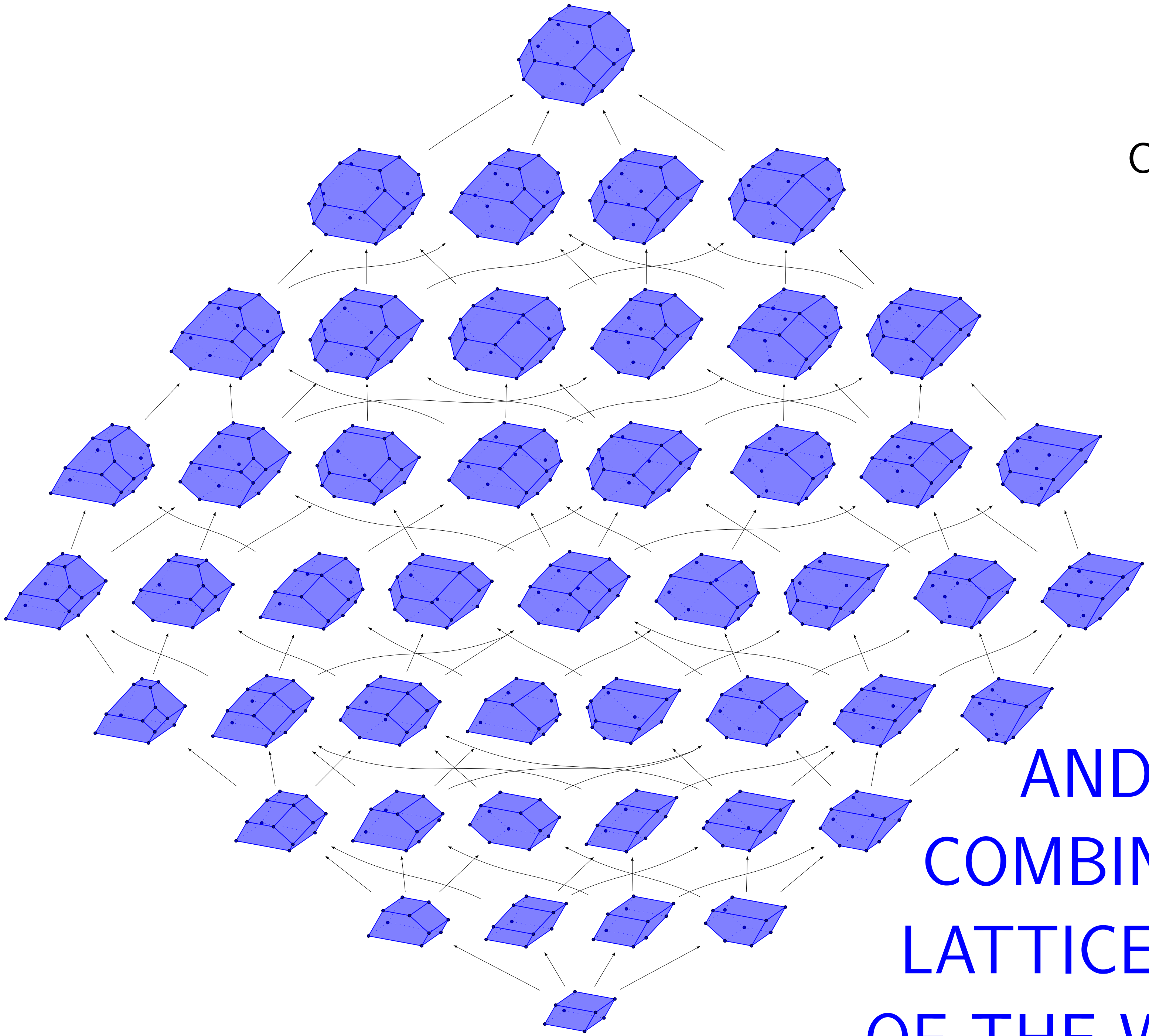
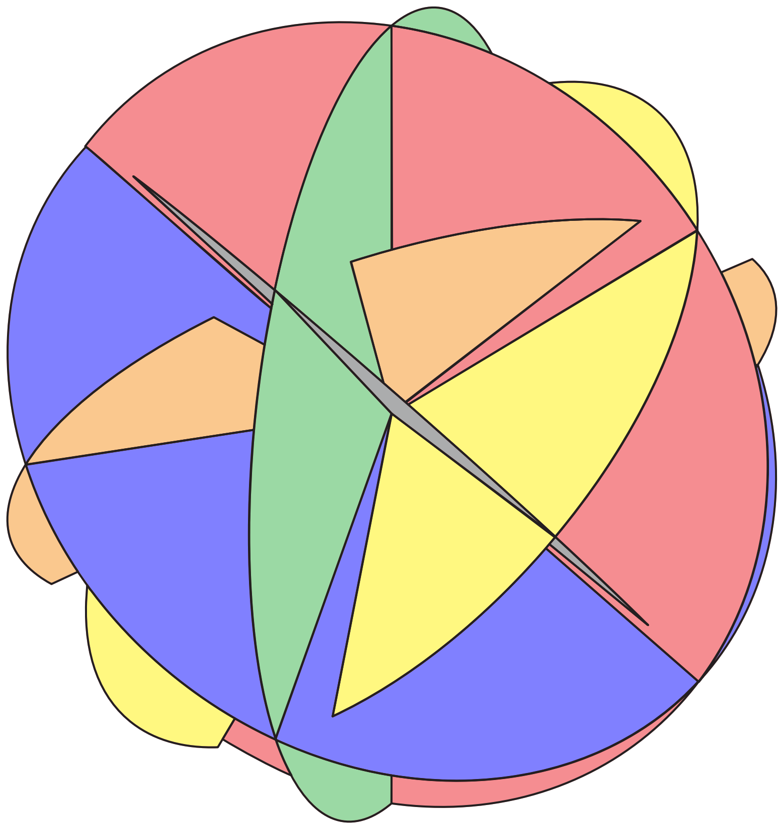


Journées du GT
Combinatoire Algébrique
26–27 juin 2018

V. PILAUD
(CNRS & LIX)

ALGEBRAIC AND GEOMETRIC COMBINATORICS OF LATTICE QUOTIENTS OF THE WEAK ORDER

LATTICE QUOTIENTS AND GEOMETRY



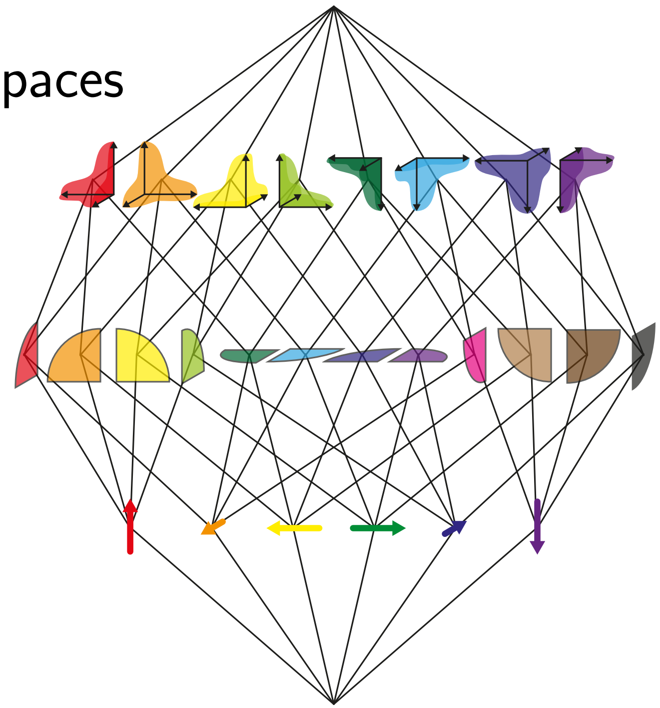
FANS & POLYTOPES

Ziegler, *Lectures on polytopes* ('95)
Matoušek, *Lectures on Discrete Geometry* ('02)

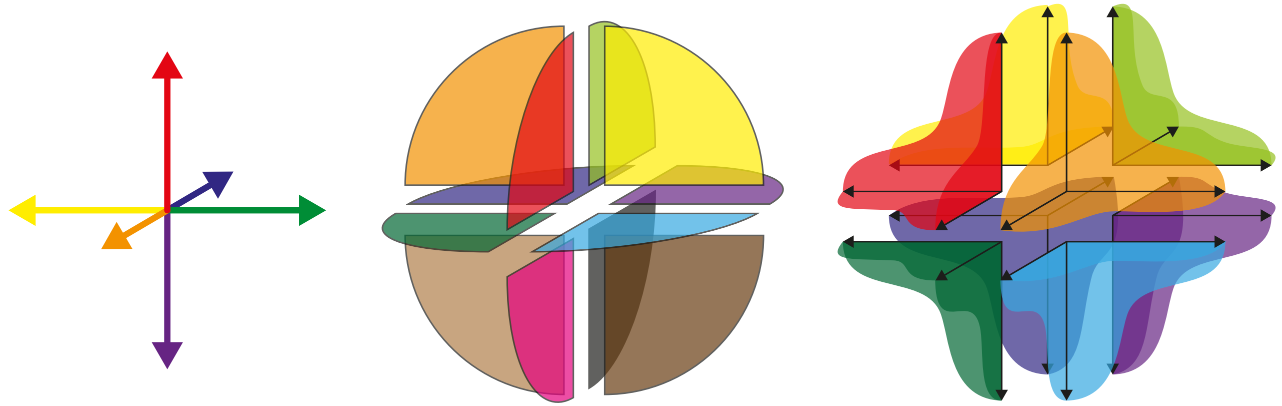
FANS

polyhedral cone $=$ positive span of a finite set of $\mathbb{R}^d$

$=$ intersection of finitely many linear half-spaces



fan $=$ collection of polyhedral cones closed by faces and where any two cones intersect along a face

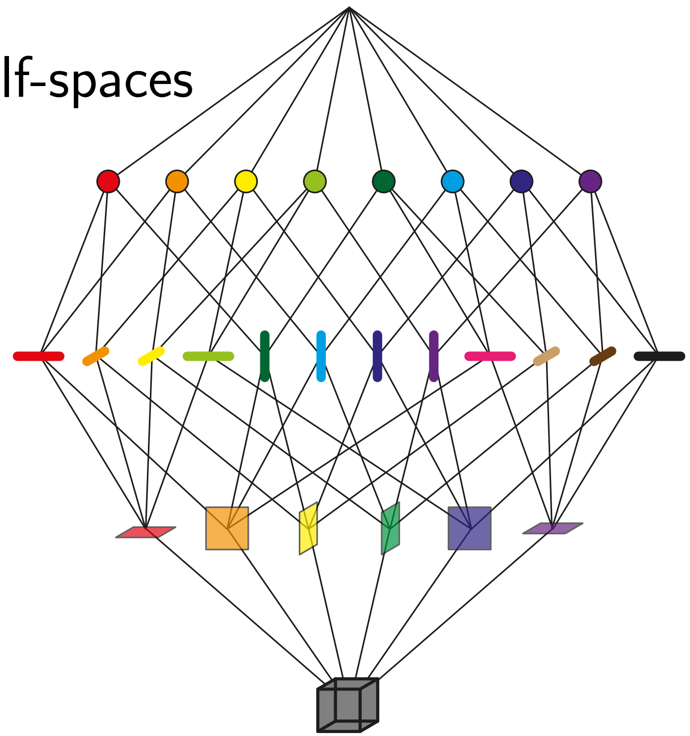


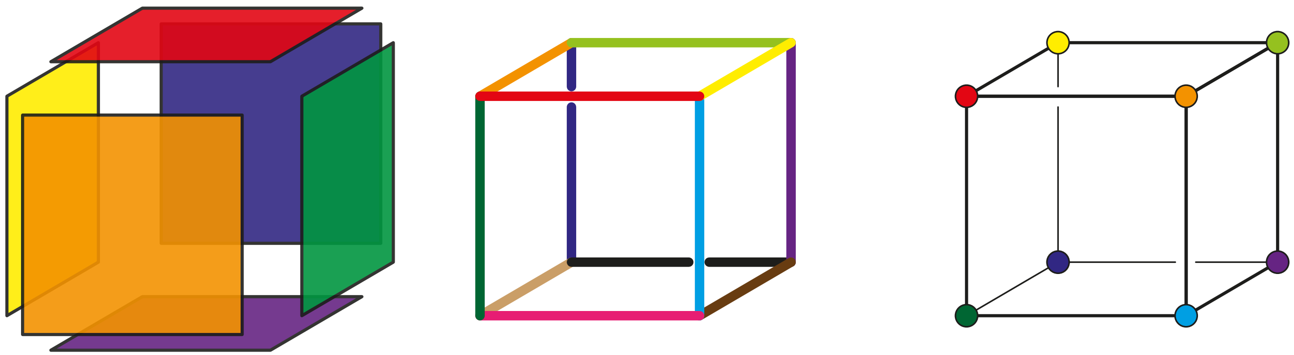
simplicial fan $=$ maximal cones generated by d rays

POLYTOPES

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many affine half-spaces

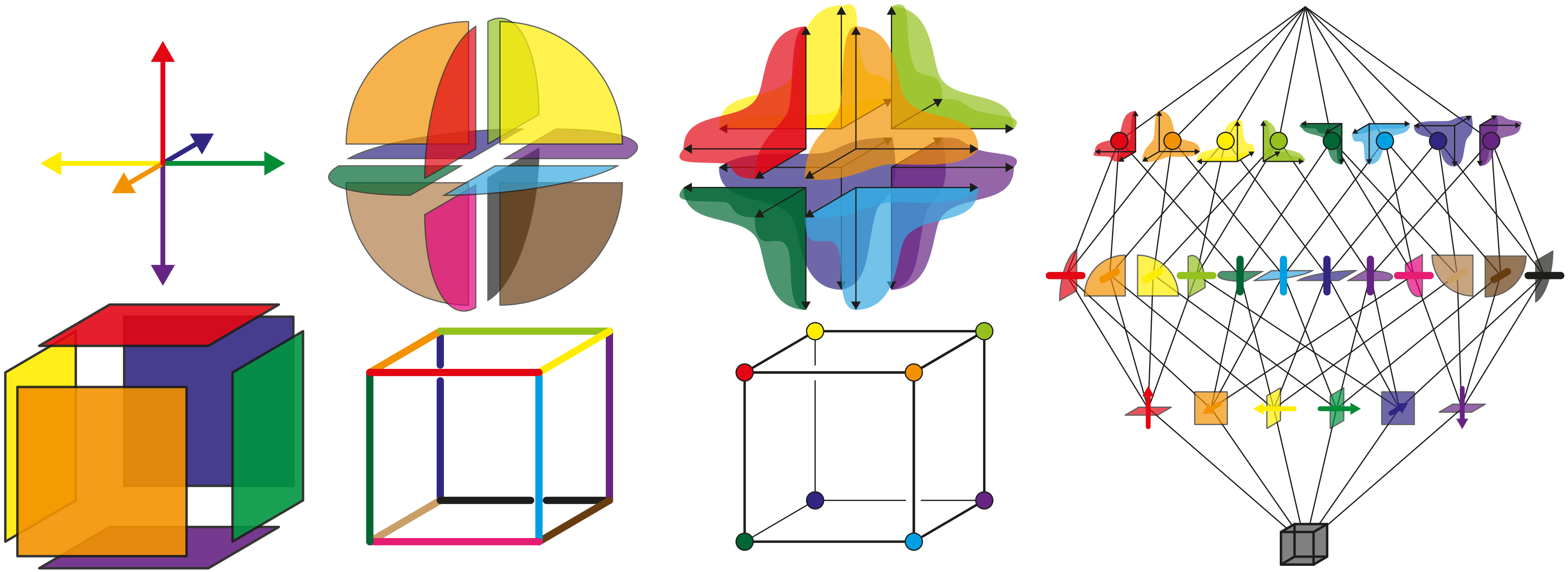
face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations





simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

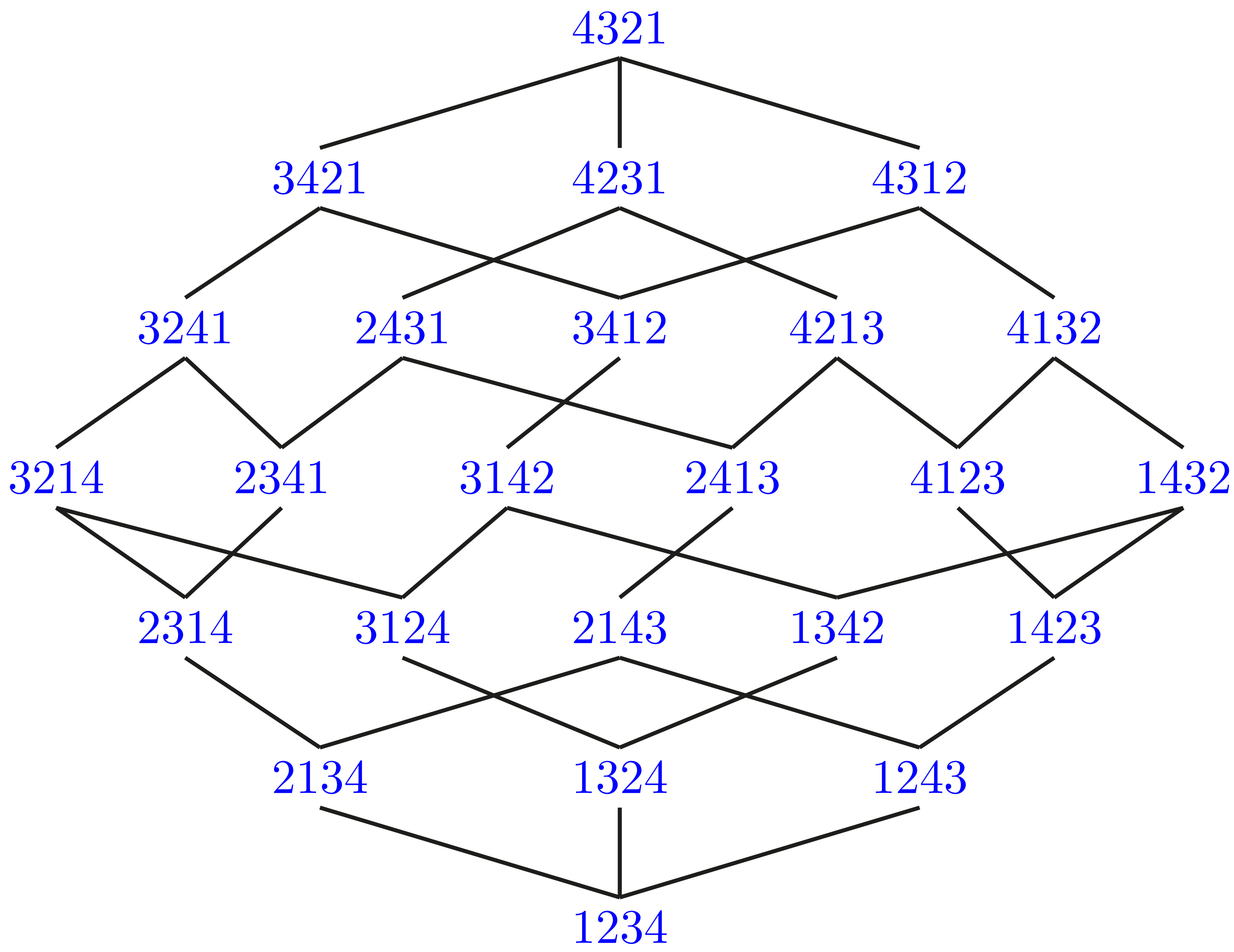
normal fan of P = $\{$ normal cone of F $|$ F face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

WEAK ORDER AND PERMUTAHEDRON

WEAK ORDER

inversions of $\sigma \in \mathfrak{S}_n$ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$
weak order = permutations of $\mathfrak{S}_n$ ordered by inclusion of inversion sets

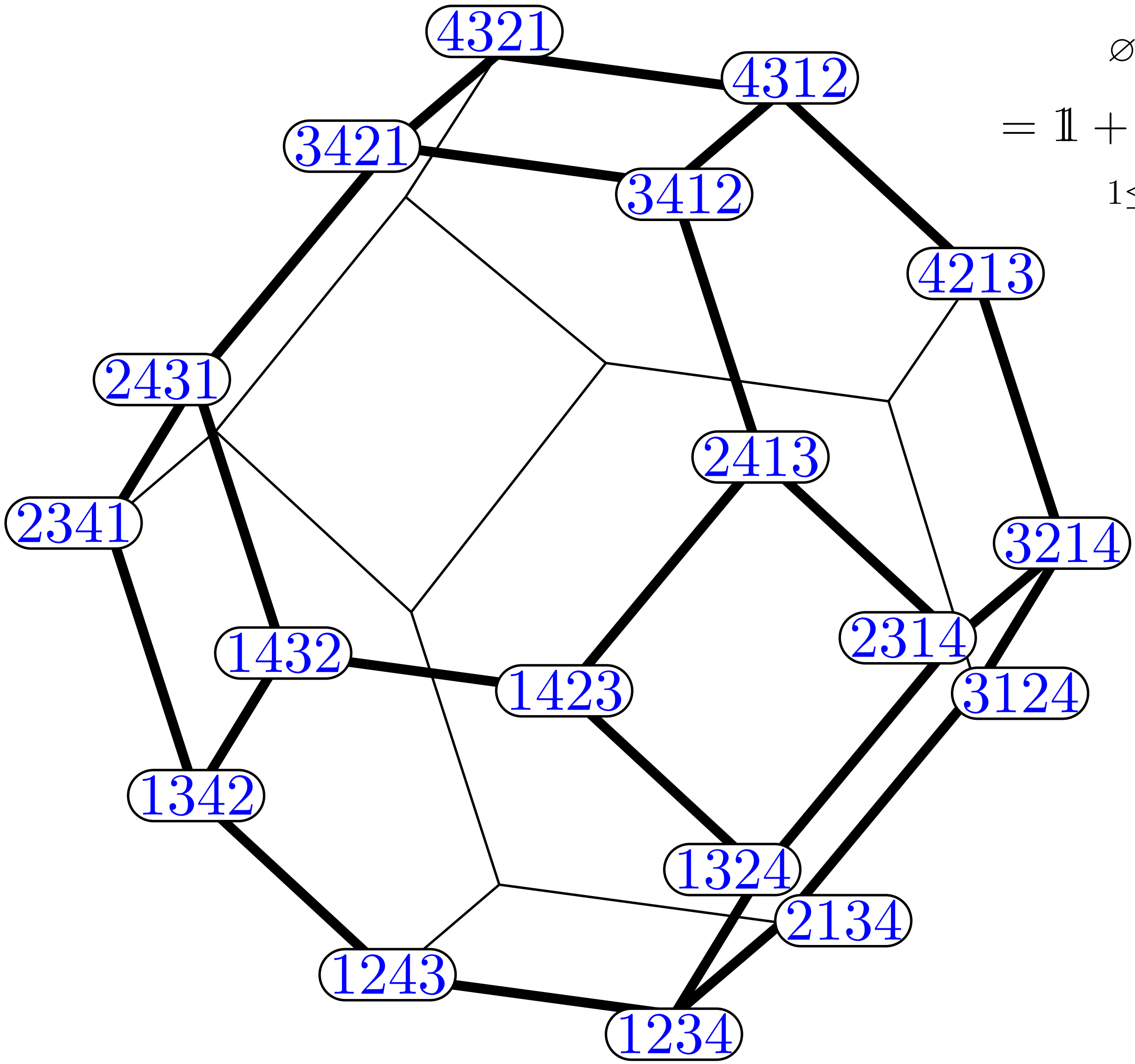


PERMUTAHEDRON

Permutahedron $\mathsf{Perm}(n) = \mathrm{conv}\left\{(\tau(1), \dots, \tau(n)) \mid \tau \in \mathfrak{S}_n\right\}$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

$$= \mathbb{1} + \sum_{1 \leq i < j \leq n} [\mathbf{e}_i, \mathbf{e}_j]$$



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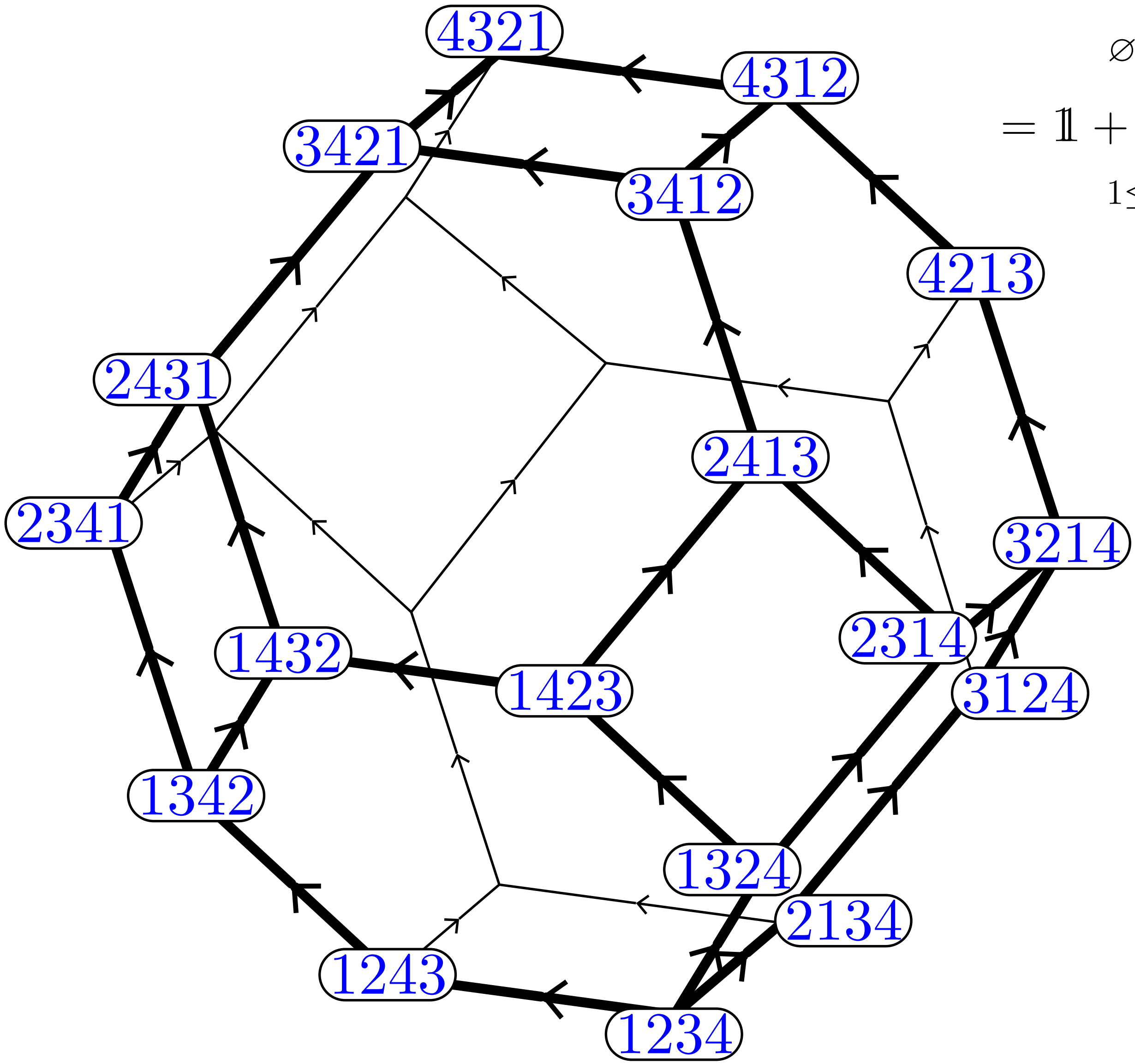
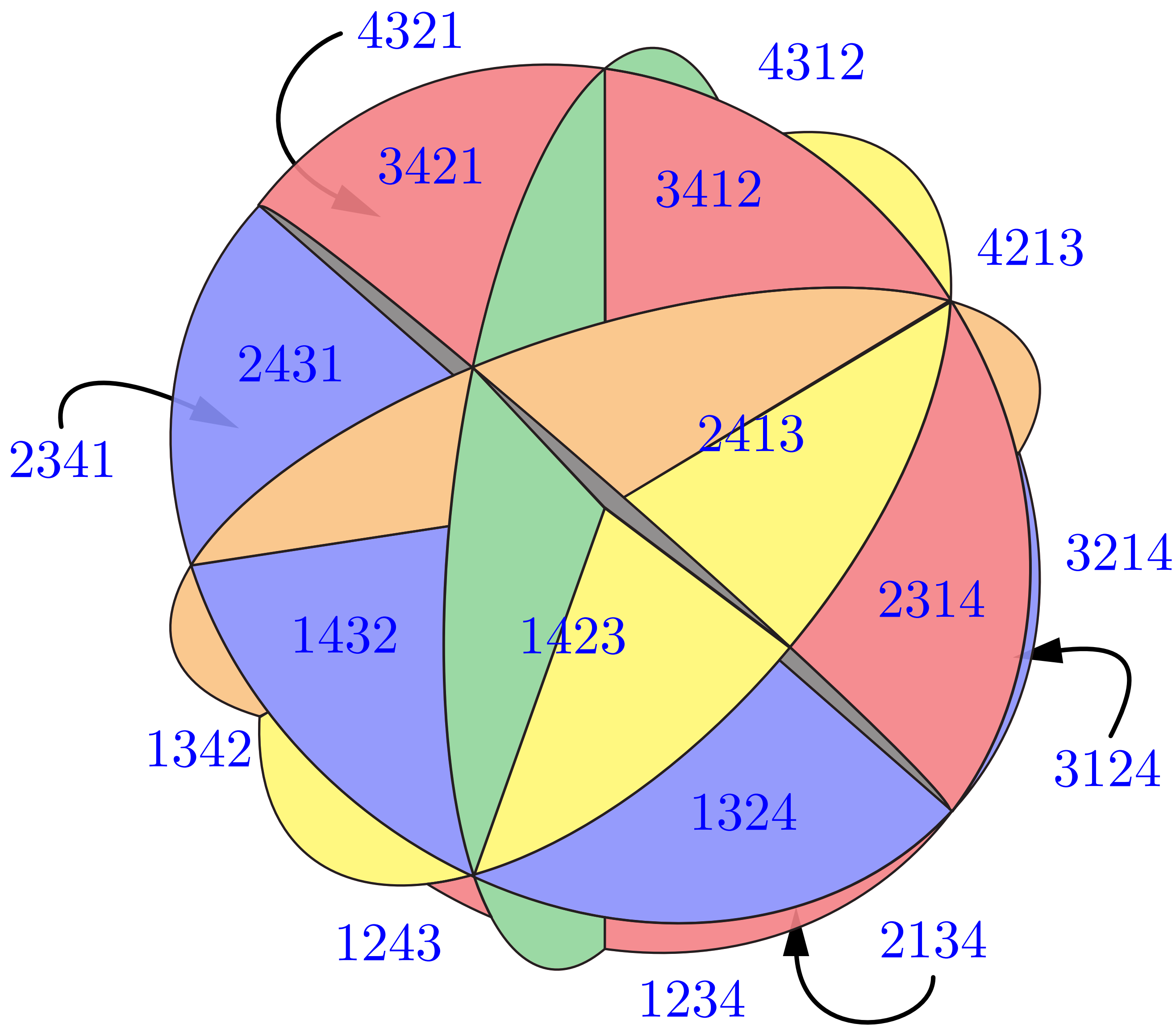


weak order = orientation of the graph of $\mathsf{Perm}(n)$

connections to

- reduced expressions
- braid moves
- cosets of the symmetric group

Coxeter fan = fan defined by the hyperplane arrangement $\{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}_{1 \le i < j \le n}$



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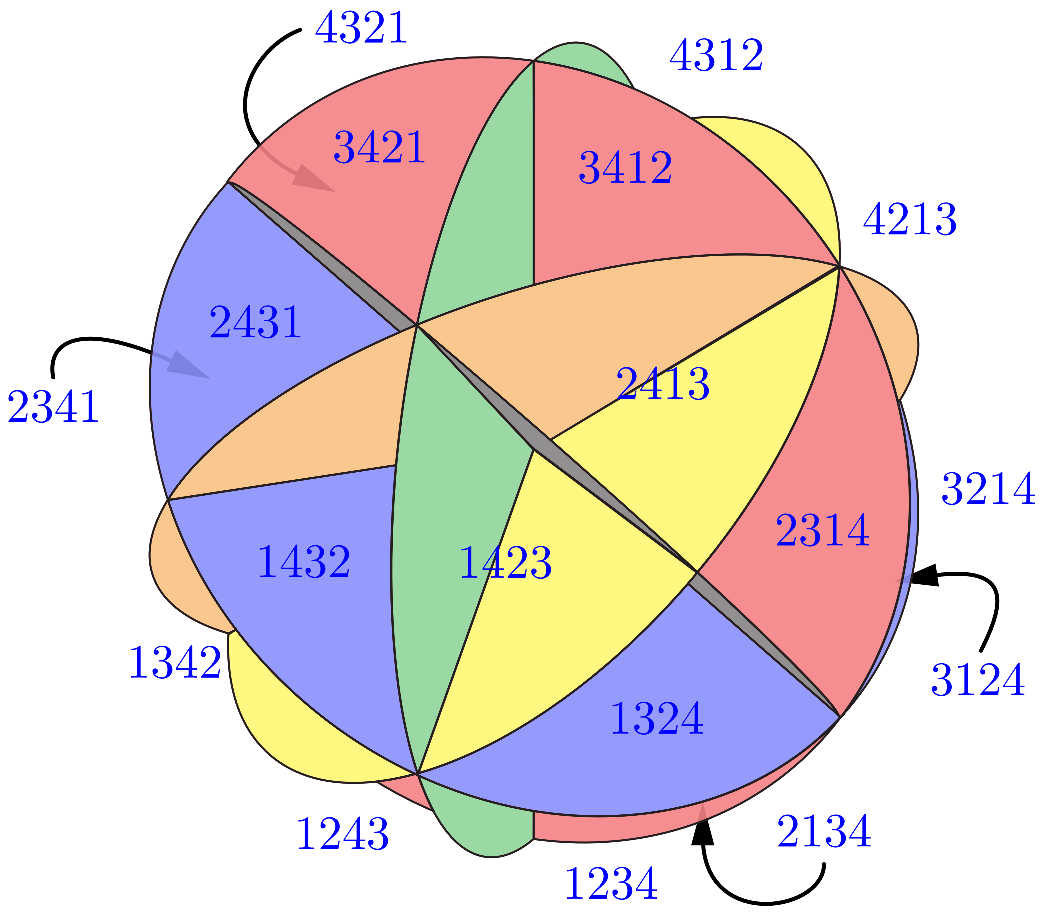
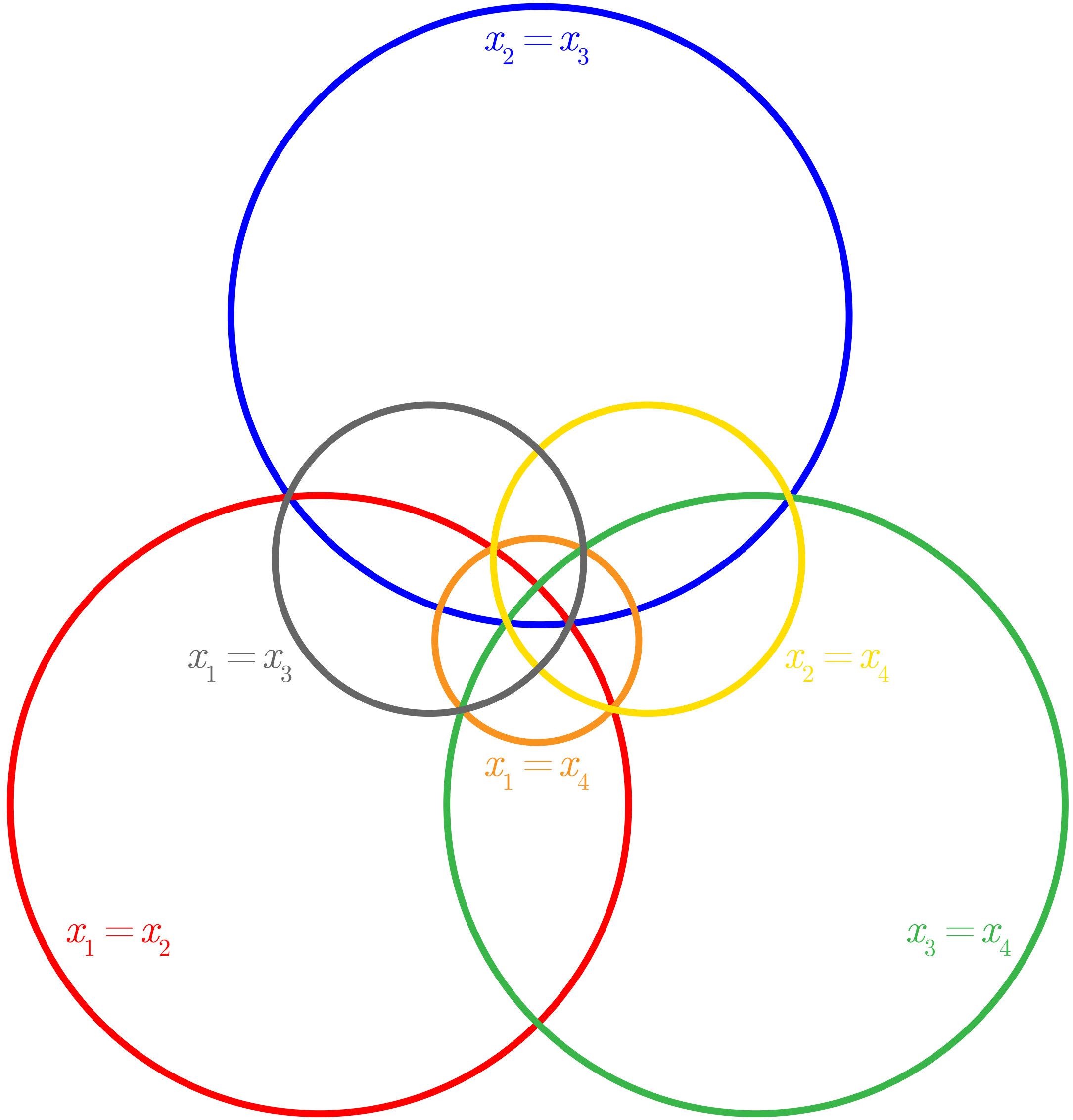




MY ZOO OF LATTICE QUOTIENTS

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Novelli–Reutenauer–Thibon. *Generalized descent patterns in permutations* ('11)
Hivert, Novelli, Thibon. *The algebra of binary search trees* ('05)
P., *Brick polytopes, lattice quotients, and Hopf algebras* ('18)
Chatel–P., *Cambrian algebras* ('17)
P.–Pons, *Permutrees* ('18)
Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)
Giraudo, *Algebraic and combinatorial structures on pairs of twin binary trees* ('12)

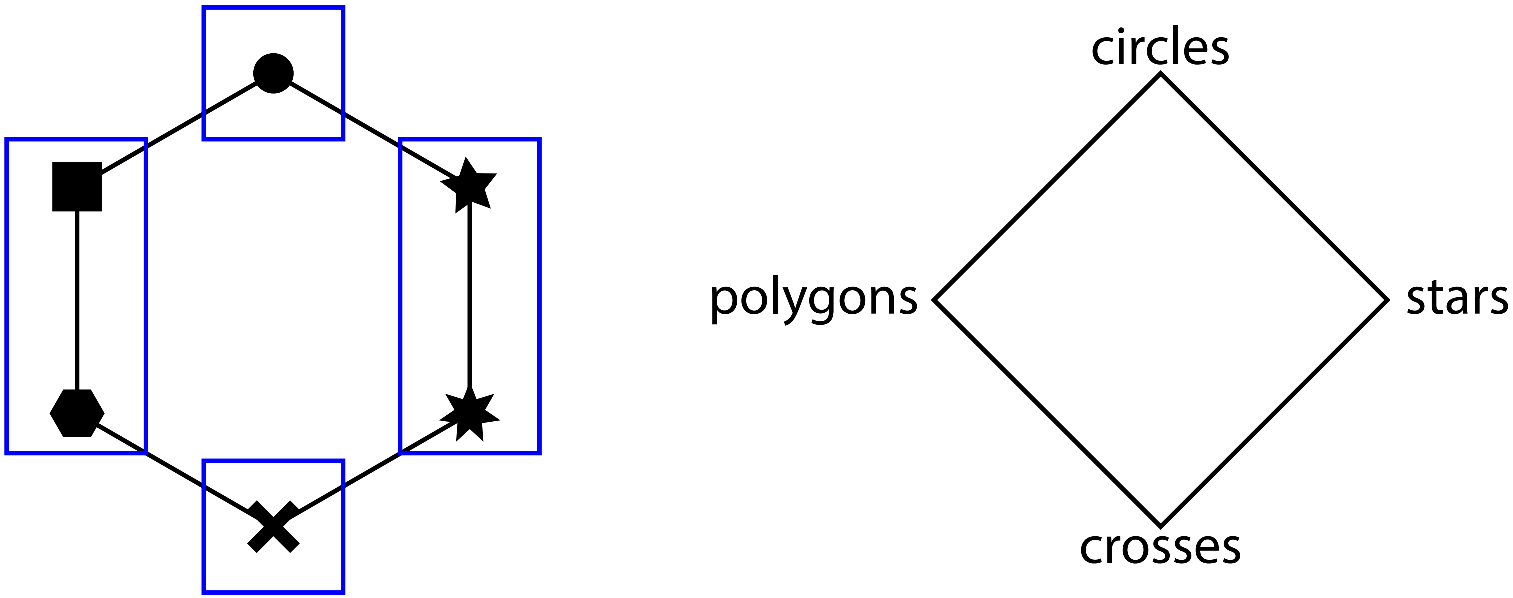
LATTICE CONGRUENCES

lattice congruence $=$ equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \quad \Longrightarrow \quad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

lattice quotient of $L/\!\equiv$ $=$ lattice on equiv. classes of L under $\equiv$ where

- $X \le Y \iff \exists x \in X,\ y \in Y,\ \ x \le y$
- $X \wedge Y$ $=$ equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ $=$ equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$



LATTICE CONGRUENCES

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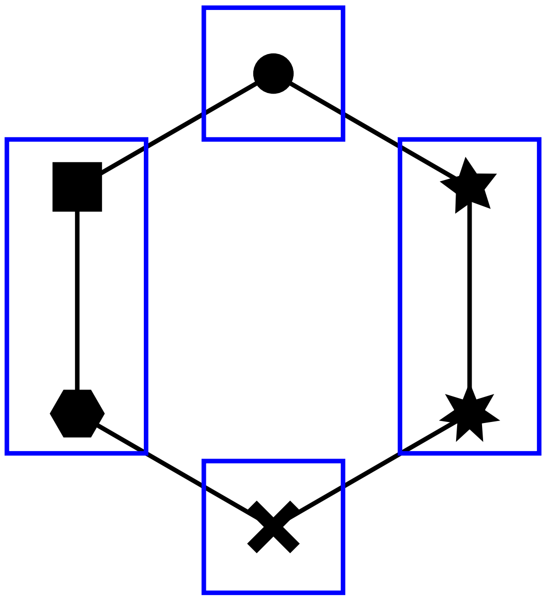
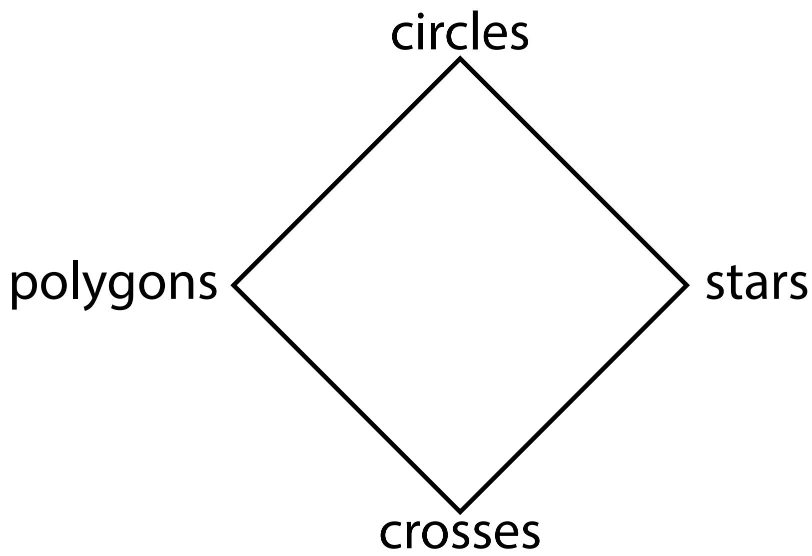
characterization:

- each equivalence class is an interval,
- the up and down projection maps $\sigma \to \pi^{\uparrow}(\sigma)$ and $\sigma \to \pi_{\downarrow}(\sigma)$ are order-preserving.

