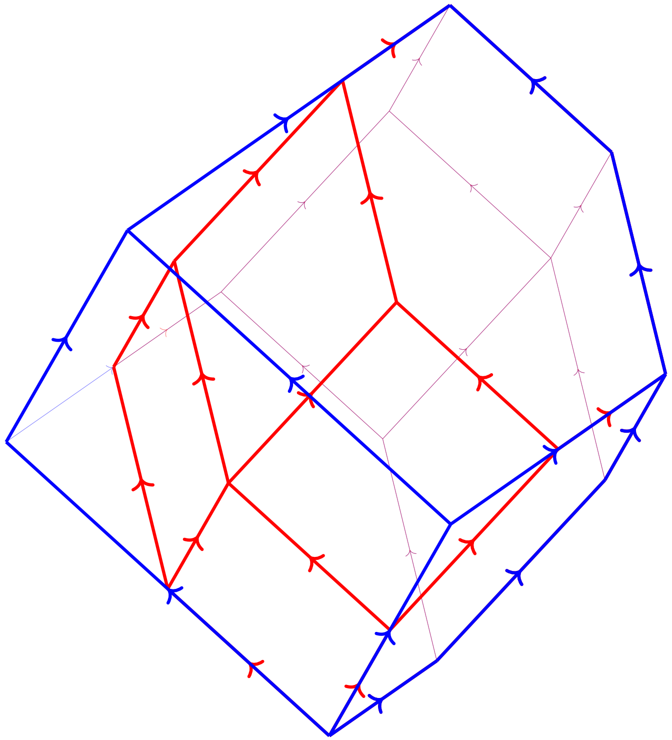
Acyclic reorientation lattices and their lattice quotients

V. PILAUD (CNRS & LIX, École Polytechnique)



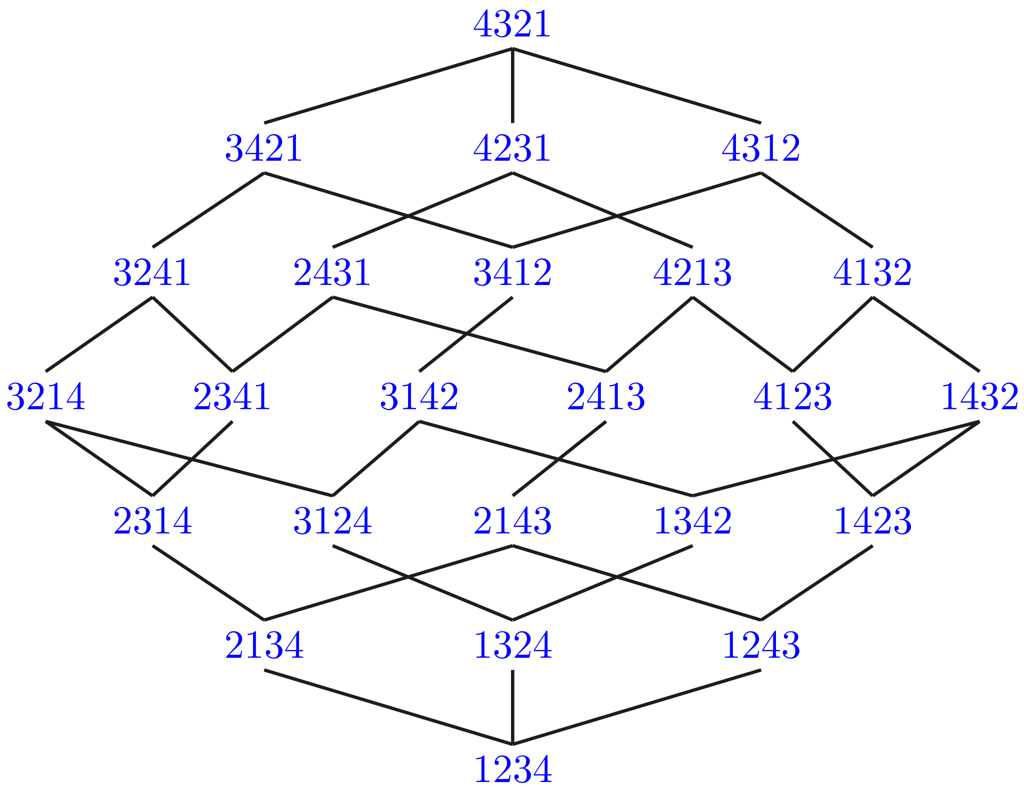
Philippe Flajolet Seminar
Thursday November 24th, 2022

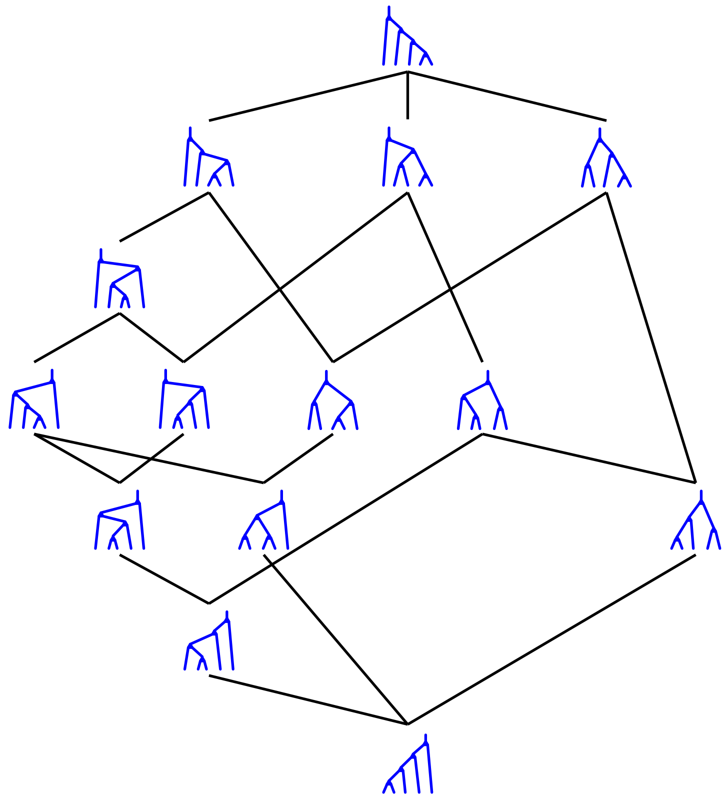
slides: http://www.lix.polytechnique.fr/~pilaud/documents/presentations/acyclicReorientationLattices.pdf
preprint: http://www.arxiv.org/abs/2111.12387

PERMUTAHEDRA & ASSOCIAHEDRA

LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice = partially ordered set L where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$





weak order = permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

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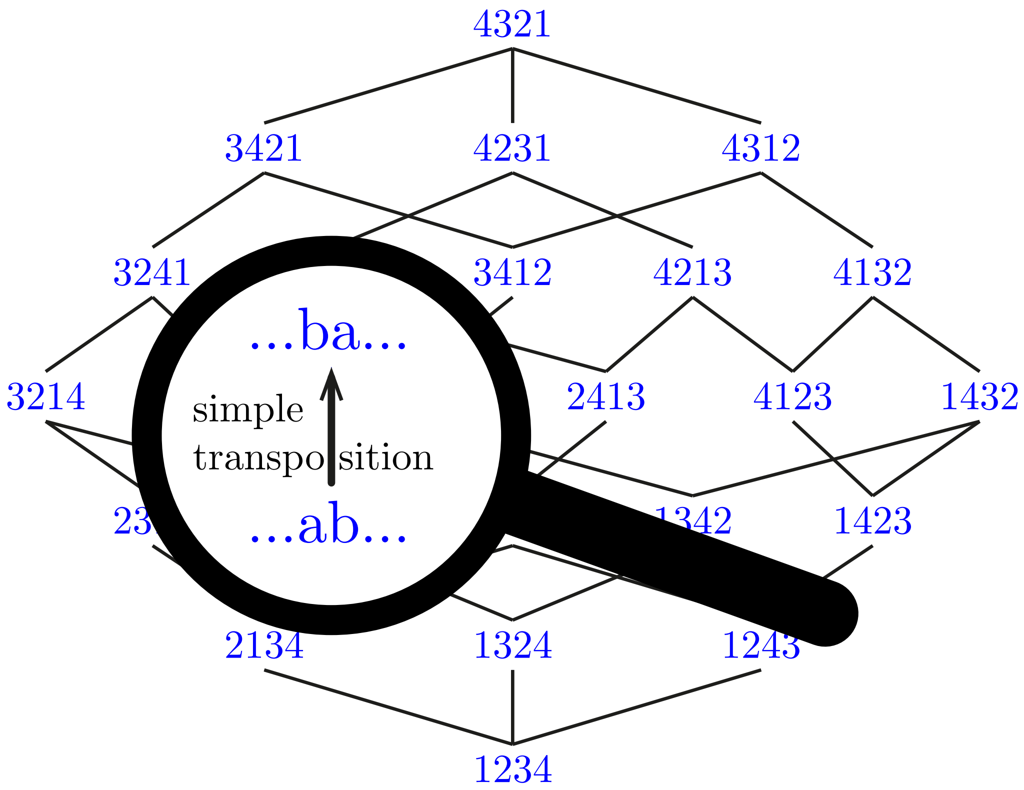

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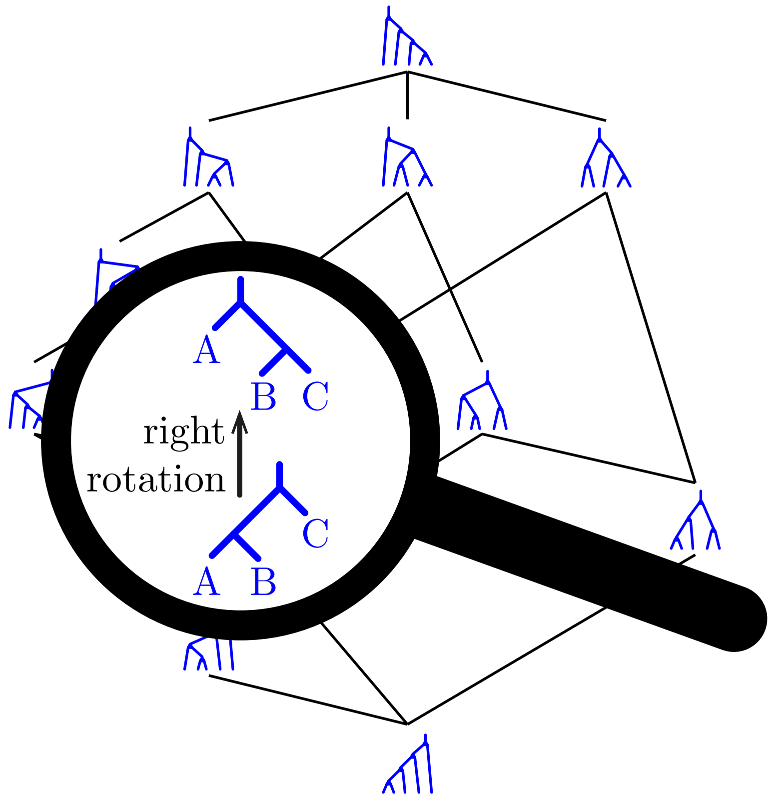

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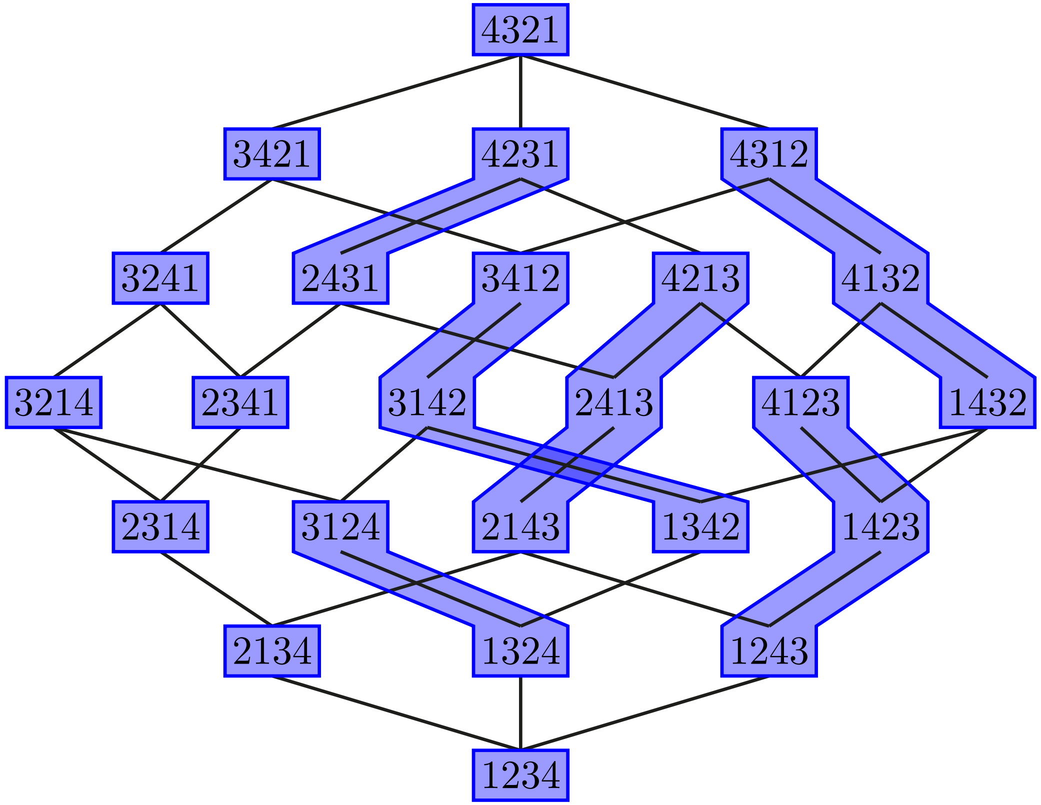

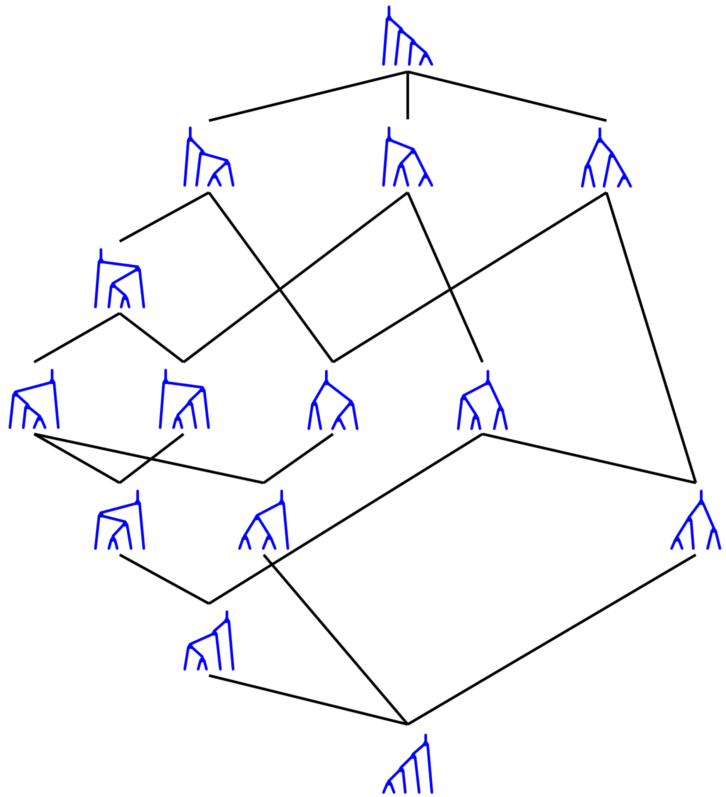

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Tamari lattice $=$ binary trees on $[n]$
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sylvester congruence $=$ equivalence classes are sets of linear extensions of binary trees
$=$ equivalence classes are fibers of BST insertion
$=$ rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

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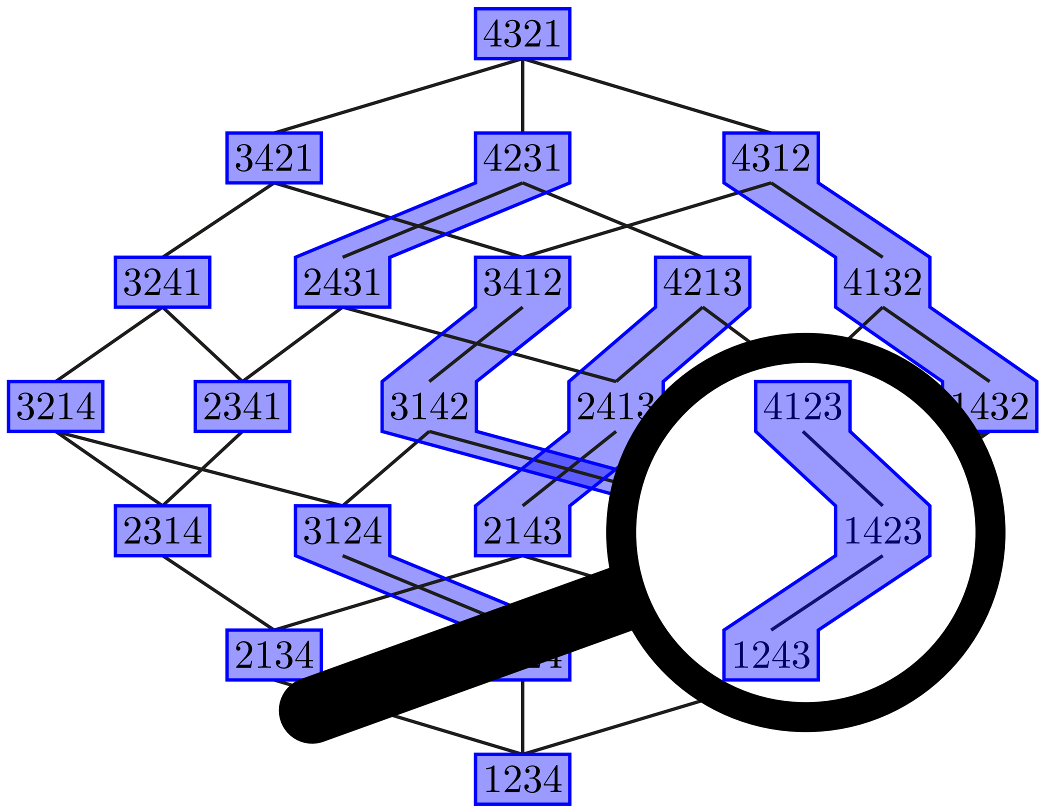

