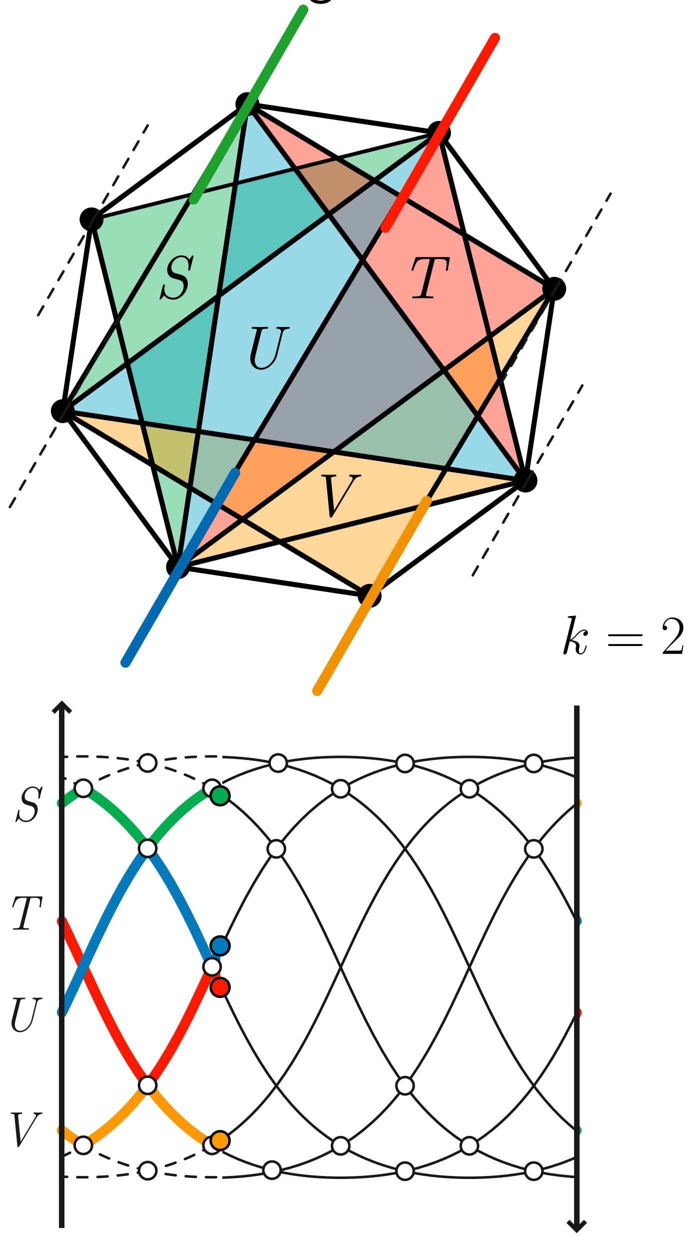

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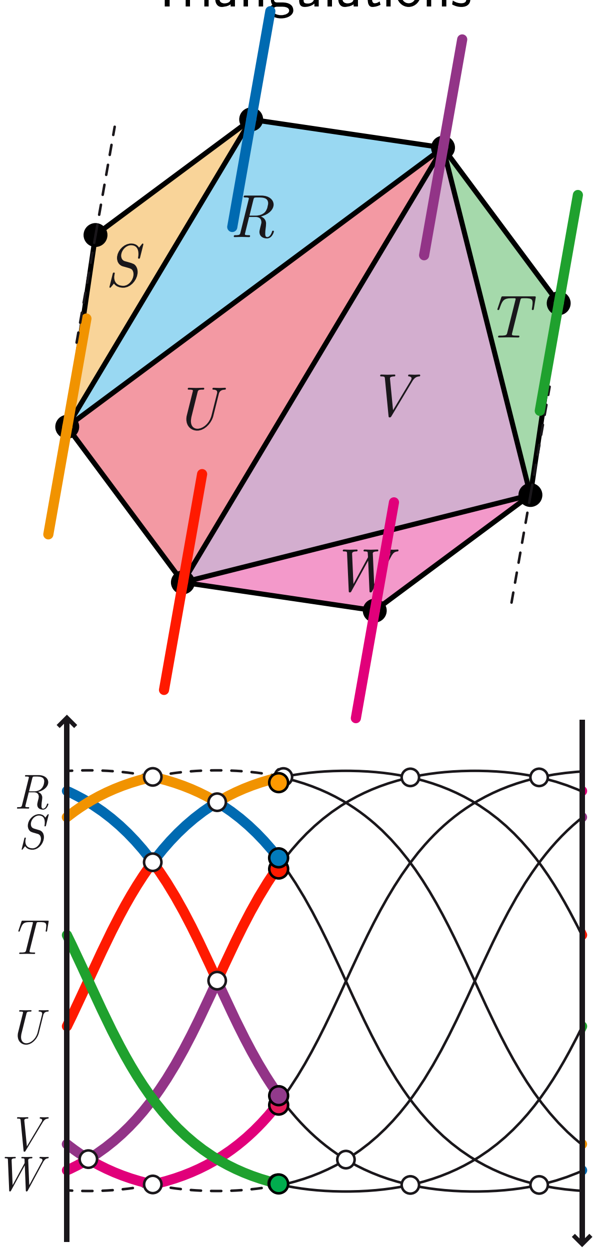
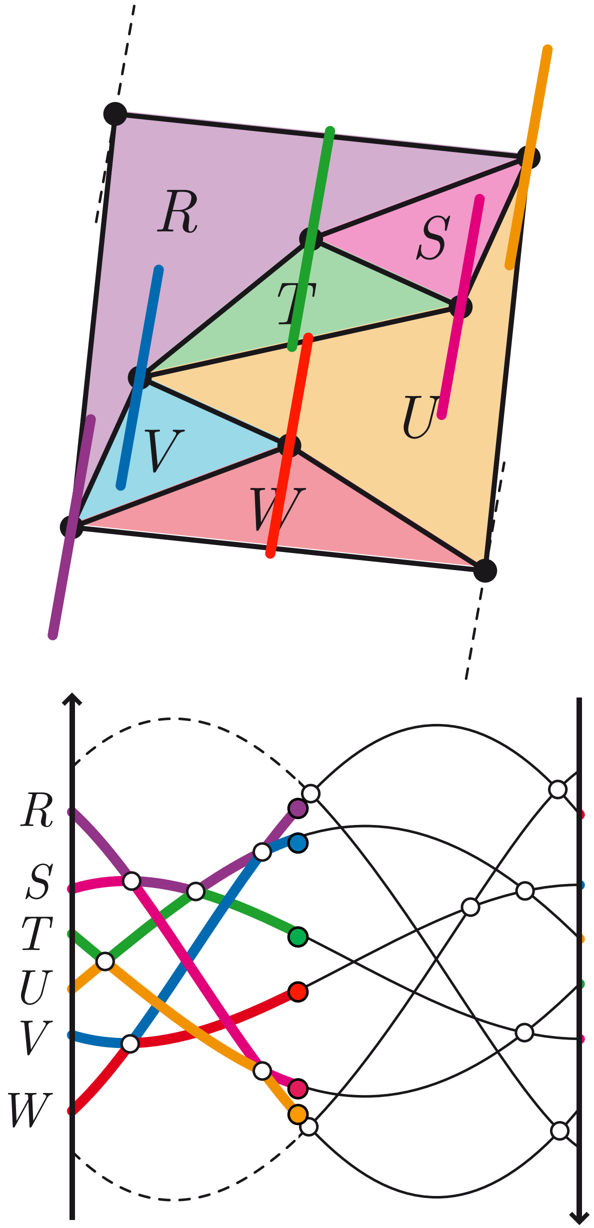
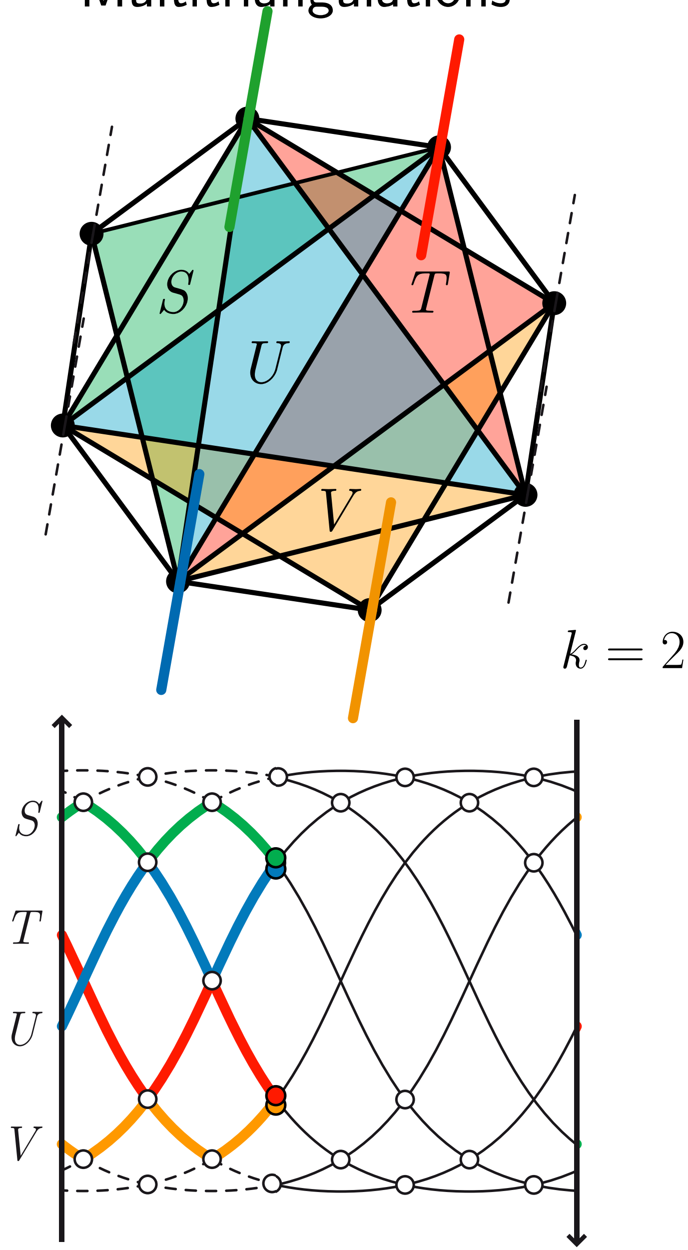
Triangulations



Pseudotriangulations



Multitriangulations



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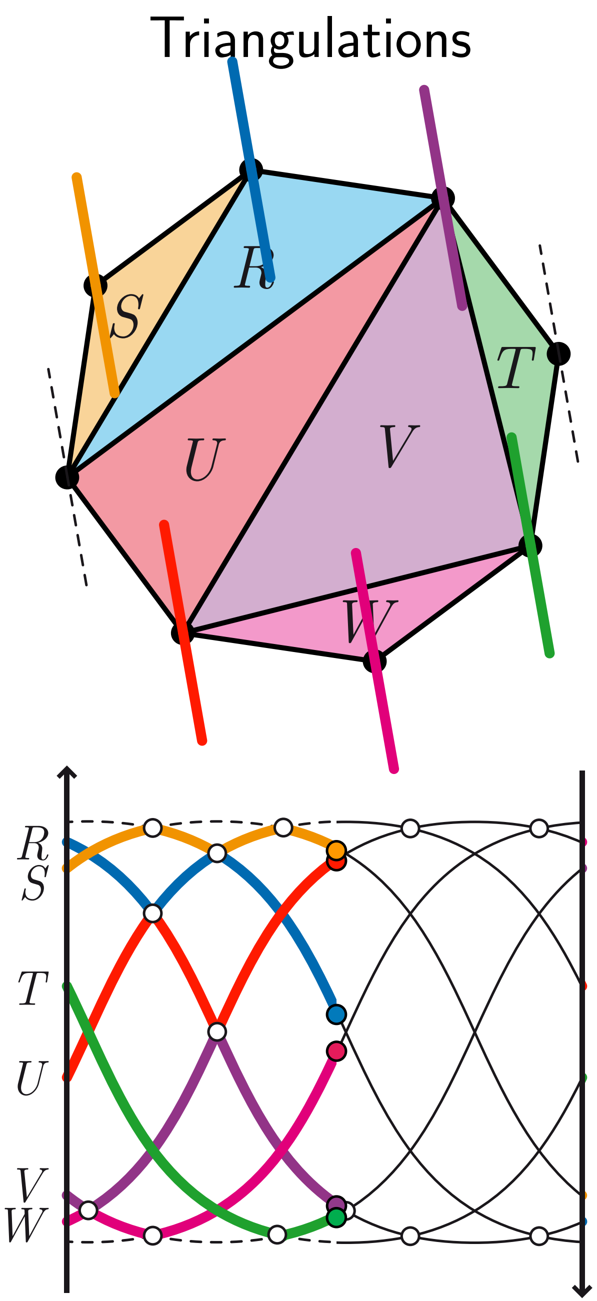
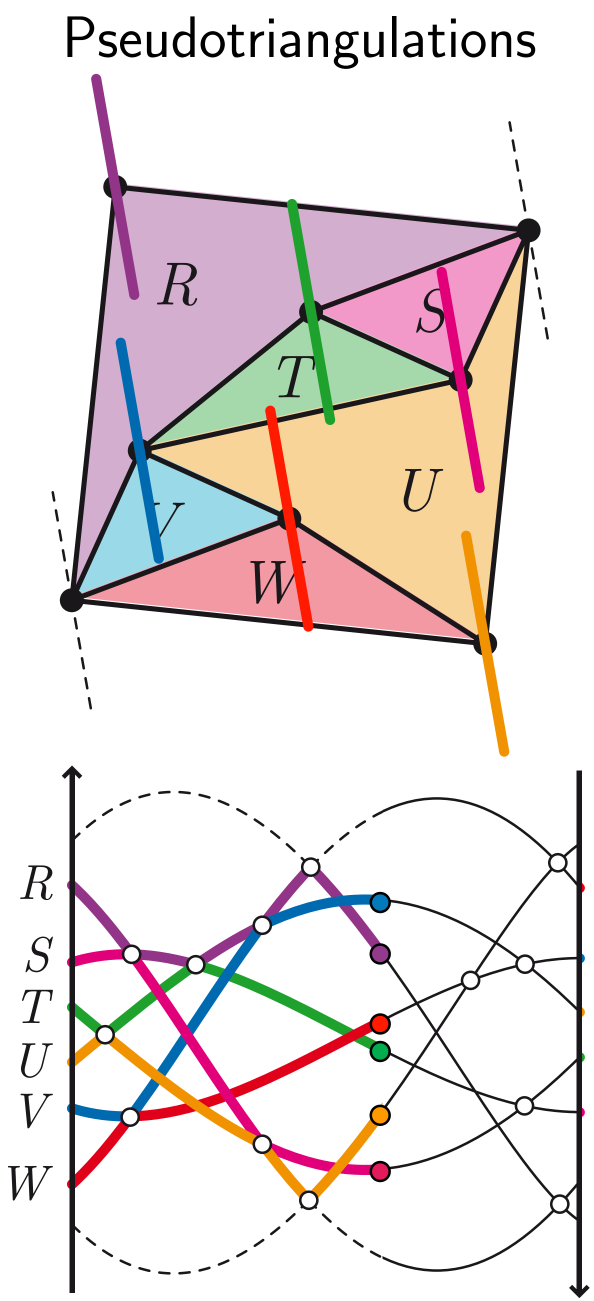
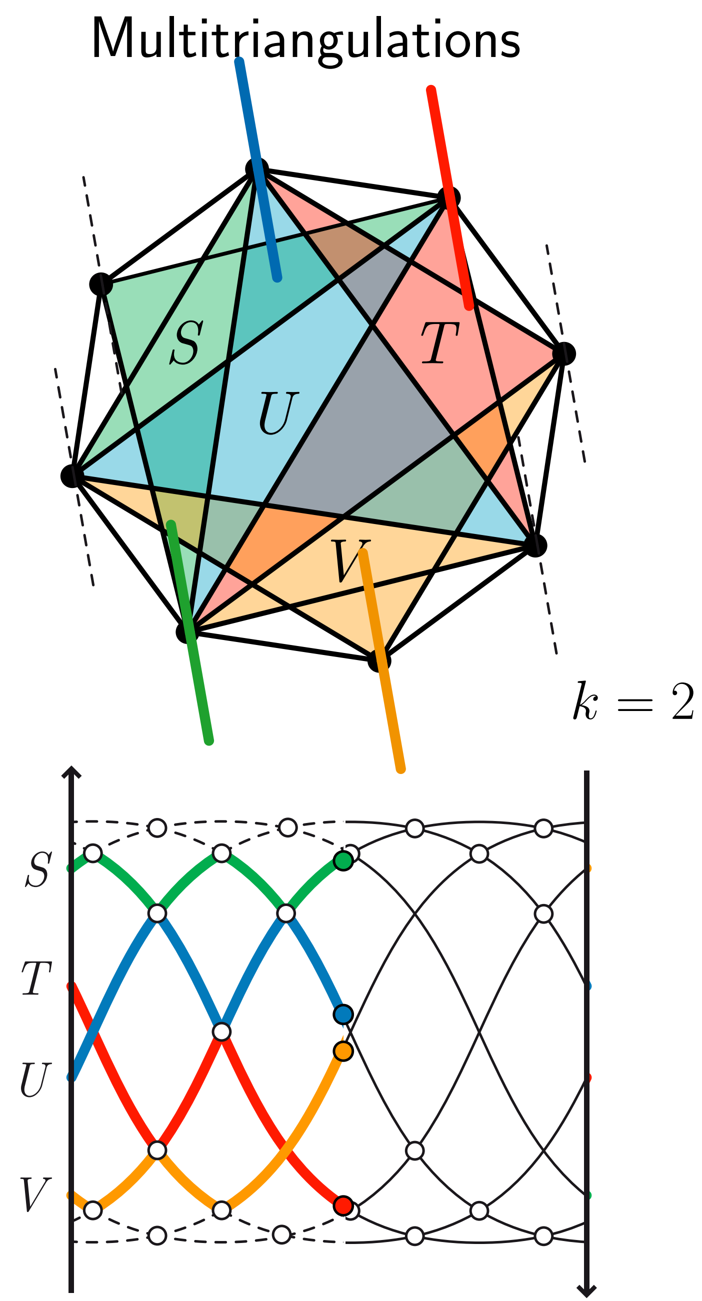