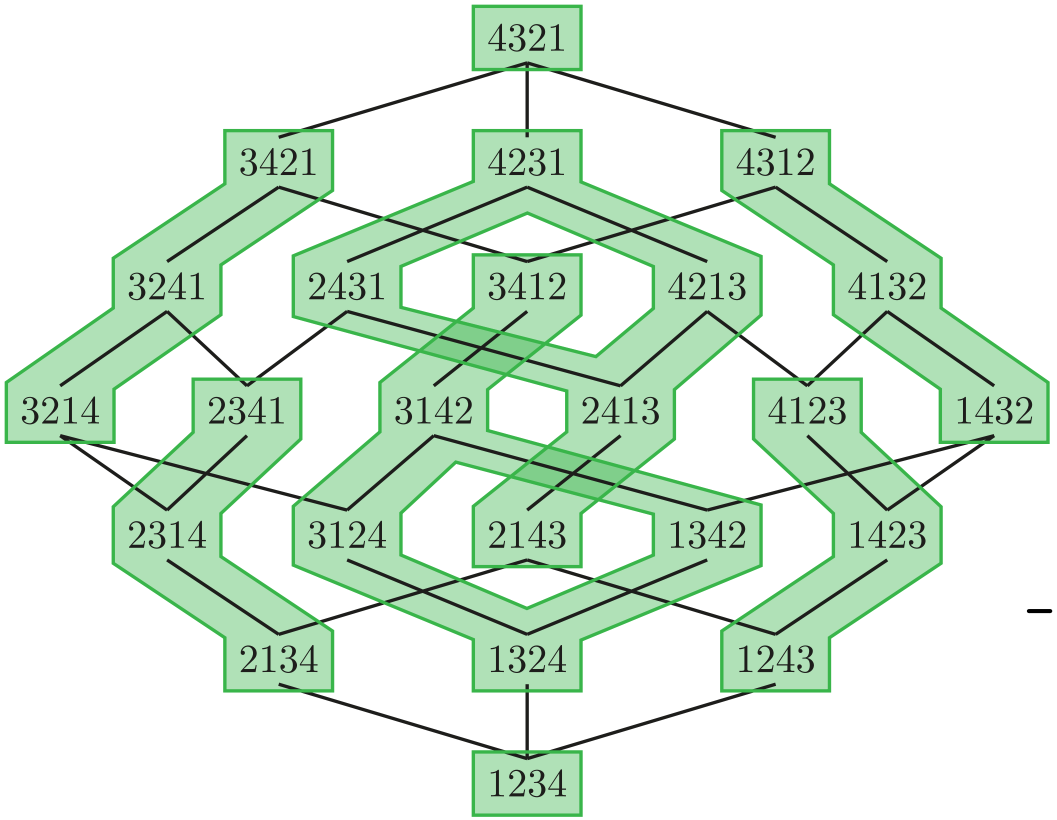


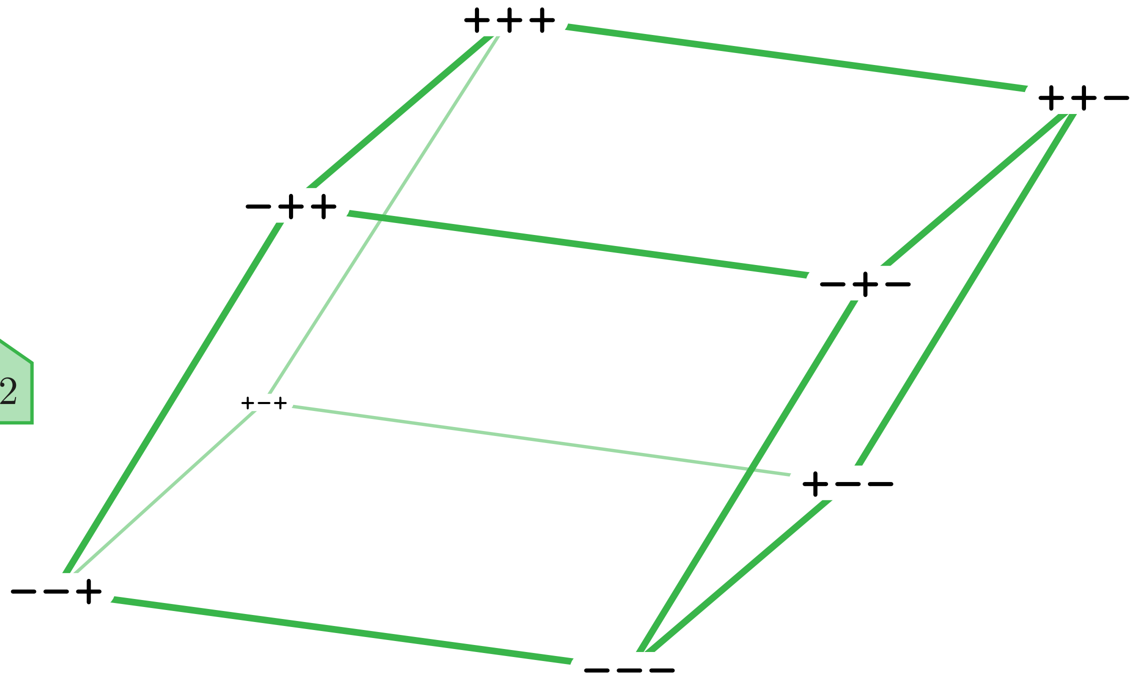


EXM 1: BOOLEAN LATTICE & CUBE

Recoils of a permutation $\sigma = i \in [n-1]$ such that $\sigma^{-1}(i) > \sigma^{-1}(i+1)$

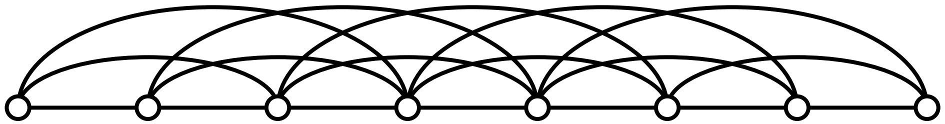
recoils lattice $=$ lattice quotient of the weak order by the relation “same recoils”





EXM 2: K-RECOIL SCHEME LATTICE & ZONOTOPES

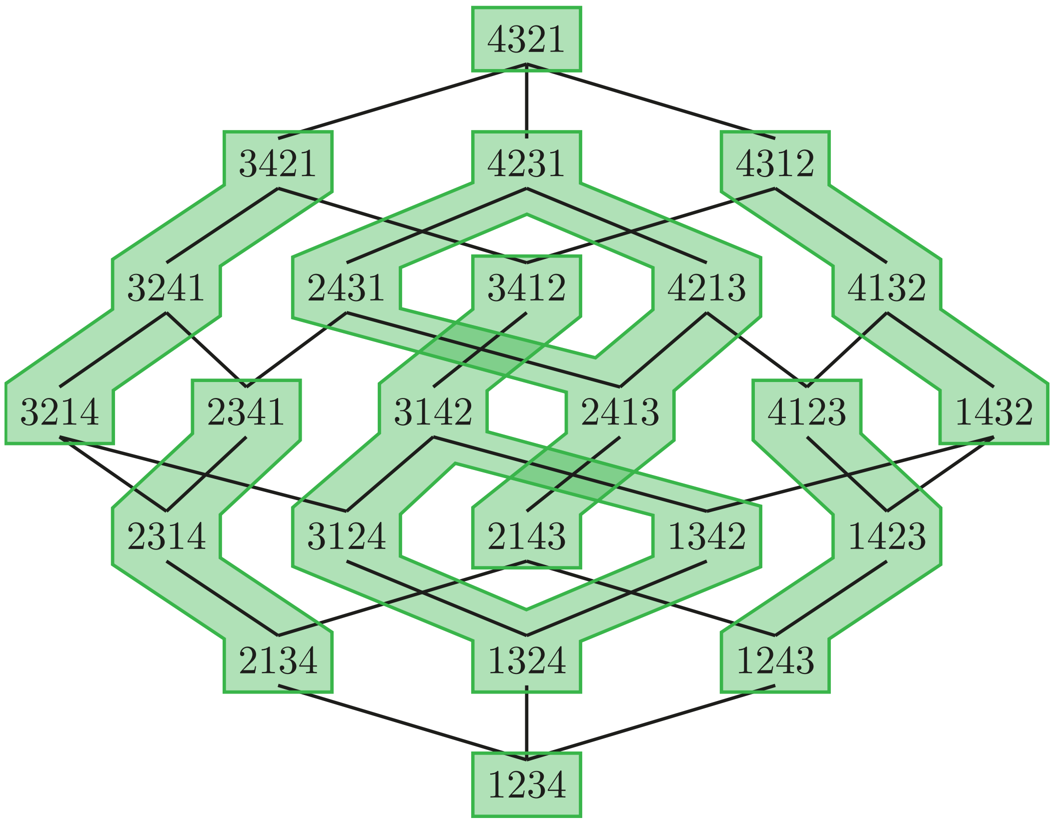
$\mathrm{G}^k(n)$ = graph with vertex set $[n]$ and edge set $\big\{\{i,j\} \in [n]^2 \;\big|\; i < j \le i+k\big\}$

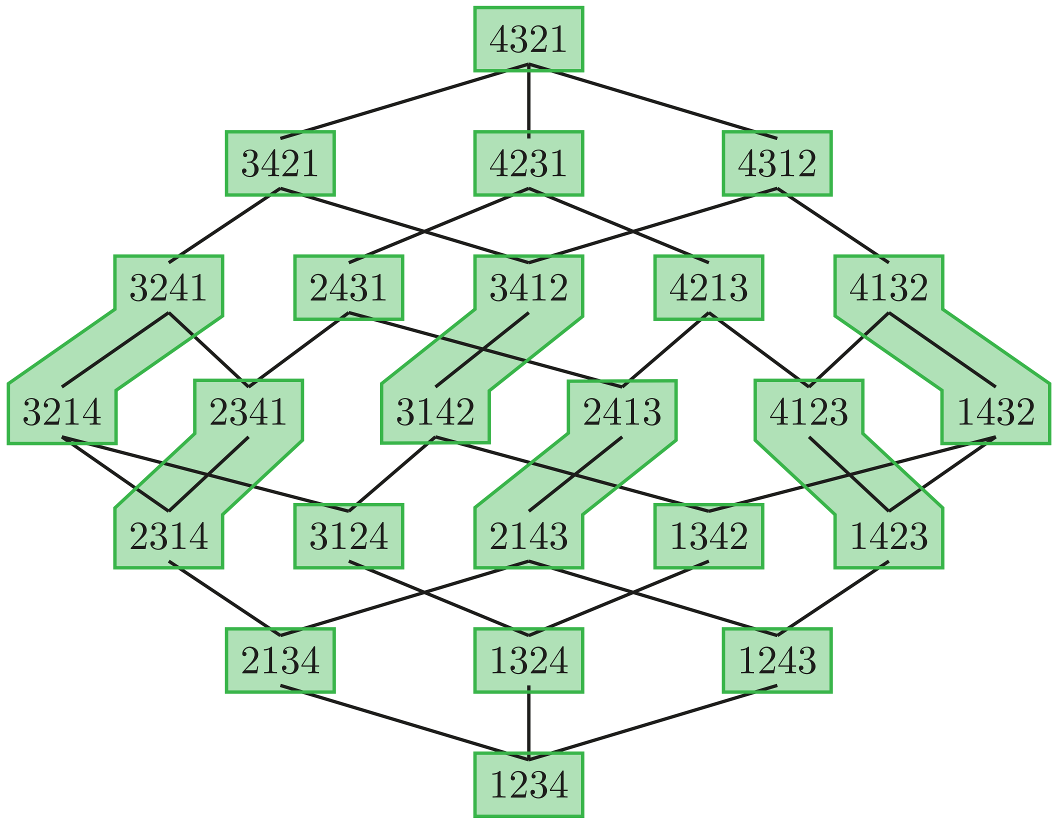


k-recoils insertion of $\tau \in \mathfrak{S}_n$ = acyclic orientation of $\mathrm{G}^k(n)$ with edge $i \to j$ for all $i, j \in [n]$ such that $|i-j| \le k$ and $\tau^{-1}(i) < \tau^{-1}(j)$

Novelli–Reutenauer–Thibon. *Generalized descent patterns in permutations and associated Hopf Algebras* ('11)

k-recoils lattice = lattice quotient of the weak order by the relation "same k-recoils"





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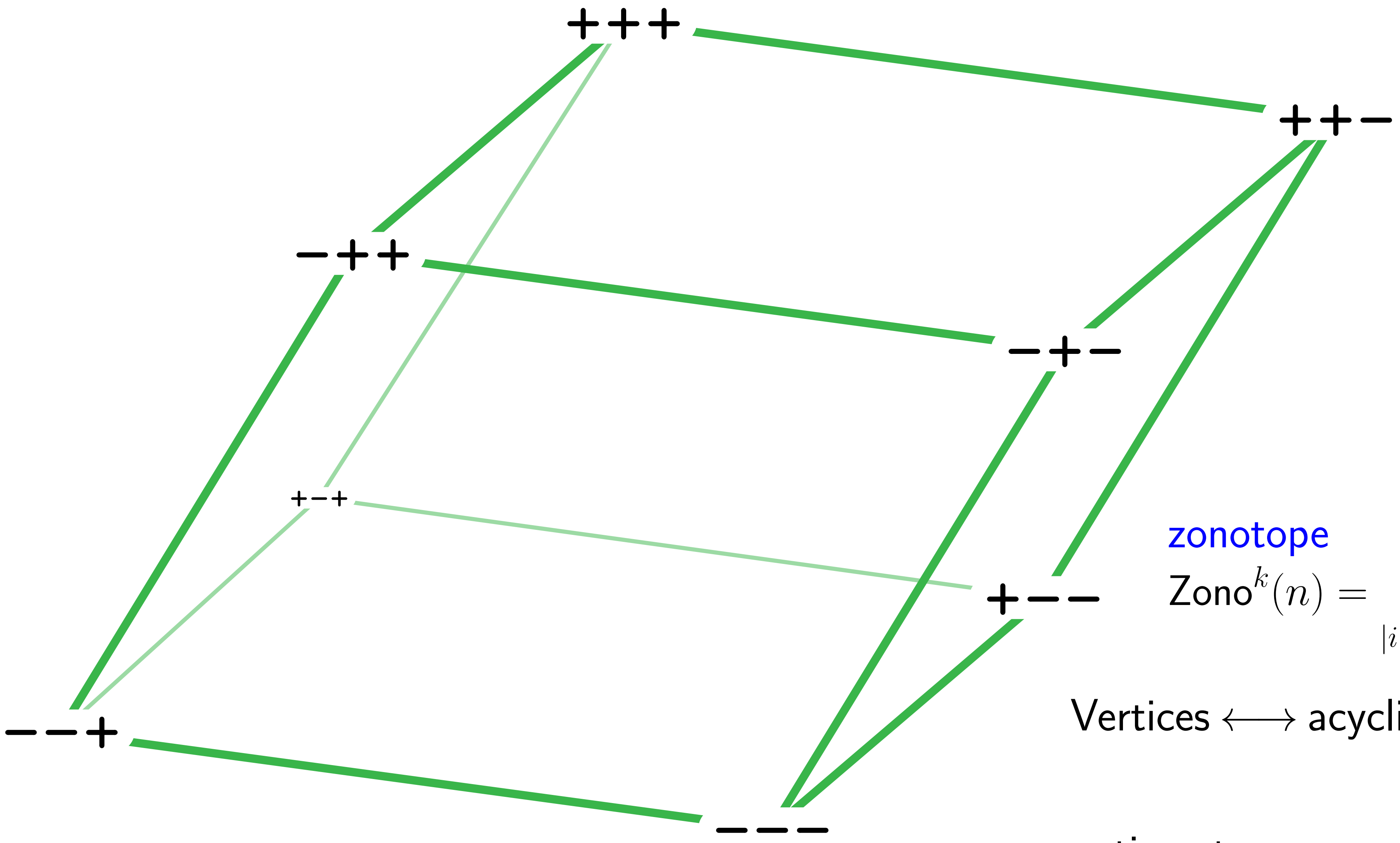


zonotope

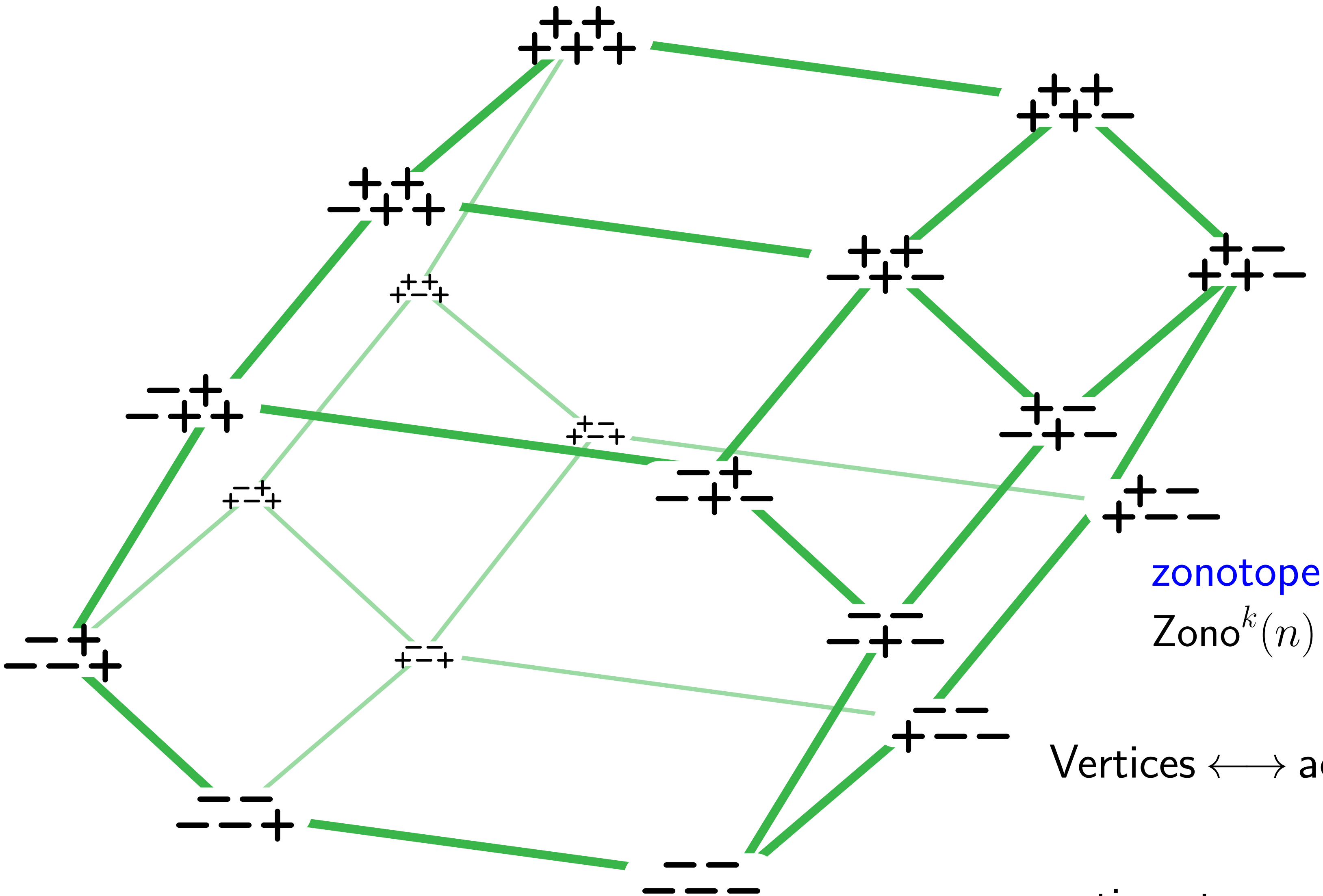
$$\text{Zono}^k(n) = \sum_{|i-j|\leq k} [\mathbf{e}_i, \mathbf{e}_j]$$

Vertices $\longleftrightarrow$ acyclic orientations of $\mathrm{G}^k(n)$

connections to

- matroids and oriented matroids
- hyperplane arrangements

EXM 2: K-RECOIL SCHEME LATTICE & ZONOTOPES



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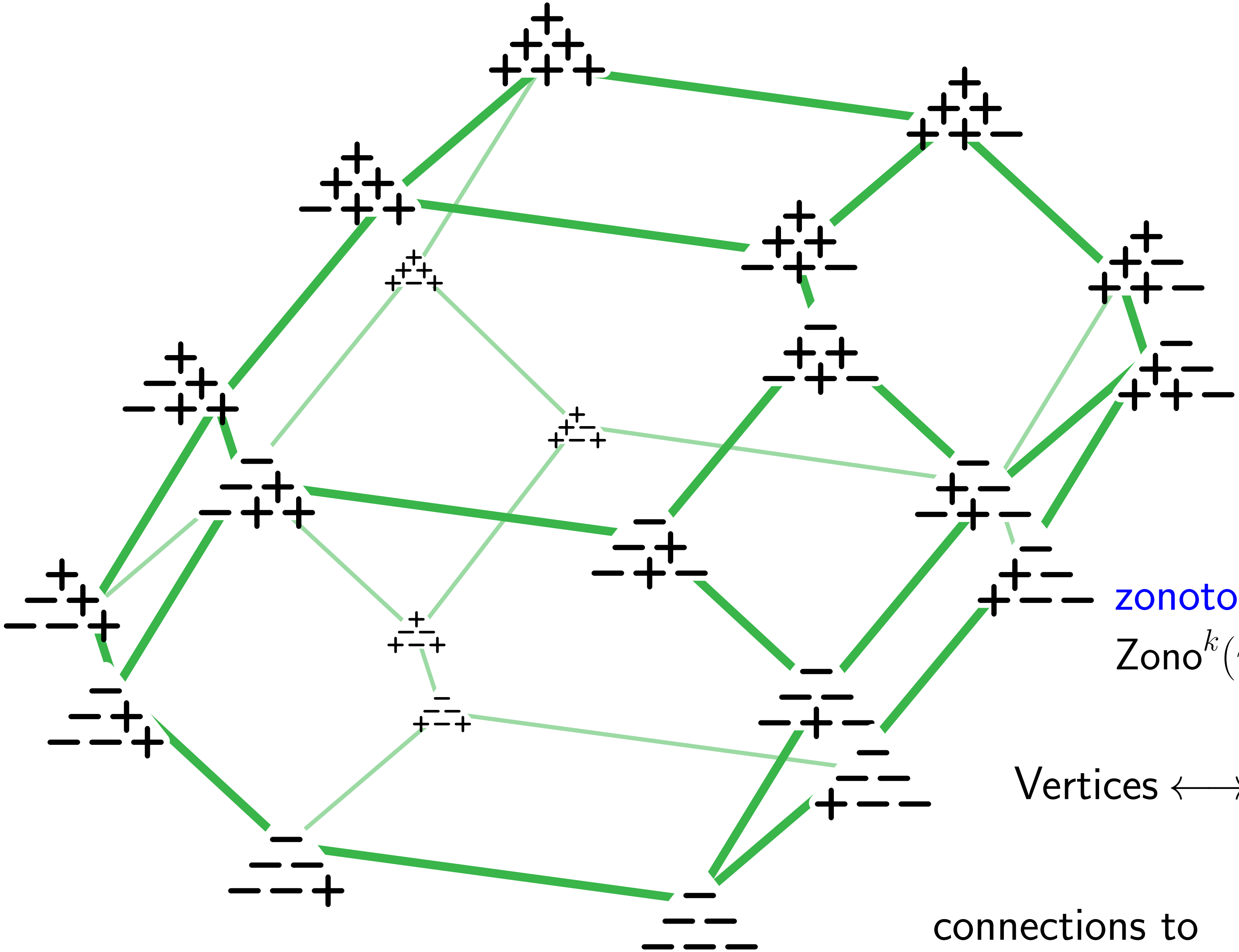

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connections to

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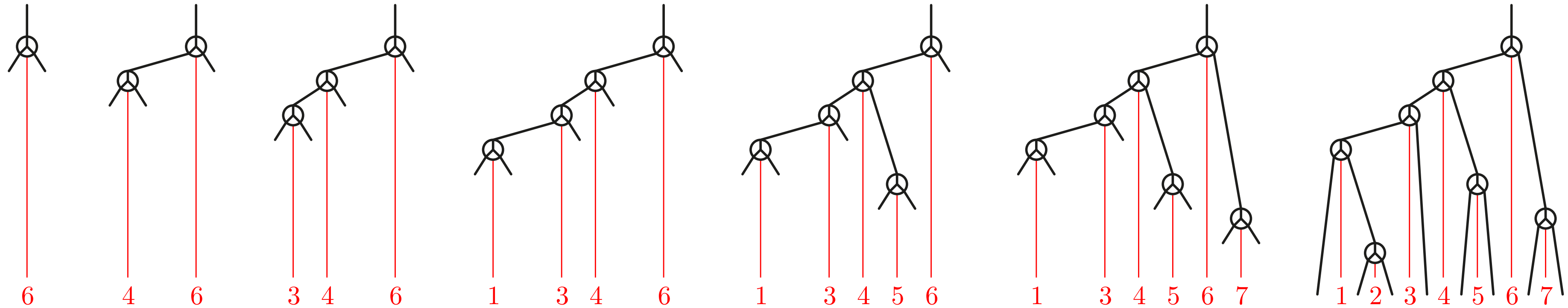
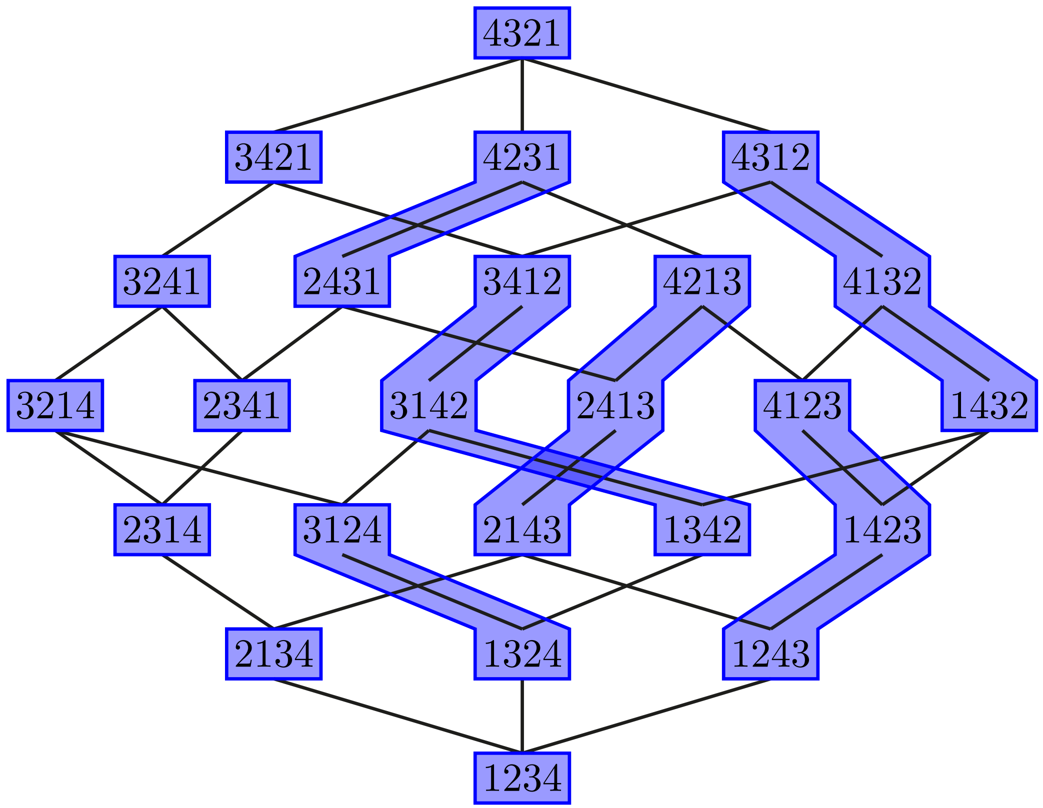
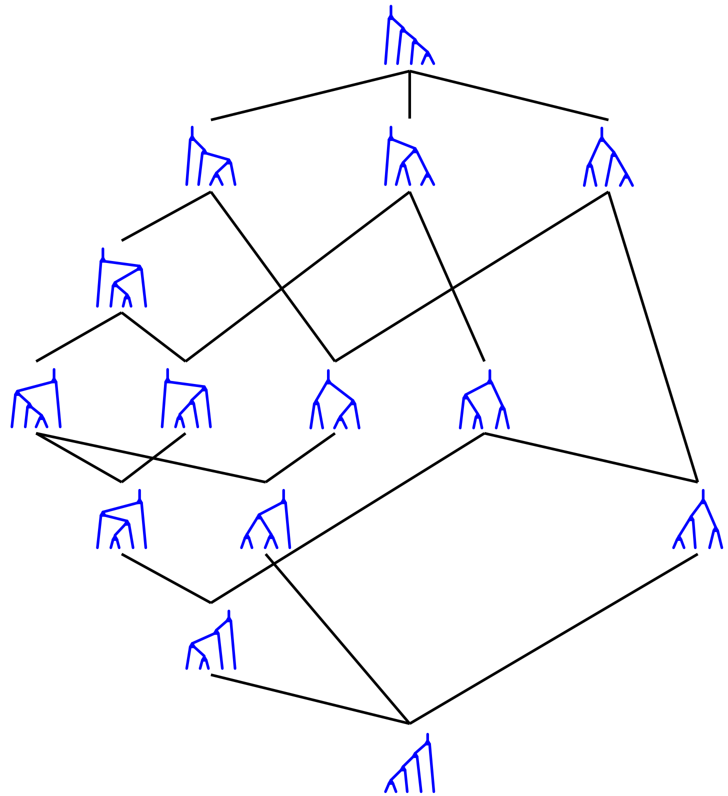
EXM 3: TAMARI LATTICE & LODAY'S ASSOCIAHEDRON

binary search tree insertion of 2751346



Tamari lattice $=$ lattice quotient of the weak order by the relation "same binary tree"





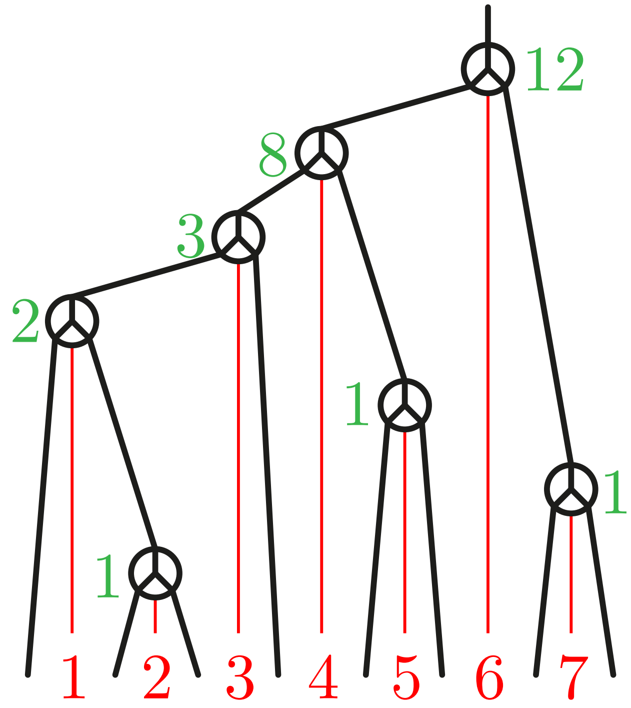
Loday's associahedron

$$\mathsf{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i) \cdot r(\mathrm{T},i)\big]_{i \in [n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Shnider–Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)

Loday, *Realization of the Stasheff polytope* ('04)



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Shnider–Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)
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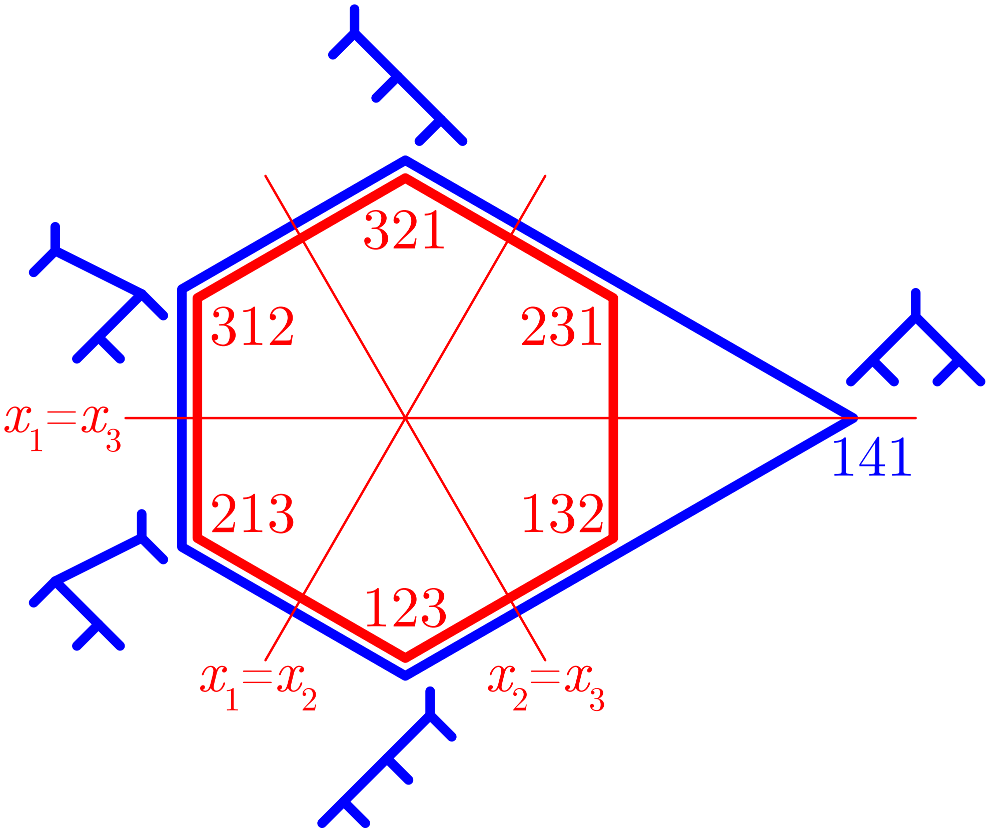

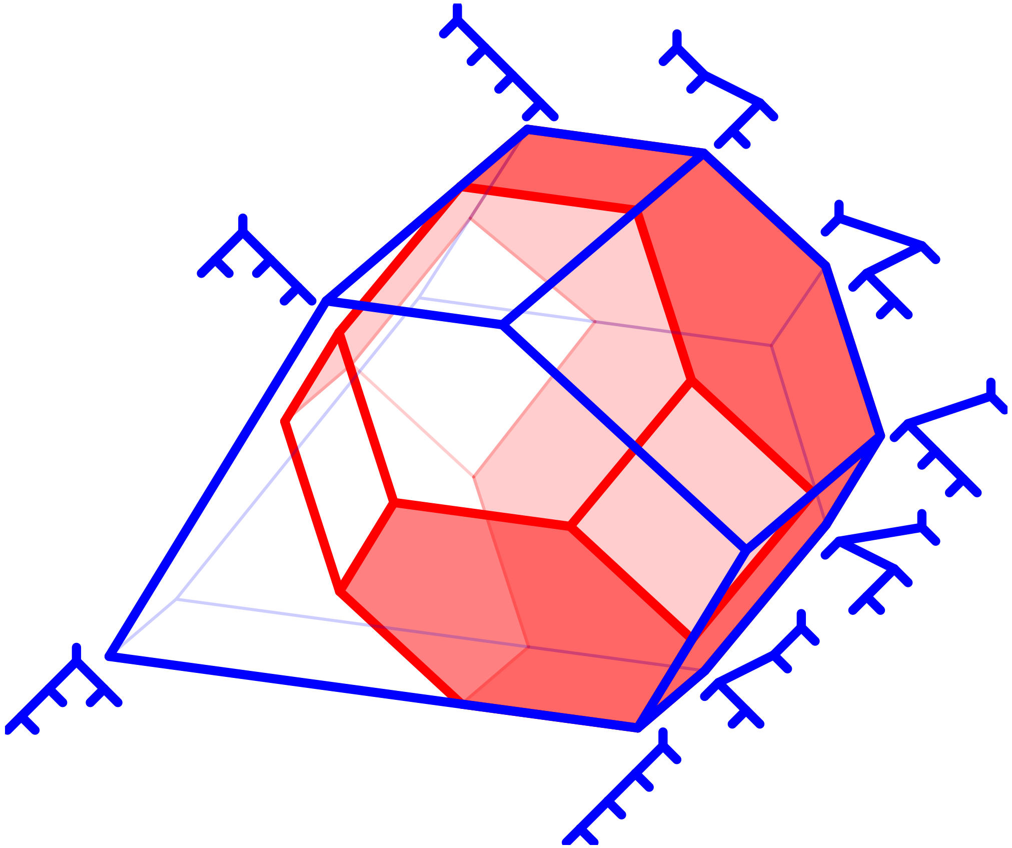

EXM 3: TAMARI LATTICE & LODAY'S ASSOCIAHEDRON

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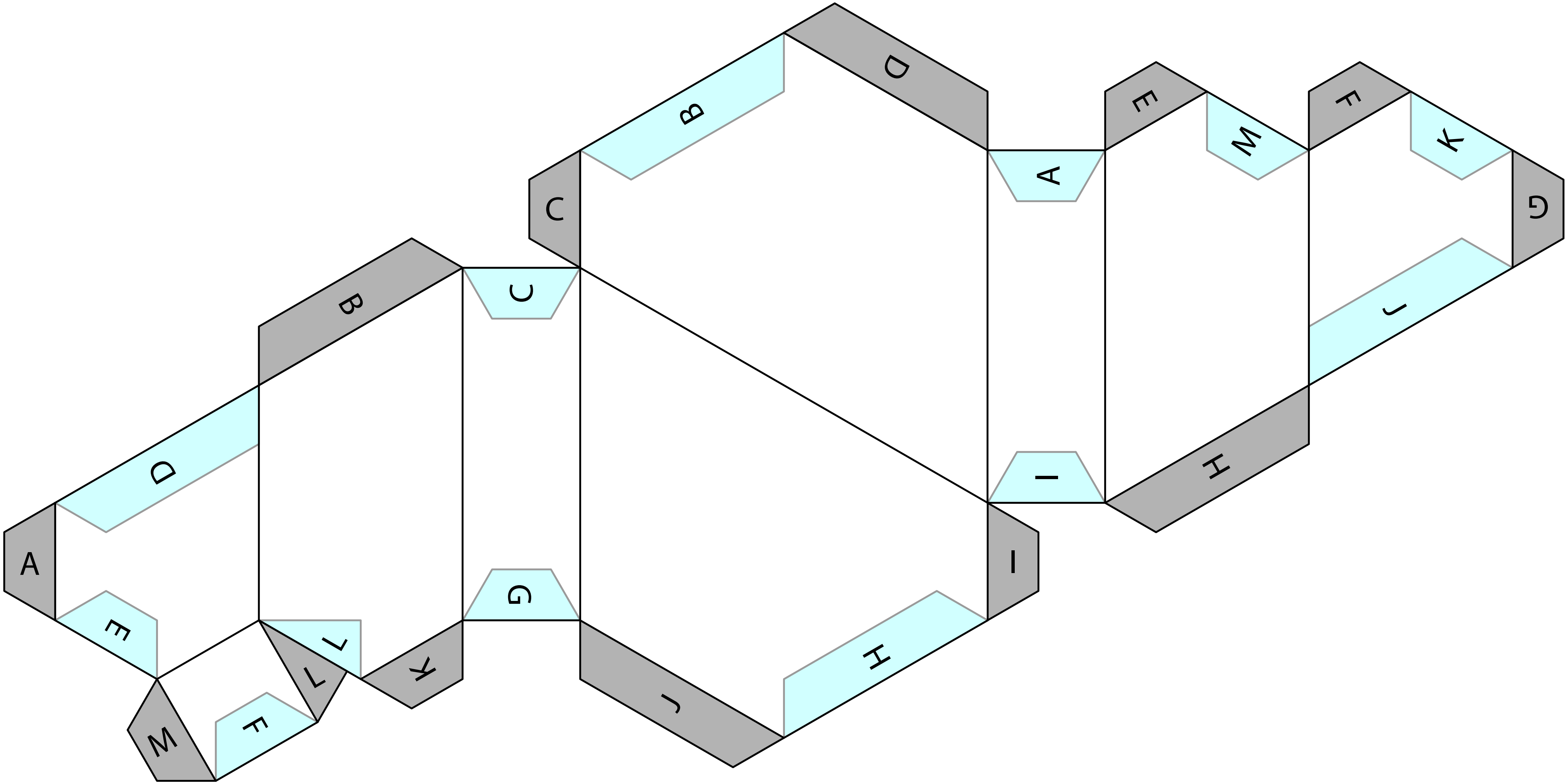
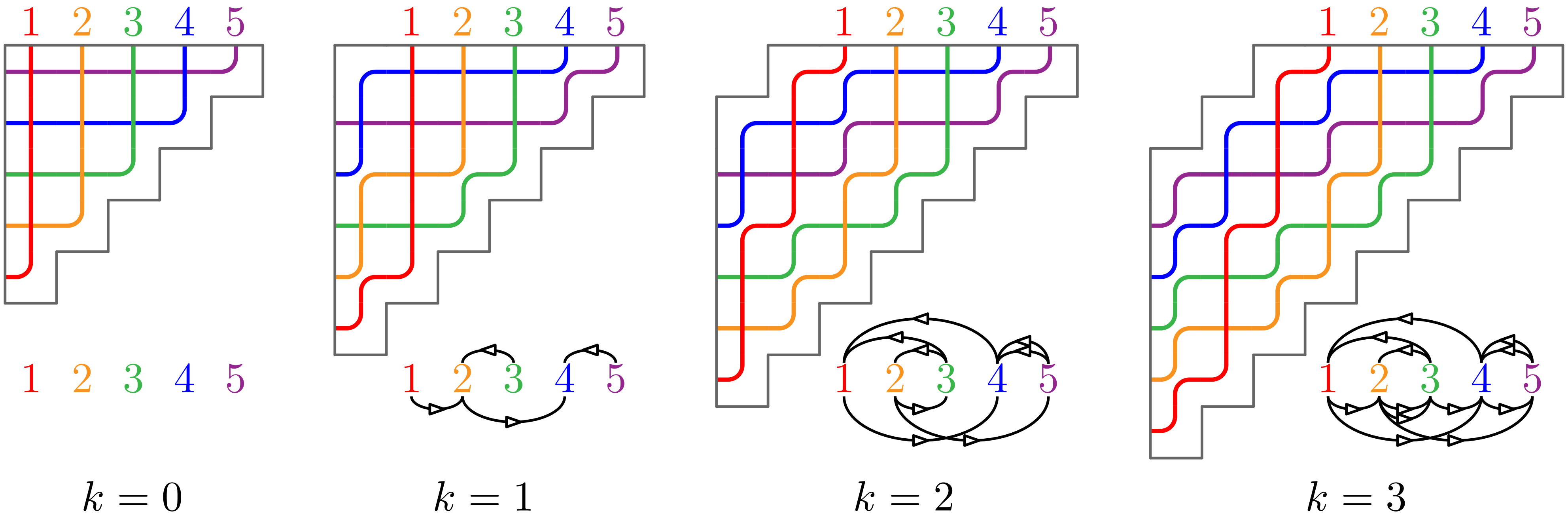


EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



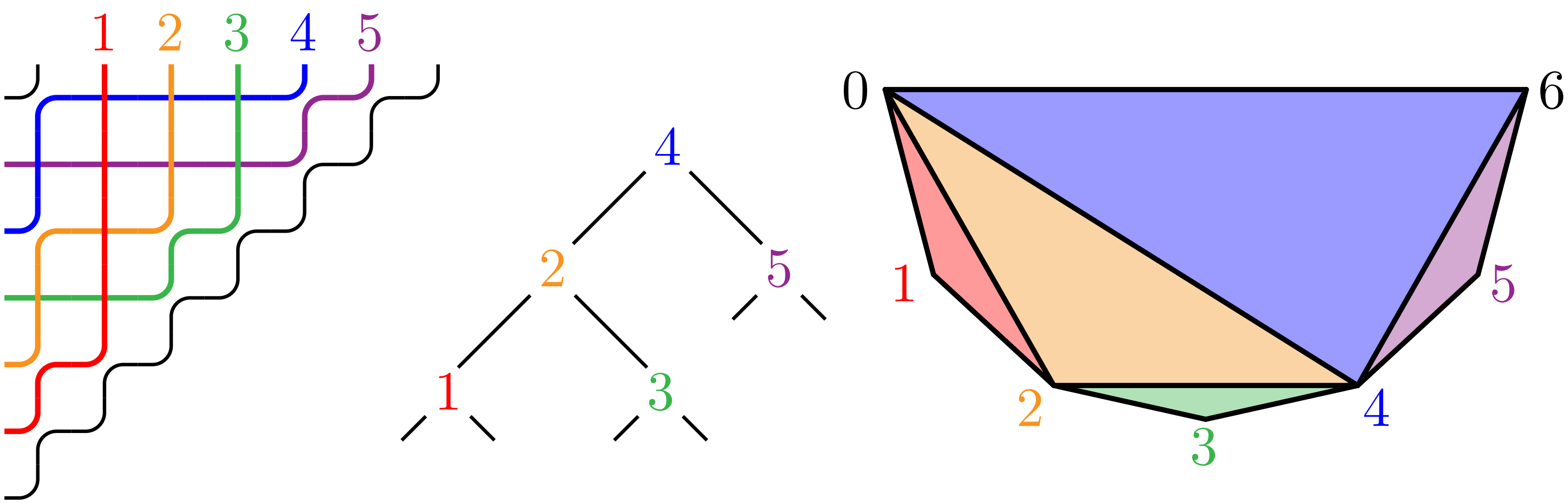
(k, n)-twist $=$ pipe dream in the trapezoidal shape of height n and width k

contact graph of a twist T $=$ vertices are pipes of T and arcs are elbows of T

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i,j]$ of the $(n+2)$-gon
$(1,n)$-twist T	$\longleftrightarrow$	triangulation $\mathrm{T}^\star$ of the $(n+2)$-gon
pth relevant pipe of T	$\longleftrightarrow$	pth triangle of $\mathrm{T}^\star$
contact graph of T	$\longleftrightarrow$	dual binary tree of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$