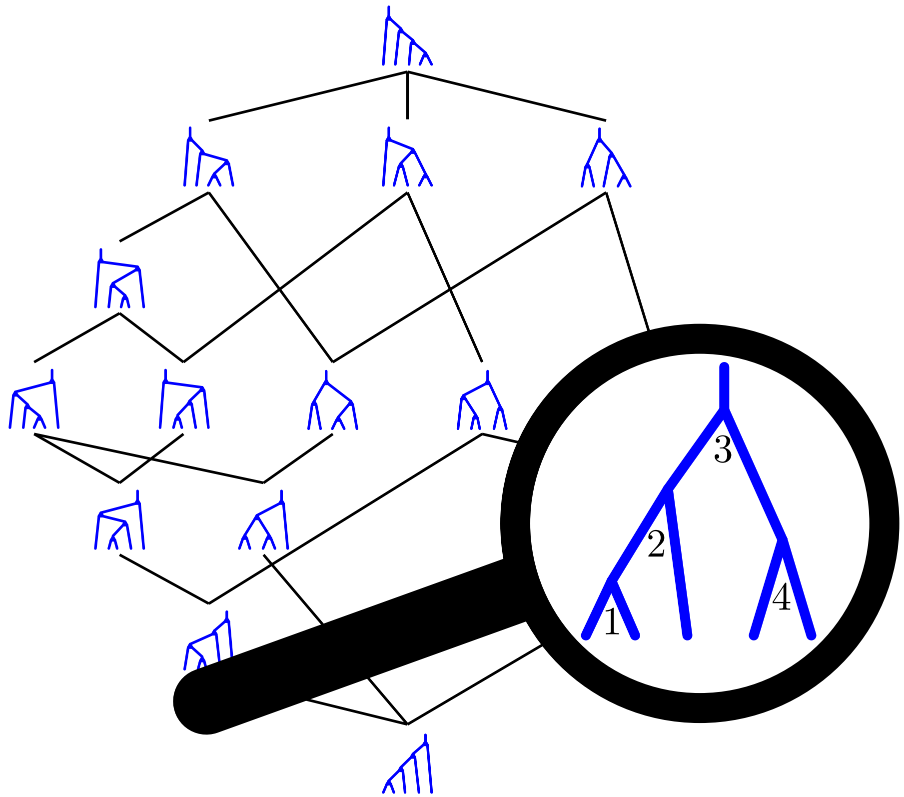

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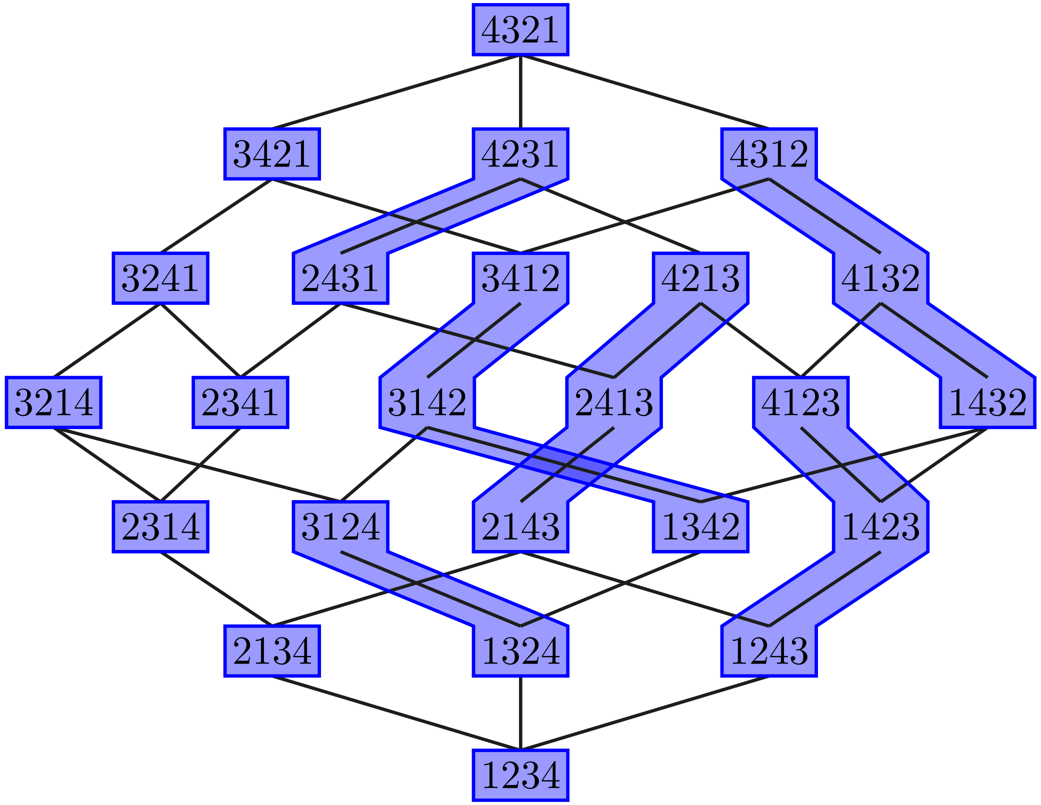

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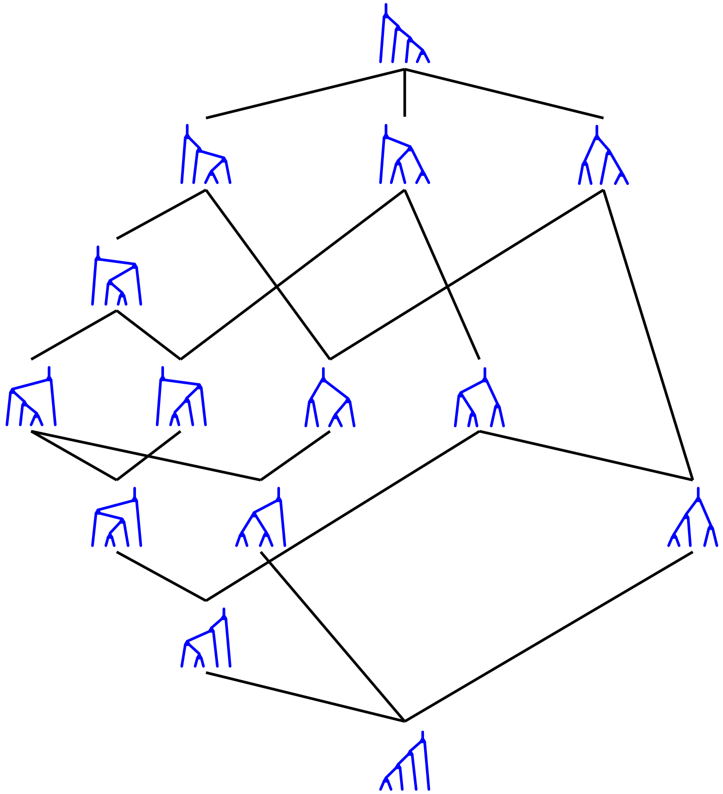


Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

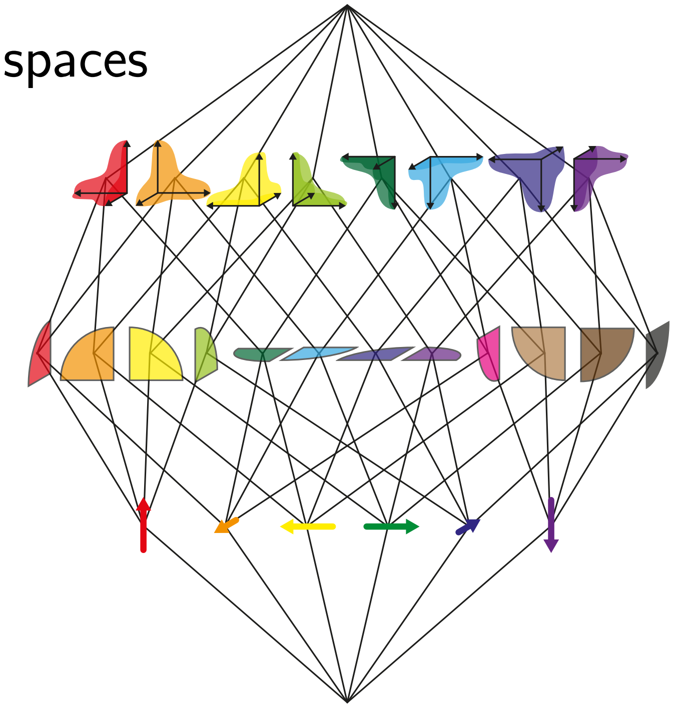
lattice congruence = equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \implies x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

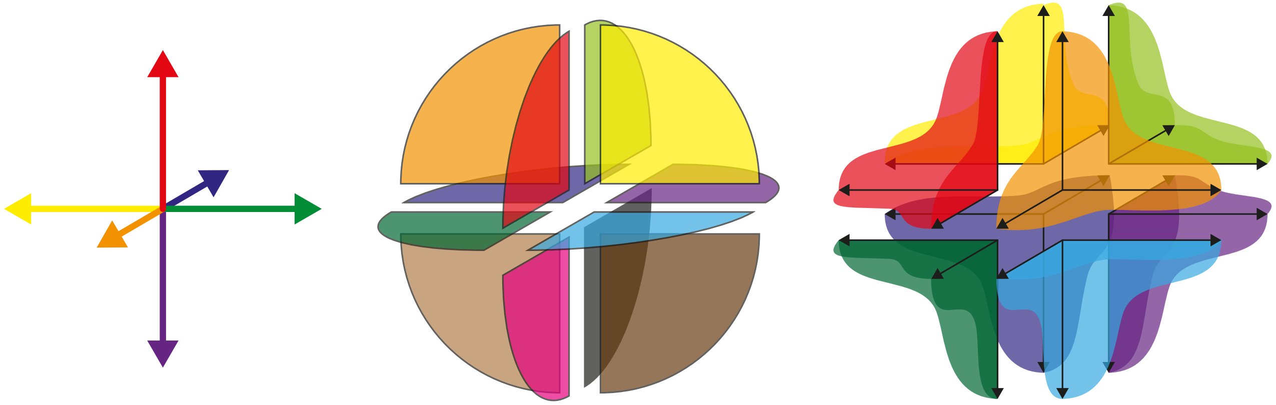
quotient lattice = lattice on classes with $X \leq Y \iff \exists\, x \in X,\ y \in Y, x \leq y$

FANS: BRAID FAN AND SYLVESTER FAN

polyhedral cone = positive span of a finite set of vectors
= intersection of a finite set of linear half-spaces

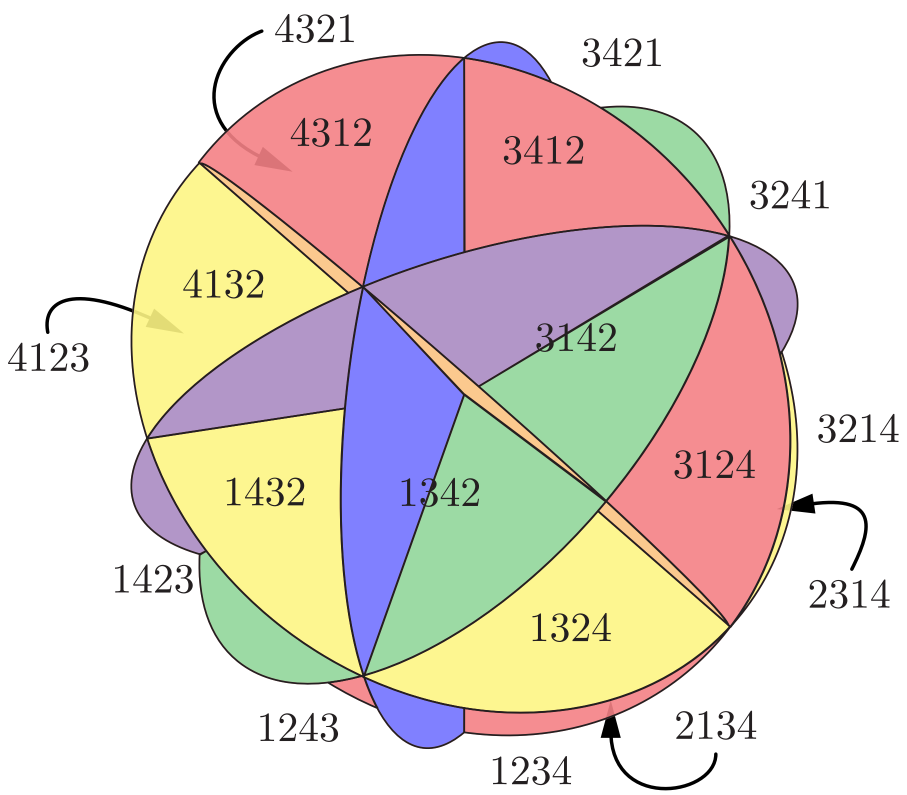


fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face



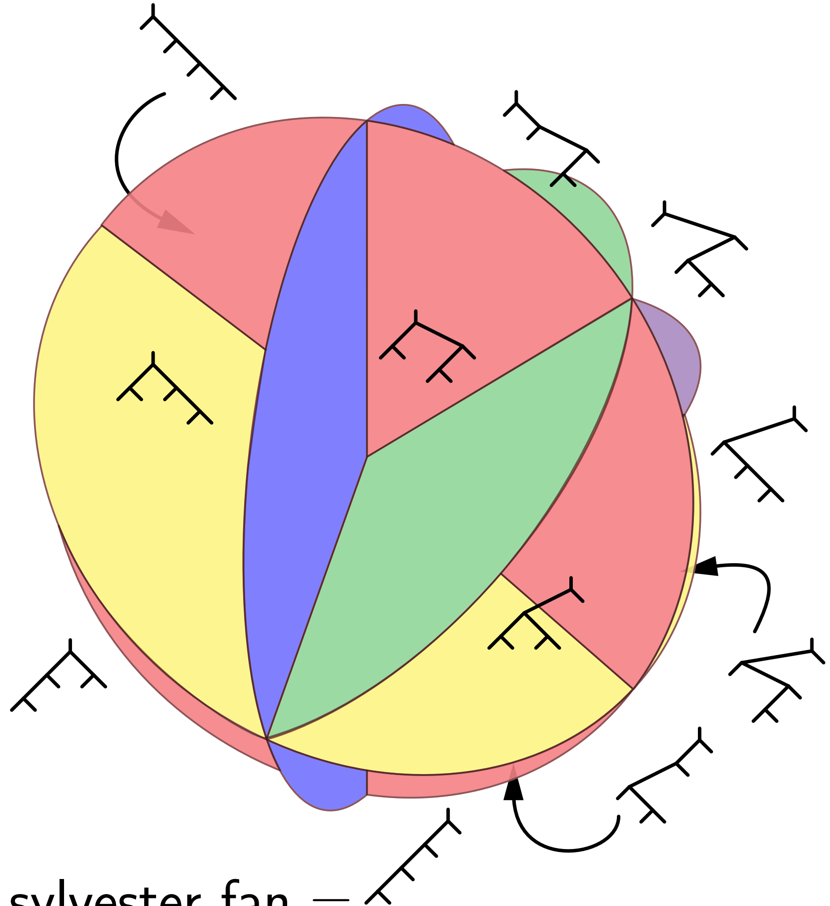
FANS: BRAID FAN AND SYLVESTER FAN

fan = collection of polyhedral cones closed by faces and intersecting along faces



braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$



sylvester fan =

$\mathbb{C}(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j$ if $i \to j$ in $T\}$

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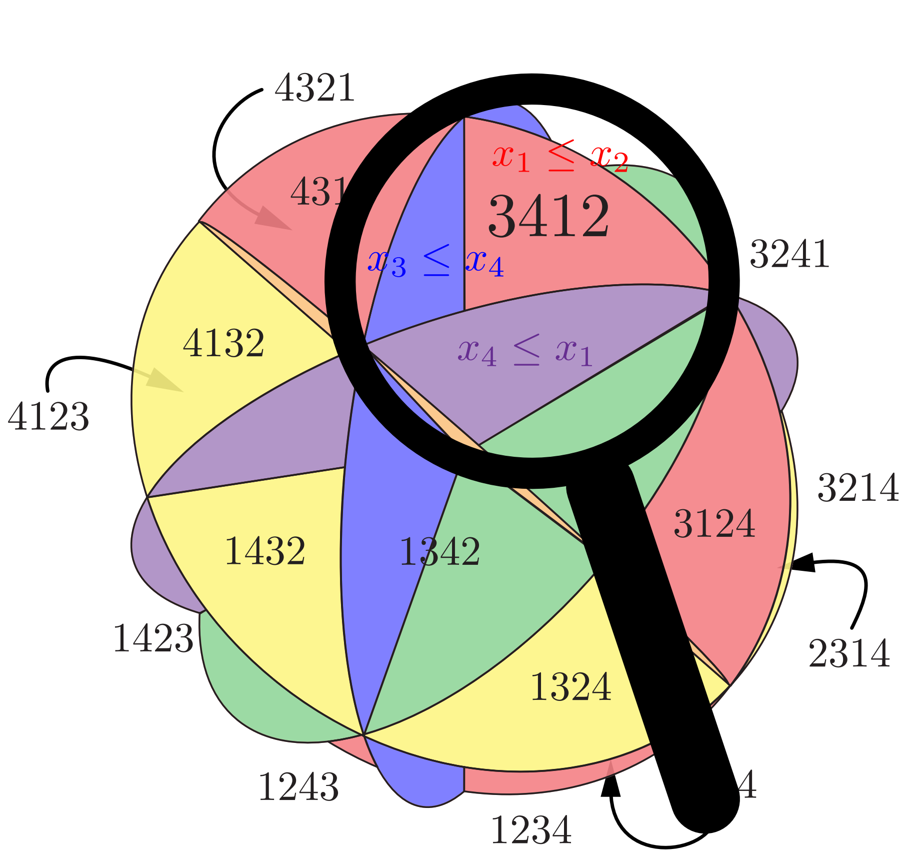

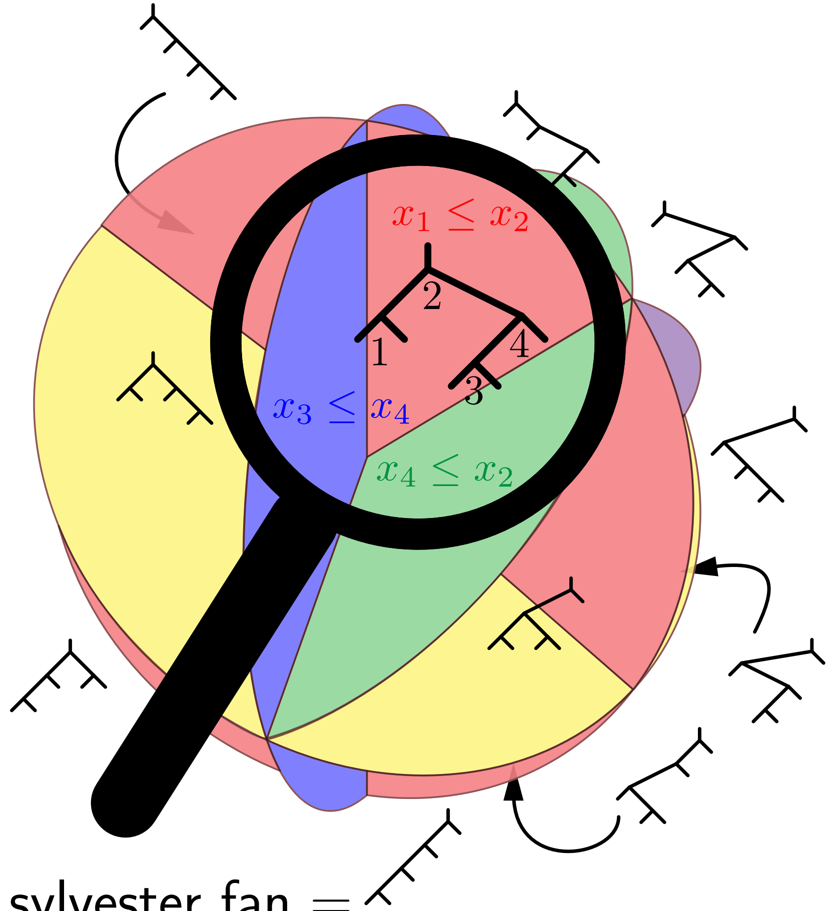

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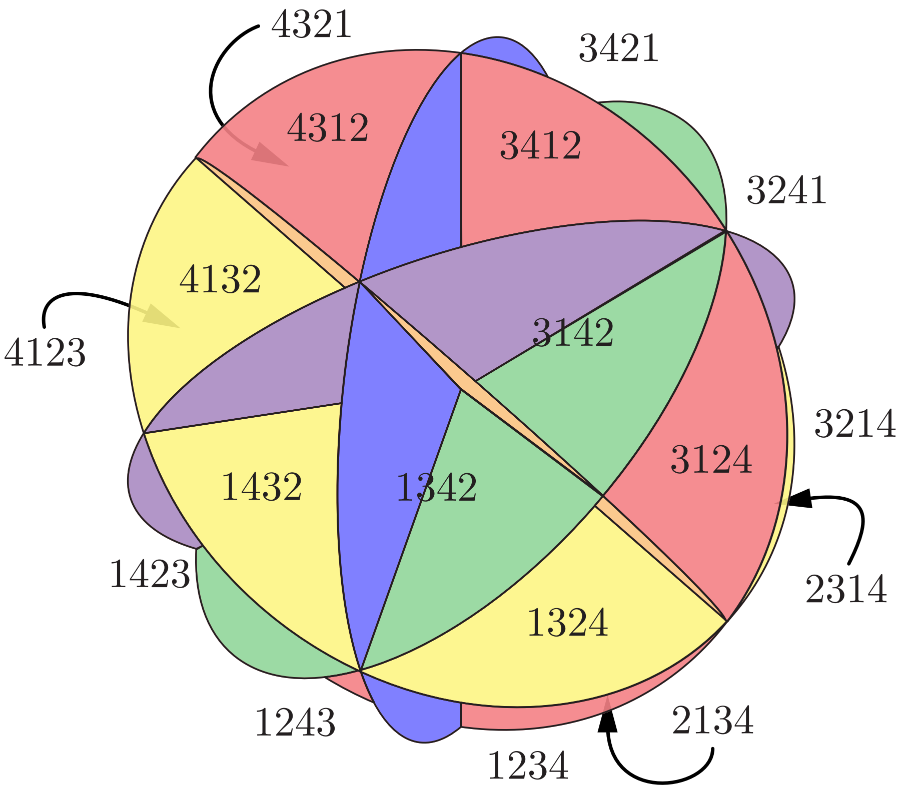