Triangulations
R
S
T
U
V
W
R
S
T
U
V
W

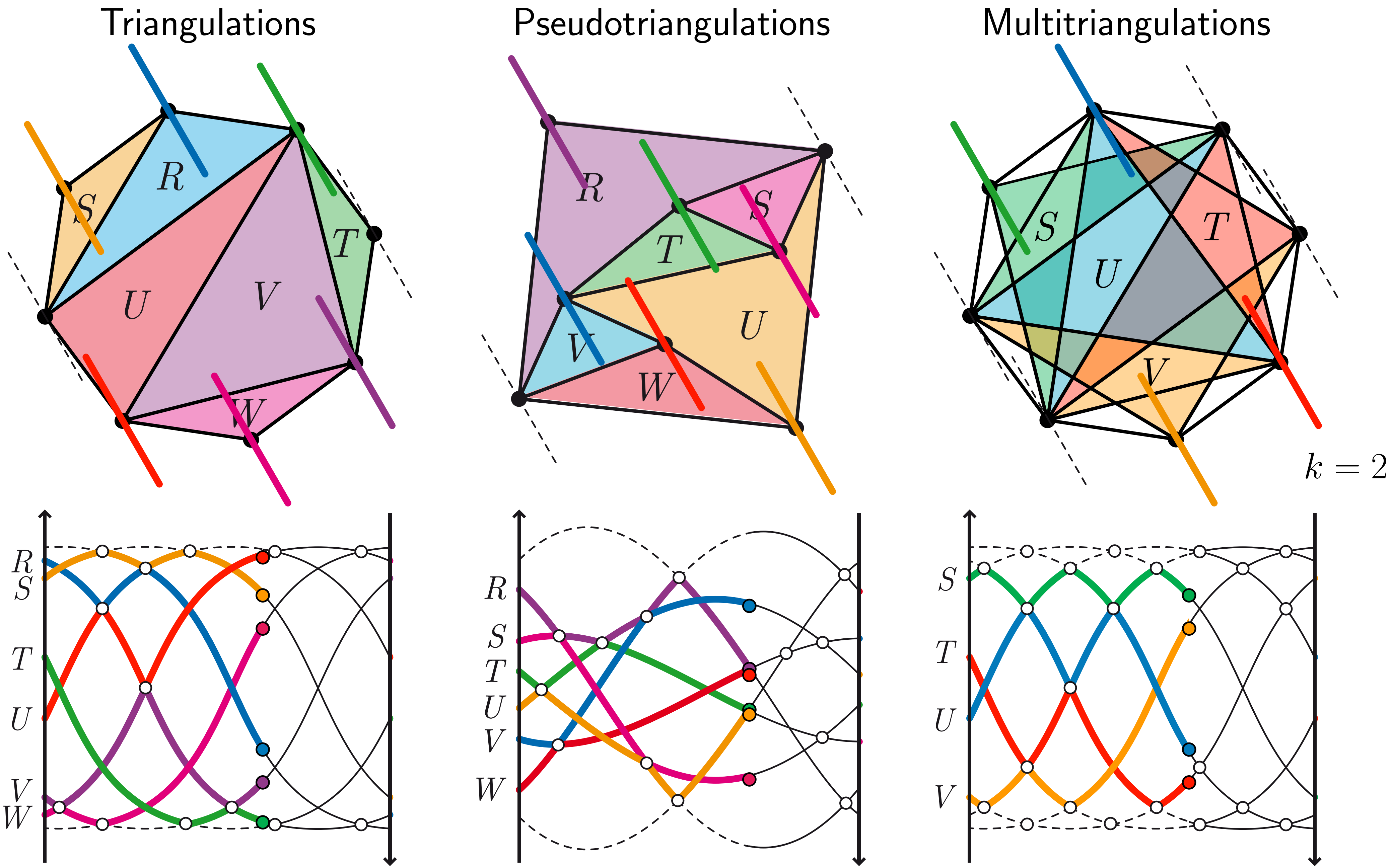


Pseudotriangulations
R
S
T
U
V
W
R
S
T
U
V
W



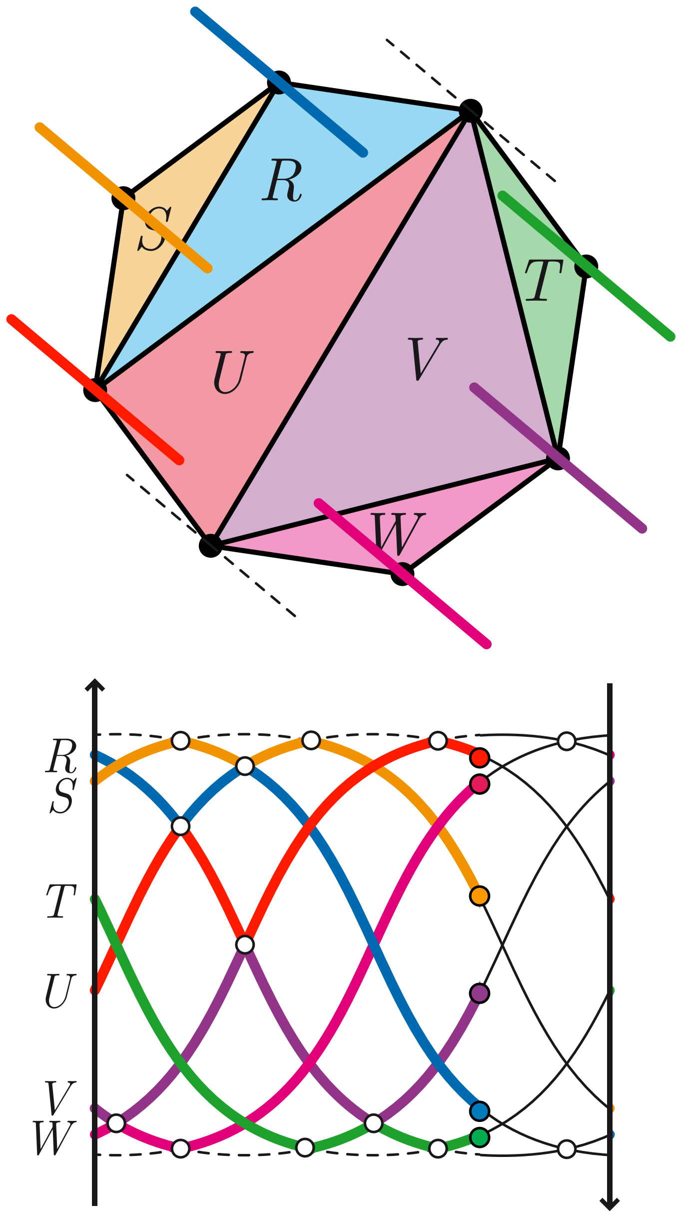
Multitriangulations
S
T
U
V
k = 2
S
T
U
V

