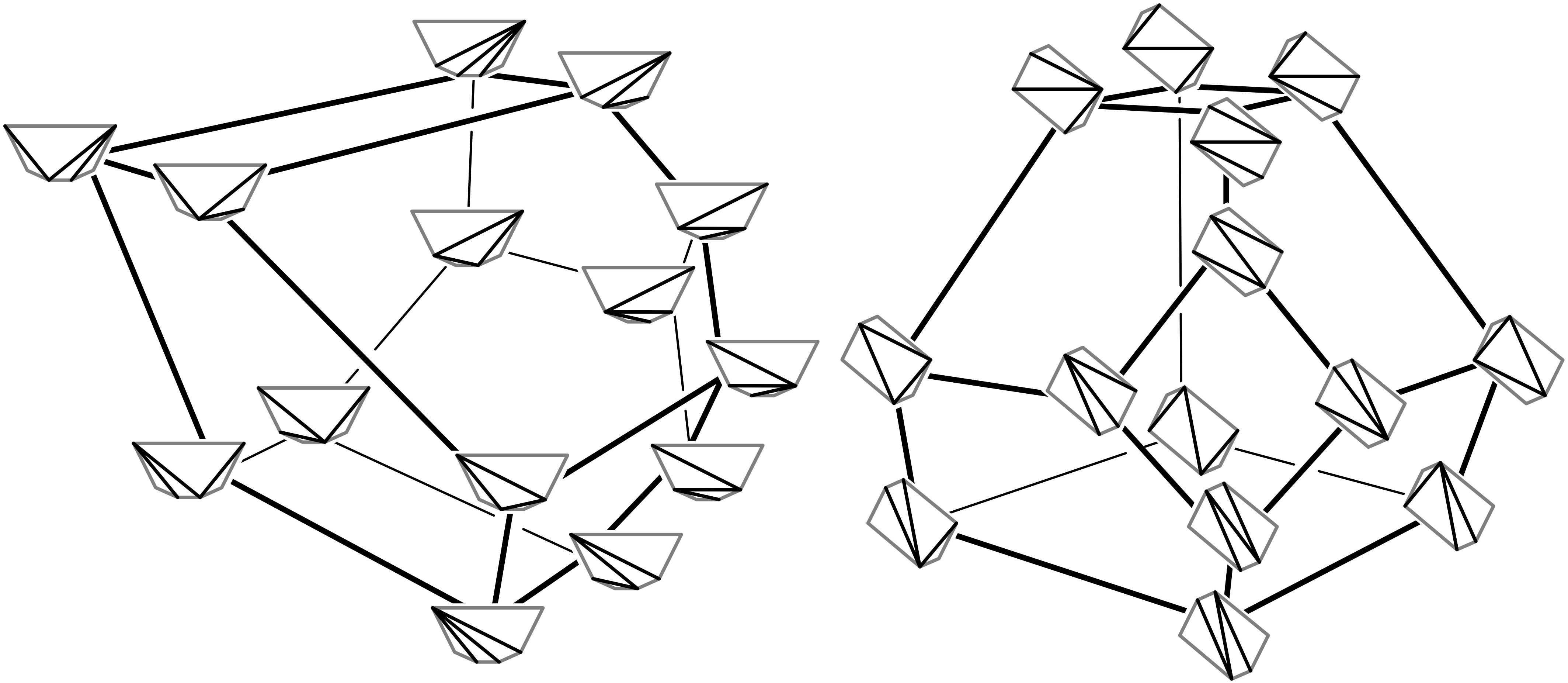

VERTEX BARYCENTERS OF GENERALIZED ASSOCIAHEDRA

Vincent PILAUD (CNRS & LIX, École Polytechnique)
Christian STUMP (Universität Hannover)

ASSOCIAHEDRON

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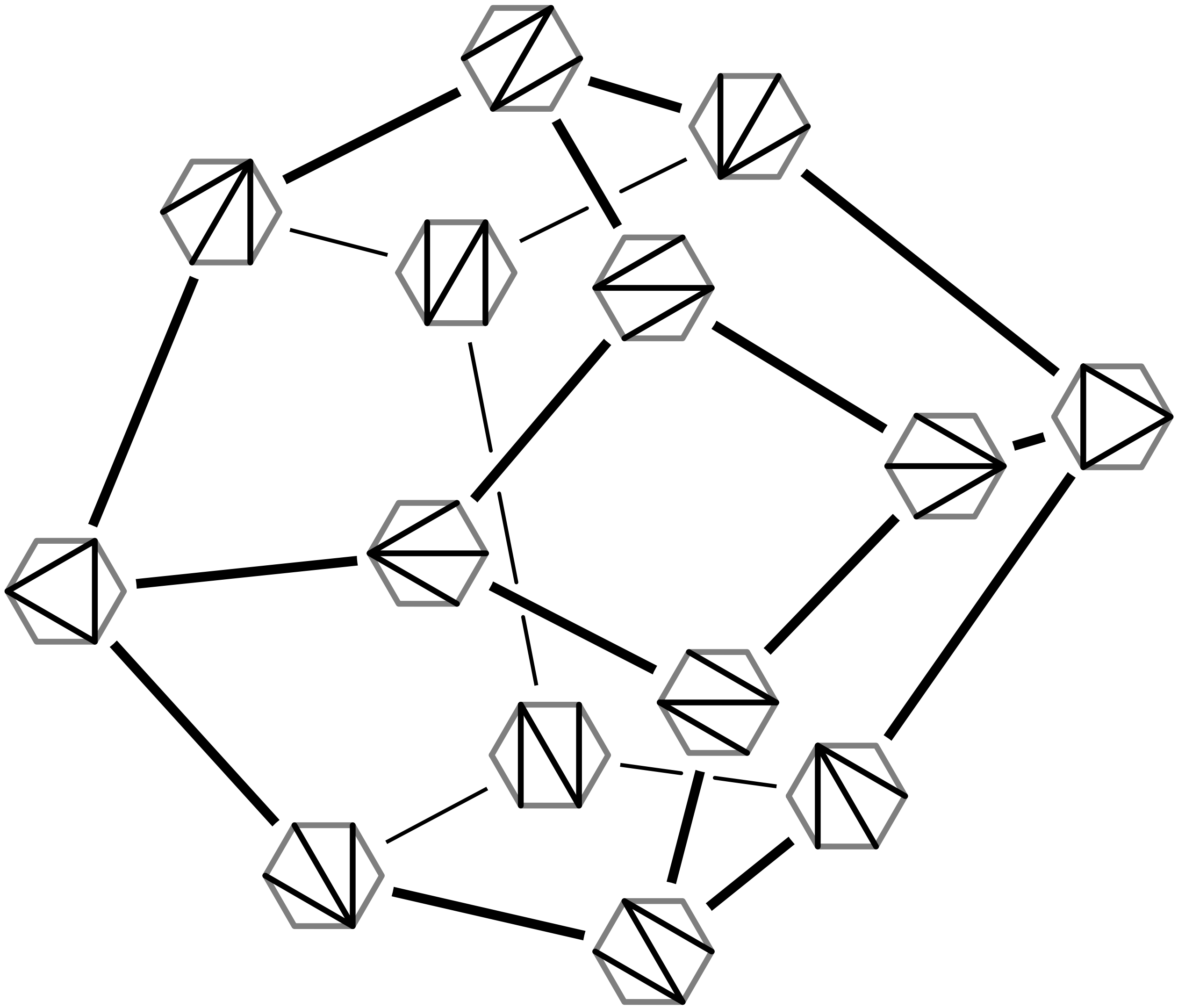
RELATED STRUCTURES

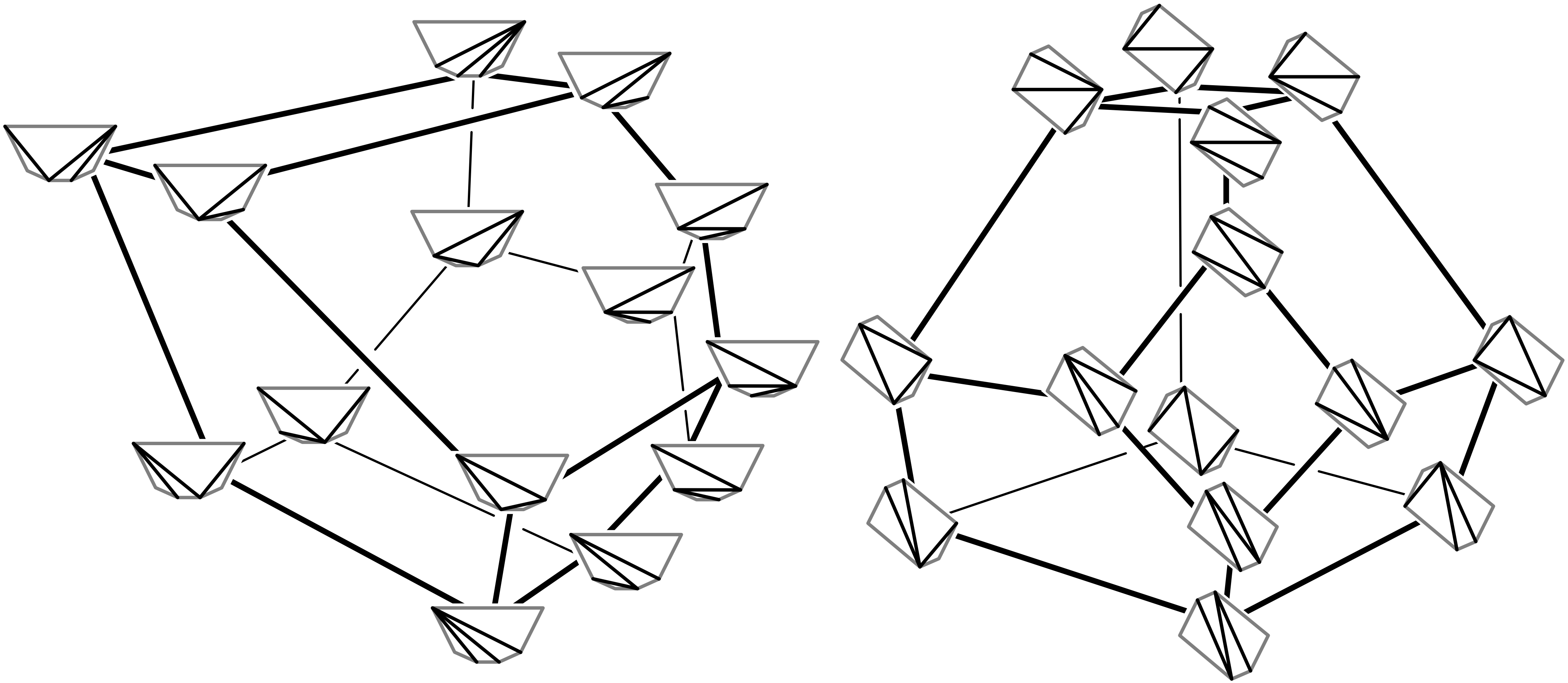
ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex n-gon, ordered by reverse inclusion.



VARIOUS ASSOCIAHEDRA

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex n-gon, ordered by reverse inclusion.

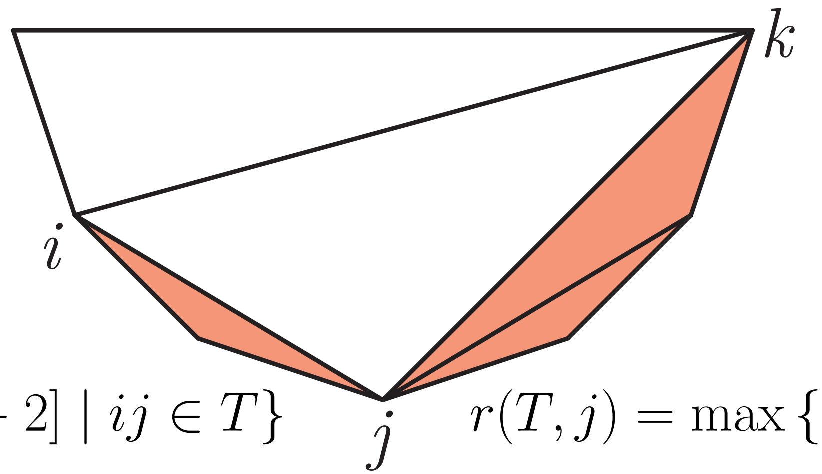


Lee ('89), Gel'fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), ..., Ceballos-Santos-Ziegler ('11)
Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12$^+$)

LODAY'S ASSOCIAHEDRON

Loday's associahedron $= \operatorname{conv}\{L(T) \mid T$ triangulation of the $(n+3)$-gon$\}$, where

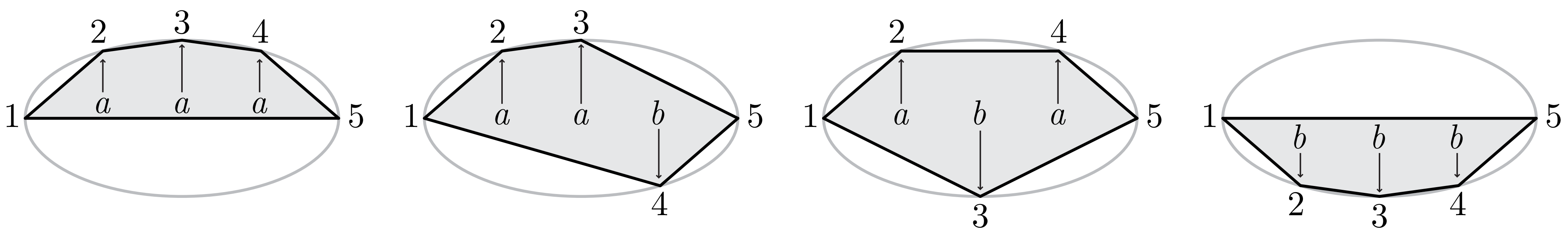
$$L(T) = \big(\ell(T,j)\cdot r(T,j)\big)_{j\in[n+1]}$$



$\ell(T,j) = j - \min\{i \in [0, j-2] \mid ij \in T\}$ $\quad$ $r(T,j) = \max\{k \in [j+2, n+2] \mid jk \in T\} - j$

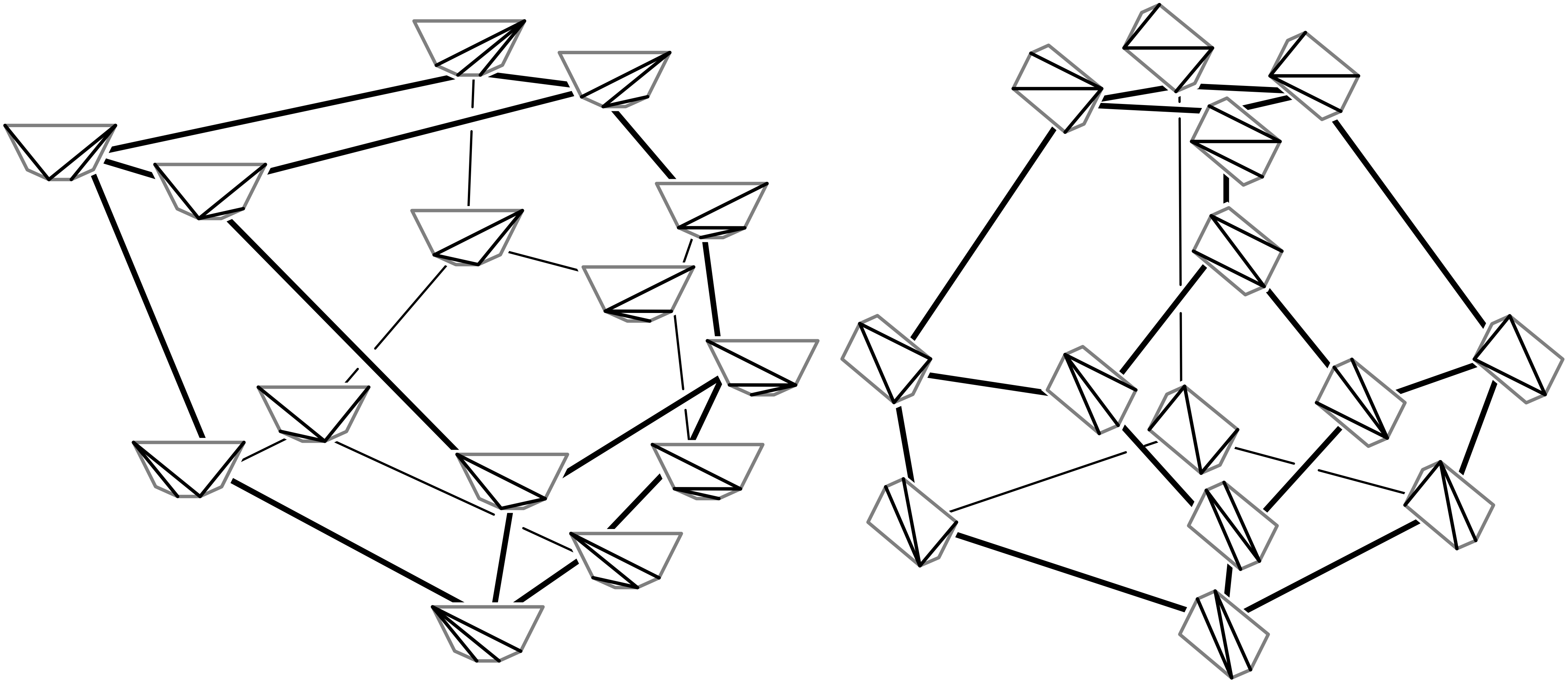
Loday, *Realization of the Stasheff polytope* ('04)

Can also replace this $(n+3)$-gon by others:



Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

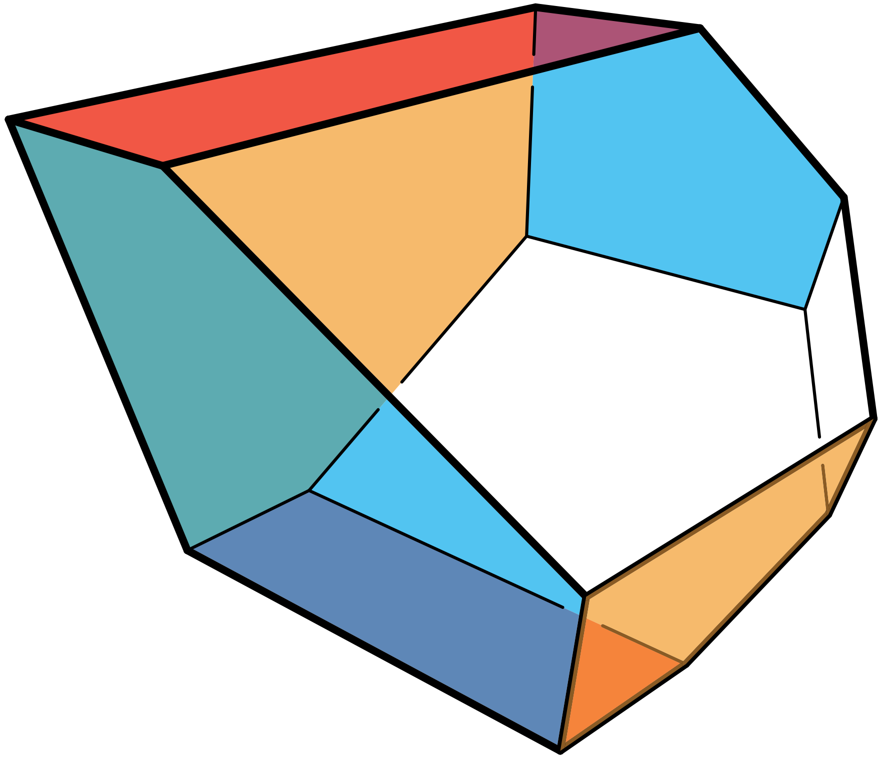
HOHLWEG & LANGE'S ASSOCIAHEDRA

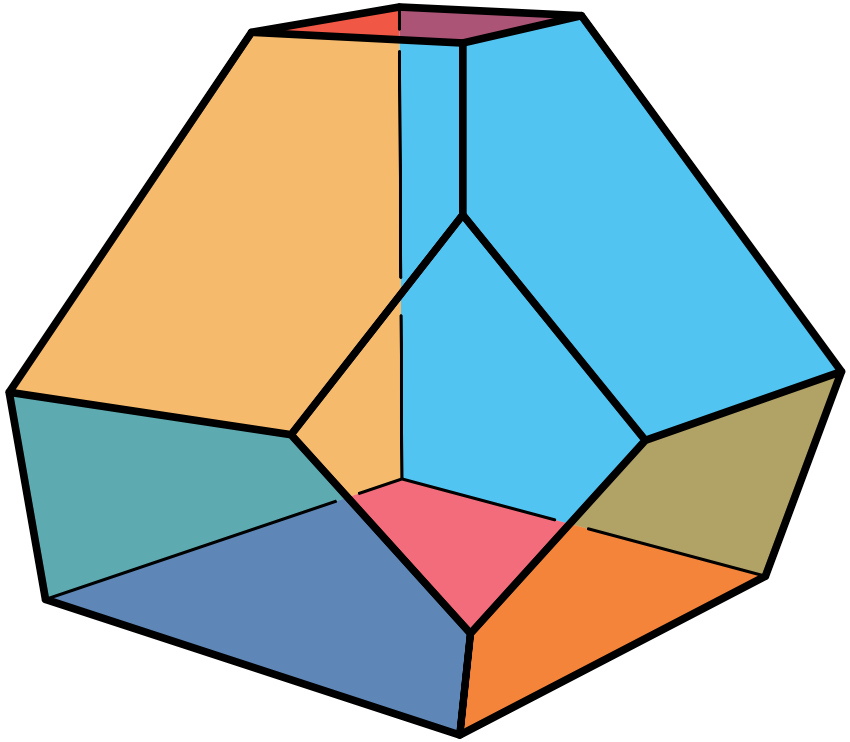


Loday, *Realization of the Stasheff polytope* ('04)