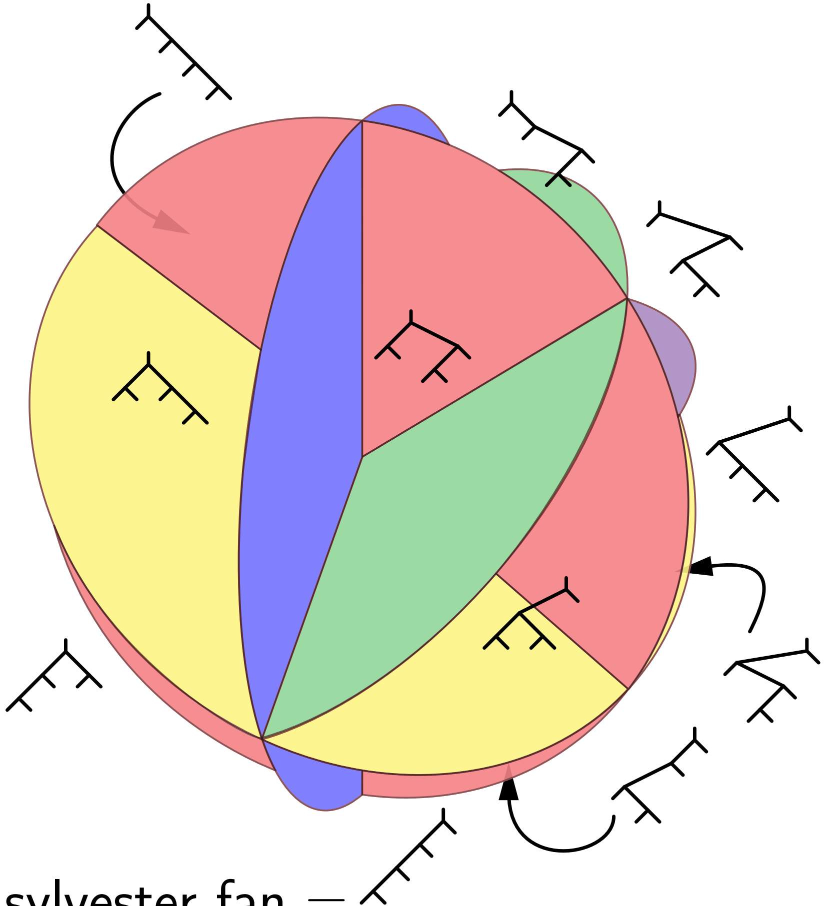

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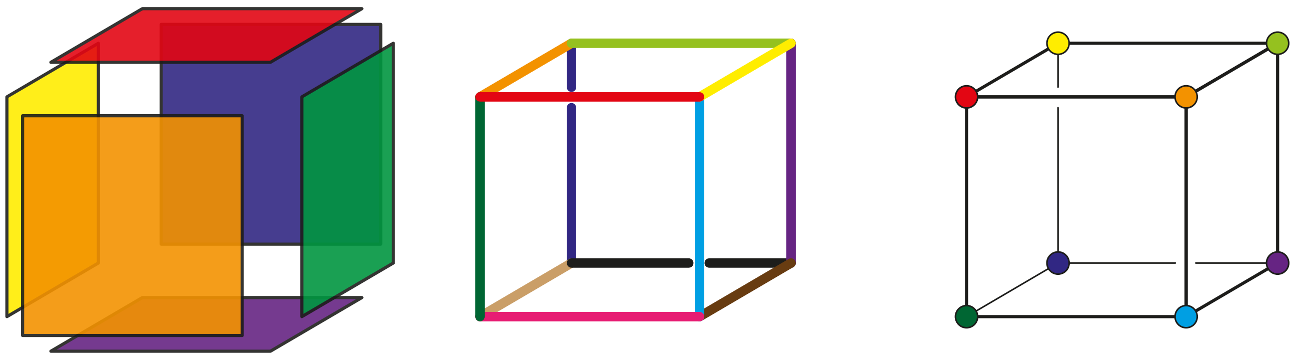
$$\mathbb{C}(T) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T \}$$

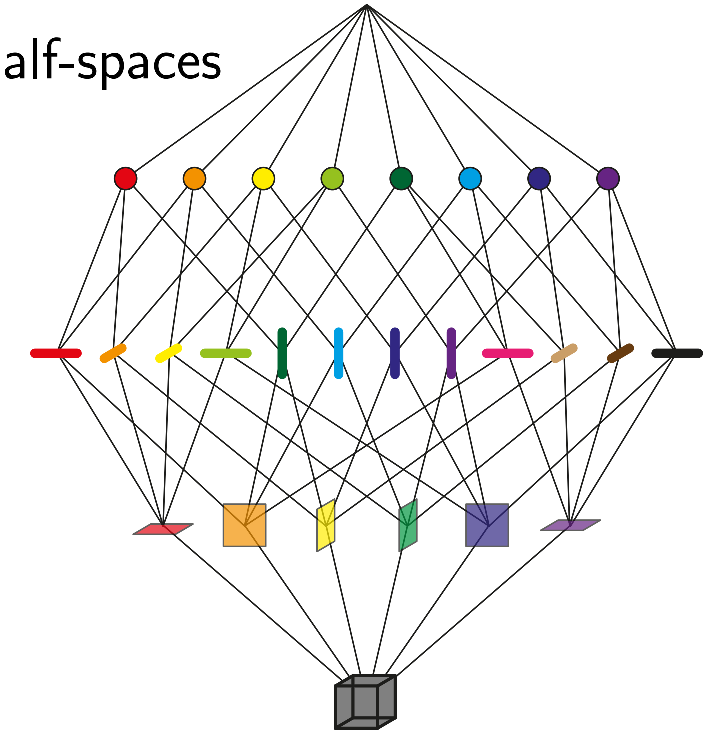
quotient fan = $\mathbb{C}(T)$ is obtained by glueing $\mathbb{C}(\sigma)$ for all linear extensions σ of T

POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

polytope = convex hull of a finite set of points
= bounded intersection of a finite set of affine half-spaces

face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations





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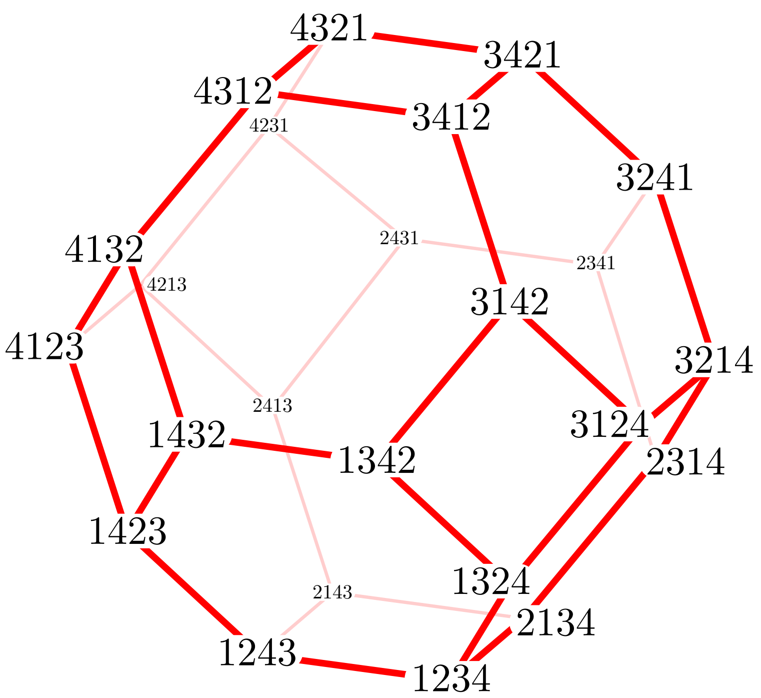

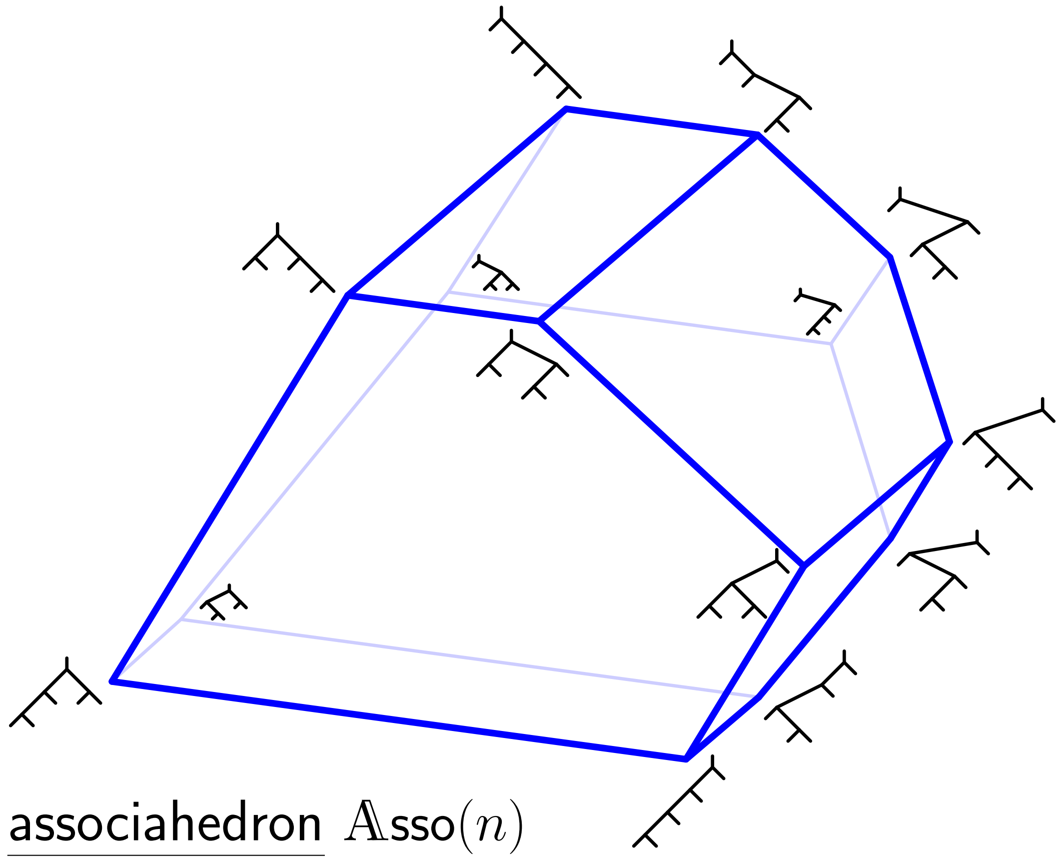

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$= \operatorname{conv}\left\{ [\sigma^{-1}(i)]_{i \in [n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$

$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$= \operatorname{conv}\left\{ [\ell(T,i) \cdot r(T,i)]_{i \in [n]} \;\middle|\; T \text{ binary tree} \right\}$

$= \mathbb{H} \cap \bigcap_{1 \leq i < j \leq n} \mathbb{H}_{[i,j]}$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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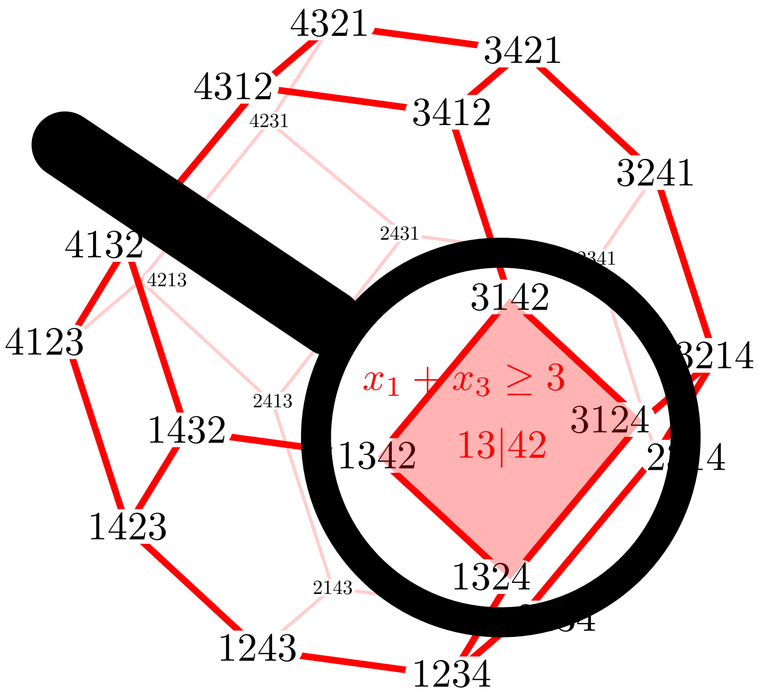

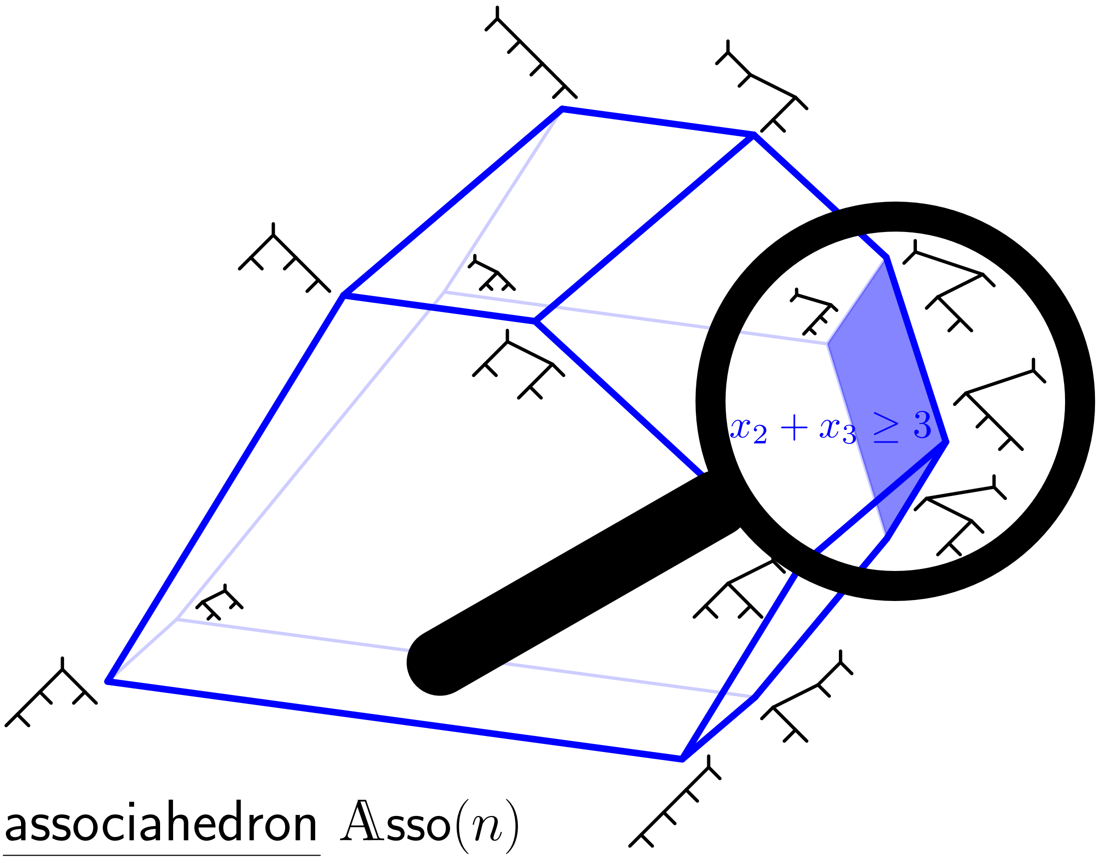

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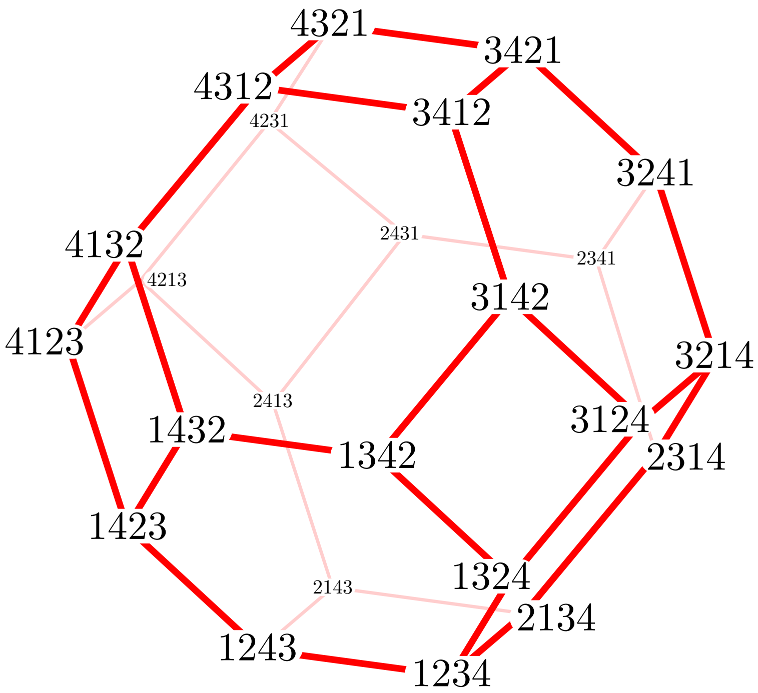

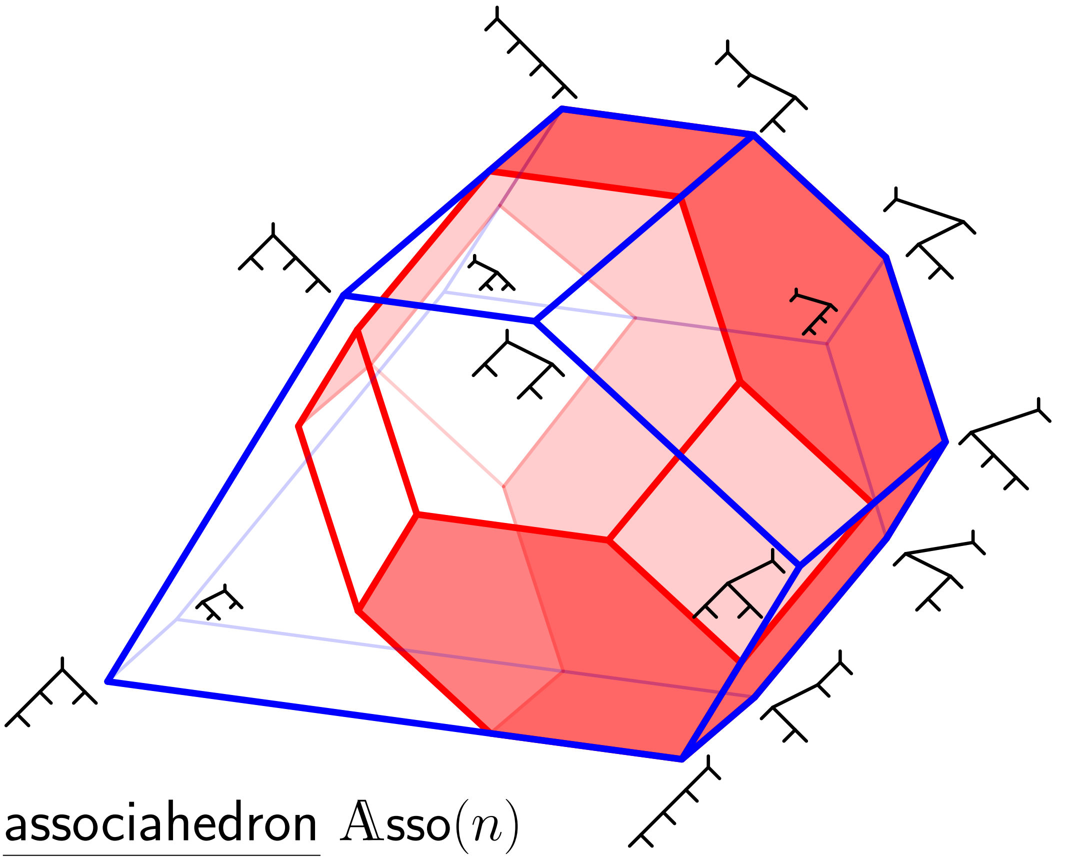

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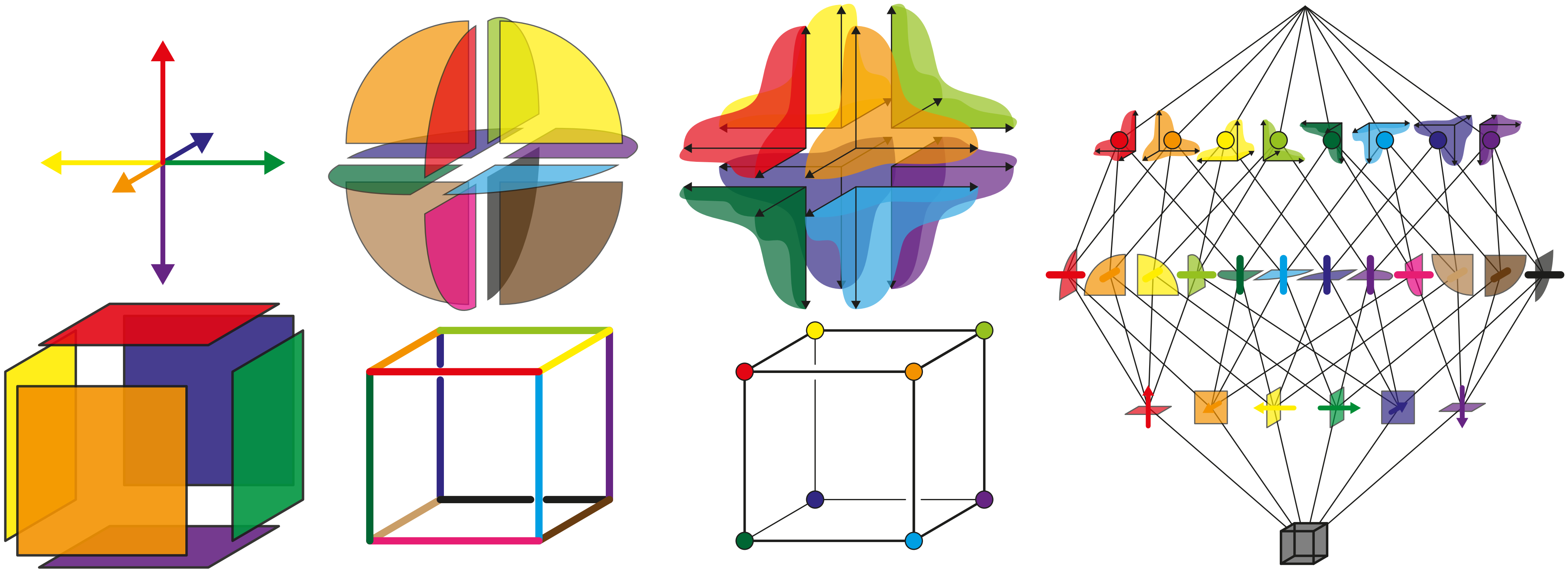
LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$\Longrightarrow$ braid fan

associahedron $\mathbb{A}\mathrm{sso}(n)$

$\Longrightarrow$ Sylvester fan



face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ $=$ positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ $= \{$ normal cone of $\mathbb{F}$ $\mid$ $\mathbb{F}$ face of $\mathbb{P}$ $\}$