DUALITY

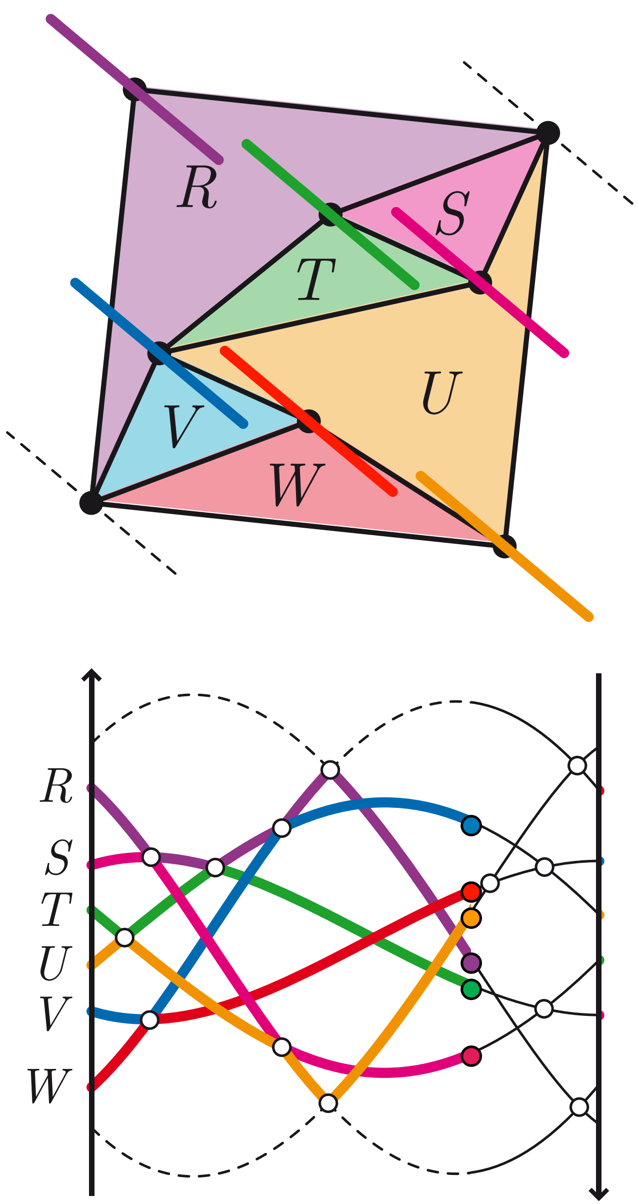

Triangulations
Pseudotriangulations
Multitriangulations
R
S
T
U
V
W
$k = 2$

DUALITY

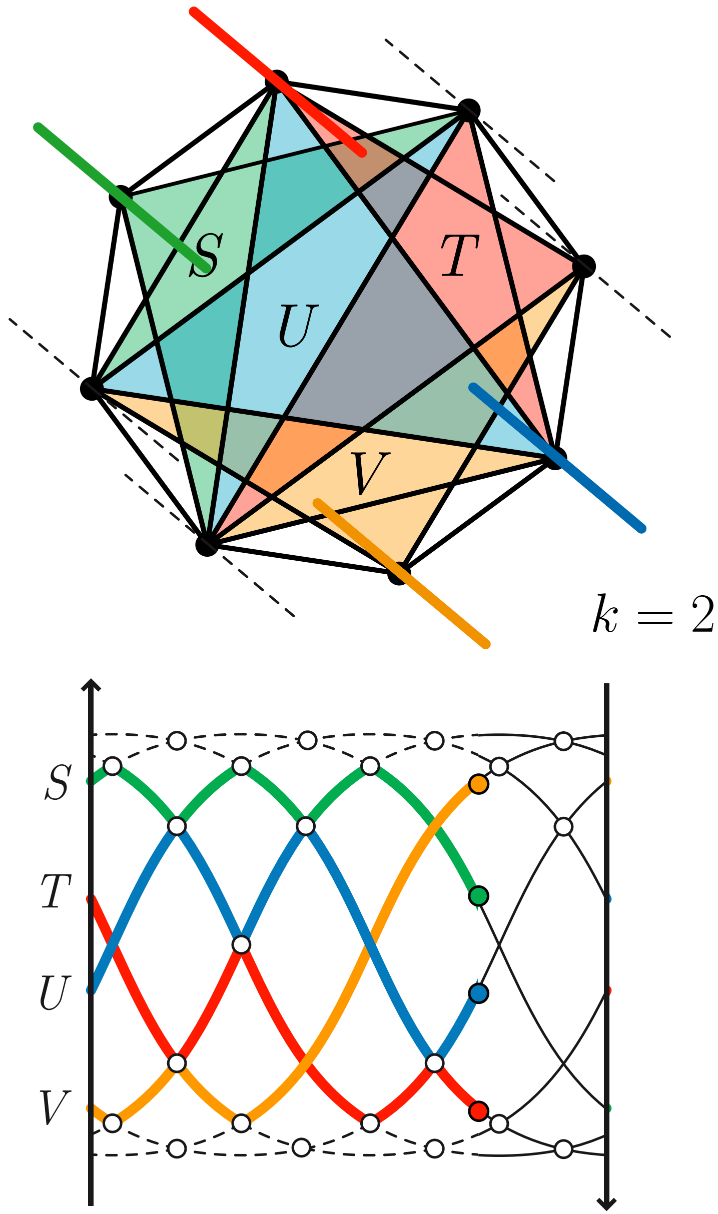
Triangulations



Pseudotriangulations

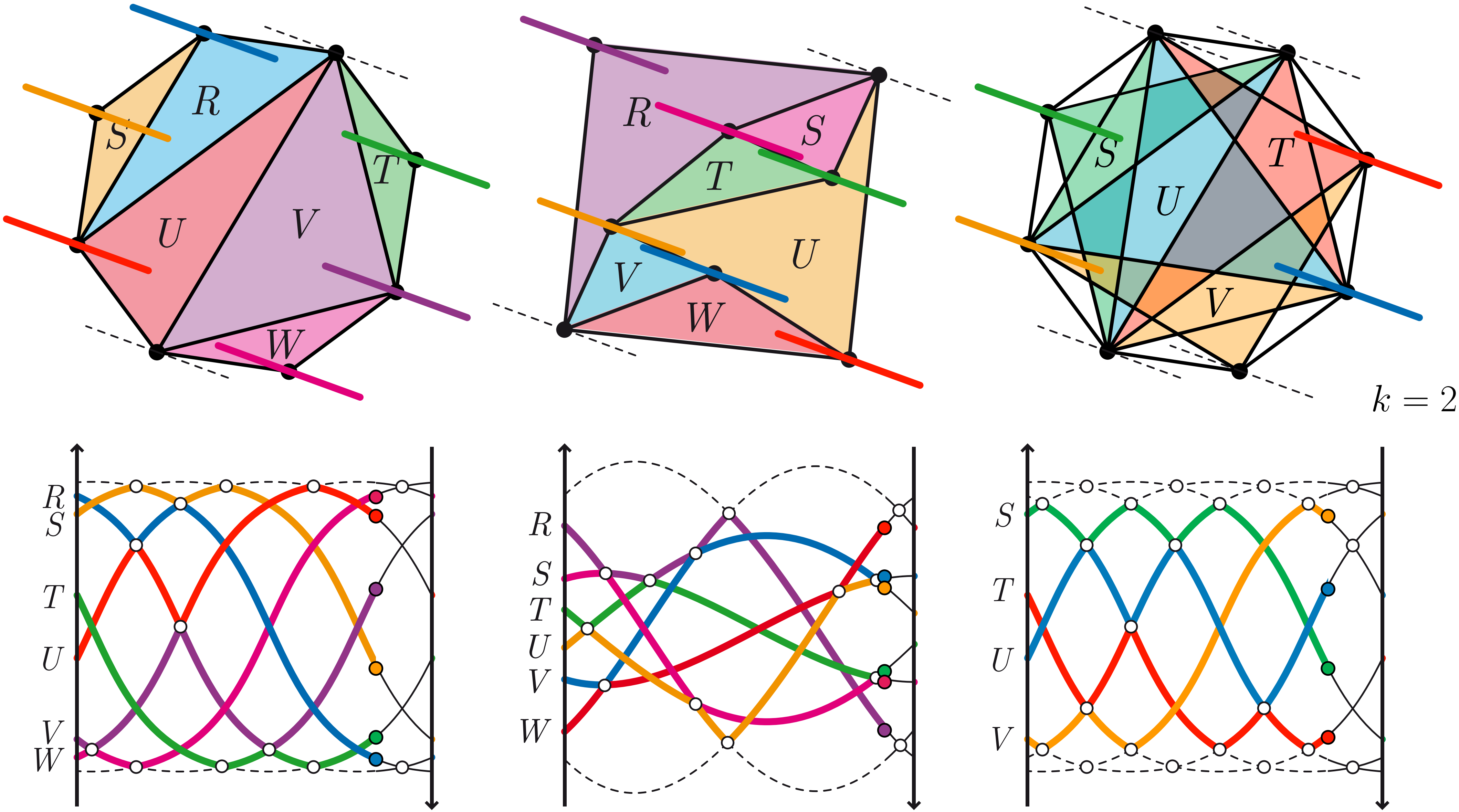


Multitriangulations



DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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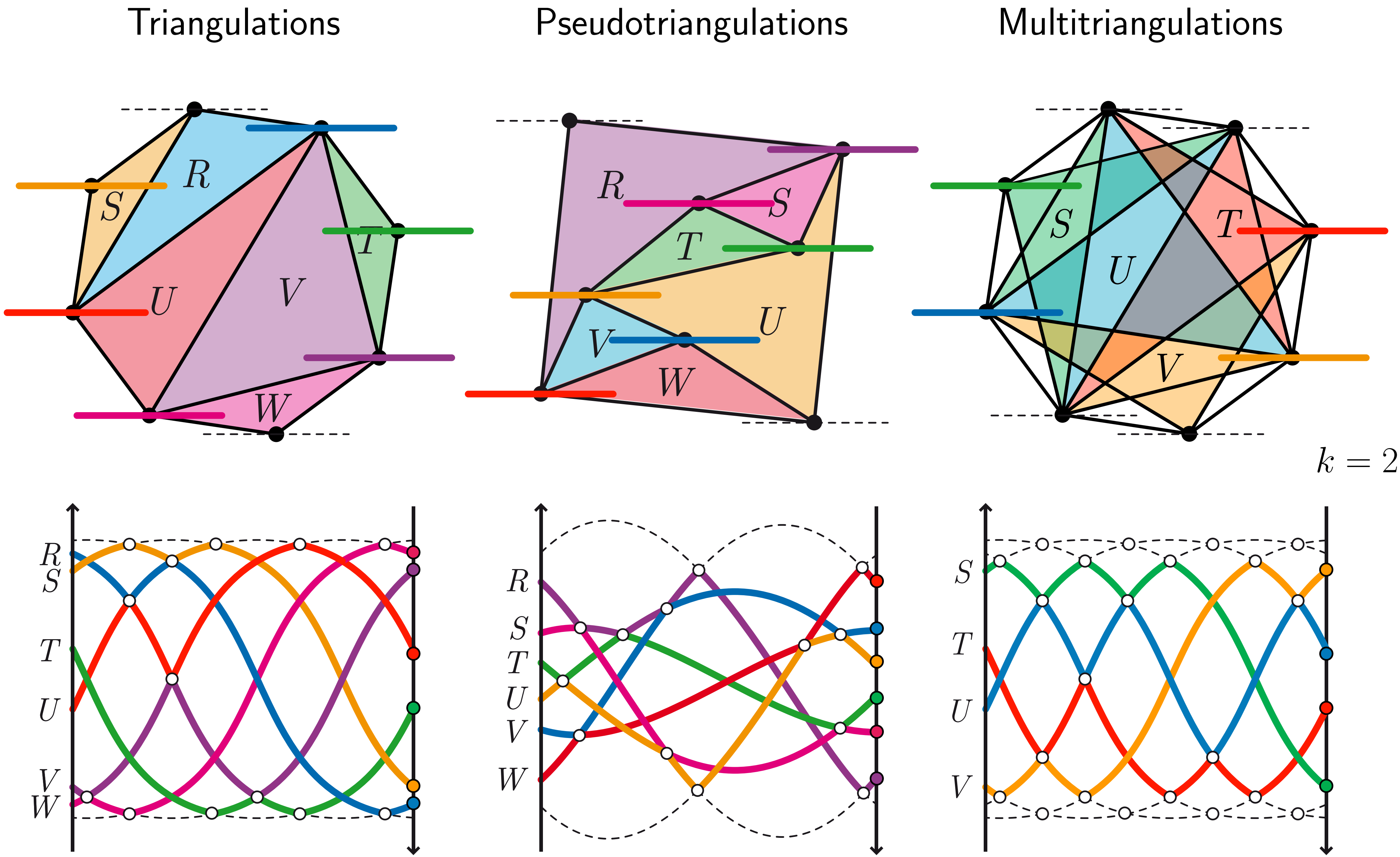
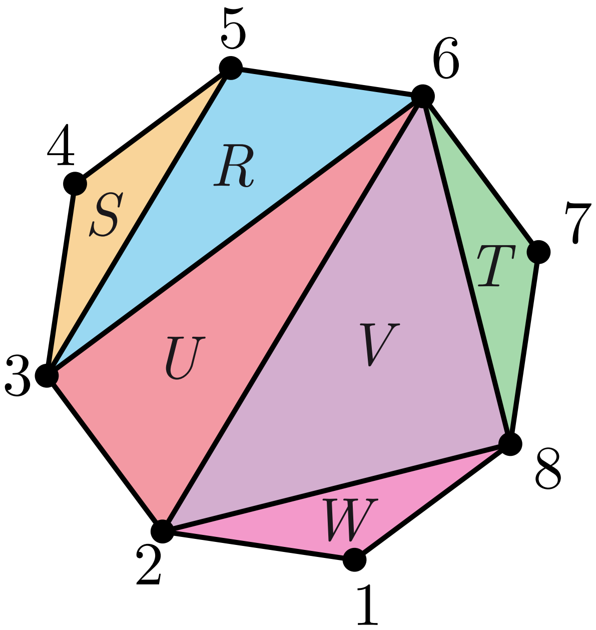
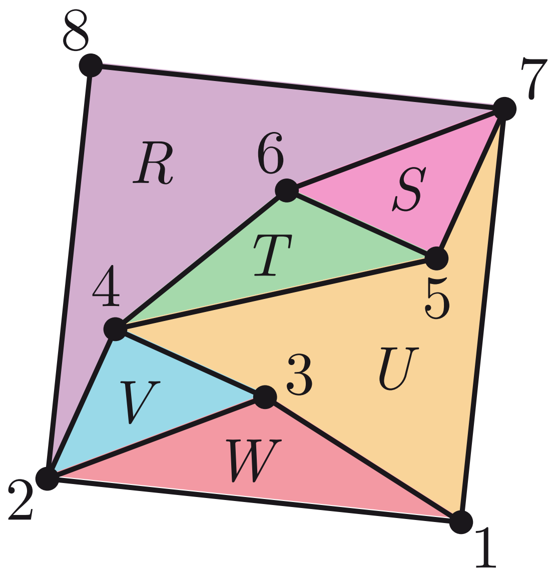
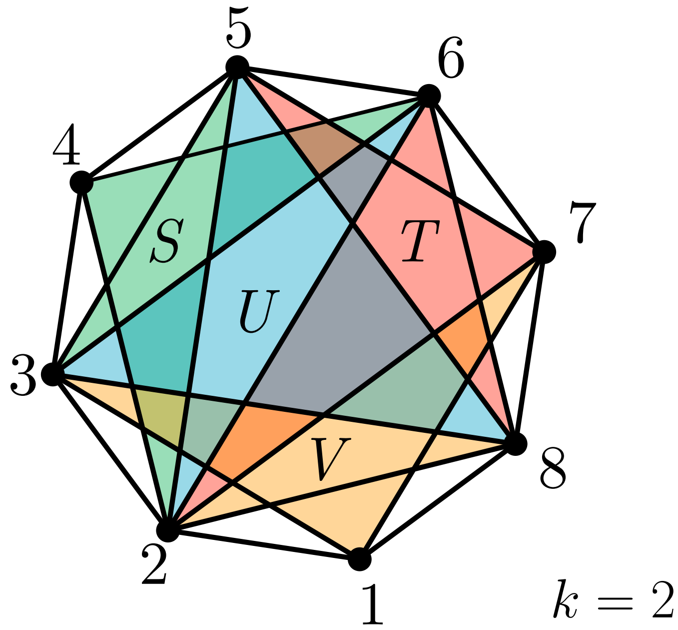

Triangulations
Pseudotriangulations
Multitriangulations
R
S
T
U
V
W
$k = 2$

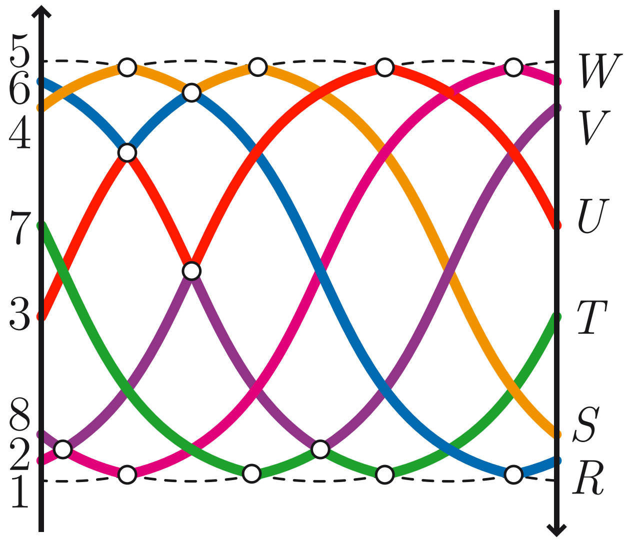
DUALITY

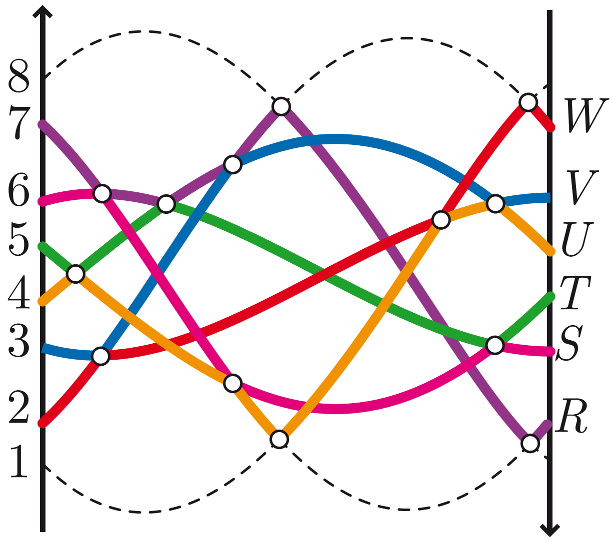
Triangulations	Pseudotriangulations	Multitriangulations

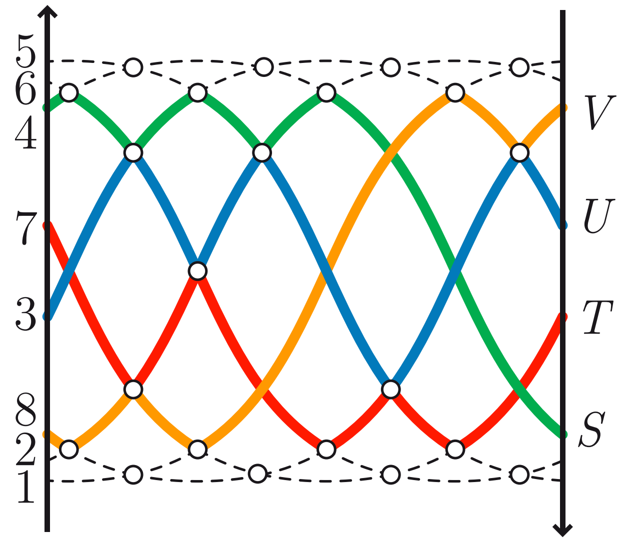






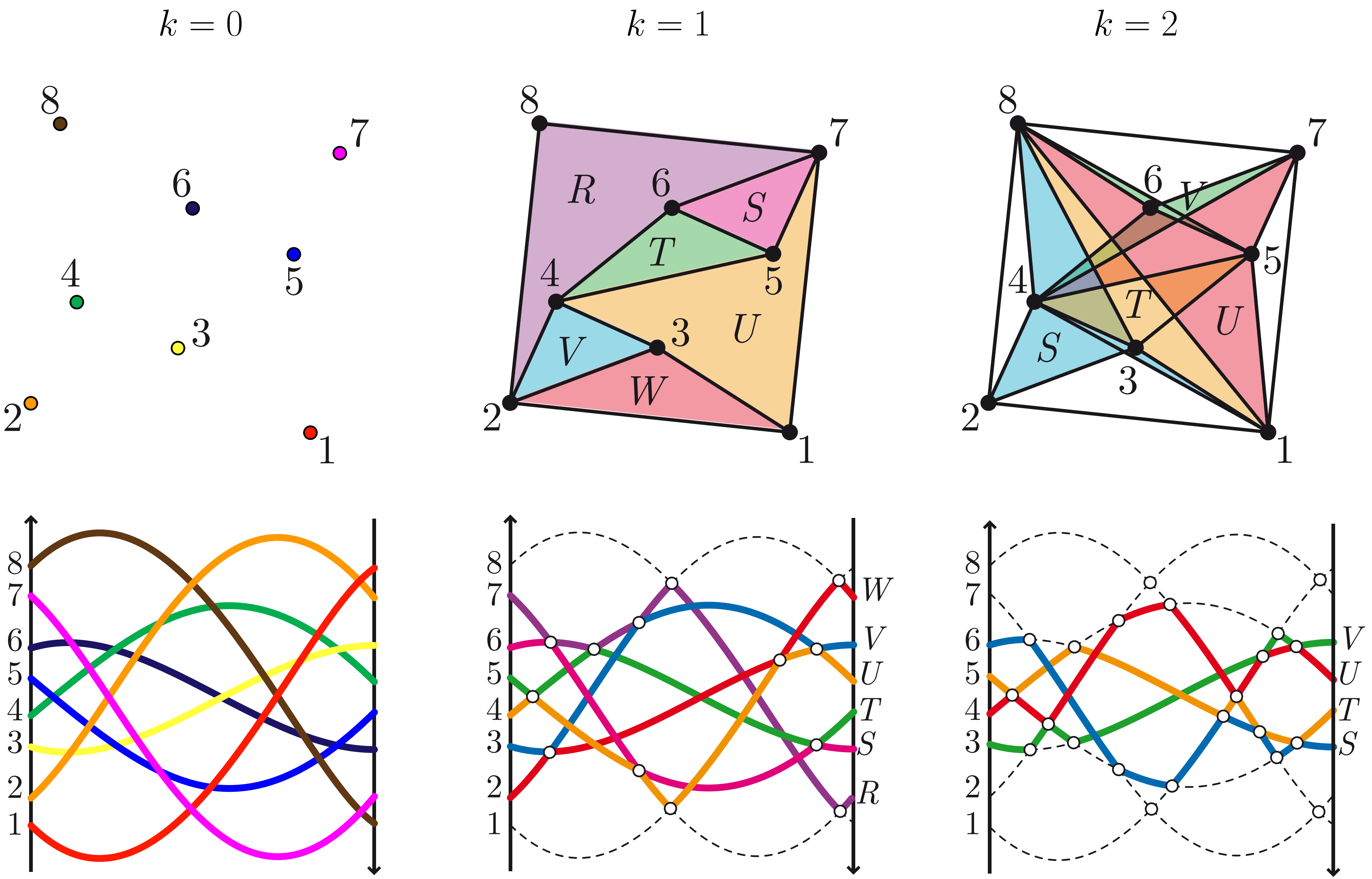






VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

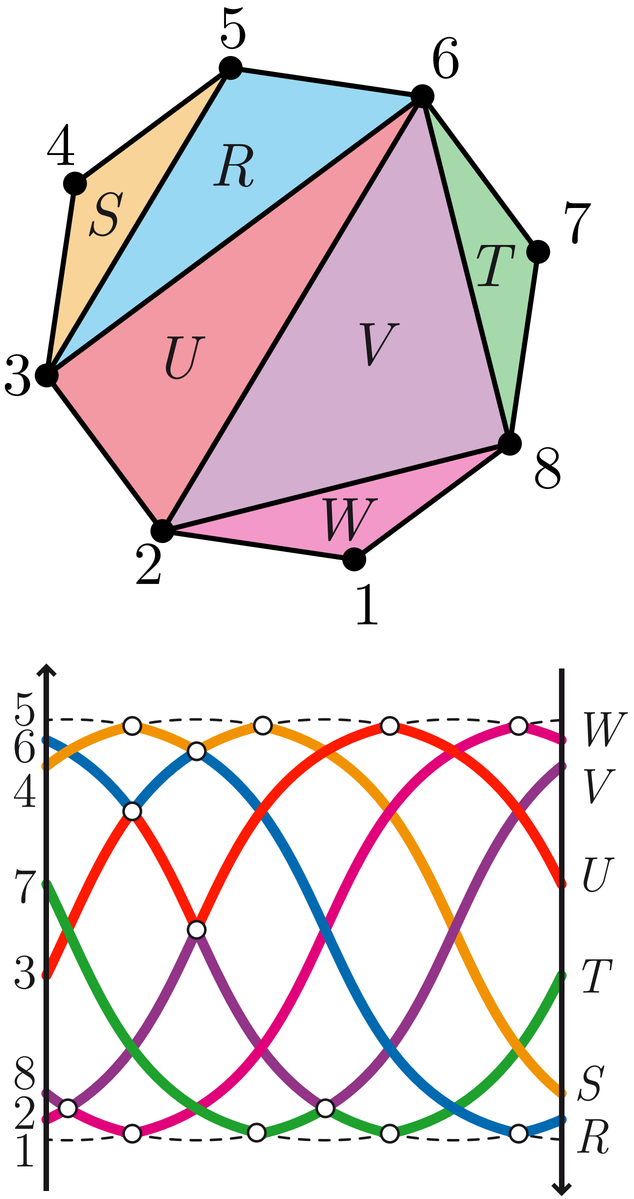
MULTIPSEUDOTRIANGULATIONS



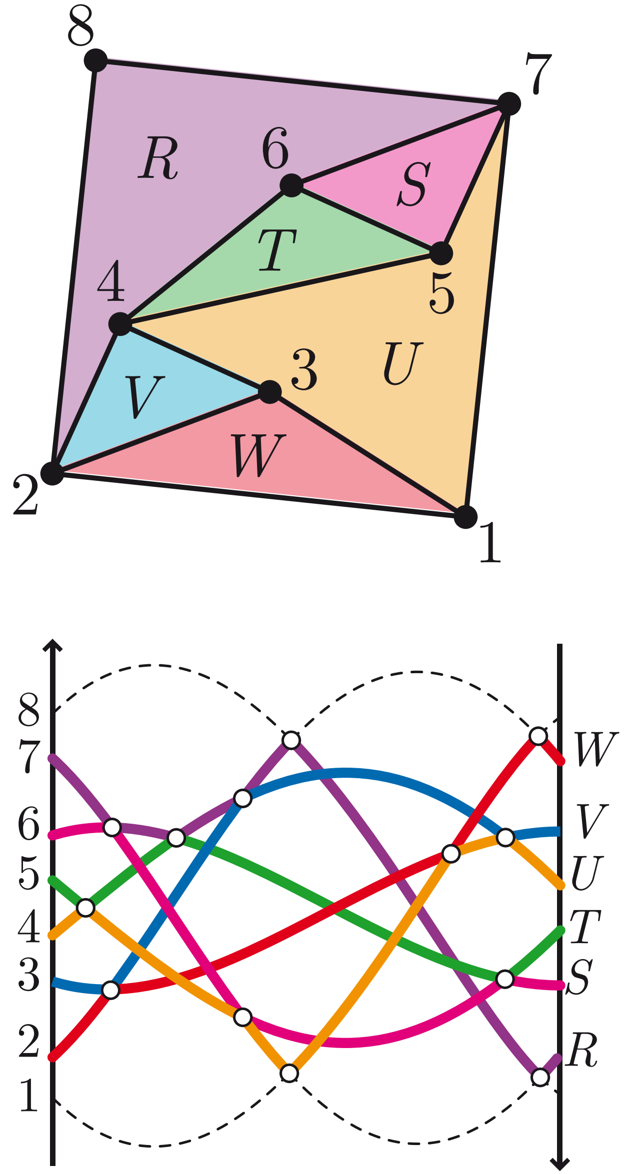
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

