18th Fall Workshop on Computational Geometry
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ENUMERATING DOUBLE PSEUDOLINE ARRANGEMENTS

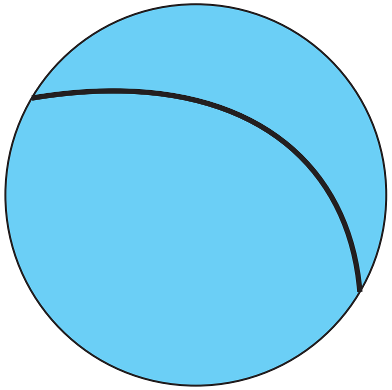
Julien Ferté, Vincent Pilaud, and Michel Pocchiola

École Normale Supérieure, Paris

INTRODUCTION

Projective plane $\mathcal{P}$ = disk with antipodal boundary points identified

A simple closed curve of $\mathcal{P}$ is a pseudoline if it is not contractible



The complement of a pseudoline is a topological disk

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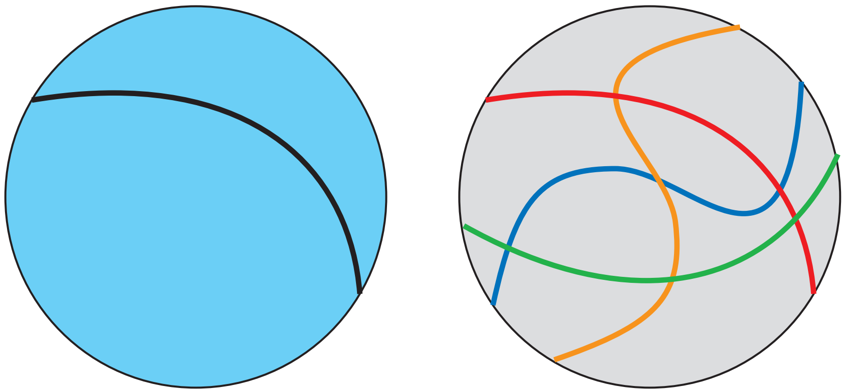

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Pseudoline arrangements correspond via duality to points configurations in geometric projective planes

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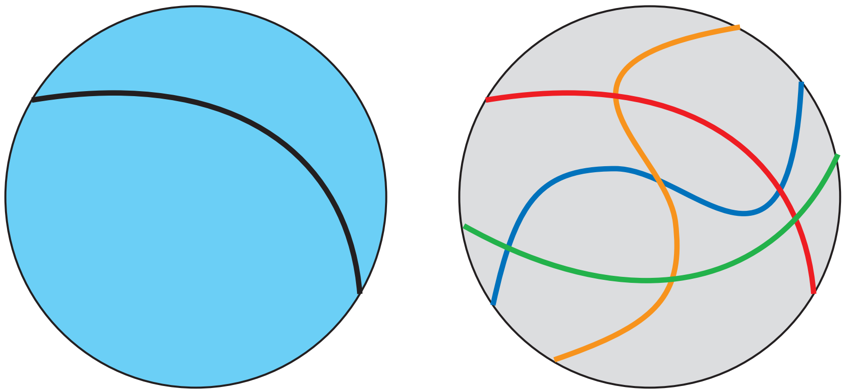

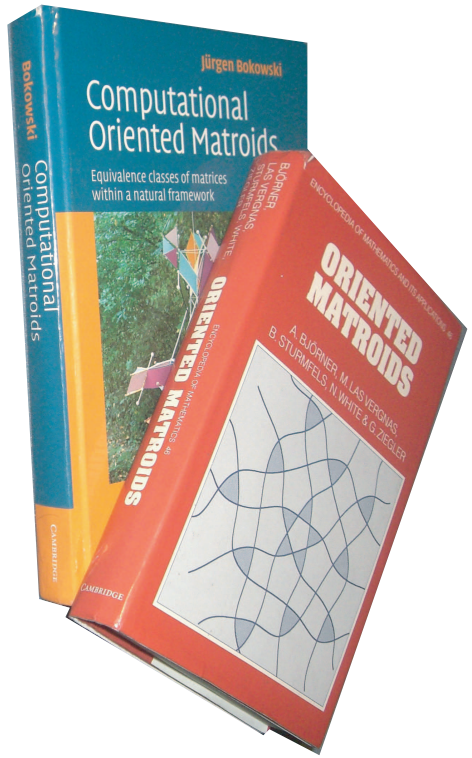

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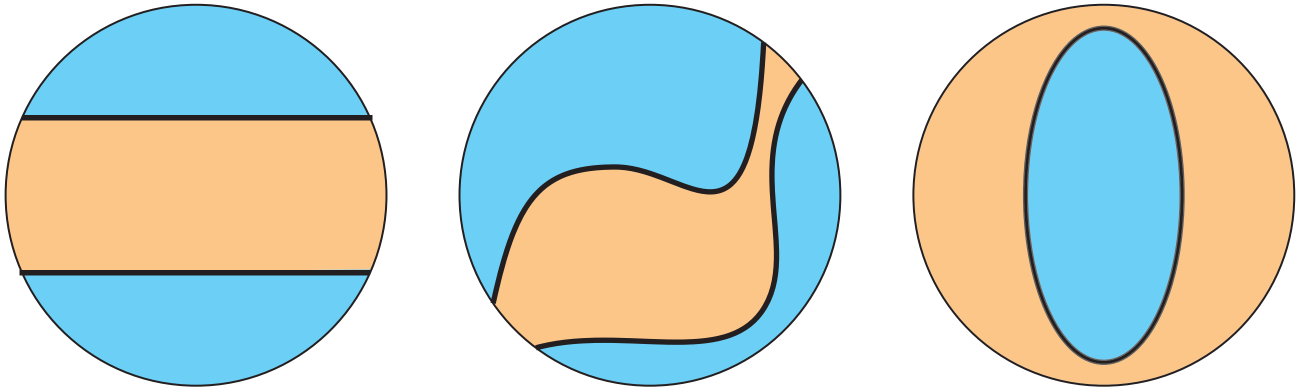

The complement of a double pseudoline ℓ has two connected components : a Möbius strip M_ℓ and a topological disk D_ℓ