LATTICES – FANS – POLYTOPES

<u>permutahedron</u> $\mathbb{P}\mathrm{erm}(n)$

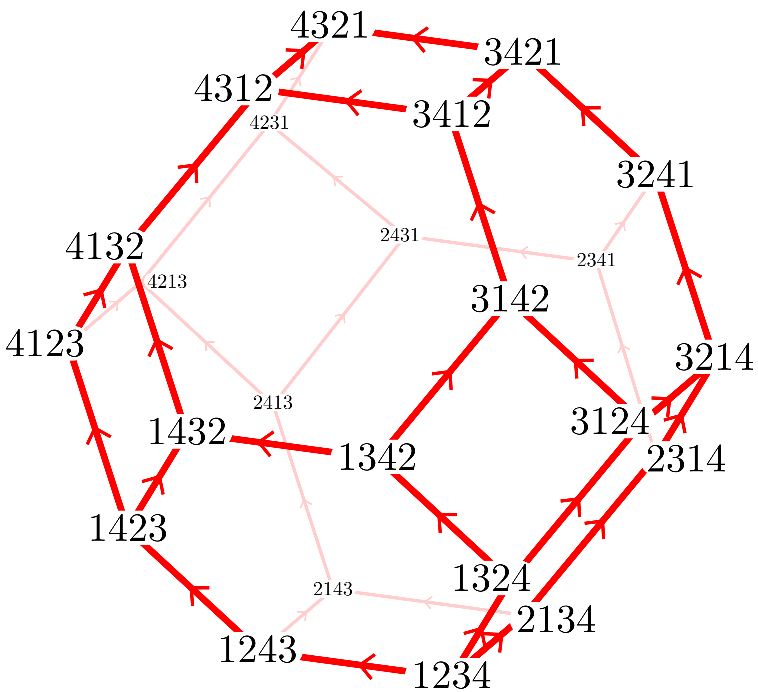
$\Longrightarrow$ braid fan

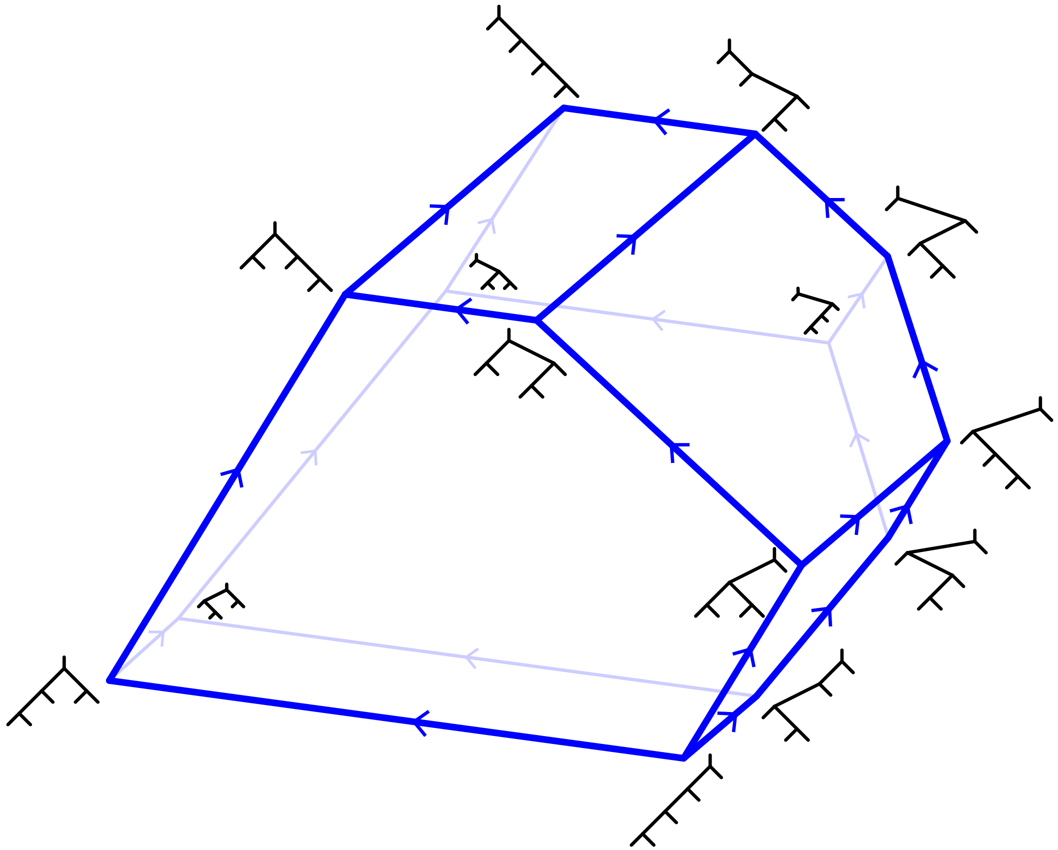
$\Longrightarrow$ weak order on permutations

<u>associahedron</u> $\mathbb{A}\mathrm{sso}(n)$

$\Longrightarrow$ Sylvester fan

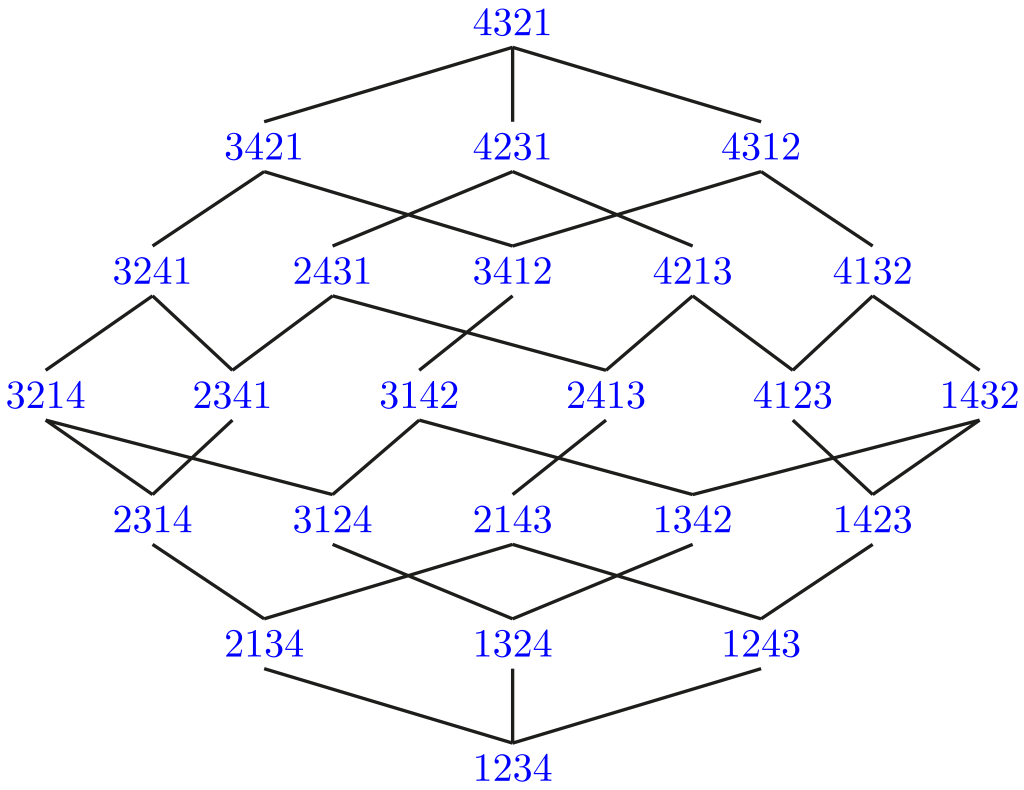
$\Longrightarrow$ Tamari lattice on binary trees





LATTICE THEORY OF THE WEAK ORDER

INVERSION SETS



weak order = permutations of $[n]$ ordered by paths of simple transpositions

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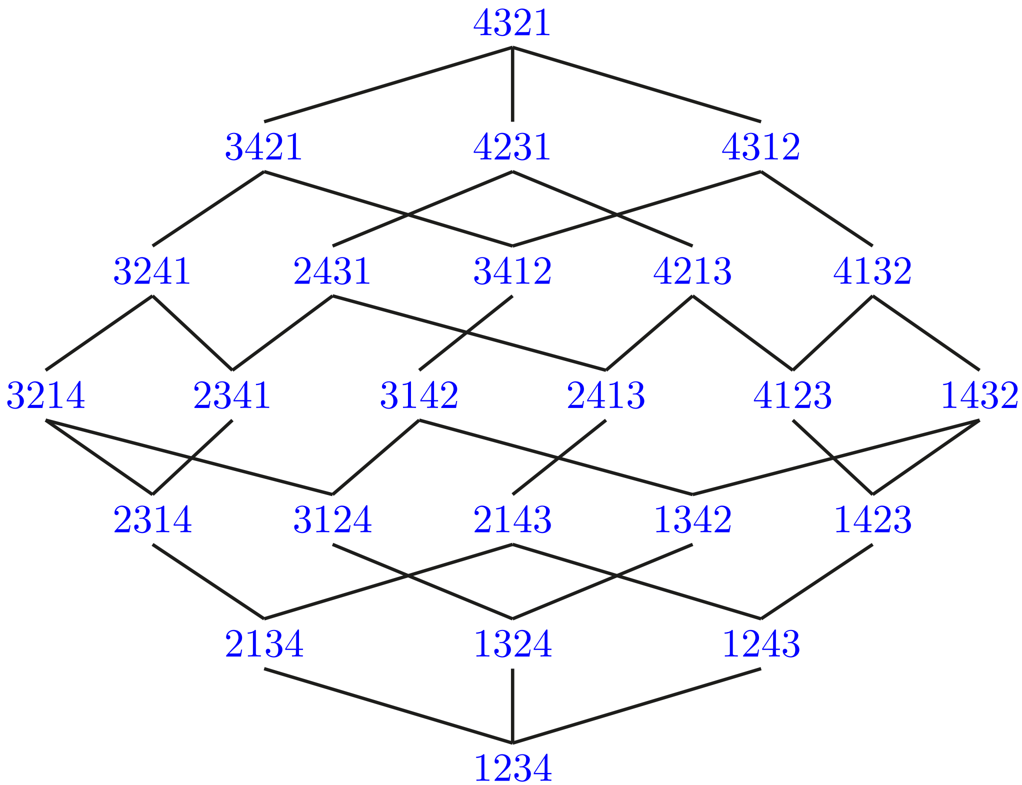
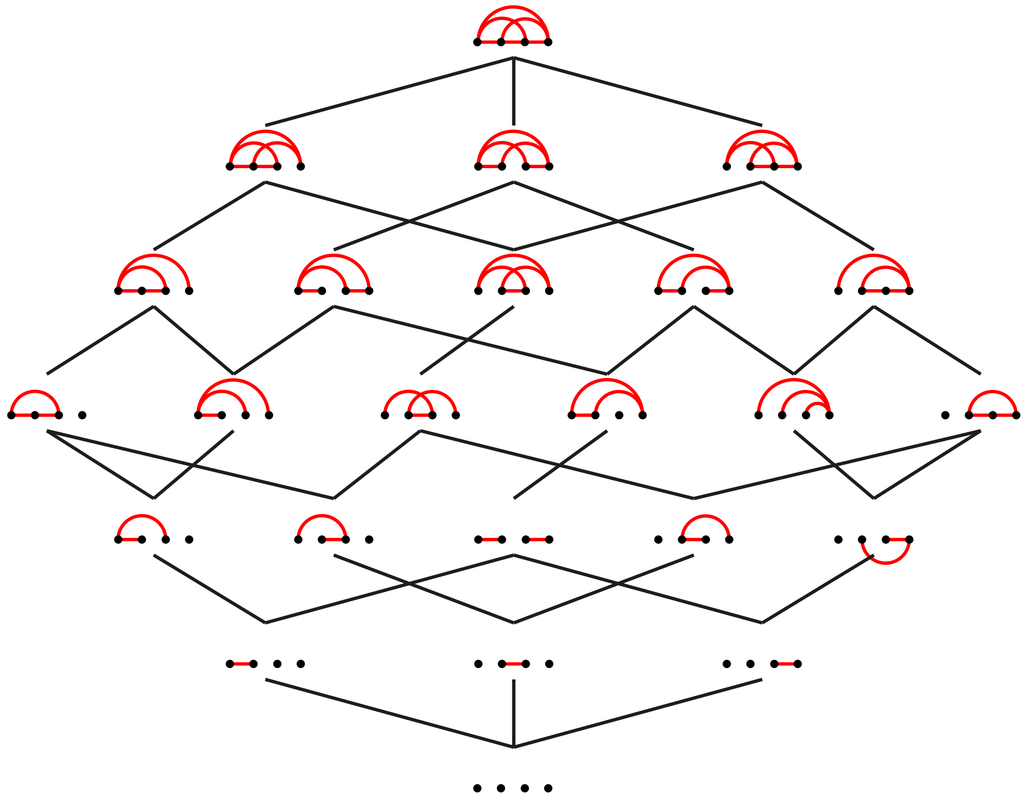




weak order = permutations of $[n]$ ordered by paths of simple transpositions
permutations of $[n]$ ordered by inclusion of inversion sets

inversion of σ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

PROP. inversion sets = transitive and cotransitive subsets of $\{(b, a) \mid 1 \le a < b \le n\}$

$\mathrm{inv}(\sigma_1 \vee \ldots \vee \sigma_k)$ = transitive closure of $\mathrm{inv}(\sigma_1) \cup \cdots \cup \mathrm{inv}(\sigma_k)$

$\mathrm{ninv}(\sigma_1 \wedge \ldots \wedge \sigma_k)$ = transitive closure of $\mathrm{ninv}(\sigma_1) \cup \cdots \cup \mathrm{ninv}(\sigma_k)$

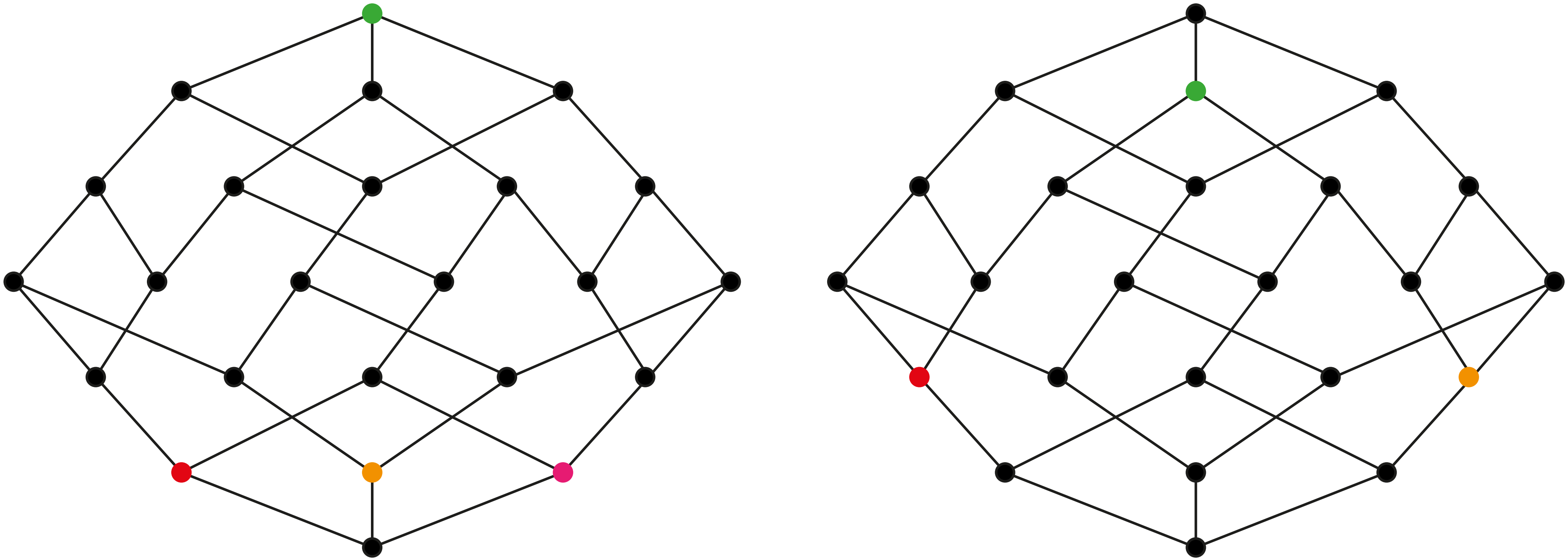
CANONICAL JOIN REPRESENTATIONS

join representation of $y \in L$ = subset $J \subseteq L$ such that $y = \bigvee J$

$y = \bigvee J$ irredundant if $\nexists\, J' \subsetneq J$ with $y = \bigvee J'$

ordered by containement of order ideals: $J \le J' \iff \forall\, z \in J,\ \exists\, z' \in J',\ z \le z'$

canonical join representation of y = minimal irredundant join representation of y
= lowest way to write y as a join



$\Longrightarrow$ a canonical join representation is an antichain of join irreducible elements of L

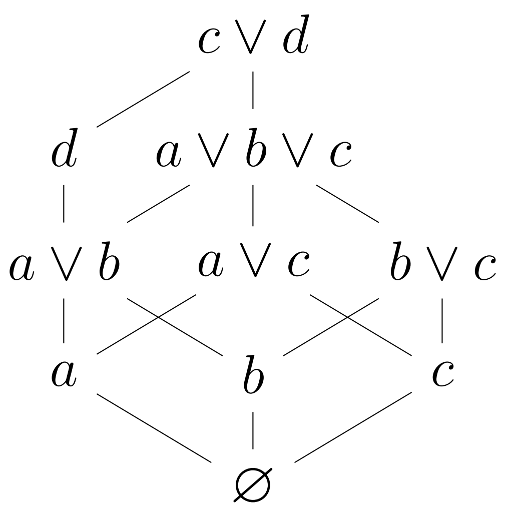
DISTRIBUTIVE AND SEMIDISTRIBUTIVE LATTICES

$(L, \leq, \wedge, \vee)$ finite lattice is

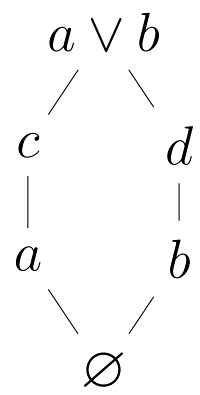
- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

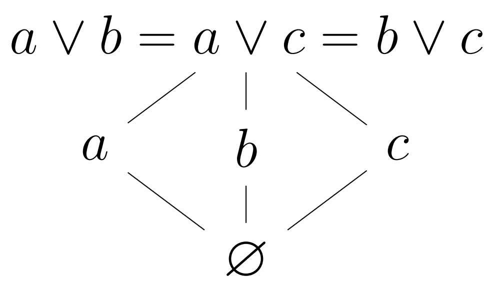
- semidistributive if both join and meet semidistributive



distributive



semidistributive



not semidistributive

DISTRIBUTIVE AND SEMIDISTRIBUTIVE LATTICES

$(L, \leq, \wedge, \vee)$ finite lattice is

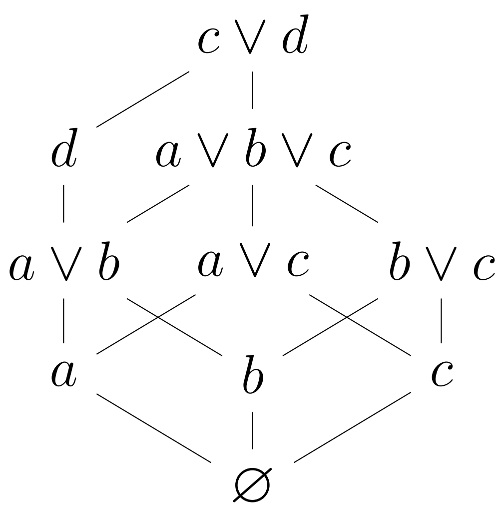
- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