Woo. Catalan numbers and Schubert Polynomials for $w = 1(n+1)\dots 2$. Unpub 2004

P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

Correspondence

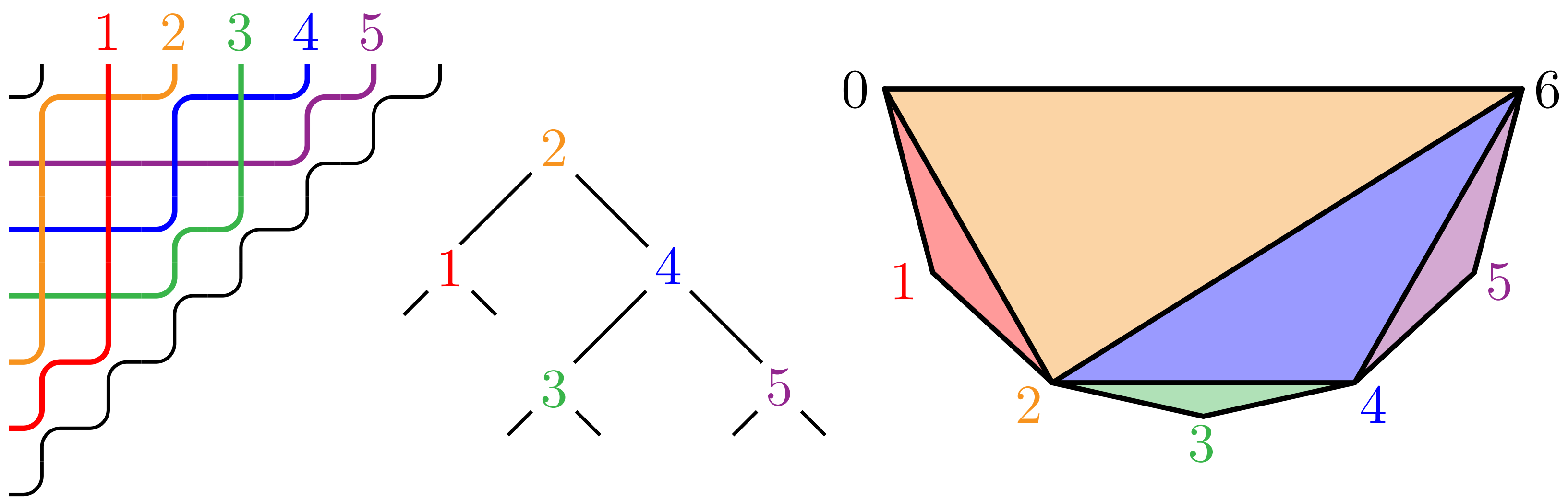
elbow in row i and column j $\longleftrightarrow$ diagonal $[i, j]$ of the $(n+2)$-gon

$(1, n)$-twist T $\longleftrightarrow$ triangulation $\mathrm{T}^\star$ of the $(n+2)$-gon

pth relevant pipe of T $\longleftrightarrow$ pth triangle of $\mathrm{T}^\star$

contact graph of T $\longleftrightarrow$ dual binary tree of $\mathrm{T}^\star$

elbow flips in T $\longleftrightarrow$ diagonal flips in $\mathrm{T}^\star$



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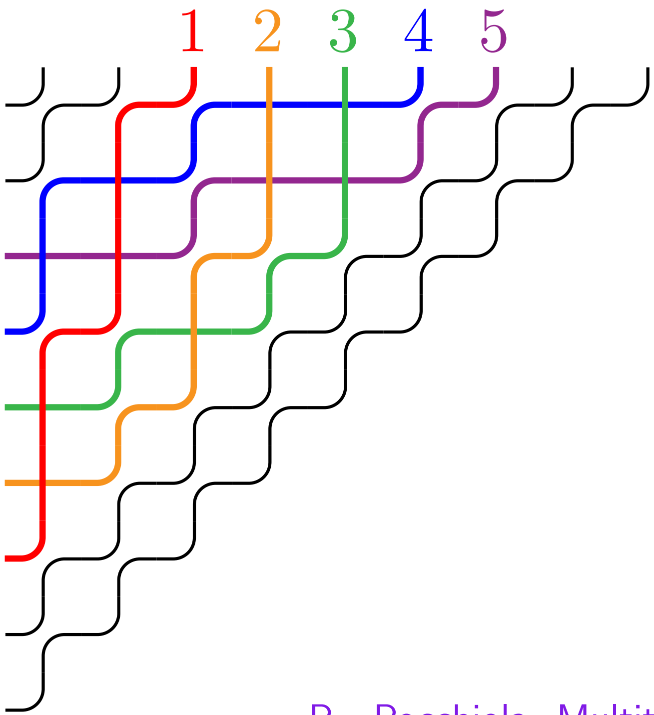
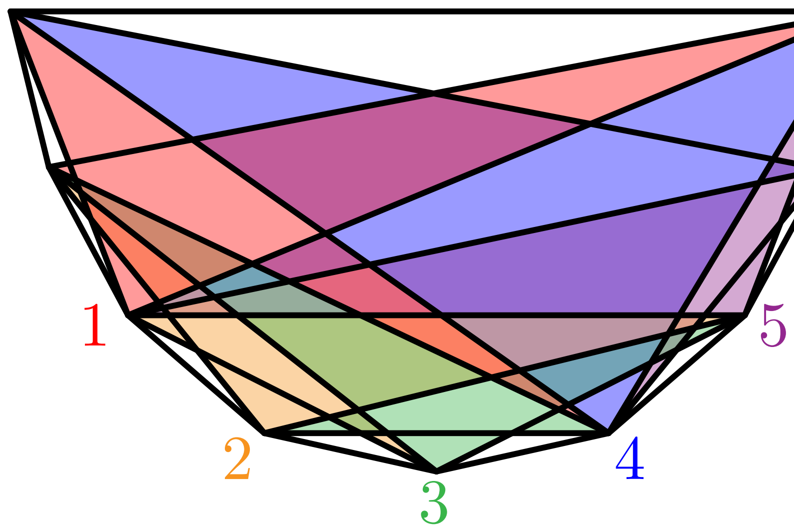
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i, j]$ of the $(n+2k)$-gon
(k, n)-twist T	$\longleftrightarrow$	k-triangulation $\mathrm{T}^\star$ of the $(n+2k)$-gon
pth relevant pipe of T	$\longleftrightarrow$	pth k-star of $\mathrm{T}^\star$
contact graph of T	$\longleftrightarrow$	dual graph of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$

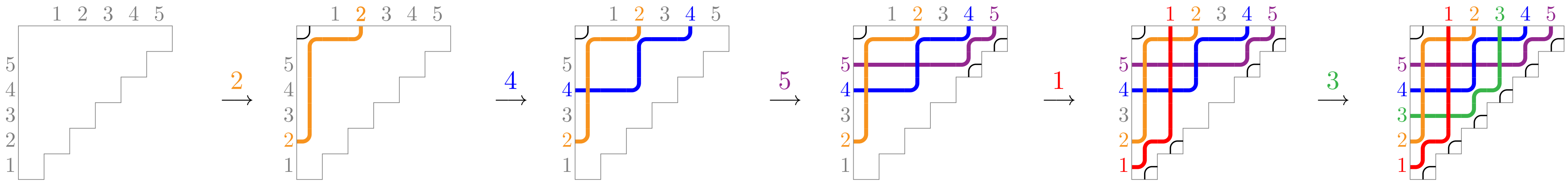




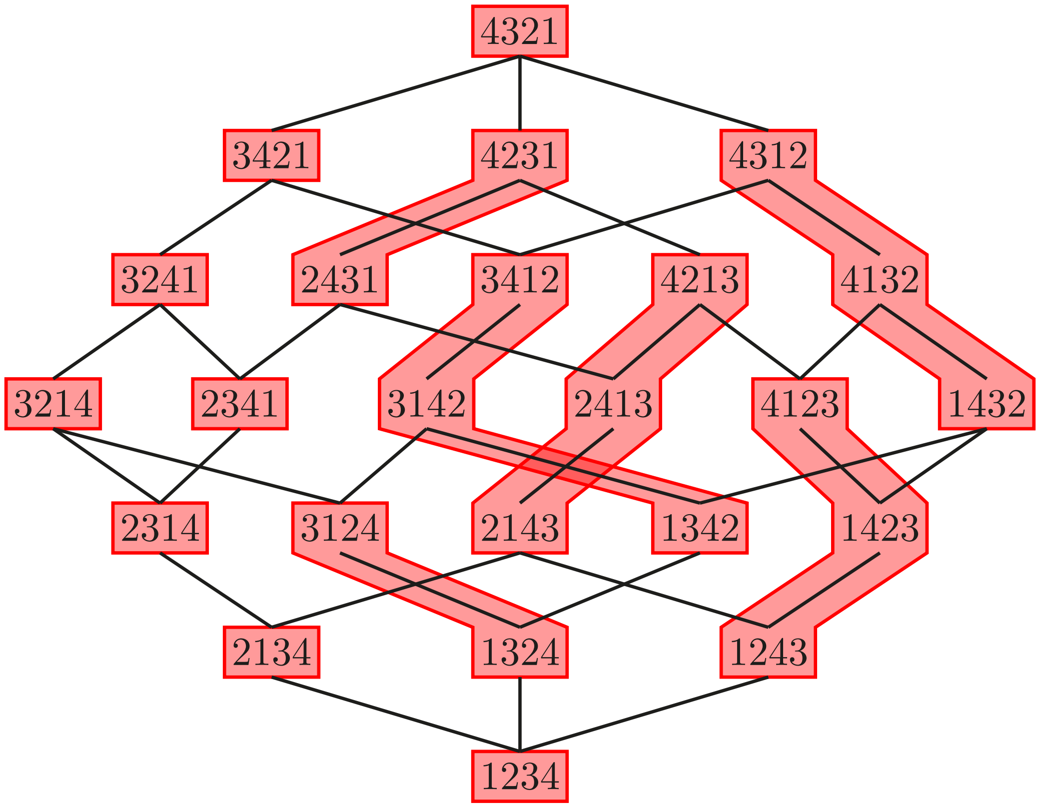
P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

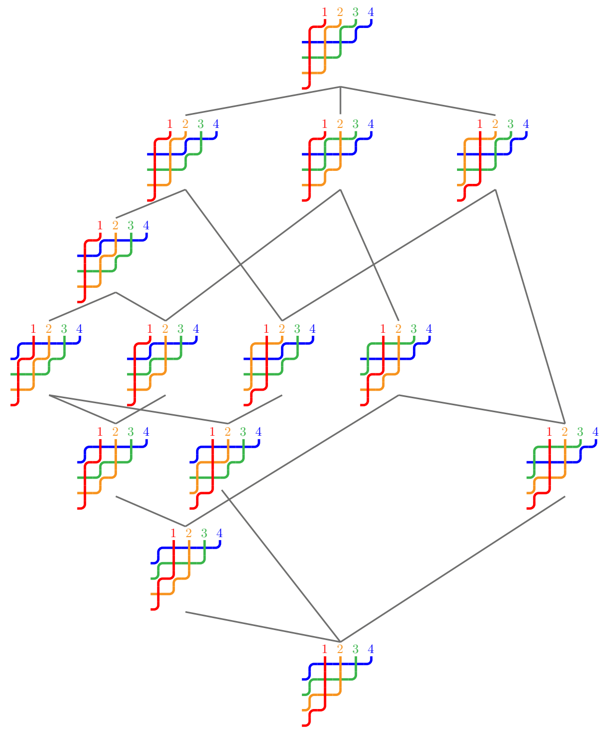
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

k-twist insertion of 31542



k-twist lattice = lattice quotient of the weak order by the relation "same k-twist"





P., *Brick polytopes, lattice quotients, and Hopf algebras* ('18)

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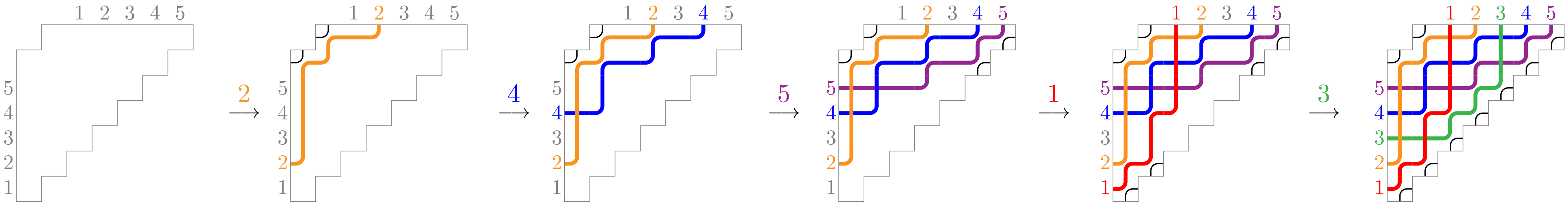

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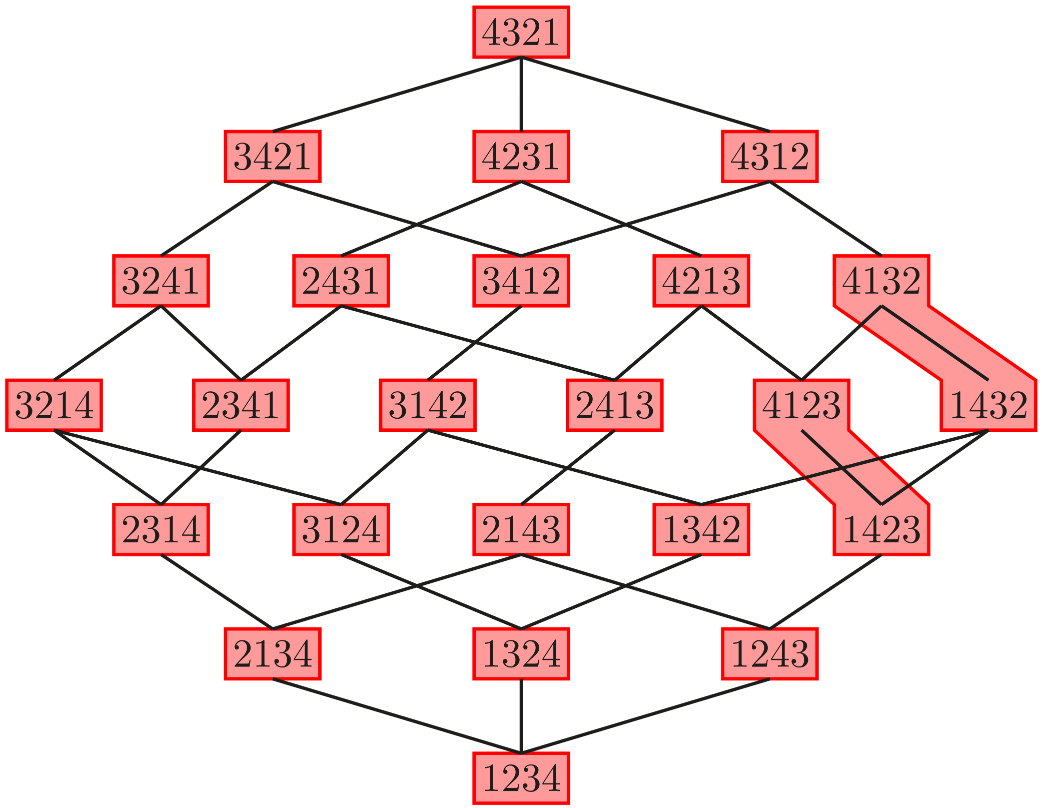
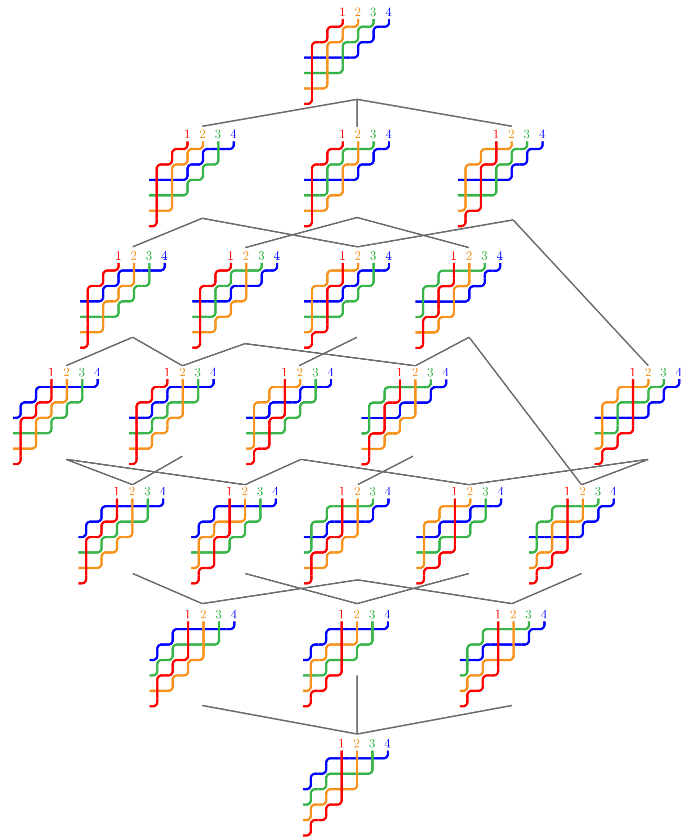




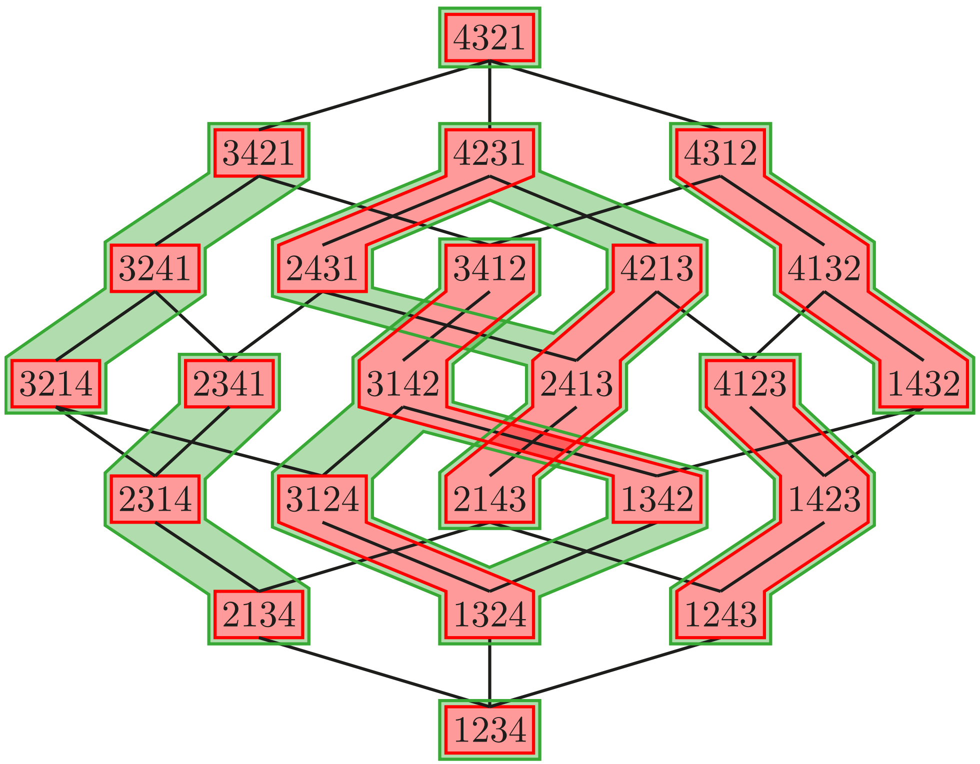
P., *Brick polytopes, lattice quotients, and Hopf algebras* ('18)

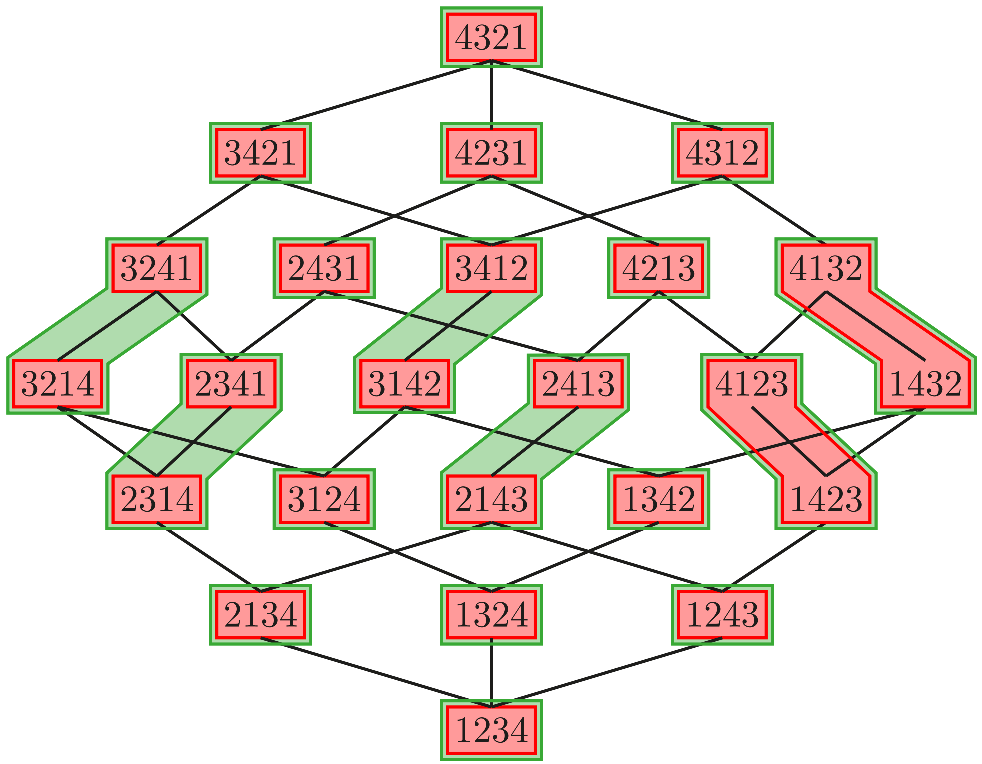
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

There is a commutative diagram of lattice homomorphisms:

permutations of $[n]$ ↠ acyclic orientations of $\mathrm{G}^k(n)$

permutations of $[n]$ ↠ acyclic k-twists on $[n]$ ↠ acyclic orientations of $\mathrm{G}^k(n)$





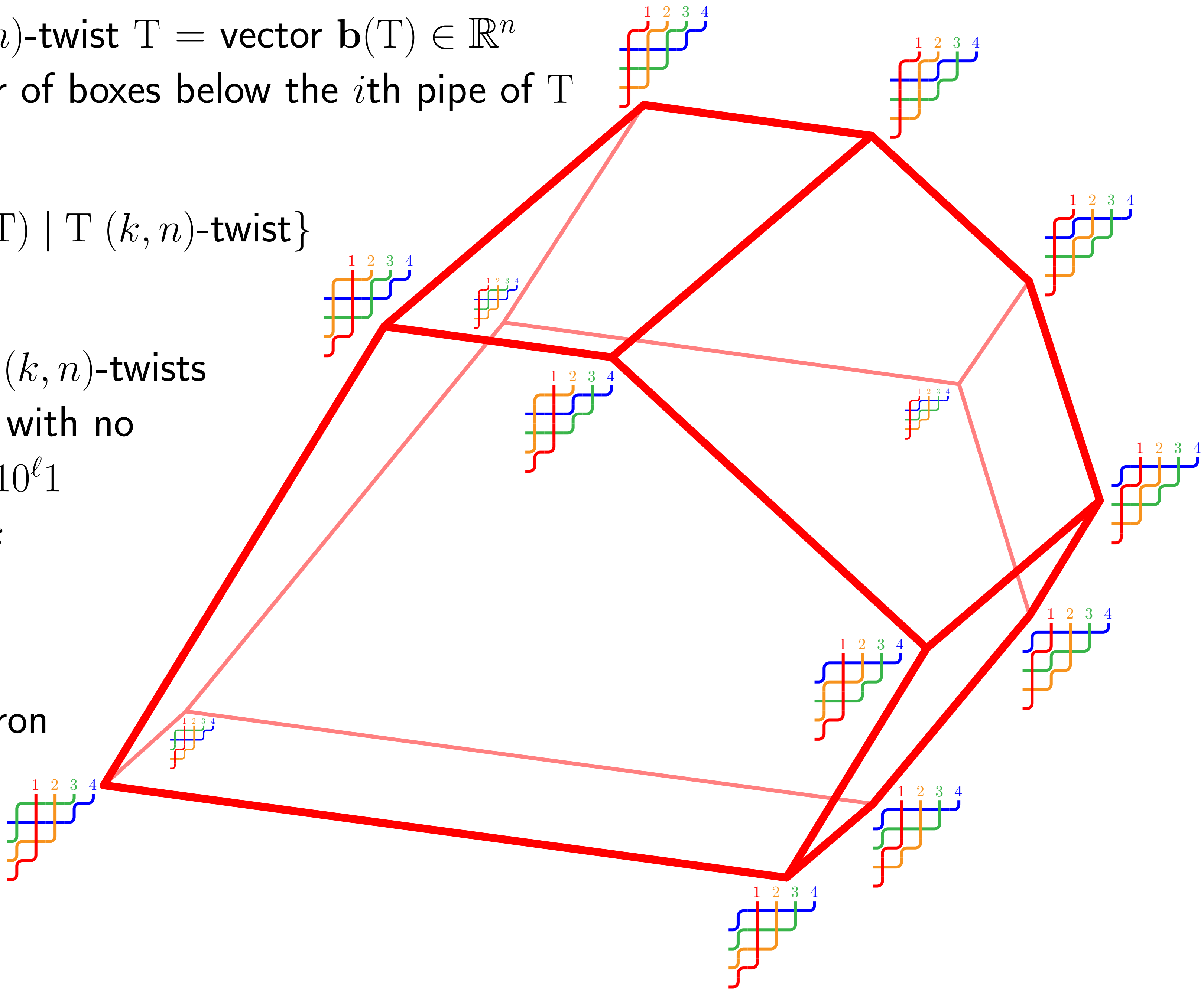
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

brick vector of a (k,n)-twist T = vector $\mathbf{b}(\mathrm{T}) \in \mathbb{R}^n$
with $\mathbf{b}(\mathrm{T})_i$ = number of boxes below the ith pipe of T

brick polytope
$\mathrm{Brick}^k(n) = \mathrm{conv}\,\{\mathbf{b}(\mathrm{T}) \mid \mathrm{T}\ (k,n)\text{-twist}\}$

Vertices $\longleftrightarrow$ acyclic (k,n)-twists
Facets $\longleftrightarrow$ $0/1$-seqs with no subseqs $10^\ell 1$ for $\ell \geq k$

connections to
- Loday associahedron
- incidence cones of binary trees
- Tamari lattice



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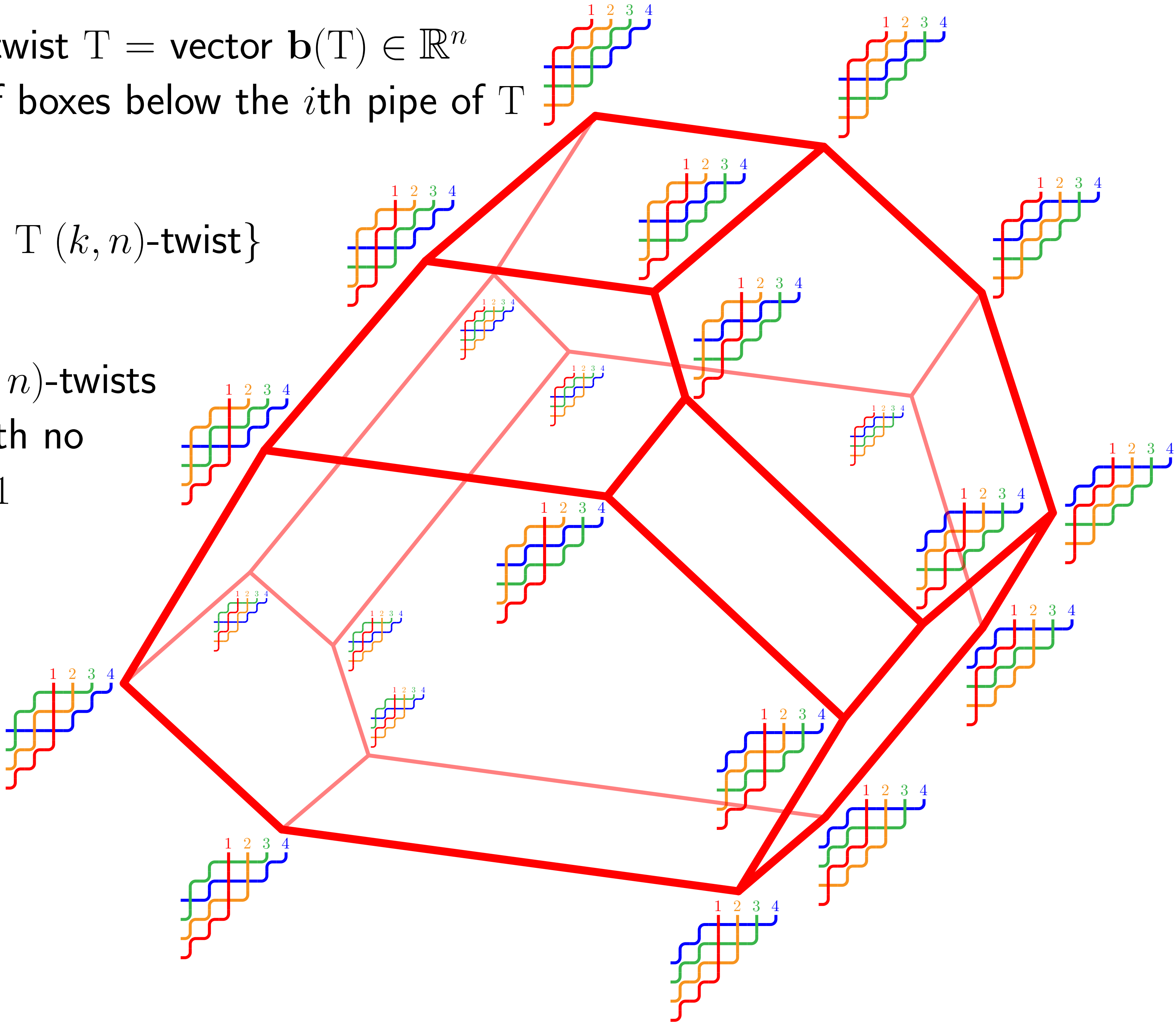

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$\mathrm{Brick}^k(n) = \mathrm{conv}\,\{\mathbf{b}(\mathrm{T}) \mid \mathrm{T}\ (k,n)\text{-twist}\}$

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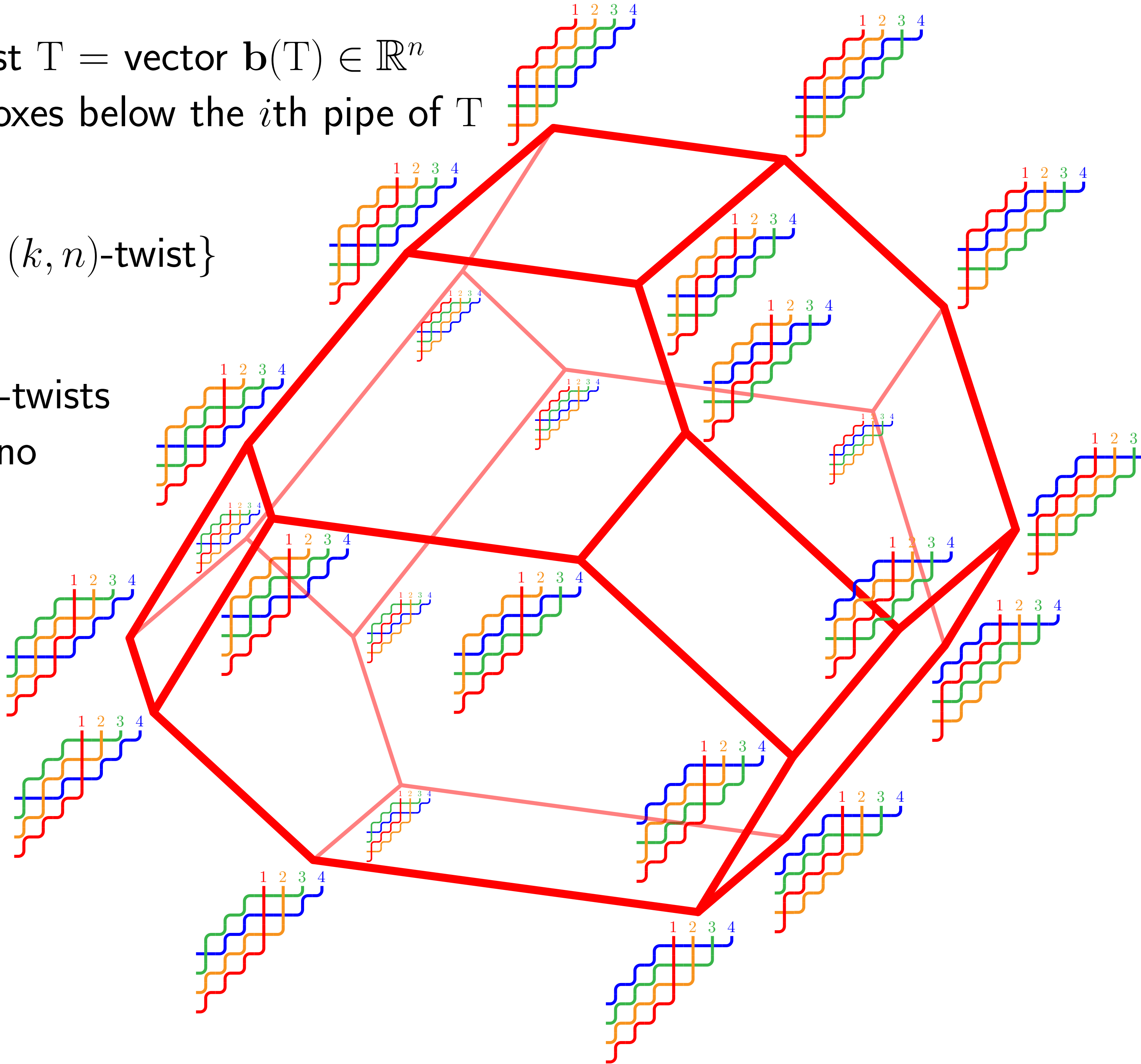

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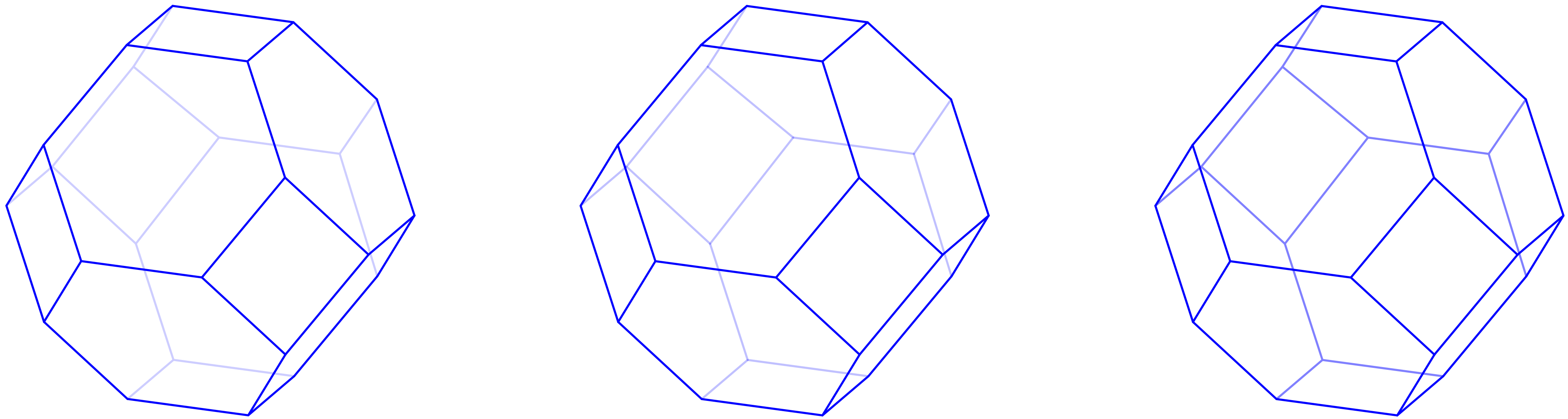
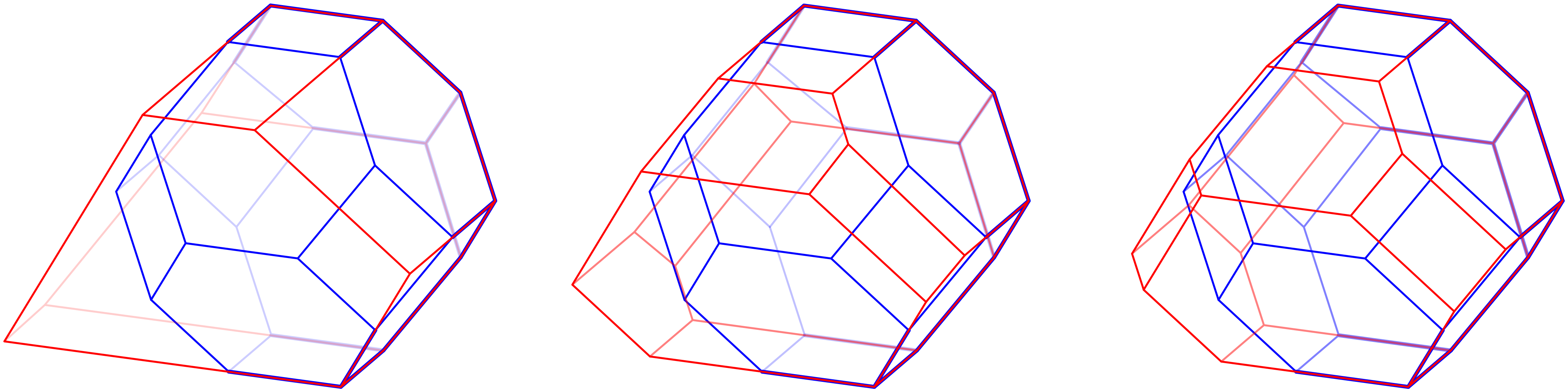


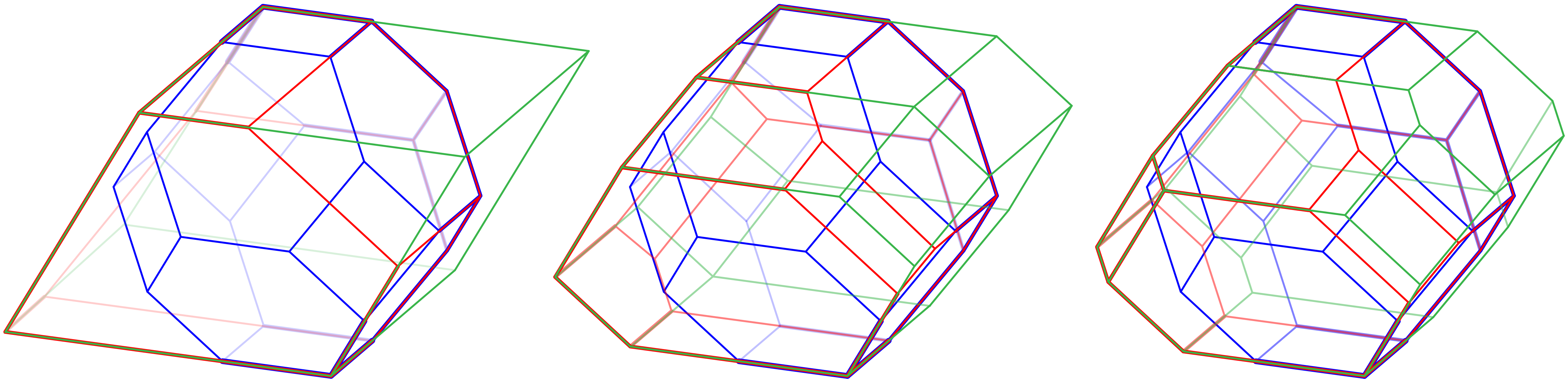
Permutahedron Perm(n)

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



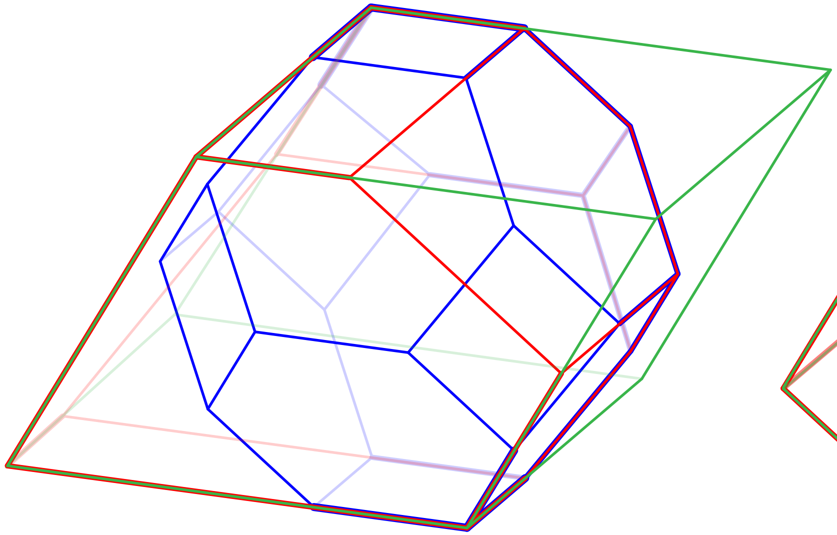
Permutahedron Perm(n) $\subset$ Brick polytope Brick$^k(n)$

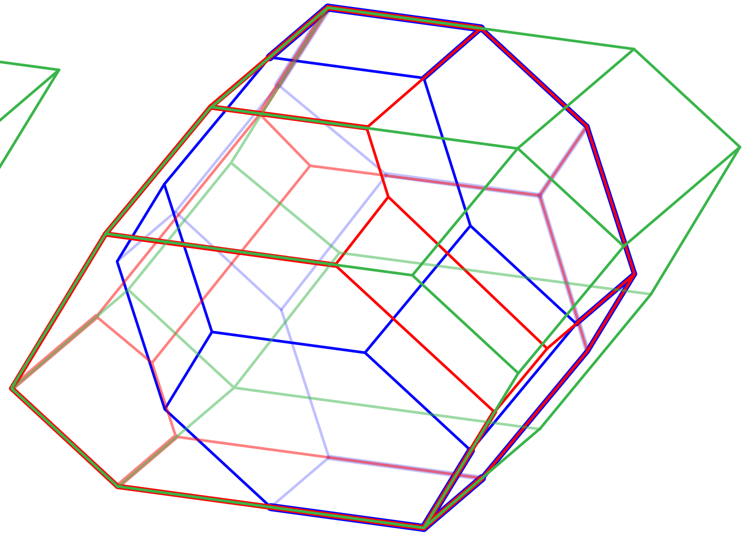
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

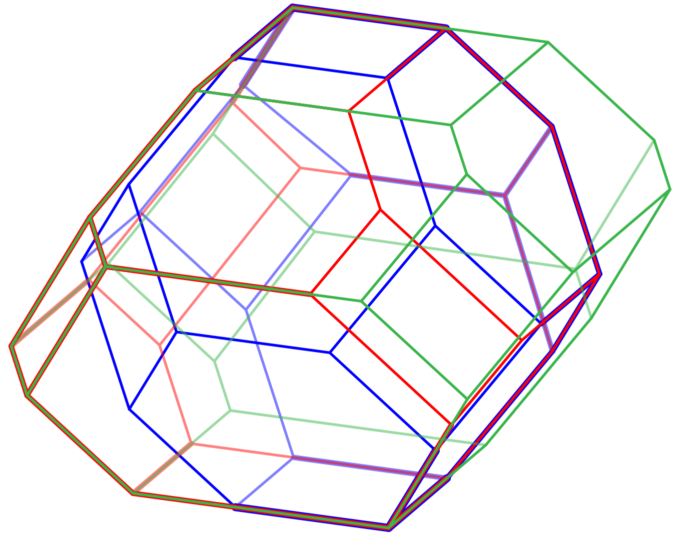


Permutahedron $\mathsf{Perm}(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