DUALITY

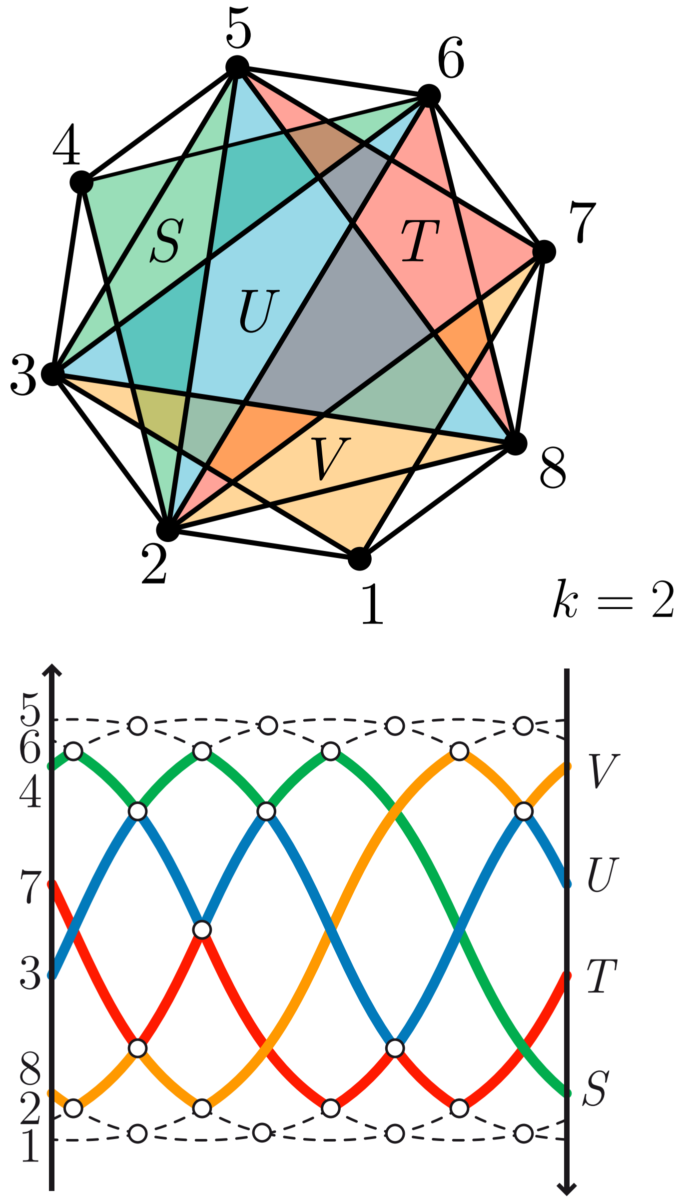
Triangulations



Pseudotriangulations



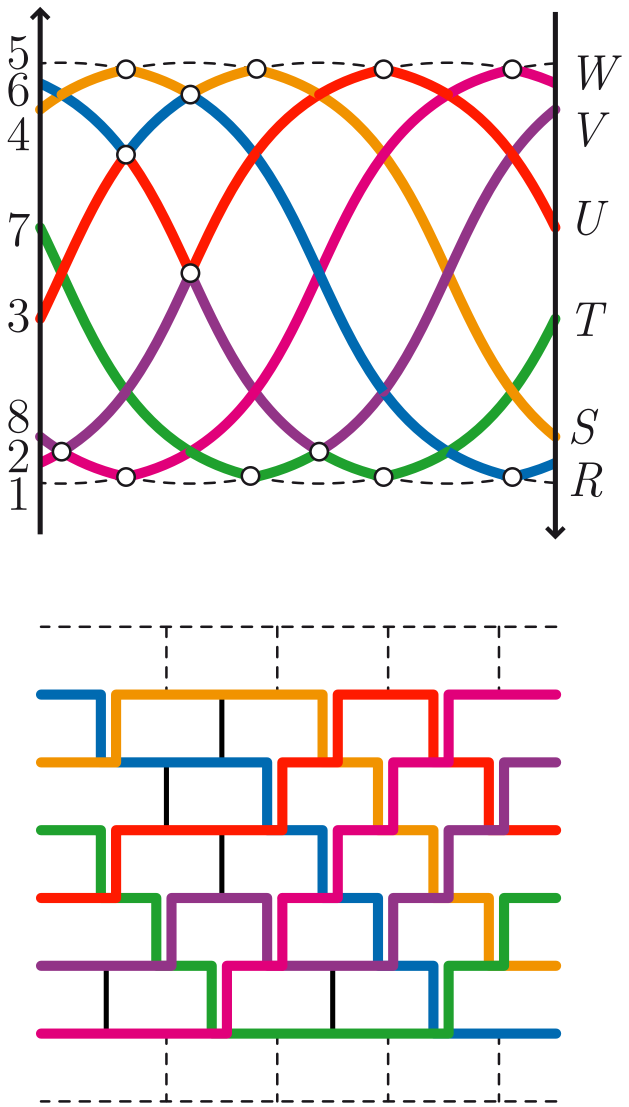
Multitriangulations



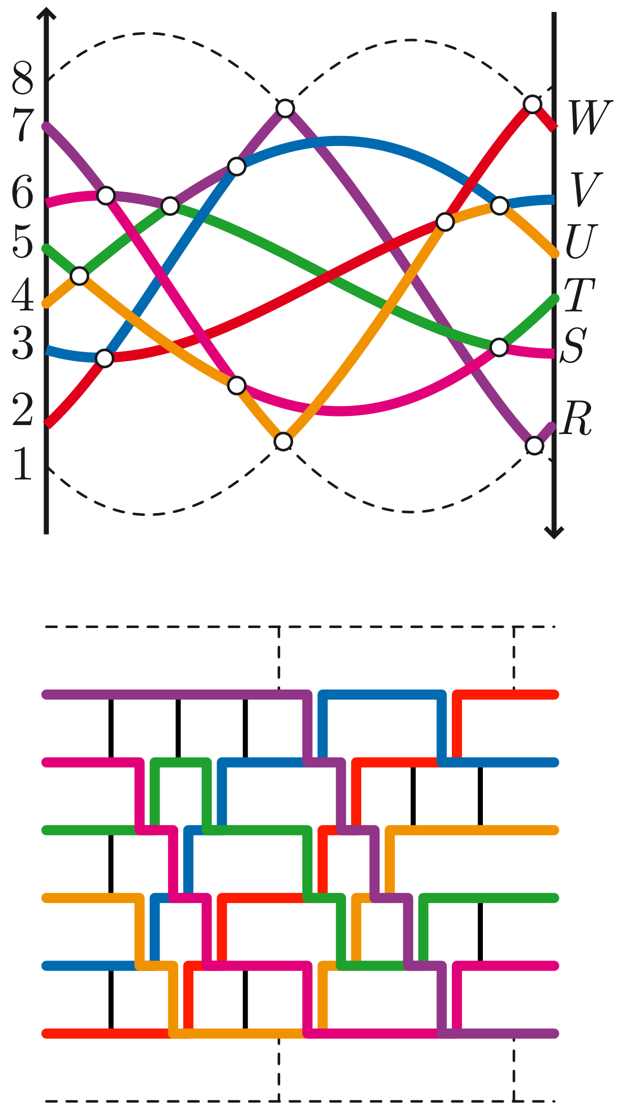
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

SORTING NETWORKS

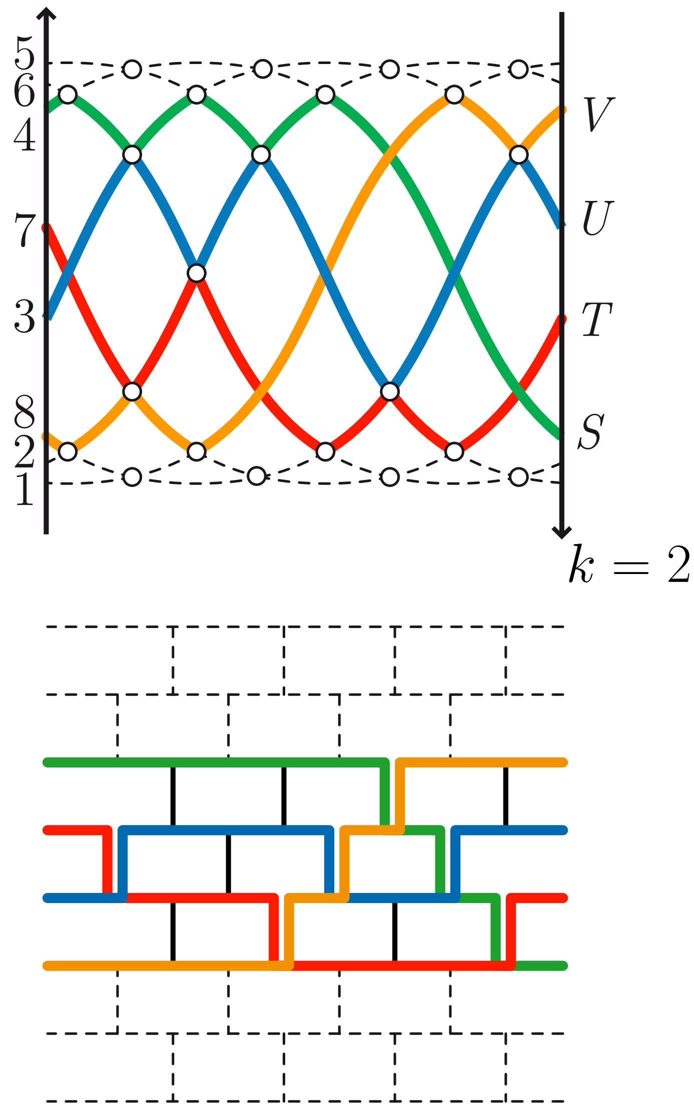
Triangulations



Pseudotriangulations



Multitriangulations



SUBWORD COMPLEXES

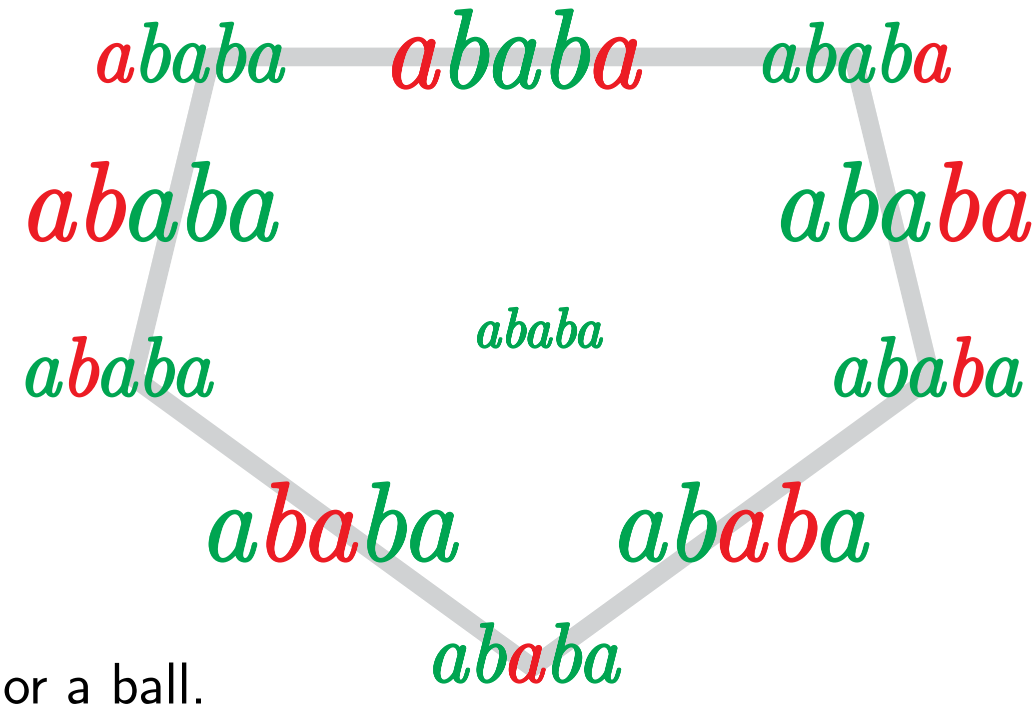
SUBWORD COMPLEX

(W, S) a finite Coxeter system, $Q = q_1 q_2 \cdots q_m$ a word on S, ρ an element of W.

Subword complex $\mathcal{S}(Q, \rho)$ = simplicial complex of subsets of positions of Q whose complement contains a reduced expression of ρ.

A. Knutson & E. Miller, Subword complexes in Coxeter groups, 2004.

$W = \mathfrak{S}_3$
$S = \{(1\ 2), (2\ 3)\} = \{a, b\}$
$Q = ababa$
$\rho = aba = bab$



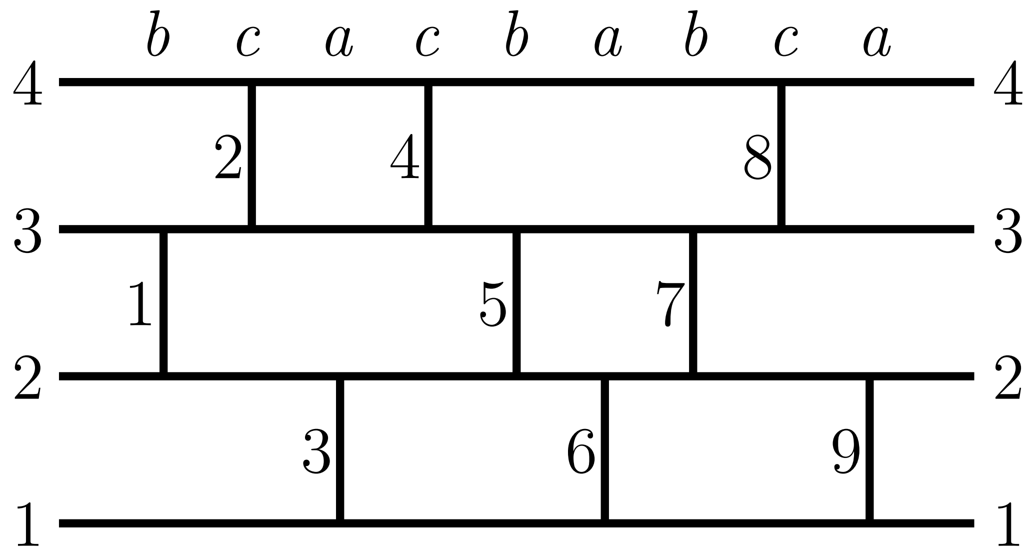
The subword complex is either a sphere (when the Demazure product of Q is ρ) or a ball.

QUESTION. Are all spherical subword complexes polytopal?

TYPE A: PRIMITIVE SORTING NETWORKS

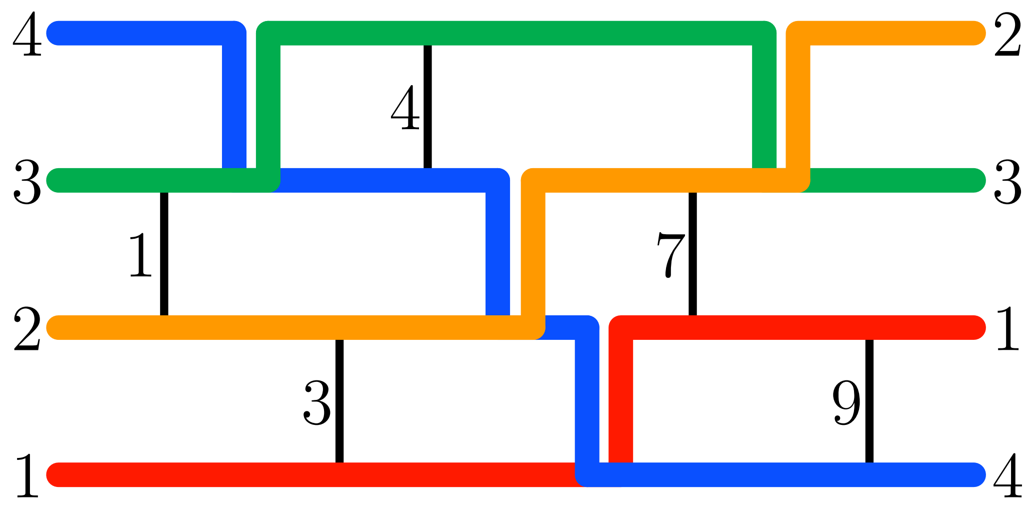
Classical situation of type A:

- Coxeter group $W = \mathfrak{S}_{n+1}$
- simple system $S = \{\tau_i \mid i \in [n]\}$, where $\tau_i = (i \;\; i+1)$
- word $Q = q_1 q_2 \cdots q_m$ on S
- ρ element of W