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

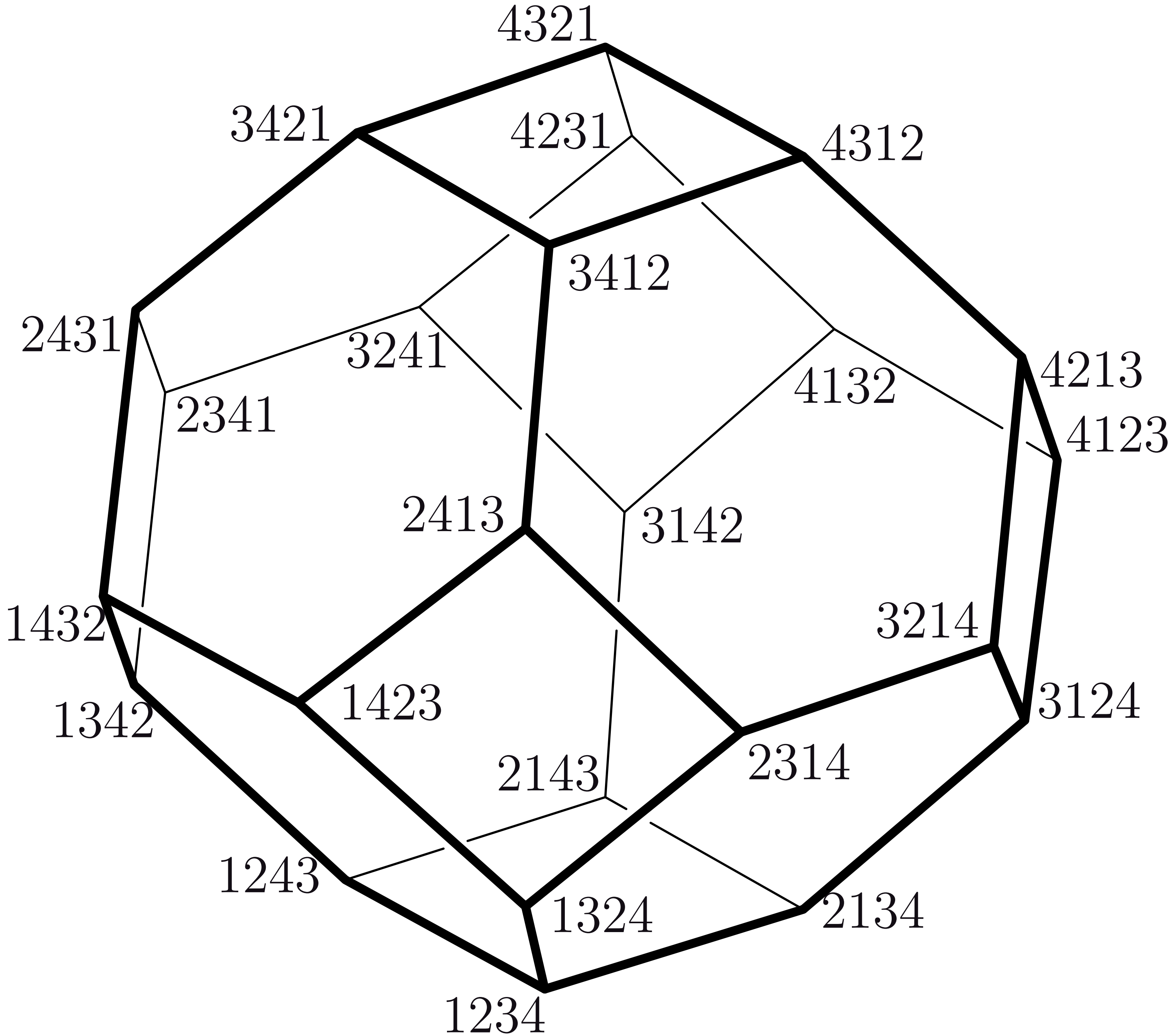
HOHLWEG & LANGE'S ASSOCIAHEDRA



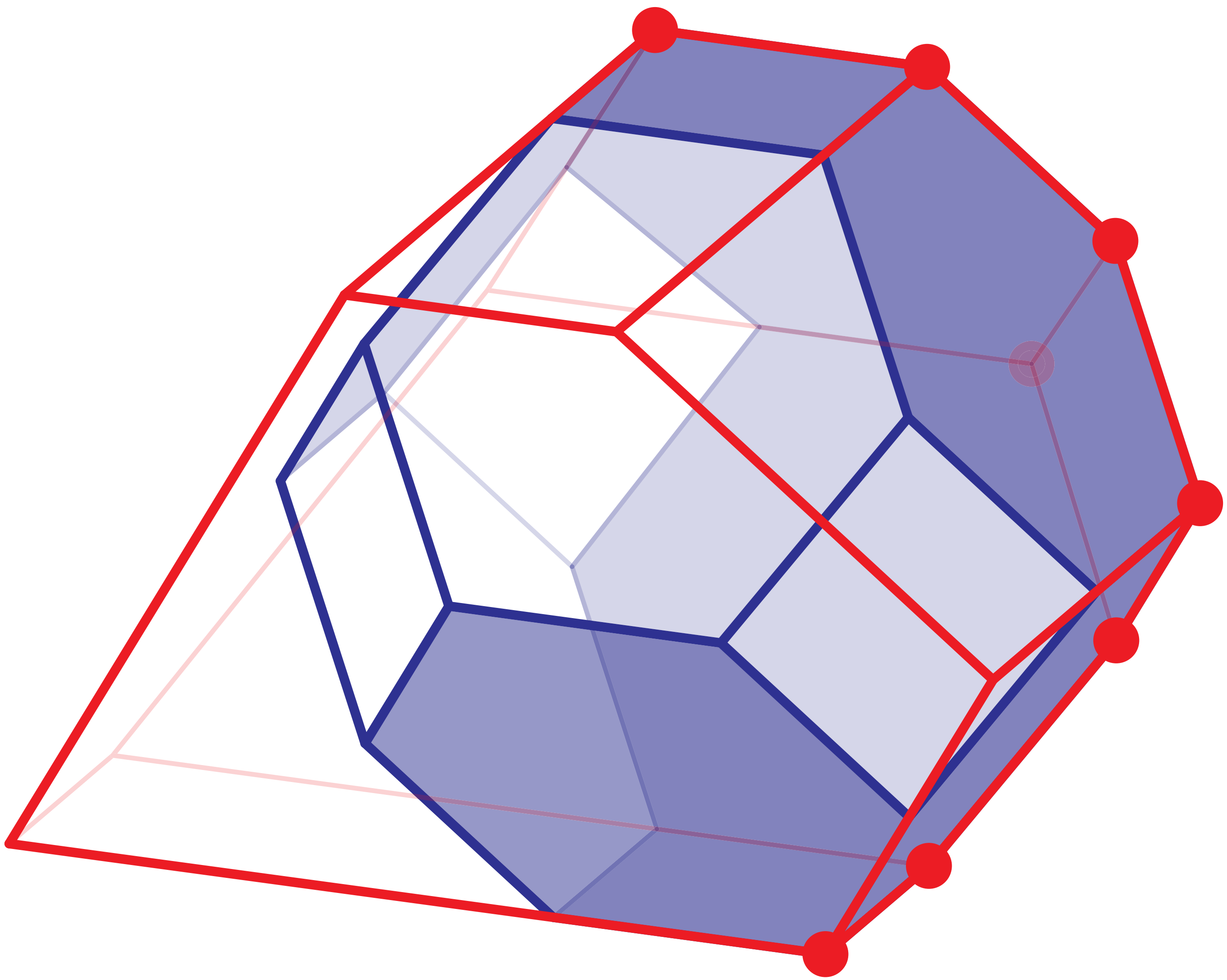


Loday, *Realization of the Stasheff polytope* ('04)
Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11$^+$)

PERMUTAHEDRON



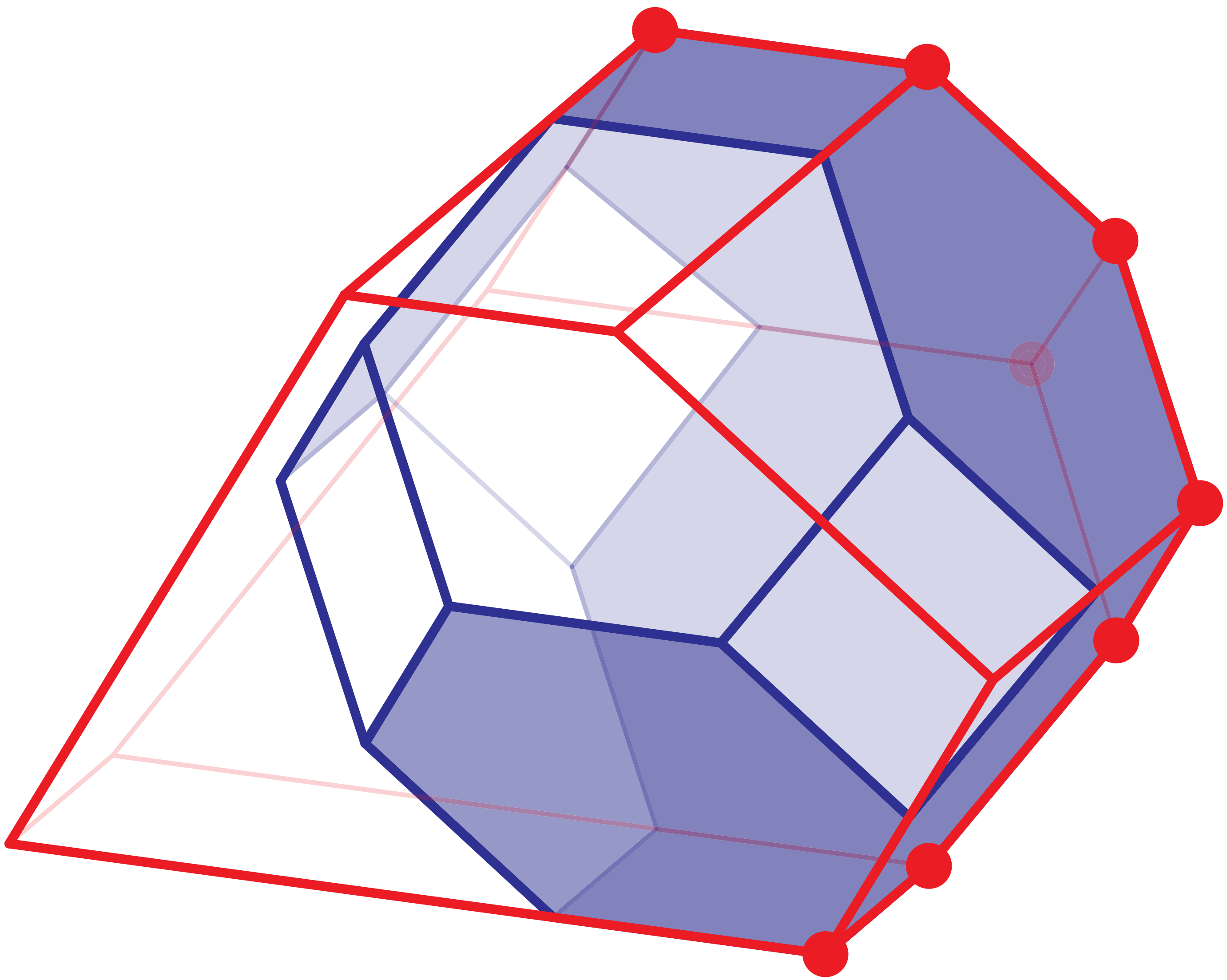
ASSOCIAHEDRA FROM THE PERMUTAHEDRON



Associahedron from permutahedron = remove facets not containing “singletons”.

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

BARYCENTERS



THEOREM. All Hohlweg & Lange's associahedra have the origin for vertex barycenter.

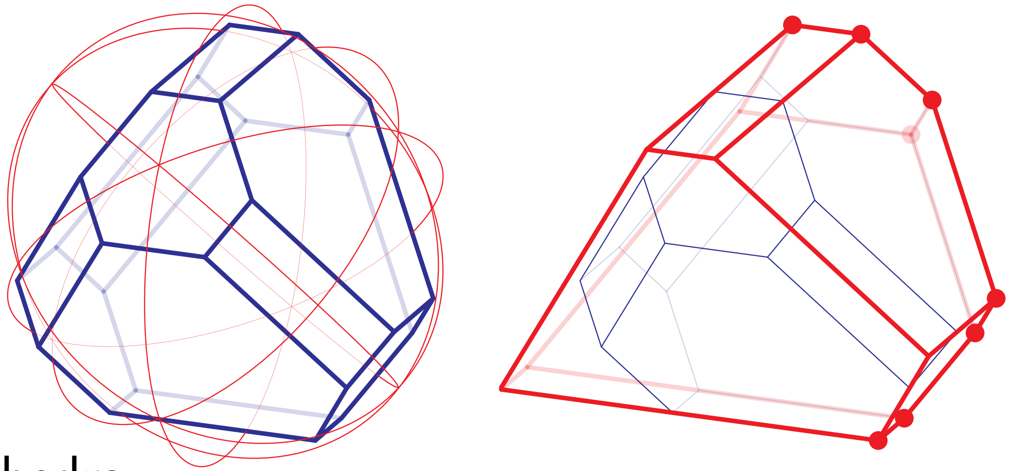
Hohlweg-Lortie-Raymond, *The center of gravity of the associahedron and of the permutahedron are the same* ('07)

BARYCENTERS

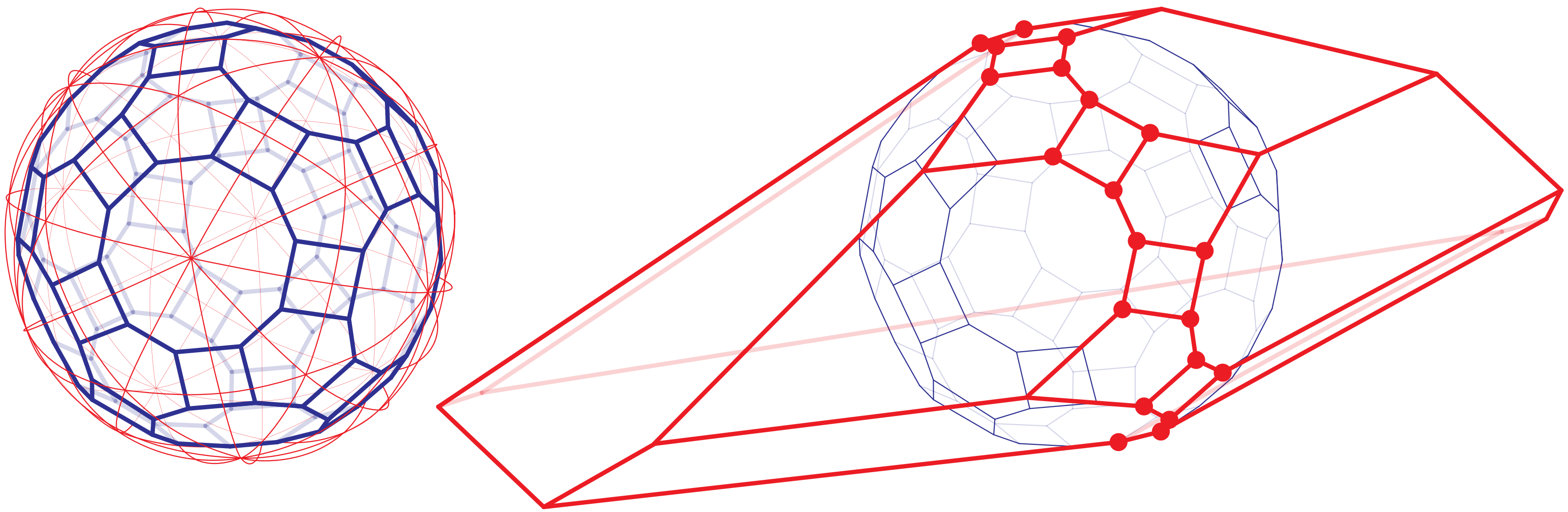
THEOREM. All Hohlweg & Lange's associahedra have the origin for vertex barycenter.