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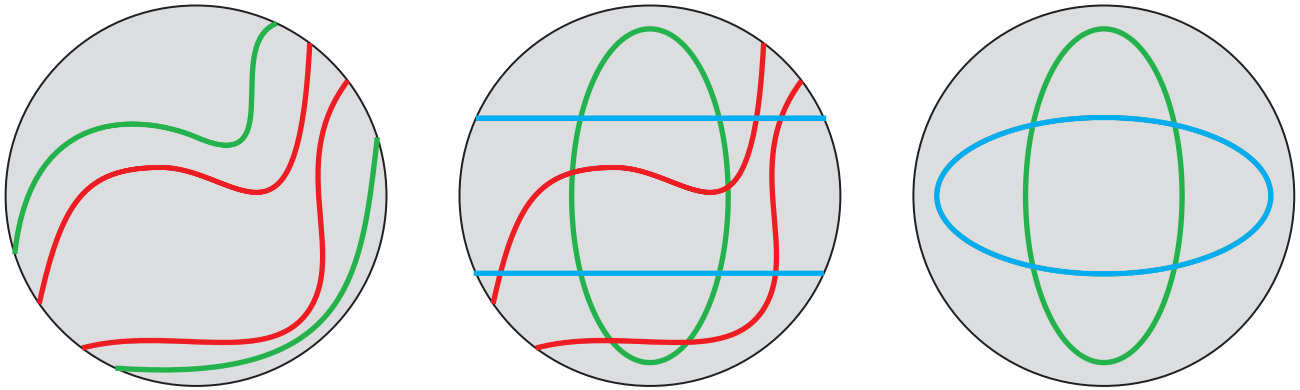

A double pseudoline arrangement is a finite set of double pseudolines such that any two of them

(i) have exactly four intersection points (and cross transversally at these points), and

(ii) induce a cell decomposition of $\mathcal{P}$

Double pseudoline arrangements correspond via duality to configurations of disjoint convex bodies in geometric projective planes

L. Habert & M. Pocchiola, Arrangements of double pseudolines (2006)

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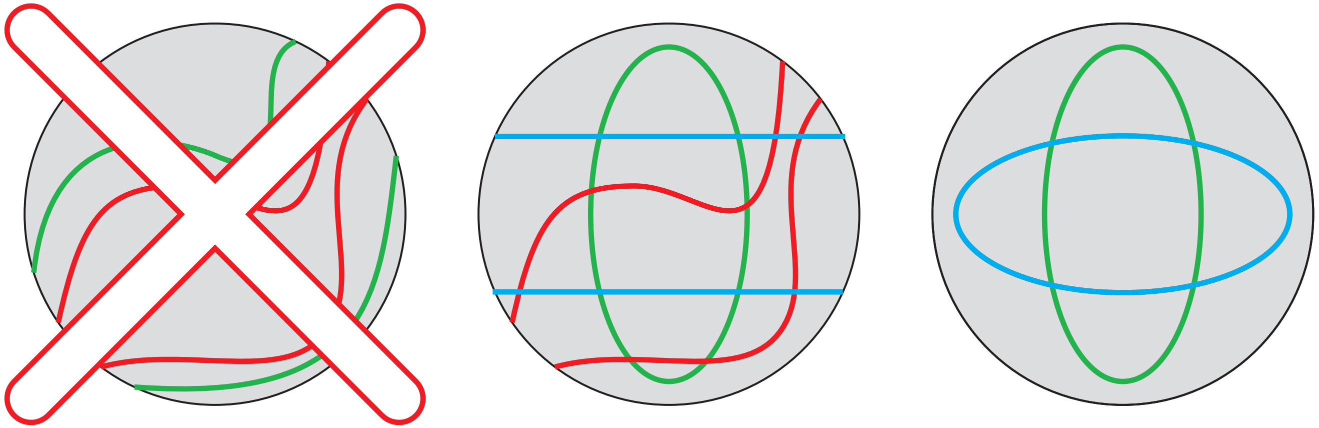

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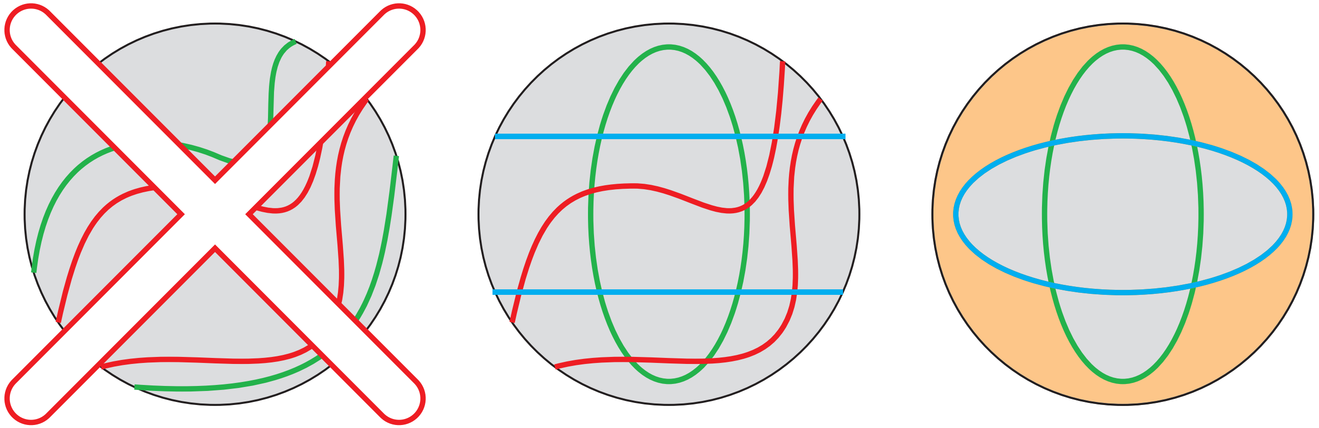