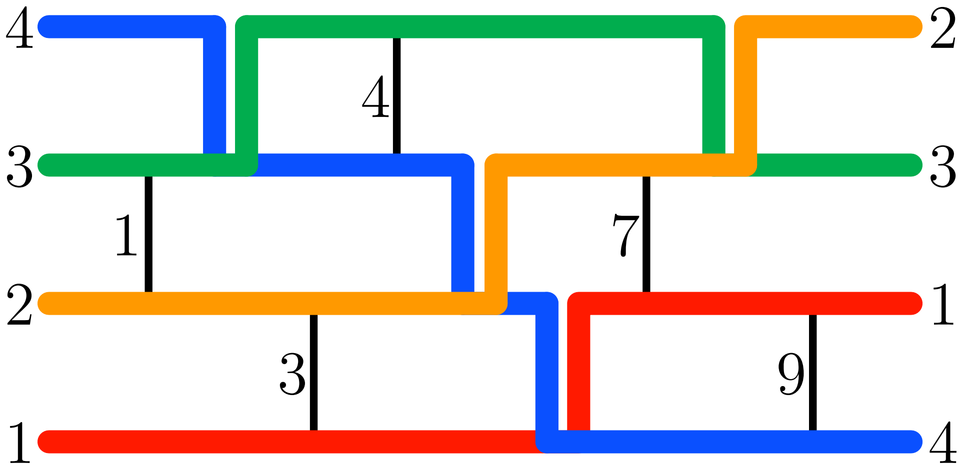
The subword complex can be interpreted with a primitive sorting network:

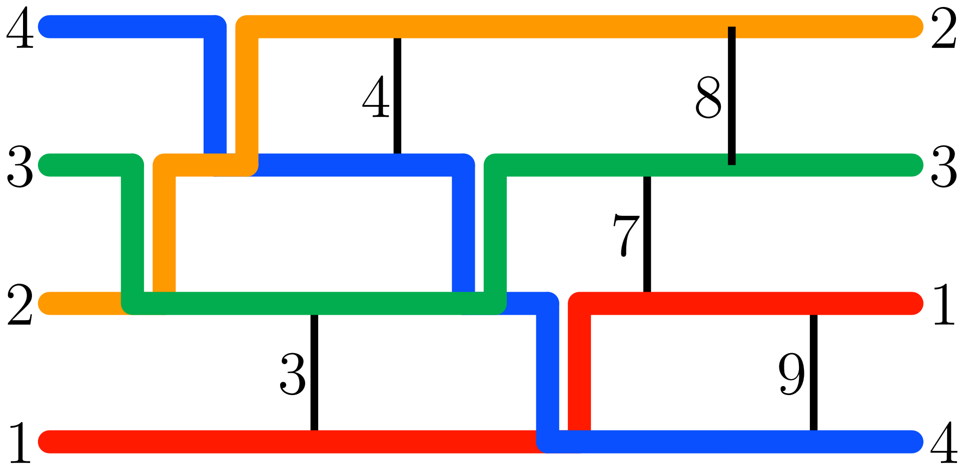
- $\mathcal{N}_Q$ formed by $n+1$ levels and m commutators
- facets of $\mathcal{S}(Q, \rho)$ $\longleftrightarrow$ pseudoline arrangements on $\mathcal{N}_Q$



FLIPS

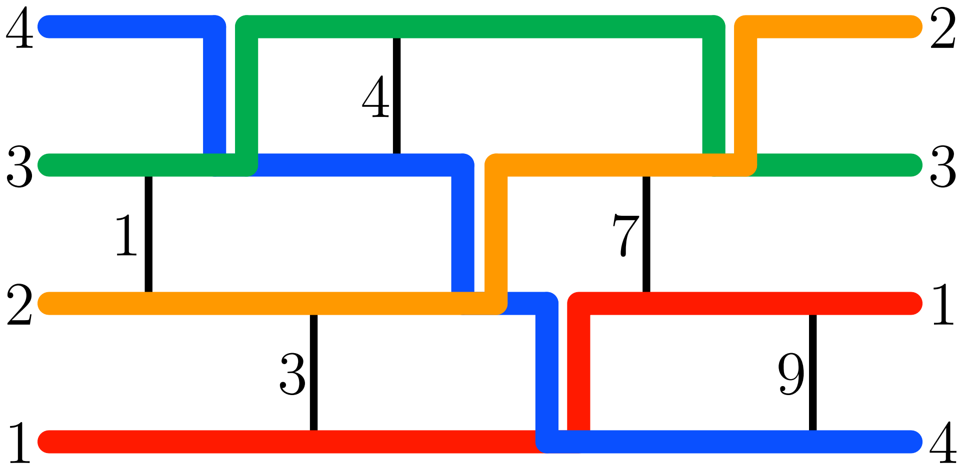
flip $=$ exchange a contact with the corresponding crossing

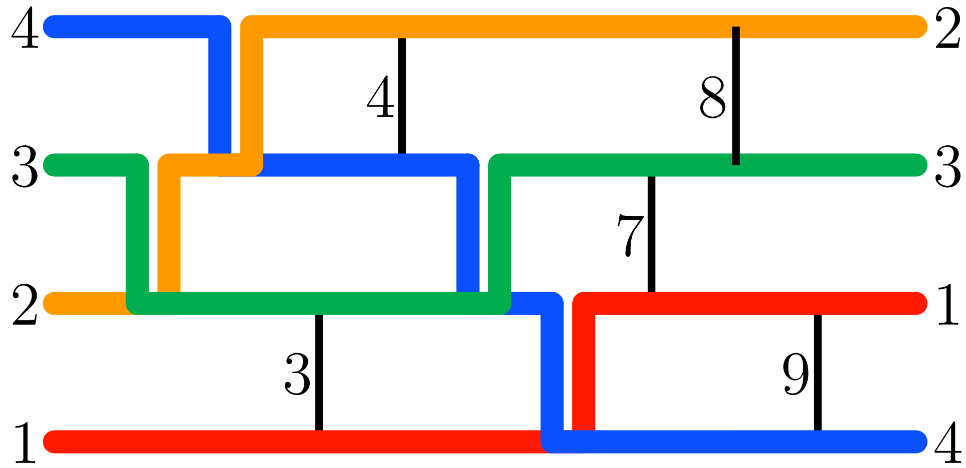


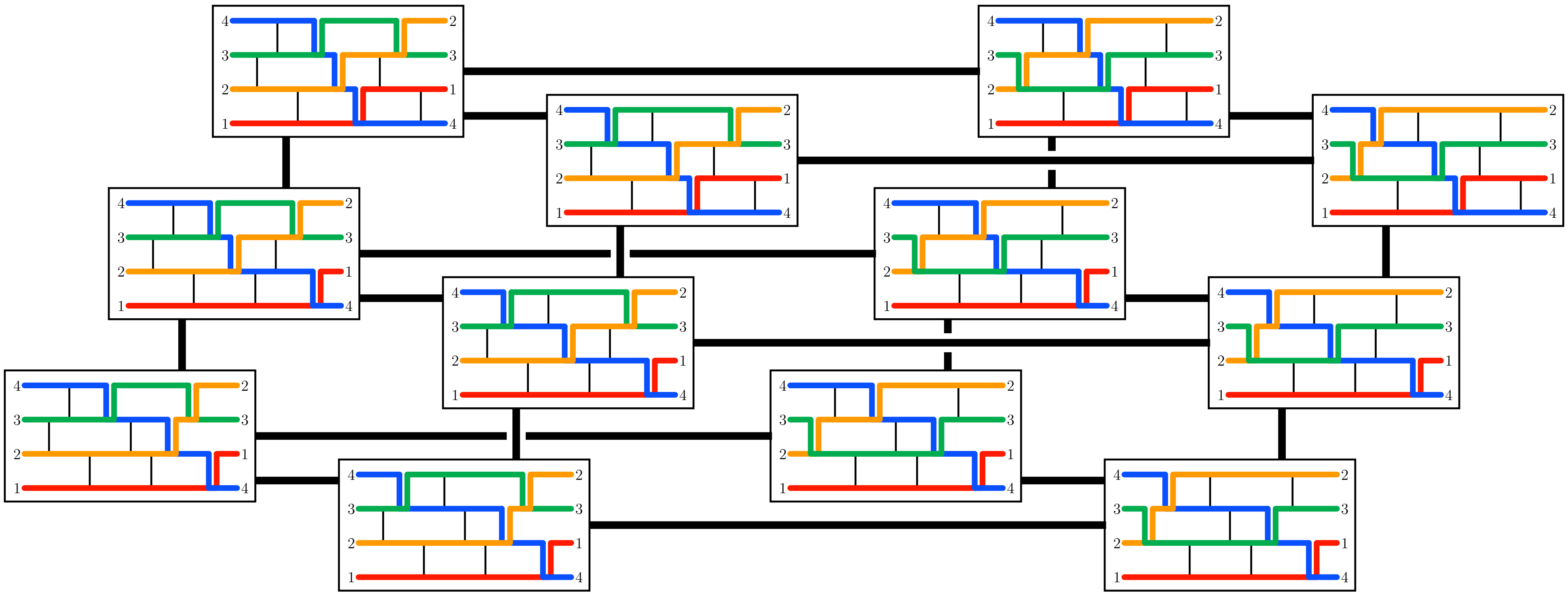


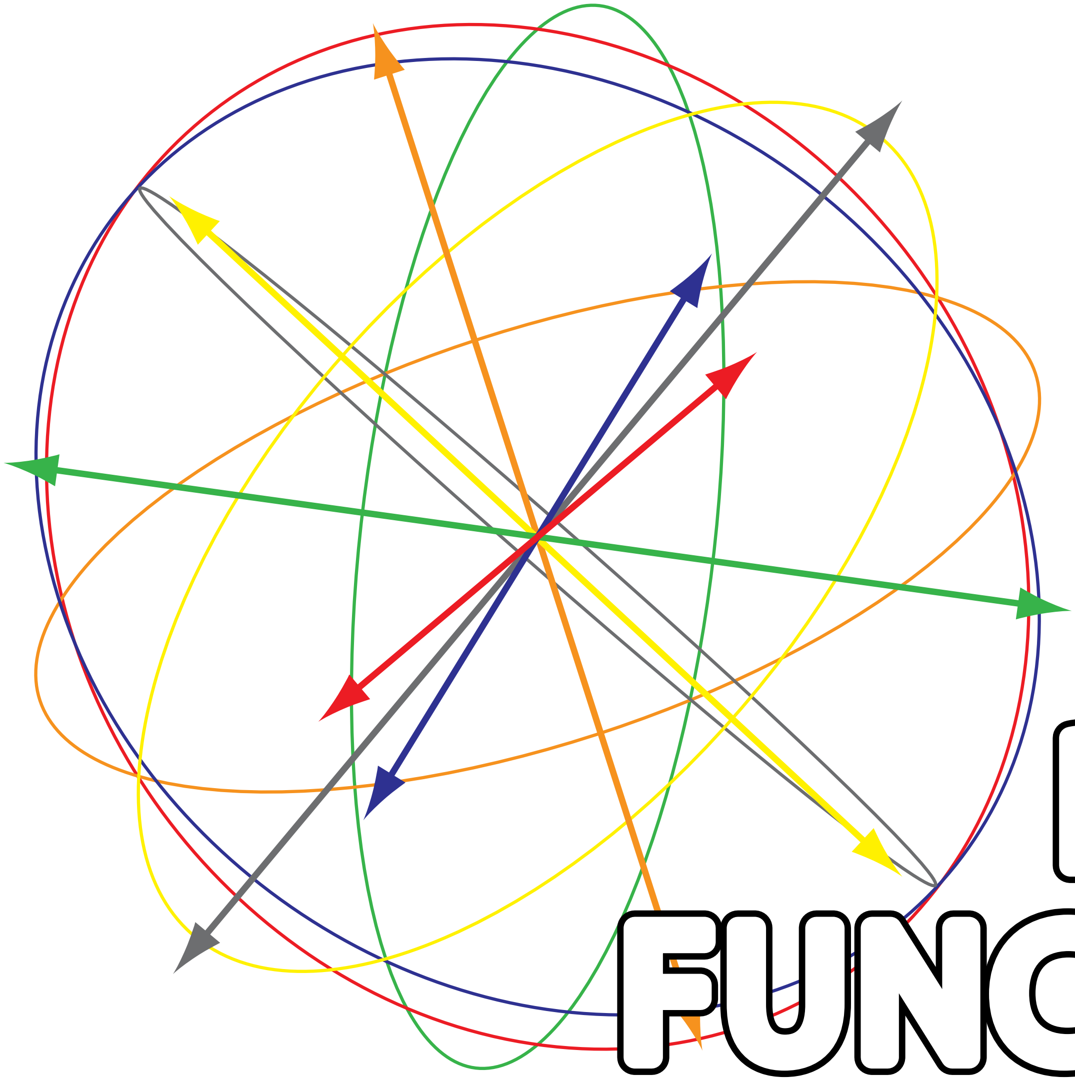
FLIPS

flip = exchange a contact with the corresponding crossing







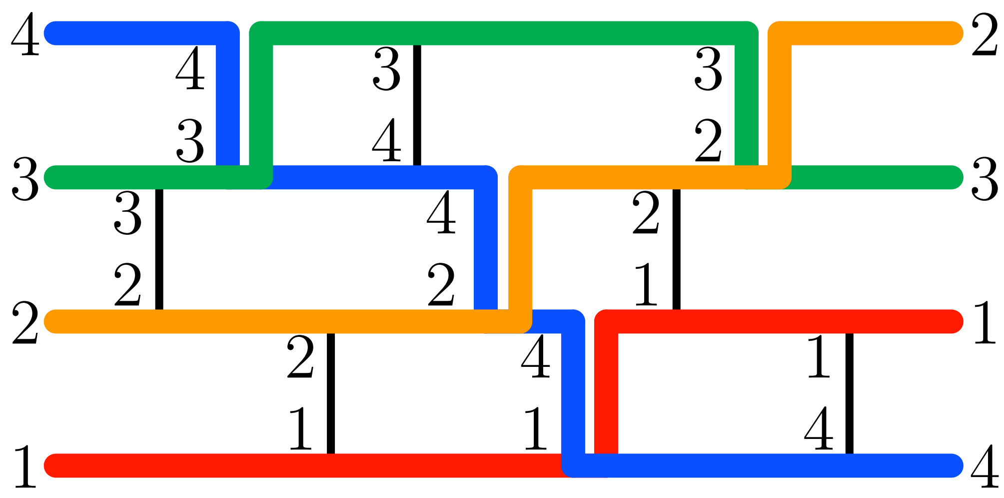


ROOT FUNCTION

C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.

VP & C. Stump, Brick polytopes of spherical subword complexes, 2015.

ROOT FUNCTION

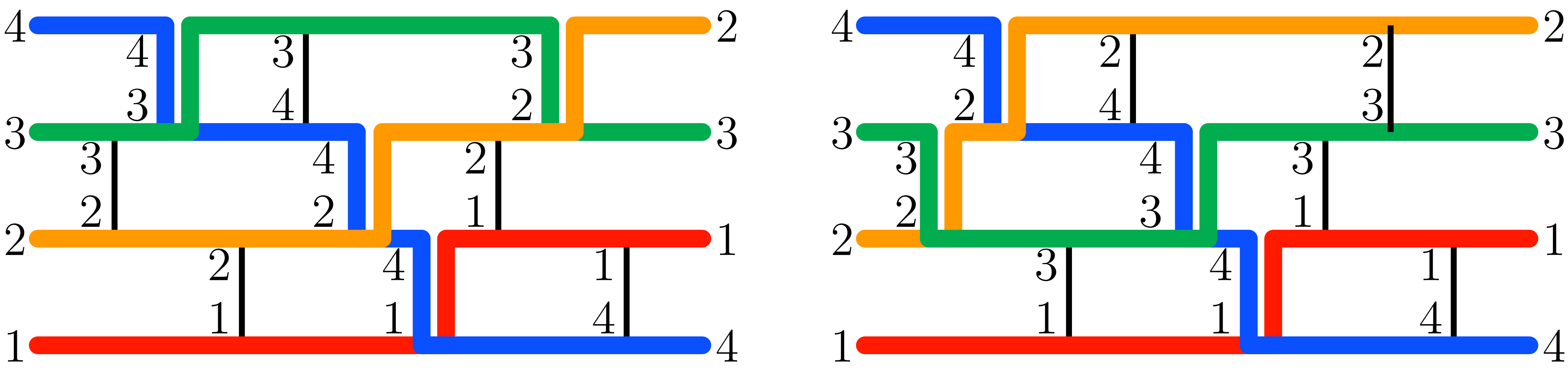


For a facet I of $\mathcal{S}(\mathrm{Q}, \rho)$ and a position $k \in [m]$, define the root $\mathsf{r}(I, k) = \mathrm{Q}_{[k-1] \setminus I}(\alpha_{q_k})$, where $\mathrm{Q}_{[k-1] \setminus I}$ is the product of all reflections q_j for j from 1 to $k-1$ but not in I.