We give an alternative proof of this result, which extends in two directions:

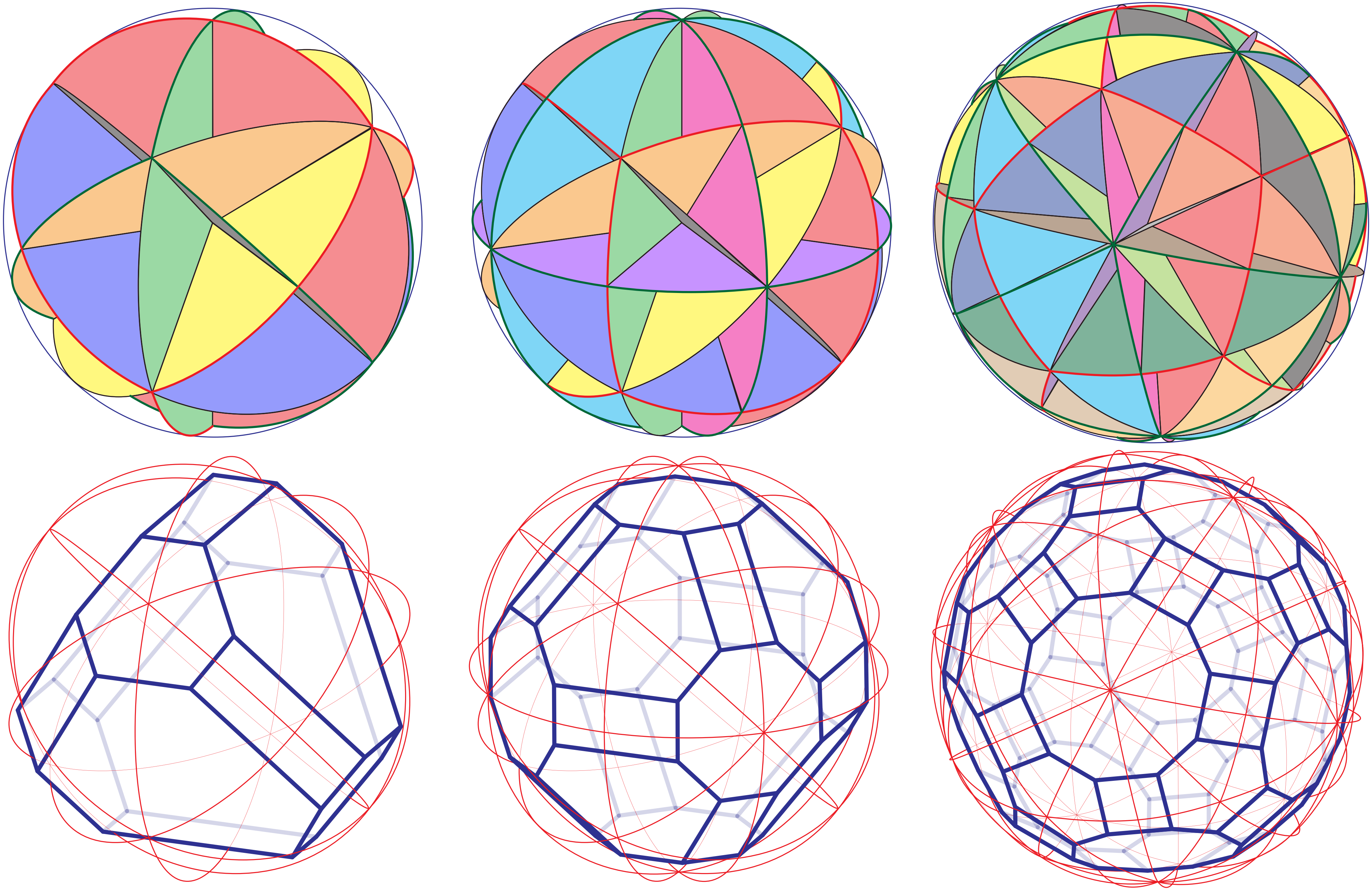
1. Fairly balanced associahedra:



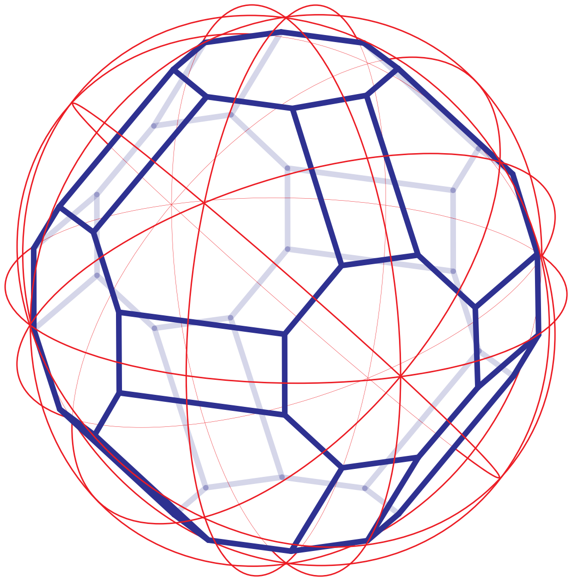
2. Generalized associahedra:

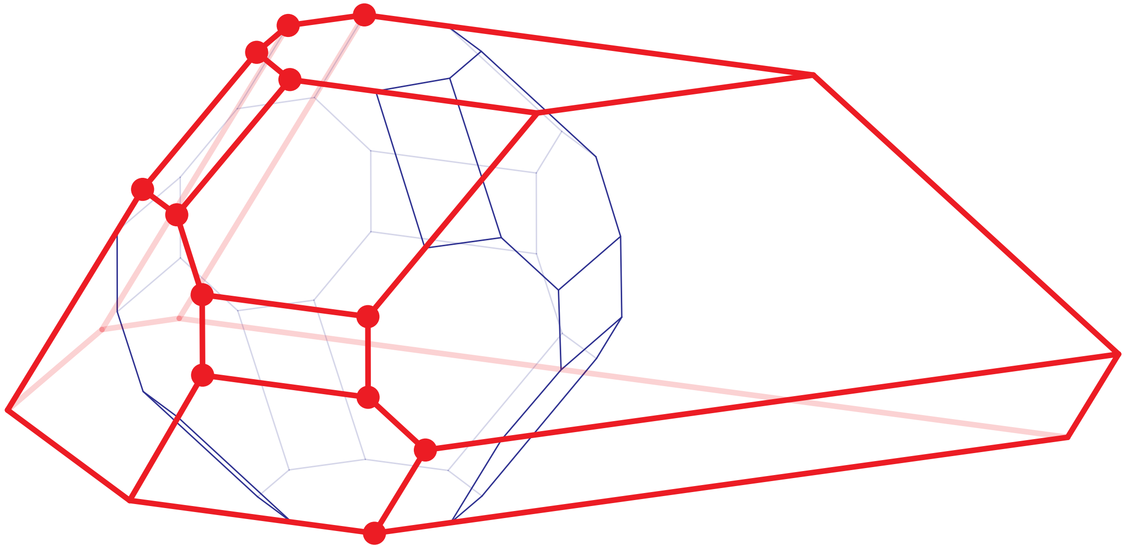


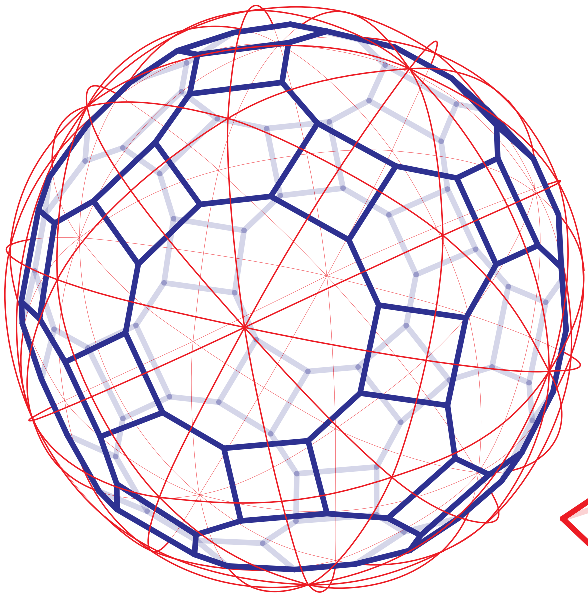
FINITE COXETER GROUPS

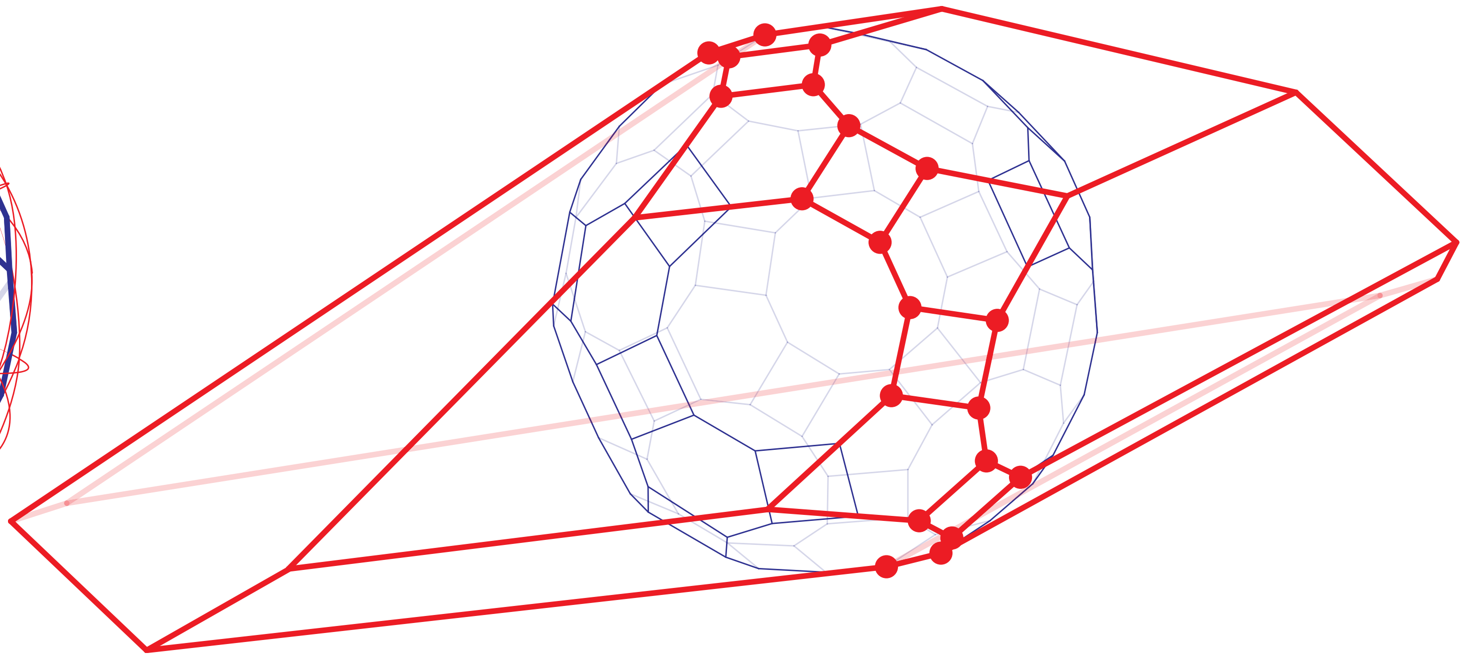


GENERALIZED ASSOCIAHEDRA







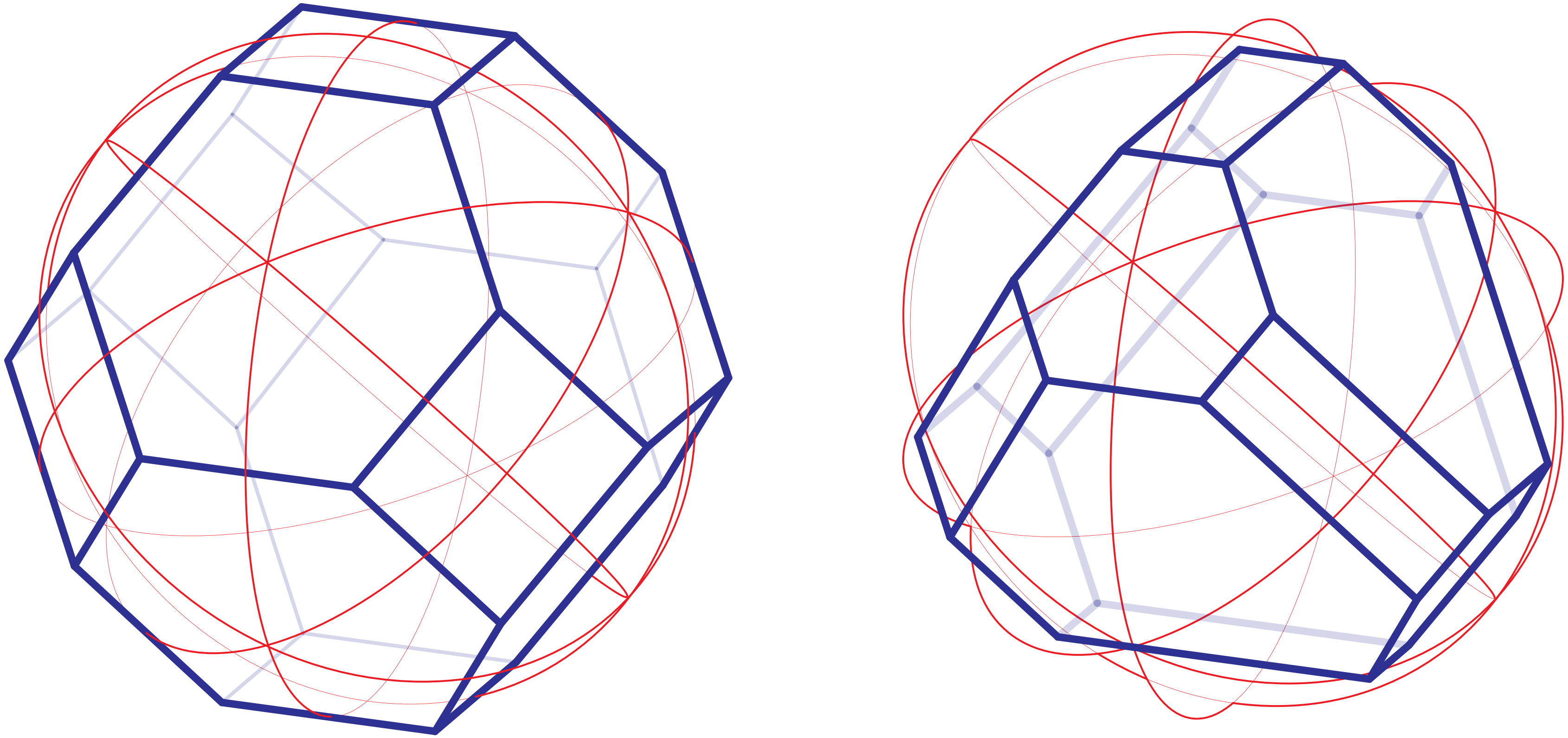


Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized associahedra* ('02)

Hohlweg-Lange-Thomas, *Permutahedra and generalized associahedra* ('11)

OUR RESULT

THEOREM. For any finite Coxeter group W, any Coxeter element c, any fairly-balanced point u, the vertex barycenters of the generalized associahedron $\mathrm{Asso}_c^u(W)$ and of the permutahedron $\mathrm{Perm}^u(W)$ coincide.



The point u is fairly balanced if $w_\circ(u) = -u$, where $w_\circ$ is the longest element in W.

ASSOCIAHEDRON

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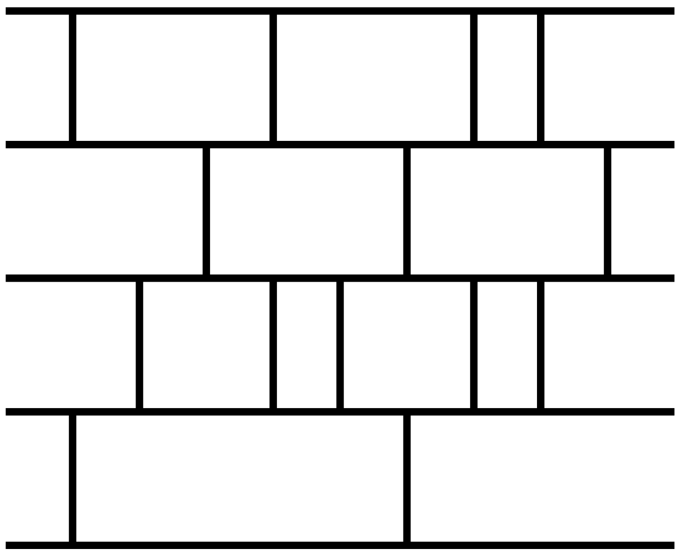
SORTING NETWORKS

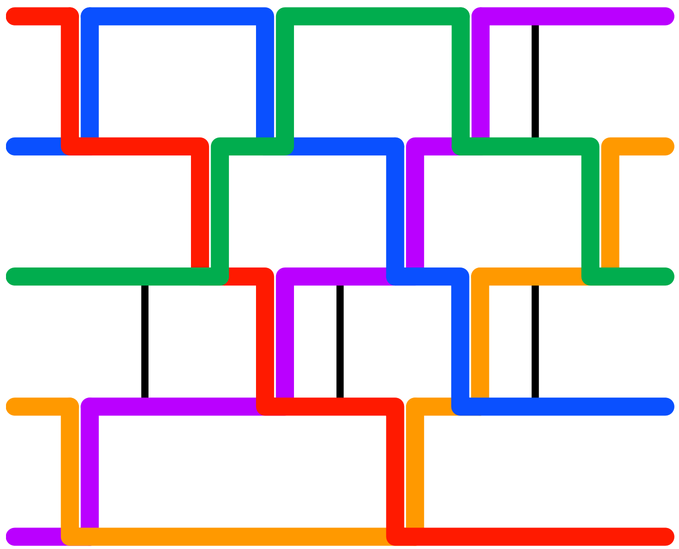
PRIMITIVE SORTING NETWORKS



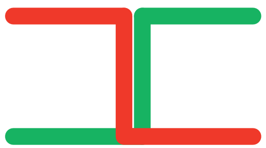
network $\mathcal{N} = n$ horizontal levels and m vertical commutators

bricks of $\mathcal{N} =$ bounded cells

PSEUDOLINE ARRANGEMENTS ON A NETWORK



pseudoline = abscissa-monotone path

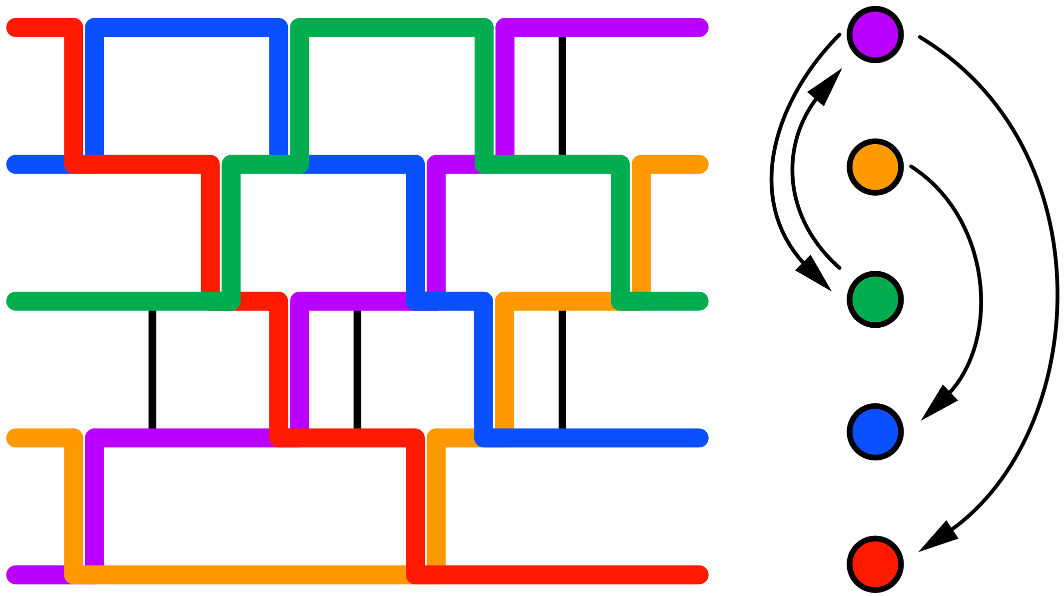
crossing =  contact = 

pseudoline arrangement (with contacts) = n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, possibly some contacts, and no other intersection

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

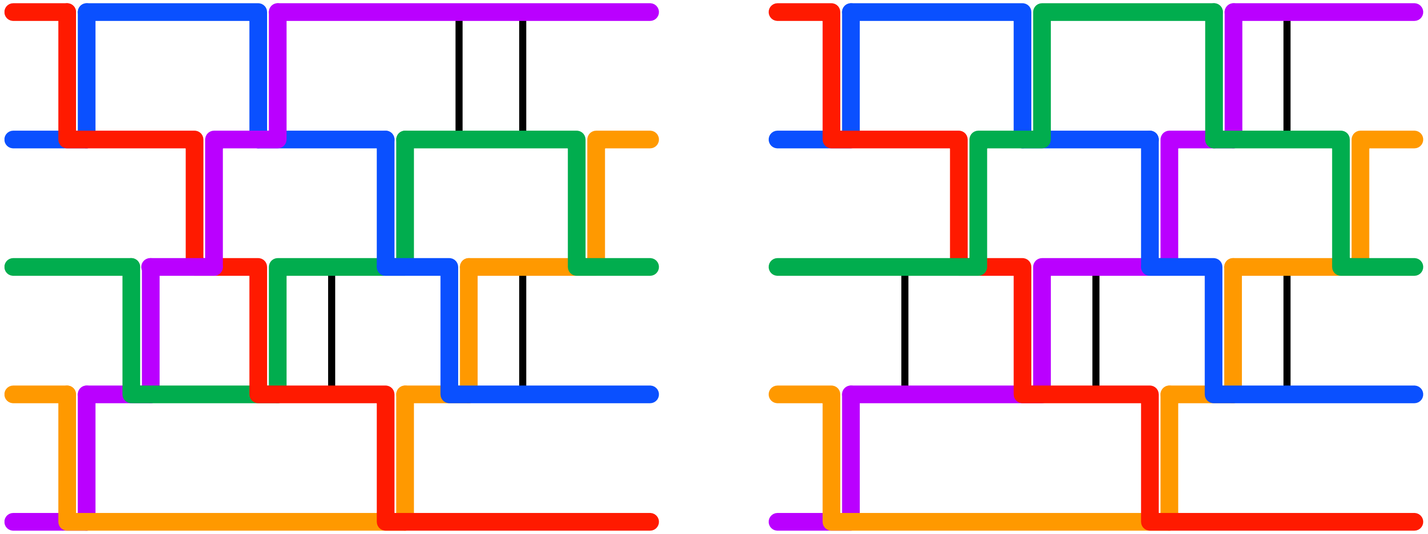
contact graph $\Lambda^{\#}$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ, and
- an arc for each contact of Λ oriented from top to bottom



FLIPS

flip = exchange an arbitrary contact with the corresponding crossing



Combinatorial and geometric properties of the graph of flips $G(\mathcal{N})$?