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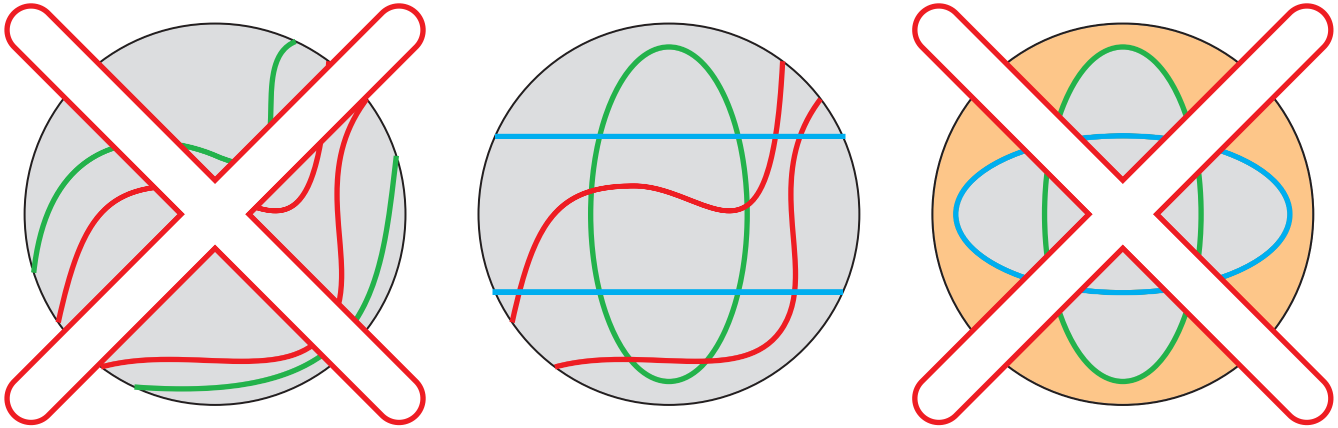

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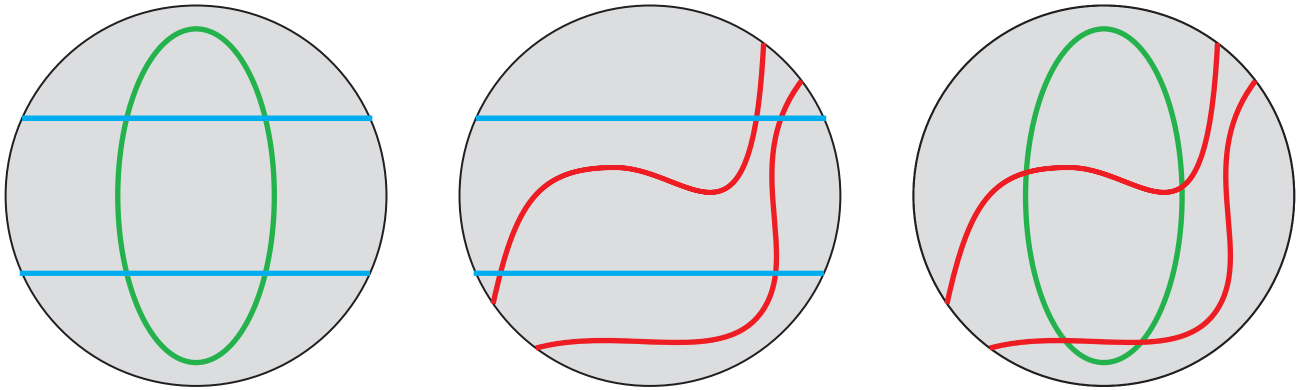

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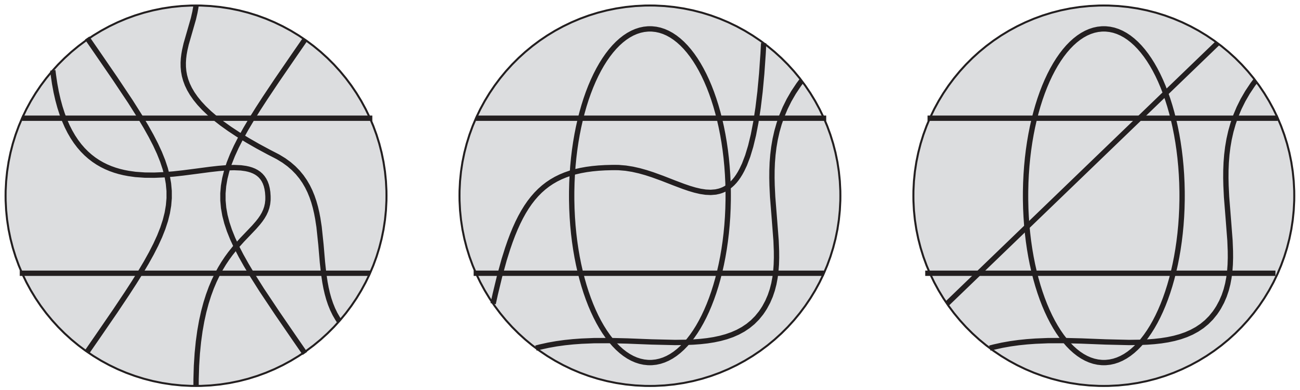
Isomorphism

Two arrangements A and B are isomorphic if there is a homeomorphism of $\mathcal{P}$ that sends A on B



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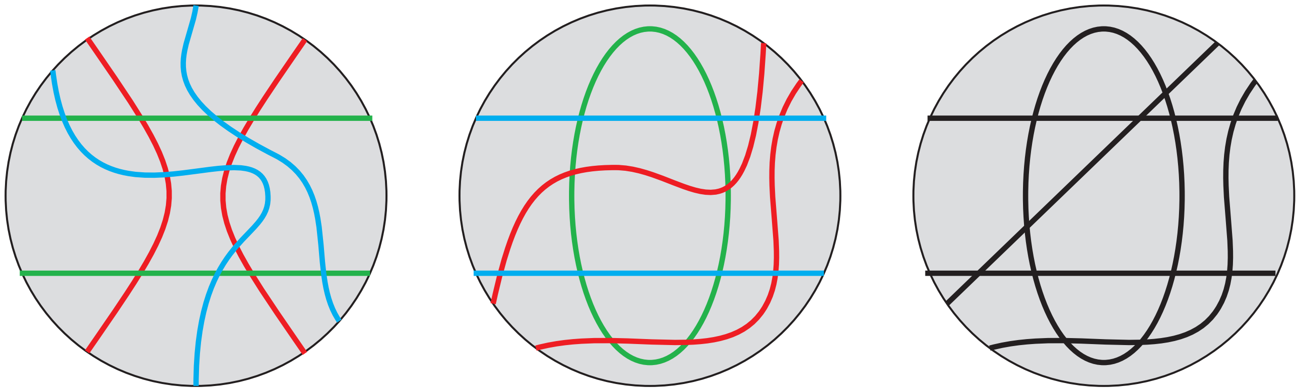