  $\Longrightarrow$ canonical join representations $=$ antichains of join irreducibles
  $\Longrightarrow$ $L \simeq$ inclusion poset of lower ideals of $JI(L)$

- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ admits the canonical join representation $y = \bigvee_{x \lessdot y} k_\vee(x, y)$
  where $k_\vee(x, y)$ is the unique minimal element of $\{z \in L \mid x \vee z = y\}$

- semidistributive if both join and meet semidistributive



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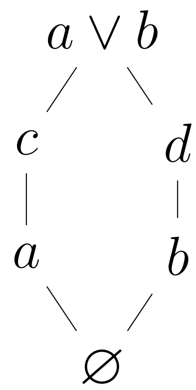

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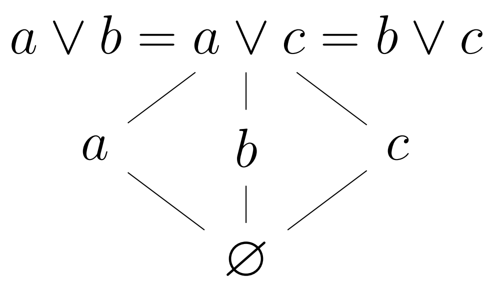
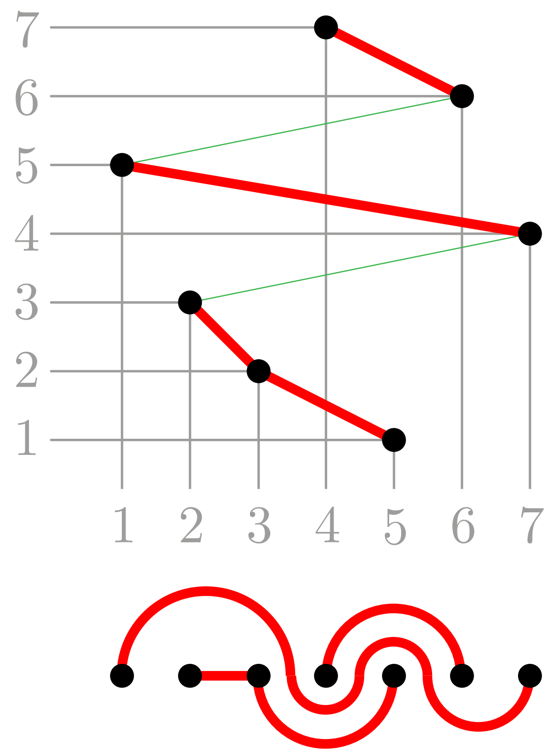


not semidistributive

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

draw all points (σ_i, i) and all segments
from (σ_i, i) to $(\sigma_{i+1}, i+1)$ with $\sigma_i > \sigma_{i+1}$
and project down to an horizontal line allowing
arcs to bend but not to cross or pass points

permutation $\sigma = 2537146$



noncrossing arc diagram $\delta(\sigma)$

arc $=$ x-monotone curve joining two points
and wiggling around the horizontal axis
(up to deformations)

compatible arcs $=$

- left$(\alpha) \neq$ left(α') and right$(\alpha) \neq$ right(α'),
- α and α' are not crossing.

noncrossing arc diagrams $=$ set of pairwise compatible arcs

THM. δ is a bijection from permutations to noncrossing arc diagrams — Reading ('15)

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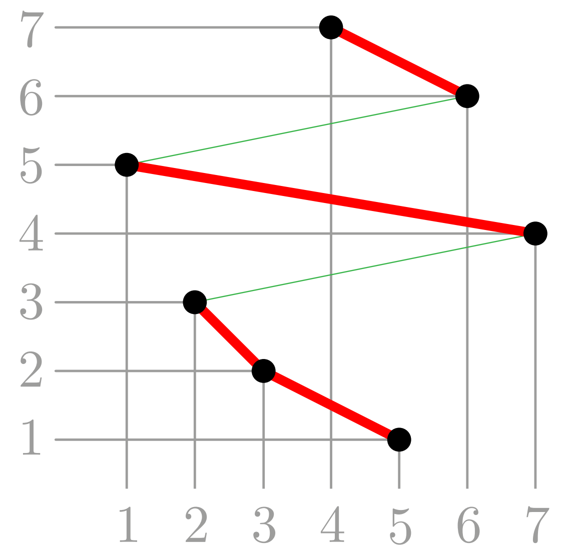
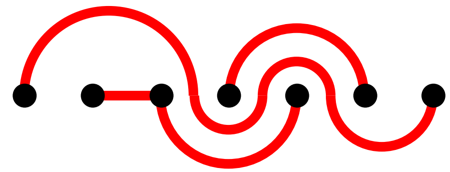
permutation $\sigma = 5327164$





noncrossing arc diagram $\delta(\sigma)$

arc $=$ x-monotone curve joining two points and wiggling around the horizontal axis (up to deformations)

$\iff$ quadruple (a, b, A, B) with $a < b$ and $]a, b[= A \sqcup B$

compatible arcs $=$

- left$(\alpha) \neq$ left(α') and right$(\alpha) \neq$ right(α'),
- α and α' are not crossing.

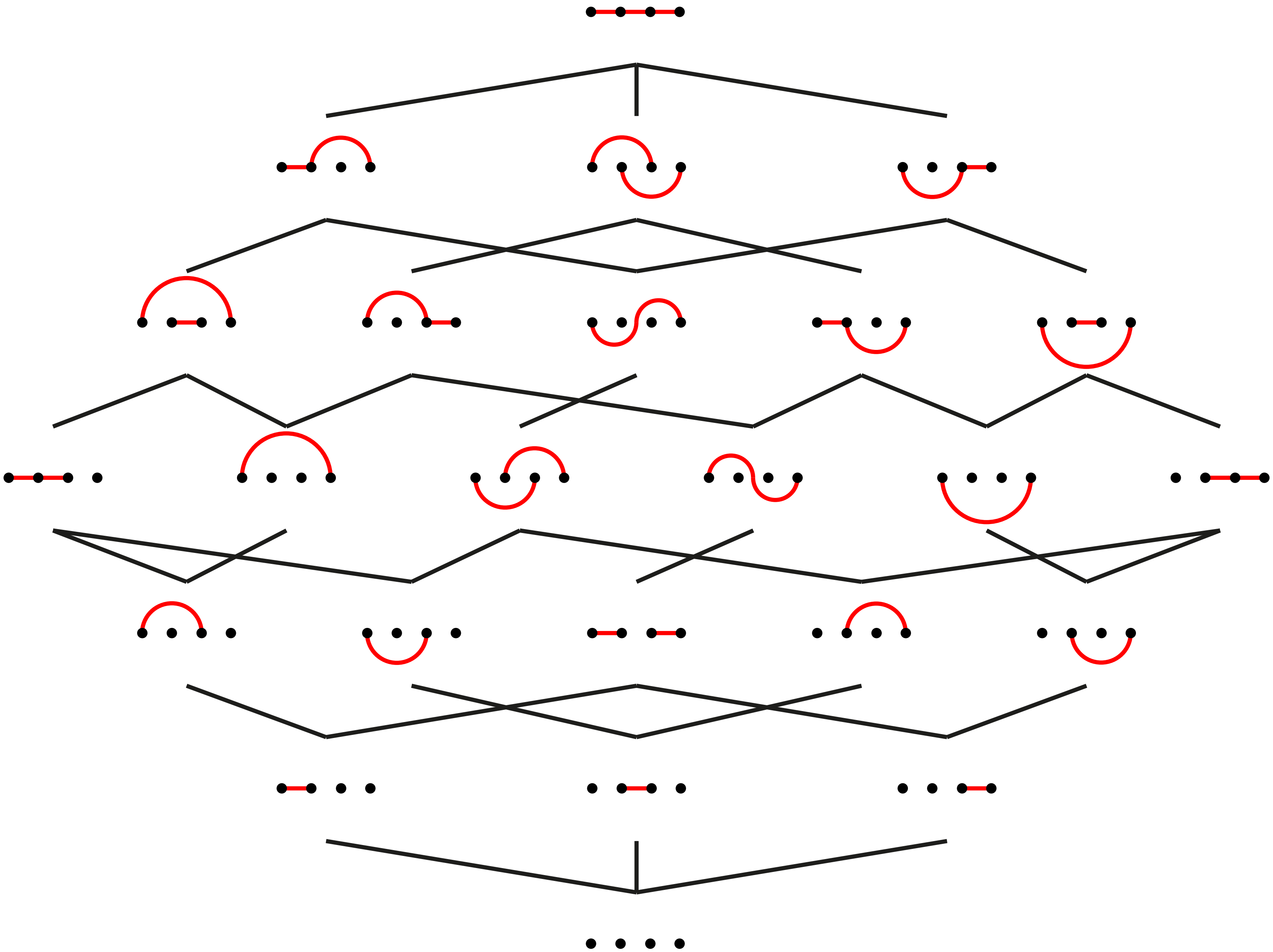
$\iff$ $\alpha = (a, b, A, B)$ and $\alpha' = (a', b', A', B')$ such that there is no $x \neq x'$ with

$$x \in (A \cup \{a, b\}) \cap (B' \cup \{a', b'\}) \text{ and } x' \in (B \cup \{a, b\}) \cap (A' \cup \{a', b'\})$$

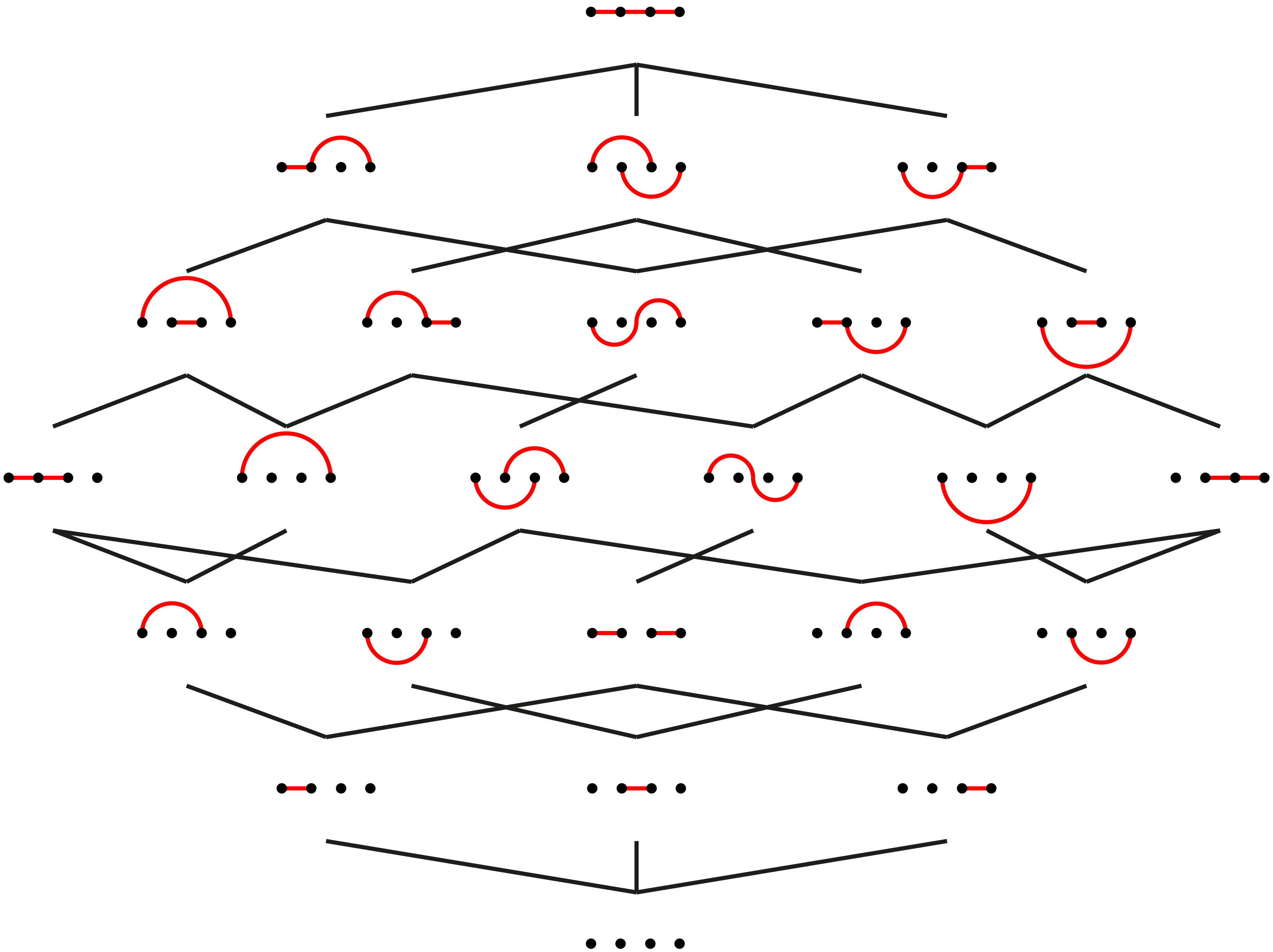
noncrossing arc diagrams $=$ set of pairwise compatible arcs

THM. δ is a bijection from permutations to noncrossing arc diagrams — Reading ('15)

WEAK ORDER ON NONCROSSING ARC DIAGRAMS



CANONICAL JOIN REPRESENTATIONS AND NONCROSSING ARC DIAGRAMS



THM. $\sigma = \bigvee_{\alpha \in \delta(\sigma)} \delta^{-1}(\{\alpha\})$ is the canonical join representation — Reading ('15)

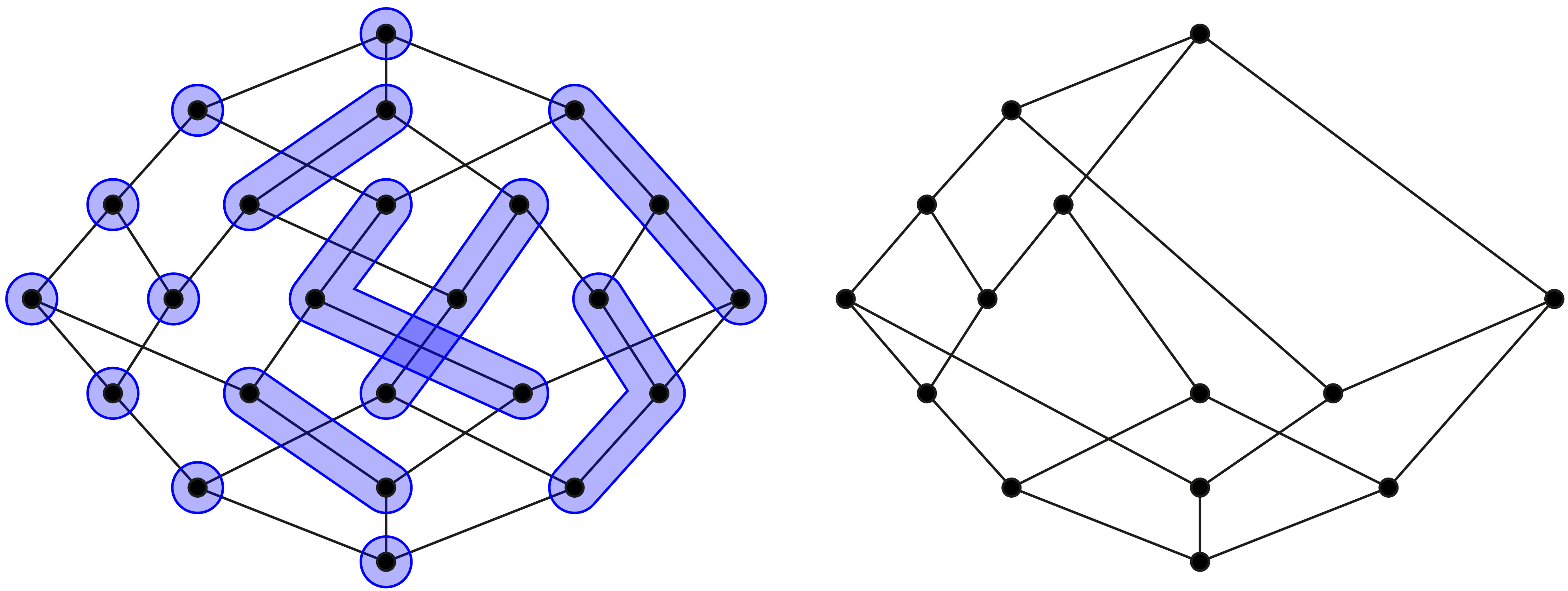
LATTICE CONGRUENCES AND LATTICE QUOTIENTS

lattice congruence of L = equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \Longrightarrow x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

lattice quotient of $L/\equiv$ = lattice on equivalence classes of L under $\equiv$ where

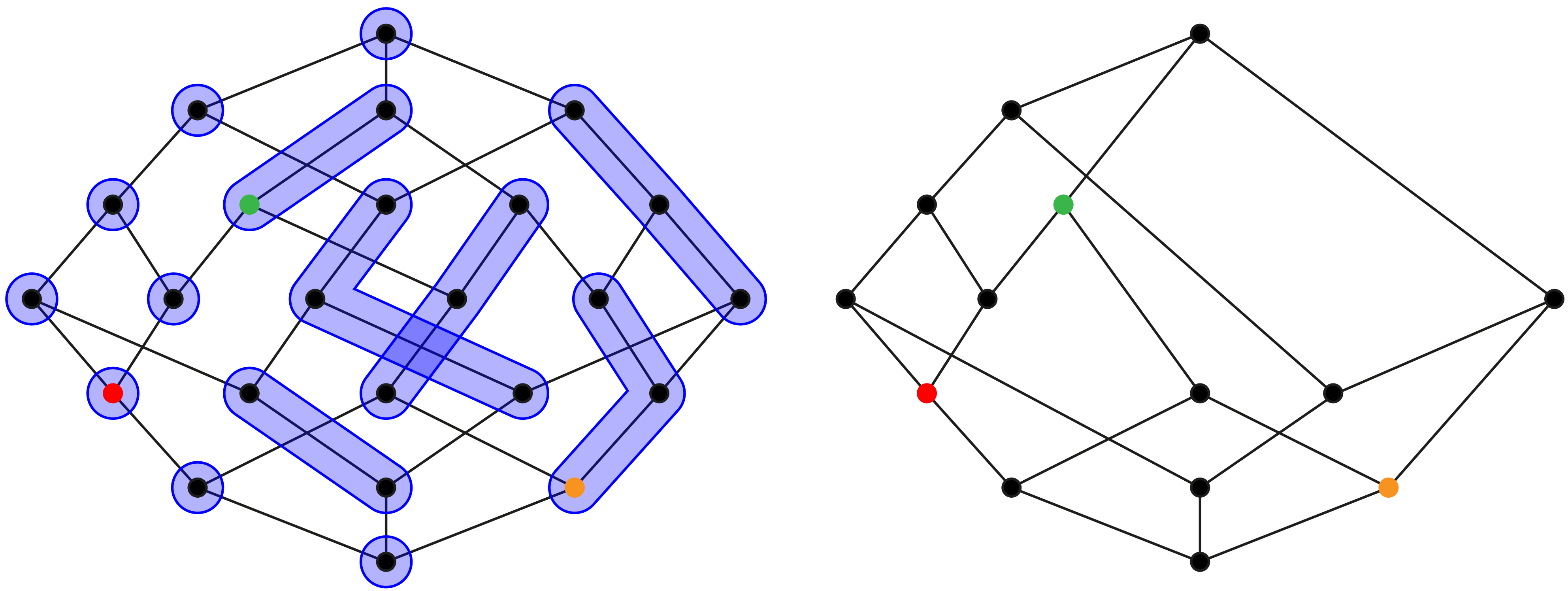
- $X \le Y \iff \exists\, x \in X,\ y \in Y, \quad x \le y$
- $X \wedge Y$ = equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ = equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$