Knutson-Miller, *Subword complexes in Coxeter groups* ('04)

P.-Pocchiola, *Multitriangulations, pseudotriangulations and sorting networks* ('12)

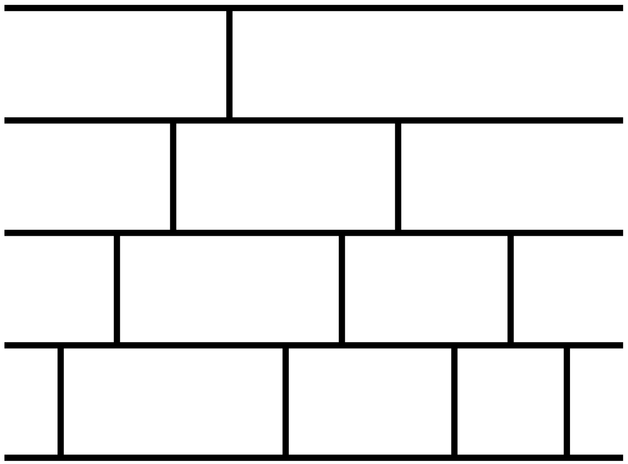
P.-Santos, *The brick polytope of a sorting network* ('12)

Ceballos-Labbé-Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra* ('12$^+$)

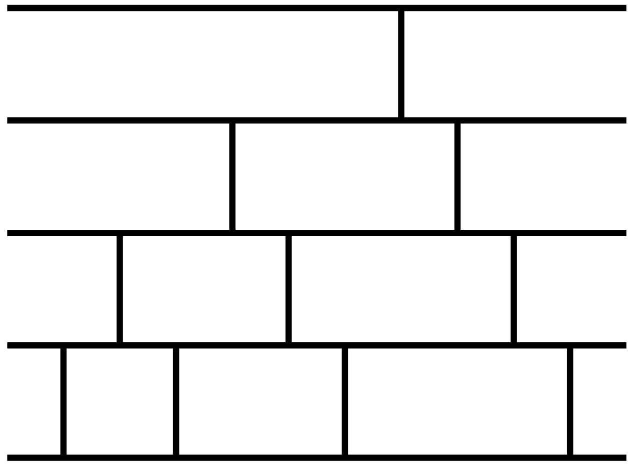
P.-Stump, *Brick polytopes of spherical subword complexes: a new approach to generalized associahedra* ('12$^+$)

P.-Stump, *EL-labelings and canonical spanning trees for subword complexes* ('12$^+$)

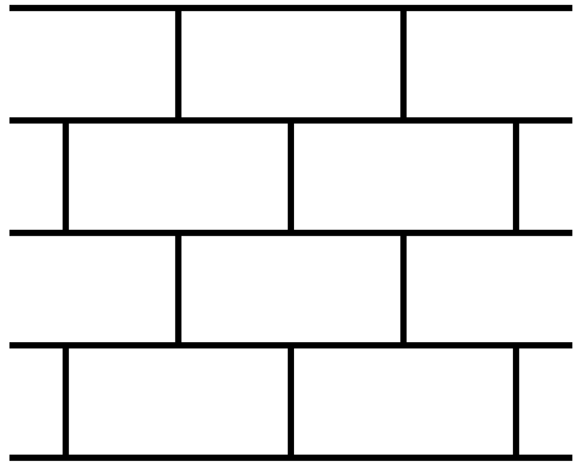
MINIMAL SORTING NETWORKS



bubble sort



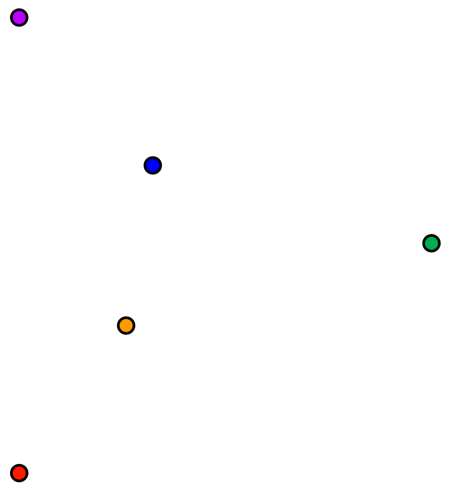
insertion sort



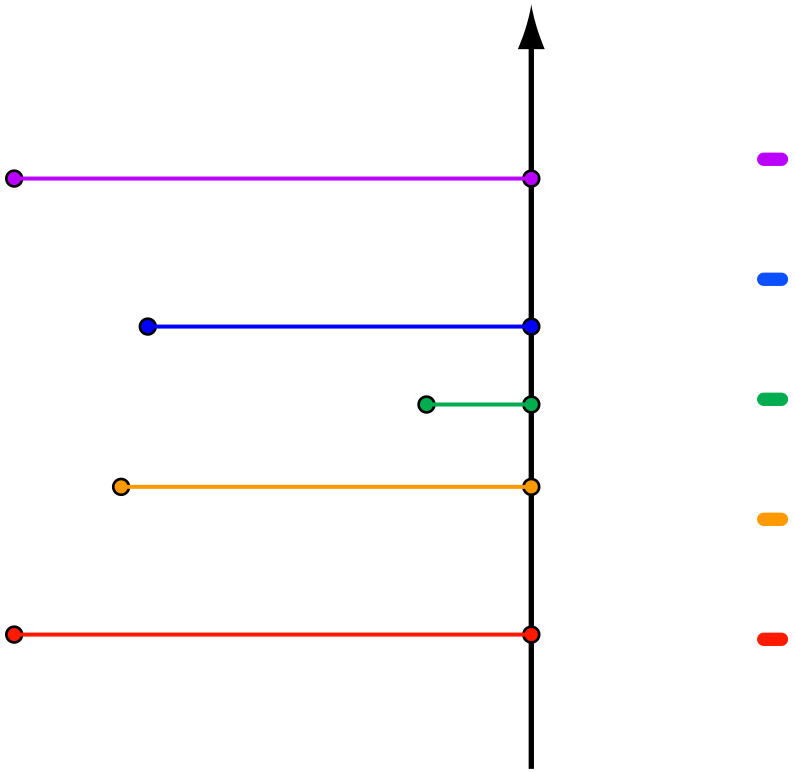
even-odd sort

Knuth, *The art of Computer Programming*, Vol. 3 Sorting and Searching ('97)