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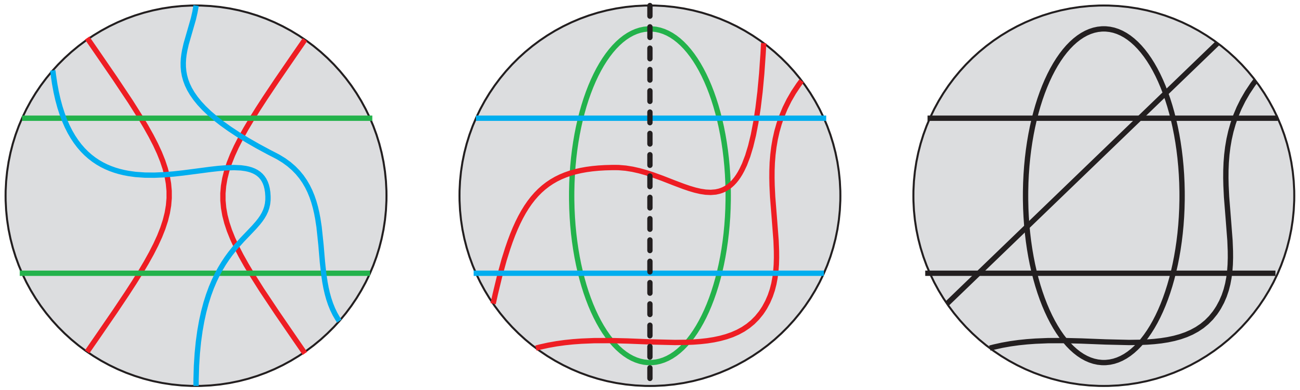

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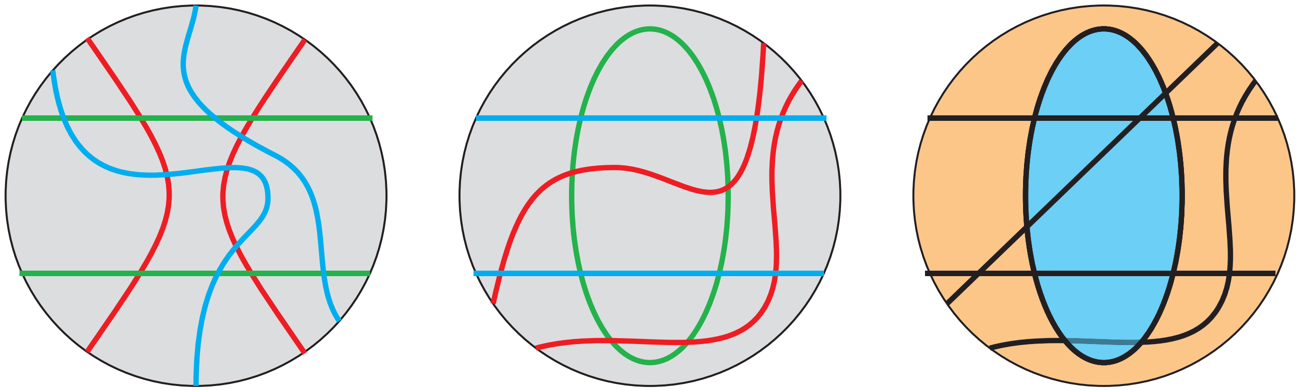

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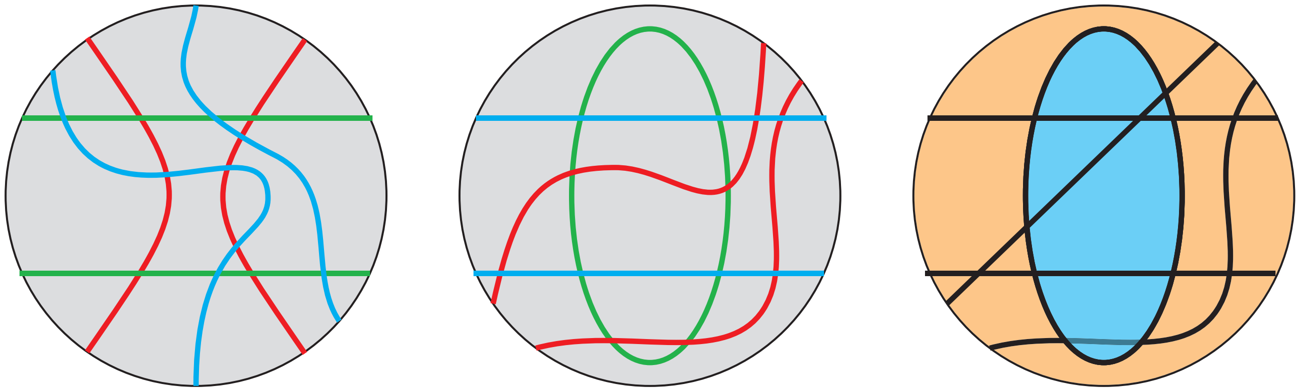

Two arrangements are isomorphic if and only if their face lattices are isomorphic

INTRODUCTION

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The number of isomorphism classes of arrangements of n pseudolines is

n	1	2	3	4	5	6	7	8	9	10	11
a_n	1	1	1	1	1	4	11	135	4 382	312 356	41 848 591

On-line Encyclopedia of Integer Sequences Identification A006248

J. Bokowski & A. G. de Oliveira, On the generation of oriented matroids (2000)

L. Finschi & K. Fukuda, Generation of oriented matroids - a graph theoretical approach (2002)

O. Aichholzer, F. Aurenhammer, & H. Krasser, Enumerating order types for small point sets with applications (2002)

The value a_{11} is due to F. Aurenhammer (2002)

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Our result : The number of isomorphism classes of arrangements of n double pseudolines is