LATTICE QUOTIENTS AND CANONICAL JOIN REPRESENTATIONS

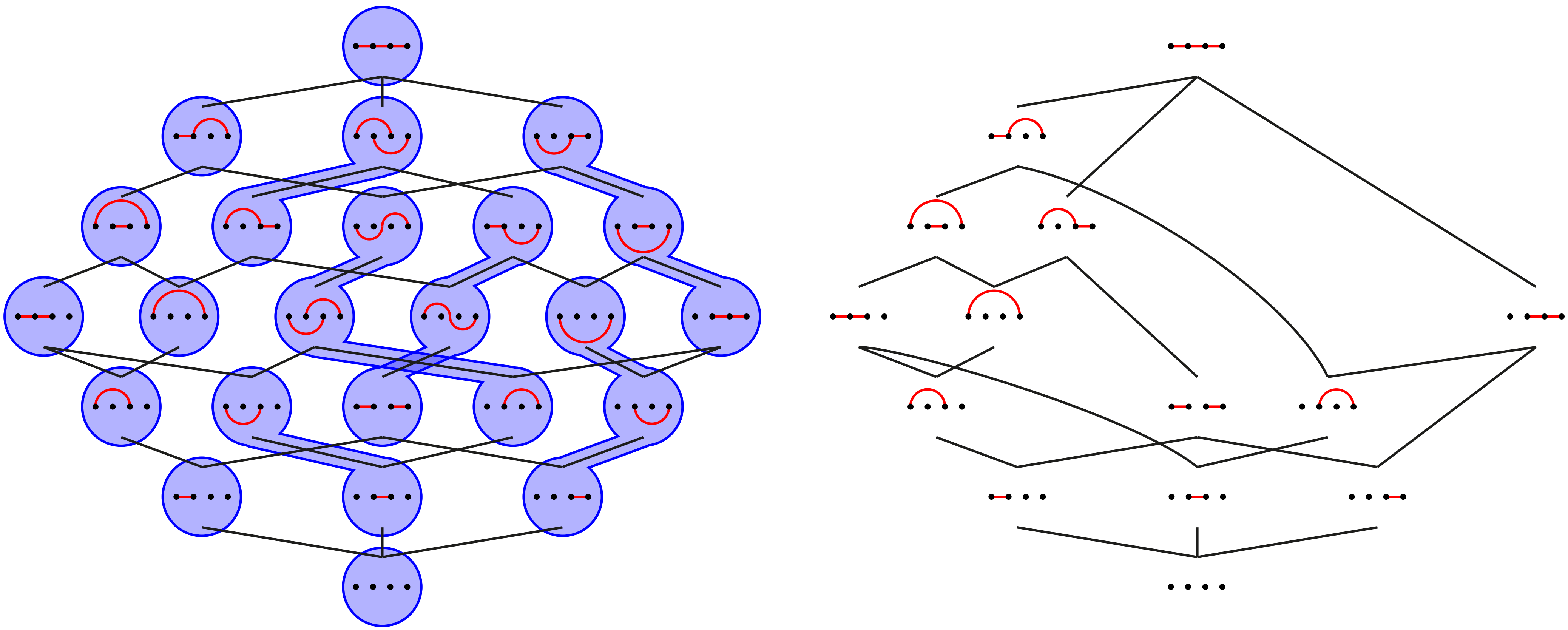
$\equiv$ lattice congruence on L, then

- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/\equiv$ is isomorphic (as poset) to the restriction of L to the elements x with $\pi_\downarrow(x) = x$
- $\pi_\downarrow(x) = x$ if and only if $\pi_\downarrow(j) = j$ for all canonical joinands j of x
- canonical join representations in $L/\equiv$ are canonical join representations in L that only involve join irreducibles j with $\pi_\downarrow(j) = j$



LATTICE QUOTIENTS OF THE WEAK ORDER

THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
$\mathcal{A}_{\equiv}$ = arcs corresponding to join irreducibles σ with $\pi_{\downarrow}(\sigma) = \sigma$
$\mathfrak{S}_n/\!\equiv\; \simeq$ subposet induced by noncrossing arc diagrams with all arcs in $\mathcal{A}_{\equiv}$



Reading ('15)

SUBARC ORDER

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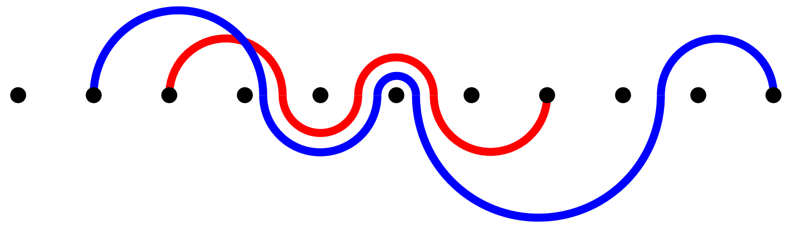
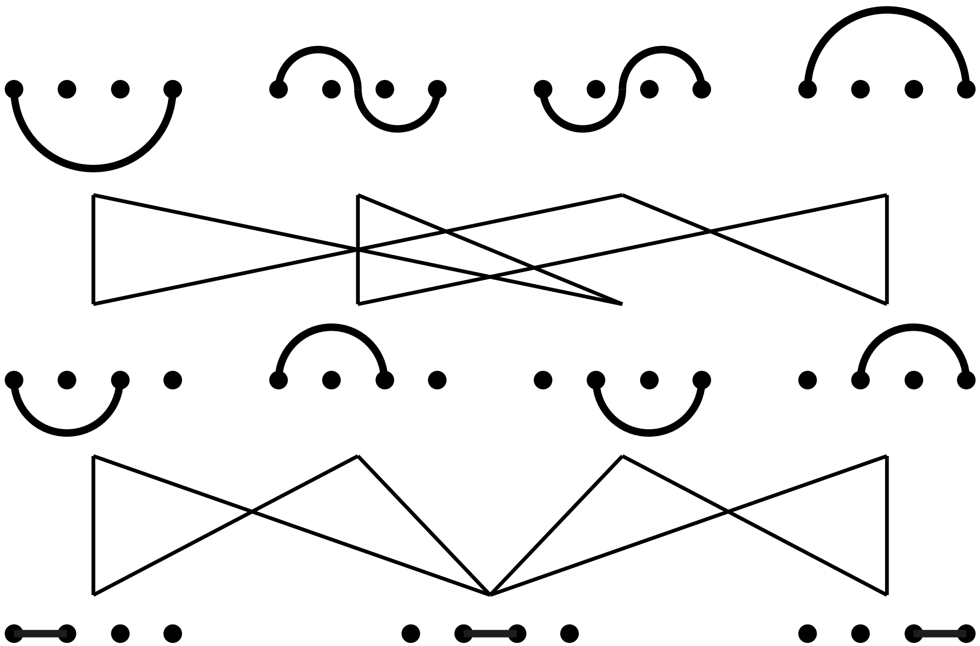
THM. The following are equivalent for a set of arcs $\mathcal{A}$:

- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{A} = \mathcal{A}_\equiv$
- $\mathcal{A}$ is a lower ideal of the subarc order





Reading ('15)

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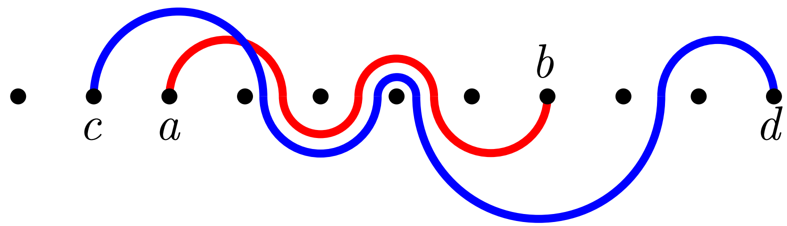
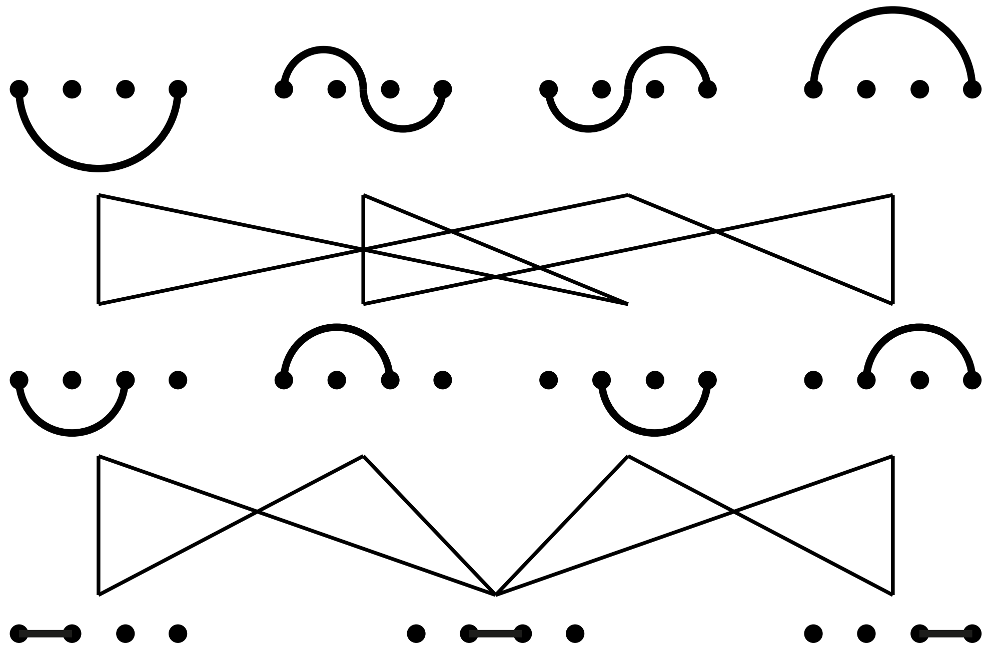


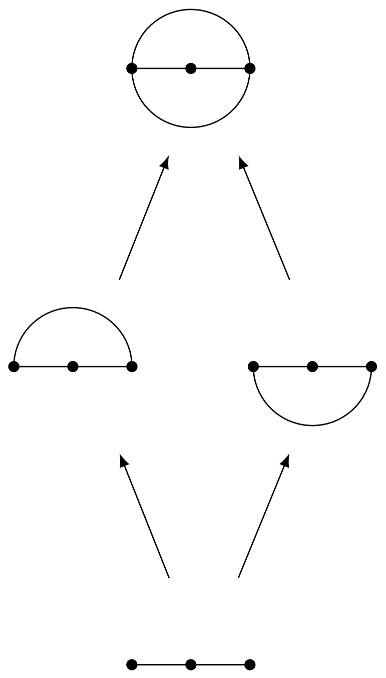
(a, b, A, B) subarc of (c, d, C, D) $\iff$
$c < a < b < d$ and $A \subseteq C$ and $B \subseteq D$



Reading ('15)

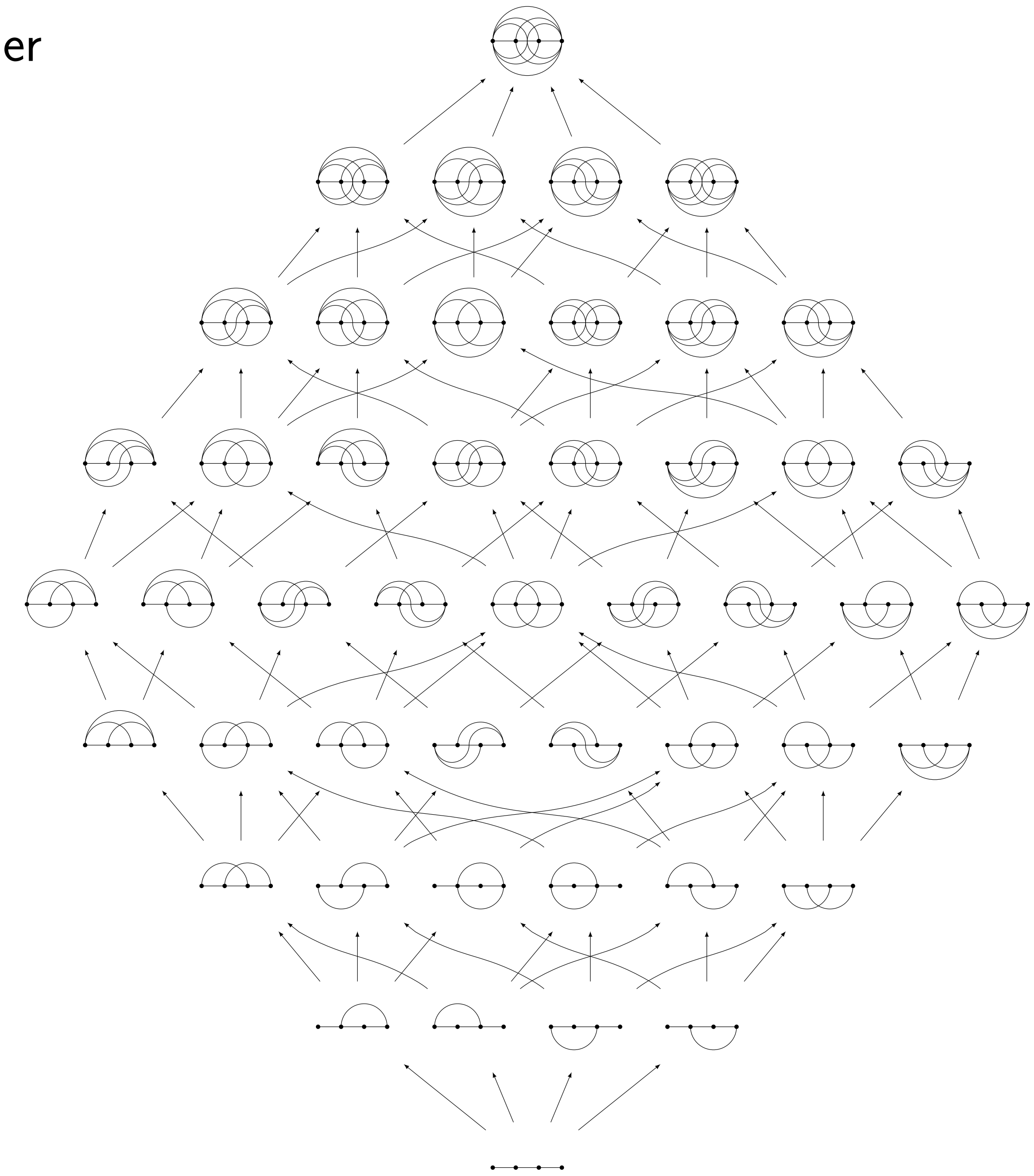
ARC IDEALS

arc ideal = lower ideal of the subarc order



essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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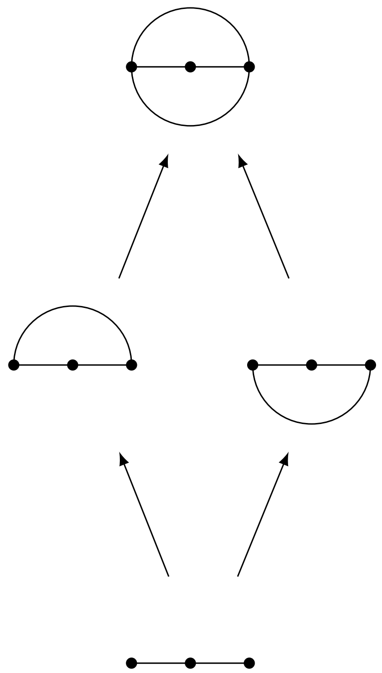

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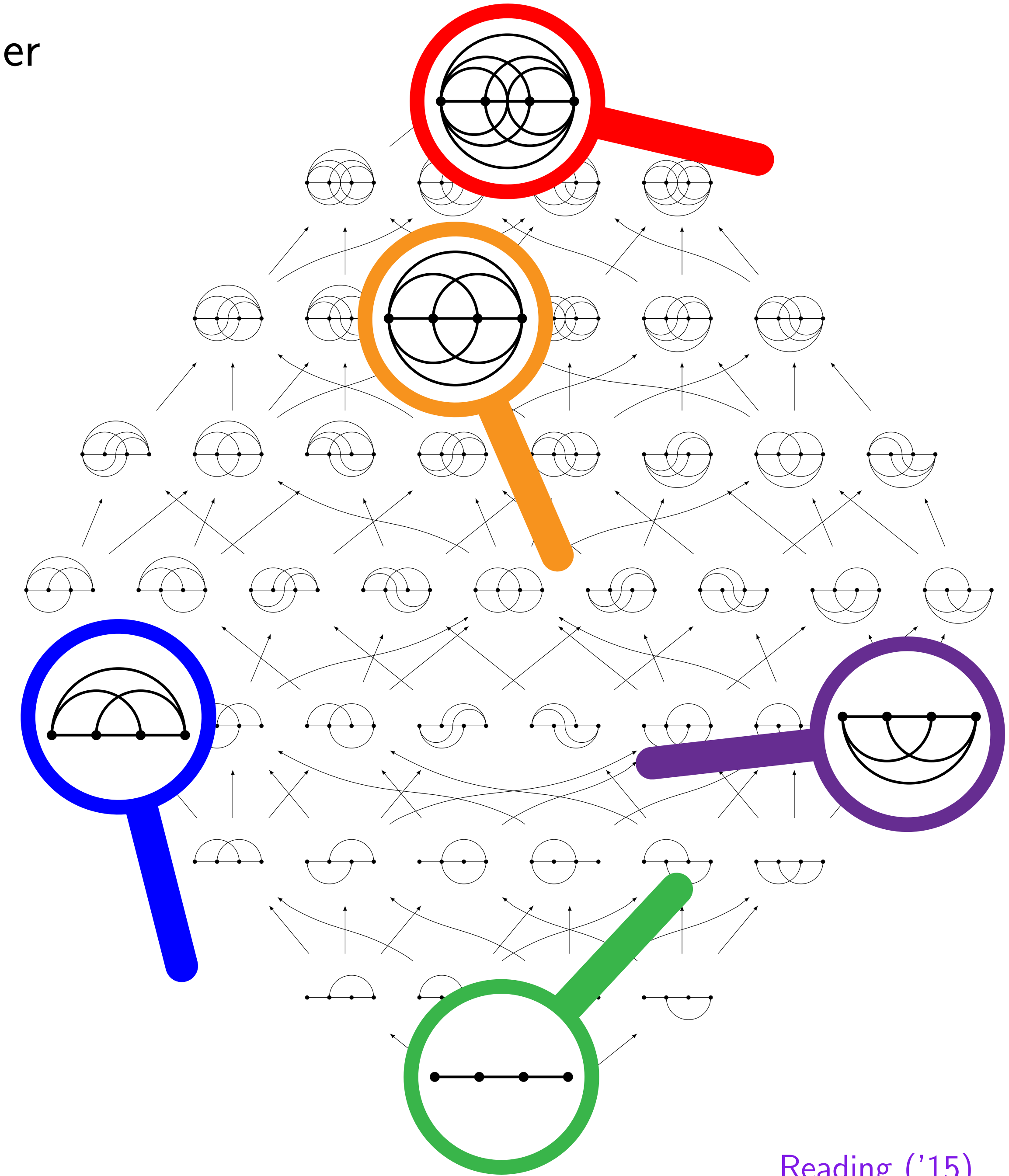
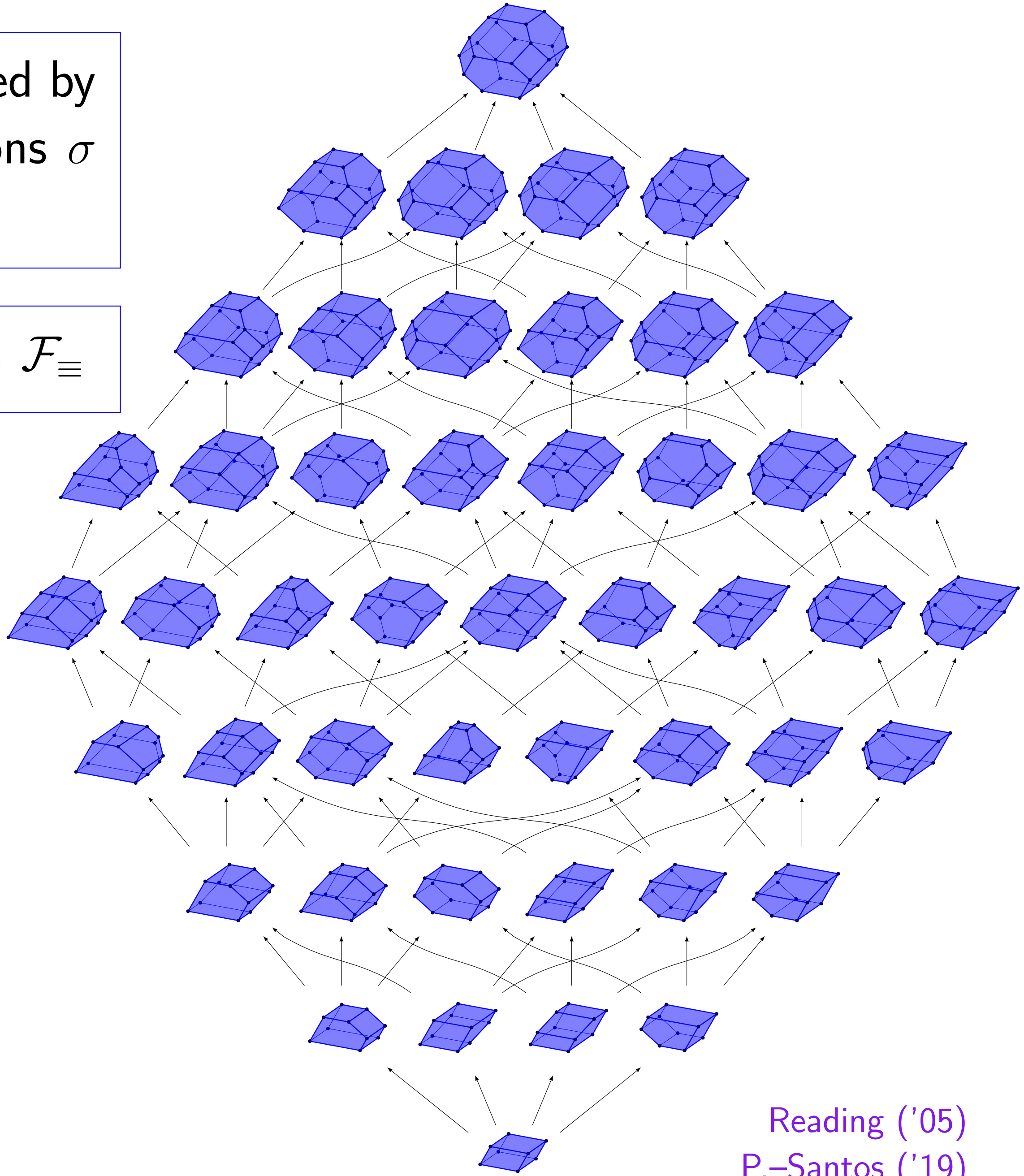