The root function of the facet I is $\mathsf{r}(I, \cdot) : [m] \longrightarrow \Phi$

The root configuration of I is $\mathsf{R}(I) = \{\mathsf{r}(I, i) \mid i \in I\}$

ROOT FUNCTION & FLIPS

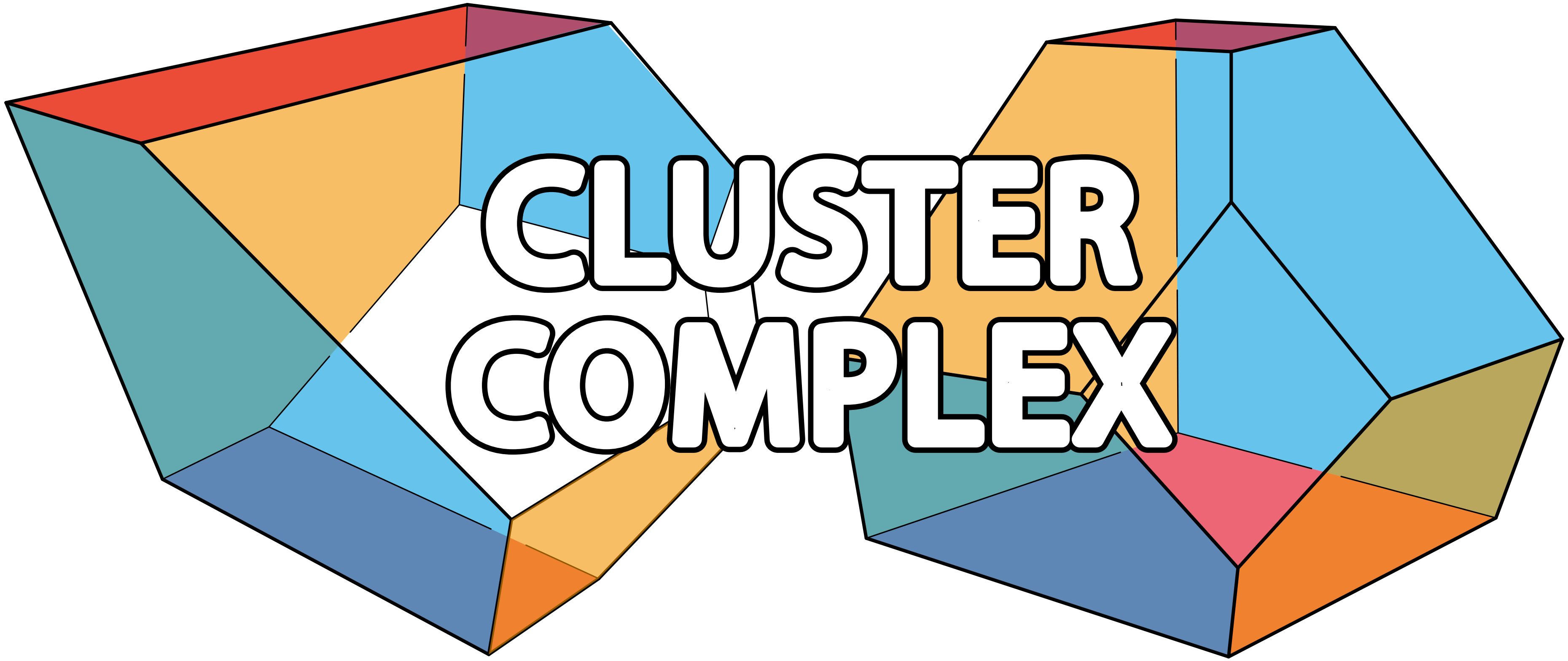


PROPOSITION. The root function encodes flips in subword complexes:

1. The map $\mathsf{r}(I, \cdot)$ is a bijection from the complement of I to $\mathrm{inv}(\rho)$.
2. If I and J are two adjacent facets of $\mathcal{S}(\mathrm{Q}, \rho)$ with $I \smallsetminus i = J \smallsetminus j$, then j is the unique position in the complement of I such that $\mathsf{r}(I, i) = \pm\mathsf{r}(I, j)$.
3. In the situation of 2, the root function of J is obtained from that of I by

$$\mathsf{r}(J, k) = \begin{cases} s_{\mathsf{r}(I,i)}(\mathsf{r}(I, k)) & \text{if } \min(i, j) < k \leq \max(i, j), \\ \mathsf{r}(I, k) & \text{otherwise.} \end{cases}$$

C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.



S. Fomin & A. Zelevinsky, Cluster Algebras I, II, III, IV, 2002 – 2007.
C. Hohlweg, C. Lange & H. Thomas, Permutahedra and generalized associahedra, 2011.
C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.
VP & C. Stump, Brick polytopes of spherical subword complexes, 2015.
C. Hohlweg, Permutahedra and associahedra, 2013.

CLUSTER ALGEBRAS

cluster algebra $=$ commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process :

cluster seed $=$ algebraic data $\{x_1, \dots, x_n\}$, combinatorical data B (matrix or quiver)

cluster mutation $= (\{x_1, \dots, x_k, \dots, x_n\}, B) \overset{\mu_k}{\longleftrightarrow} (\{x_1, \dots, x'_k, \dots, x_n, \mu_k(B))$

$$x_k \cdot x'_k = \prod_{i,\, b_{ik}>0} x_i^{b_{ik}} + \prod_{i,\, b_{ik}<0} x_i^{-b_{ik}}$$

$$\big(\mu_k(B)\big)_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i,j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

cluster complex $=$ simplicial complex w/ vertices $=$ cluster variables & facets $=$ clusters

S. Fomin & A. Zelevinsky, Cluster Algebras I, II, III, IV, 2002 – 2007.

CLUSTER ALGEBRAS

THEOREM. (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.

S. Fomin & A. Zelevinsky, Cluster algebras I: Fundations, 2002.

THEOREM. (Classification)
Finite type cluster algebras are classified by the Cartan-Killing classification for crystallographic root systems.

S. Fomin & A. Zelevinsky, Cluster algebras II: Finite type classification, 2003.

In fact, for a root system Φ, there is a bijection

cluster variables	$\longleftrightarrow$	$\Phi_{\geq -1} = \Phi^+ \cup -\Delta$
$y = \dfrac{F(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$	$\longleftrightarrow$	$\beta = d_1\alpha_1 + \cdots + d_n\alpha_n$
cluster	$\longleftrightarrow$	c-cluster
cluster complex	$\longleftrightarrow$	c-cluster complex

FINITE CLUSTER COMPLEXES ARE SUBWORD COMPLEXES

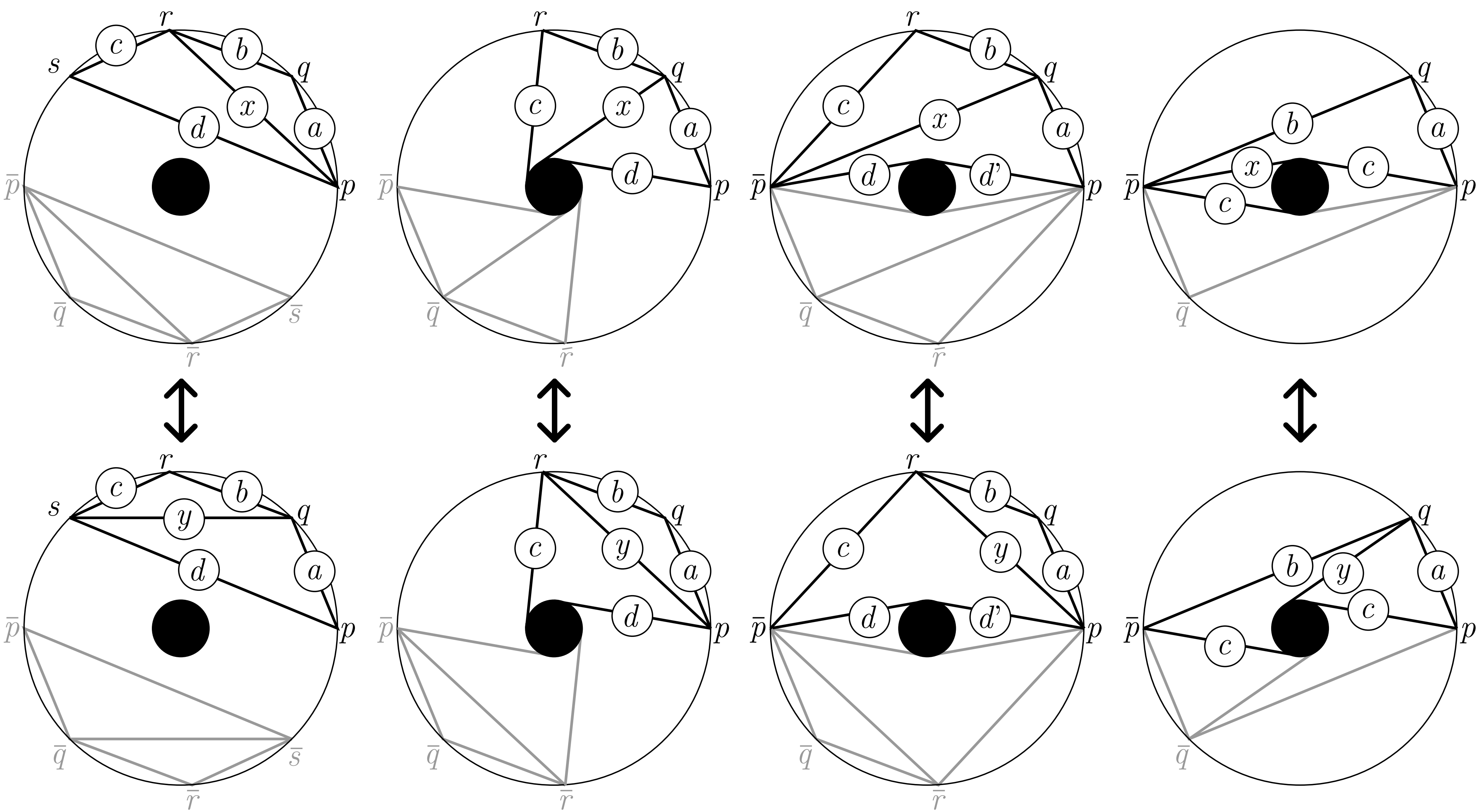
New approach to the combinatorics and geometry of the cluster complex:

THEOREM. The subword complex $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$ is isomorphic to the cluster complex.

C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.

cluster variables	$\longleftrightarrow$	$\Phi_{\geq -1} = \Phi^+ \cup -\Delta$	$\longleftrightarrow$	position in $\mathrm{cw}_\circ(\mathrm{c})$
$y = \dfrac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$	$\longleftrightarrow$	$\beta = d_1\alpha_1 + \cdots + d_n\alpha_n$	$\longleftrightarrow$	$\begin{cases} i & \text{if } \beta = -\alpha_{c_i} \\ j & \text{if } \beta = \mathsf{r}([n], j) \end{cases}$
cluster	$\longleftrightarrow$	c-cluster	$\longleftrightarrow$	facet of $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$
cluster complex	$\longleftrightarrow$	c-cluster complex	$\longleftrightarrow$	subword complex $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$

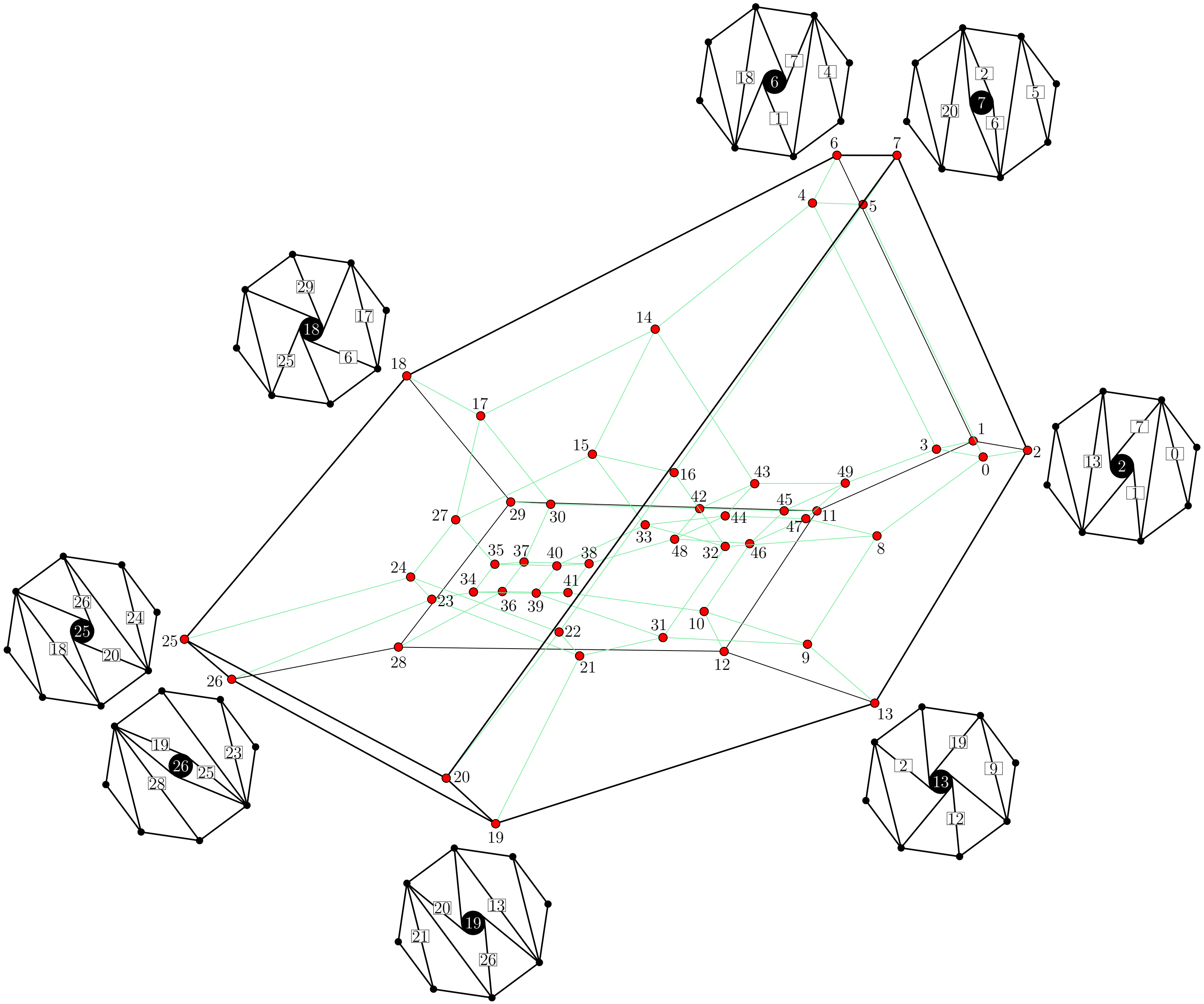
TYPE D_n AS PSEUDOTRIANGULATIONS



$xy = ac + bd$ | $xy = ac + bd$ | $xy = ac + bdd'$ | $(xc)y = ac + bc$

C. Ceballos & VP, Cluster algebras of type D: pseudotriangulations approach, 2015$^+$.

TYPE D_n



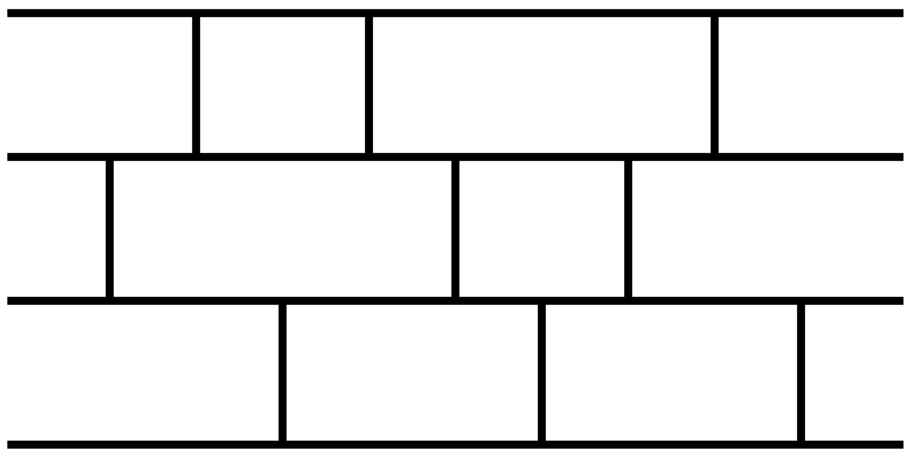
C. Ceballos & VP, Cluster algebras of type D: pseudotriangulations approach, 2015^+.

BRICK POLYTOPE

VP & F. Santos, The brick polytope of a sorting network, 2012.

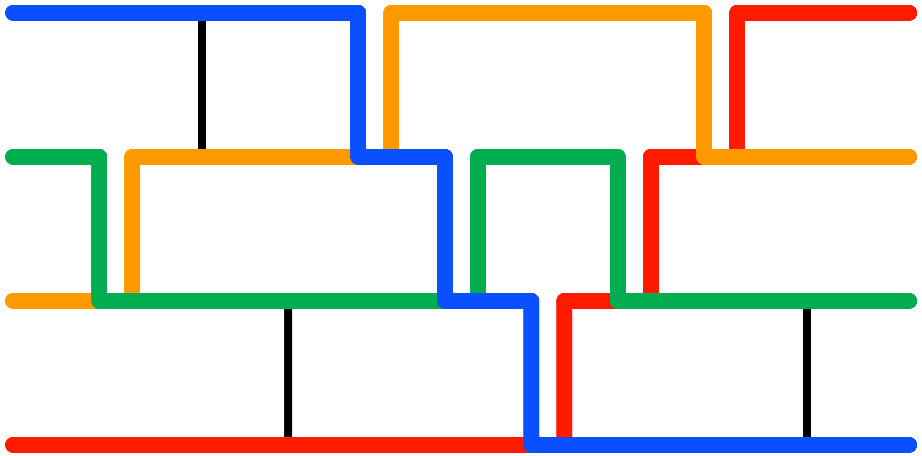
VP & C. Stump, Brick polytopes of spherical subword complexes, 2015.

BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels

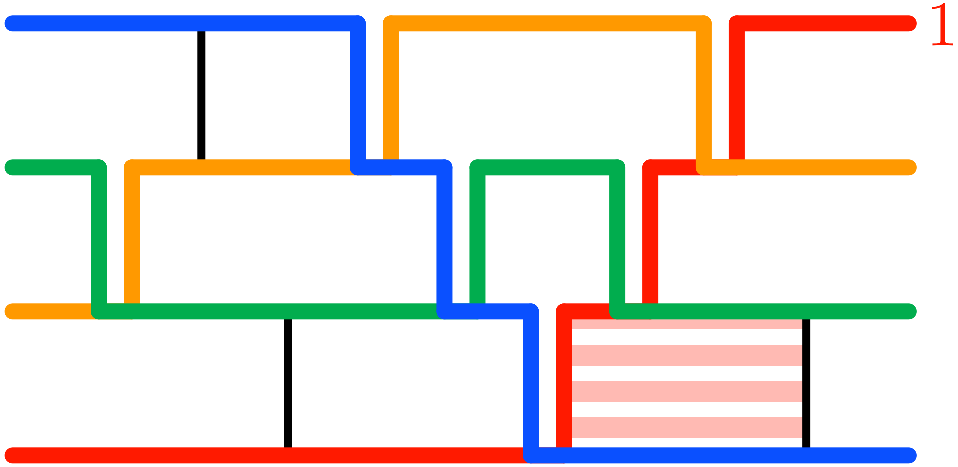
BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels

Λ pseudoline arrangement supported by $\mathcal{N}$ $\quad\longmapsto\quad$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

BRICK POLYTOPE

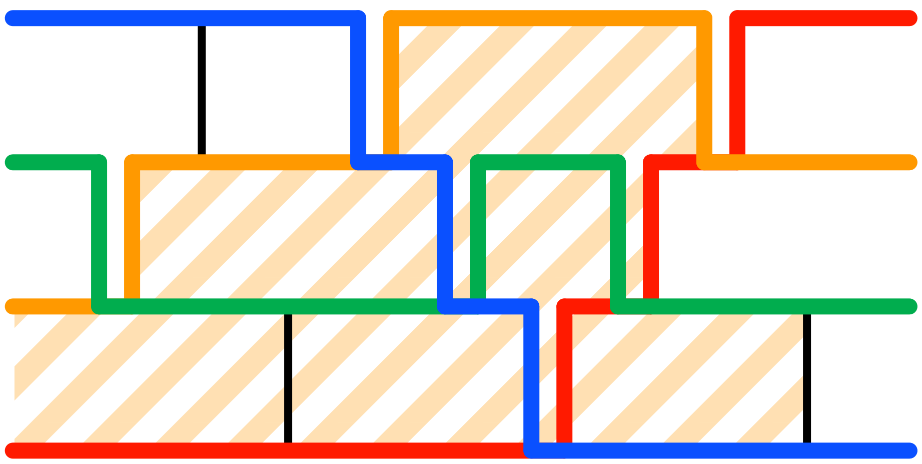


$\mathcal{N}$ a sorting network with $n+1$ levels

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

$\mathsf{B}(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

BRICK POLYTOPE



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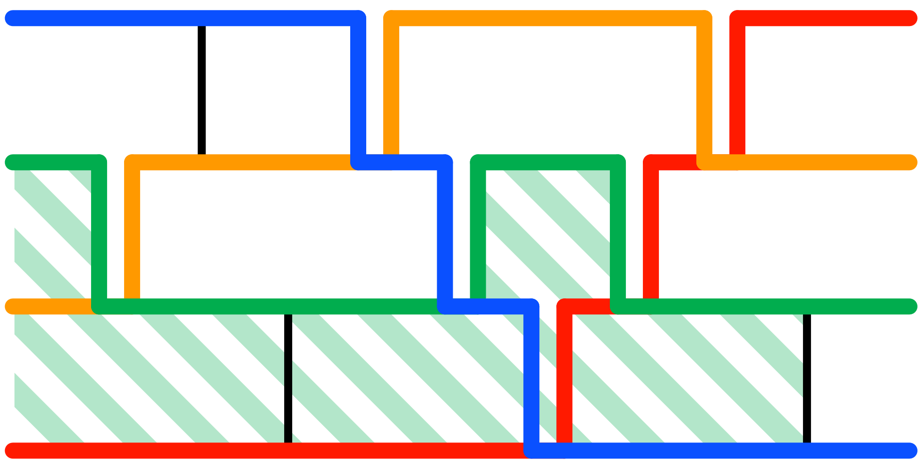

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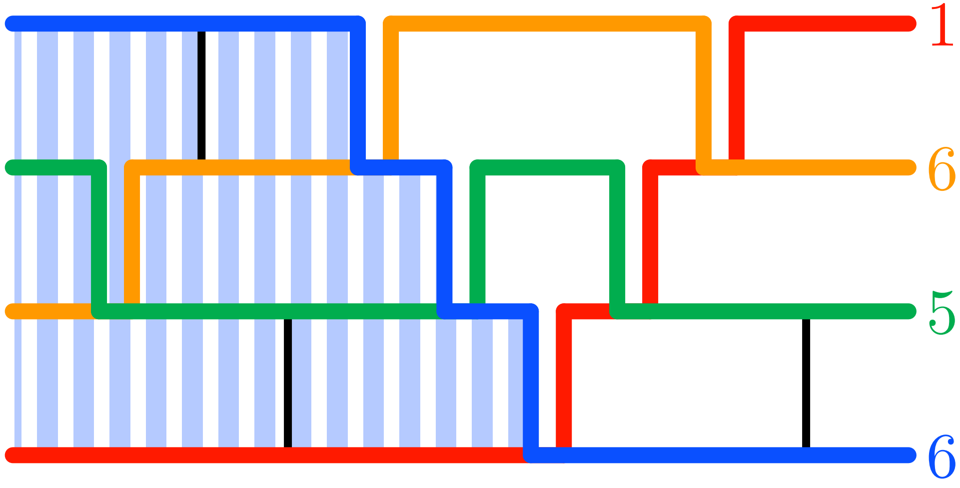

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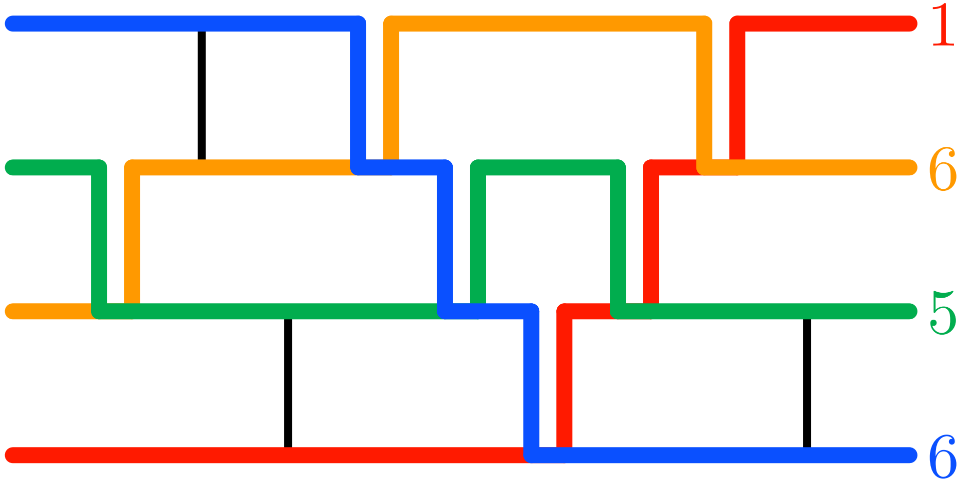


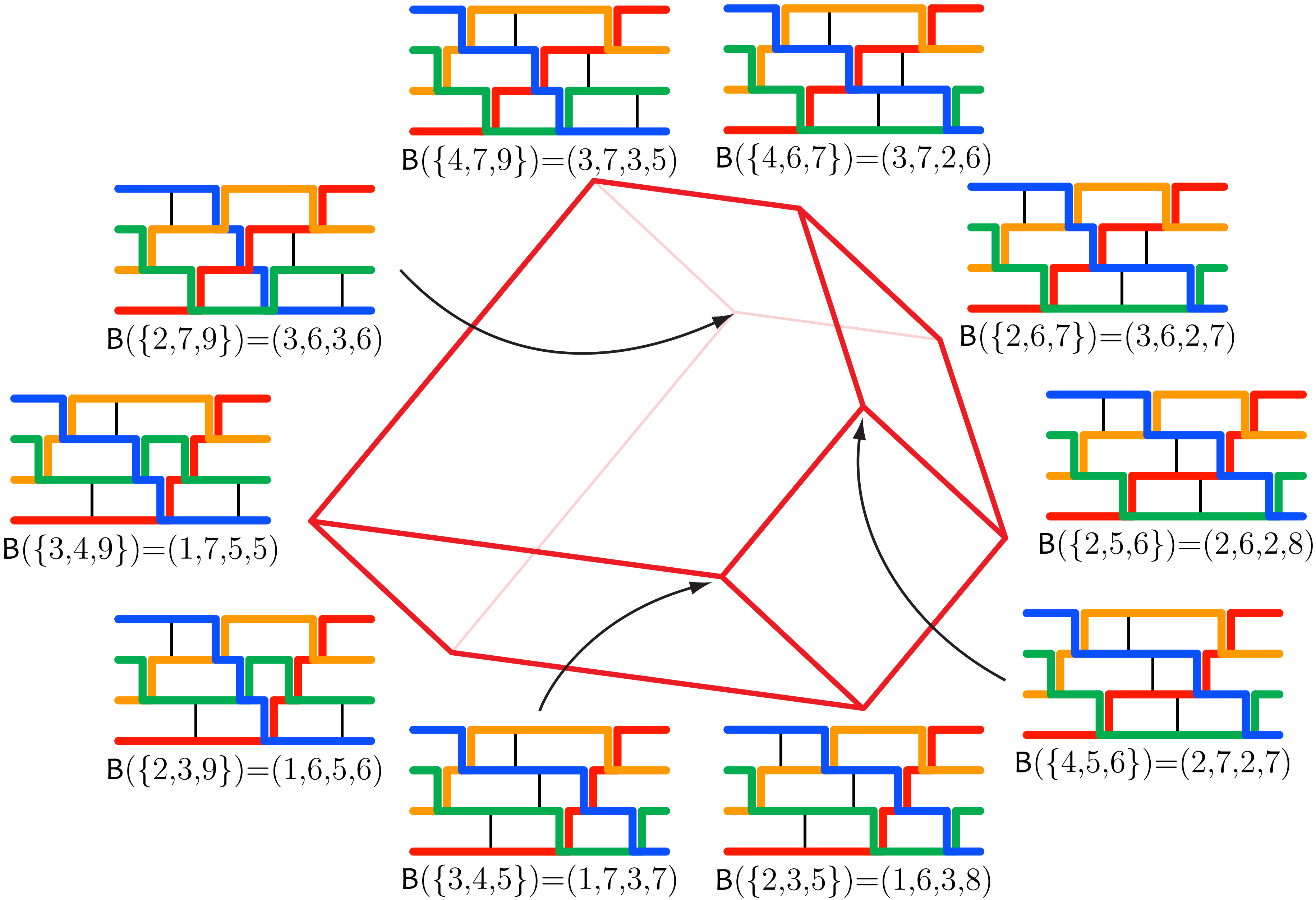
$\mathcal{N}$ a sorting network with $n+1$ levels

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathrm{B}(\Lambda) \in \mathbb{R}^{n+1}$

$\mathrm{B}(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

Brick polytope $\mathcal{B}(\mathcal{N}) = \mathrm{conv}\,\{\mathrm{B}(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE



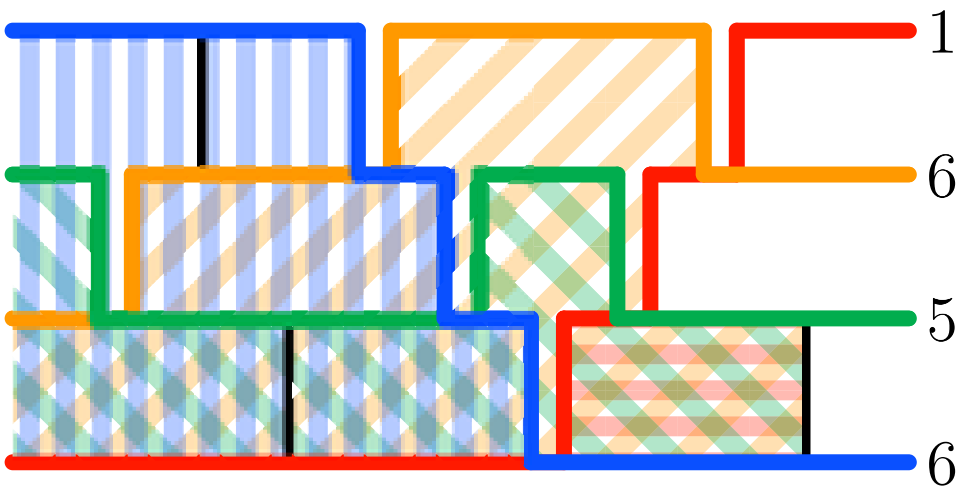
WEIGHT FUNCTION, BRICK VECTOR & BRICK POLYTOPE

(W, S) a finite Coxeter system, $\mathrm{Q} = q_1 q_2 \cdots q_m$ a word on S, $w_\circ$ longest element of W.
$\mathcal{S}(\mathrm{Q}) = \mathcal{S}(\mathrm{Q}, w_\circ)$ spherical subword complex.

To a facet I of $\mathcal{S}(\mathrm{Q})$ and a position $k \in [m]$, associate a weight $\mathsf{w}(I, k) = \mathrm{Q}_{[k-1] \smallsetminus I}(\omega_{q_k})$, where $\mathrm{Q}_{[k-1] \smallsetminus I}$ is the product of all reflections q_j for j from 1 to $k-1$ but not in I.

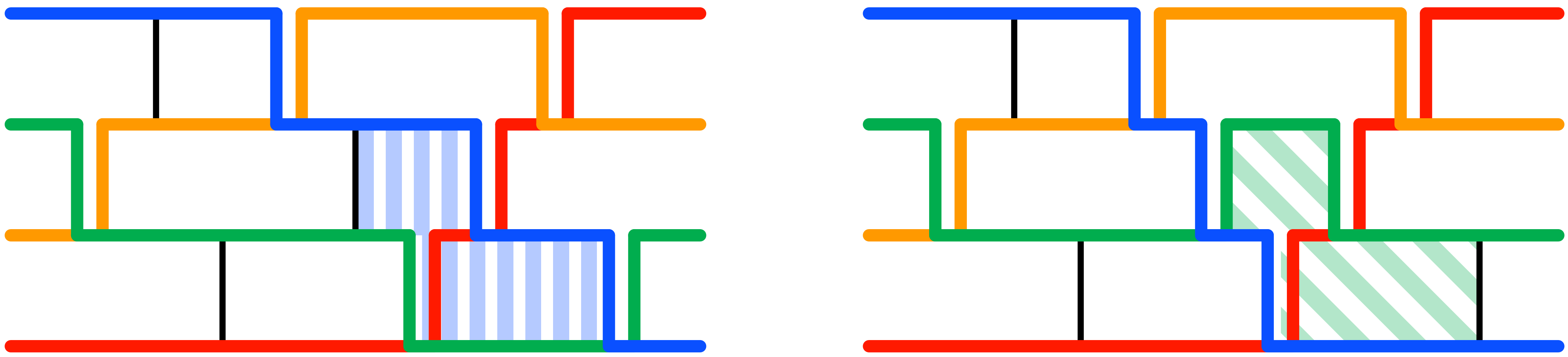
The brick vector of I is the vector $\mathsf{B}(I) = \sum_{k \in [m]} \mathsf{w}(I, k)$.

The brick polytope is the convex polytope $\mathcal{B}(\mathrm{Q}) = \mathrm{conv}\,\{\mathsf{B}(I) \mid I \text{ facet of } \mathcal{S}(\mathrm{Q})\}$.



In type A, $\mathsf{w}(I, k) =$ characteristic vector of the pseudolines passing above the kth brick.
$\mathsf{B}(I) = (\text{number of bricks below the } j\text{th pseudoline of } I)_{j \in [n+1]}$

BRICK VECTORS AND FLIPS



If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their ith and jth pseudolines, then $\mathsf{B}(\Lambda) - \mathsf{B}(\Lambda') \in \mathbb{N}_{>0}\,(e_j - e_i)$.

THEOREM. The cone of the brick polytope $\mathcal{B}(\mathrm{Q})$ at the brick vector $\mathsf{B}(I)$ is generated by $-\mathsf{R}(I)$, for any facet I of $\mathcal{S}(\mathrm{Q})$.

BRICK POLYTOPE

The brick polytope is the convex polytope $\mathcal{B}(\mathrm{Q}) = \mathrm{conv}\,\{\mathsf{B}(I) \mid I \text{ facet of } \mathcal{S}(\mathrm{Q})\}$.

THEOREM. The polar of the brick polytope $\mathcal{B}(\mathrm{Q})$ realizes the subword complex $\mathcal{S}(\mathrm{Q})$
$\iff$ Q is such that $\mathsf{R}(I)$ is linearly independent, for I facet of $\mathcal{S}(\mathrm{Q})$.