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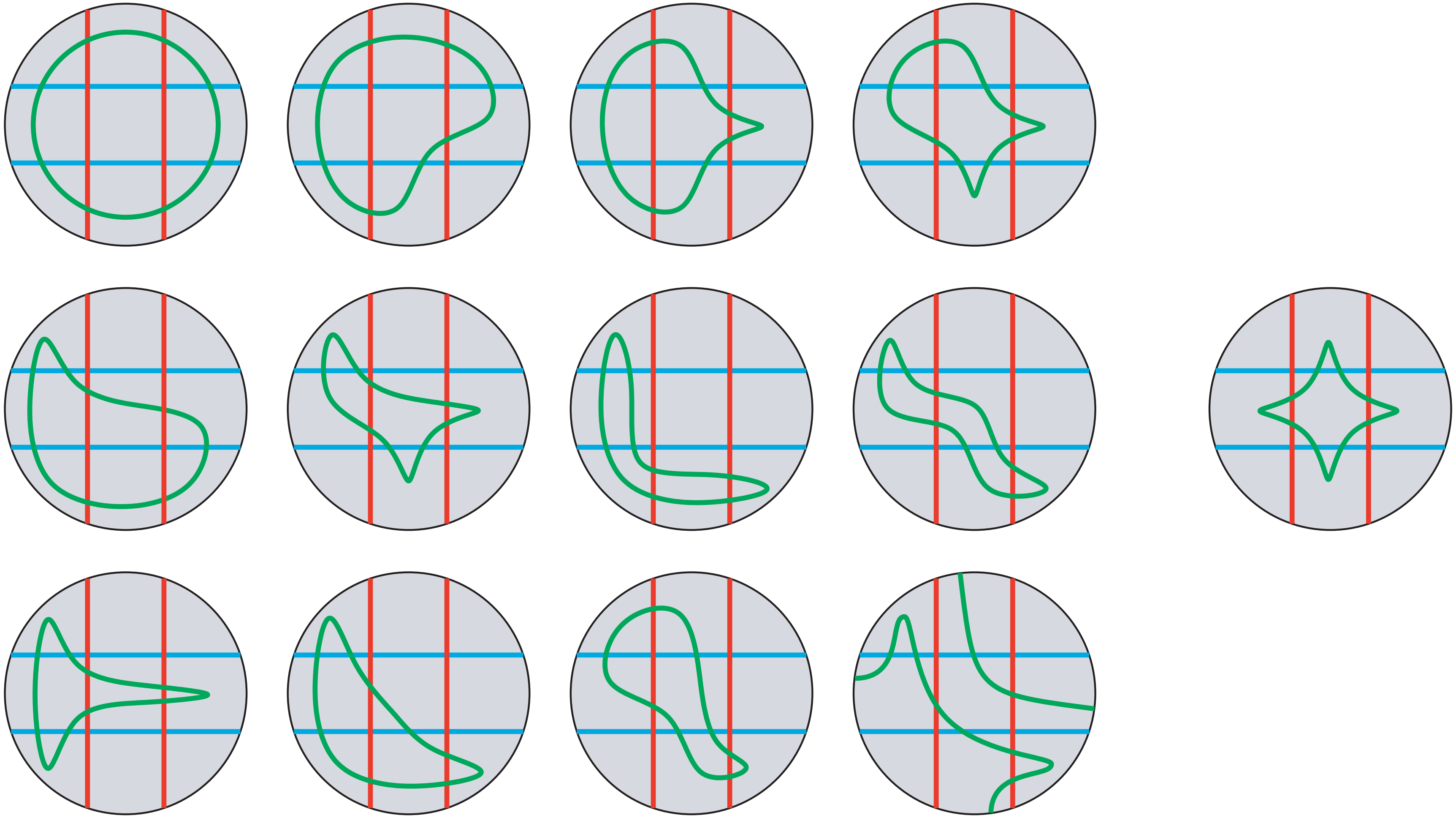
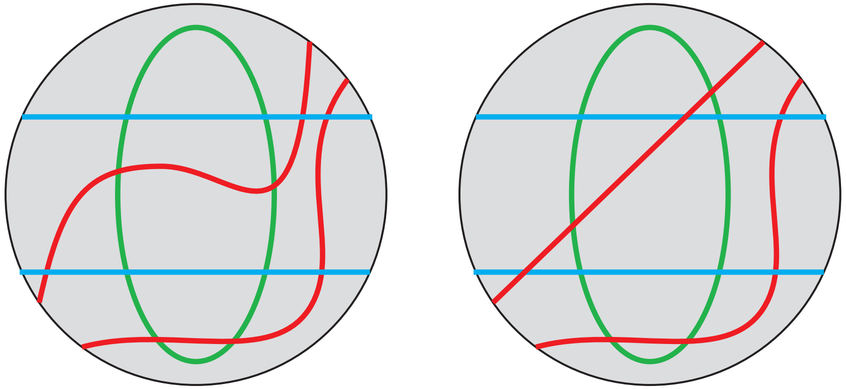
Comments on the computation of A_5 :

1. RUNNING TIME : $\simeq$ 3 weeks on 4 processors of 2 GHz
2. RESULT SIZE : complete data base represents $\simeq$ 15 Go

Example



MUTATIONS



A mutation is a homotopy of arrangements in which only one curve ℓ moves, sweeping a single vertex of the remaining arrangement $L \smallsetminus \{\ell\}$

THEOREM. Any two double pseudoline arrangements (with the same number of double pseudolines) are homotopic via a finite sequence of mutations, followed by a homeomorphism