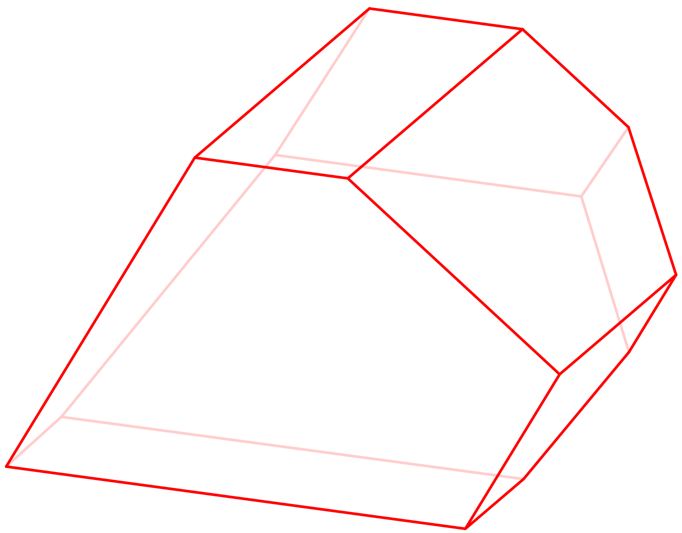


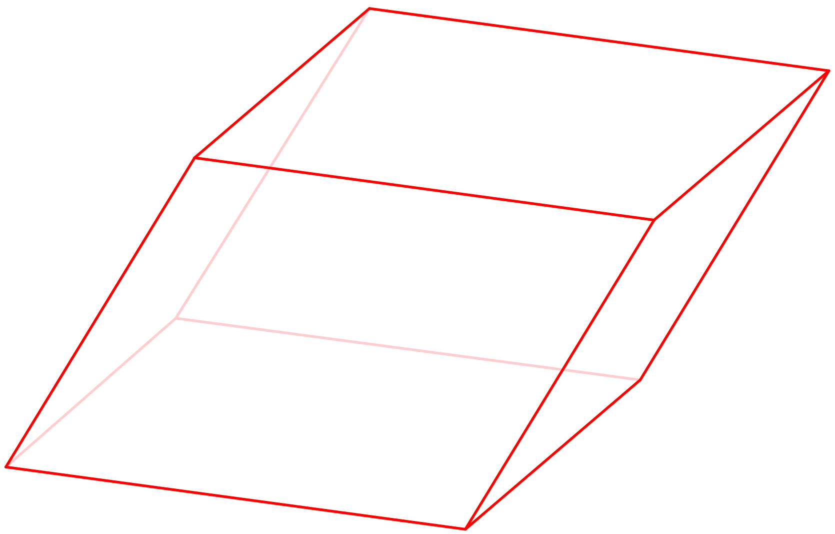




Permutahedron $\mathsf{Perm}(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

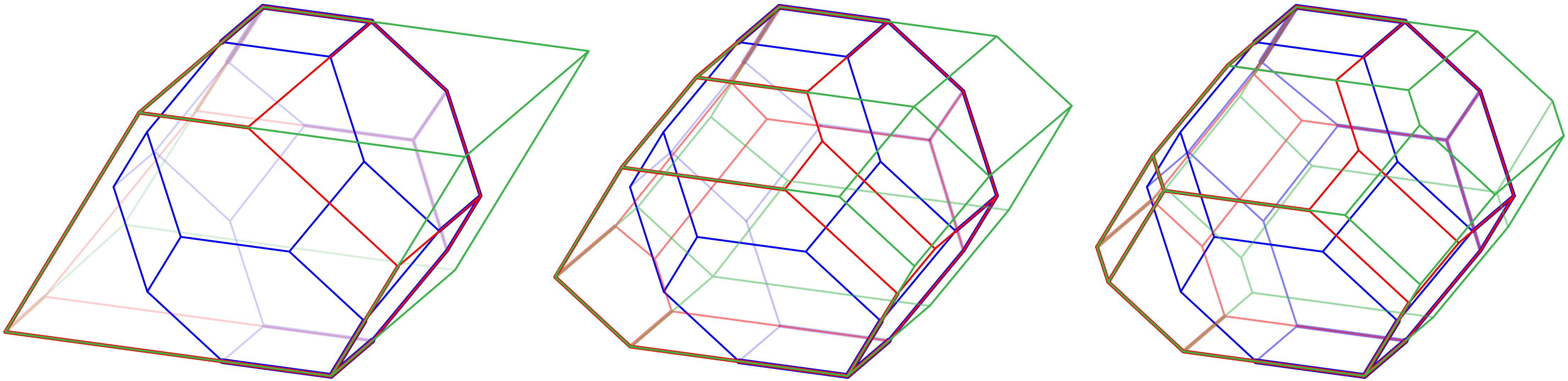
$\mathsf{Brick}^1(n)$





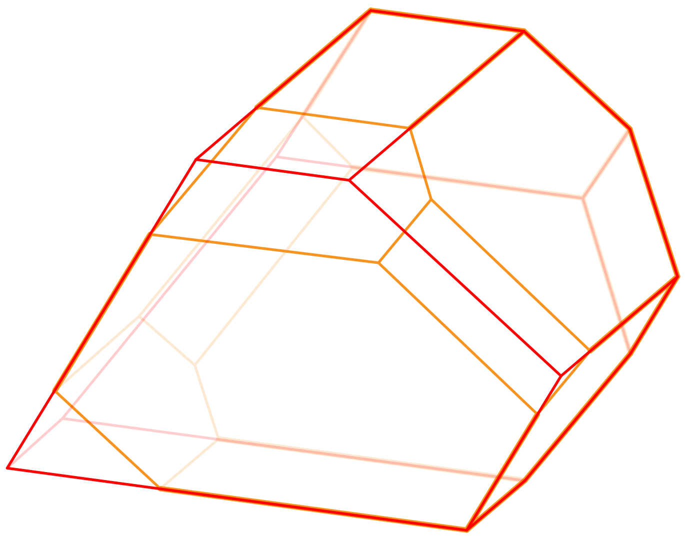
$\mathsf{Zono}^1(n)$

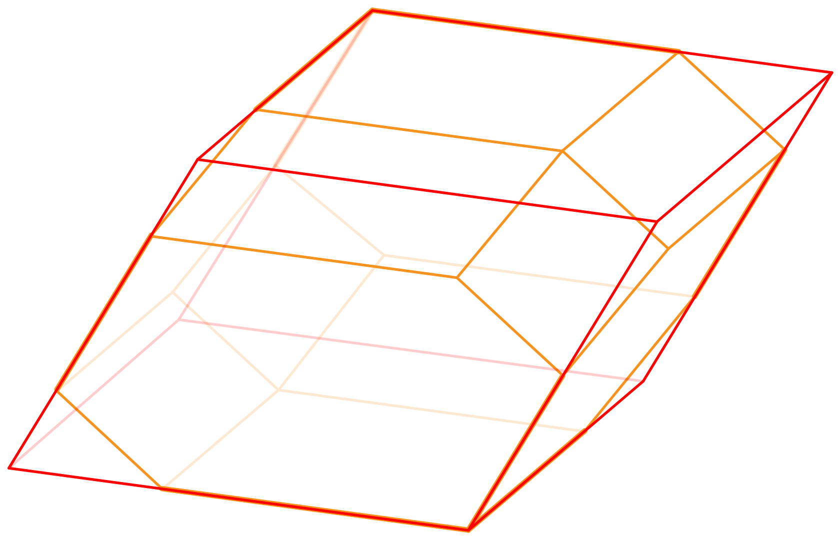
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



Permutahedron Perm(n) $\subset$ Brick polytope Brick$^k(n)$ $\subset$ Zonotope Zono$^k(n)$

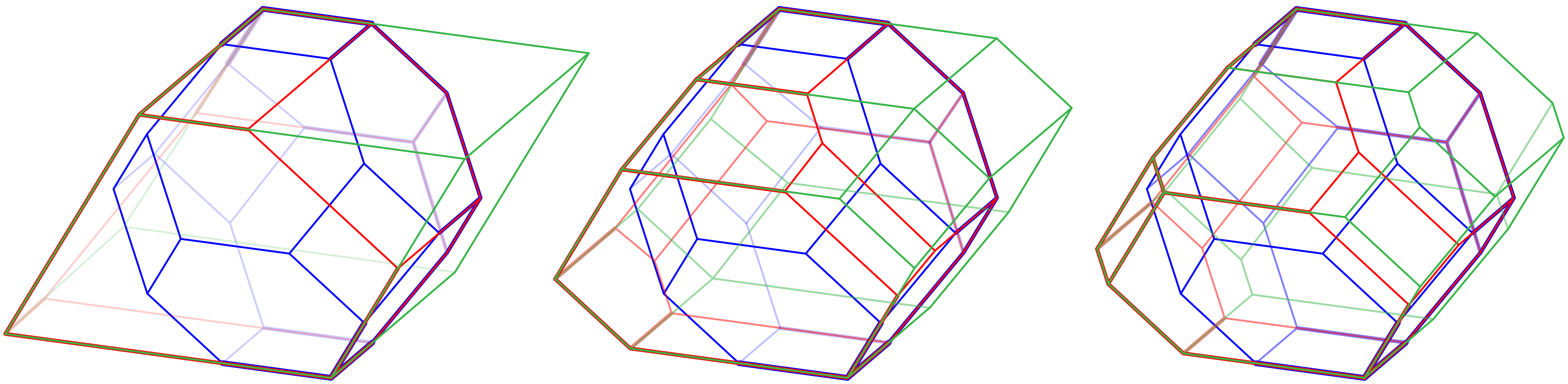
Brick$^1(n)$
$\cap$
Brick$^2(n)$





Zono$^1(n)$
$\cap$
Zono$^2(n)$

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



Permutahedron $\mathsf{Perm}(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

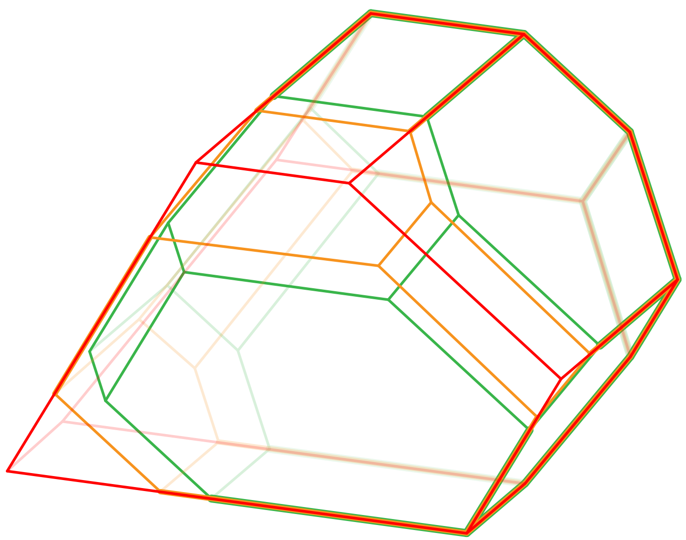
$\mathsf{Brick}^1(n)$

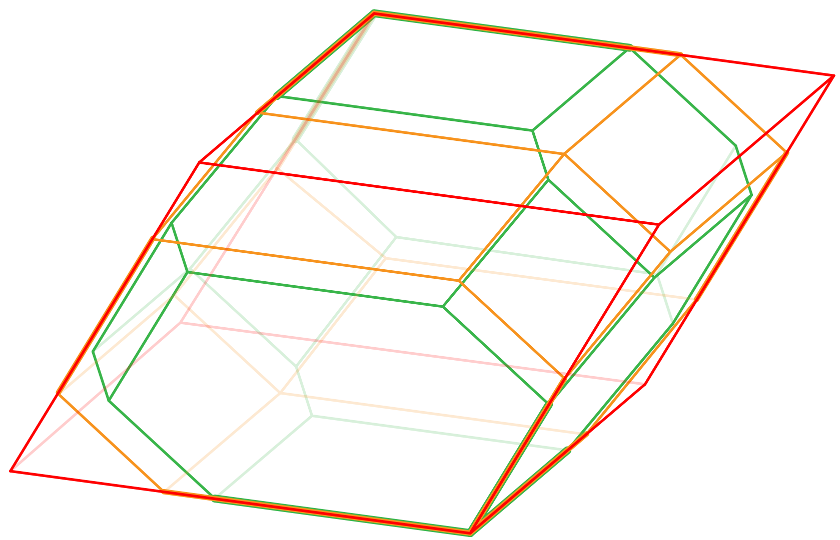
$\cap$

$\mathsf{Brick}^2(n)$

$\cap$

$\mathsf{Brick}^3(n)$





$\mathsf{Zono}^1(n)$

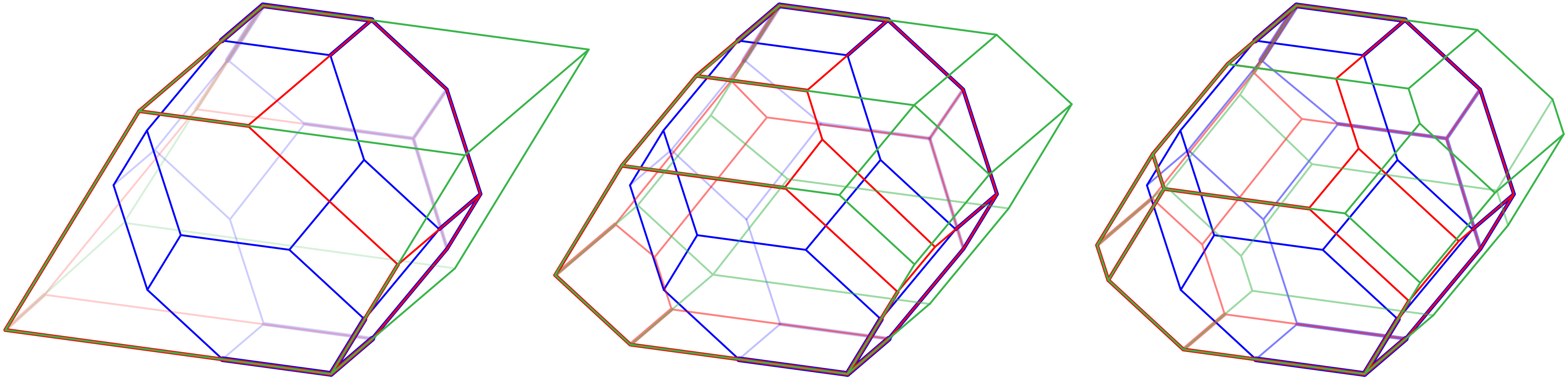
$\cap$

$\mathsf{Zono}^2(n)$

$\cap$

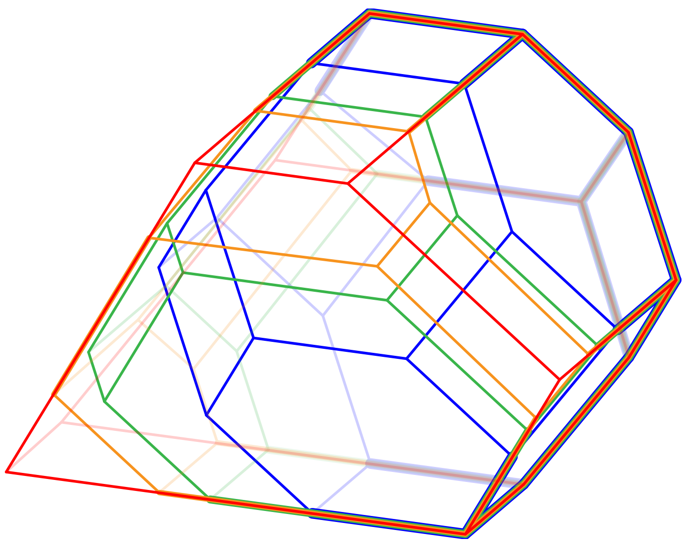
$\mathsf{Zono}^3(n)$

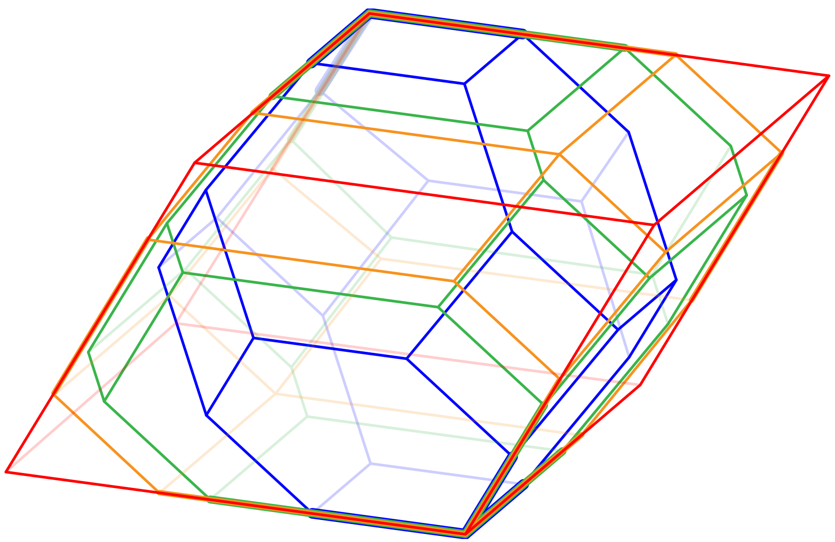
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



Permutahedron $\mathsf{Perm}(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

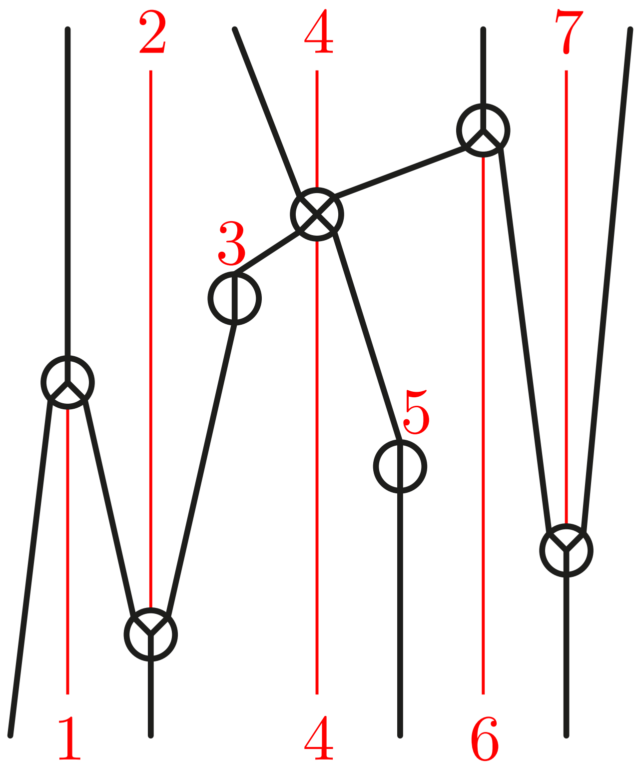
$\mathsf{Brick}^1(n)$ $\cap$ $\mathsf{Brick}^2(n)$ $\cap$ $\mathsf{Brick}^3(n)$ $\cap$ $\vdots$ $\cap$ $\mathsf{Perm}(n)$



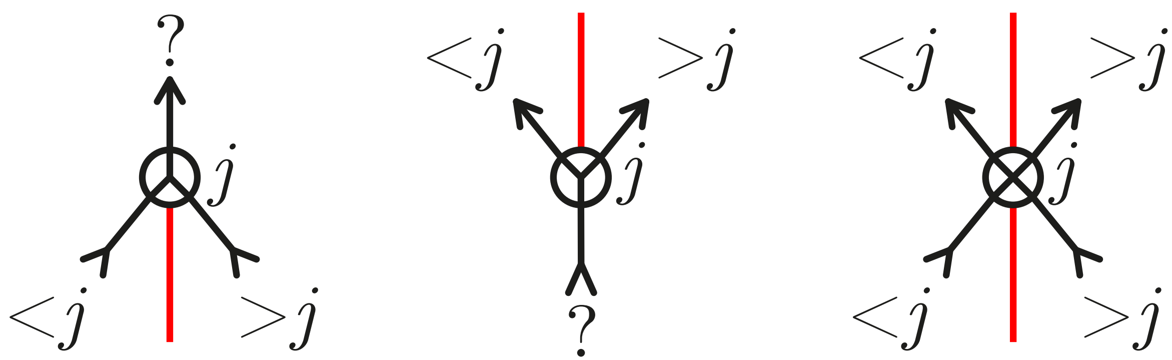


$\mathsf{Zono}^1(n)$ $\cap$ $\mathsf{Zono}^2(n)$ $\cap$ $\mathsf{Zono}^3(n)$ $\cap$ $\vdots$ $\cap$ $\mathsf{Perm}(n)$

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA



permutree = directed (bottom to top) and labeled (bijectively by $[n]$) tree such that

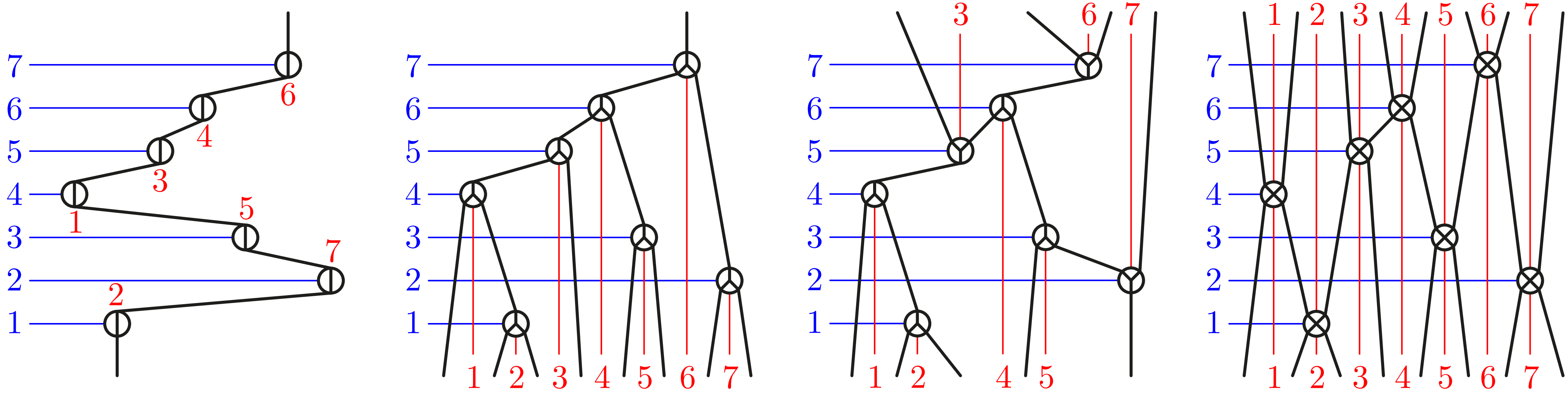


P.–Pons, *Permutrees* ('18)

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

Examples.

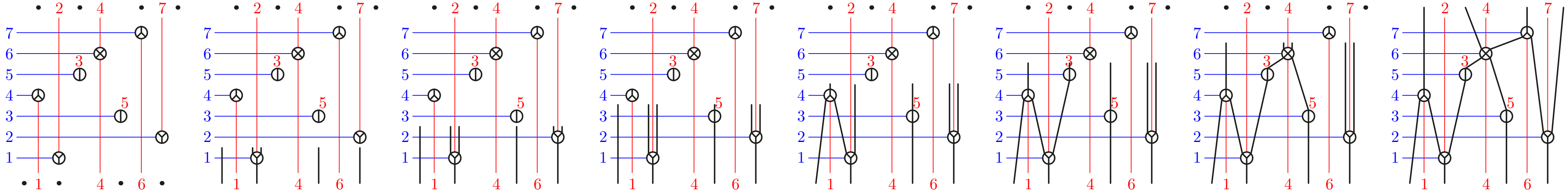
decoration		permutrees
$⦶^n$	$\longleftrightarrow$	permutations of $[n]$
$☮^n$	$\longleftrightarrow$	standard binary search trees
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian trees
$\otimes^n$	$\longleftrightarrow$	binary sequences



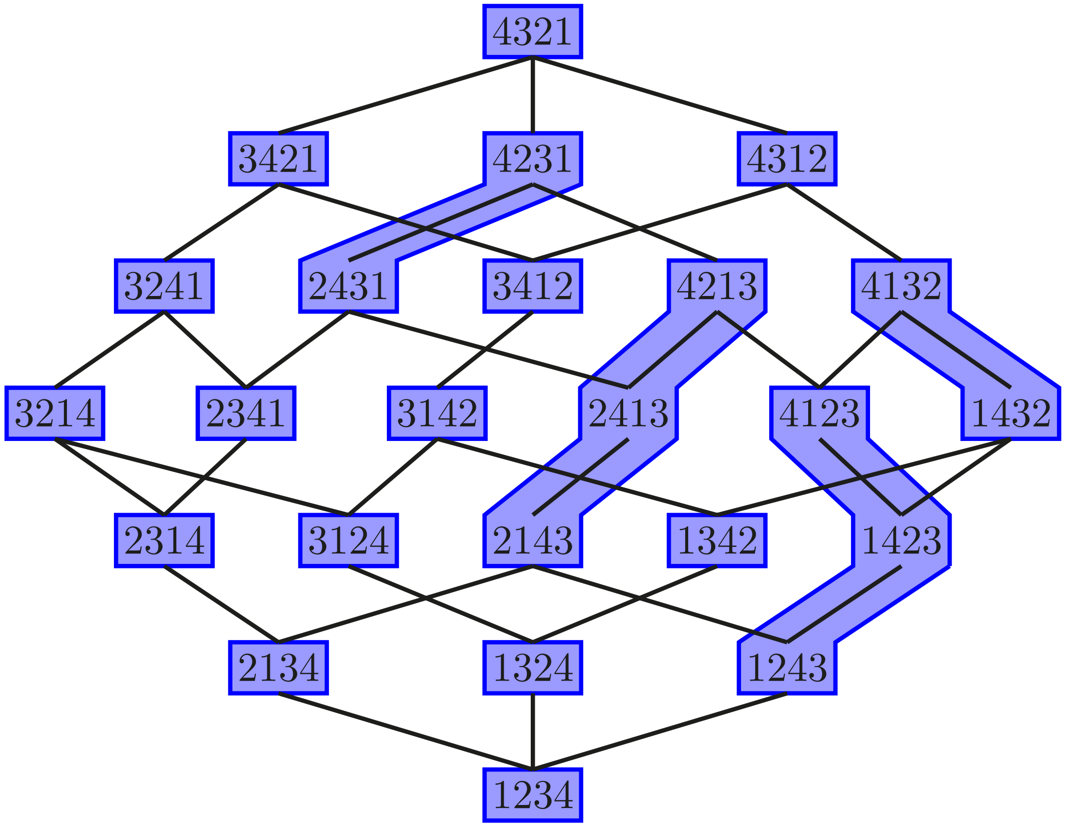
P.–Pons, *Permutrees* ('18)

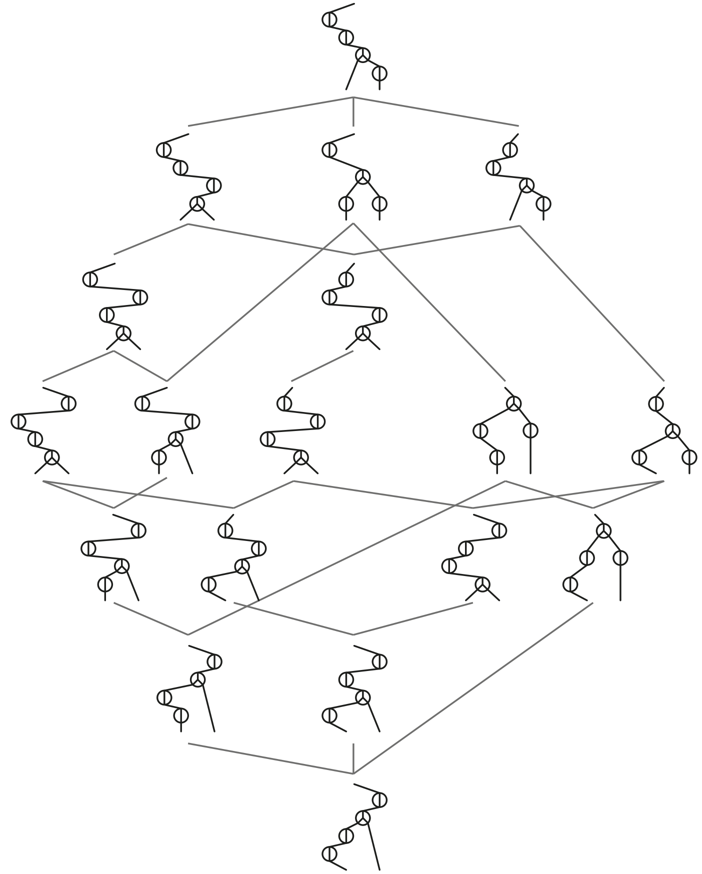
EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

permutree insertion of 2751346



permutree lattice = lattice quotient of the weak order by the relation "same permutree"





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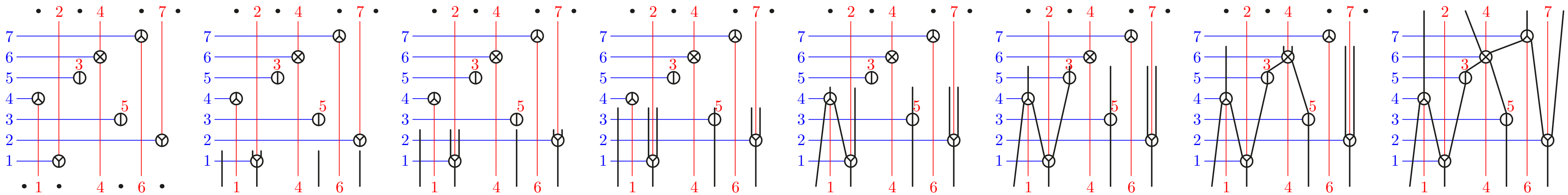
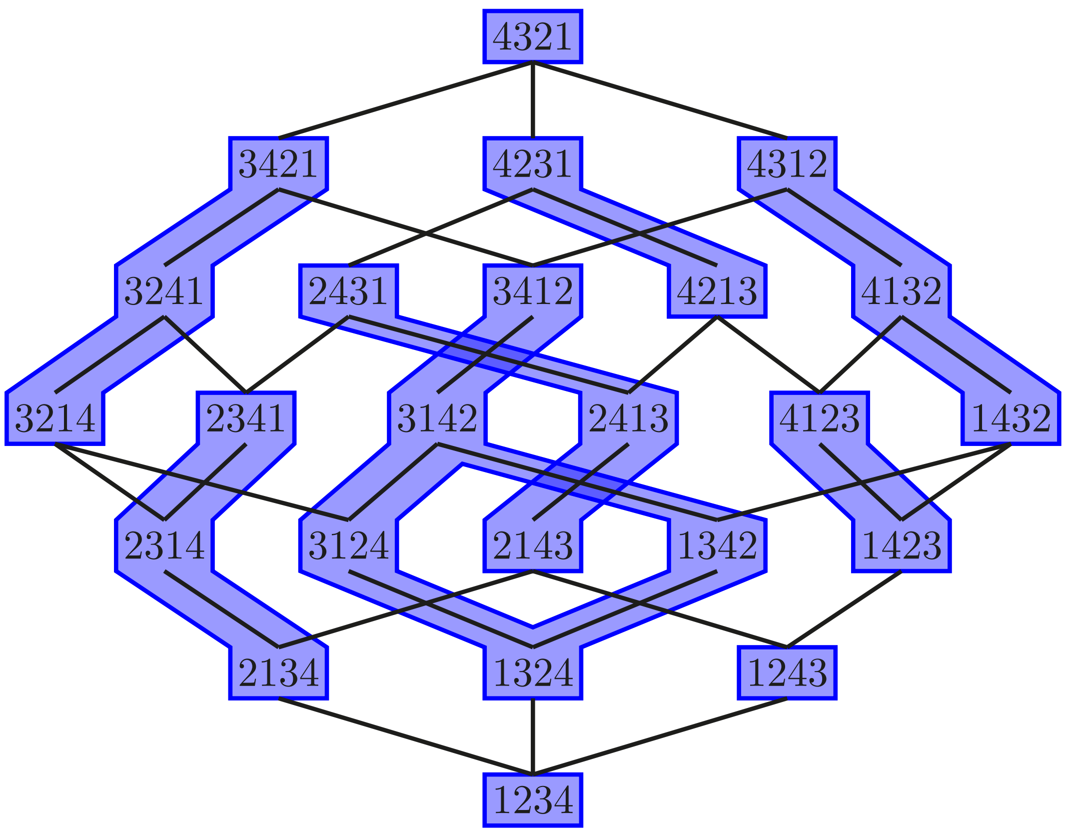
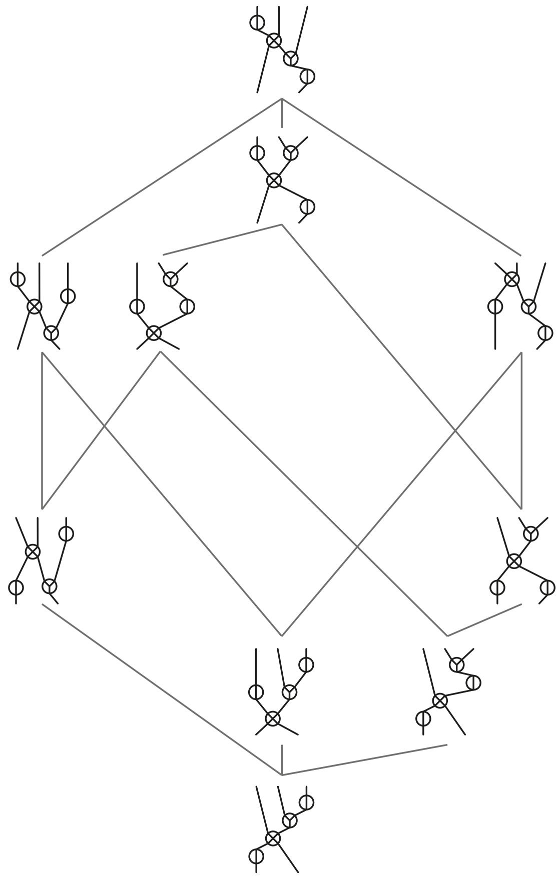


permutree lattice $=$ lattice quotient of the weak order by the relation “same permutree”



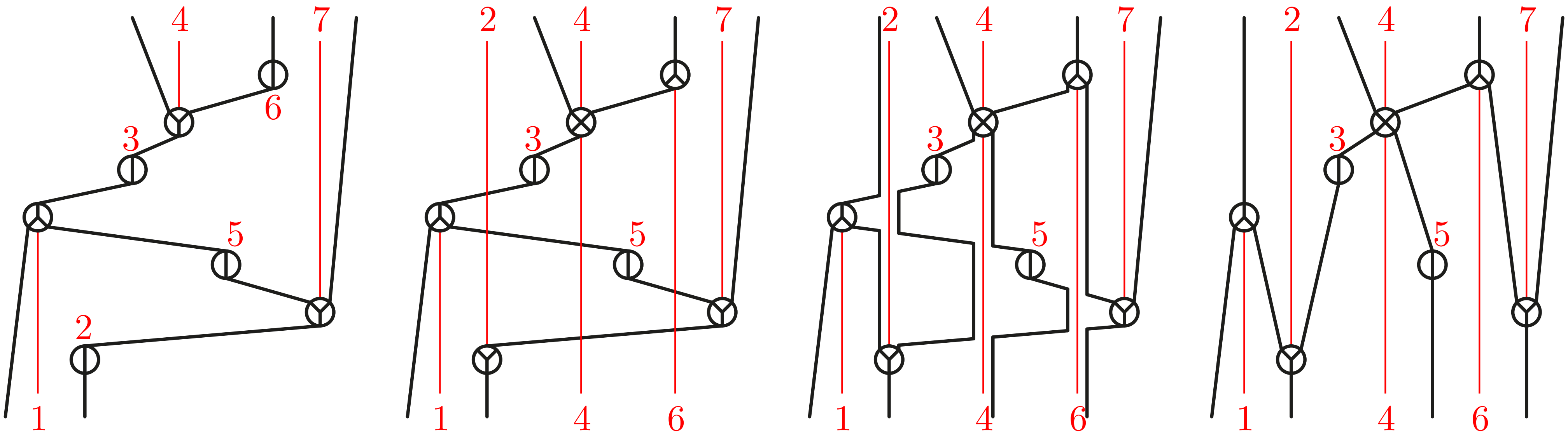


P.–Pons, *Permutrees* ('18)

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

δ **refines** δ' when $\delta_i \preccurlyeq \delta'_i$ for all $i \in [n]$ for the order ⦶ $\preccurlyeq$ Ⓨ, ☮ $\preccurlyeq$ $\otimes$

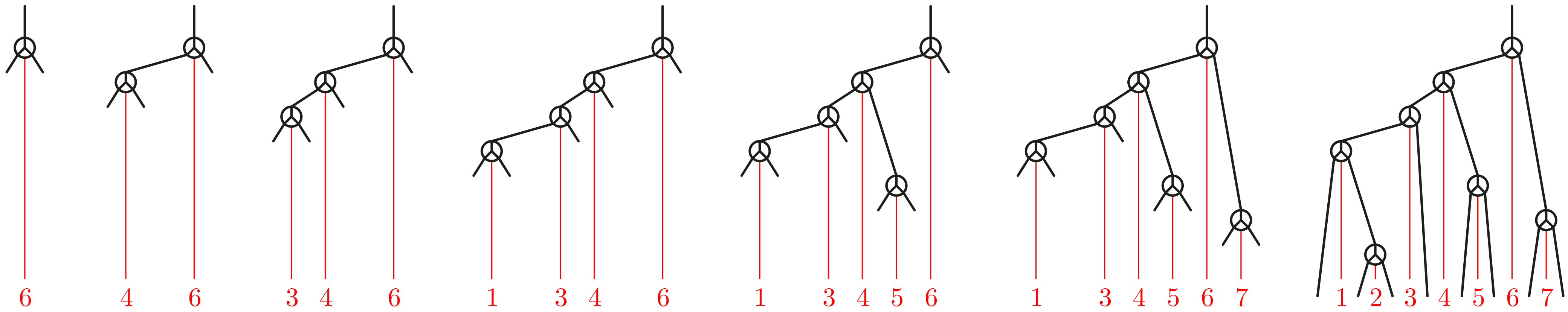
When δ refines δ', the δ-permutree congruence classes refine the δ-permutree congruence classes: $\sigma \equiv_\delta \tau \Longrightarrow \sigma \equiv_{\delta'} \tau$.

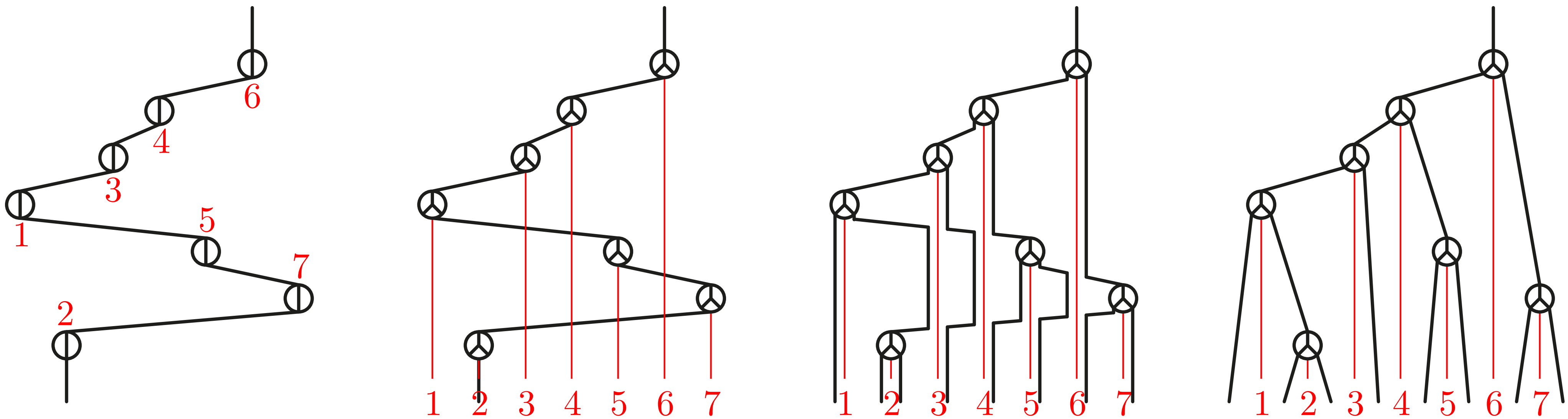


P.–Pons, *Permutrees* ('18)

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

Binary search tree insertion with cisors and elastics





P.–Pons, *Permutrees* ('18)

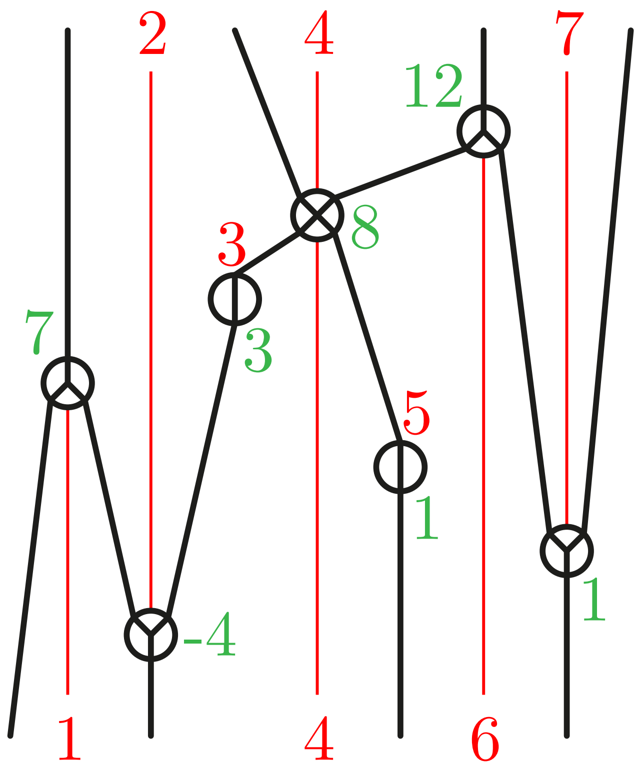
EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

permutreehedron

$$\mathsf{PT}(\delta) := \operatorname{conv}\{\mathbf{PP}(T) \mid T\ \delta\text{-permutree}\} = \mathbb{H} \cap \bigcap_{I \text{ cut}} \mathbf{H}^{\geq}(I)$$

$$\mathbf{PP}(T) := \left[1 + d_i + \underline{\ell}_i \underline{r}_i - \overline{\ell}_i \overline{r}_i\right]_{i \in [n+1]} \qquad \mathbf{H}^{\geq}(I) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \right\}$$

P.–Pons, *Permutrees* ('18)