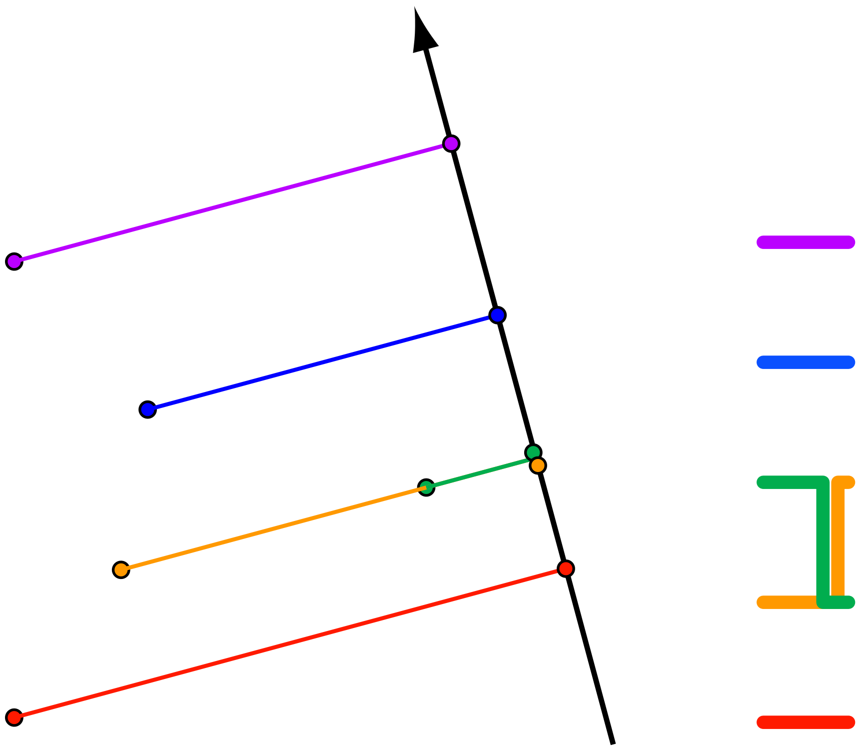
POINT SETS & MINIMAL SORTING NETWORKS



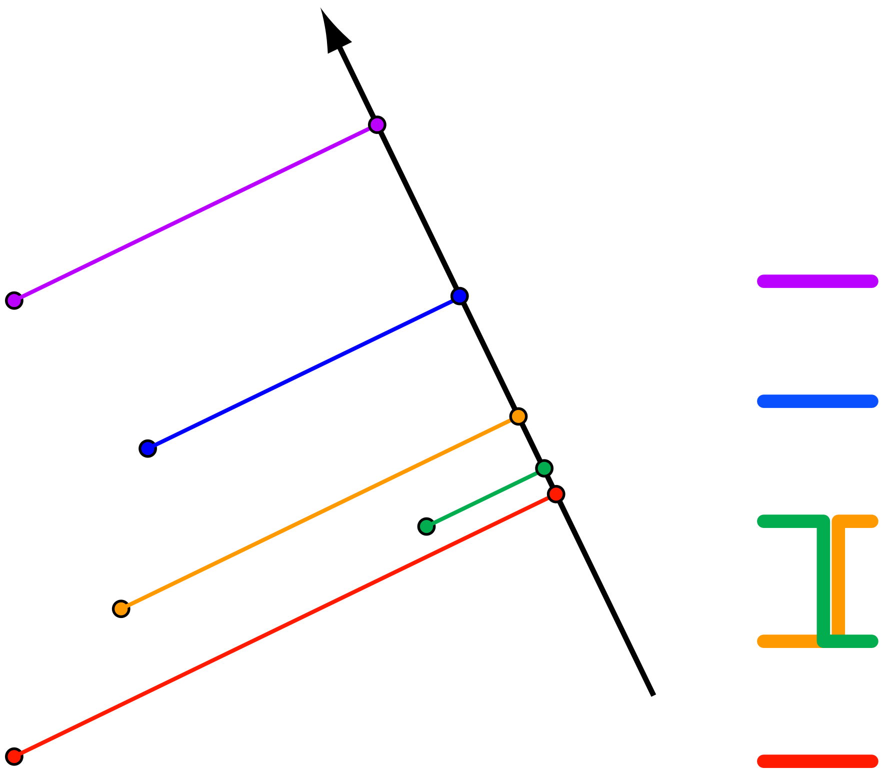
POINT SETS & MINIMAL SORTING NETWORKS



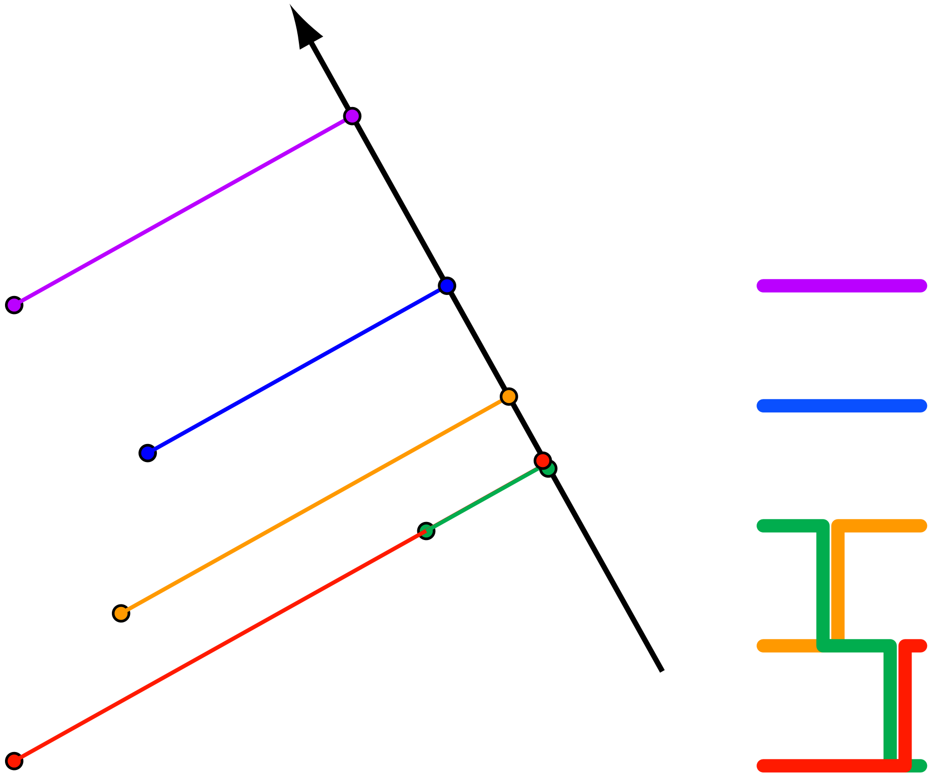
POINT SETS & MINIMAL SORTING NETWORKS



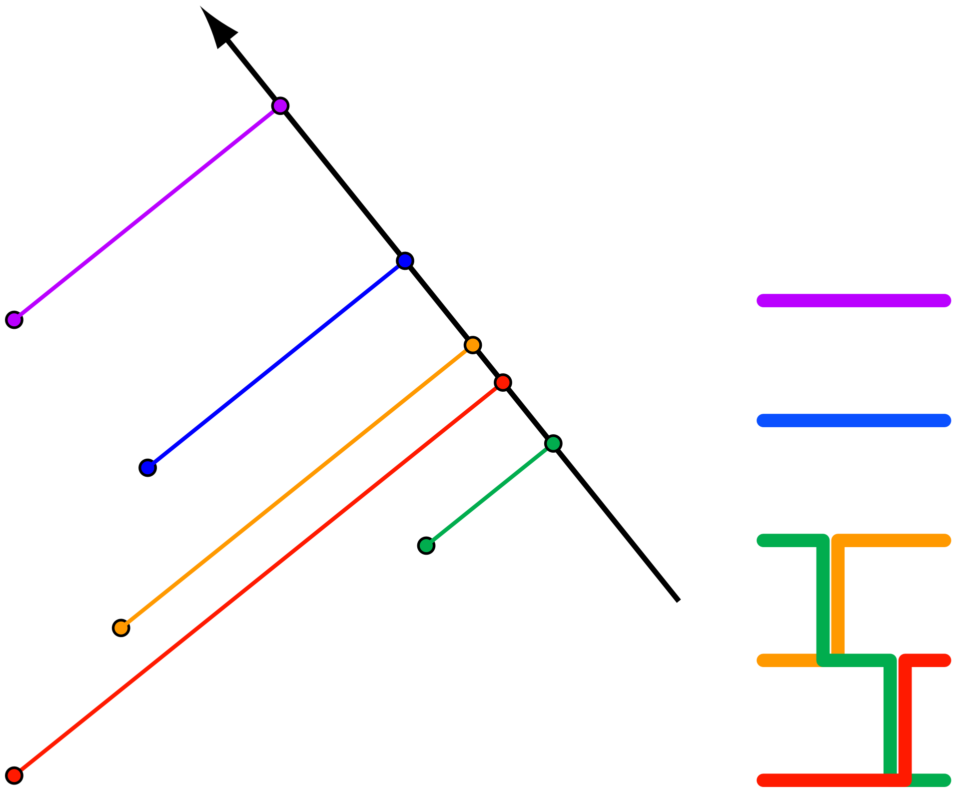
POINT SETS & MINIMAL SORTING NETWORKS



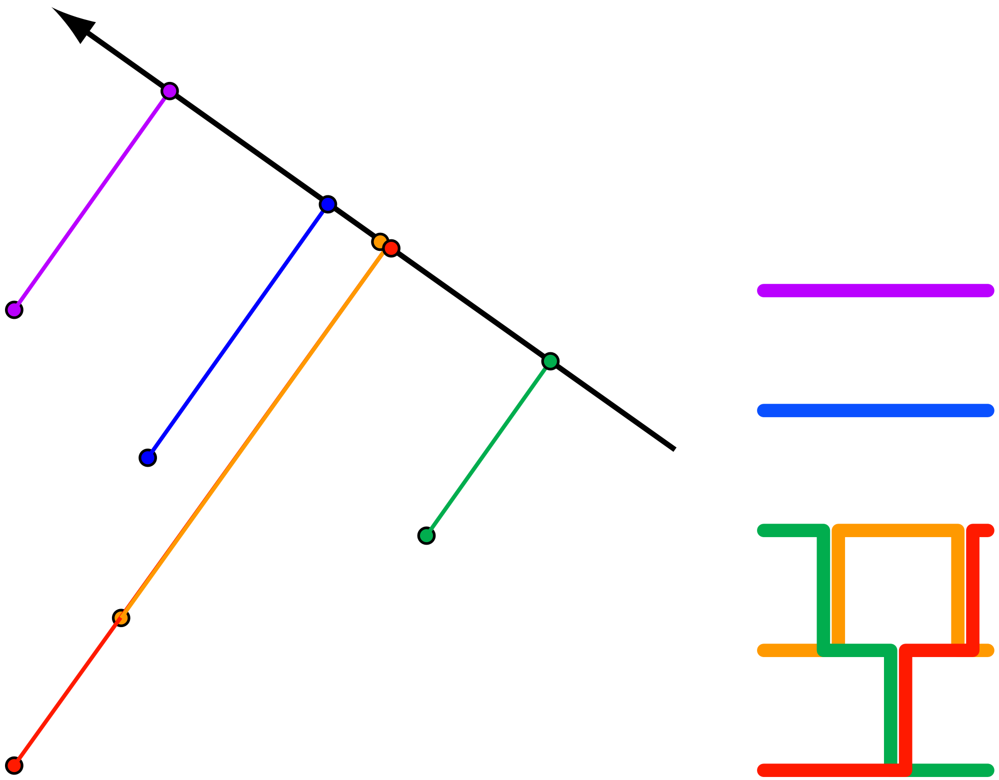
POINT SETS & MINIMAL SORTING NETWORKS



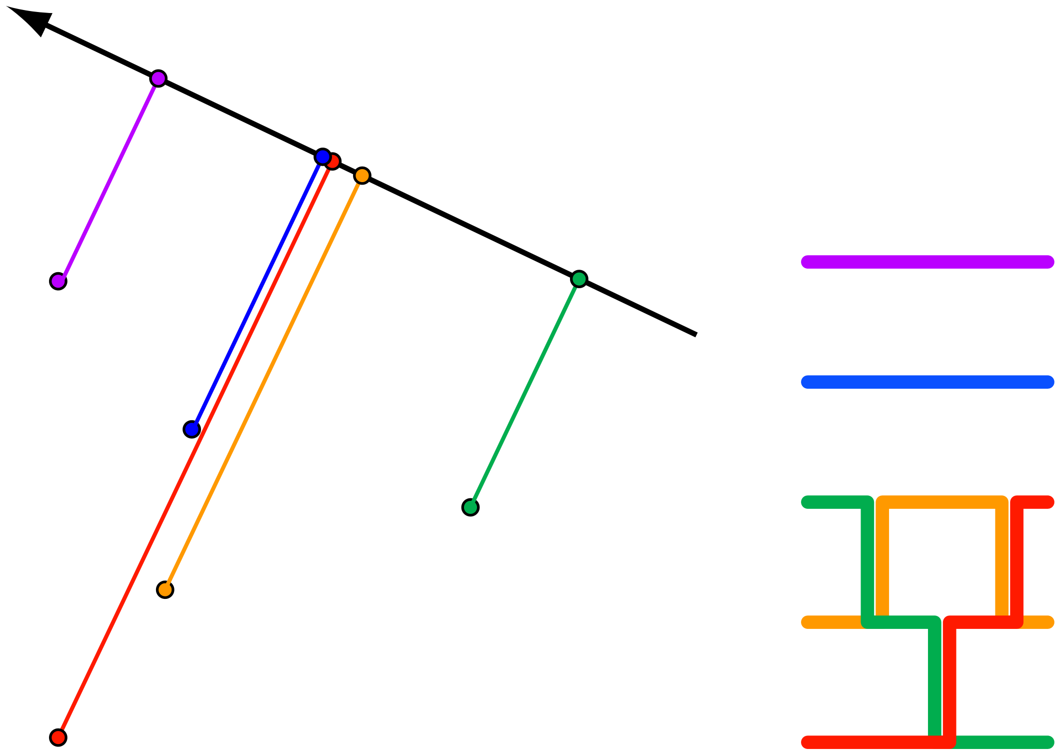
POINT SETS & MINIMAL SORTING NETWORKS



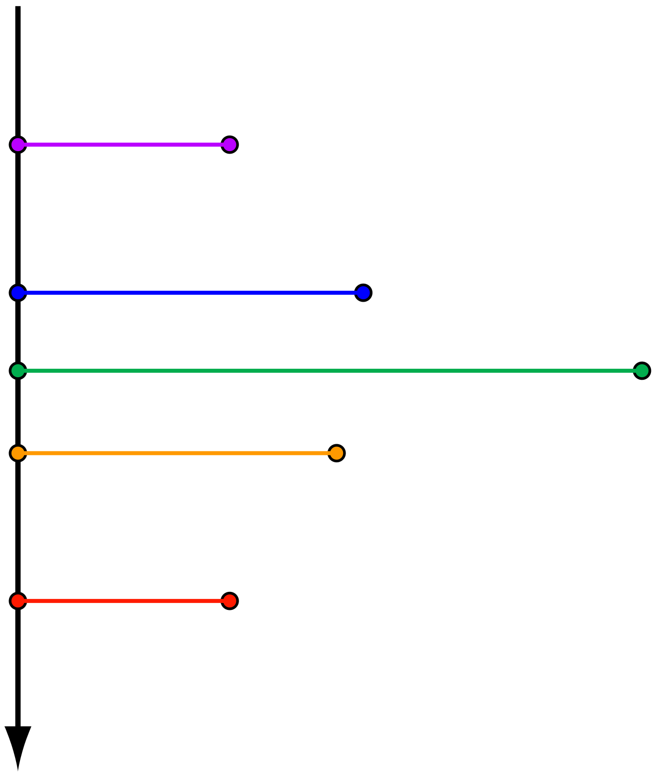
POINT SETS & MINIMAL SORTING NETWORKS

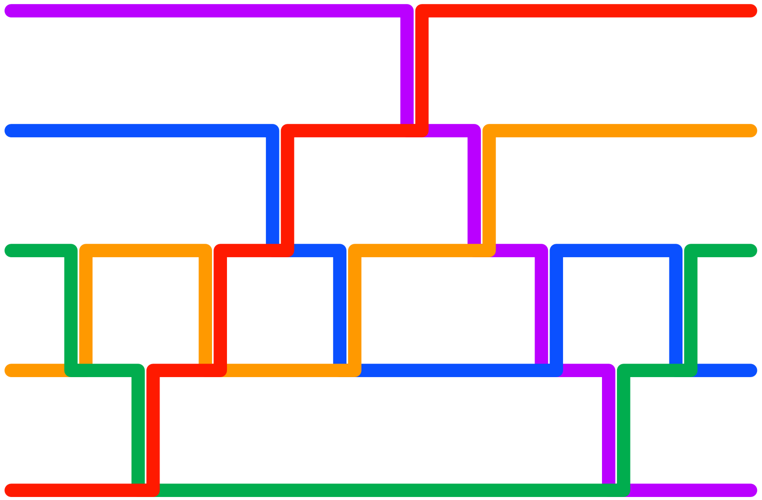


POINT SETS & MINIMAL SORTING NETWORKS



POINT SETS & MINIMAL SORTING NETWORKS





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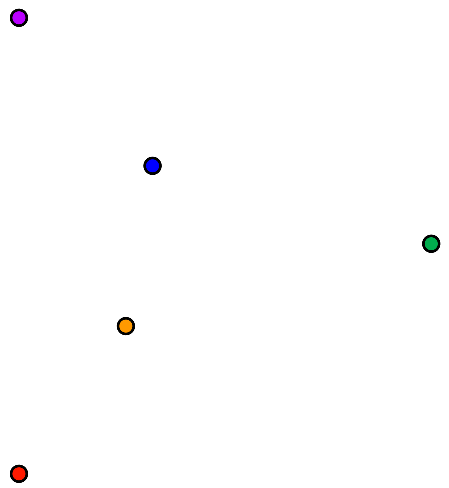
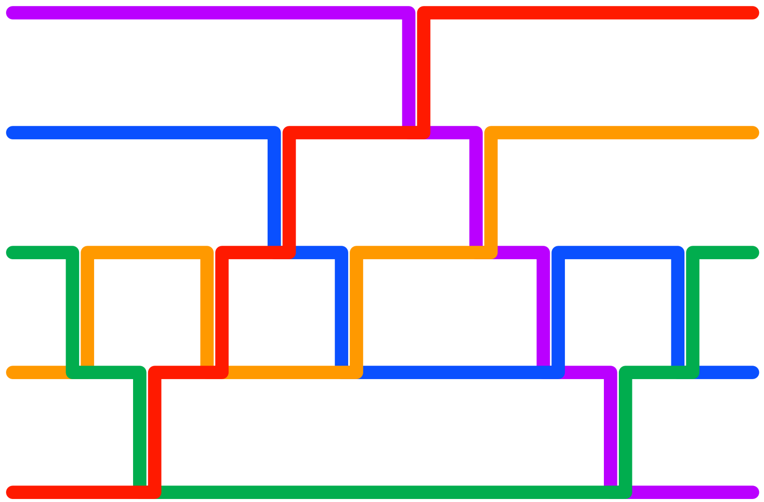




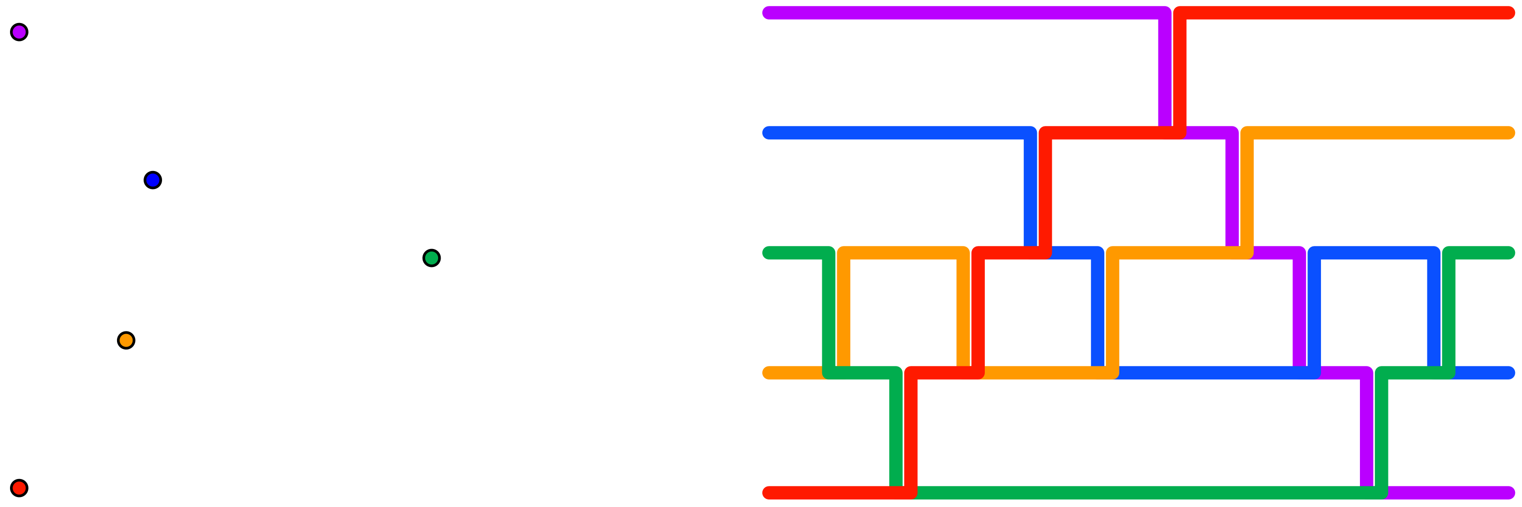
n points in $\mathbb{R}^2$ $\Longrightarrow$ minimal primitive sorting network with n levels

point $\longleftrightarrow$ pseudoline

edge $\longleftrightarrow$ crossing

boundary edge $\longleftrightarrow$ external crossing

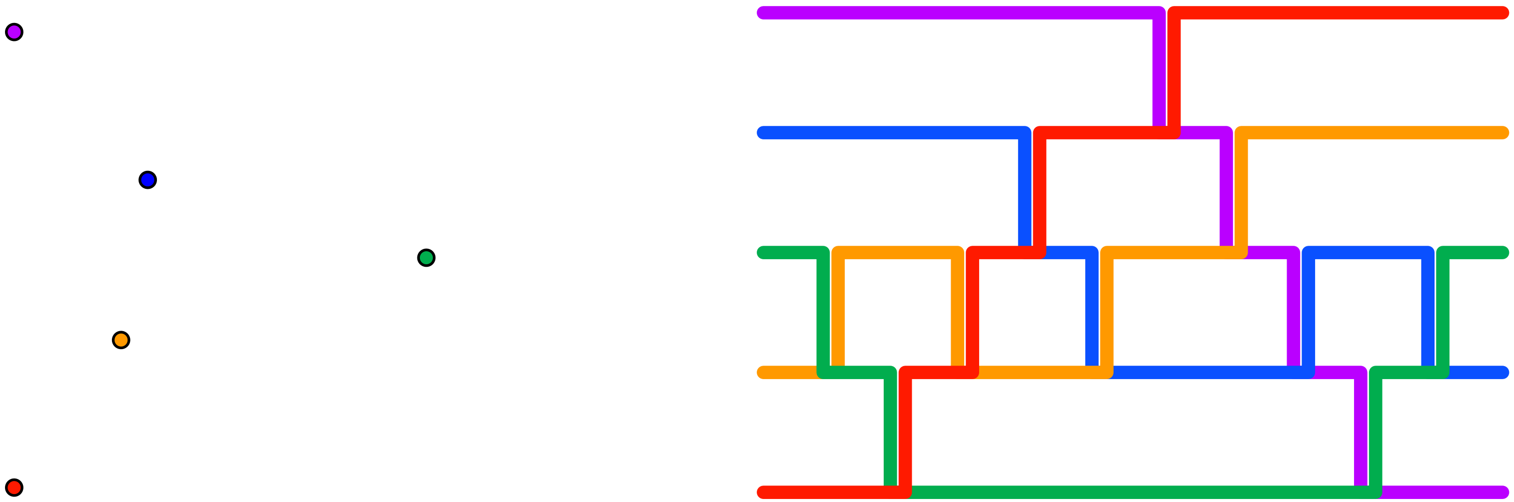
POINT SETS & MINIMAL SORTING NETWORKS



n points in $\mathbb{R}^2 \implies$ minimal primitive sorting network with n levels

not all minimal primitive sorting networks correspond to points sets of $\mathbb{R}^2$
$\implies$ realizability problems

POINT SETS & MINIMAL SORTING NETWORKS



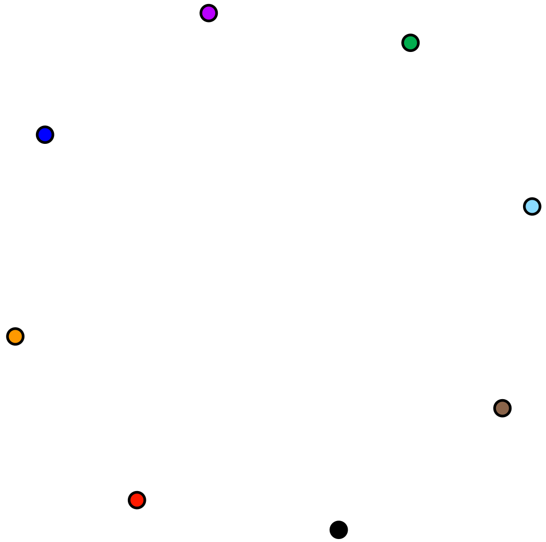
Goodmann-Pollack, *On the combinatorial classification of nondegenerate configurations in the plane* ('80)

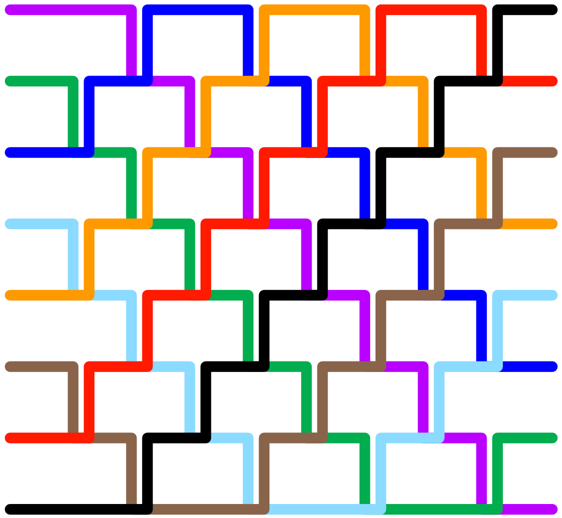
Knuth, *Axioms and Hulls* ('92)

Björner-Las Vergnas-Sturmfels-White-Ziegler, *Oriented Matroids* ('99)

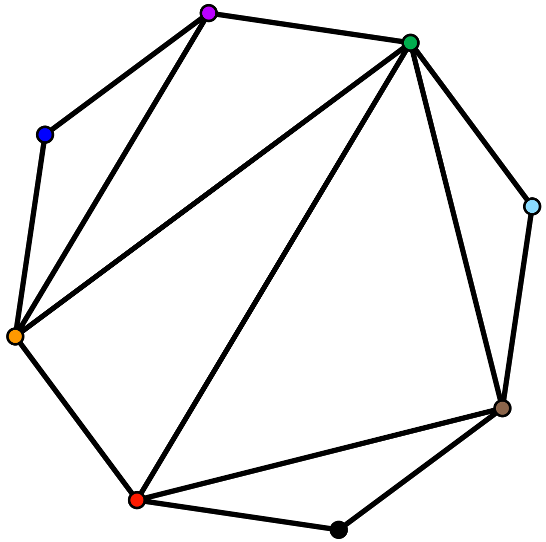
Bokowski, *Computational oriented matroids* ('06)

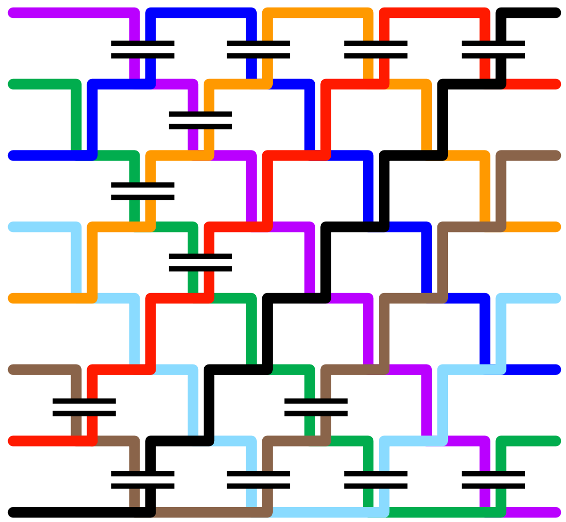
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



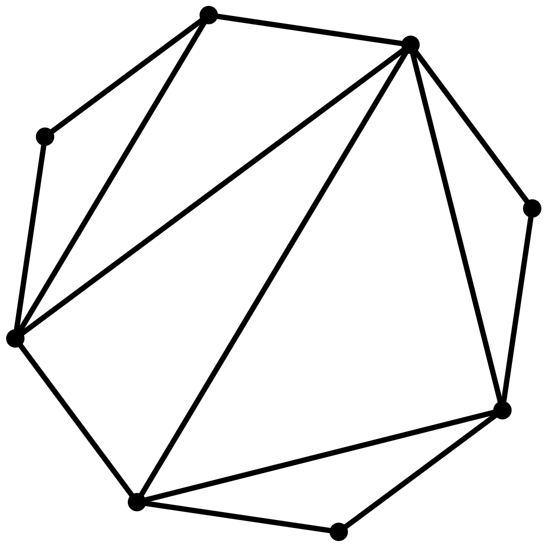


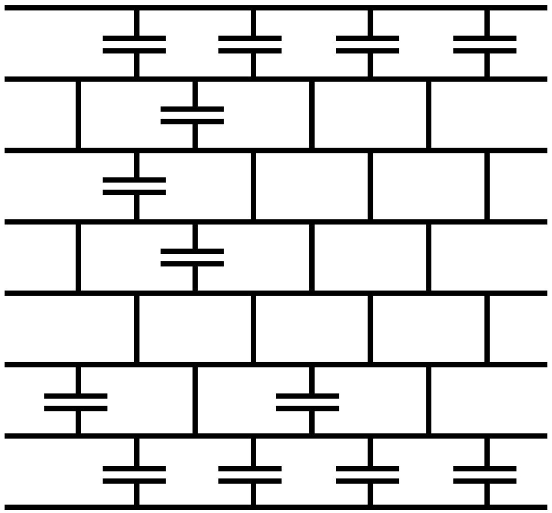
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



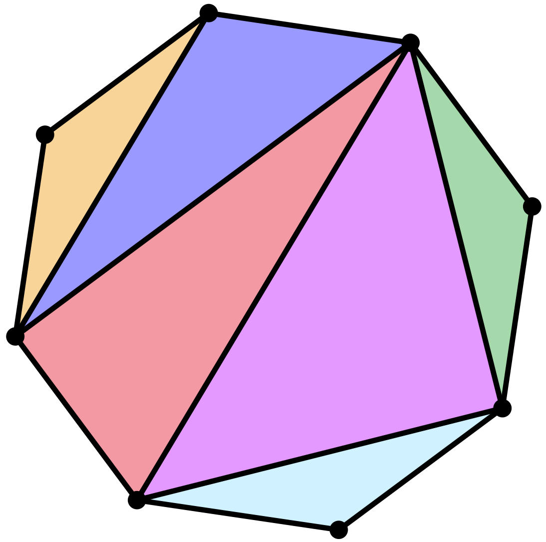


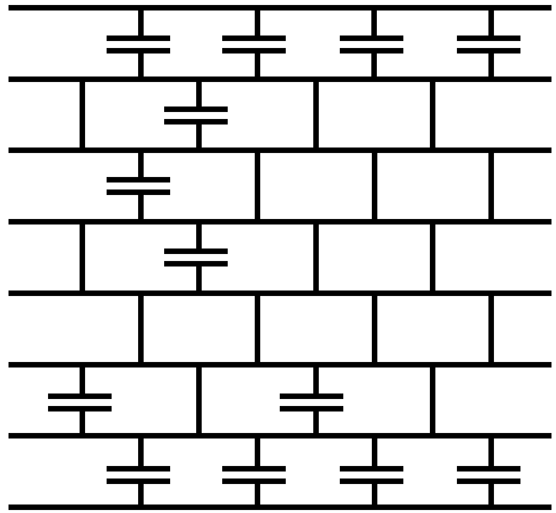
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