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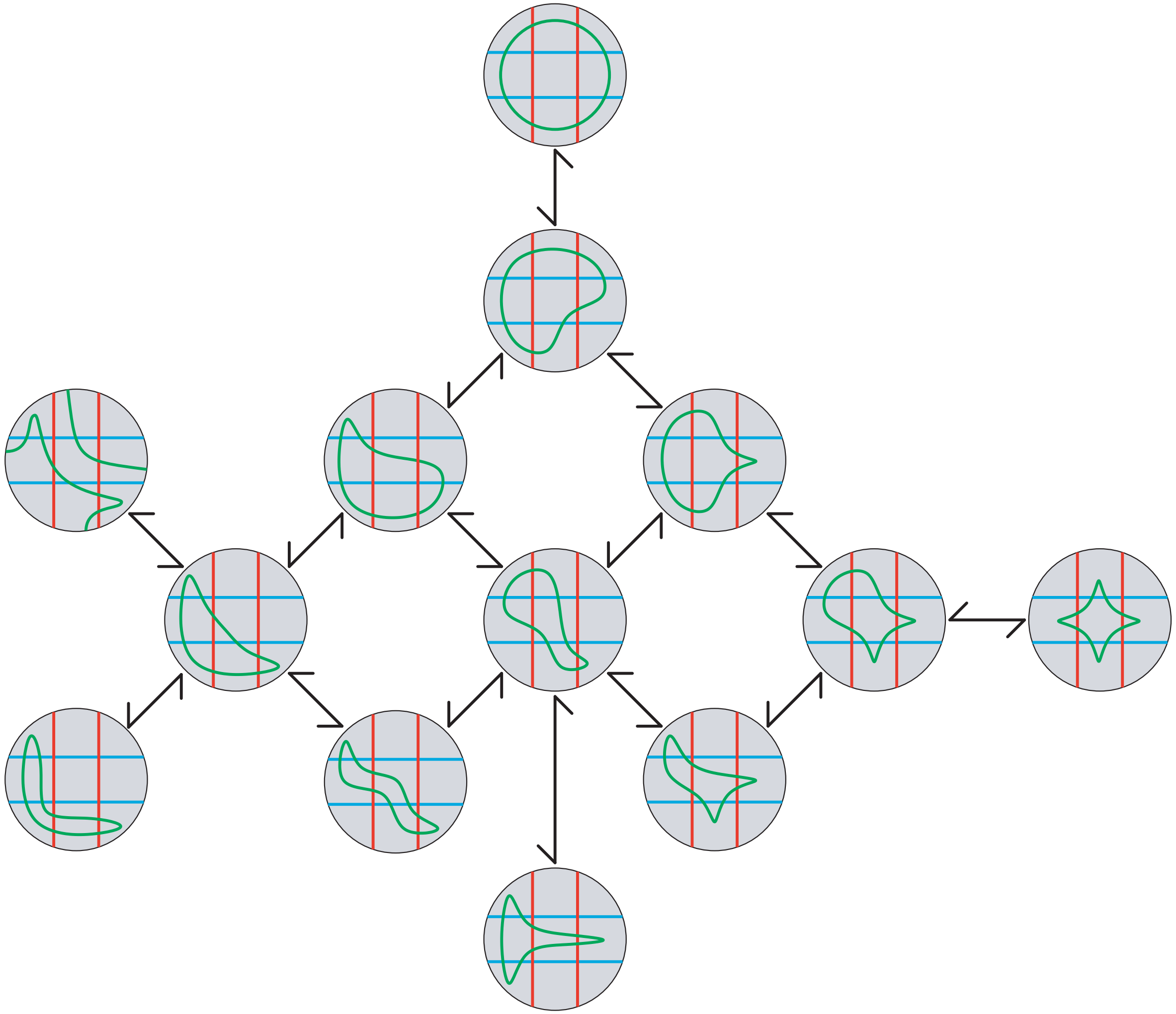

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$\Rightarrow$ first enumeration algorithm : exploring the graph of mutations
Fails for arrangements of five double pseudolines (RAM memory limitation)

MUTATIONS

Connectivity

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⇒ first enumeration algorithm : exploring the graph of mutations
Fails for arrangements of five double pseudolines (RAM memory limitation)

THEOREM. Any two double pseudoline arrangements containing a subarrangement L (and with the same number of double pseudolines) are homotopic via a finite sequence of mutations where L remains fixed, followed by a homeomorphism

V. PILAUD & M. POCCHIOLA, A relative homotopy theorem for arrangements of double pseudolines

INCREMENTAL ALGORITHM

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Notations

$\mathcal{A}_n$ = the set of isomorphism classes of arrangements of n double pseudolines

pointed arrangement $A^\bullet$ = arrangement A with a distinguished double pseudoline

$\mathcal{A}_n^\bullet$ = the set of isomorphism classes of pointed arrangements

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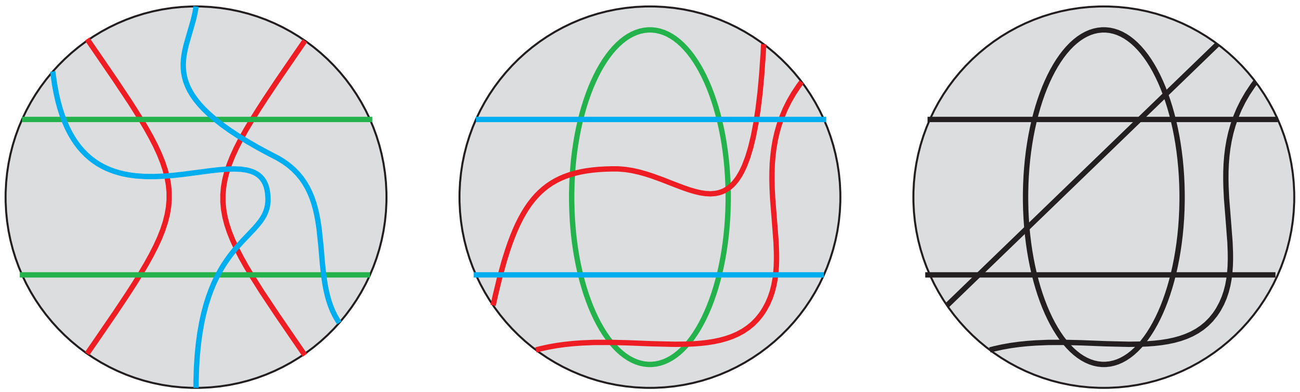
An isomorphism between two pointed arrangements $A^\bullet$ and $B^\bullet$ is a homeomorphism of $\mathcal{P}$ that sends $A^\bullet$ on $B^\bullet$ respecting the distinguished double pseudoline