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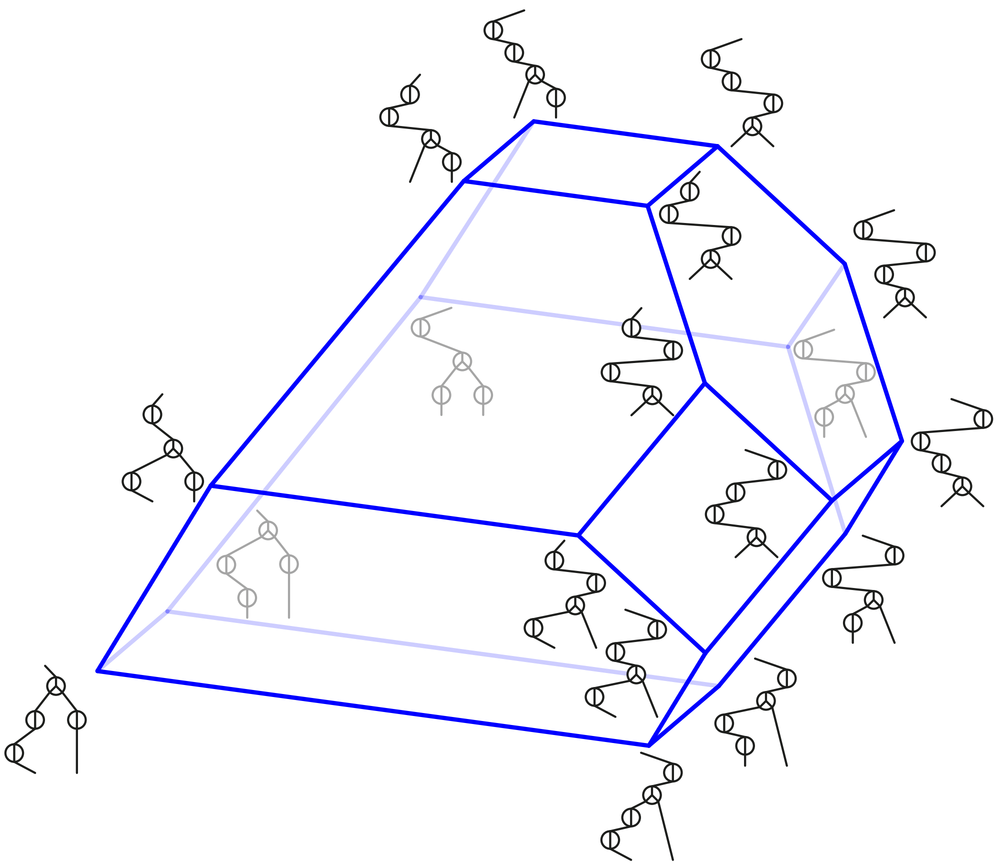
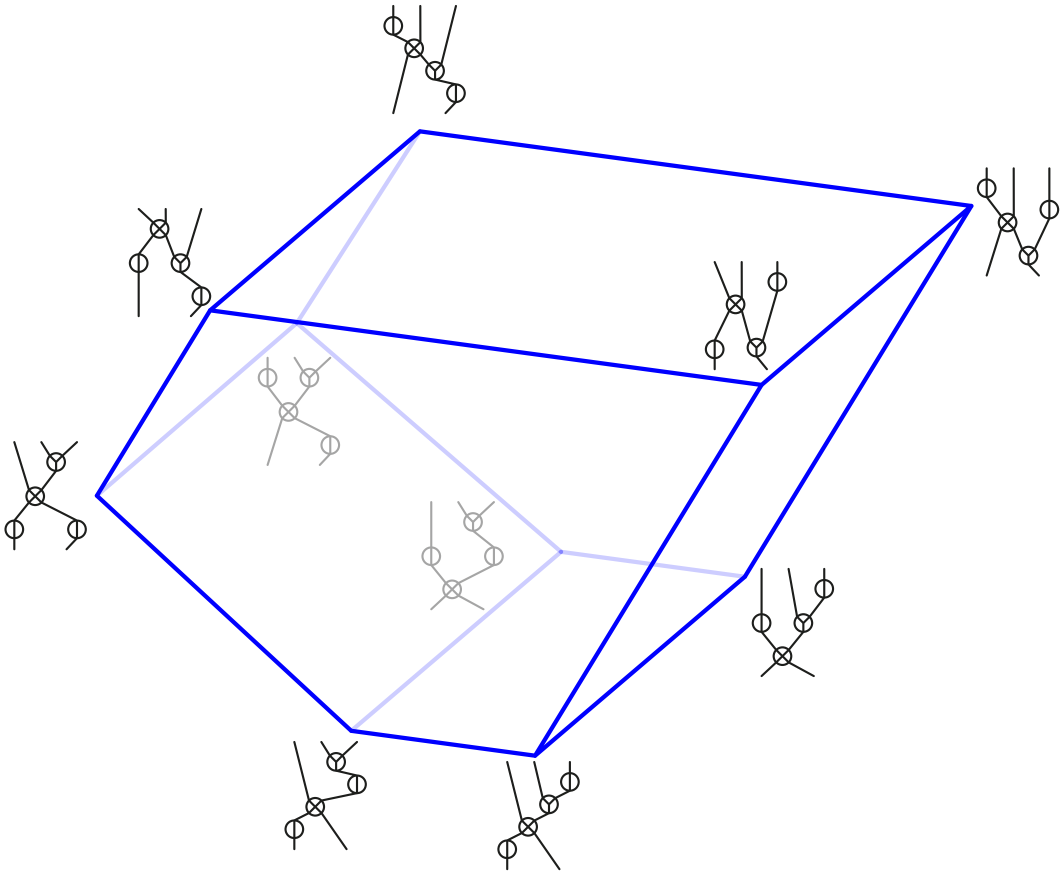
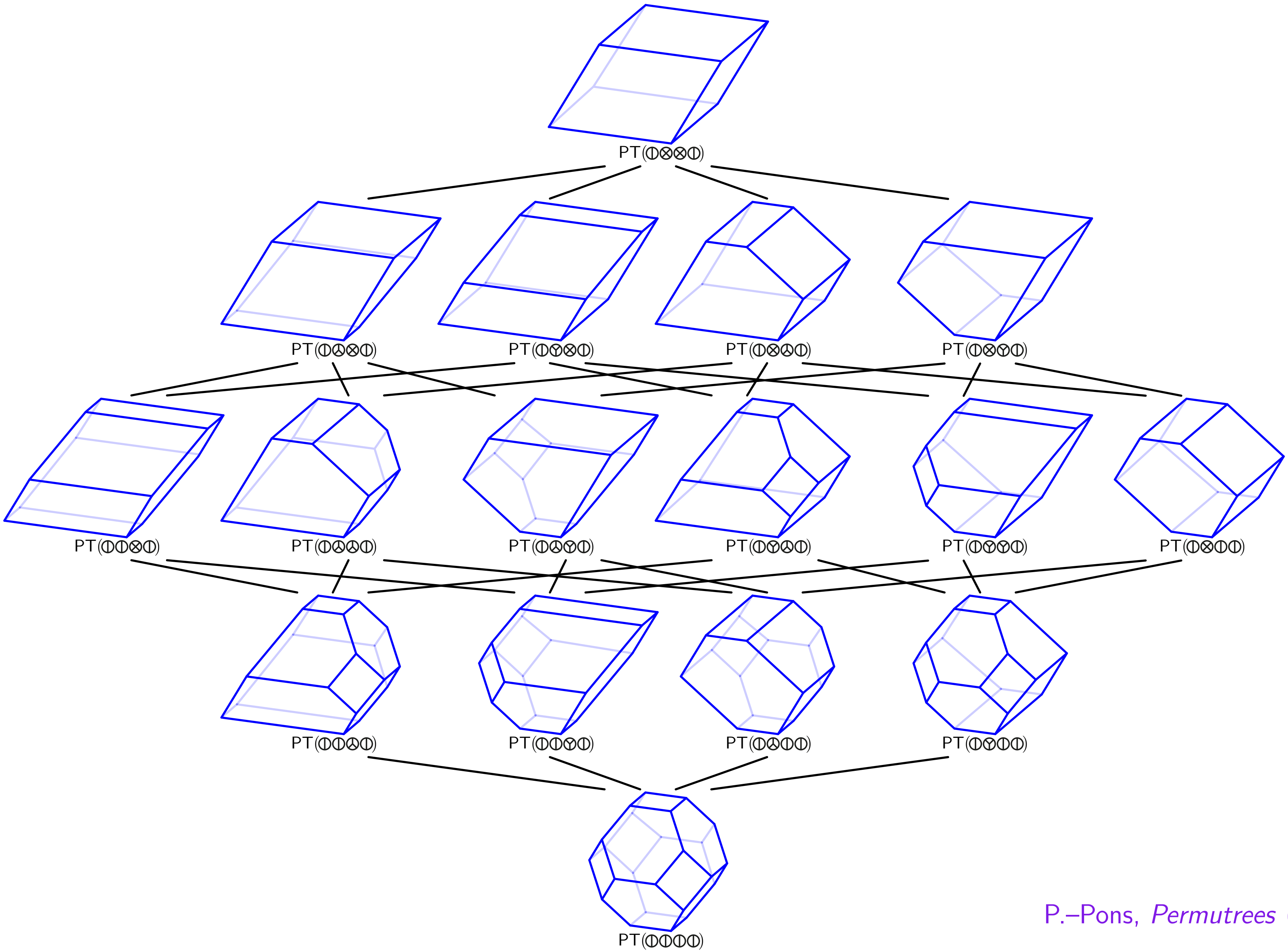
P.–Pons, *Permutrees* ('18)





EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA



P.–Pons, *Permutrees* ('18)

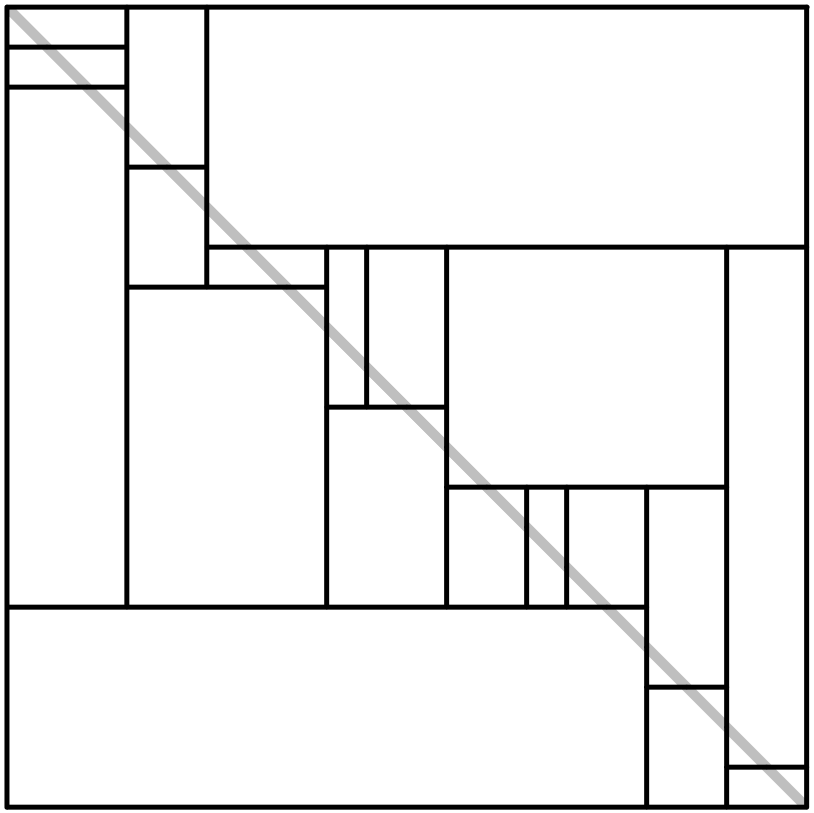
EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

Twin binary trees = pair of binary trees with opposite canopy
= $(\mathrm{T}, \mathrm{T}')$ where T and $\mathrm{T}'^{\mathsf{op}}$ have a common linear extension

Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)

in bijection with diagonal rectangulations

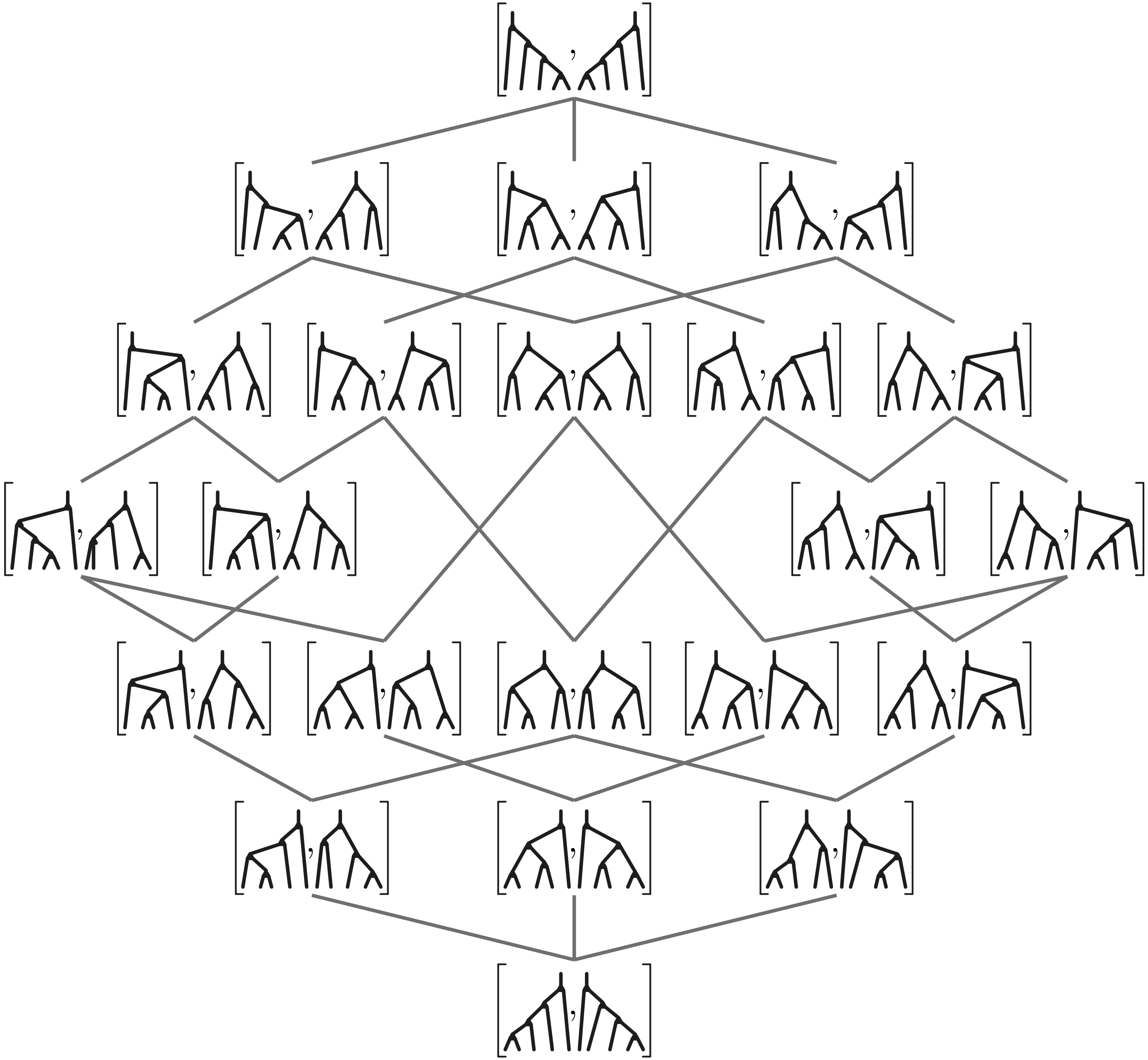


Giraudo, *Algebraic and combinatorial structures on pairs of twin binary trees* ('12)

Baxter insertion = insert σ in a binary tree and σ^{op} in another binary tree

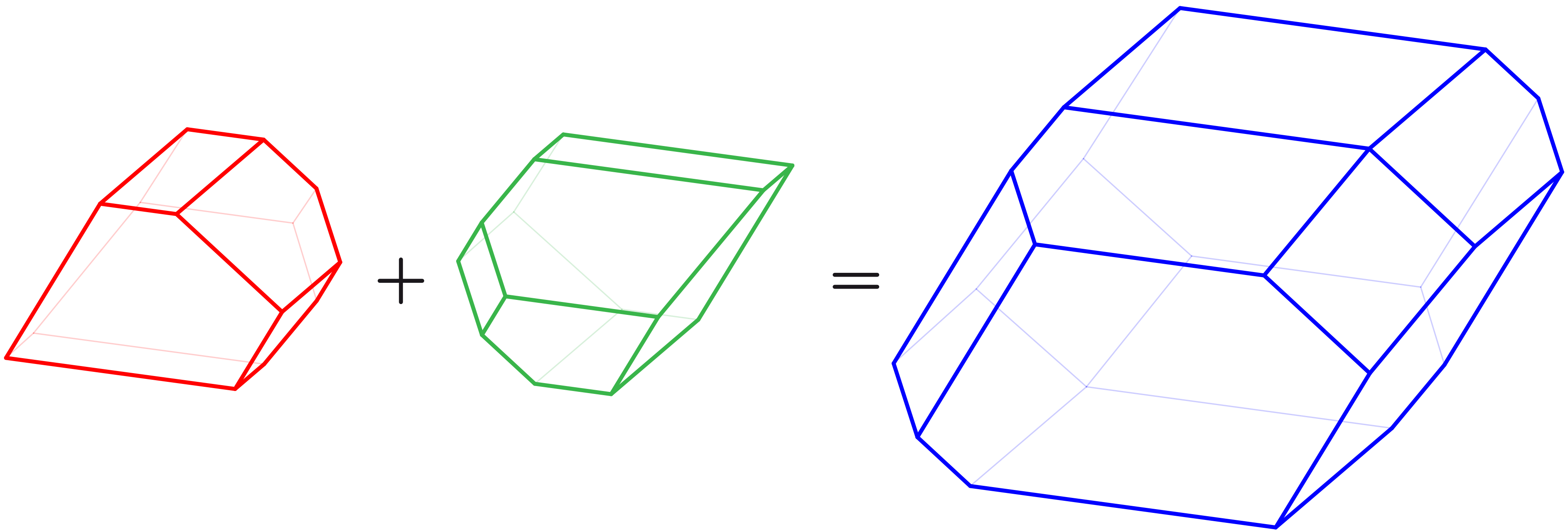
EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

Baxter lattice = lattice quotient of the weak order by the relation "same twin binary tree"



EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

Baxter associahedron = Minkowski sum of two opposite associahedra



EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

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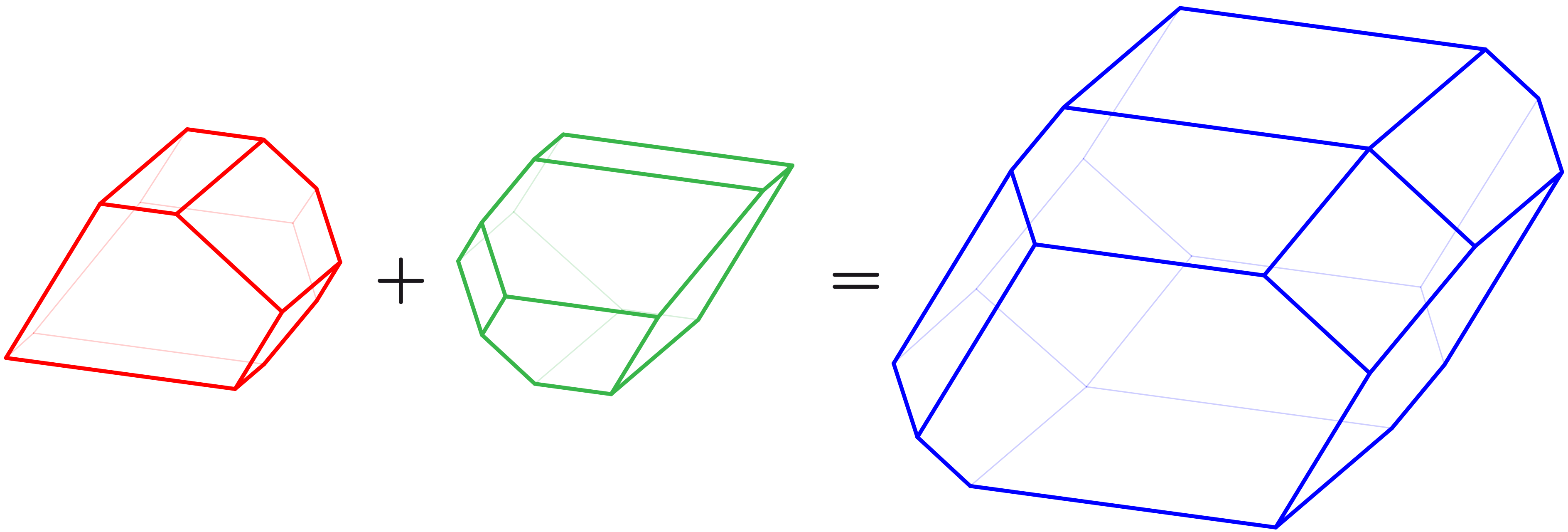
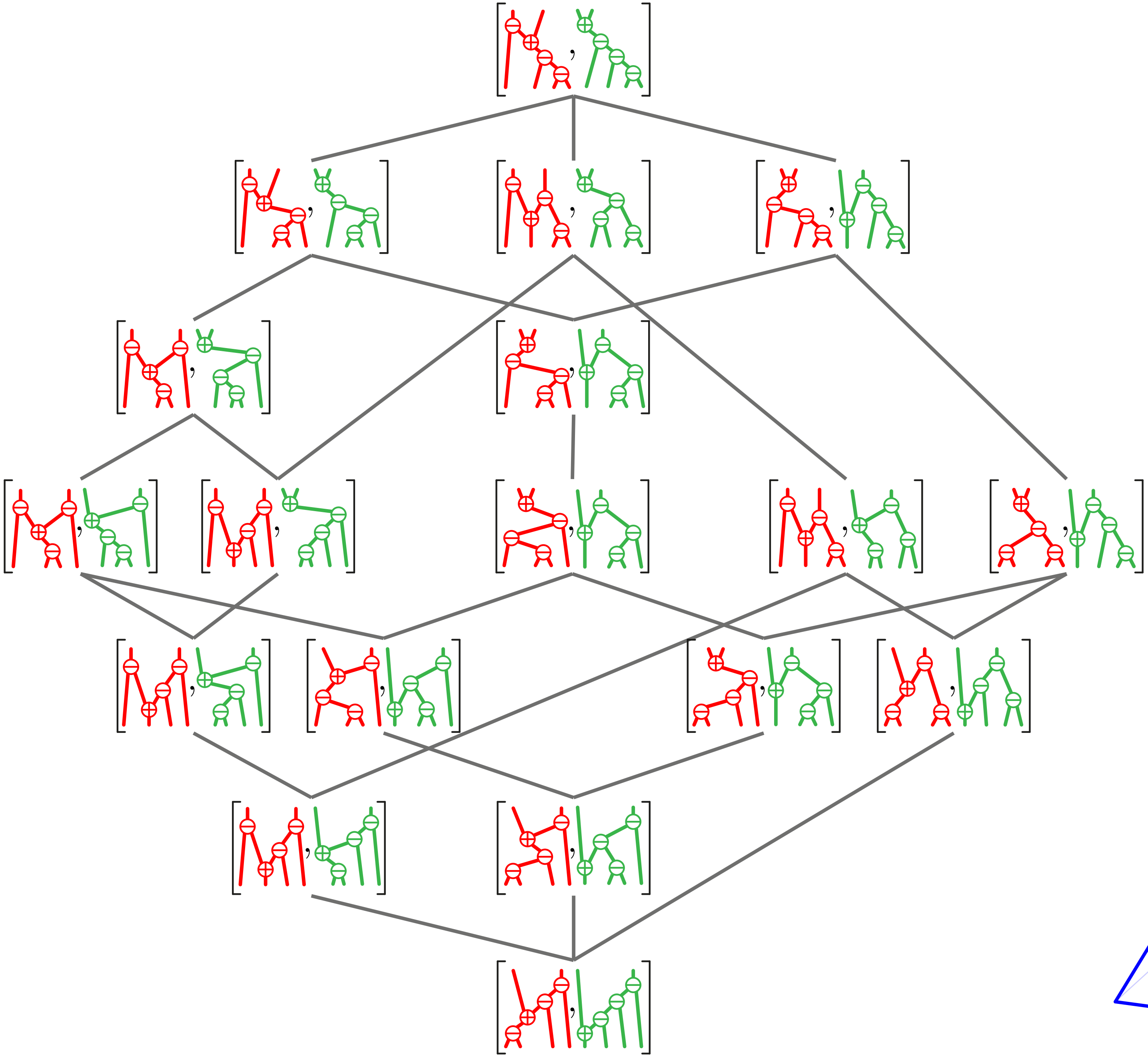
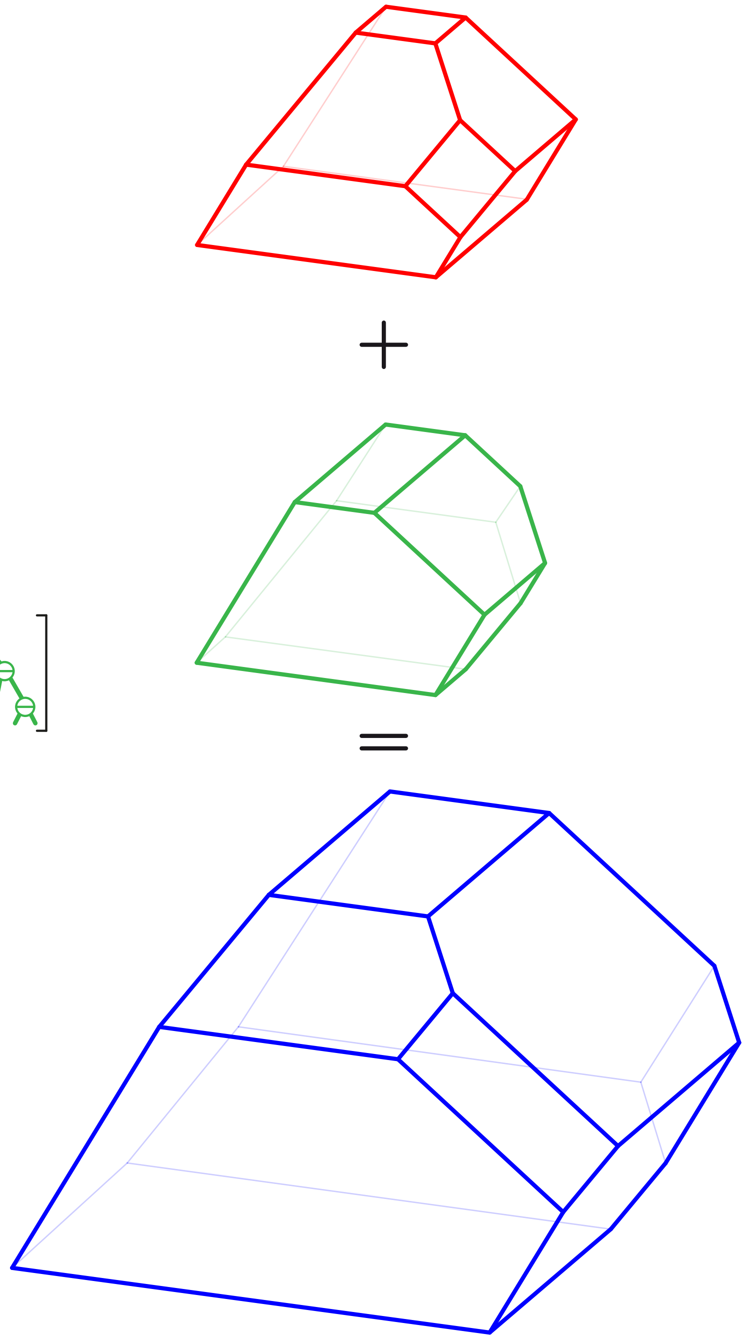


General tuplization process:

- tuples of objects representing classes
- intersection of lattice congruences
- Minkowski sum of polytopes

EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS



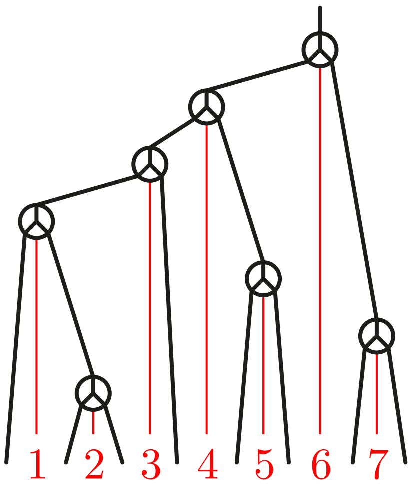


SUMMARY

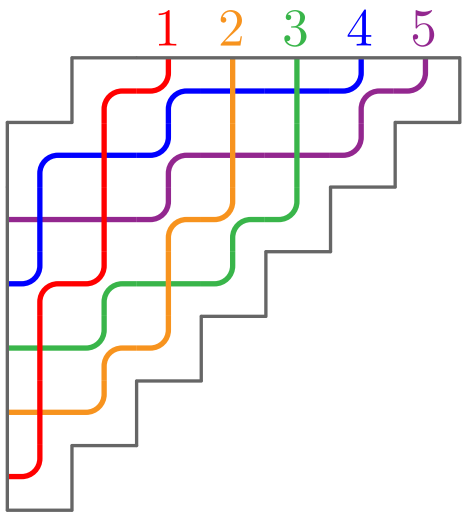
lattice congruence = equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \quad \Longrightarrow \quad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

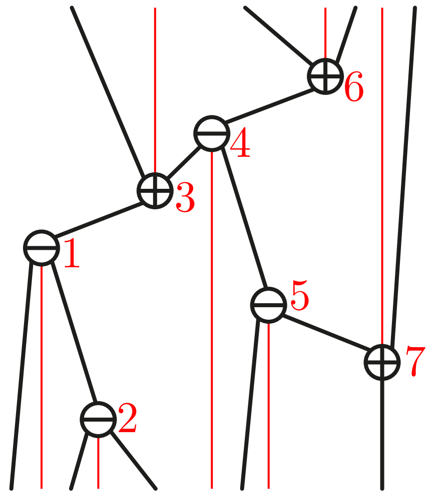
sylvester congruence	multiplization	Cambrianization	tuplization
$\cdots ac \cdots b \cdots$	$\cdots ac \cdots b_1 \cdots b_k \cdots$	left if $b \in \{\ovee, \otimes\}$	intersection
$\equiv \cdots ca \cdots b \cdots$	$\equiv \cdots ca \cdots b_1 \cdots b_k \cdots$	right if $b \in \{\owedge, \otimes\}$	of congruences
if $a < b < c$	if $a < b_i < c$		



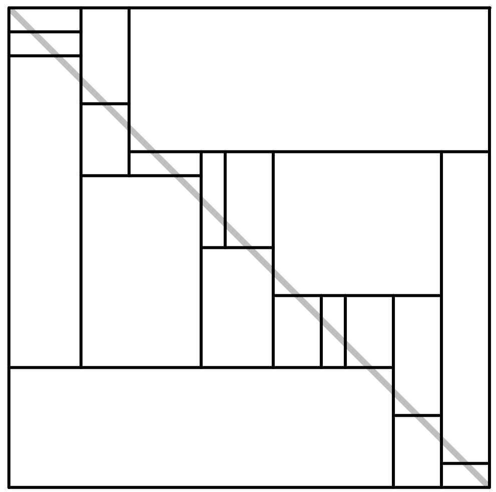
binary tree



multitriangulation



permutree



diagonal rectangulations

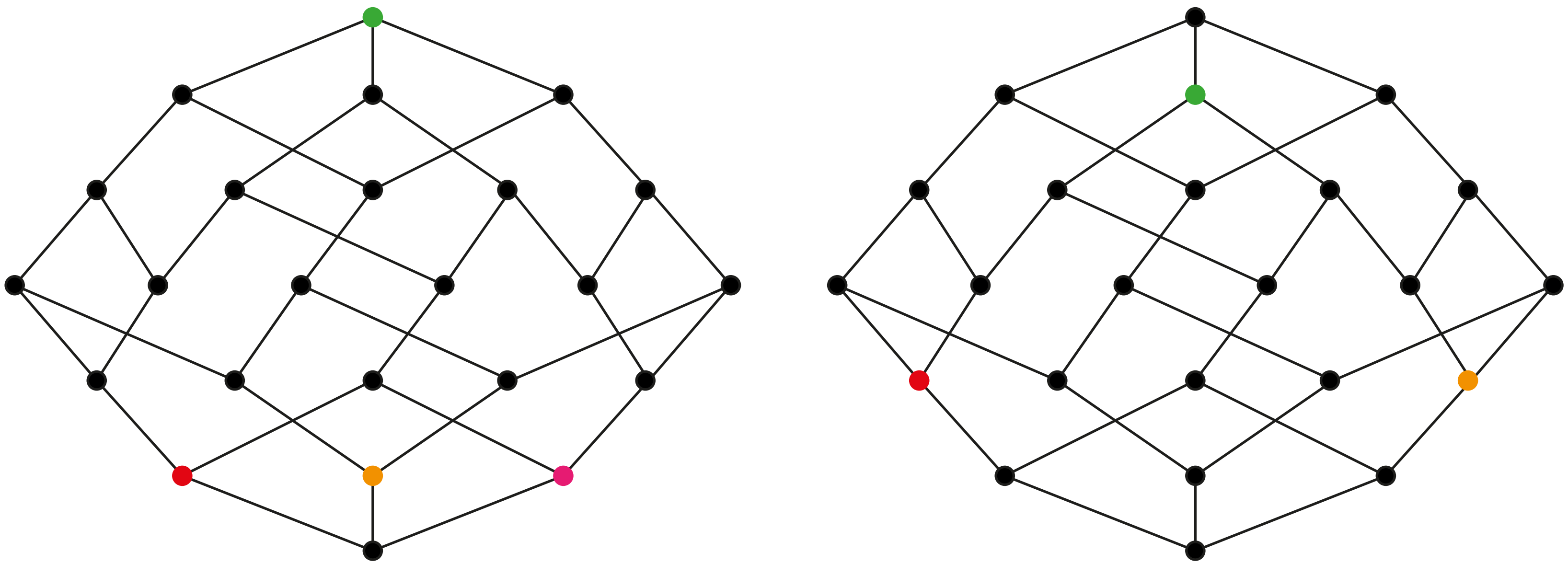
CANONICAL JOIN REPRESENTATIONS

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

CANONICAL JOIN REPRESENTATIONS

lattice $=$ poset $(L, \leq)$ with a meet $\wedge$ and a join $\vee$

join representation of $x \in L =$ subset $J \subseteq L$ such that $x = \bigvee J$.
$x = \bigvee J$ irredundant if $\nexists\, J' \subsetneq J$ with $x = \bigvee J'$
JR are ordered by containement of order ideals: $J \leq J' \iff \forall\, y \in J,\ \exists\, y' \in J',\ y \leq y'$
canonical join representation of $x =$ minimal irred. join representation of x (if it exists)



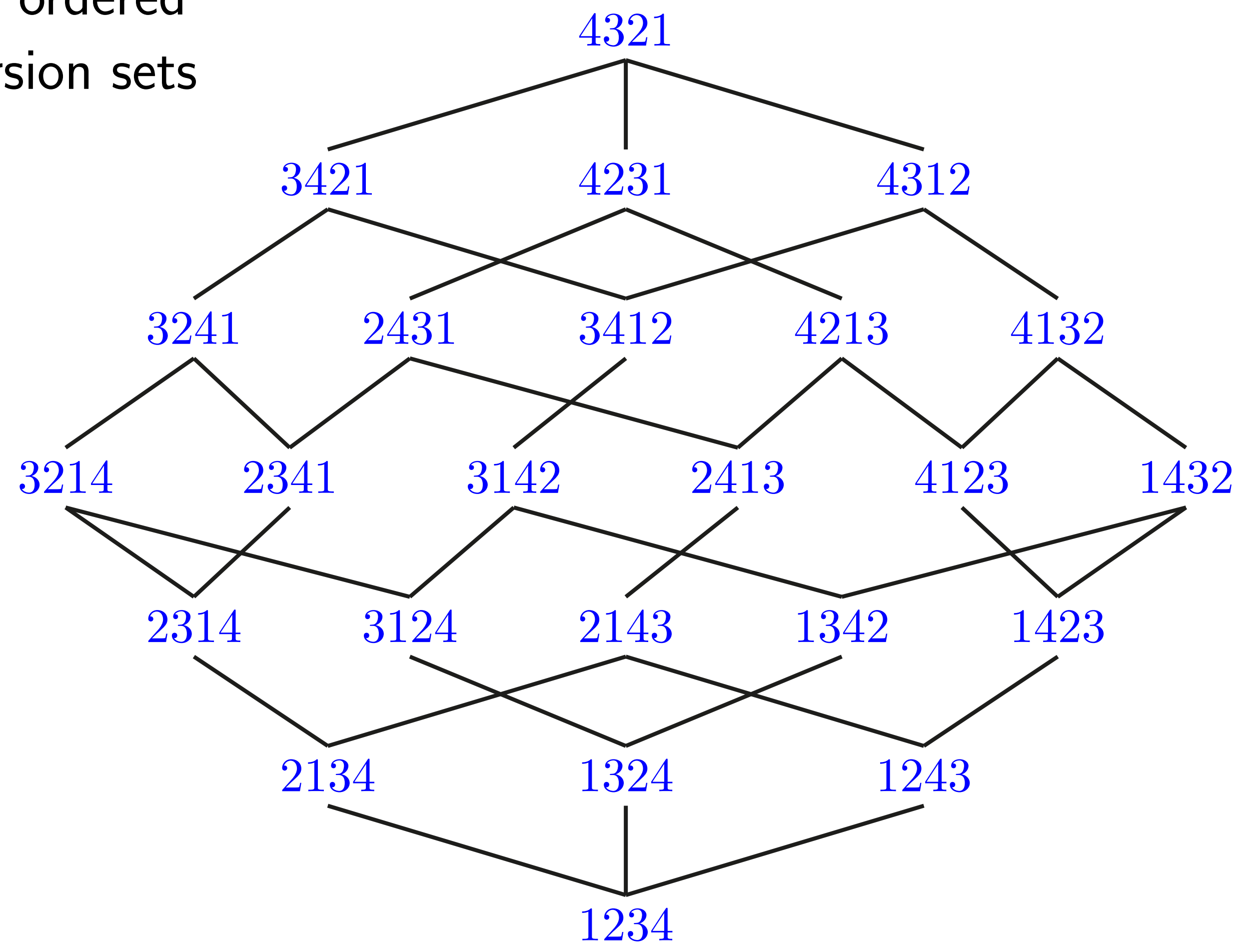
$\Longrightarrow$ "lowest way to write x as a join"

CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

σ permutation

inversions of σ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered by inclusion of inversion sets



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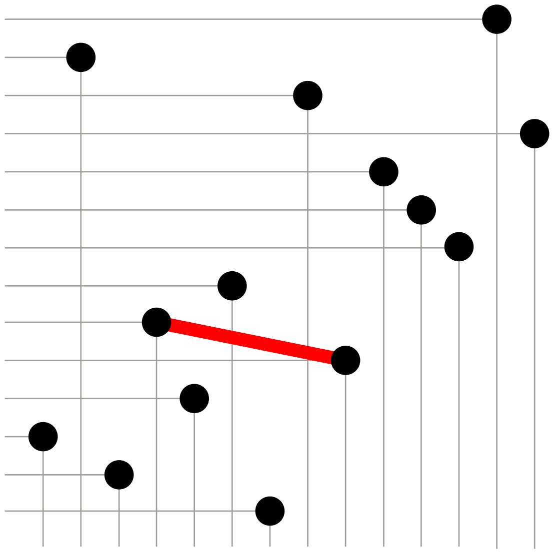
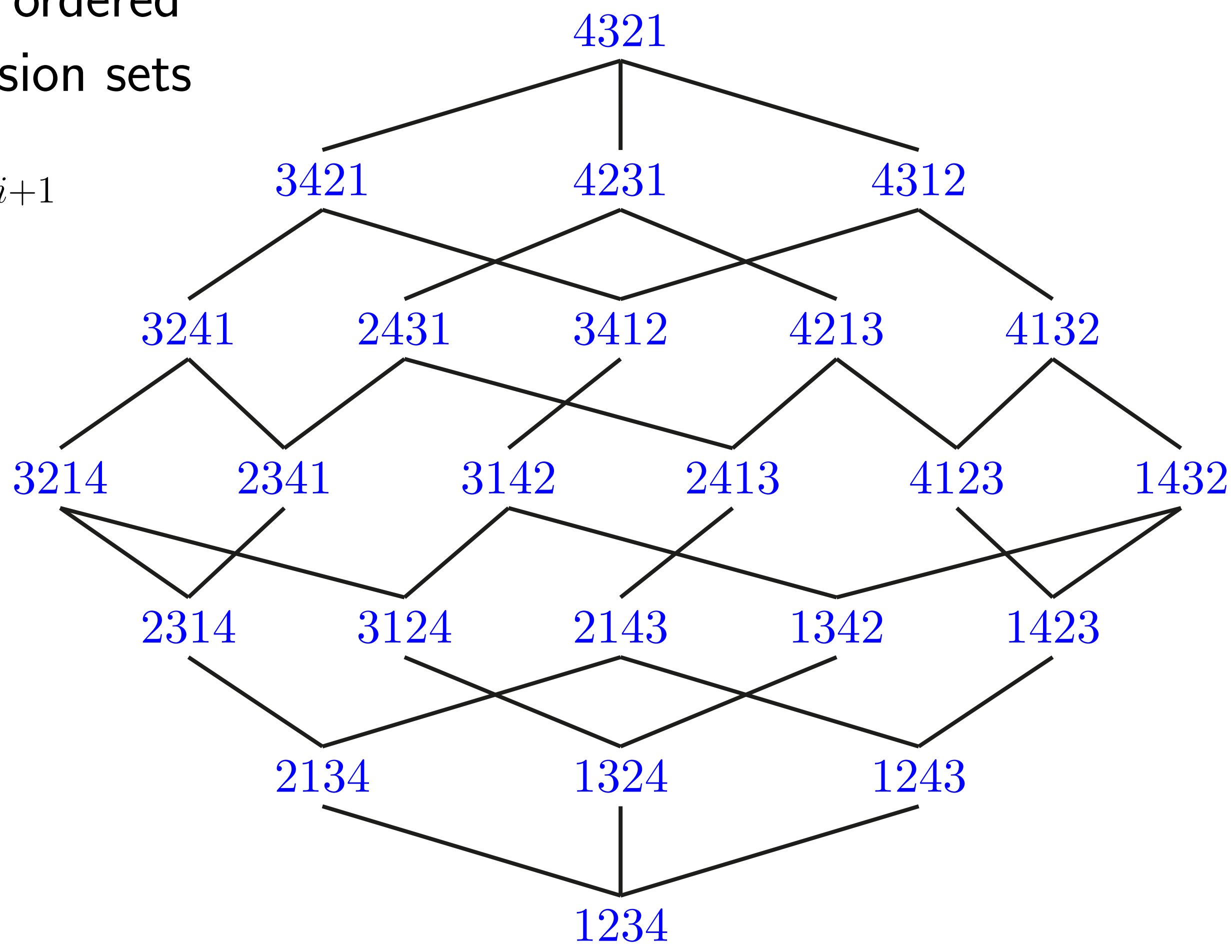
descent of σ = i such that $\sigma_i > \sigma_{i+1}$

$\sigma =$





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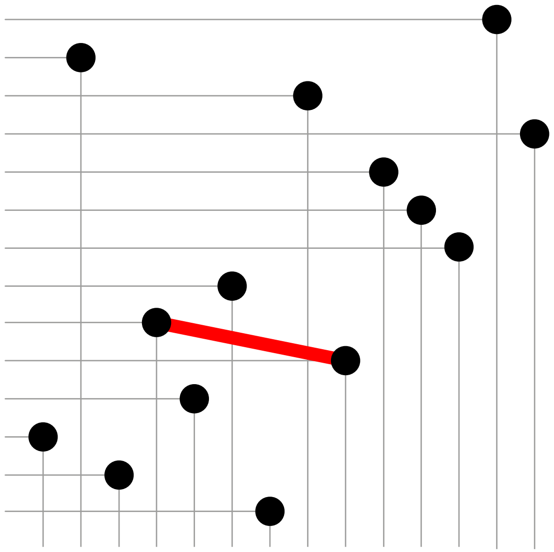
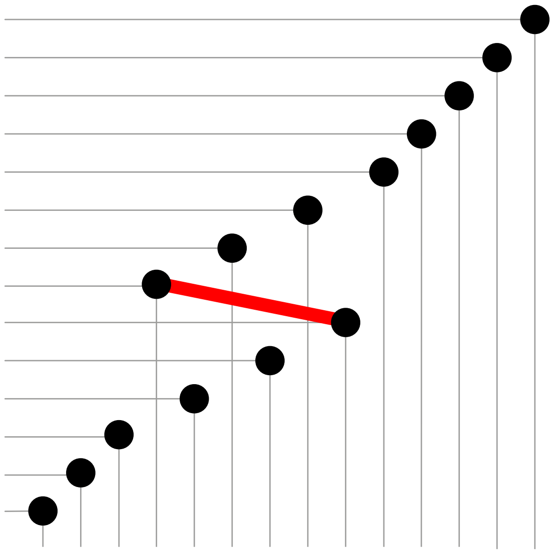
join-irreducible $\lambda(\sigma, i)$

$\sigma =$  $\longrightarrow$ $\lambda(\sigma, i) =$ 

THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$.