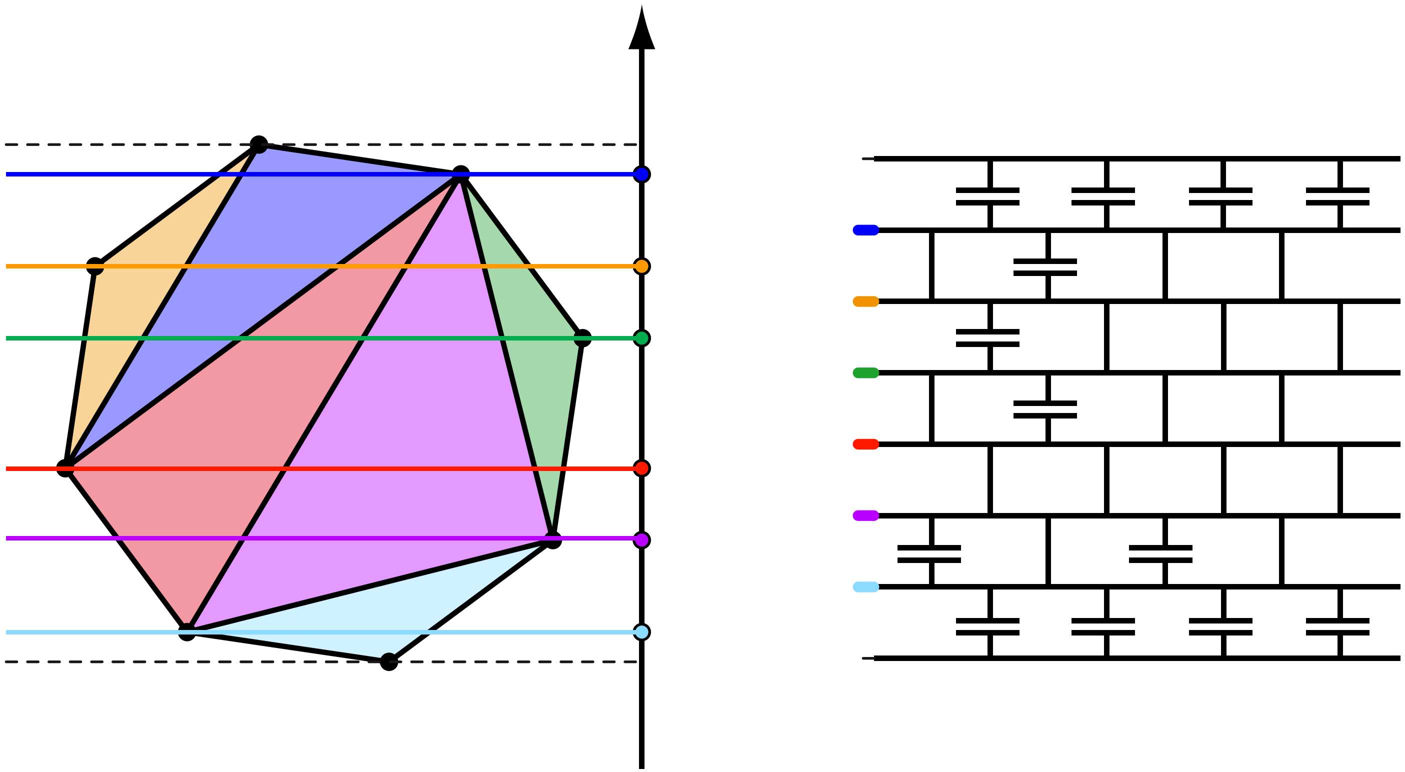


TRIANGULATIONS & ALTERNATING SORTING NETWORKS





TRIANGULATIONS & ALTERNATING SORTING NETWORKS



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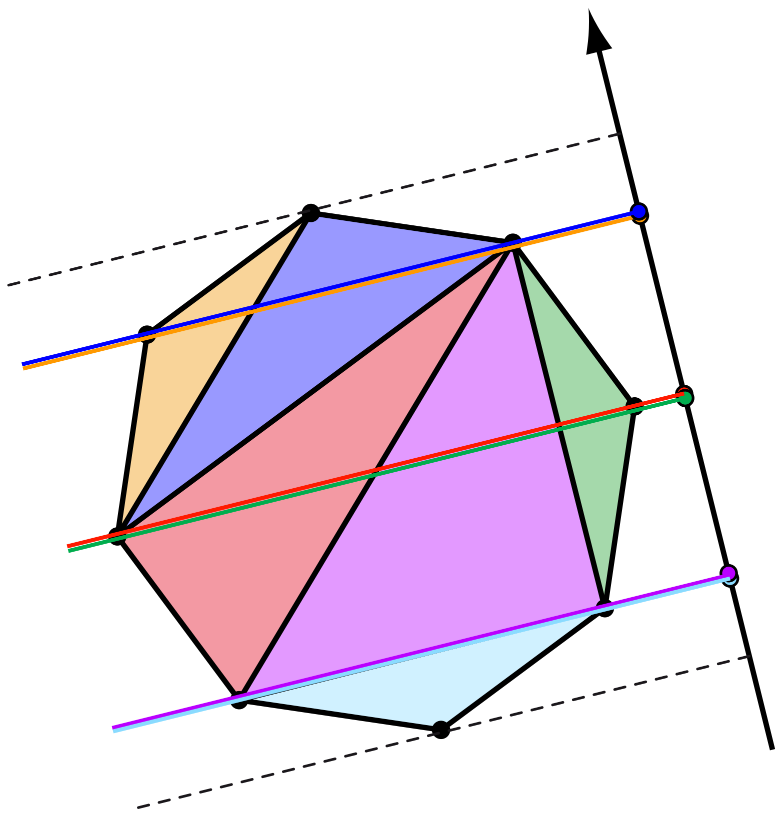

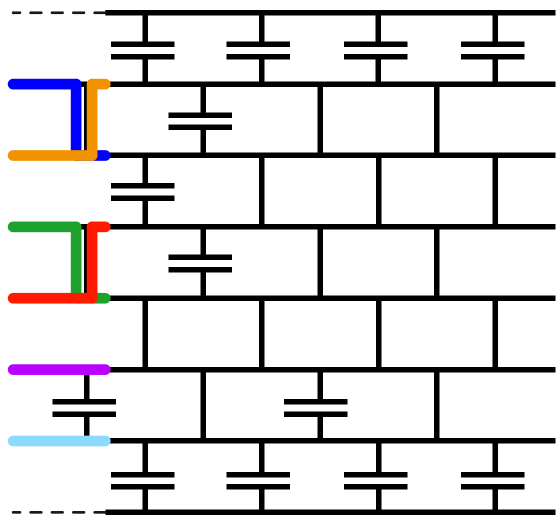

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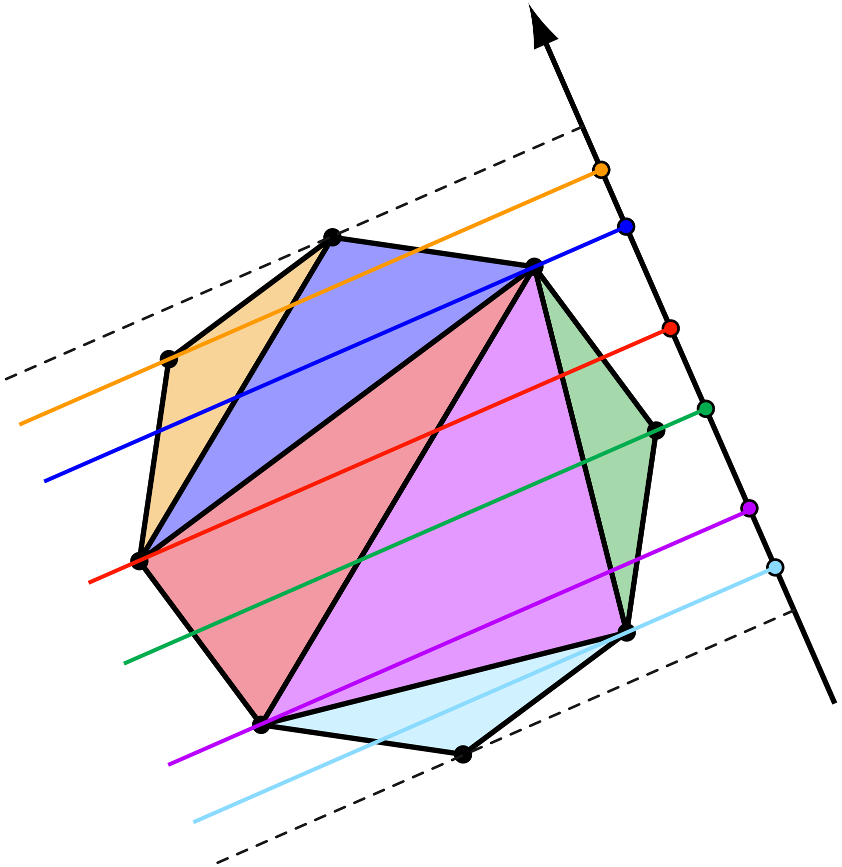
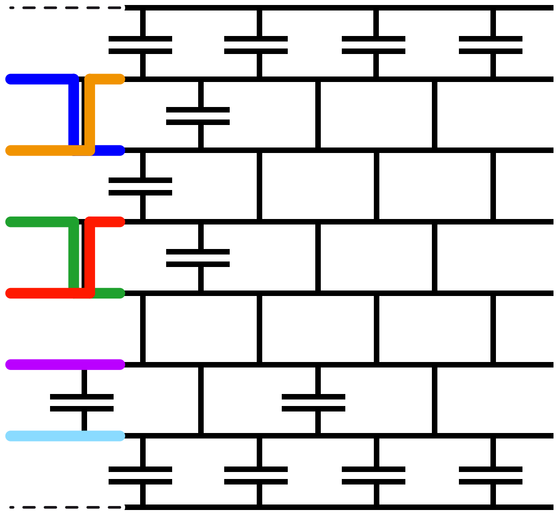
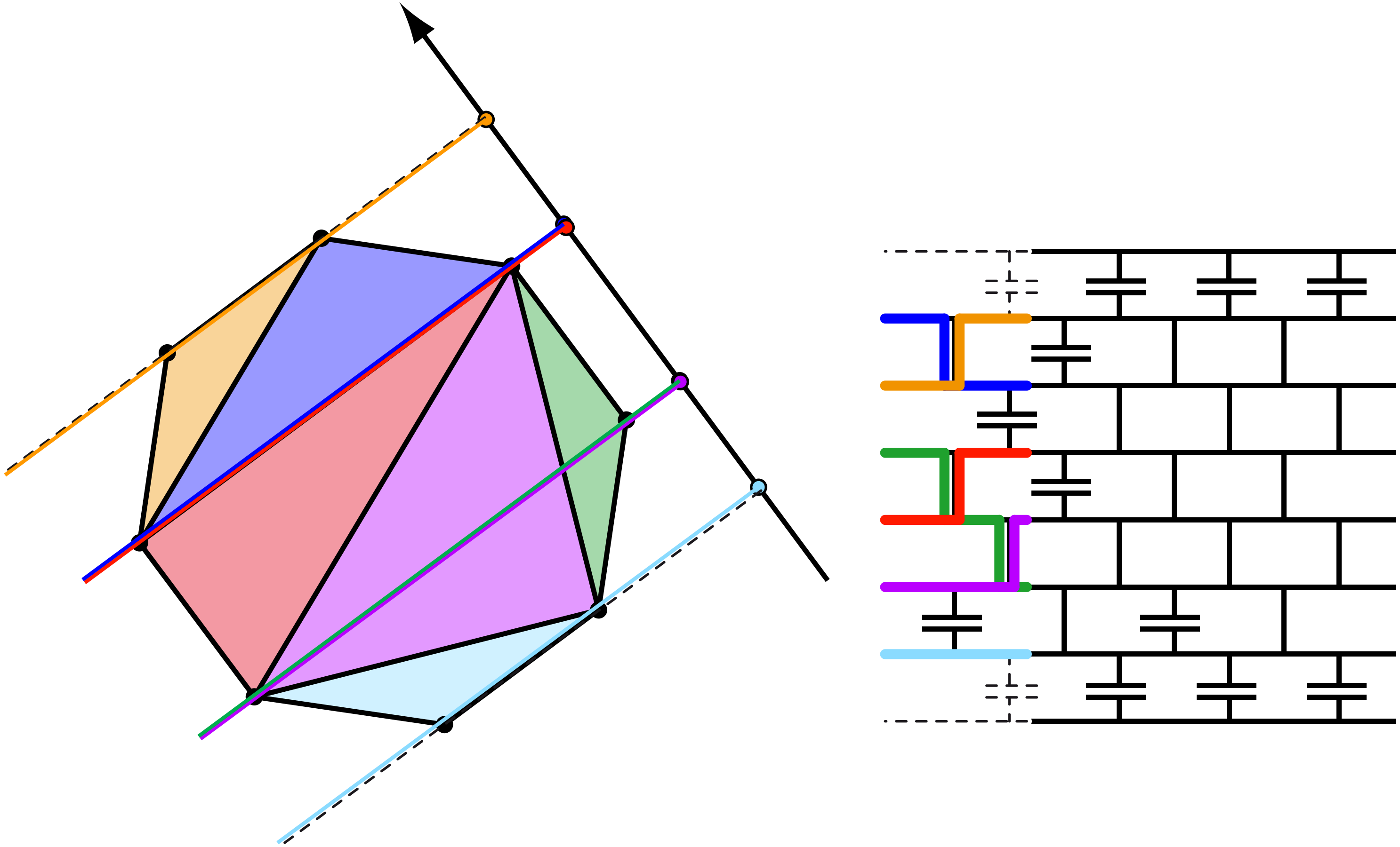
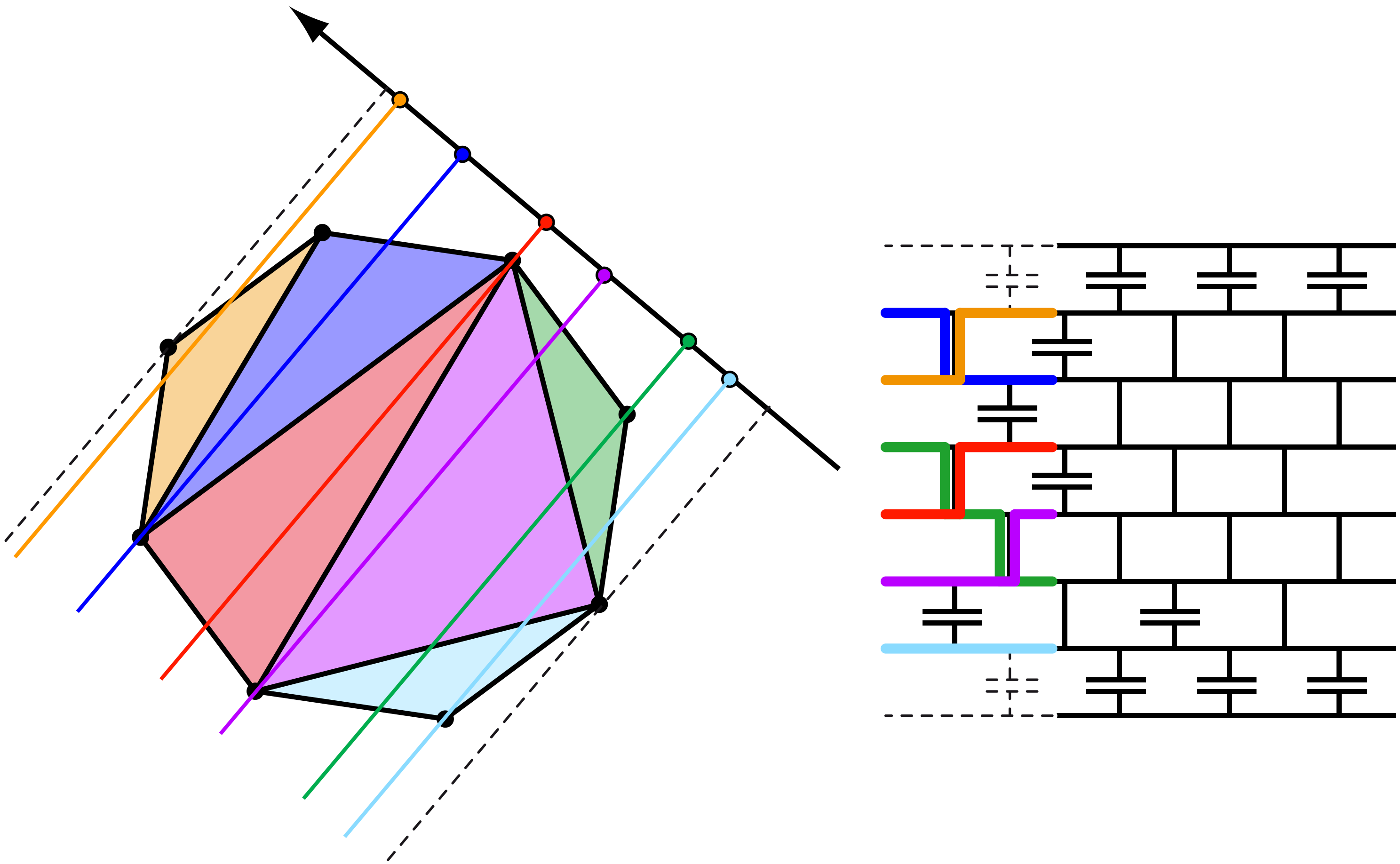