Reading ('15)

QUOTIENT FANS & QUOTIENTOPES

quotient fan $\mathcal{F}_{\equiv}$ = chambers are obtained by glueing the chambers of the permutations σ in the same congruence class of $\equiv$

quotientope = polytope with normal fan $\mathcal{F}_{\equiv}$



Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('20^{+})

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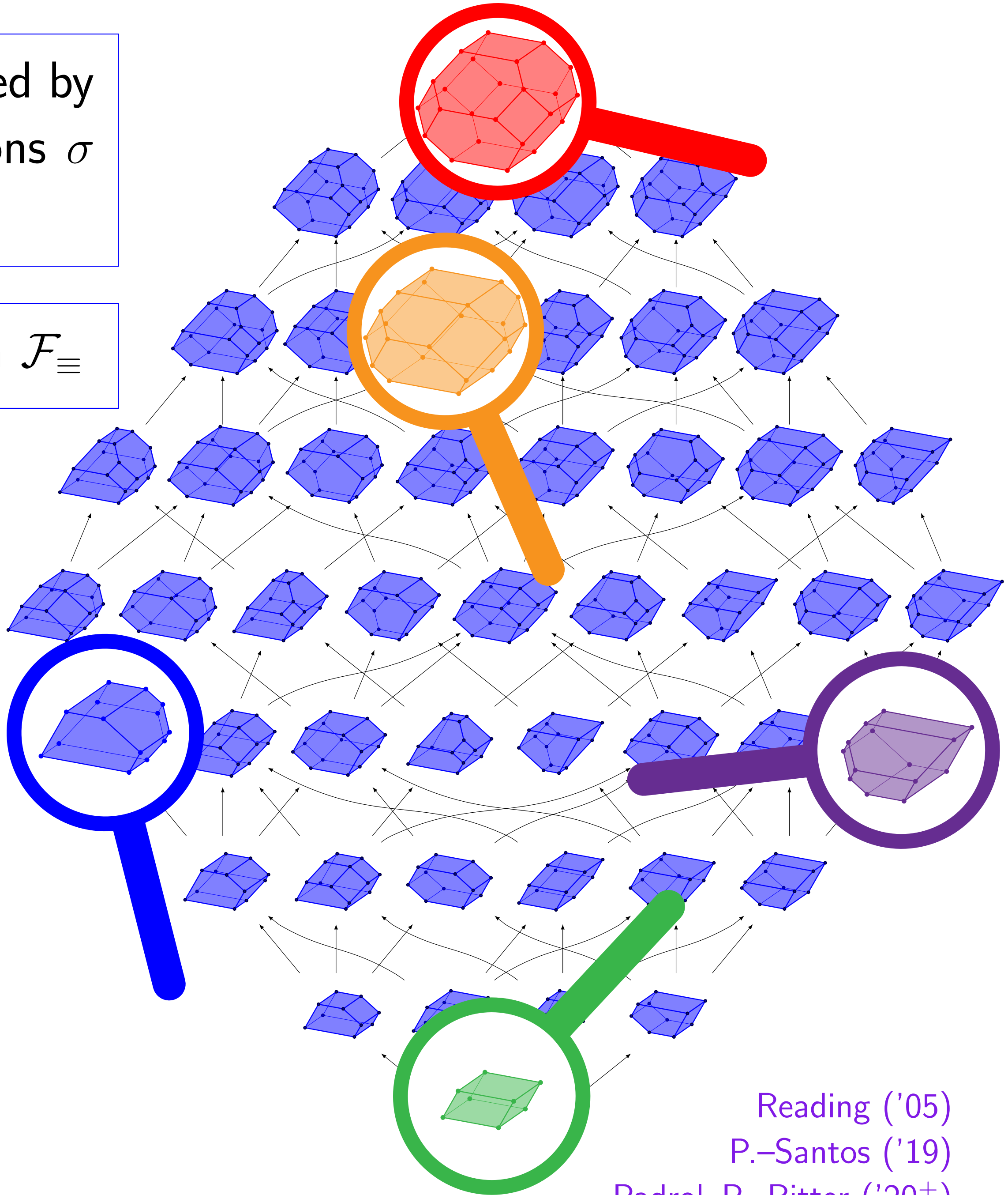

Reading ('05)
P.–Santos ('19)
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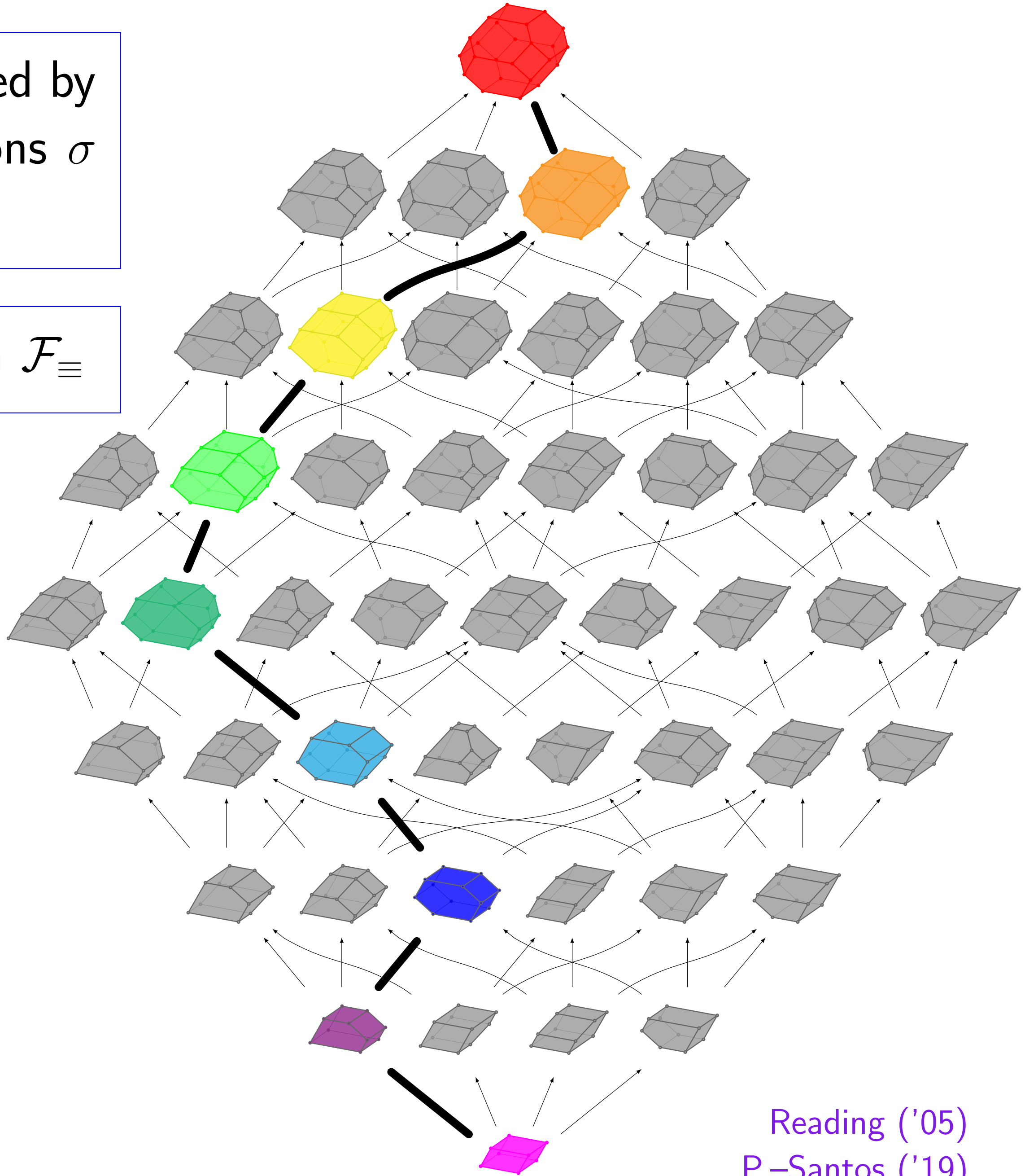


POLYWOOD

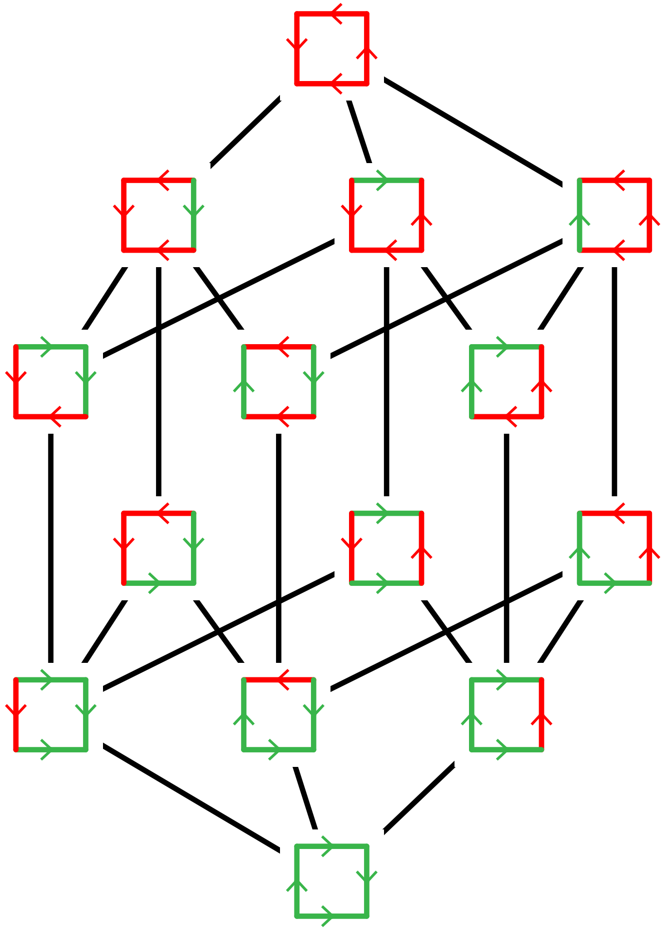
Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('20$^+$)

ACYCLIC REORIENTATION LATTICES

ACYCLIC REORIENTATION POSETS

D directed acyclic graph

$\mathcal{AR}_D$ = all acyclic reorientations of D, ordered by inclusion of their sets of reversed arcs



minimal element D

maximal element $\bar{D}$

self-dual under reversing all arcs

cover relations = flipping a single arc

flippable arcs of E = transitive reduction of E

$= E \smallsetminus \{(u,v) \in E \mid \exists$ directed path $u \rightsquigarrow v$ in $E\}$

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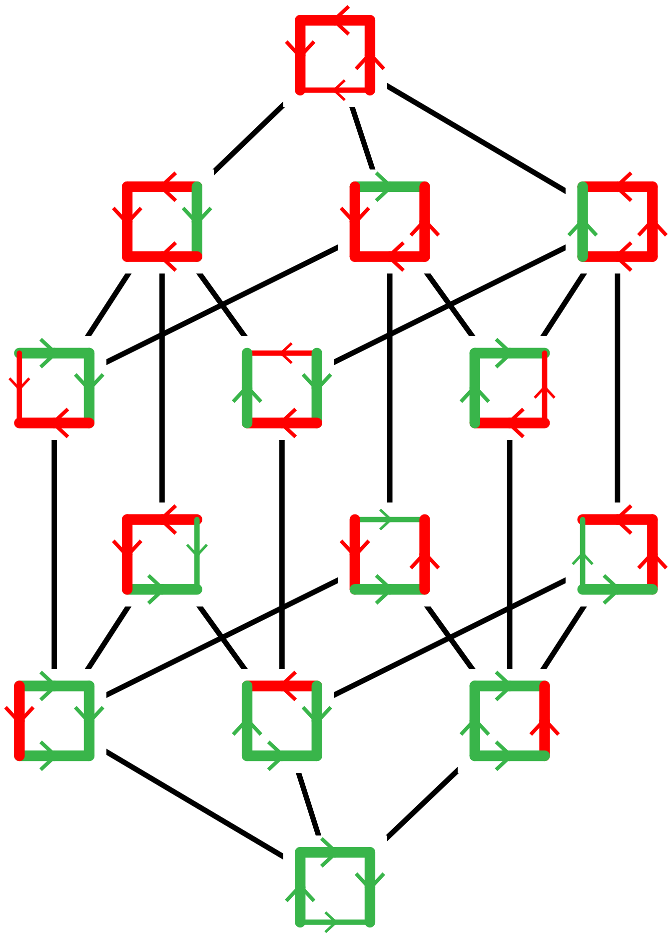

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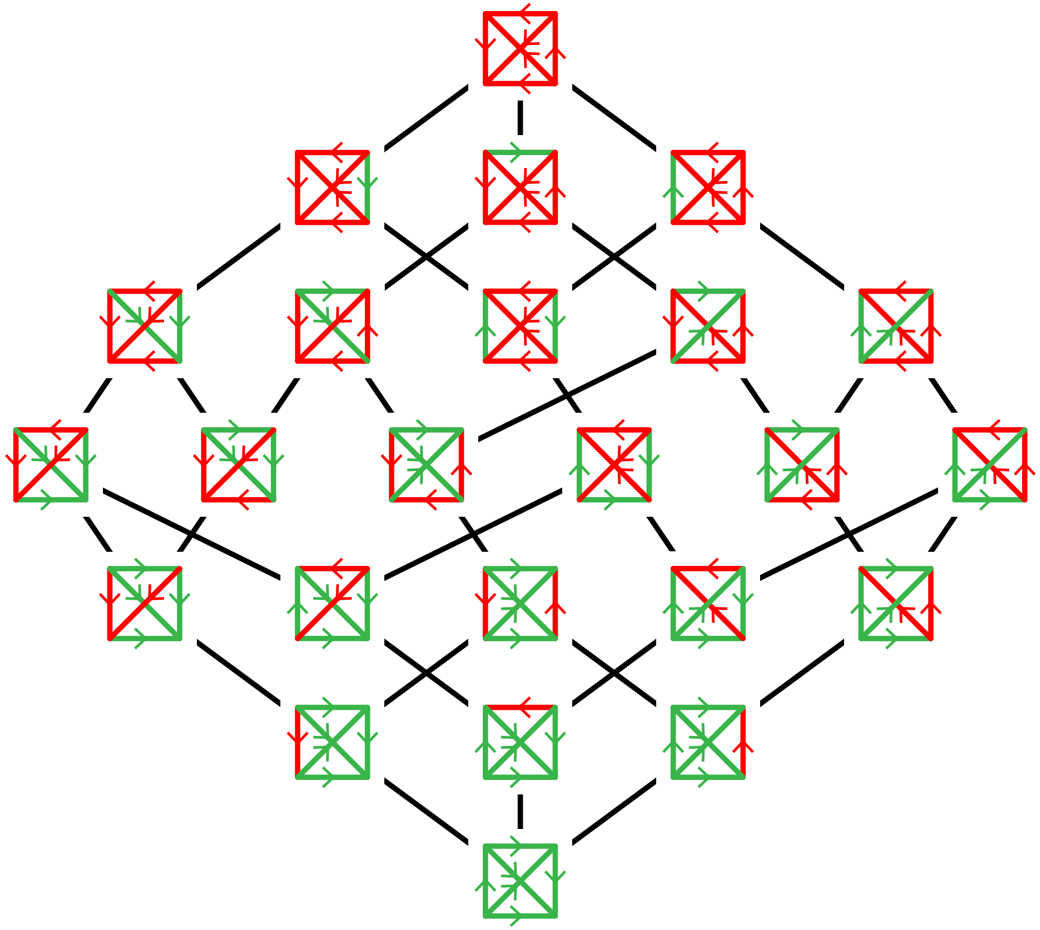
D forest

D tournament



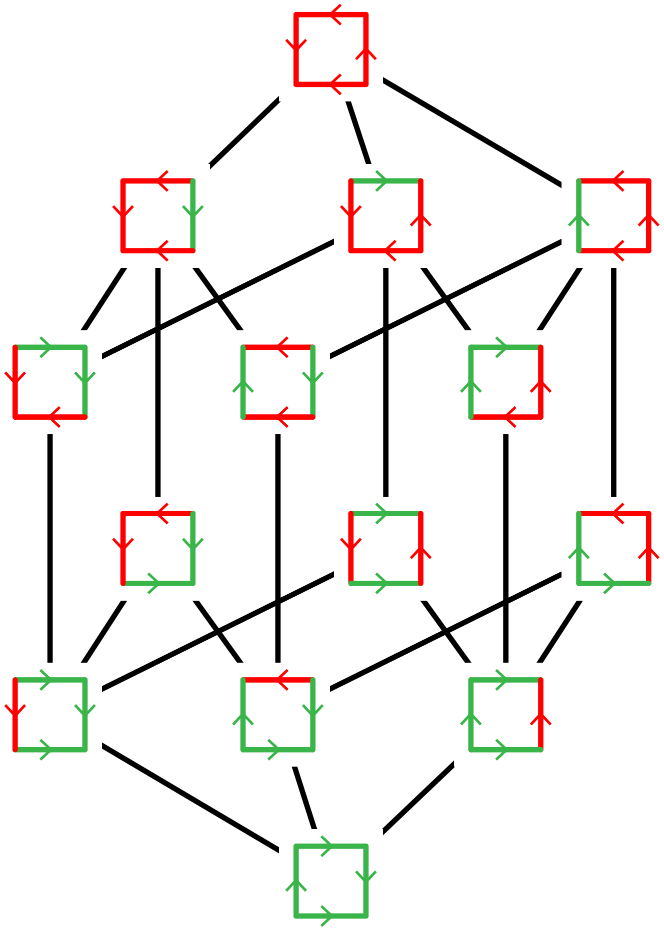
boolean lattice

weak order

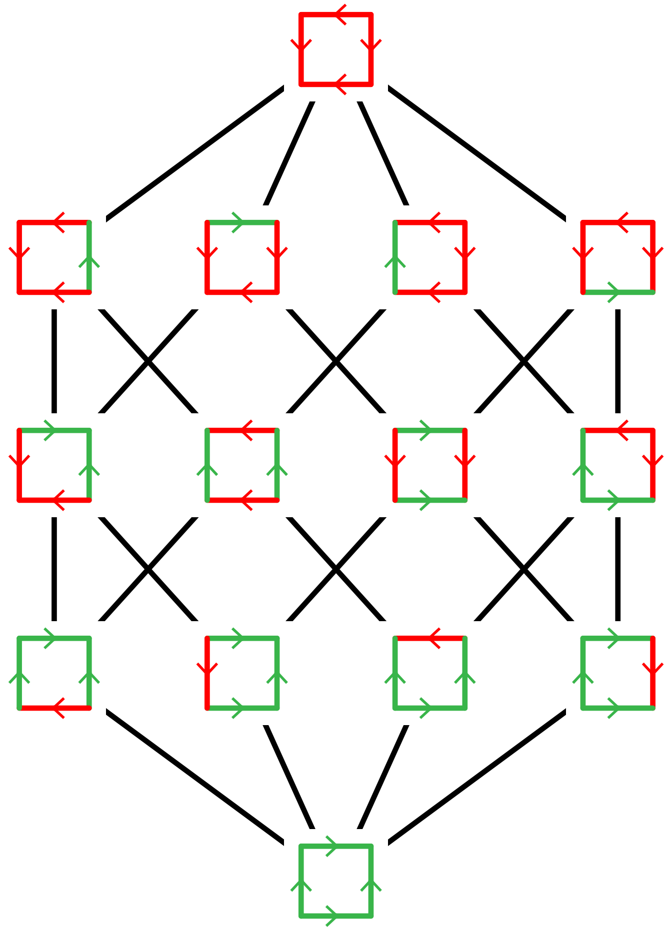
ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate