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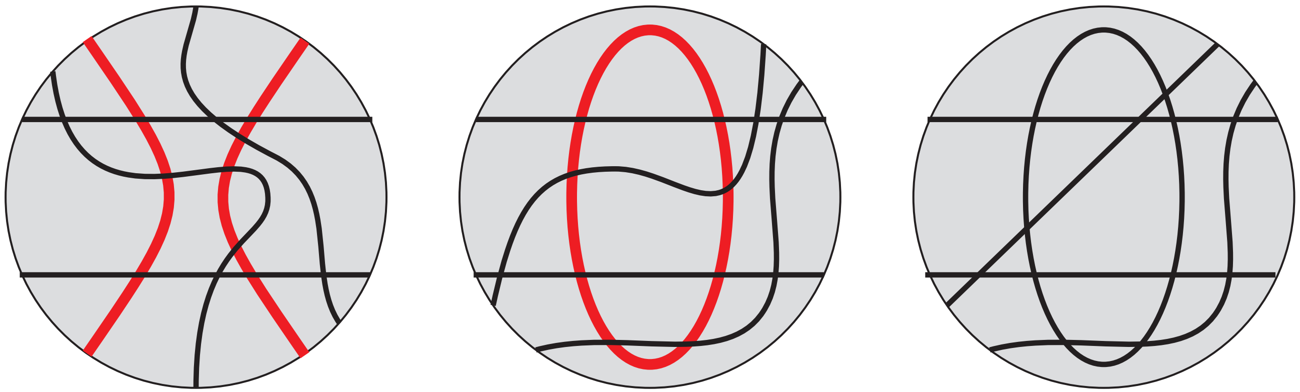

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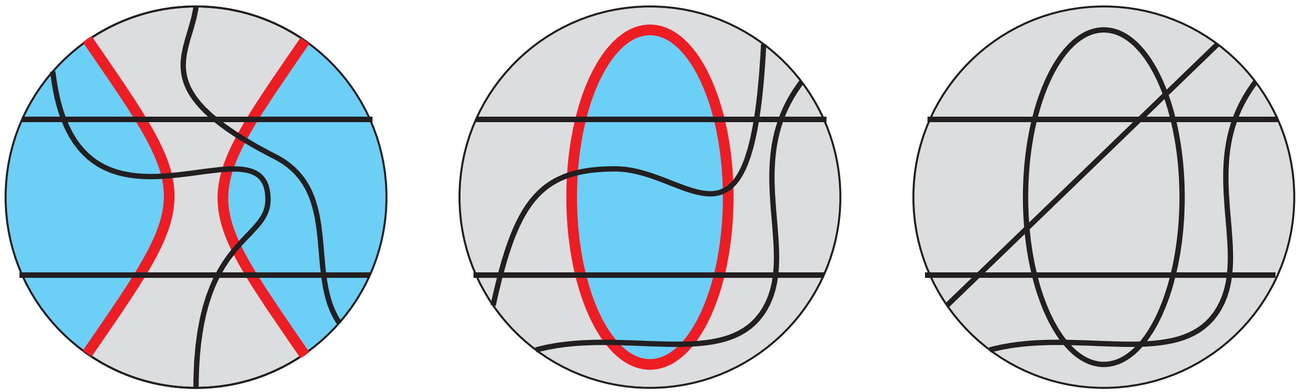

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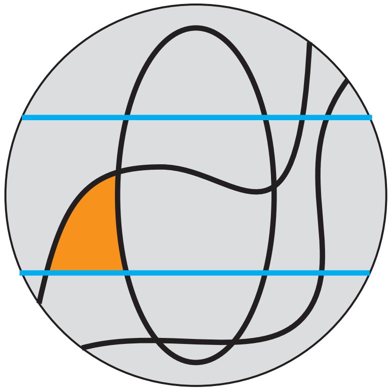
R_i is the set of arrangements of $\mathcal{A}_{n+1}$ whose first subarrangement among $\{a_1, \ldots, a_p\}$ is a_i.

$$\boxed{\mathcal{A}_{n+1} = \bigsqcup_{i=1}^{p} R_i}$$

INCREMENTAL ALGORITHM

Adding a double pseudoline

How can we add a double pseudoline to an arrangement A?



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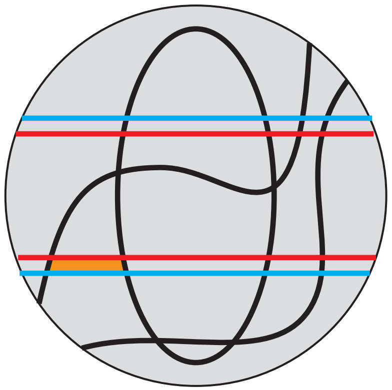

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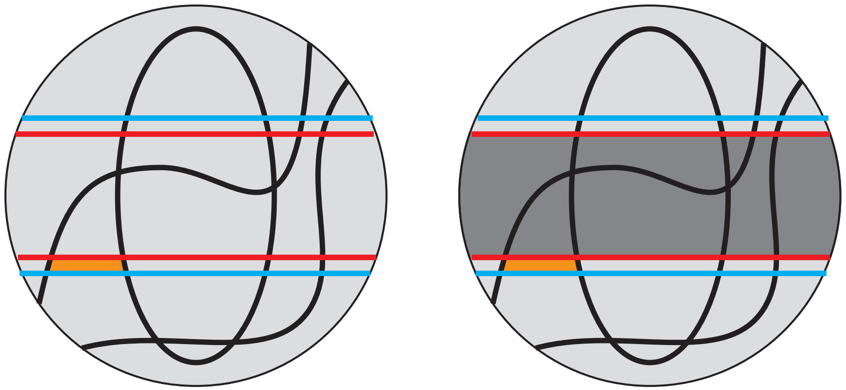

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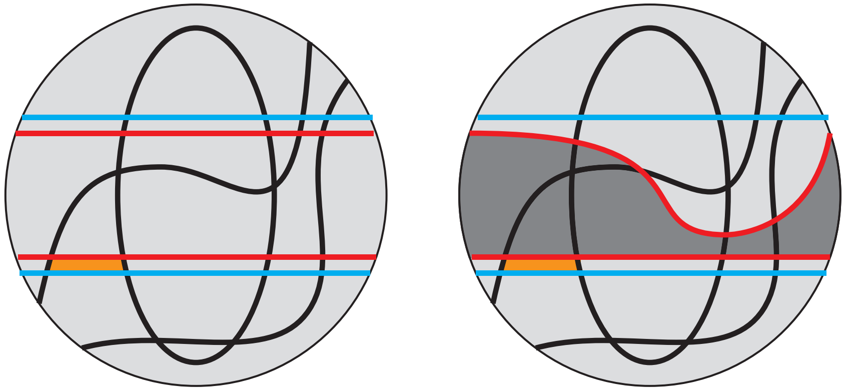