Reading, *Noncrossing arc diagrams and canonical join representations* ('15)

CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

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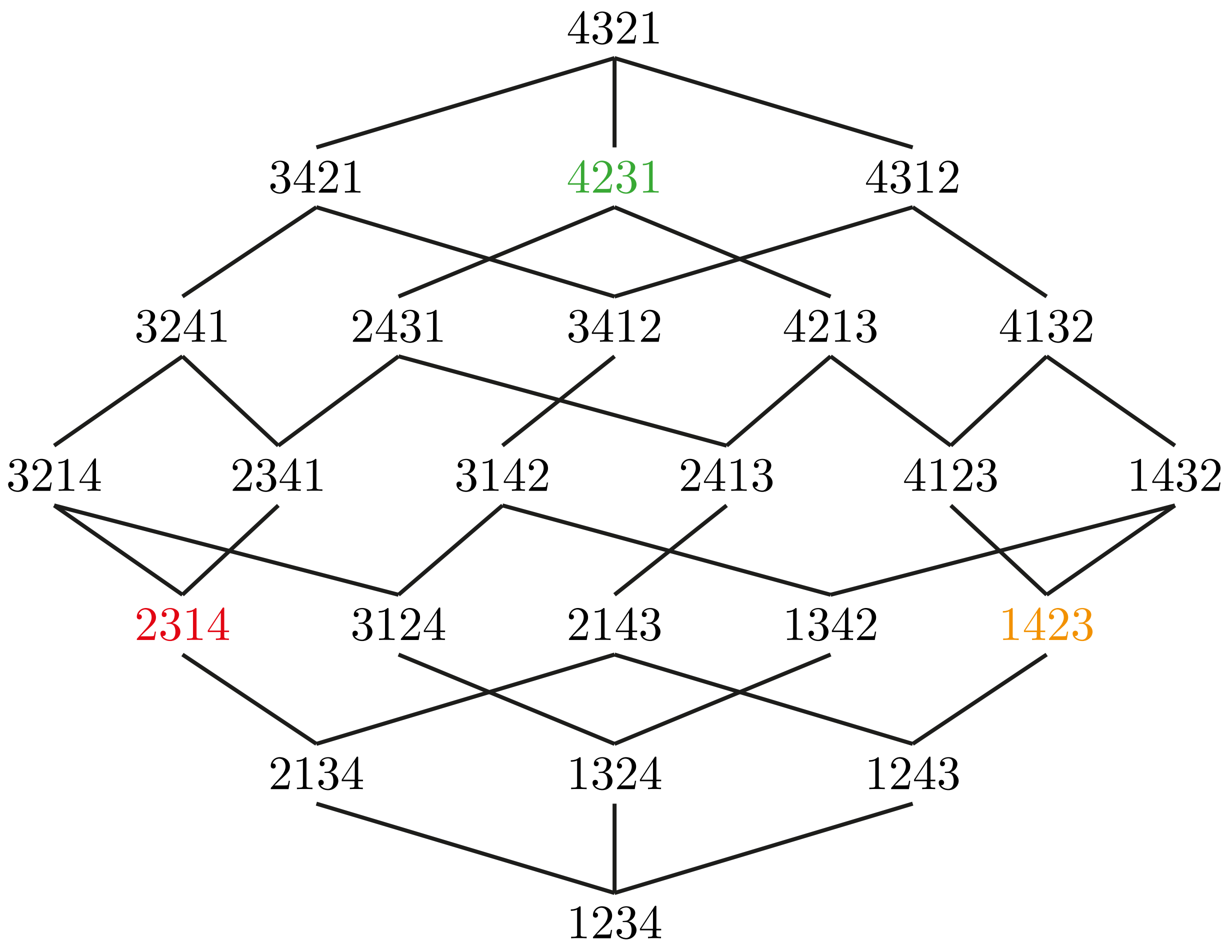
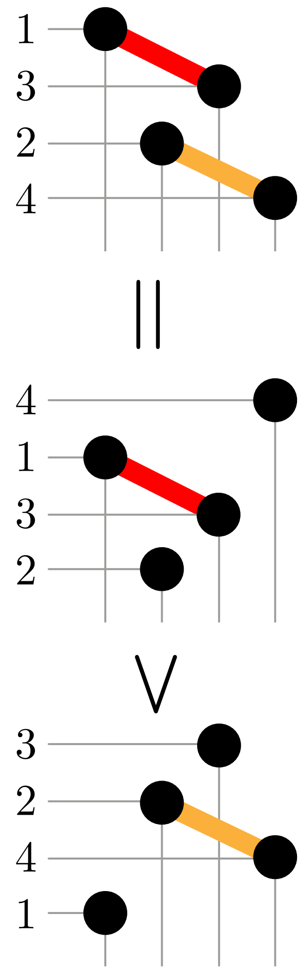
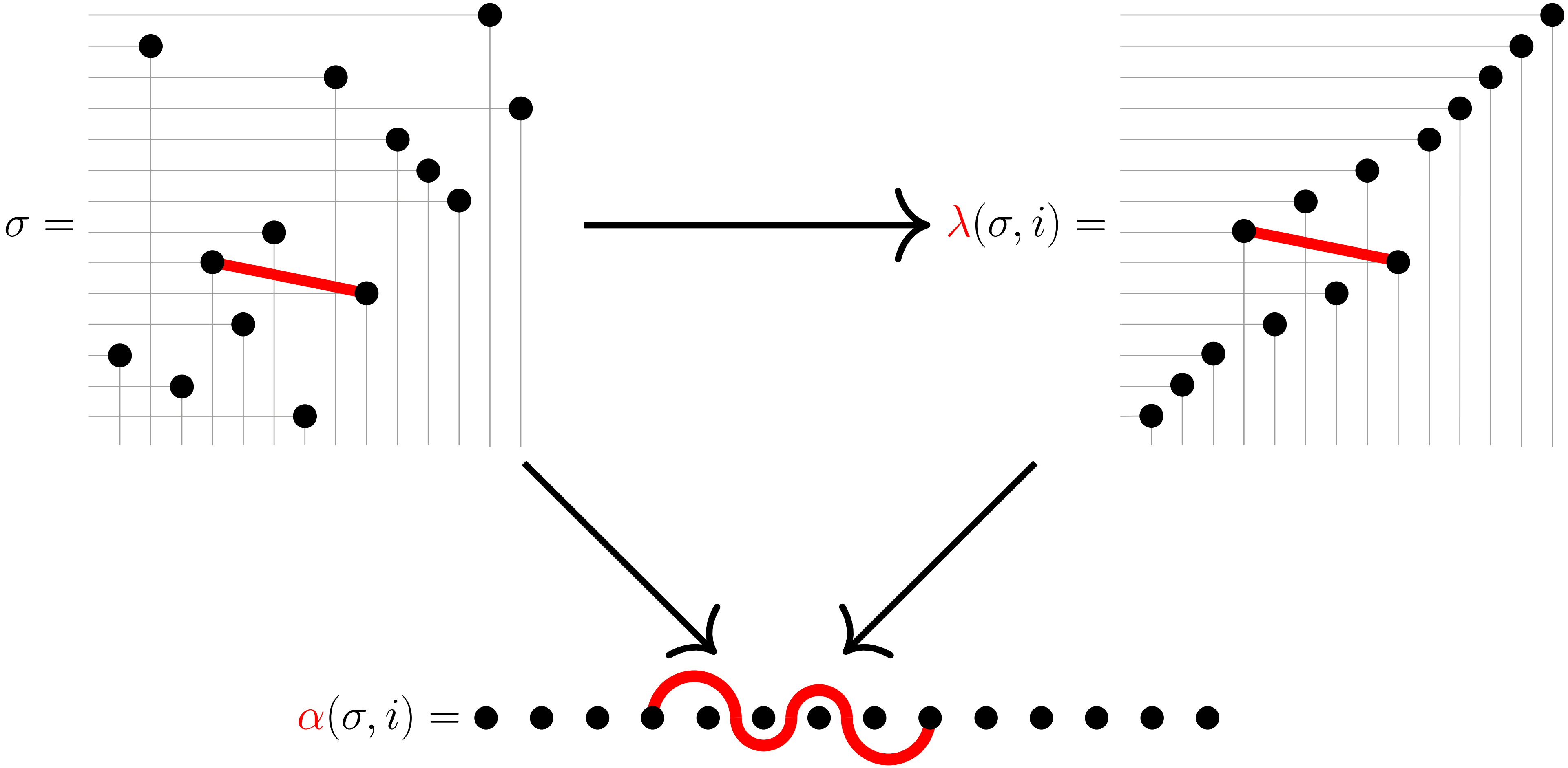




ARCS



arc $= (a, b, n, S)$ with $1 \leq a < b \leq n$ and $S \subseteq \,]a, b[$

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

$\sigma = 2537146$

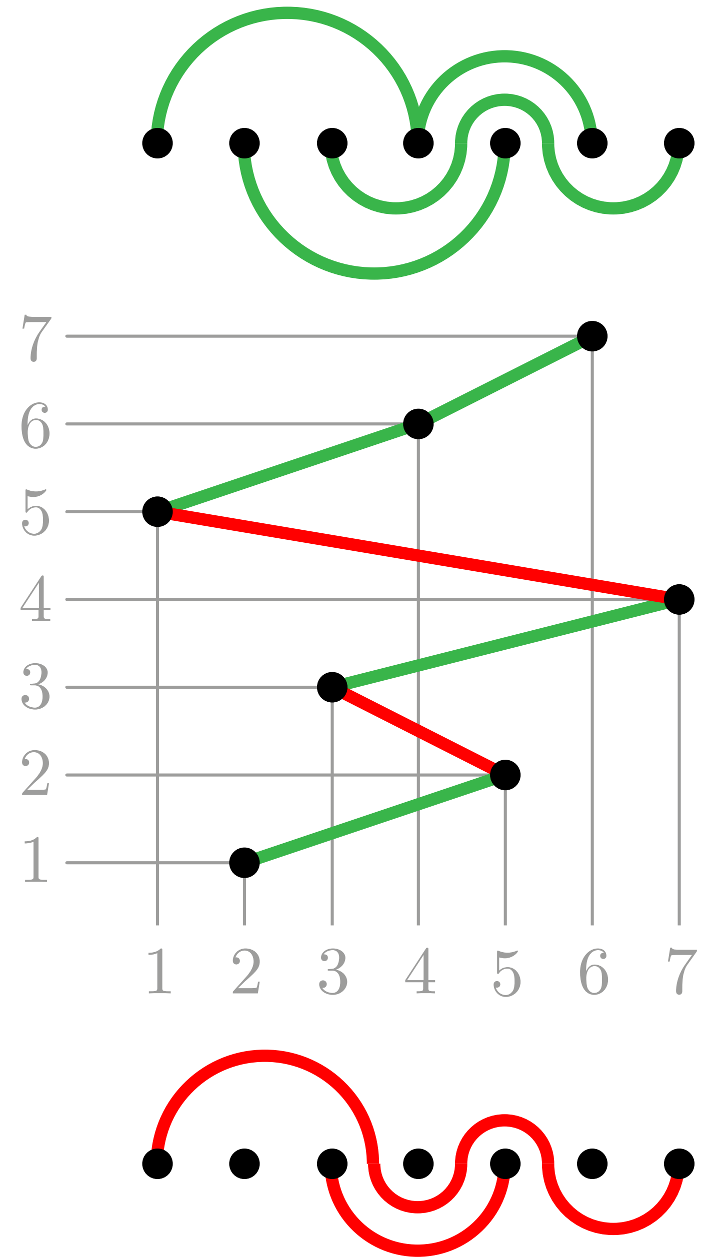
draw the table of points (σ_i, i)
draw all arcs (σ_i, i) — $(\sigma_{i+1}, i+1)$ with
descents in red and ascent in green

project down the red arcs and up the green arcs
allowing arcs to bend but not to cross or pass points

$\delta(\sigma)$ = projected red arcs
$\delta(\sigma)$ = projected green arcs

noncrossing arc diagrams = set $\mathcal{D}$ of arcs st. $\forall\, \alpha, \beta \in \mathcal{D}$:

- left$(\alpha) \neq$ left(β) and right$(\alpha) \neq$ right(β),
- α and β are not crossing.

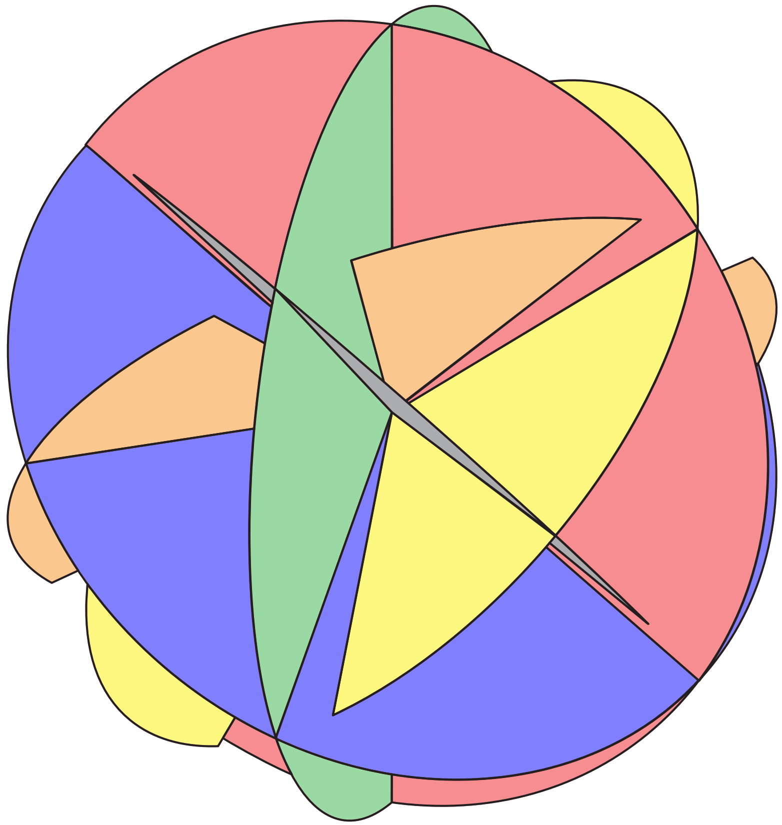


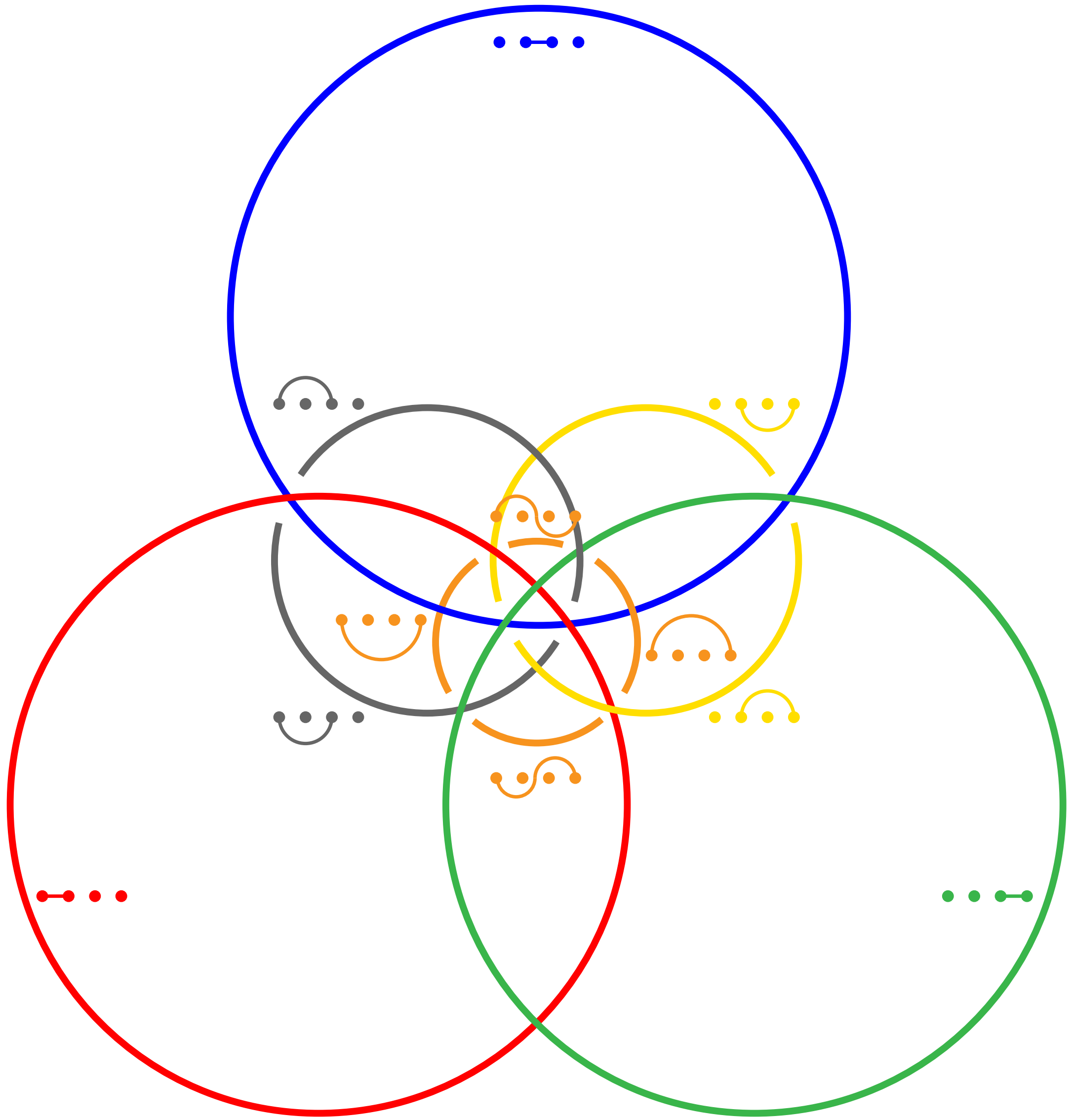
THM. $\sigma \to \delta(\sigma)$ and $\sigma \to \delta(\sigma)$ are bijections from permutations to noncrossing arc diagrams.

Reading, *Noncrossing arc diagrams and can. join representations* ('15)

SHARDS

shard $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{array} \right. \right\}$





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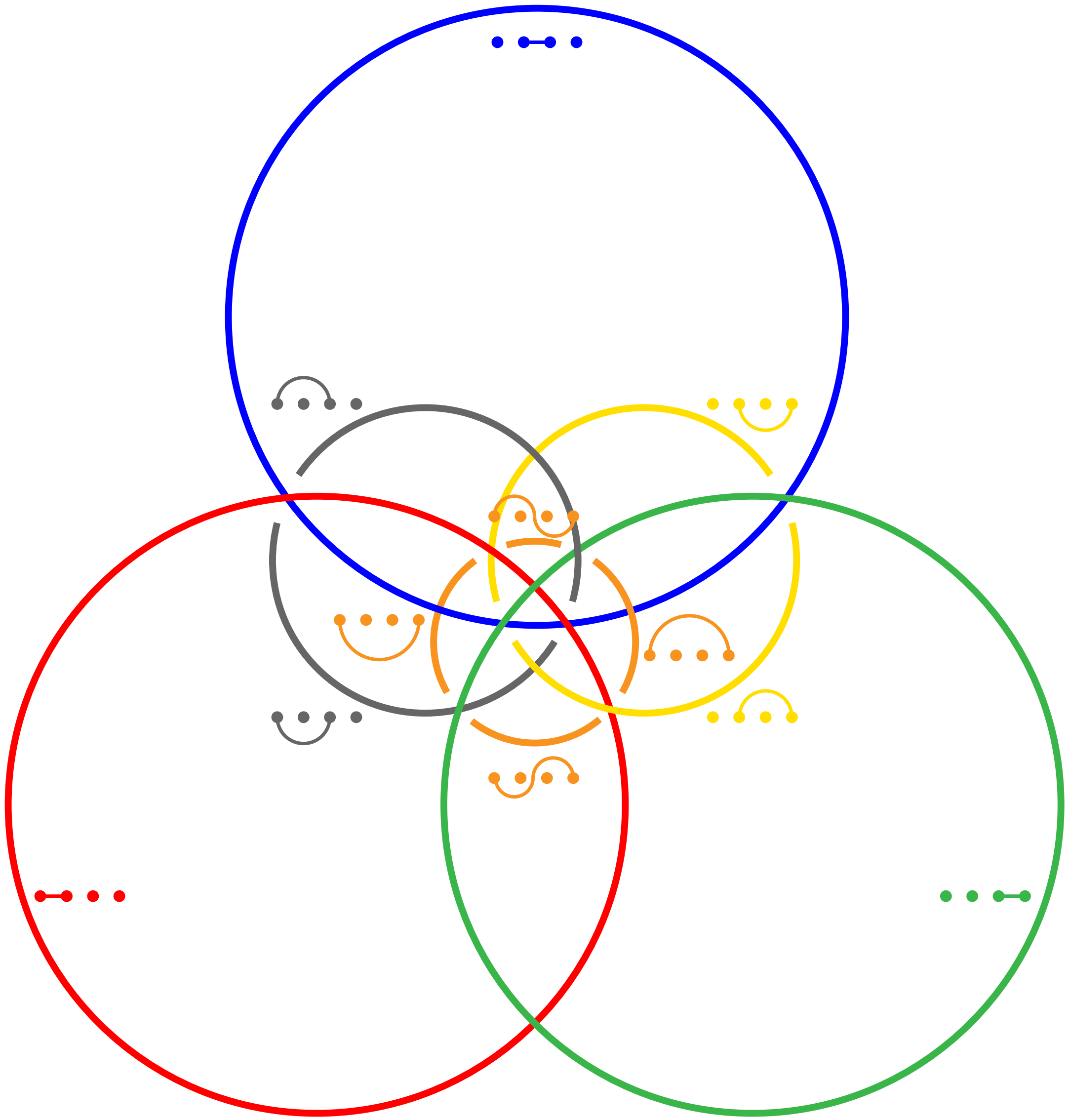


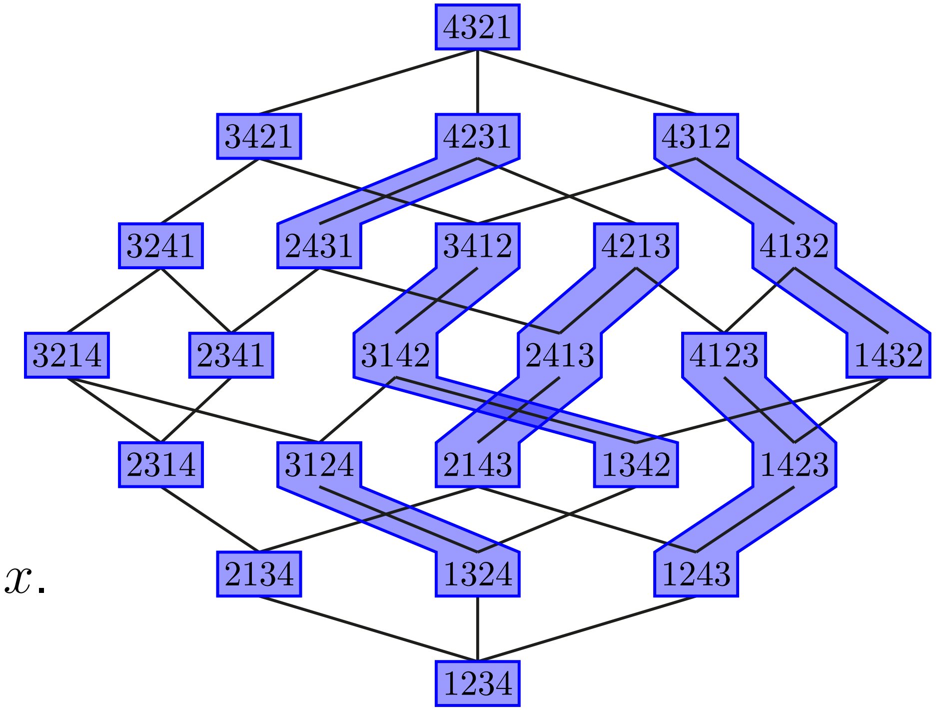
REM. The shards $\Sigma(i,j,n,S)$ for all subsets $S \subseteq \,]i,j[$ decompose the hyperplane $x_i = x_j$ into 2^{j-i-1} pieces.

REM. A chamber of the Coxeter fan is characterized by the shards below it.

LATTICE QUOTIENTS AND CANONICAL JOIN REPRESENTATIONS

$\equiv$ lattice congruence on L, then

- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/\equiv$ is isomorphic to $\pi_\downarrow(L)$ (as poset)
- canonical join representations in $L/\equiv$ are canonical join representations in L that only involve join irreducibles x with $\pi_\downarrow(x) = x$.

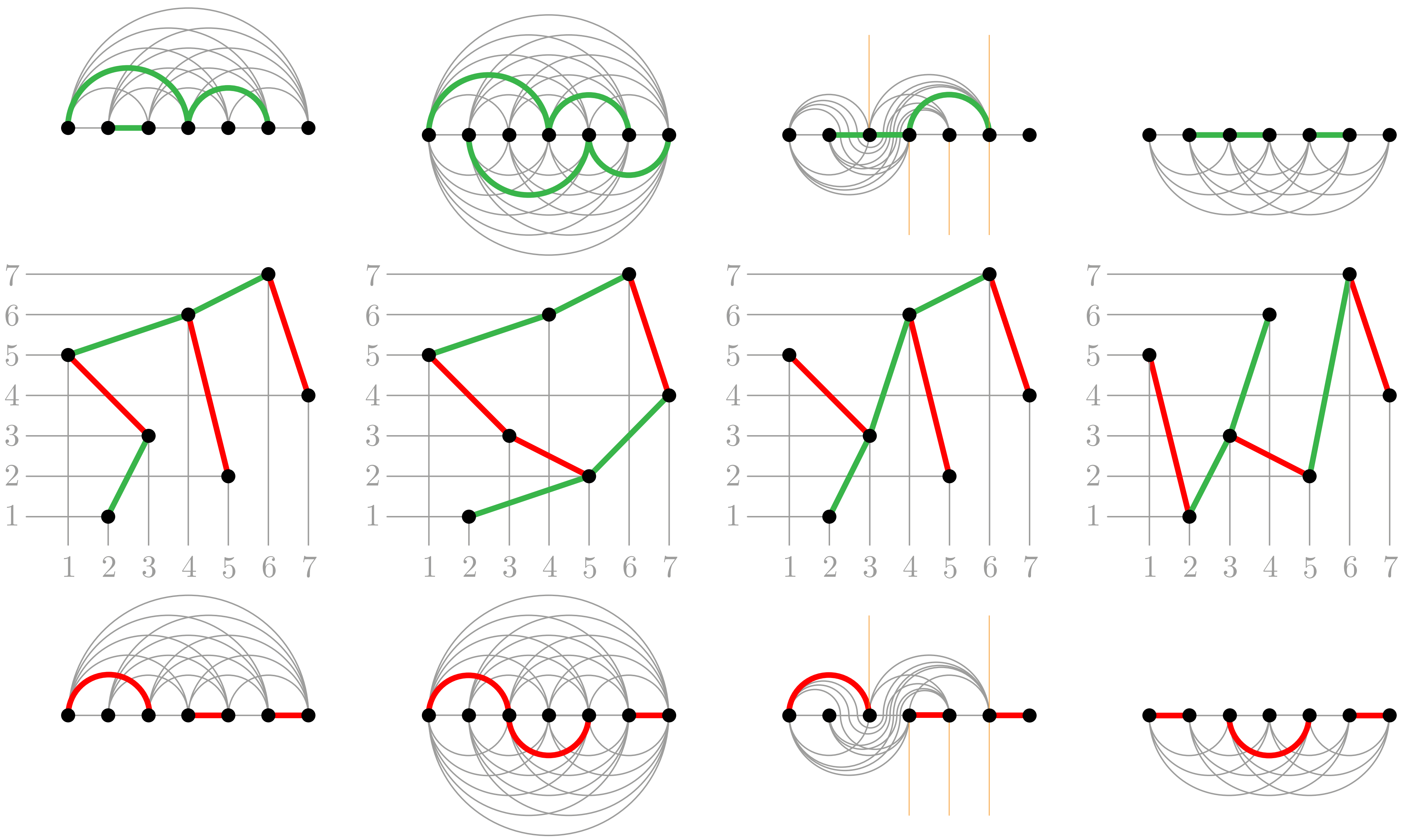


THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
Let $\mathcal{I}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$
Then

- $\pi_\downarrow(\sigma) = \sigma \iff \delta(\sigma) \subseteq \mathcal{I}_\equiv$.
- the map $\mathfrak{S}_n/\equiv \longrightarrow$ {nc arc diagrams in $\mathcal{I}_\equiv$} is a bijection.
  $X \longmapsto \delta(\pi_\downarrow(X))$

Reading, *Noncrossing arc diagrams and can. join representations* ('15)

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN



binary trees | diagonal quadrangulations | permutrees | k-sashes

FORCING AND ARC IDEALS

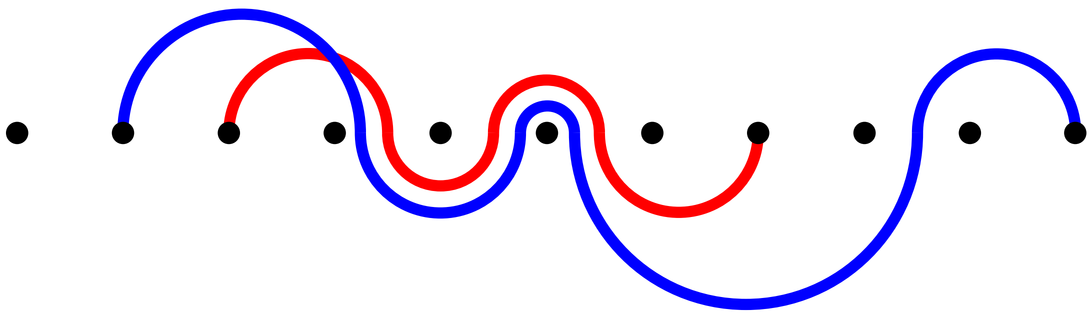
THM. $\mathcal{I}_{\equiv}$ = arcs corresponding to join irreducibles σ with $\pi_{\downarrow}(\sigma) = \sigma$.
Bijection $\mathfrak{S}_n/\equiv \;\longleftrightarrow\;$ {nc arc diagrams in $\mathcal{I}_{\equiv}$}.

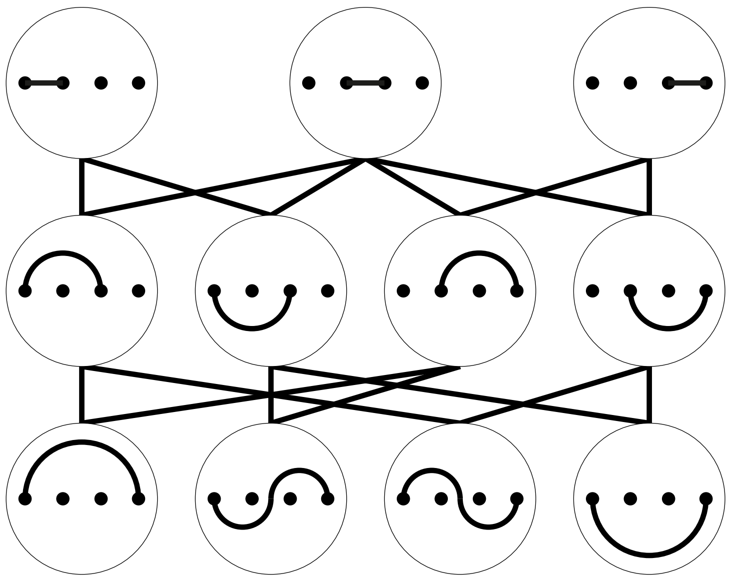
What sets of arcs can be $\mathcal{I}_{\equiv}$?

THM. The following are equivalent for a set of arcs $\mathcal{I}$:

- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{I} = \mathcal{I}_{\equiv}$,
- $\mathcal{I}$ is an upper ideal for the order $(a, d, n, S) \prec (b, c, n, T) \iff a \le b < c \le d$ and $T = S \cap \,]b, c[$.

Reading, *Noncrossing arc diagrams and can. join representations* ('15)

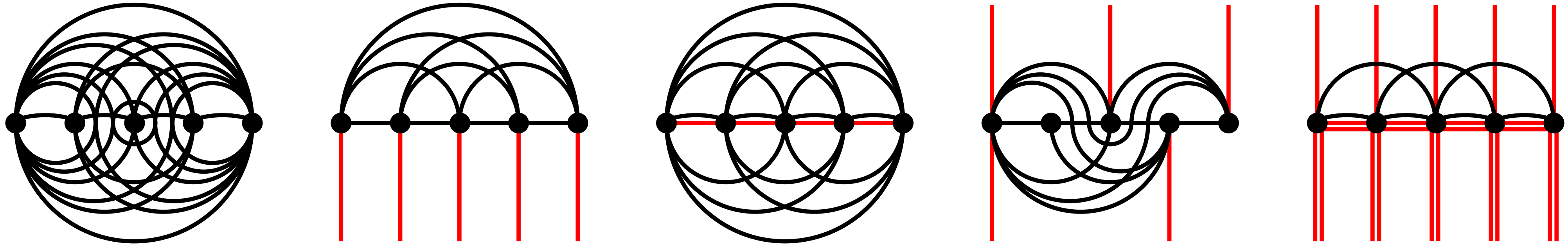




BOUNDED CROSSINGS ARC IDEALS

arc ideal $=$ ideal of the forcing poset on arcs $=$ subsets of arcs closed by forcing

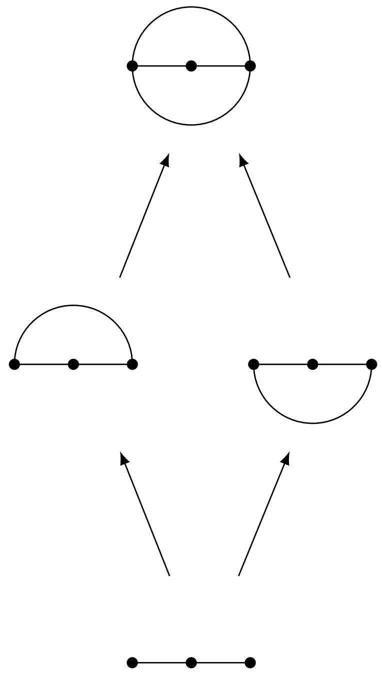
fix $k \geq 0$ and some red walls above, below and in between the points
allow arcs that cross at most k walls



weak order | Tamari lattice | diagonal rectangulations | permutree lattices | k-sashes lattices

ARC IDEALS

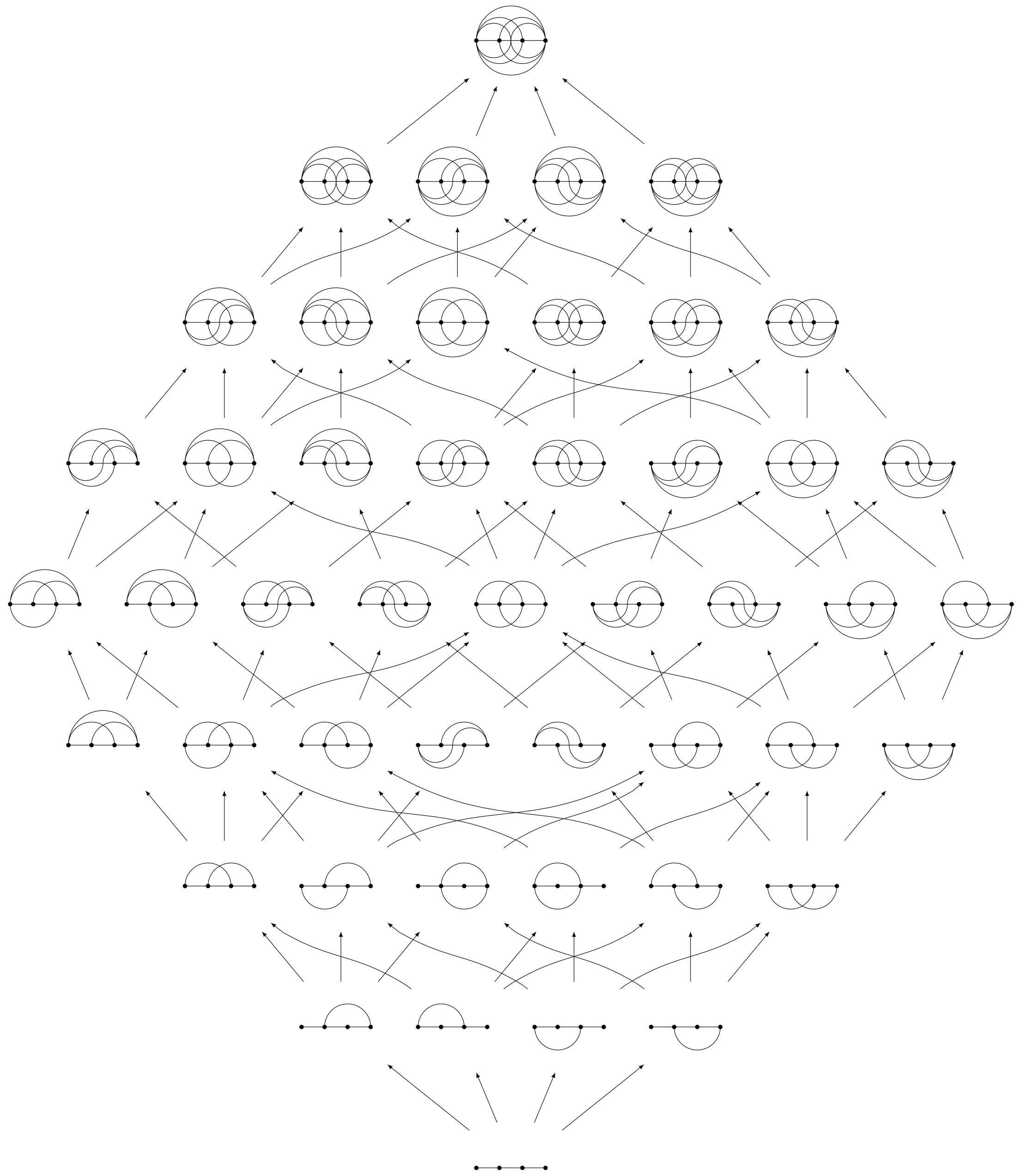
arc ideal = ideal of the forcing poset on arcs = subsets of arcs closed by forcing



1, 1, 4, 47, 3322, ...

1, 2, 7, 60, 3444, ...

OEIS A091687



QUOTIENTOPES

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Pilaud-Santos, *Quotientopes* ('17$^+$)

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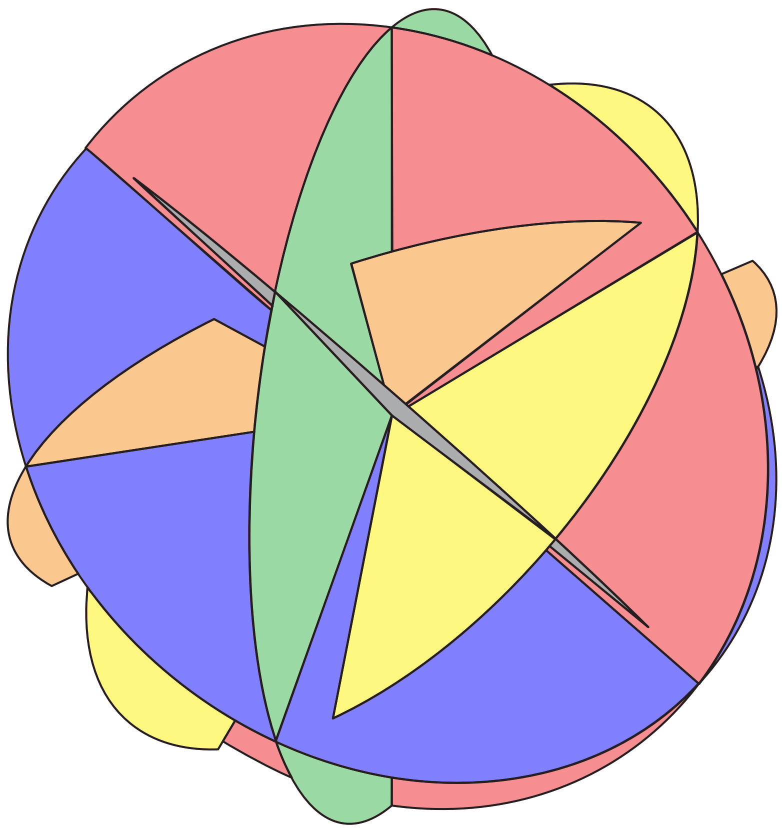
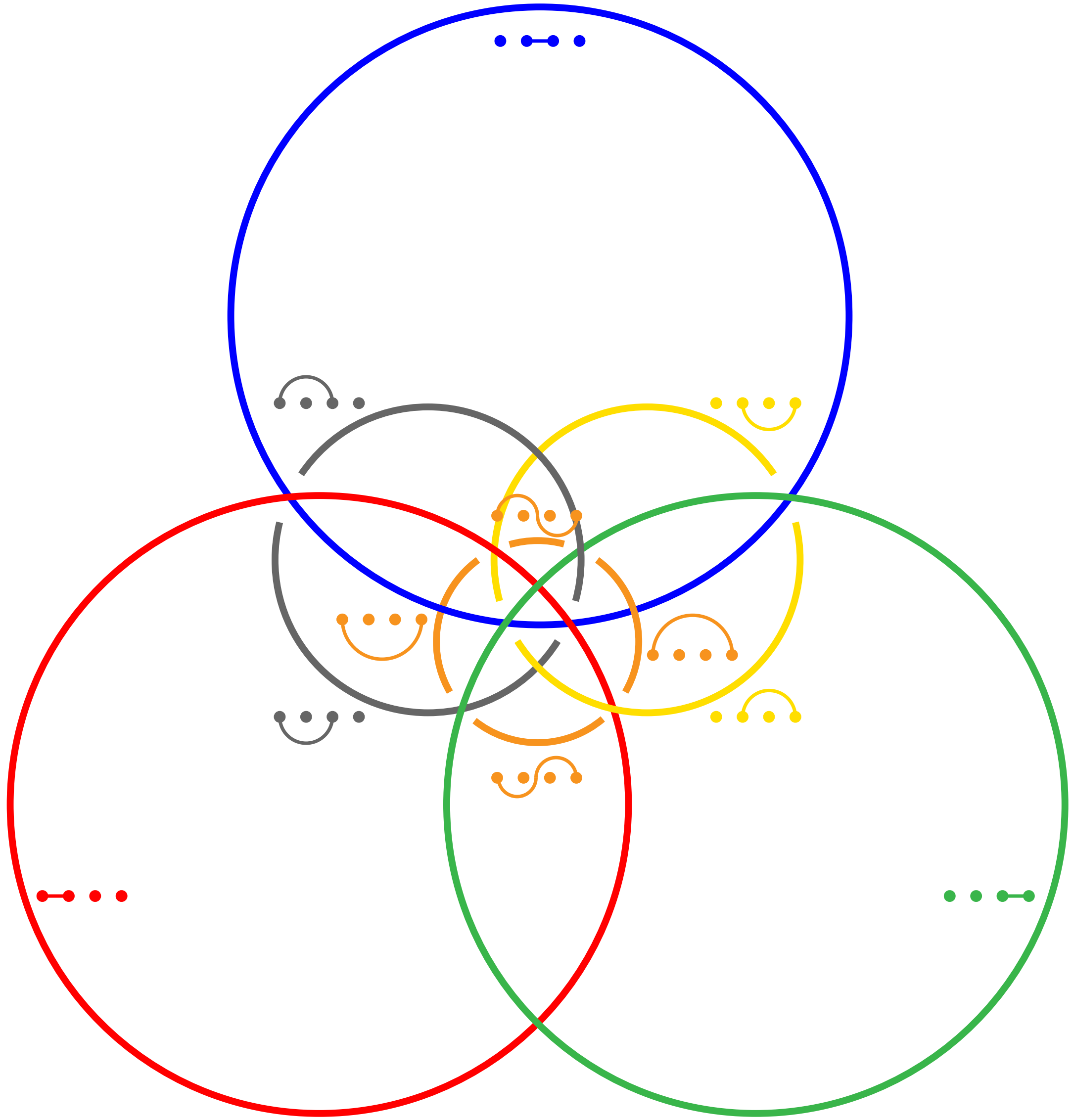
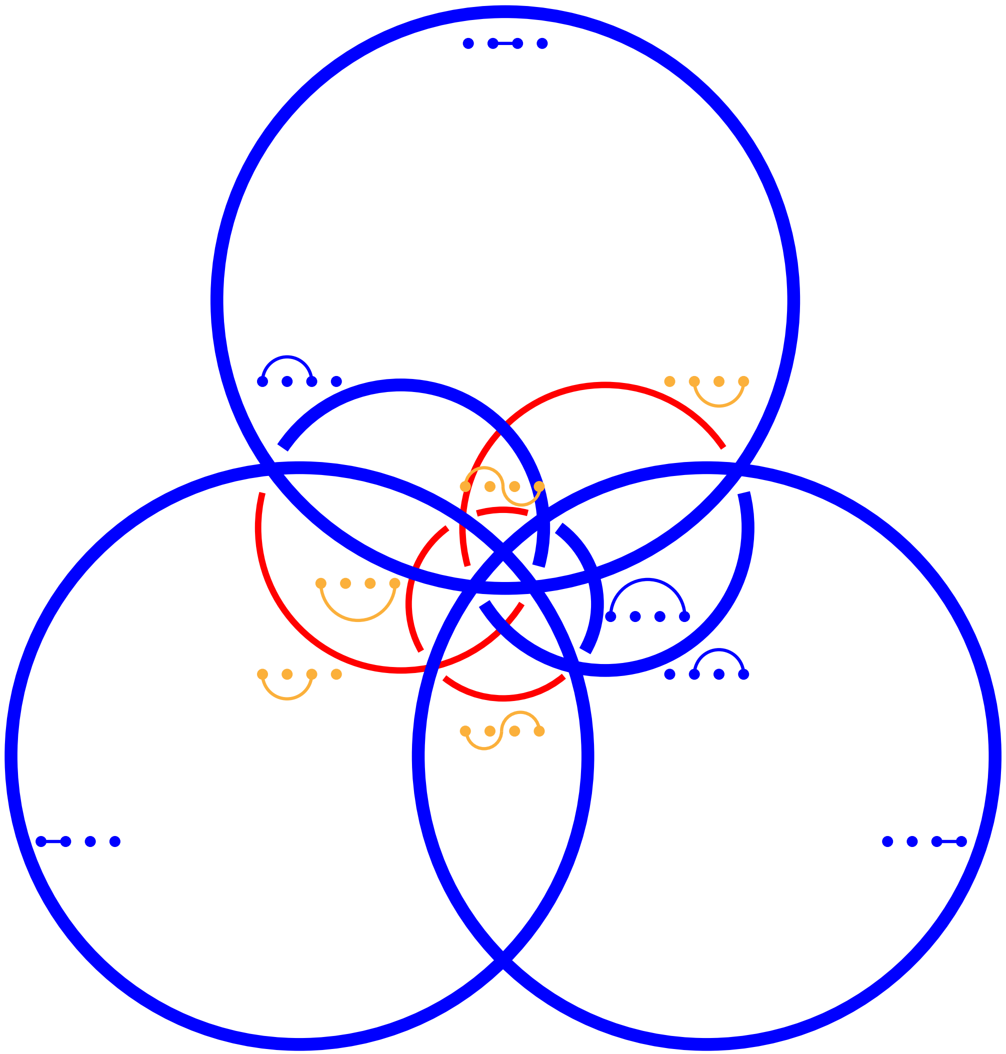
shard $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{array}\right. \right\}$





SHARDS AND QUOTIENT FAN

shard $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{array} \right. \right\}$



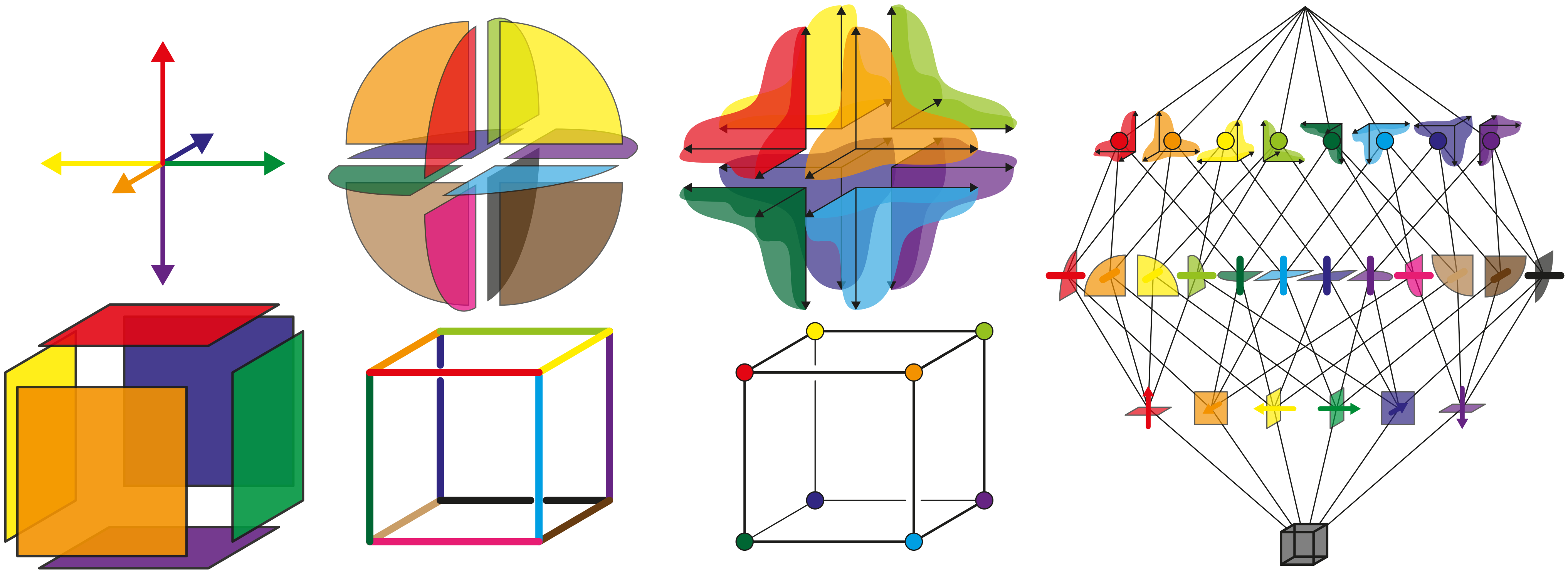
THM. For a lattice congruence $\equiv$ on $\mathfrak{S}_n$, the cones obtained by glueing the Coxeter regions of the permutations in the same congruence class of $\equiv$ form a fan $\mathcal{F}_\equiv$ of $\mathbb{R}^n$ whose dual graph realizes the lattice quotient $\mathfrak{S}_n/\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

THM. Each lattice congruence $\equiv$ on $\mathfrak{S}_n$ corresponds to a set of shards $\boldsymbol{\Sigma}_\equiv$ such that the cones of $\mathcal{F}_\equiv$ are the connected components of the complement of the union of the shards in $\boldsymbol{\Sigma}_\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = $\{$ normal cone of F $\mid$ F face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

QUOTIENTOPE

fix a forcing dominant function $f : \sigma \to \mathbb{R}_{>\mathbf{0}}$ ie. st. $f(\Sigma) > \sum_{\Sigma' \succ \Sigma} f(\Sigma')$ for any shard Σ.

for a shard $\Sigma = (i, j, n, S)$ and a subset $\varnothing \neq R \subsetneq [n]$ define the contribution

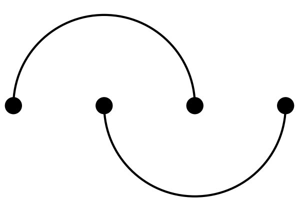
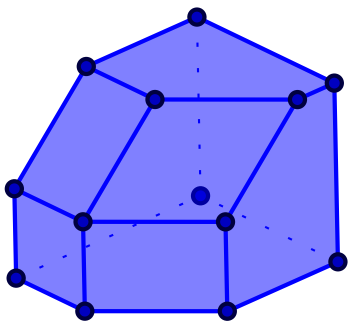
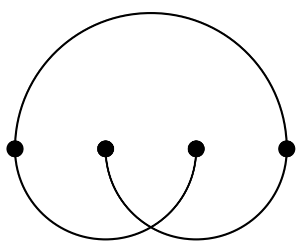
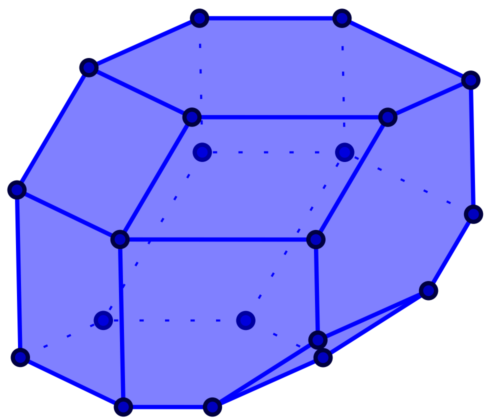
$$\gamma(\Sigma, R) := \begin{cases} 1 & \text{if } |R \cap \{i,j\}| = 1 \text{ and } S = R \cap \,]i,j[, \\ 0 & \text{otherwise} \end{cases}$$

define height function h for $\varnothing \neq R \subsetneq [n]$ by $h^f_{\equiv}(R) := \sum_{\Sigma \in \mathbf{\Sigma}_{\equiv}} f(\Sigma)\, \gamma(\Sigma, R)$.