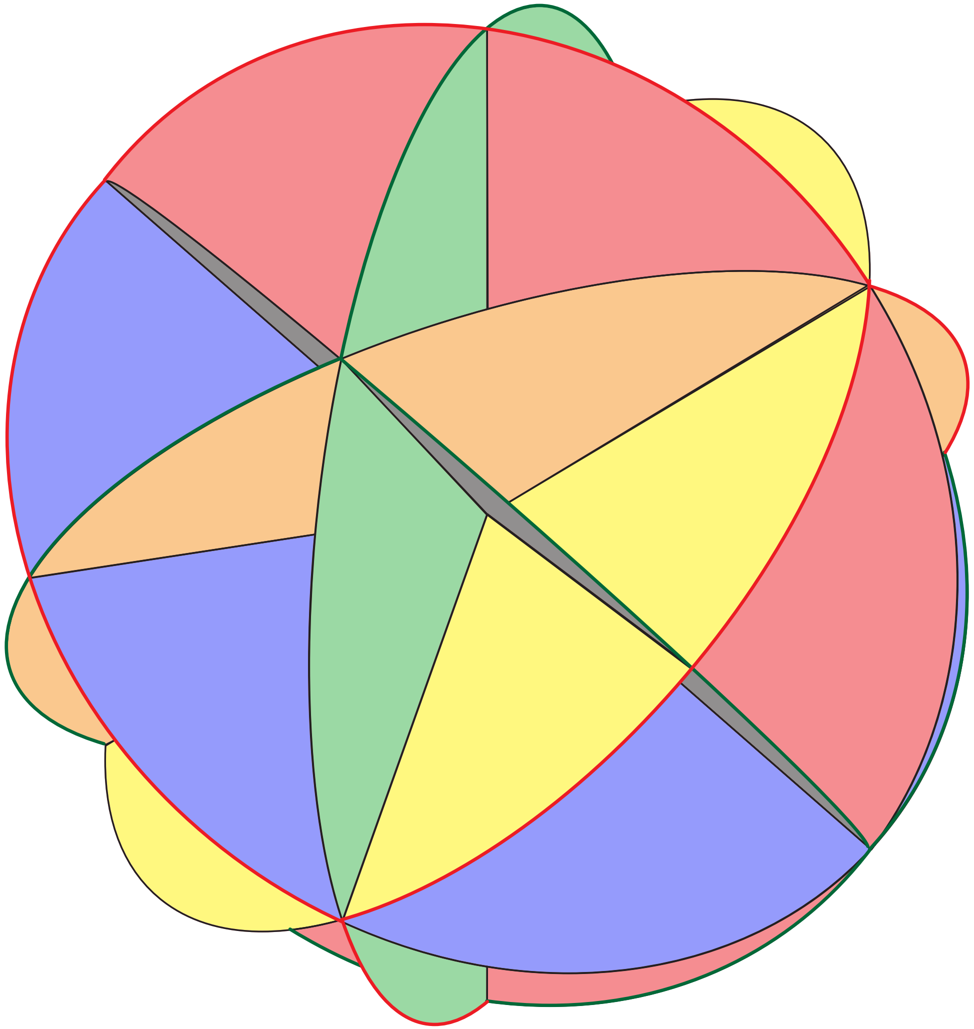
THEOREM. If Q is root-independent, the cone of the brick polytope $\mathcal{B}(\mathrm{Q})$ at the brick vector $\mathsf{B}(I)$ is generated by $-\mathsf{R}(I)$, for any facet I of $\mathcal{S}(\mathrm{Q})$.

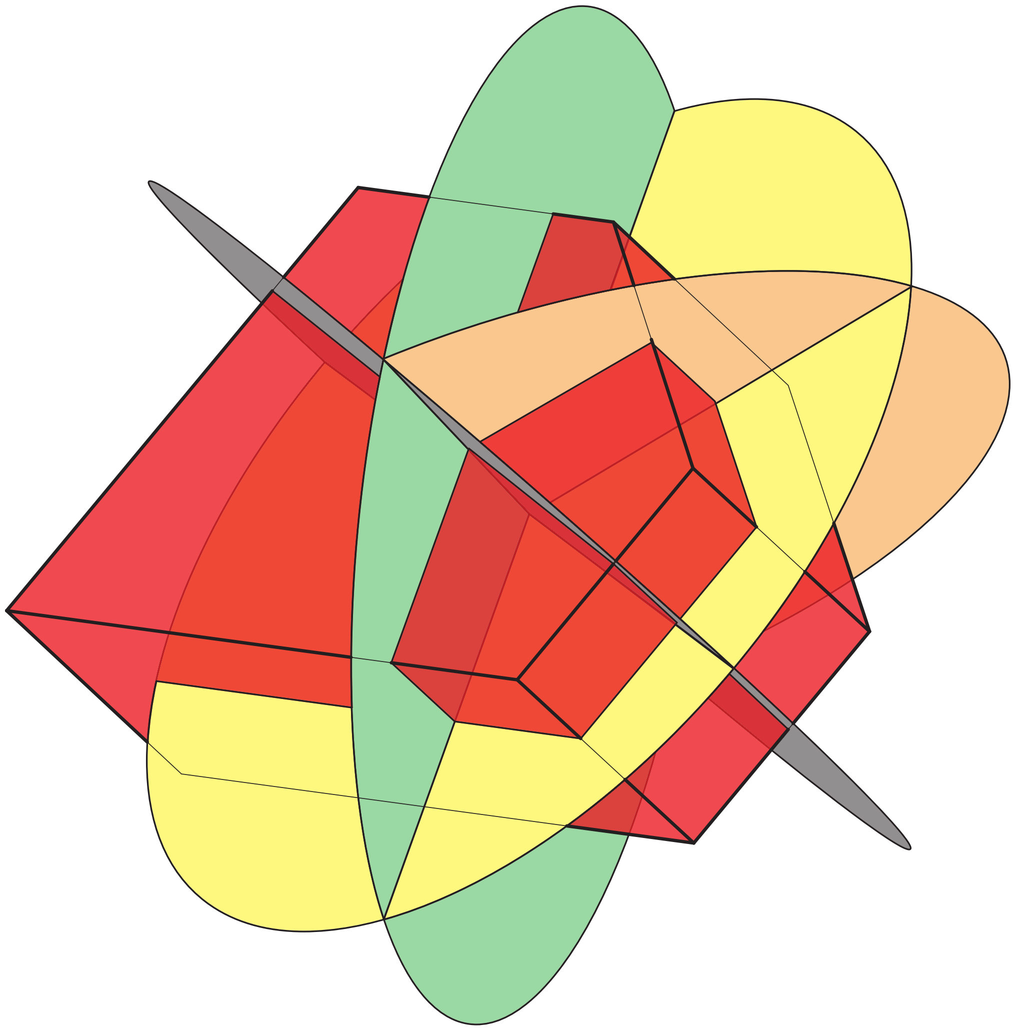
THEOREM. If Q is root-independent, the Coxeter fan refines the normal fan of the brick polytope. More precisely,

$$\text{normal cone of } \mathsf{B}(I) \text{ in } \mathcal{B}(\mathrm{Q}) = \bigcup_{\substack{w \in W \\ \mathsf{R}(I) \subset w(\Phi^+)}} w(\text{ fundamental cone }).$$

NORMAL FAN

THEOREM. The Coxeter fan refines the normal fan of the brick polytope.





GENERALIZED ASSOCIAHEDRA

THEOREM. The brick polytope $\mathcal{B}(\mathrm{cw}_\circ(\mathrm{c}))$ realizes the subword complex $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$.

THEOREM. The brick polytope $\mathcal{B}(\mathrm{cw}_\circ(\mathrm{c}))$ is a translate of the known realizations of the generalized associahedron.

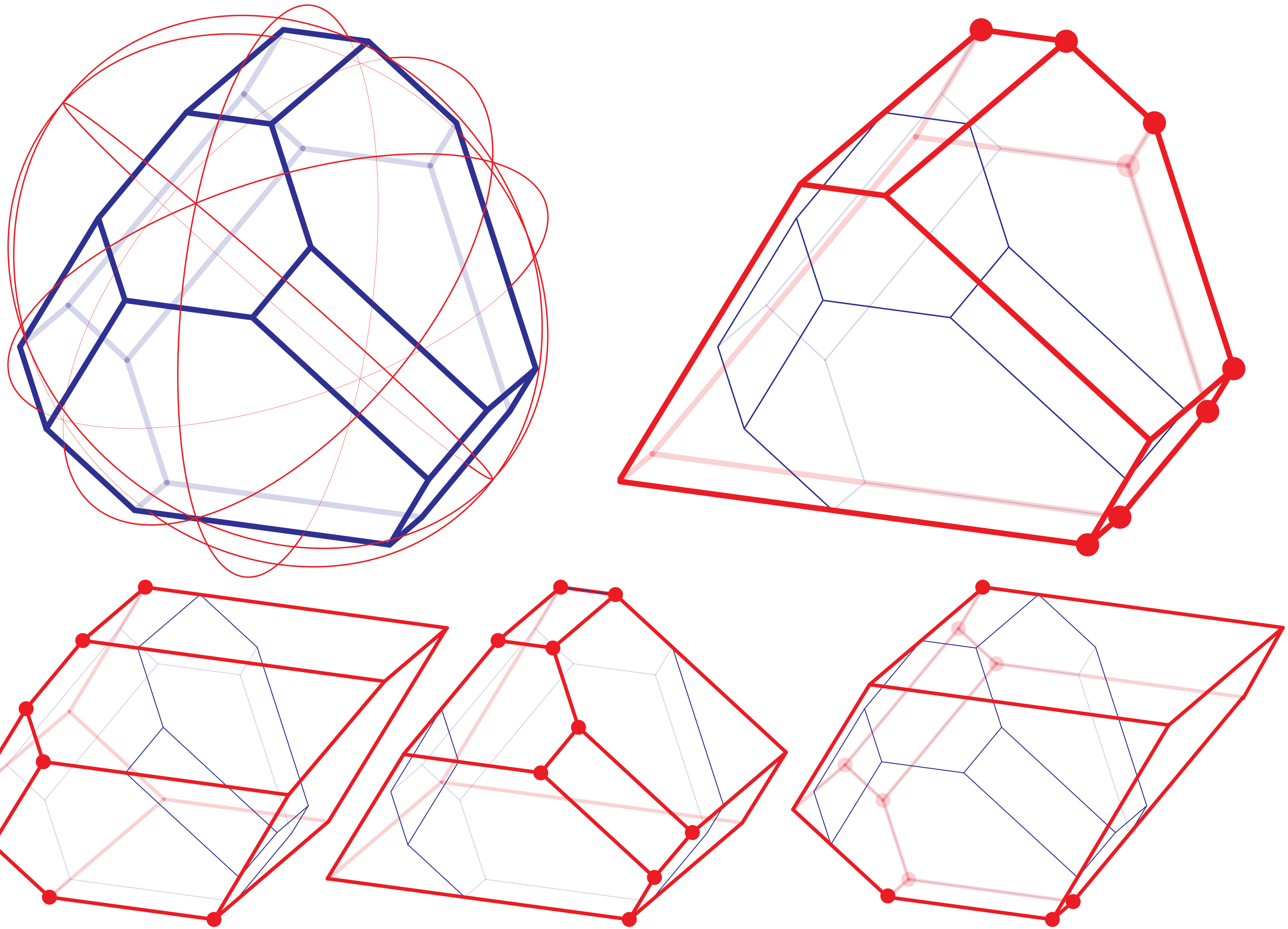
F. Chapoton, S. Fomin & A. Zelevinsky, Polytopal realizations of generalized associahedra, 2002.

C. Hohlweg, C. Lange & H. Thomas, Permutahedra and generalized associahedra, 2011.

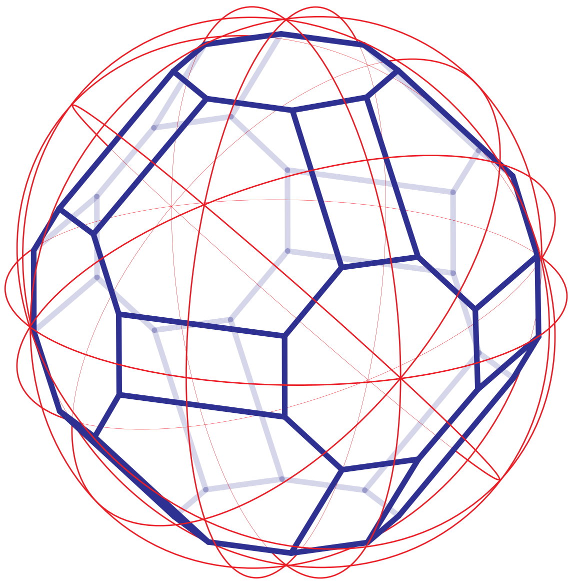
S. Stella, Polyhedral models for generalized associahedra via Coxeter elements, 2013.

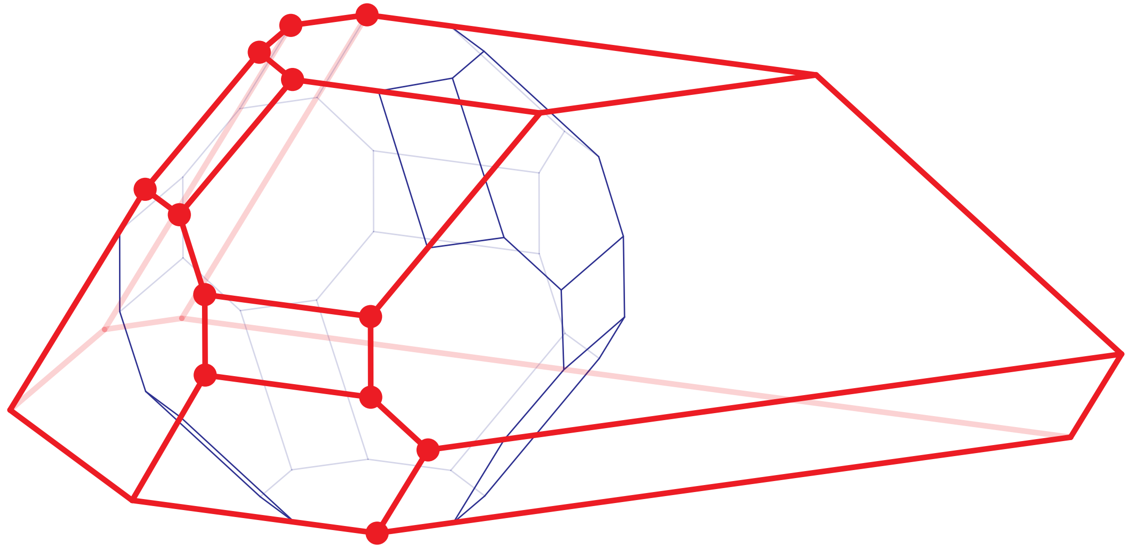
C. Hohlweg, Permutahedra and associahedra, 2013.

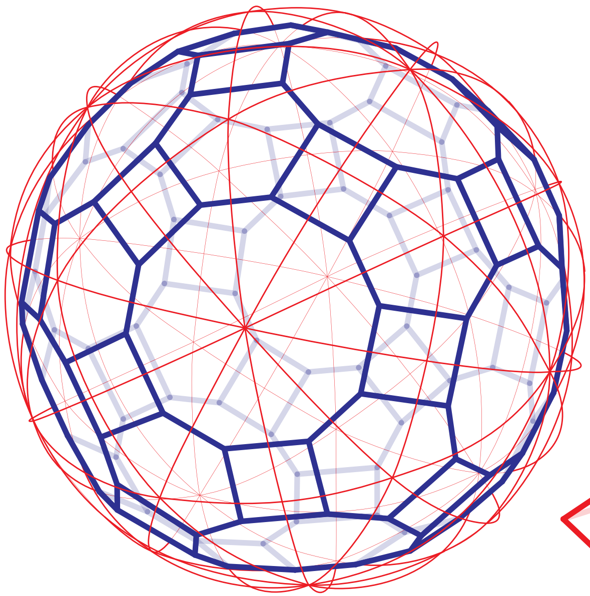
GENERALIZED ASSOCIAHEDRA

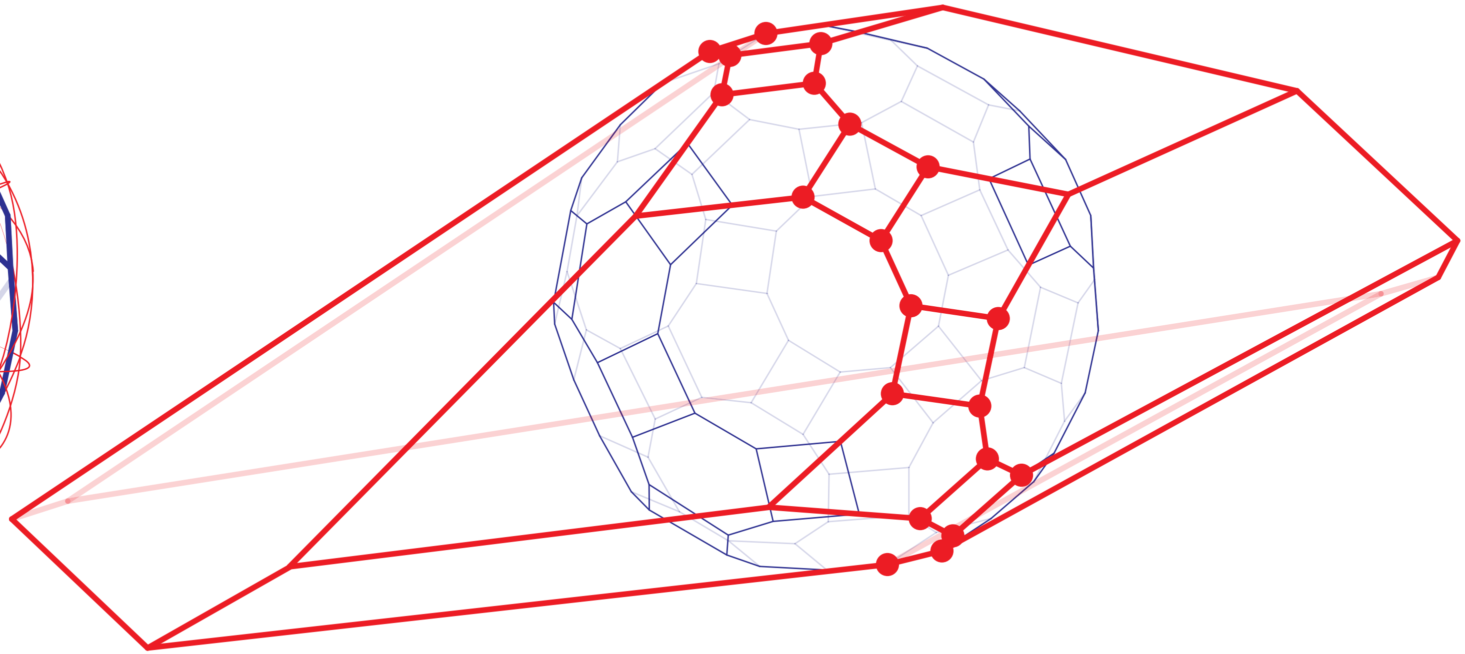


GENERALIZED ASSOCIAHEDRA









THANK YOU