TRIANGULATIONS & ALTERNATING SORTING NETWORKS



TRIANGULATIONS & ALTERNATING SORTING NETWORKS



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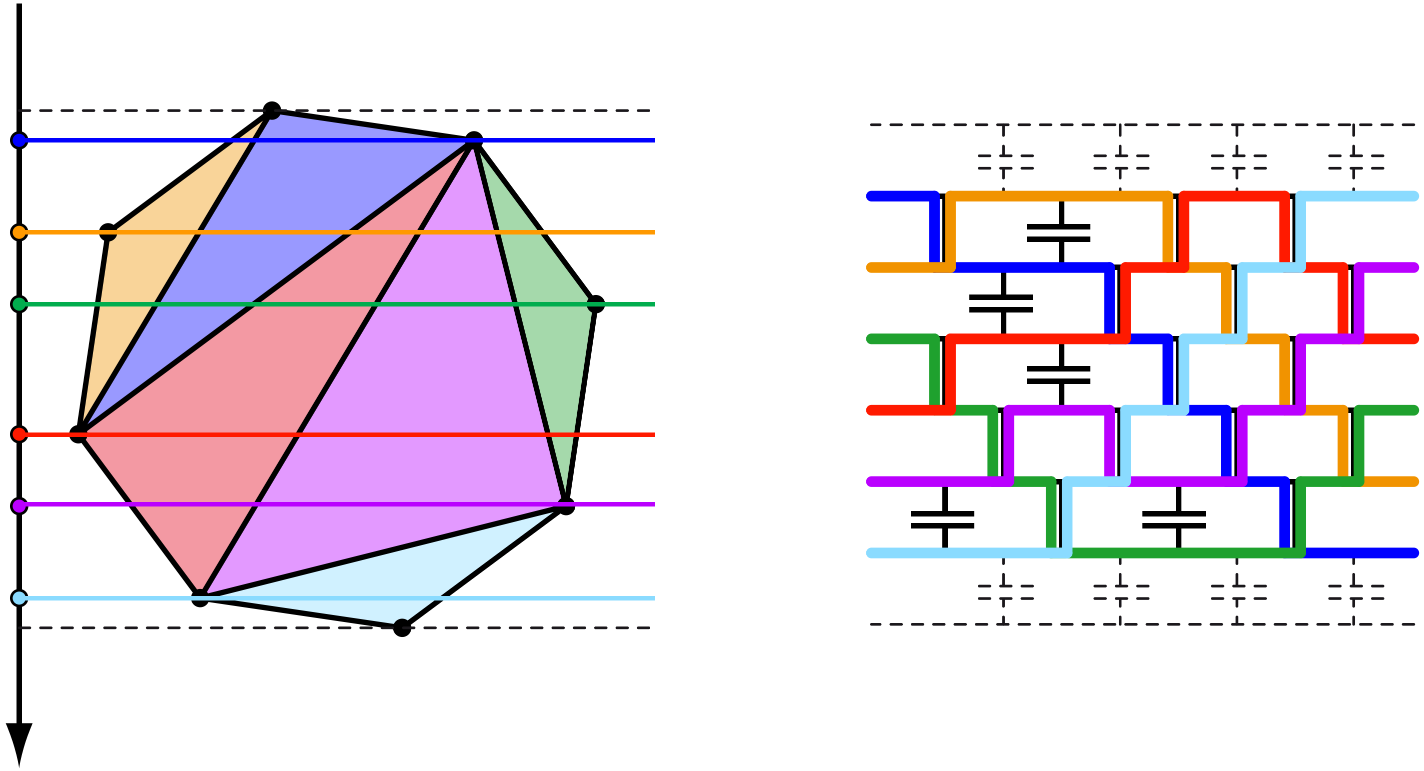
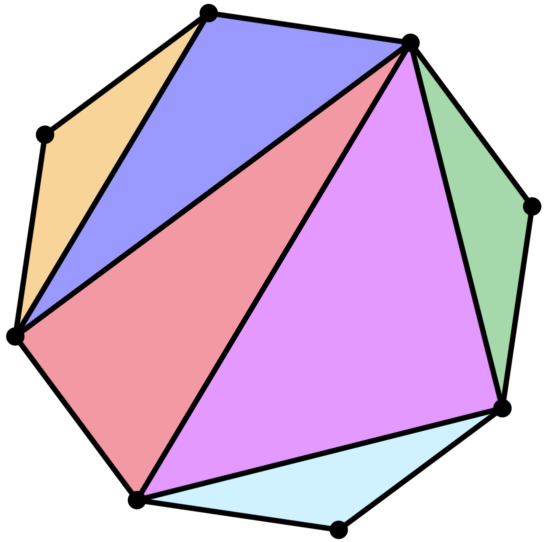
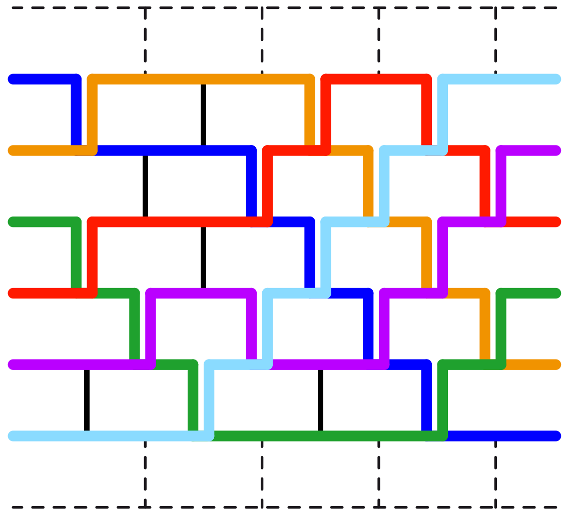


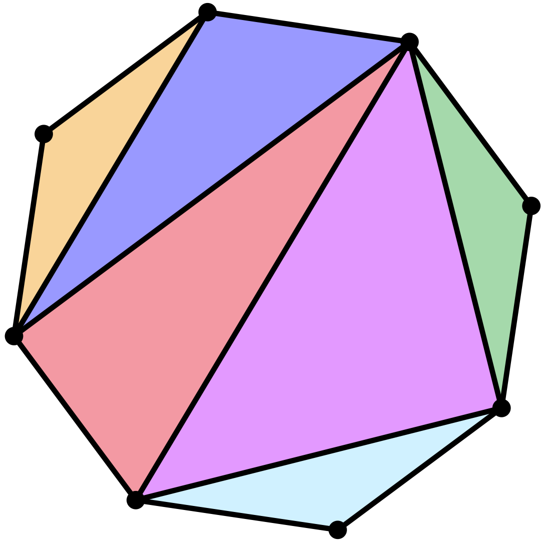
TRIANGULATIONS & ALTERNATING SORTING NETWORKS

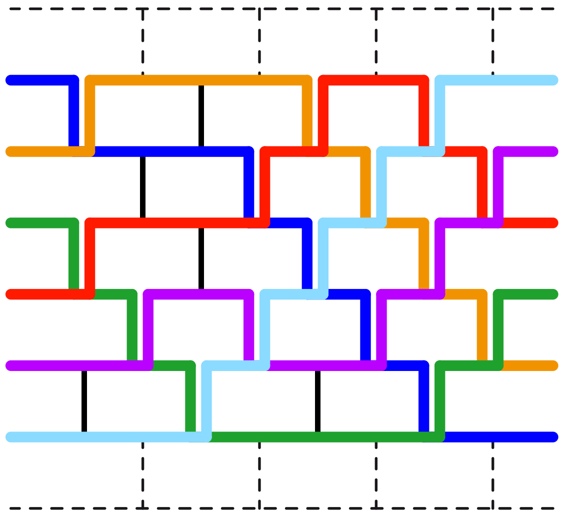


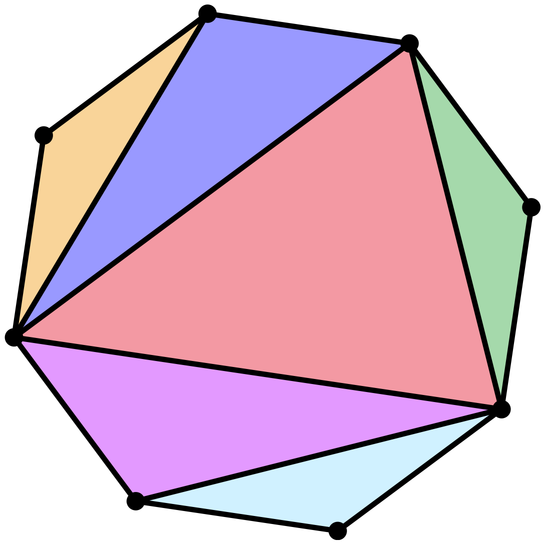


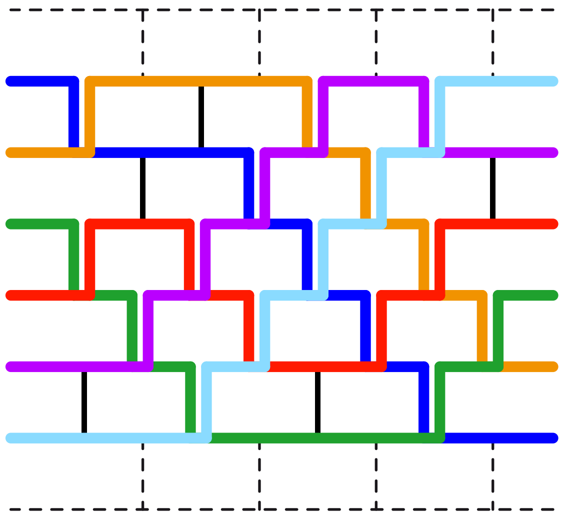
triangulation of the n-gon	$\longleftrightarrow$	pseudoline arrangement
triangle	$\longleftrightarrow$	pseudoline
edge	$\longleftrightarrow$	contact point
common bisector	$\longleftrightarrow$	crossing point
dual binary tree	$\longleftrightarrow$	contact graph

FLIPS







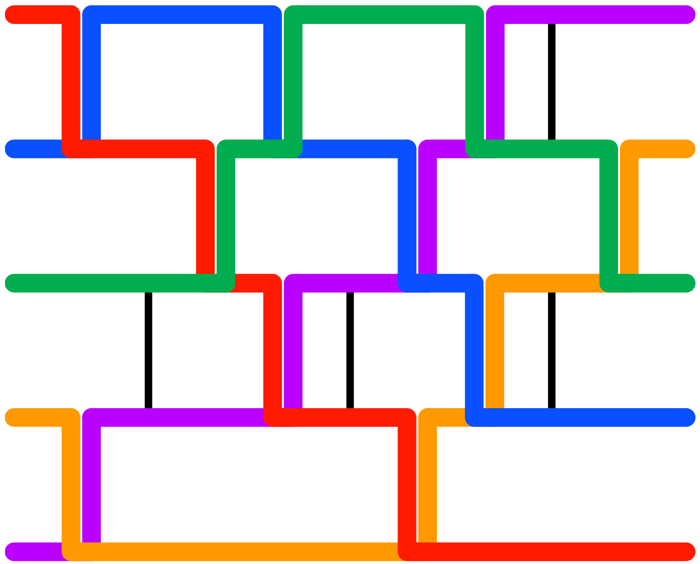


ASSOCIAHEDRON
— & —
BRICK POLYTOPE

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $b(\Lambda) \in \mathbb{R}^n$

$b(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ



Brick polytope $\mathcal{B}(\mathcal{N}) = \operatorname{conv}\{b(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

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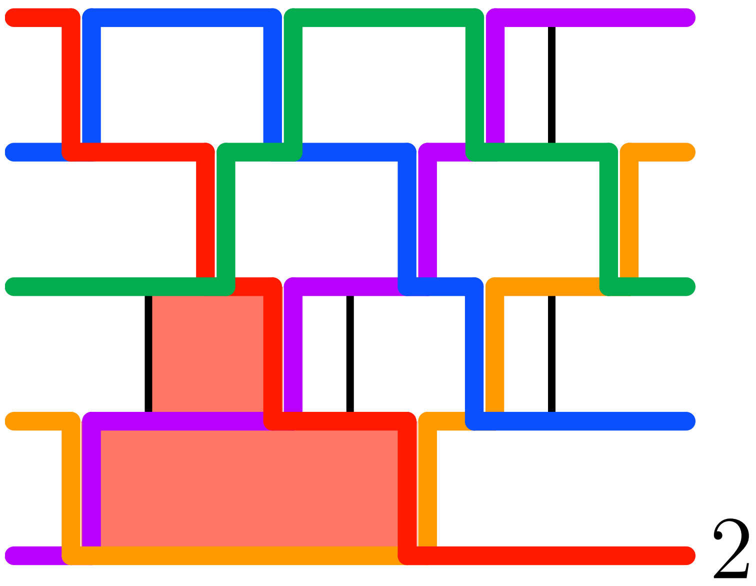

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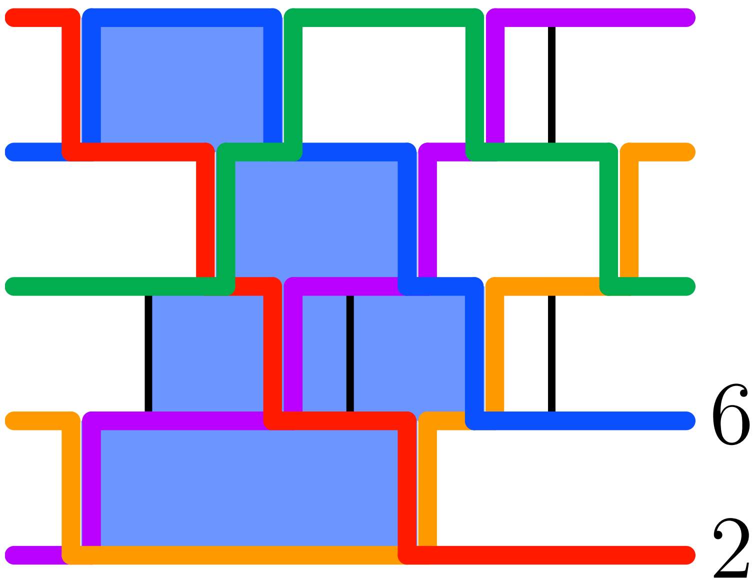

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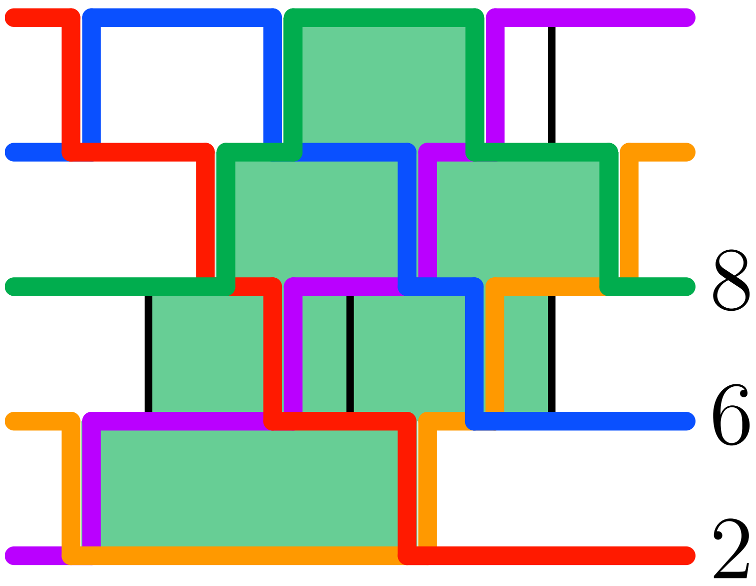

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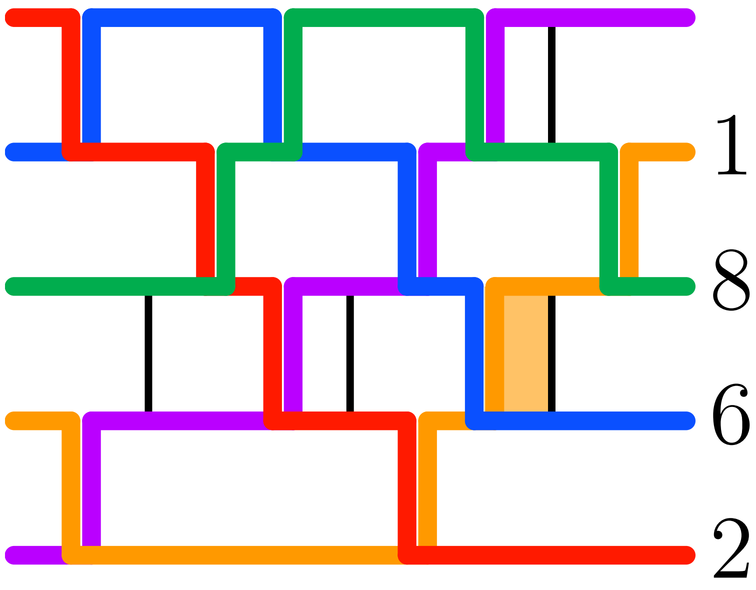

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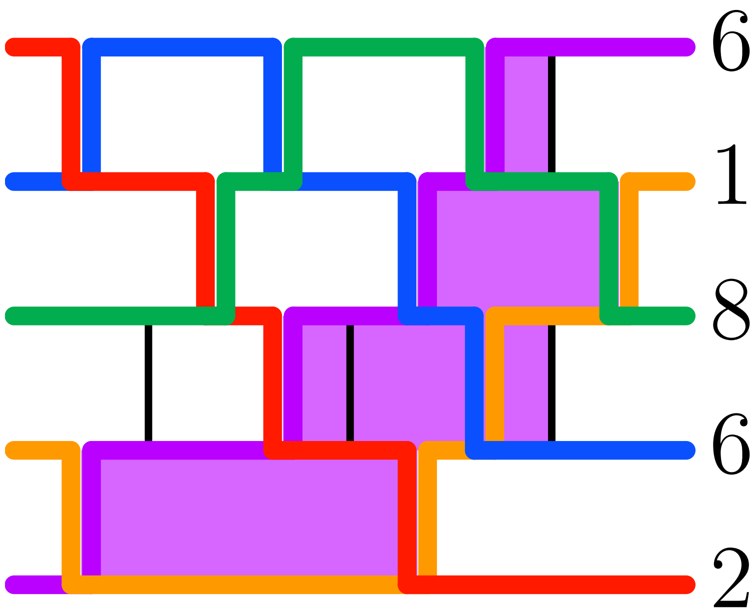

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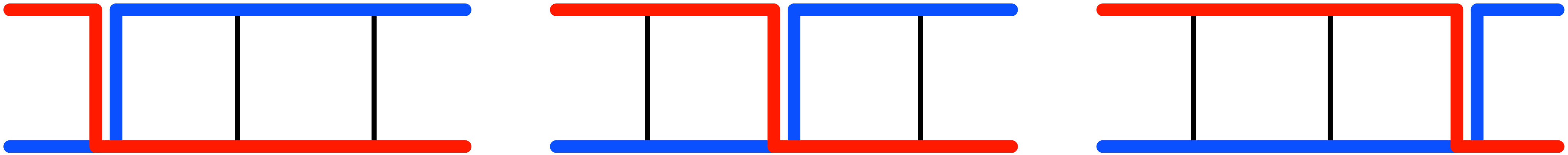
BRICK POLYTOPE

$\mathcal{X}_m$ = network with two levels and m commutators

graph of flips $G(\mathcal{X}_m)$ = complete graph K_m

brick polytope $\mathcal{B}(\mathcal{X}_m) = \mathrm{conv}\left\{ \begin{pmatrix} m-i \\ i-1 \end{pmatrix} \;\middle|\; i \in [m] \right\} = \left[\begin{pmatrix} m-1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ m-1 \end{pmatrix} \right]$



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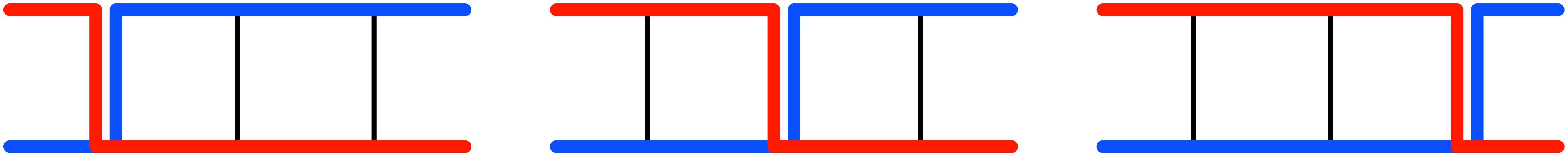
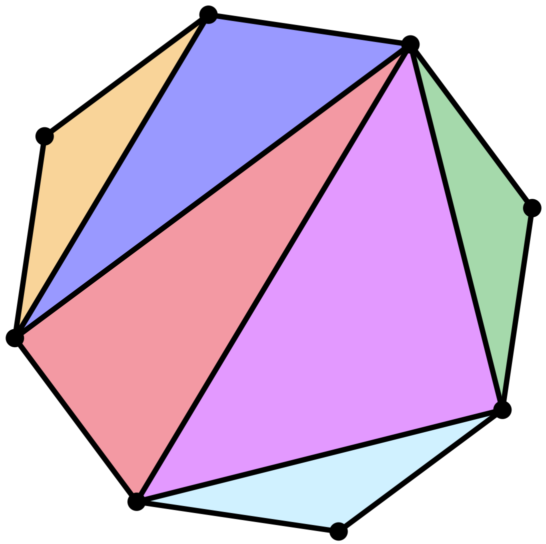
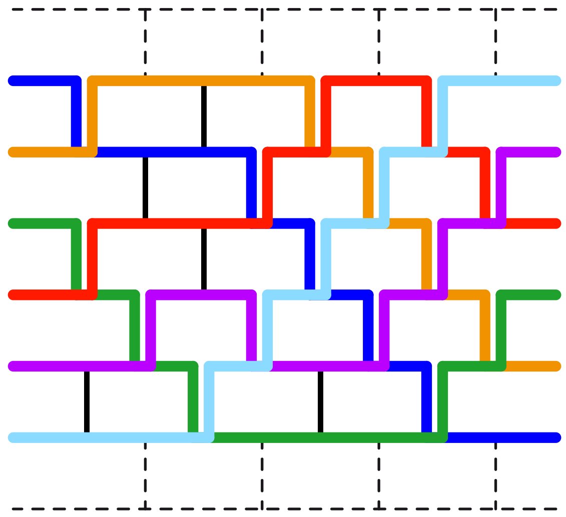