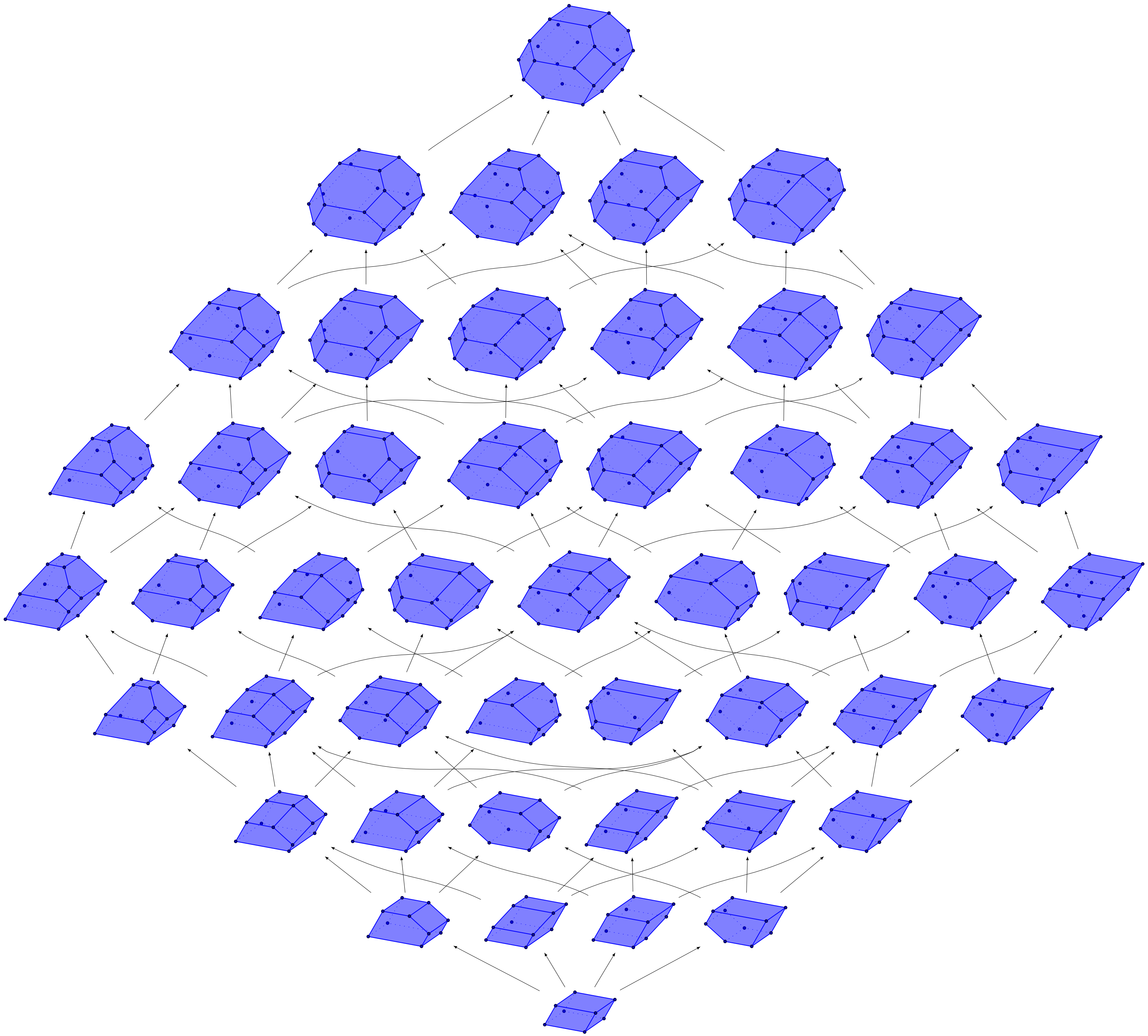
THM. For a lattice congruence $\equiv$ on $\mathfrak{S}_n$ and a forcing dominant function $f : \mathbf{\Sigma} \to \mathbb{R}_{>\mathbf{0}}$, the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of the polytope

$$P^f_{\equiv} := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \langle\, \mathbf{r}(R) \mid \mathbf{x} \,\rangle \leq h^f_{\equiv}(R) \text{ for all } \varnothing \neq R \subsetneq [n] \right\}.$$

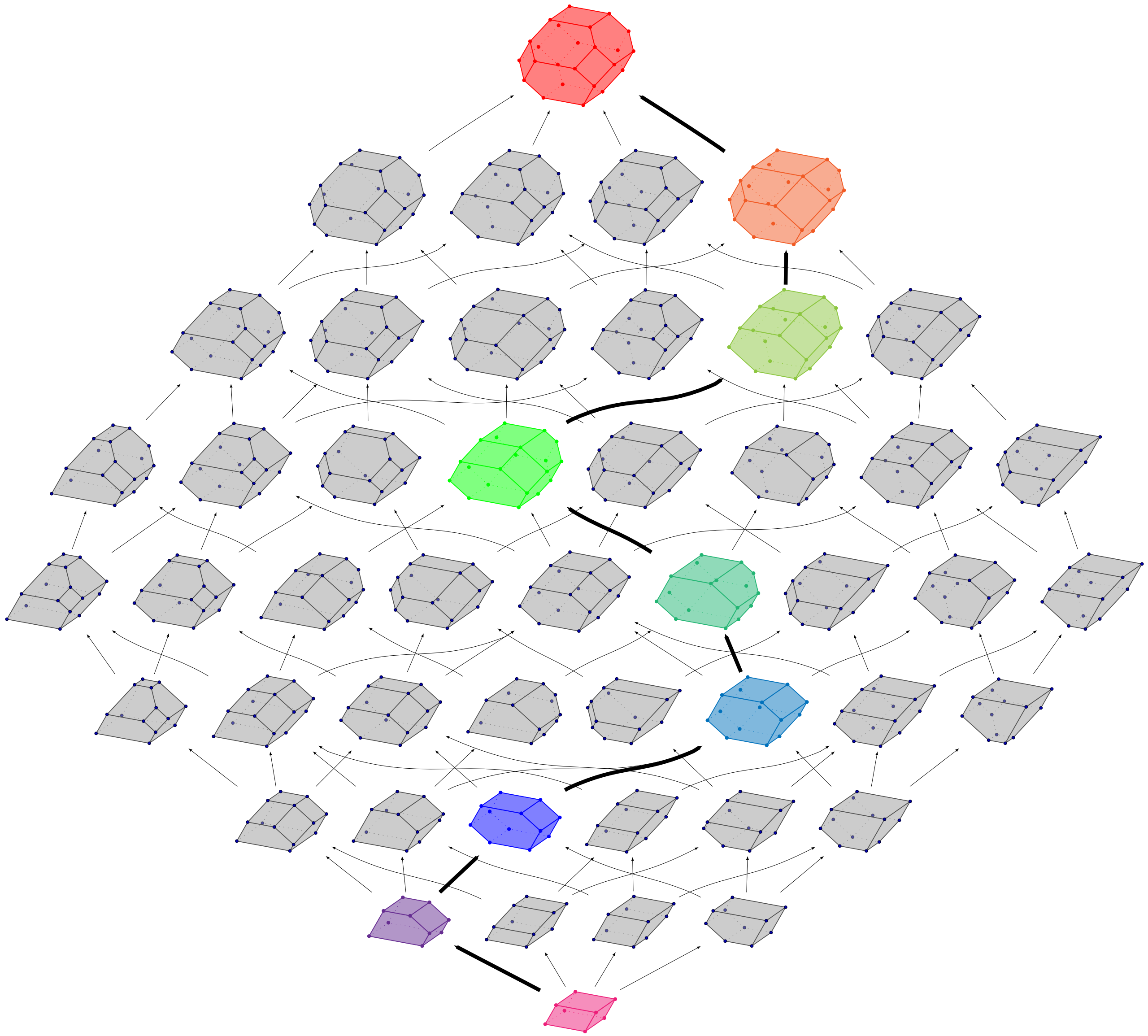
P.-Santos, *Quotientopes* ('17+)

QUOTIENTOPE LATTICE



QUOTIENTOPE LATTICE



POLYWOOD

INSIDAHEDRA / OUTSIDAHEDRA

outsidahedra	insidahedra
permutrees	quotientopes

POLYWOOD

HYPERPLANE ARRANGEMENTS

Björner–Edelman–Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)
Reading, *Lattice theory of the poset of regions* ('16)
P.–Ritter, *Quotientopes for congruence uniform arrangements* ('18$^+$)

POSET OF REGIONS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
B distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$
inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C
poset of regions $\mathrm{Pos}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

THM. The poset of regions $\mathrm{Pos}(\mathcal{H}, B)$

- is never a lattice when B is not a simple region,
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement.

Björner–Edelman–Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)

LATTICE CONGRUENCES OF THE POSET OF REGIONS

THM. If $\mathrm{Pos}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a lattice congruence of $\mathrm{Pos}(\mathcal{H}, B)$, the cones obtained by glueing together the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?

LATTICE CONGRUENCES OF THE POSET OF REGIONS

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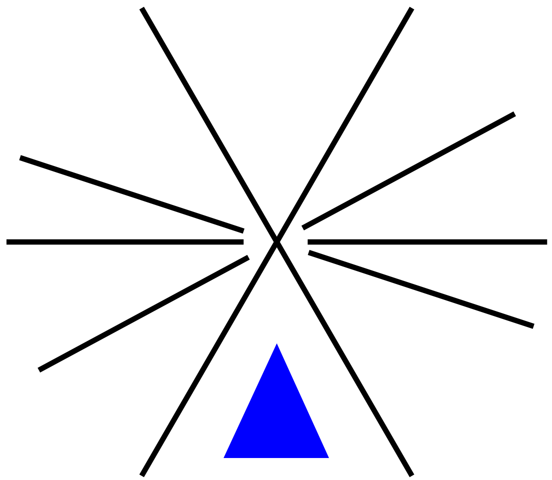
Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?

goal: construct a height function

tool: shards



QUOTIENTOPES FOR CONGRUENCE-UNIFORM ARRANGEMENTS

THM. If $\mathrm{Pos}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a lattice congruence of $\mathrm{Pos}(\mathcal{H}, B)$, the cones obtained by glueing together the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?

THM. In the situation when

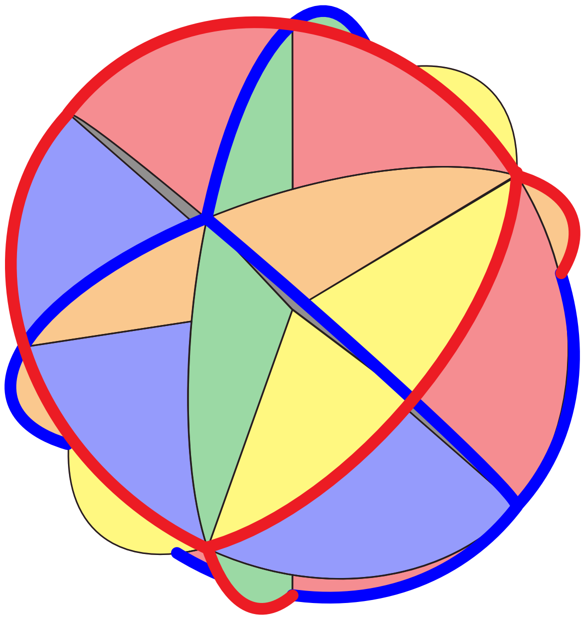
- $\mathcal{H}$ is a simplicial arrangement,
- the poset of regions $\mathrm{Pos}(\mathcal{H}, B)$ is a congruence-uniform lattice,

then the quotient fan is the normal fan of a polytope.

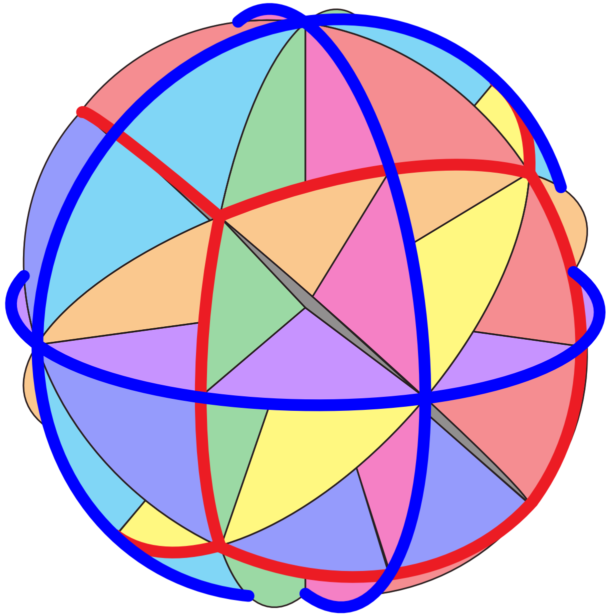
P.–Ritter, *Quotientopes for congruence uniform arrangements* ('18$^+$)

COXETER ARRANGEMENTS

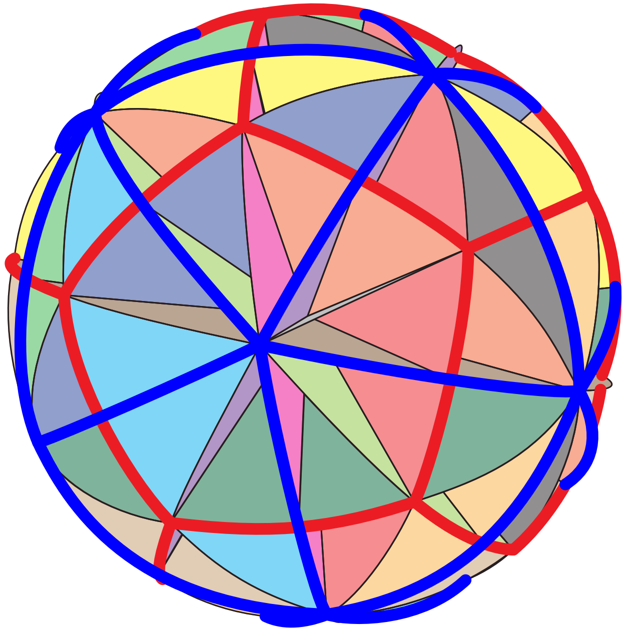
Coxeter group = group generated by reflections



Type A



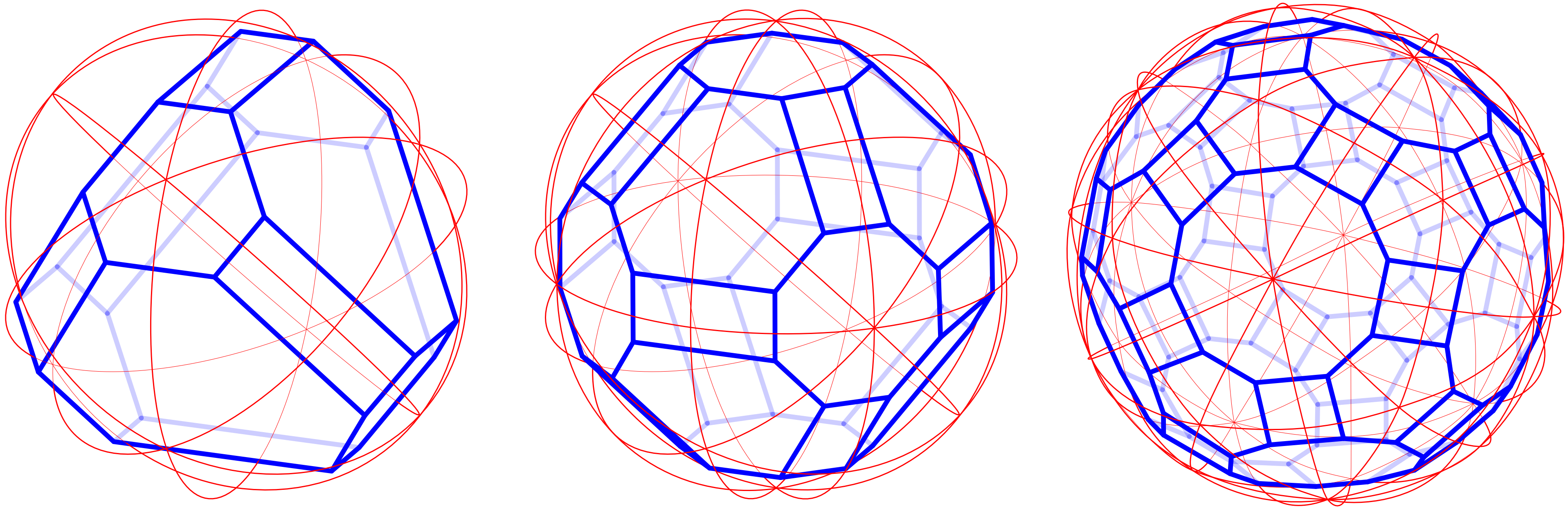
Type B



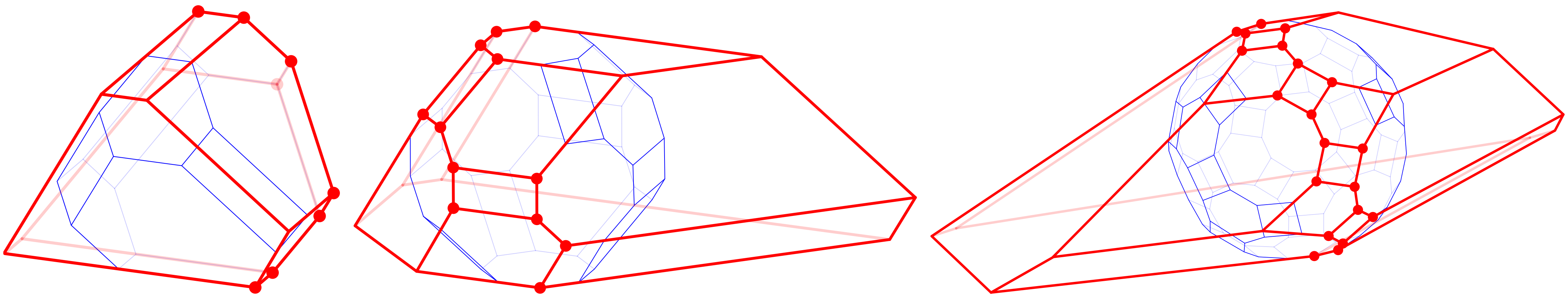
Type H

COXETER ARRANGEMENTS

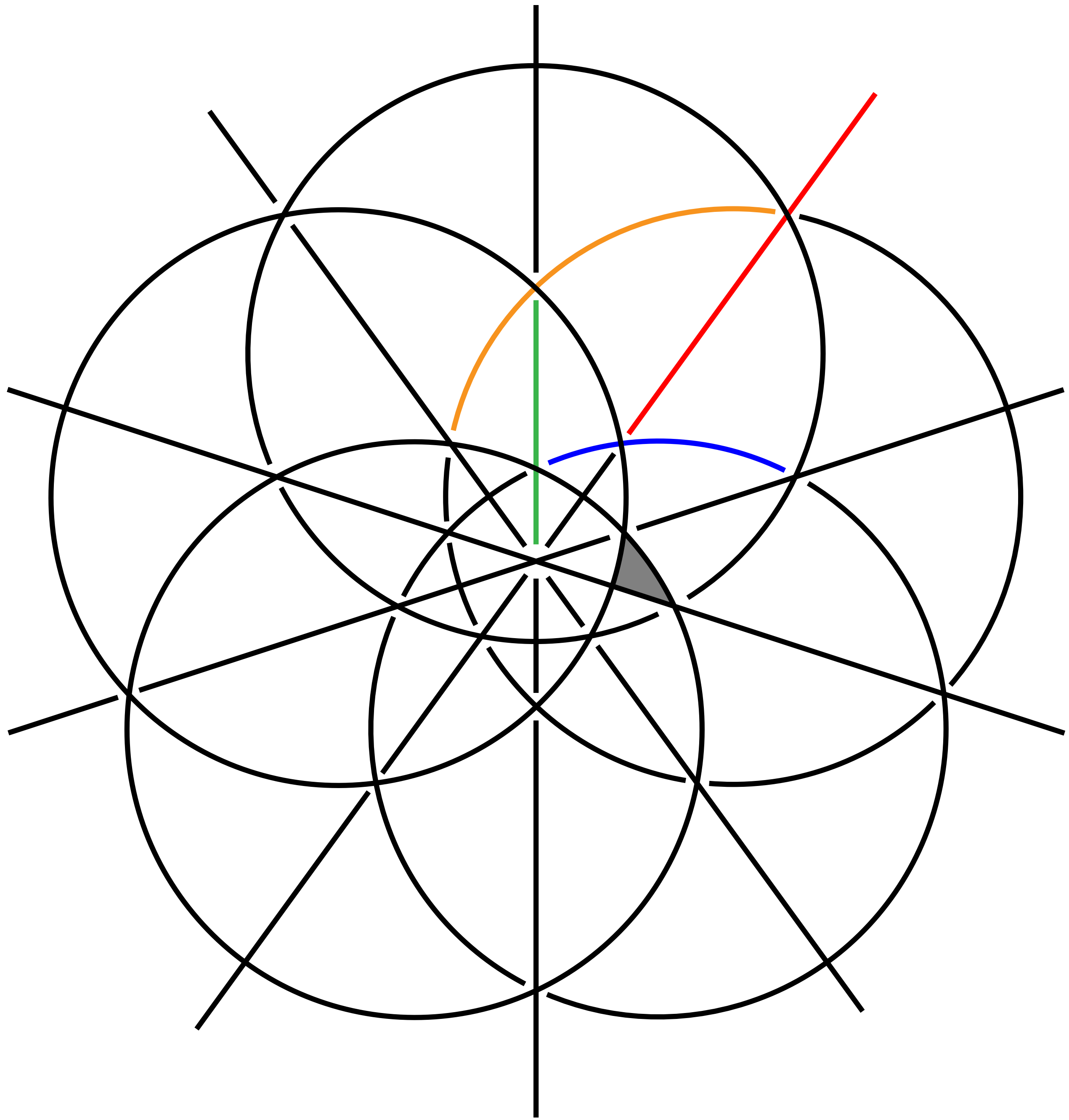
weak order = orientation of the graph of the Coxeter permutahedron



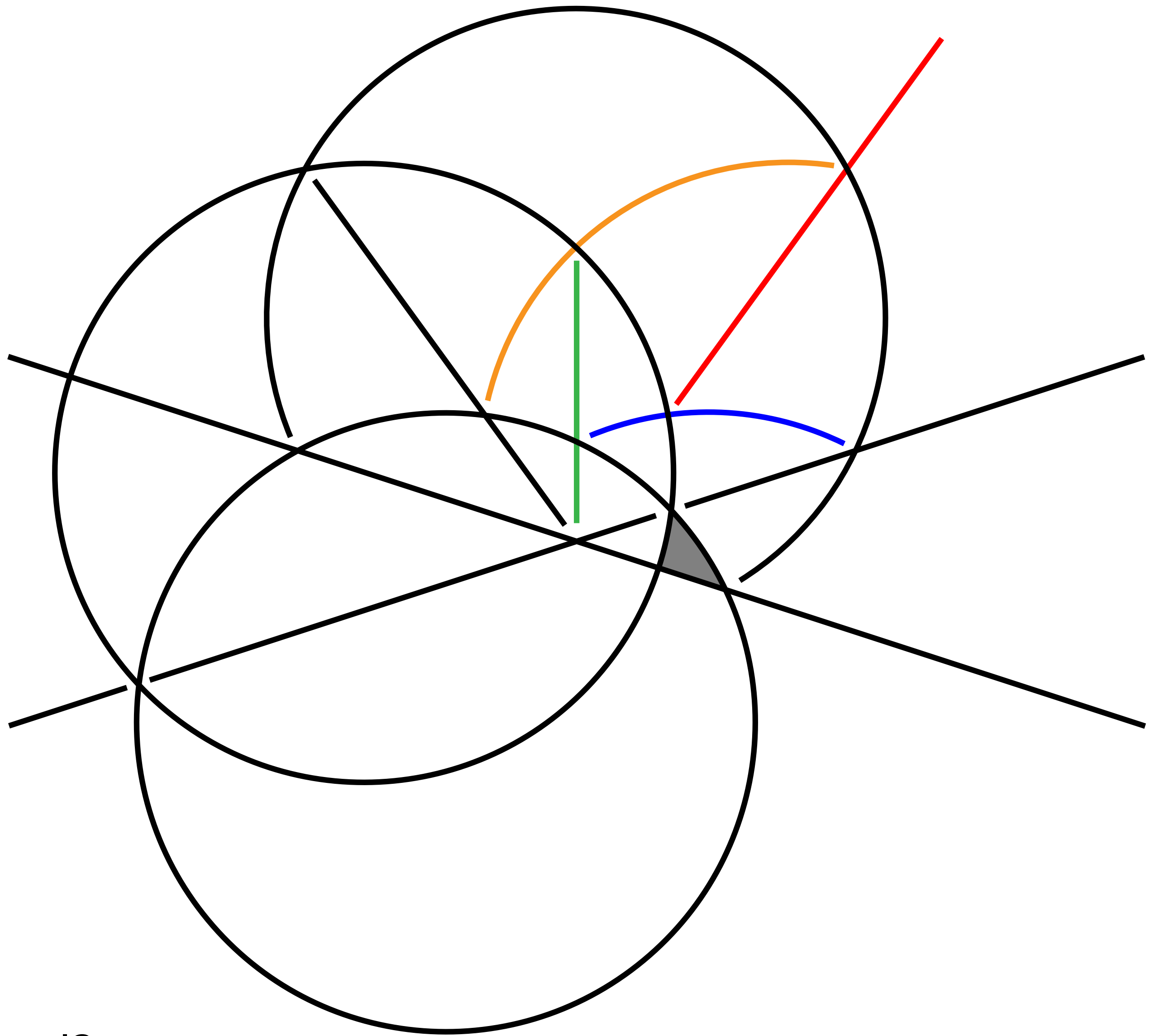
Interesting lattice quotients: Coxeter associahedra (cluster algebras)



A SIMPLICIAL BUT NOT CONGRUENCE-UNIFORM ARRANGEMENT

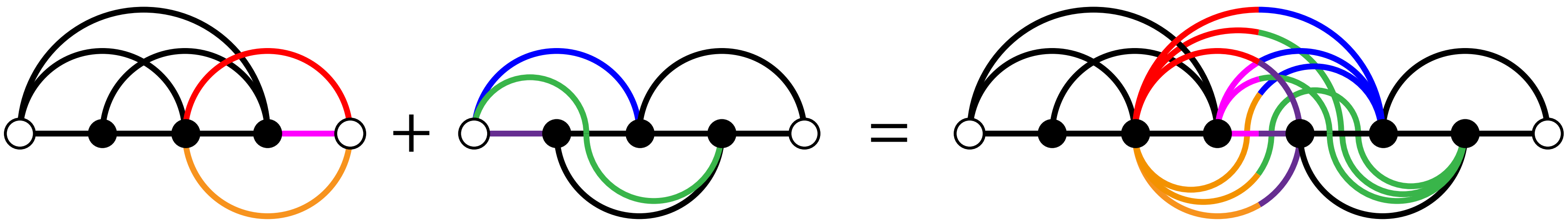


A SIMPLICIAL BUT NOT CONGRUENCE-UNIFORM ARRANGEMENT



Is this fan polytopal?

LATTICE QUOTIENTS AND HOPF ALGEBRAS



Combinatorial Hopf Algebra $=$ combinatorial vector space $\mathcal{B}$ endowed with

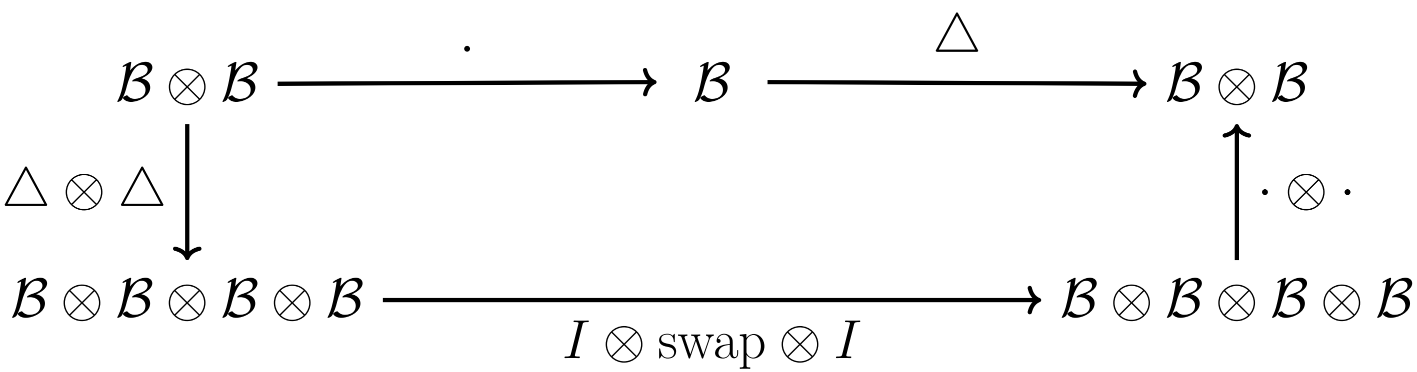
$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$

$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

which are compatible:

$$\triangle(x \cdot y) = \triangle(x) \cdot \triangle(y)$$

or more precisely the following diagram commutes:



Exm: $(G, \star)$ group. $\mathbf{k}G$ group algebra where

$$g \cdot h = g \star h \qquad \text{and} \qquad \triangle(g) = g \otimes g$$

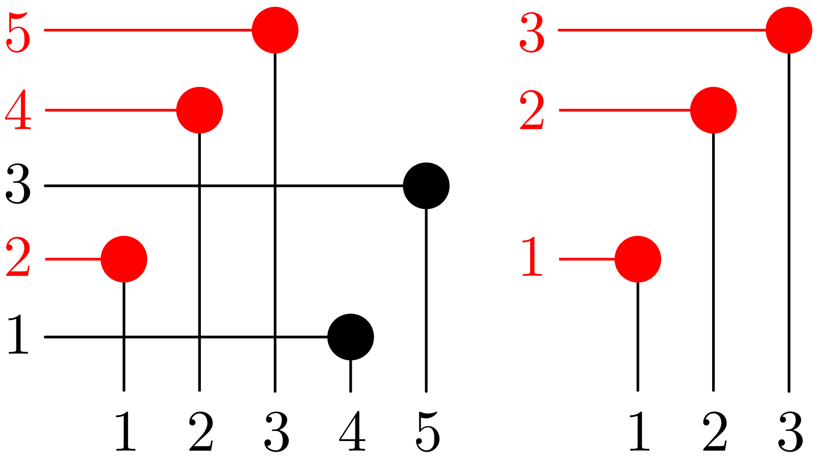
LODAY-RONCO HOPF ALGEBRA

Malvenuto–Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)
Loday–Ronco, *Hopf algebra of the planar binary trees* ('98)

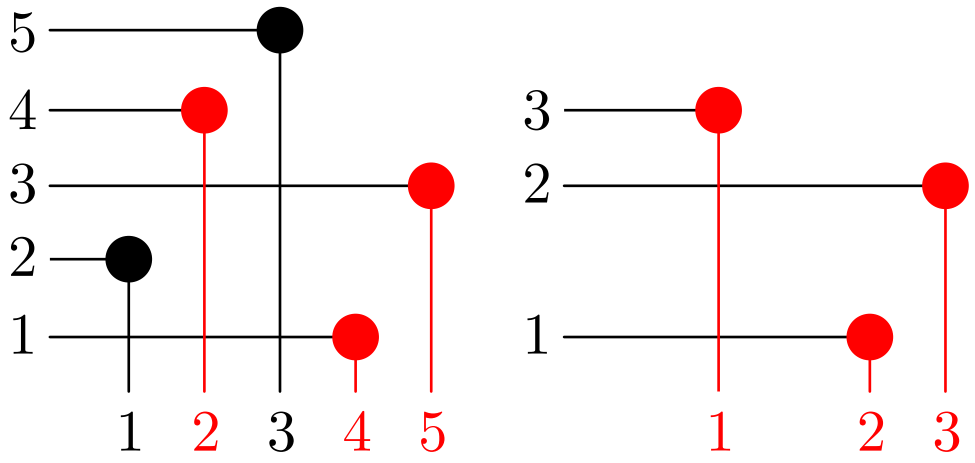
PERMUTATION RESTRICTION

For $\sigma \in \mathfrak{S}_n$ and $I \subseteq [n]$, define

- $\sigma_{|I}$ = restriction of σ to the positions in I,
- $\sigma^{|I}$ = restriction of σ to the values in I.



$41523_{|[245]} = 123$



$41523^{|[245]} = 231$

SHUFFLE AND CONVOLUTION

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

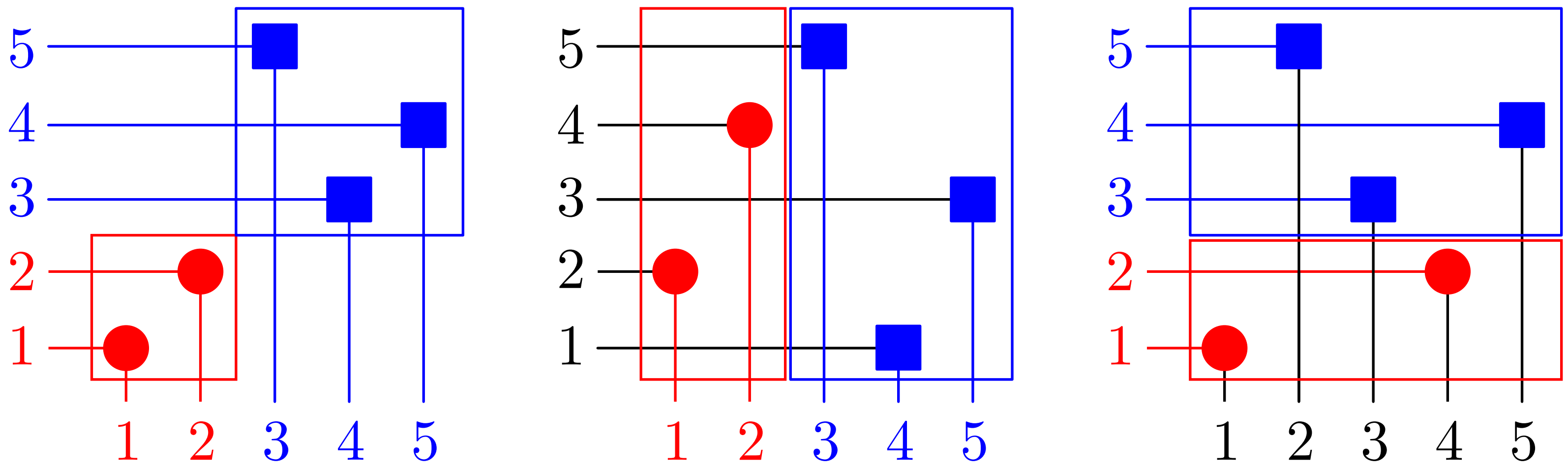
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1) + n, \dots, \tau'(n') + n] \in \mathfrak{S}_{n+n'}$

shifted shuffle $\tau \,\bar{\sqcup\!\sqcup}\, \tau' = \left\{\sigma \in \mathfrak{S}_{n+n'} \;\middle|\; \sigma^{|[n]} = \tau \text{ and } \sigma^{|[n+n'] \smallsetminus [n]} = \tau'\right\}$

convolution $\tau \star \tau' = \left\{\sigma \in \mathfrak{S}_{n+n'} \;\middle|\; \sigma_{|[n]} = \tau \text{ and } \sigma_{|[n+n'] \smallsetminus [n]} = \tau'\right\}$

Exm: $12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation — shuffle — convolution

MALVENUTO–REUTENAUER ALGEBRA

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1)+n, \dots, \tau'(n')+n] \in \mathfrak{S}_{n+n'}$

shifted shuffle $\tau \,\bar{⧢}\, \tau' = \left\{\sigma \in \mathfrak{S}_{n+n'} \;\middle|\; \sigma^{|[n]} = \tau \text{ and } \sigma^{|[n+n']\smallsetminus[n]} = \tau'\right\}$

convolution $\tau \star \tau' = \left\{\sigma \in \mathfrak{S}_{n+n'} \;\middle|\; \sigma_{|[n]} = \tau \text{ and } \sigma_{|[n+n']\smallsetminus[n]} = \tau'\right\}$

Exm: $12 \,\bar{⧢}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$

THM. The vector space $\mathbf{k}\mathfrak{S} = \bigoplus_{n\in\mathbb{N}} \mathbf{k}\mathfrak{S}_n$ with basis $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}}$ endowed with

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{⧢}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

is a combinatorial Hopf algebra.

Malvenuto–Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)

LODAY–RONCO ALGEBRA

THM. The vector space $\mathbf{k}\mathfrak{S} = \bigoplus_{n \in \mathbb{N}} \mathbf{k}\mathfrak{S}_n$ with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ endowed with

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

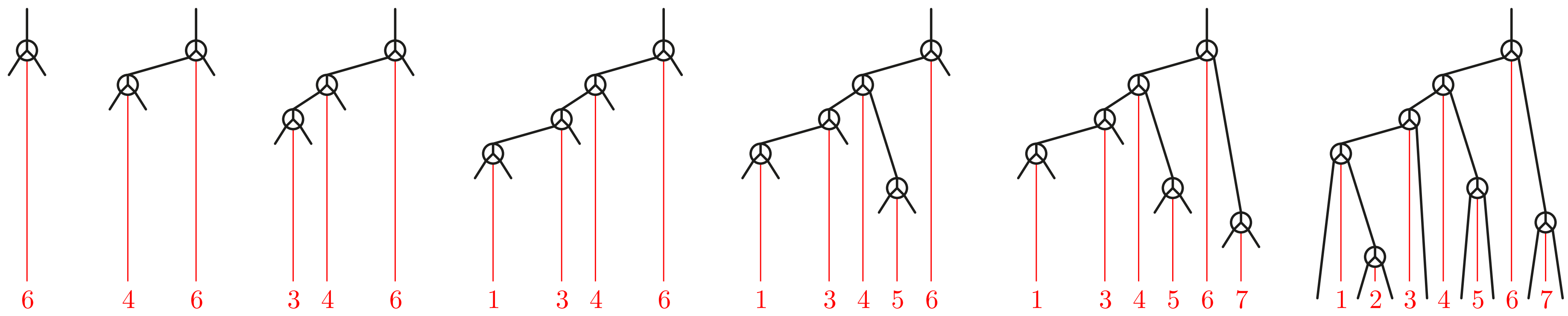
is a combinatorial Hopf algebra.

Malvenuto–Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)

THM. For a binary search tree T, consider the element $\mathbb{P}_\mathrm{T} := \sum_{\substack{\tau \in \mathfrak{S} \\ \mathrm{BST}(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_\tau$.
These elements generate a Hopf subalgebra $\mathbf{k}\mathfrak{T}$ of $\mathbf{k}\mathfrak{S}$.

Loday–Ronco, *Hopf algebra of the planar binary trees* ('98)

binary search tree insertion of 2751346



PERMUTREE ALGEBRA

P.–Pons, *Permutrees* ('18)
Chatel–P., *Cambrian algebras* ('17)

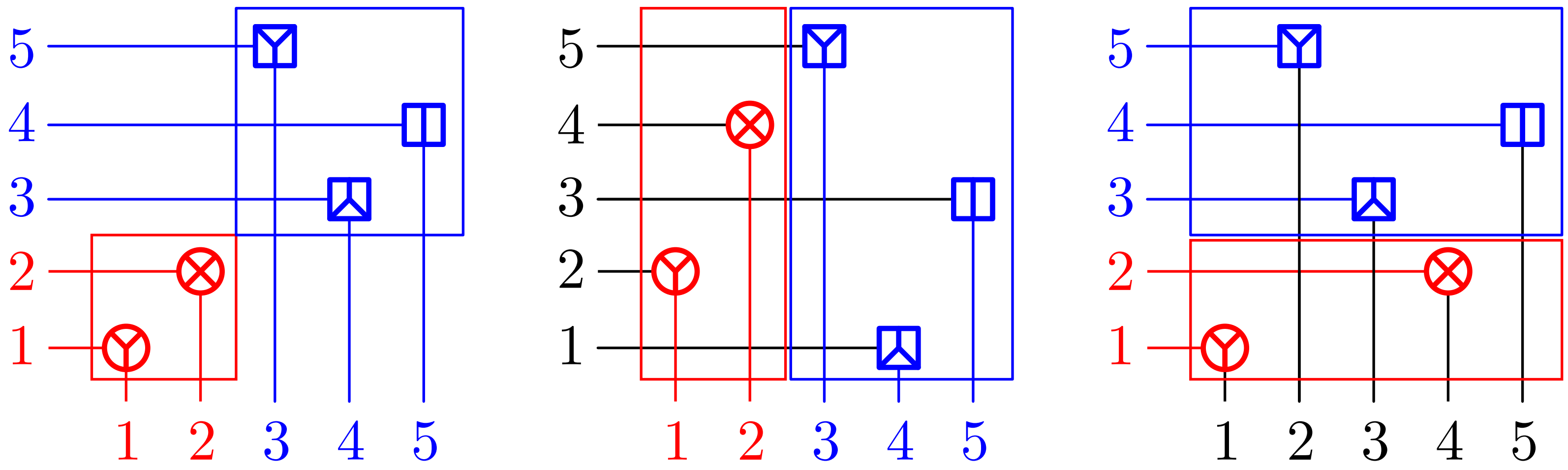
DECORATED VERSION

For decorated permutations:

- decorations are attached to values in the shuffle
- decorations are attached to positions in the convolution

Exm: $12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation — shuffle — convolution

$\mathbf{k}\mathfrak{S}_{\{\oplus,\otimes,\ominus,\otimes\}}$ = Hopf algebra with basis $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}_{\{\oplus,\otimes,\ominus,\otimes\}}}$ and where

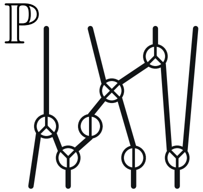
$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma\in\tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle\mathbb{F}_\sigma = \sum_{\sigma\in\tau\star\tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

PERMUTREE ALGEBRA AS SUBALGEBRA

Permutree algebra $=$ vector subspace $\mathbf{k}\mathfrak{PT}$ of $\mathbf{k}\mathfrak{S}_{\{⦶,Ⓨ,⅄,\otimes\}}$ generated by

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S}_{\{⦶,Ⓨ,⅄,\otimes\}} \\ \mathbf{P}(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_\tau,$$

for all permutrees T.