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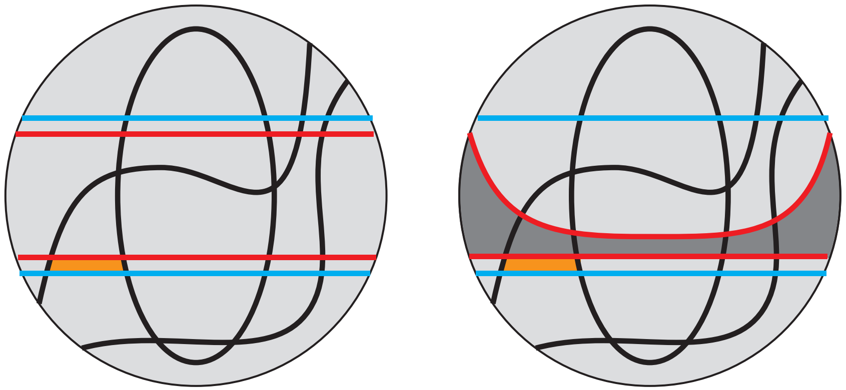

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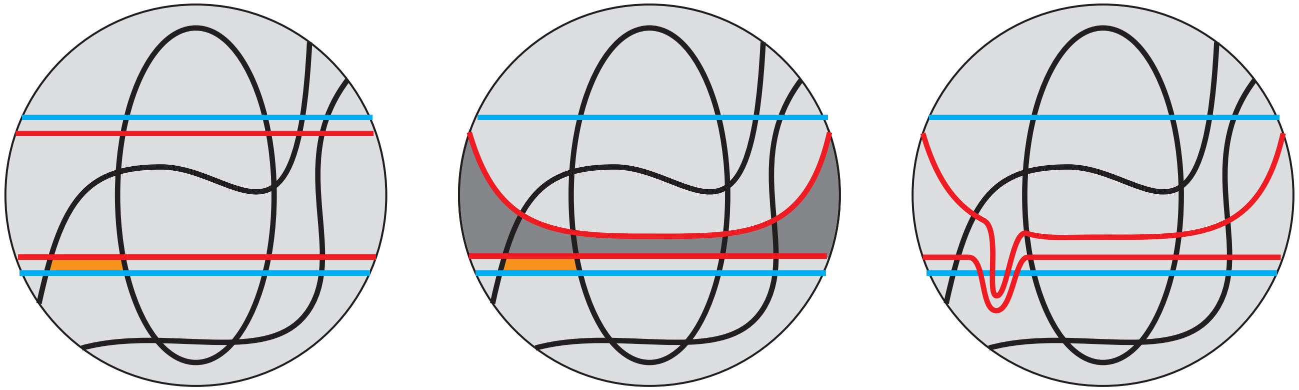

1. choose an arbitrary double pseudoline and duplicate it
2. pump the added double pseudoline ℓ until no vertex of A lies in M_ℓ
3. add four crossings

TWO OPEN PROBLEMS

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Axiomatization

Pseudoline arrangements admit simple axiomatizations :

(i) few axioms

(ii) dealing with configurations of at most five pseudolines

Enumeration = complete list of arrangements of at most five double pseudolines
= axiomatization

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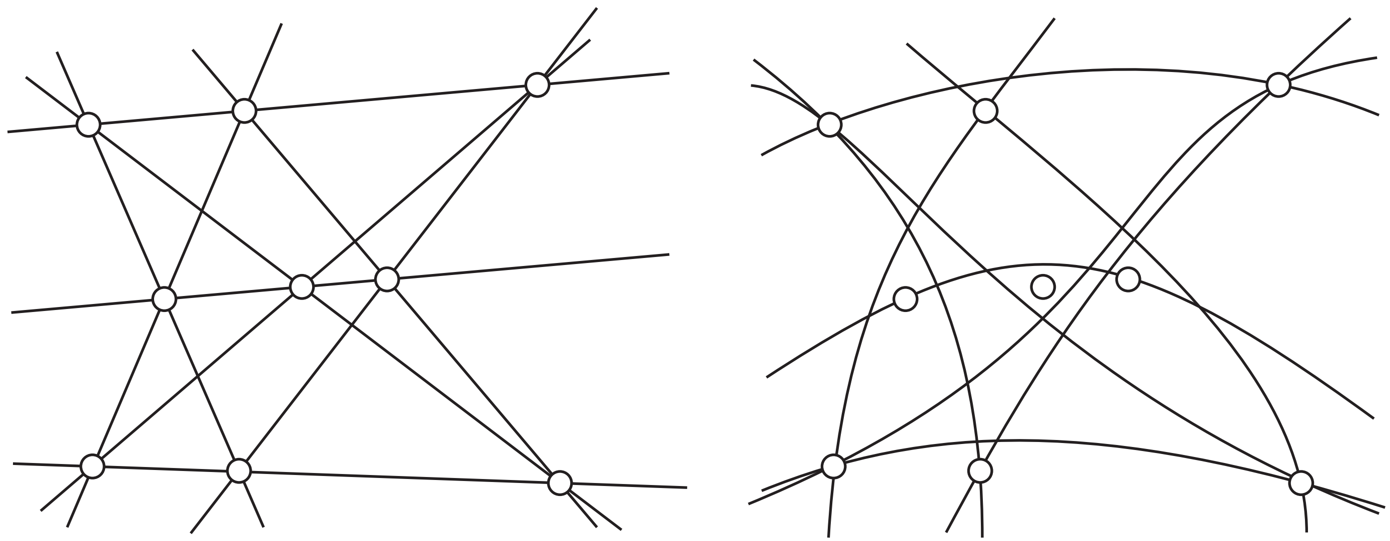
Well, we have about 200 000 000 axioms

Is it possible to algorithmically reduce our axiomatization ?

TWO OPEN PROBLEMS

Realizability

Certain pseudoline arrangements are not realizable in the Euclidean plane



Inflating pseudolines into thin double pseudolines in such an arrangement give rize to non-realizable double pseudoline arrangement.

Are there smaller examples ?

THANK YOU.