The brick vector $b(\Lambda)$ is a vertex of $\mathcal{B}(\mathcal{N}) \iff$ the contact graph $\Lambda^{\#}$ is acyclic
The graph of the brick polytope $\mathcal{B}(\mathcal{N})$ is a subgraph of the flip graph $G(\mathcal{N})$

The graph of the brick polytope $\mathcal{B}(\mathcal{N})$ coincides with the graph of flips $G(\mathcal{N})$
$\iff$ the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$ are forests

ALTERNATING NETWORKS & ASSOCIAHEDRA





triangulation of the n-gon $\longleftrightarrow$ pseudoline arrangement

triangle $\longleftrightarrow$ pseudoline

edge $\longleftrightarrow$ contact point

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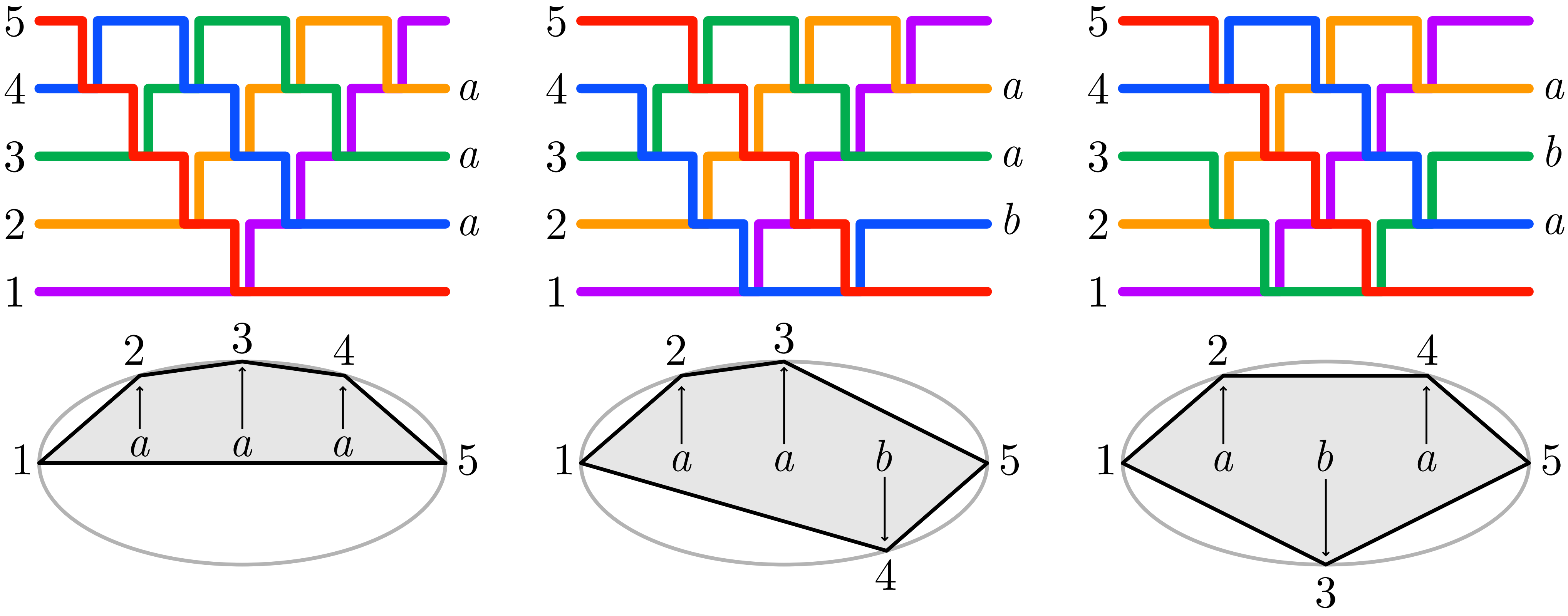
dual binary tree $\longleftrightarrow$ contact graph

The brick polytope is an associahedron.

ALTERNATING NETWORKS & ASSOCIAHEDRA

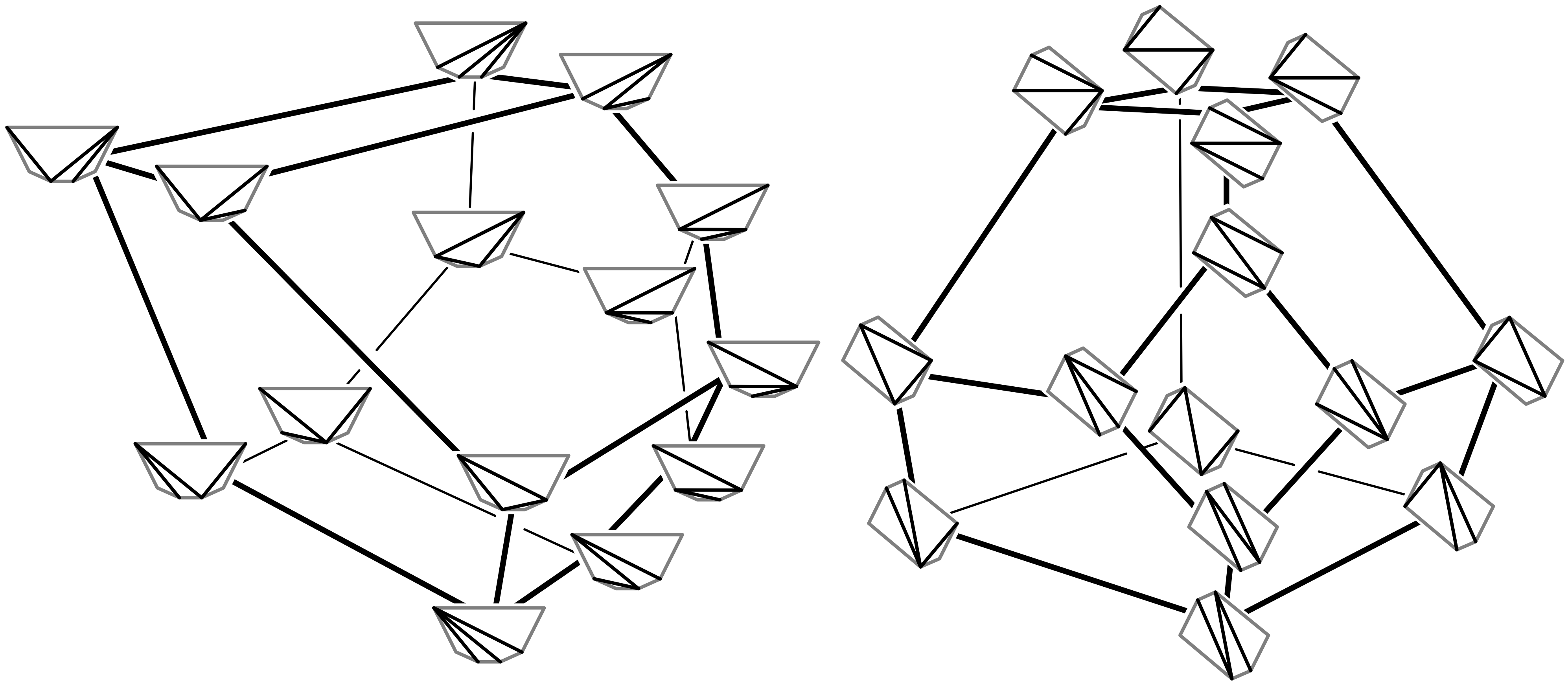
for $x \in \{a,b\}^{n-2}$, define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$



Pseudoline arrangements on $\mathcal{N}_x^1 \longleftrightarrow$ triangulations of the polygon $\mathcal{P}_x$.

ALTERNATING NETWORKS & ASSOCIAHEDRA

For any word $x \in \{a, b\}^{n-2}$, the brick polytope $\mathcal{B}_x = \mathcal{B}(\mathcal{N}_x^1)$ is an associahedron.



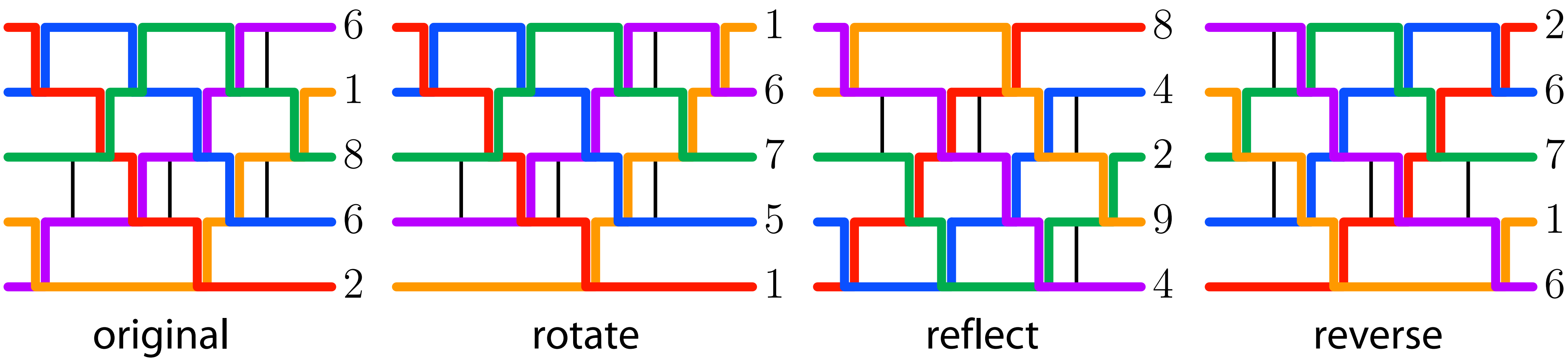
Up to a translation Ω_x, the brick polytope $\mathcal{B}_x$ coincides with the associahedron Asso_x of Hohlweg and Lange.

P.-Santos, *The brick polytope of a sorting network* ('12)

ASSOCIAHEDRON
— & —
BARYCENTER

THREE OPERATIONS

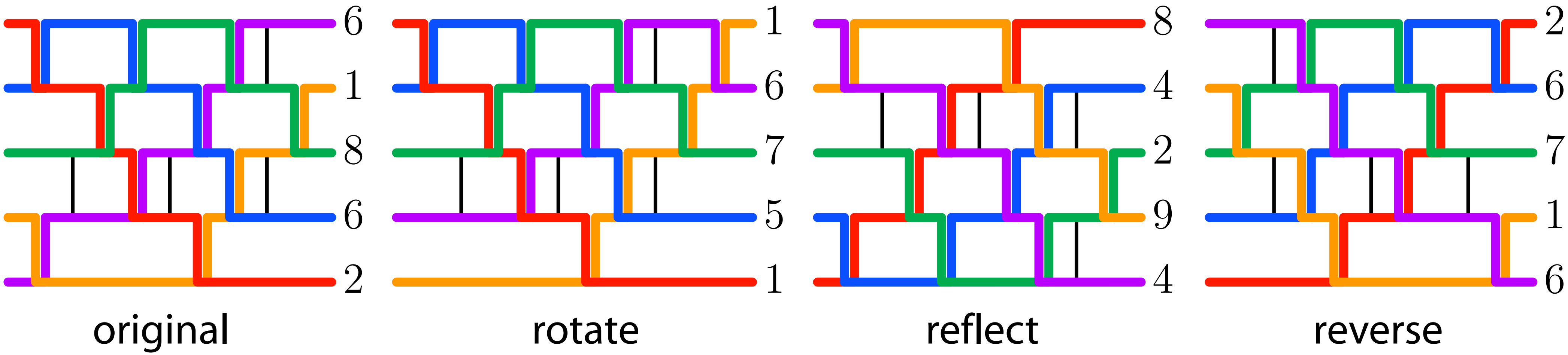
Evolution of the brick vector $b_{\mathcal{N}}(\Lambda)$ under three operations:



1. Rotate: $b_{\mathcal{N}^{\circlearrowleft}}(\Lambda^{\circlearrowleft}) - b_{\mathcal{N}}(\Lambda) \ \in \ \omega_i + \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $b_{\mathcal{N}^{\updownarrow}}(\Lambda^{\updownarrow}) = \#\{\text{bricks of } \mathcal{N}\} \,.\, \mathbb{1} - (b_{\mathcal{N}}(\Lambda))^{\leftrightarrow}$
3. Reverse: $b_{\mathcal{N}^{\leftrightarrow}}(\Lambda^{\leftrightarrow}) = (b_{\mathcal{N}}(\Lambda))^{\leftrightarrow}$

THREE OPERATIONS

Evolution of the translated brick vector $\bar{b}_x(\Lambda) = b_x(\Lambda) - \Omega_x$ under three operations:



1. Rotate: $\bar{b}_{x^{\circlearrowleft}}(\Lambda^{\circlearrowleft}) - \bar{b}_x(\Lambda) \ \in \ \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $\bar{b}_{x^{\updownarrow}}(\Lambda^{\updownarrow}) = -(\bar{b}_x(\Lambda))^{\leftrightarrow}$
3. Reverse: $\bar{b}_{x^{\leftrightarrow}}(\Lambda^{\leftrightarrow}) = (\bar{b}_x(\Lambda))^{\leftrightarrow}$

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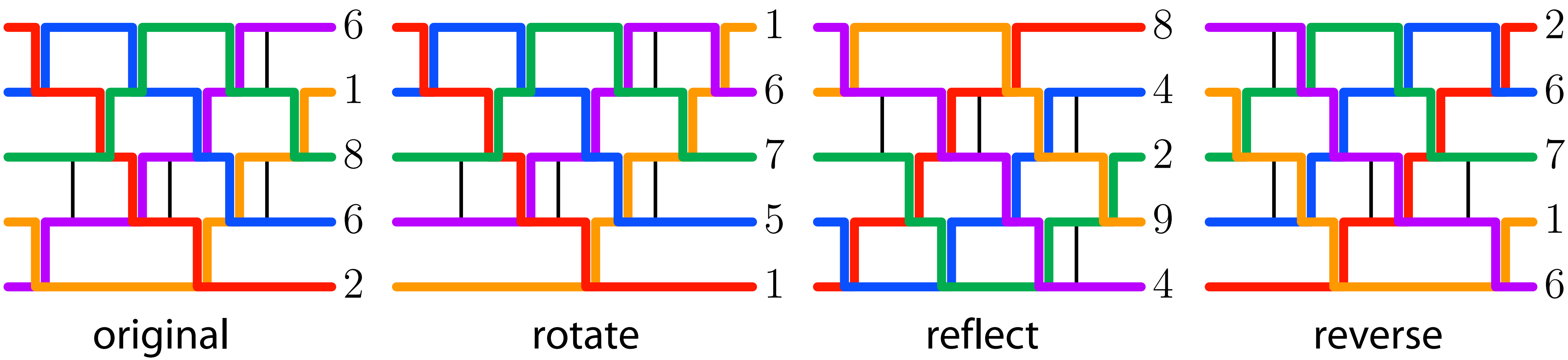

1. Rotate: $\bar{b}_{x^{\circlearrowleft}}(\Lambda^{\circlearrowleft}) - \bar{b}_x(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$

All associahedra Asso_x have the same barycenter

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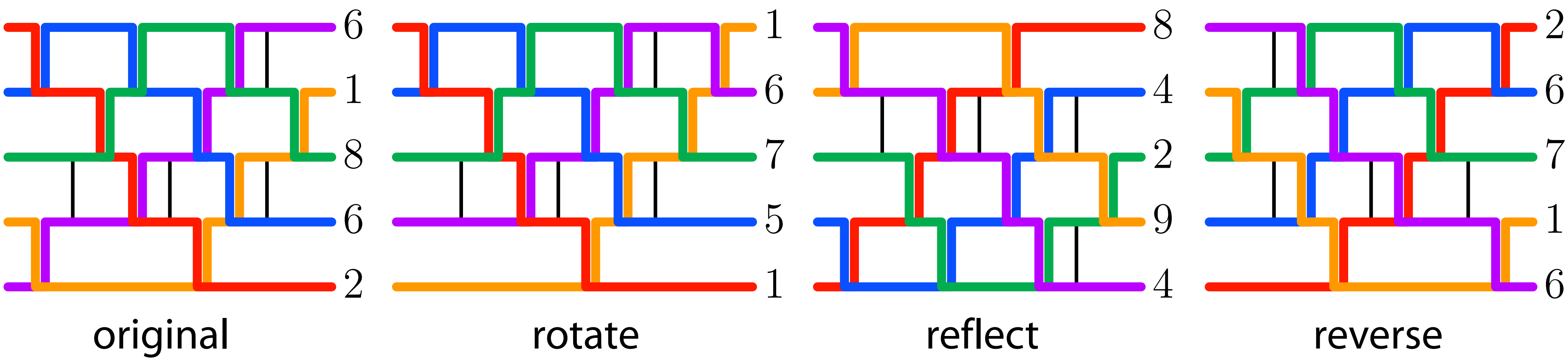

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The barycenter of the superposition of the vertices of $\mathrm{Asso}_{x^{\updownarrow}}$ and $\mathrm{Asso}_{x^{\leftrightarrow}}$ is the origin

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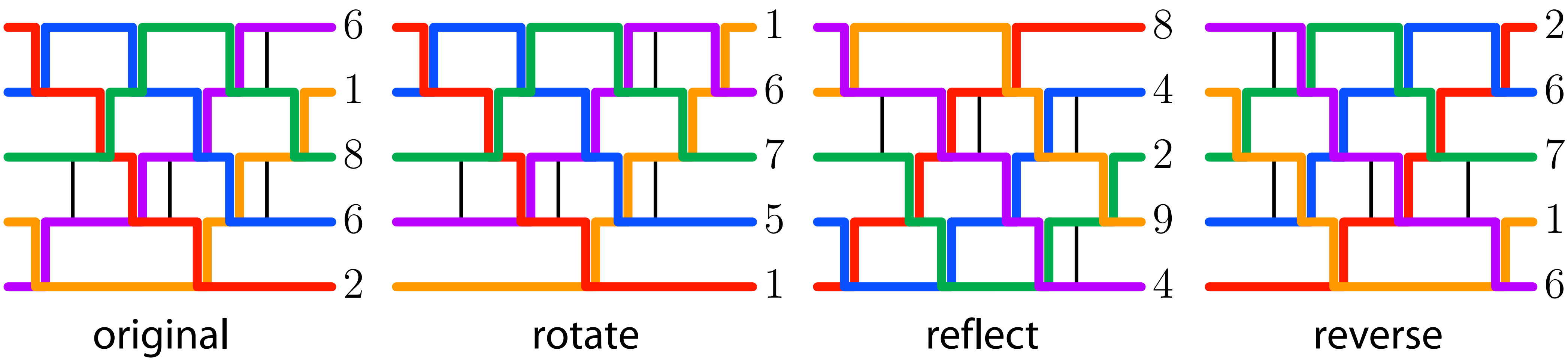

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THEOREM. All associahedra Asso_x have vertex barycenter at the origin

. . . and the same method works for fairly balanced and generalized associahedra.

THANK YOU