Exm: $\mathbb{P}$  $= \mathbb{F}_{\overline{2}\underline{1}35\overline{\underline{4}}\overline{7}\underline{6}} + \mathbb{F}_{\overline{2}\underline{1}35\overline{7}\overline{\underline{4}}\underline{6}} + \mathbb{F}_{\overline{2}\underline{1}3\overline{7}5\overline{\underline{4}}\underline{6}} + \cdots + \mathbb{F}_{\overline{7}5\overline{2}3\underline{1}\overline{\underline{4}}\underline{6}} + \mathbb{F}_{\overline{7}5\overline{2}3\overline{\underline{4}}\underline{1}\underline{6}} + \mathbb{F}_{\overline{7}5\overline{2}3\overline{\underline{4}}\underline{6}\underline{1}}$

THEO. $\mathbf{k}\mathfrak{PT}$ is a subalgebra of $\mathbf{k}\mathfrak{S}_{\{⦶,Ⓨ,⅄,\otimes\}}$

Loday–Ronco, *Hopf algebra of the planar binary trees* ('98)

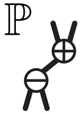
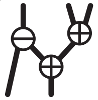
Hivert–Novelli–Thibon, *The algebra of binary search trees* ('05)

Chatel–P., *Cambrian Hopf algebras* ('14$^+$)

P.–Pons, *Permutrees* ('16$^+$)

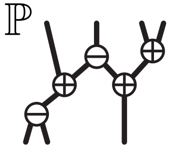
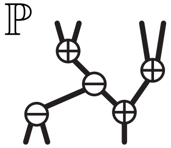
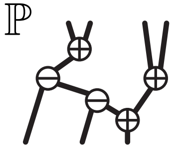
GAME: Explain the product and coproduct directly on the permutrees...

PRODUCT IN PERMUTREE ALGEBRA

$\mathbb{P}$  $\cdot\ \mathbb{P}$ 

$$= \mathbb{F}_{\underline{1}\overline{2}} \cdot \left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\right)$$

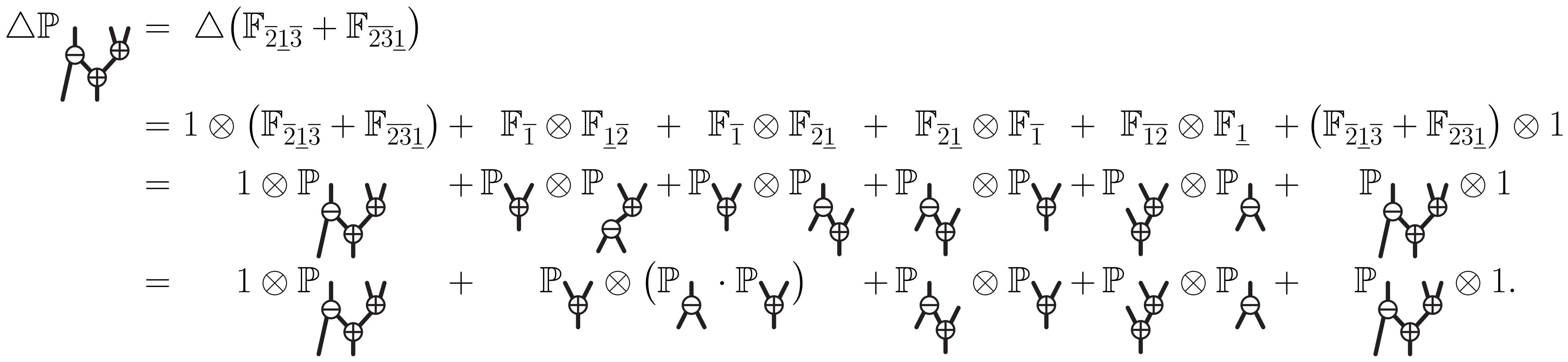
$$= \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{F}_{\underline{1}\overline{2}\overline{4}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\overline{2}\underline{3}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{2}\overline{5}} + \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{5}\overline{2}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\underline{3}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{2}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{5}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\underline{3}\overline{2}} \\ + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\underline{3}\overline{2}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{2}\overline{5}} + \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{5}\overline{2}} \\ + \mathbb{F}_{\overline{4}\underline{3}\overline{5}\underline{1}\overline{2}} + \mathbb{F}_{\overline{4}\overline{5}\underline{3}\underline{1}\overline{2}} \end{pmatrix}$$

$=\ \mathbb{P}$  $+\ \mathbb{P}$  $+\ \mathbb{P}$ 

PROP. For any permutrees T and T',

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$$

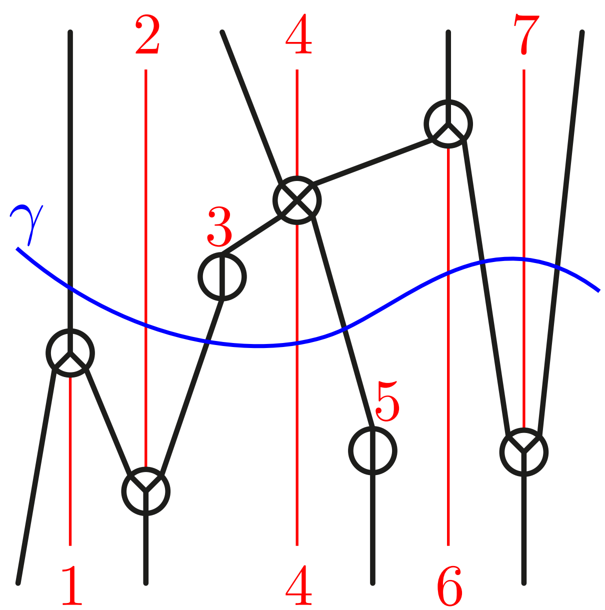
where S runs over the interval $\left[\mathrm{T} \nearrow \bar{\mathrm{T}}'\ ,\ \mathrm{T} \nwarrow \bar{\mathrm{T}}'\right]$ in the $\delta(\mathrm{T})\delta(\mathrm{T}')$-permutree lattice



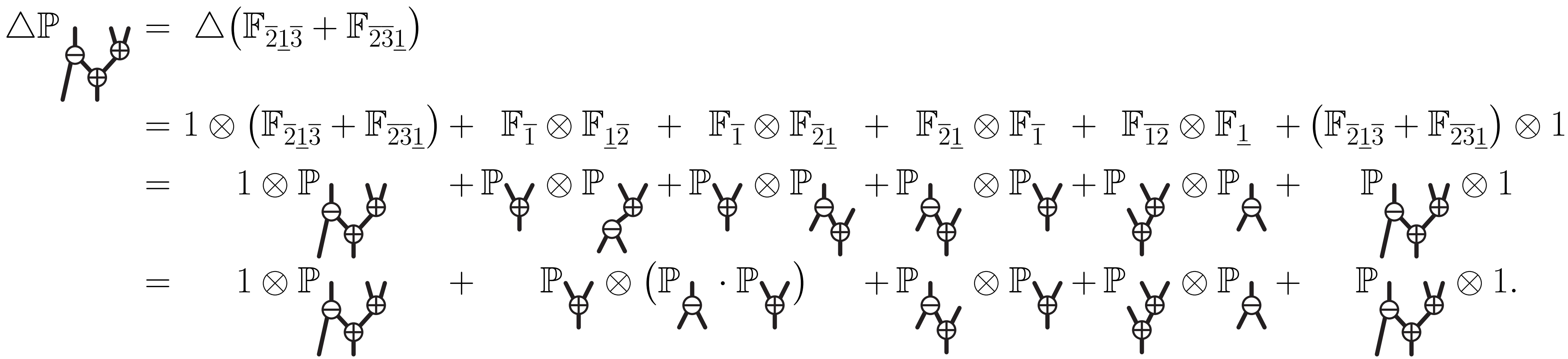
PROP. For any permutree S,

$$\triangle \mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \Big(\prod_{\mathrm{T} \in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \Big) \otimes \Big(\prod_{\mathrm{T}' \in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \Big)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the forests above and below γ respectively



COPRODUCT IN PERMUTREE ALGEBRA



PROP. For any permutree S,

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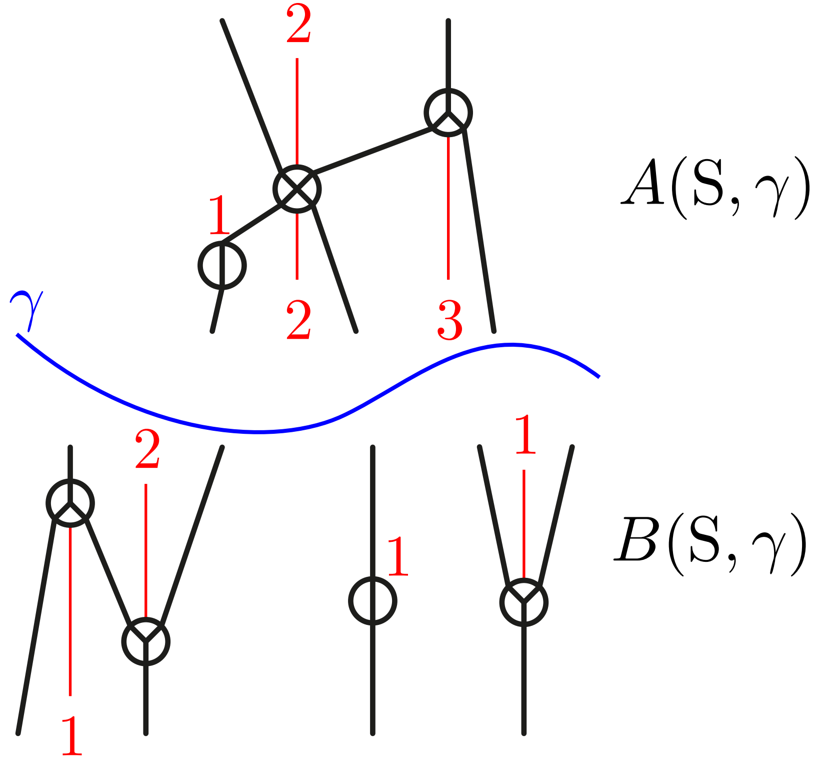
where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the forests above and below γ respectively



DUAL PERMUTREE ALGEBRA AS QUOTIENT

$\mathbf{k}\mathfrak{S}^*_{\{\mathbb{O},\mathbb{\otimes},\mathbb{\oslash},\otimes\}}$ = dual Hopf algebra with basis $(\mathbb{G}_\tau)_{\tau \in \mathfrak{S}_{\{\mathbb{O},\mathbb{\otimes},\mathbb{\oslash},\otimes\}}}$ and where

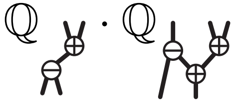
$$\mathbb{G}_\tau \cdot \mathbb{G}_{\tau'} = \sum_{\sigma \in \tau \star \tau'} \mathbb{G}_\sigma \qquad \text{and} \qquad \triangle \mathbb{G}_\sigma = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{G}_\tau \otimes \mathbb{G}_{\tau'}$$

PROP. The graded dual $\mathbf{k}\mathfrak{P}\mathfrak{T}^*$ of the permutree algebra is isomorphic to the image of $\mathbf{k}\mathfrak{S}^*_{\{\mathbb{O},\mathbb{\otimes},\mathbb{\oslash},\otimes\}}$ under the canonical projection

$$\pi : \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}\langle A \rangle / \equiv,$$

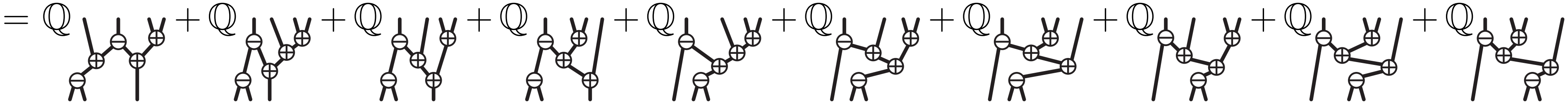
where $\equiv$ denotes the permutree congruence. The dual basis $\mathbb{Q}_{\mathrm{T}}$ of $\mathbb{P}_{\mathrm{T}}$ is expressed as $\mathbb{Q}_{\mathrm{T}} = \pi(\mathbb{G}_\tau)$, where τ is any linear extension of T

PRODUCT IN DUAL PERMUTREE ALGEBRA



$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

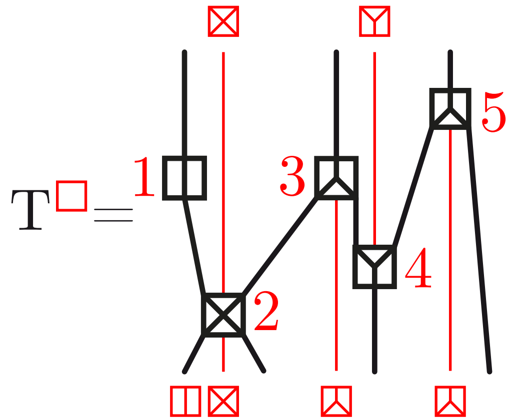
$$= \mathbb{G}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{34}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{43}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{53}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{34}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{43}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{53}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{42}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{52}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{52}\underline{1}\overline{3}}$$

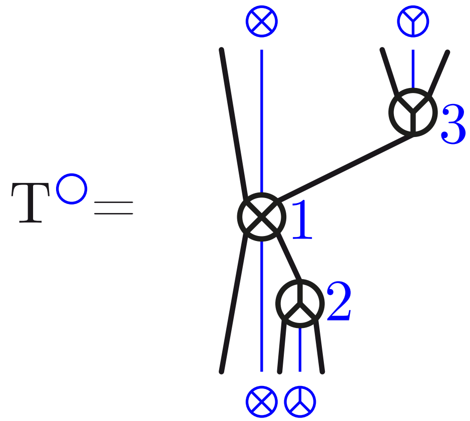


PROP. For any permutrees T and T',

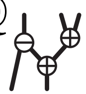
$$\mathbb{Q}_{\mathrm{T}} \cdot \mathbb{Q}_{\mathrm{T}'} = \sum_{s} \mathbb{Q}_{\mathrm{T}s\mathrm{T}'}$$

where s runs over all shuffles of $\delta(\mathrm{T})$ and $\delta(\mathrm{T}')$

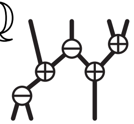
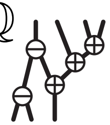
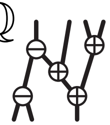
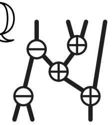
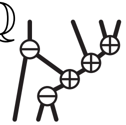
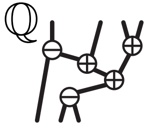
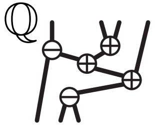
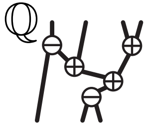
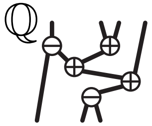
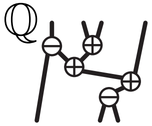




PRODUCT IN DUAL PERMUTREE ALGEBRA

$\mathbb{Q}$  $\cdot\ \mathbb{Q}$  $= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$

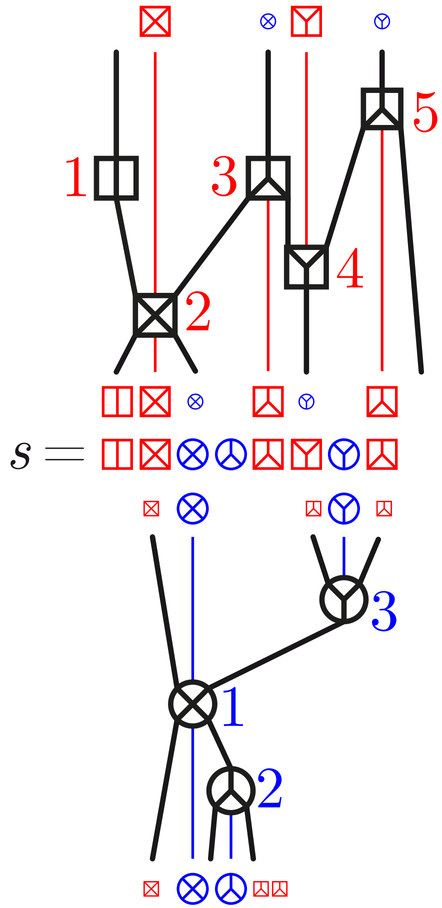
$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{3}\overline{4}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\overline{3}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\overline{3}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{4}\overline{3}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{5}\overline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{5}\overline{2}\underline{1}\overline{3}}$

$= \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$ 

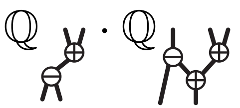
PROP. For any permutrees T and T',

$$\mathbb{Q}_{\mathrm{T}} \cdot \mathbb{Q}_{\mathrm{T}'} = \sum_{s} \mathbb{Q}_{\mathrm{T} s \mathrm{T}'}$$

where s runs over all shuffles of $\delta(\mathrm{T})$ and $\delta(\mathrm{T}')$

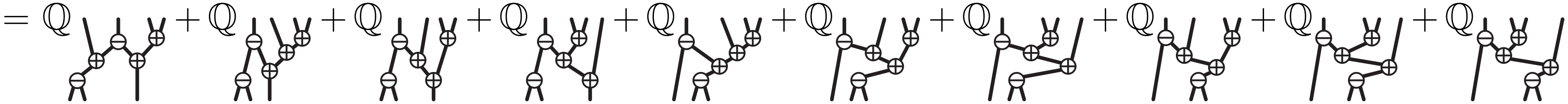


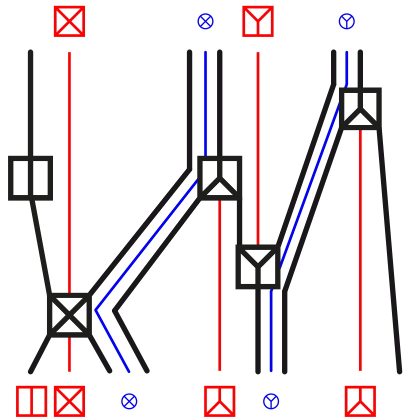
PRODUCT IN DUAL PERMUTREE ALGEBRA

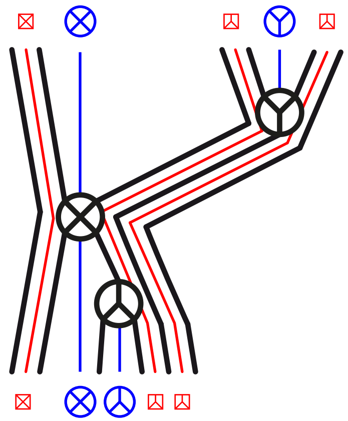


$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{3}\overline{4}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\overline{3}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\overline{3}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{4}\overline{3}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{5}\overline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{5}\overline{2}\underline{1}\overline{3}}$$







PROP. For any permutrees T and T',

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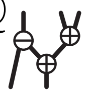
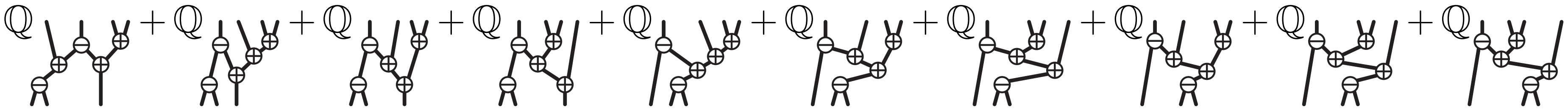
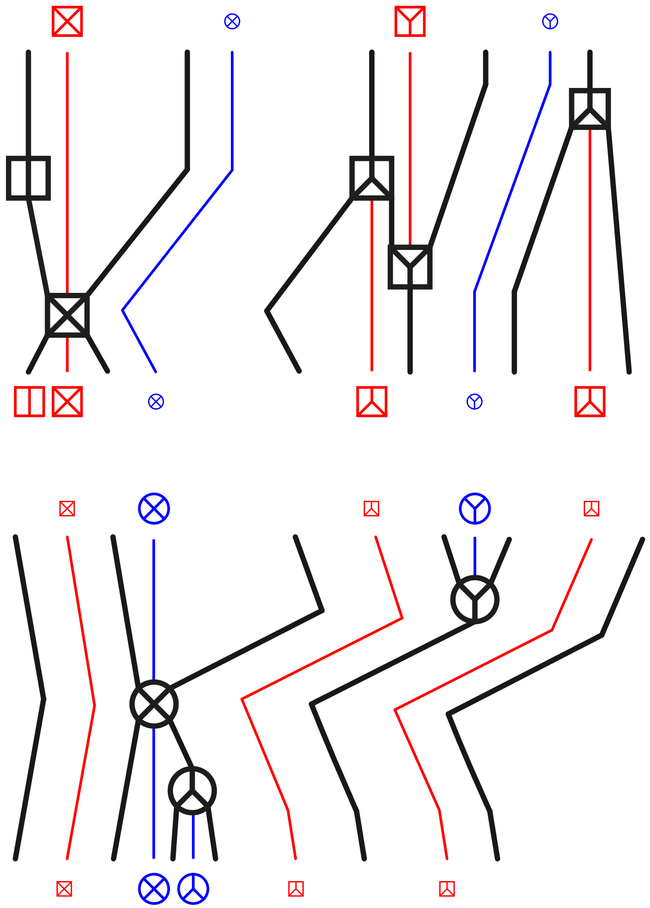
where s runs over all shuffles of $\delta(\mathrm{T})$ and $\delta(\mathrm{T}')$

PRODUCT IN DUAL PERMUTREE ALGEBRA

$\mathbb{Q}$  $\cdot\ \mathbb{Q}$  $= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$

$$= \mathbb{G}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{34}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{43}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{53}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{34}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{43}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{53}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{42}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{52}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{52}\underline{1}\overline{3}}$$

$=$ 



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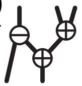
$\mathbb{Q}$  $\cdot\ \mathbb{Q}$  $= \ \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$

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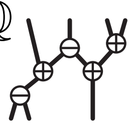
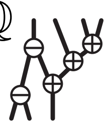
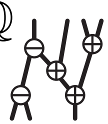
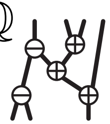
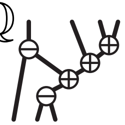
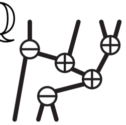
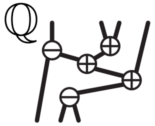
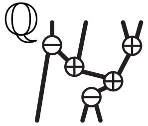
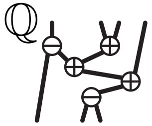
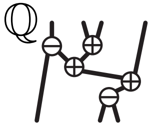
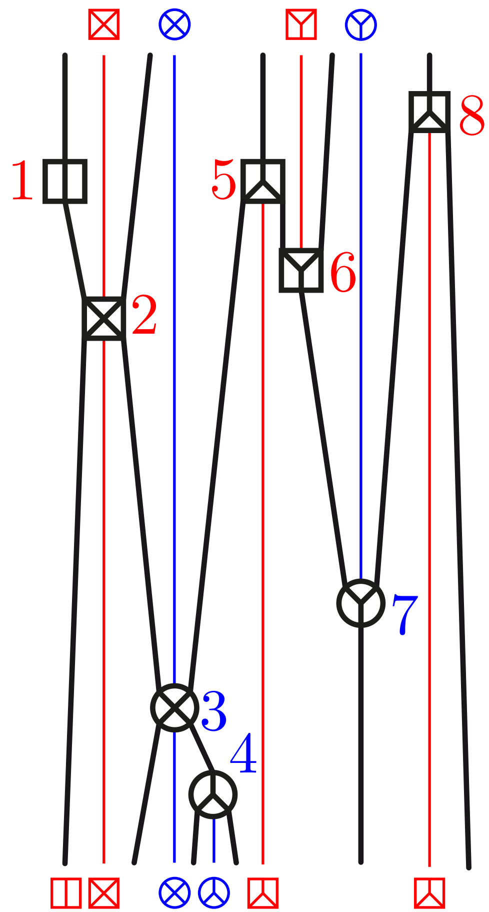
$= \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$  $+ \mathbb{Q}$ 



$= \mathrm{T}^{\bigcirc}{}_{s}\backslash \mathrm{T}^{\square}$

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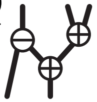
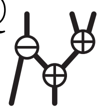
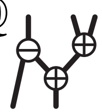
COPRODUCT IN DUAL PERMUTREE ALGEBRA

$\triangle \mathbb{Q}$  $= \triangle \mathbb{G}_{\bar{2}\underline{1}\bar{3}}$

$= 1 \otimes \mathbb{G}_{\bar{2}\underline{1}\bar{3}} + \mathbb{G}_{\underline{1}} \otimes \mathbb{G}_{\overline{12}} + \mathbb{G}_{\bar{2}\underline{1}} \otimes \mathbb{G}_{\bar{1}} + \mathbb{G}_{\bar{2}\underline{1}\bar{3}} \otimes 1$

$= 1 \otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes 1.$

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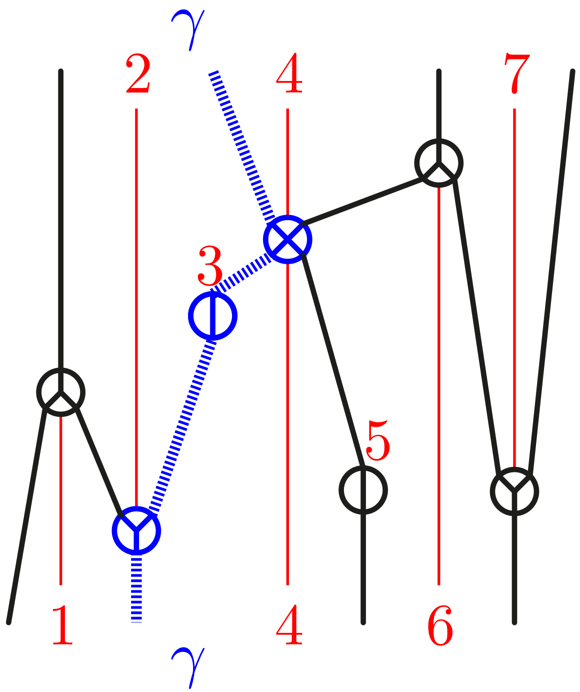
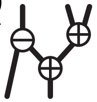


$$\triangle\mathbb{Q}$$  $$= \triangle\mathbb{G}_{\bar{2}\underline{1}\bar{3}}$$

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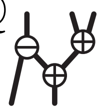
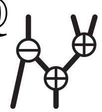
$= 1\otimes\mathbb{Q}$  $+\ \mathbb{Q}$  $\otimes\ \mathbb{Q}$  $+\ \mathbb{Q}$  $\otimes\ \mathbb{Q}$  $+\ \mathbb{Q}$  $\otimes\ 1.$

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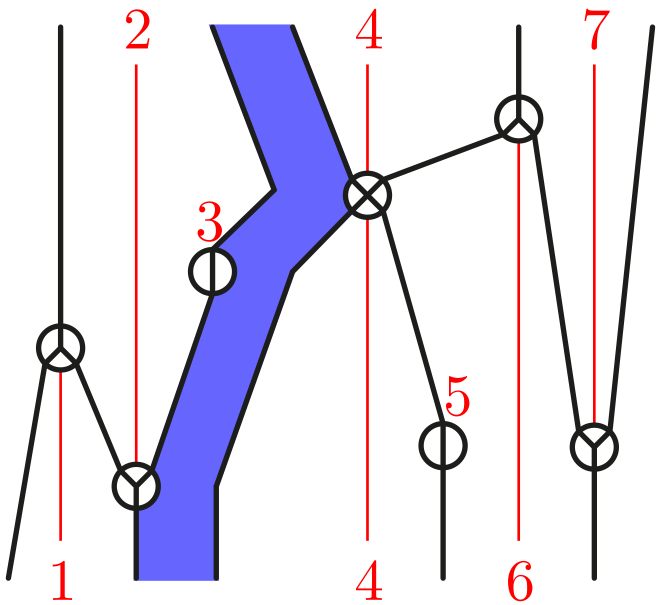
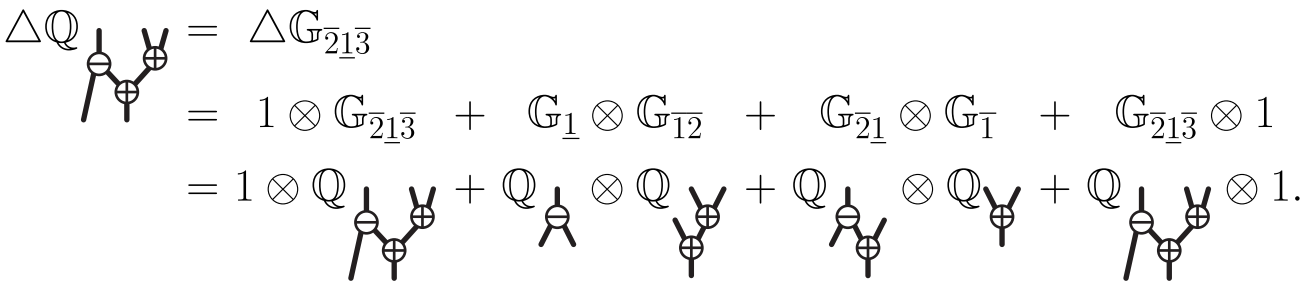


COPRODUCT IN DUAL PERMUTREE ALGEBRA



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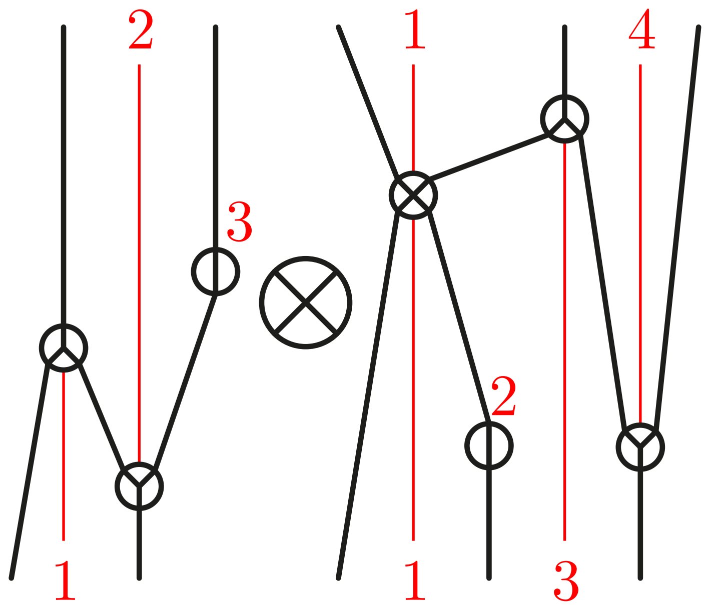

HOPF ALGEBRAS ON ARC DIAGRAMS

P., *Hopf algebras on decorated noncrossing arc diagrams* ('18^{+})

DECORATED PERMUTATION

decoration set = a graded set $\mathfrak{X} := \bigsqcup_{n \geq 0} \mathfrak{X}_n$ endowed with

- a concatenation concat : $\mathfrak{X}_m \times \mathfrak{X}_n \longrightarrow \mathfrak{X}_{m+n}$
- a selection select : $\mathfrak{X}_m \times \binom{[m]}{k} \longrightarrow \mathfrak{X}_k$

such that

(i) $\mathrm{concat}(\mathcal{X}, \mathrm{concat}(\mathcal{Y}, \mathcal{Z})) = \mathrm{concat}(\mathrm{concat}(\mathcal{X}, \mathcal{Y}), \mathcal{Z})$

(ii) $\mathrm{select}(\mathrm{select}(\mathcal{X}, R), S) = \mathrm{select}(\mathcal{X}, \{r_s \mid s \in S\})$

(iii) $\mathrm{concat}(\mathrm{select}(\mathcal{X}, R), \mathrm{select}(\mathcal{Y}, S)) = \mathrm{select}(\mathrm{concat}(\mathcal{X}, \mathcal{Y}), R \cup S^{\to m})$
where $S^{\to m} := \{s + m \mid s \in S\}$.

Exm:

- $\mathcal{A}^* =$ words on an alphabet $\mathcal{A}$, with concatenation and subwords
- labeled graphs, with shifted union and induced subgraphs
- ...

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where $S^{\rightarrow m} := \{s + m \mid s \in S\}$.

$\mathfrak{X}$-decorated permutation = pair $(\sigma, \mathcal{X})$ with $\sigma \in \mathfrak{S}_n$ and $\mathcal{X} \in \mathfrak{X}_n$.

standardization $\mathrm{std}((\rho, \mathcal{Z}), R) := \big(\mathrm{stdp}(\rho, R), \mathrm{select}(\mathcal{Z}, \rho^{-1}(R))\big)$

THM. The product $\cdot$ and coproduct $\triangle$ defined by

$$\mathbb{F}_{(\sigma,\mathcal{X})} \cdot \mathbb{F}_{(\tau,\mathcal{Y})} := \sum_{\rho \in \sigma \,\bar{\sqcup\!\sqcup}\, \tau} \mathbb{F}_{(\rho, \mathrm{concat}(\mathcal{X},\mathcal{Y}))} \quad \text{and} \quad \triangle \mathbb{F}_{(\rho,\mathcal{Z})} := \sum_{k=0}^{p} \mathbb{F}_{\mathrm{std}((\rho,\mathcal{Z}),[k])} \otimes \mathbb{F}_{\mathrm{std}((\rho,\mathcal{Z}),[p]\smallsetminus[k])}$$

endow the vector space of decorated permutations with a graded Hopf algebra structure.

P., *Hopf algebras on decorated noncrossing arc diagrams* ('18$^+$)

DECORATED NONCROSSING ARC DIAGRAMS

a graded function $\Psi : \mathfrak{X} = \bigsqcup_{n \ge 0} \mathfrak{X}_n \longrightarrow \mathfrak{I} = \bigsqcup_{n \ge 0} \mathfrak{I}_n$ is conservative if

(i) $\Psi(\mathcal{X})^{+n}$ and $\Psi(\mathcal{Y})^{\rightarrow m}$ are both subsets of $\Psi(\text{concat}(\mathcal{X}, \mathcal{Y}))$

(ii) $(r_a, r_b, p, S) \in \Psi(\mathcal{Z})$ implies $(a, b, q, \{c \mid r_c \in S\}) \in \Psi(\text{select}(\mathcal{Z}, R))$

$\mathcal{I}$ collection of arcs closed by forcing

surjection $$\begin{array}{rccc} \eta_{\mathcal{I}} : & \mathfrak{S}_n & \longrightarrow & \{\text{nc arc diagrams in } \mathcal{I}\} \\ & \sigma & \longmapsto & \eta_{\mathcal{I}}(\sigma) = \delta(\pi_{\downarrow}(\sigma)) \end{array}$$

$\mathfrak{X}$-decorated noncrossing arc diagram $= (\mathcal{D}, \mathcal{X})$ where $\mathcal{D}$ is a non crossing arc diagram contained in $\Psi(\mathcal{X})$

THM. For a decorated noncrossing arc diagram $(\mathcal{D}, \mathcal{X})$, define

$$\mathbb{P}_{(\mathcal{D},\mathcal{X})} := \sum \mathbb{F}_{(\sigma,\mathcal{X})},$$

where σ ranges over the permutations such that $\eta_{\Psi(\mathcal{X})}(\sigma) = \mathcal{D}$. The graded vector subspace $\mathbf{k}\mathfrak{D} := \bigoplus_{n \ge 0} \mathbf{k}\mathfrak{D}_n$ of $\mathbf{k}\mathfrak{P}$ generated by the elements $\mathbb{P}_{(\mathcal{D},\mathcal{X})}$, for all $\mathfrak{X}$-decorated noncrossing arc diagrams $(\mathcal{D}, \mathcal{X})$, is a Hopf subalgebra of $\mathbf{k}\mathfrak{P}$.

P., *Hopf algebras on decorated noncrossing arc diagrams* ('18$^+$)

APPLICATIONS

fix $k \geq 0$

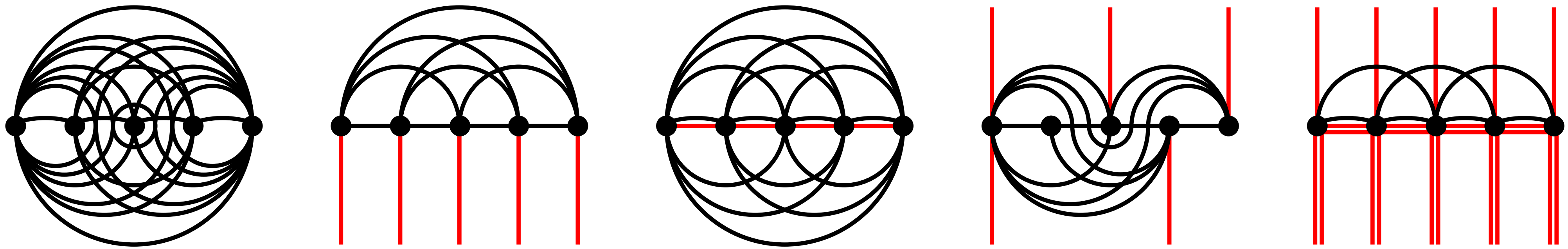
$\mathfrak{X}$ = words on $\mathbb{N}^4$ (each letter a is made of 4 numbers $\begin{smallmatrix} & u_a & \\ \ell_a & + & r_a \\ & d_a & \end{smallmatrix}$)

with concat$(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_m b_1 \cdots b_n$

select$(c_1 \cdots c_p, R) = \bar{c}_{r_1} \cdots \bar{c}_{r_q}$ where $\bar{c}_{r_i} = \min\limits_{r_{i-1} < k \leq r_i} \ell_{c_k} \overset{u_{c_{r_i}}}{\underset{d_{c_{r_i}}}{+}} \min\limits_{r_i \leq k < r_{i+1}} r_{c_k}$

$\Psi(a_1 \cdots a_m)$ = draw $\left| \begin{array}{l} u_{a_i} \\ d_{a_i} \\ \min(r_{a_i}, \ell_{a_{i+1}}) \end{array} \right.$ red walls $\left| \begin{array}{l} \text{above } i \\ \text{below } i \\ \text{between } i \text{ and } i+1 \end{array} \right.$

allow arcs that cross at most k walls

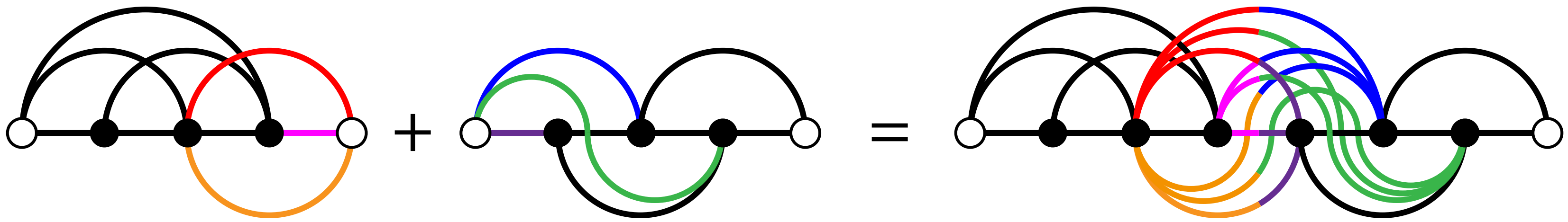


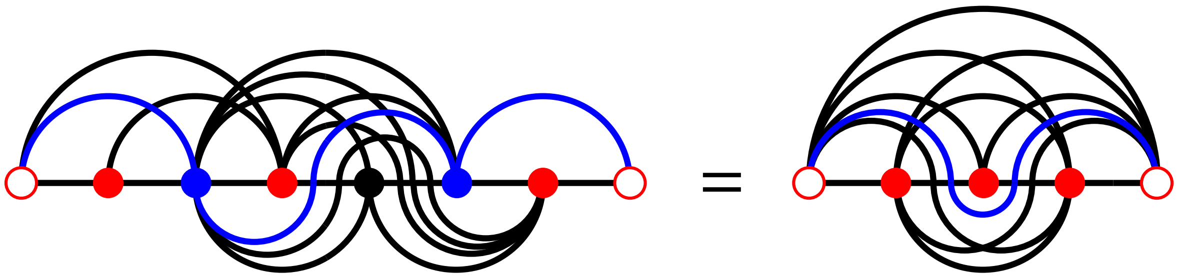
weak order | Tamari lattice | diagonal rectangulations | Cambrian lattices | k-sashes lattices

APPLICATIONS

extended arc $=$ arc allowed to start at 0 or end at $n+1$

$\mathfrak{X}$ $=$ extended arc ideals with

concatenation: 

selection: 

$\Psi(\mathcal{X})$ $=$ strict arcs in $\mathcal{X}$

$\Longrightarrow$ Hopf algebra on all arc ideals containing the permutree algebra

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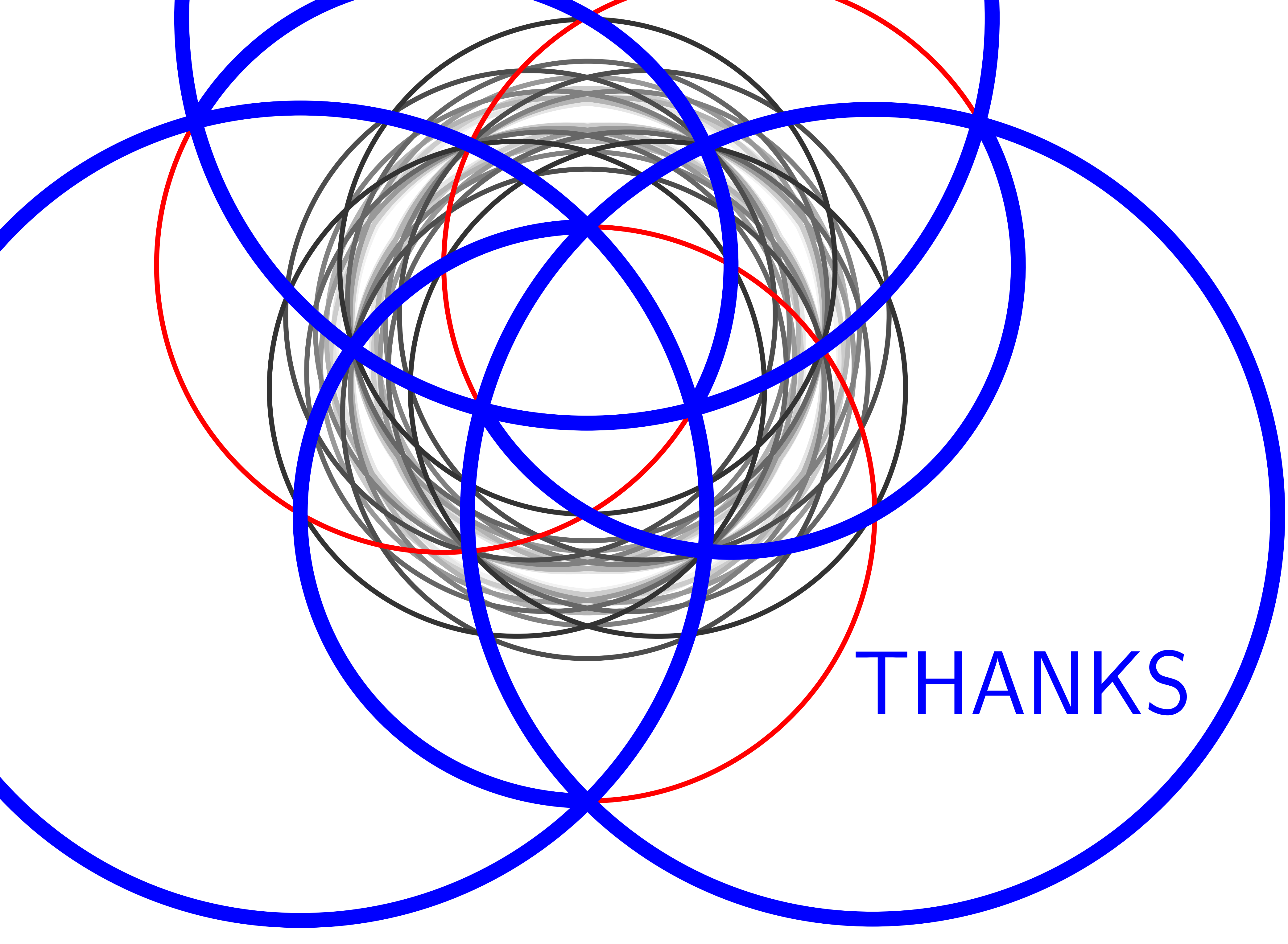
THANKS