lattice



not lattice

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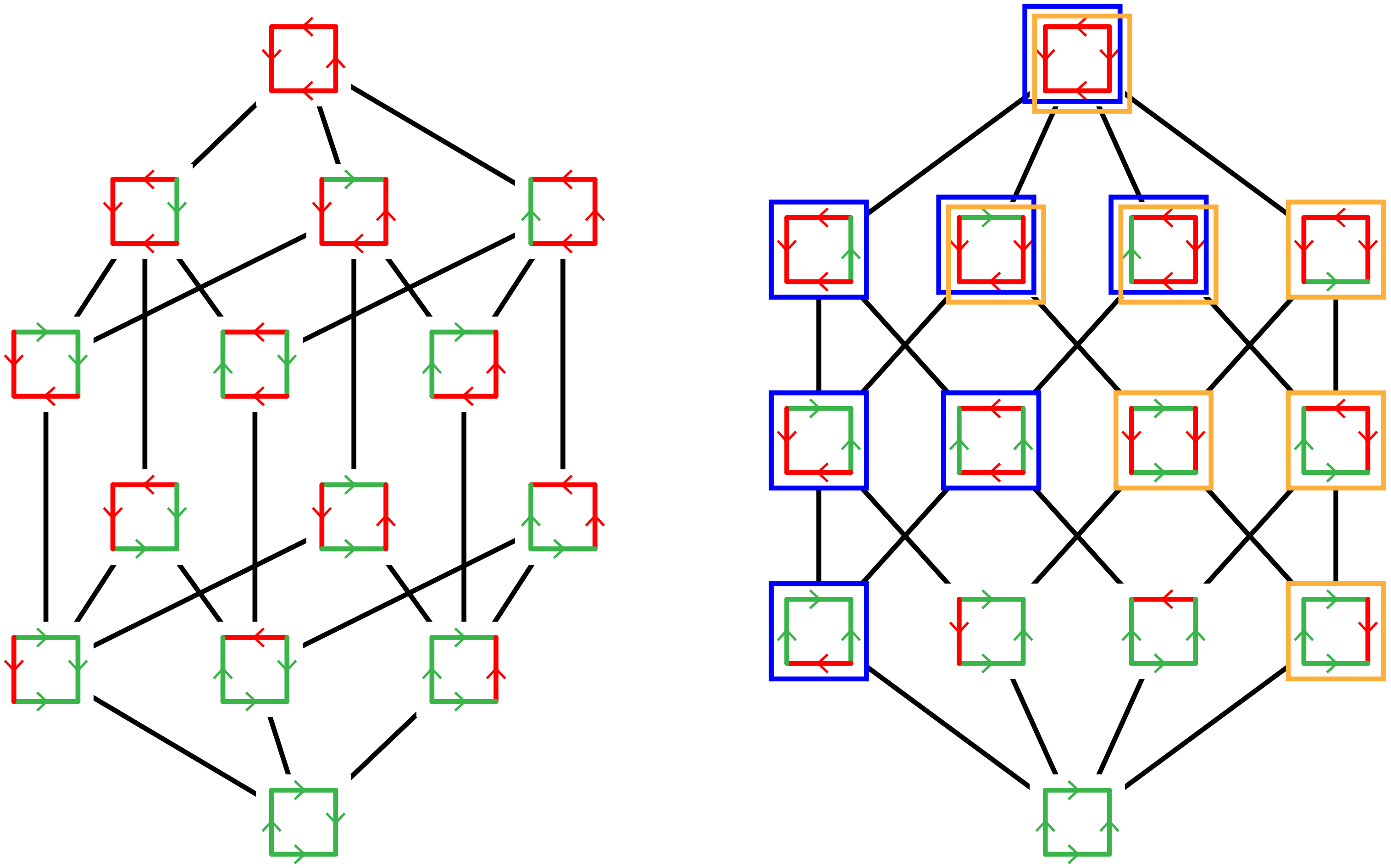

lattice

not lattice

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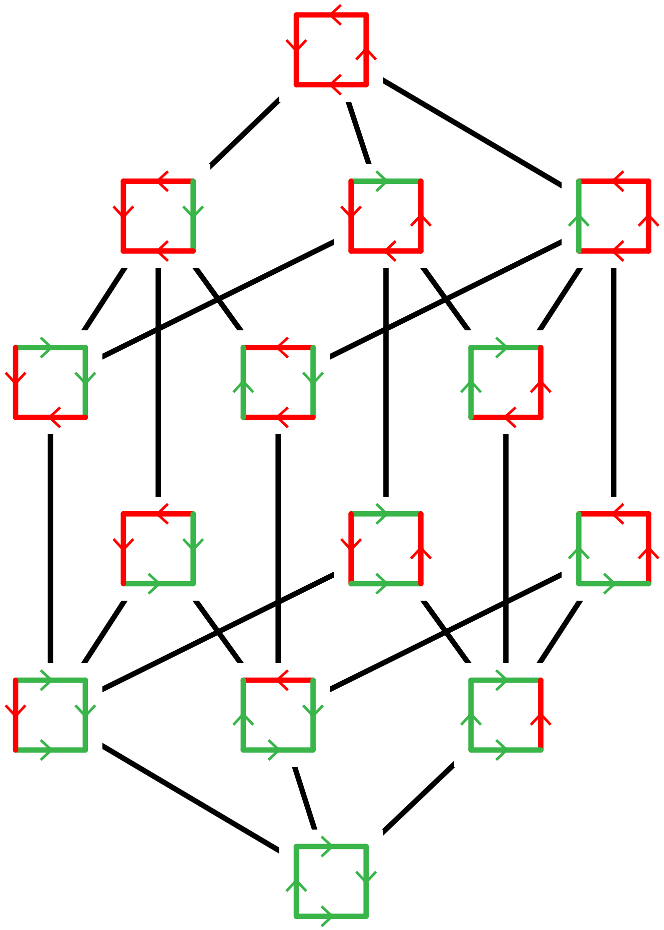

X subset of arcs of D is

- closed if all arcs of D in the transitive closure of X also belong to X
- coclosed if its complement is closed
- biclosed if it is closed and coclosed

PROP. If D vertebrate,
X biclosed $\iff$ the reorientation of X is acyclic

ACYCLIC REORIENTATION LATTICES

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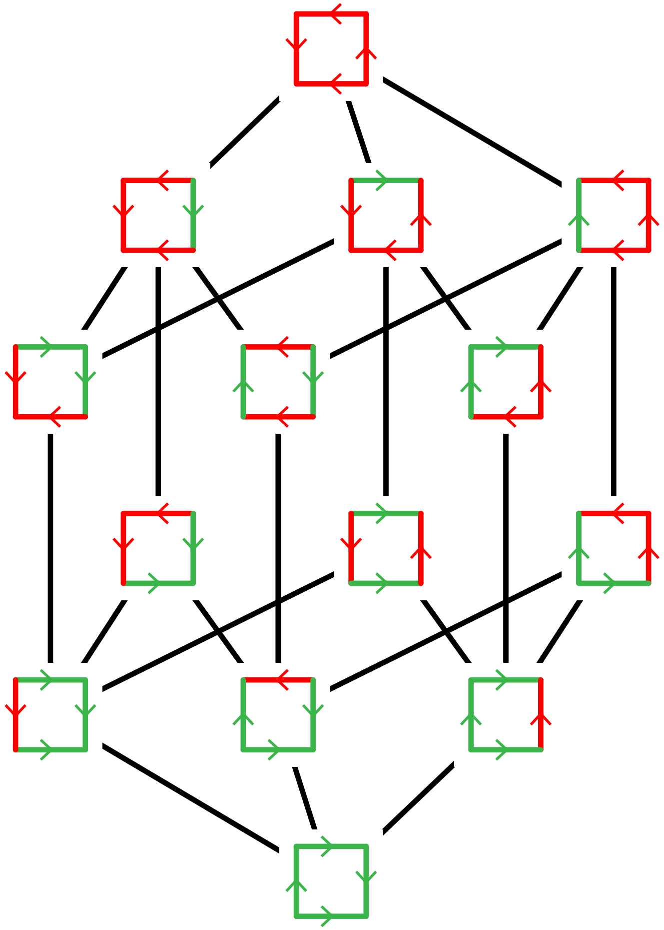
THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate



PROP. If D vertebrate,

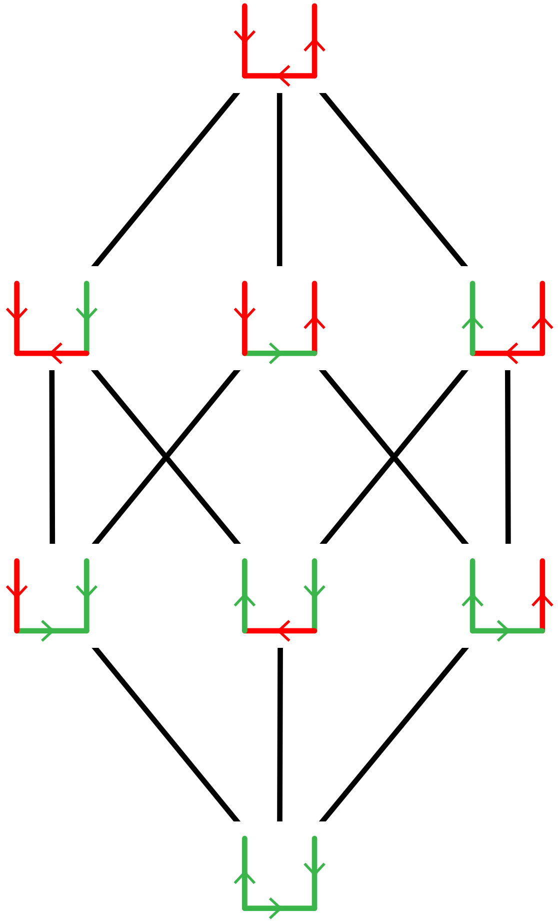
$$\mathrm{bwd}(E_1 \vee \ldots \vee E_k) =$$
$$\text{transitive closure of } \mathrm{bwd}(E_1) \cup \cdots \cup \mathrm{bwd}(E_k)$$

$$\mathrm{fwd}(E_1 \wedge \ldots \wedge E_k) =$$
$$\text{transitive closure of } \mathrm{fwd}(E_1) \cup \cdots \cup \mathrm{fwd}(E_k)$$

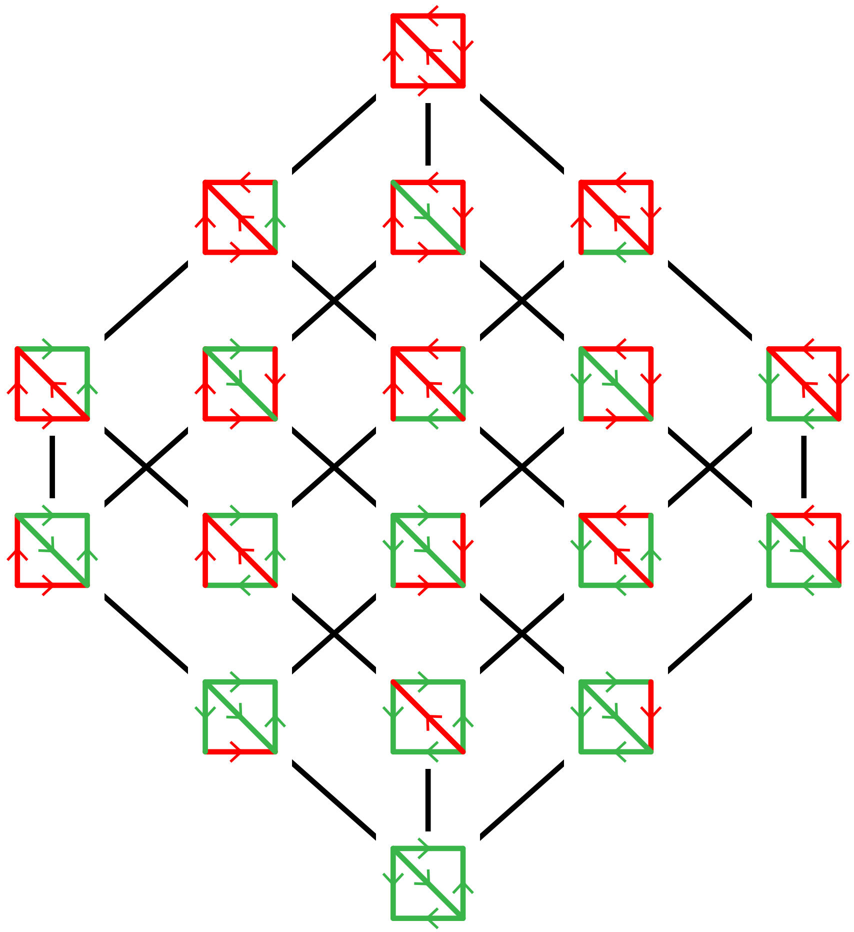
DISTRIBUTIVITY & SEMIDISTRIBUTIVITY

DISTRIBUTIVE ACYCLIC REORIENTATION POSETS

THM. $\mathcal{AR}_D$ distributive lattice $\iff$ D forest $\iff$ $\mathcal{AR}_D$ boolean lattice



distributive



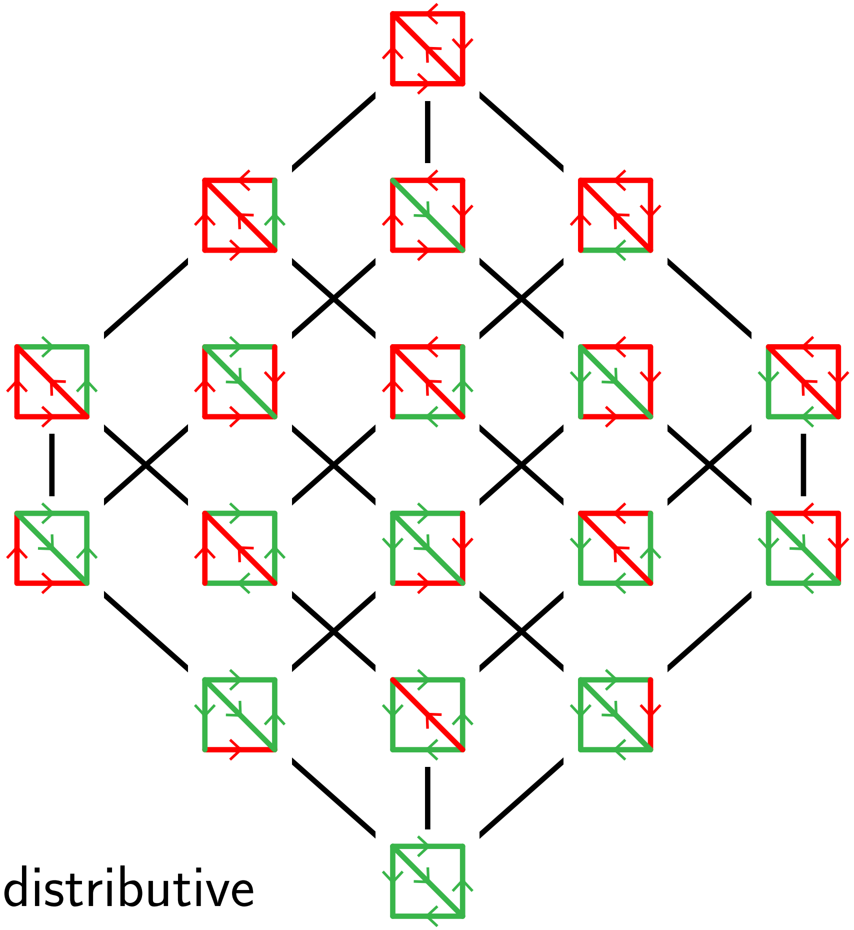
not distributive

SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

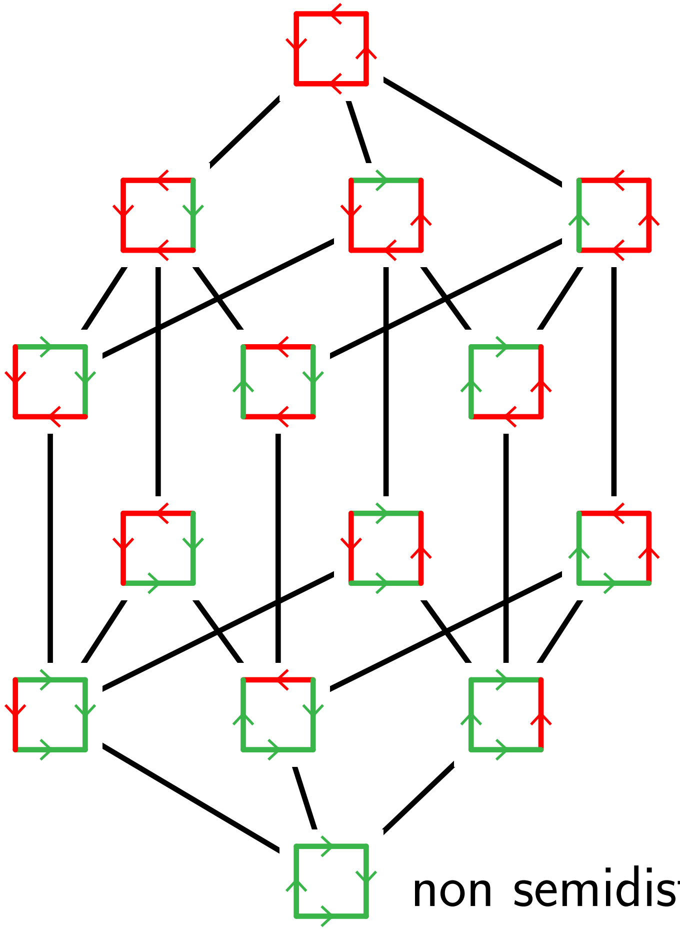
D skeletal =

- D vertebrate = transitive reduction of any induced subgraph of D is a forest
- D filled = any directed path joining the endpoints of an arc in D induces a tournament

THM. $\mathcal{AR}_D$ semidistributive lattice $\iff$ D is skeletal



semidistributive



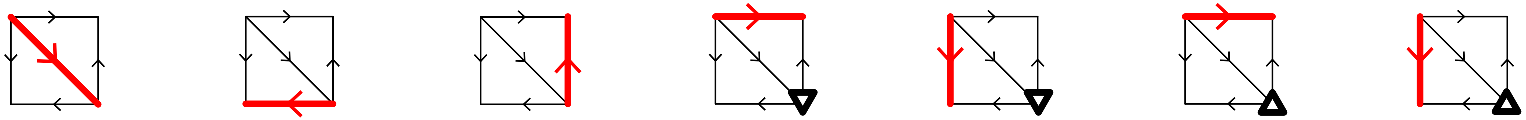
non semidistributive

ROPES & NON-CROSSING ROPE DIAGRAMS

ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \bigtriangledown, \bigtriangleup)$ where

- (u, v) is an arc of D
- $\bigtriangledown \sqcup \bigtriangleup$ partitions the transitive support of (u, v) minus $\{u, v\}$

ropes 

ROPES & NON-CROSSING ROPE DIAGRAMS

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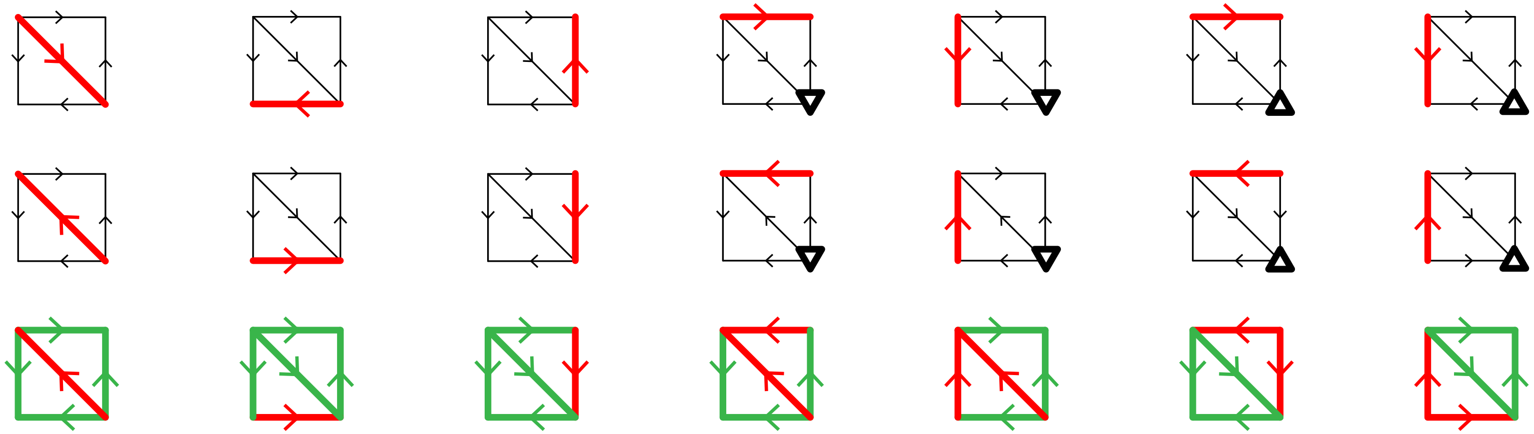
ropes

join irreducibles



THM. join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ ropes of D

ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \bigtriangledown, \bigtriangleup)$ where

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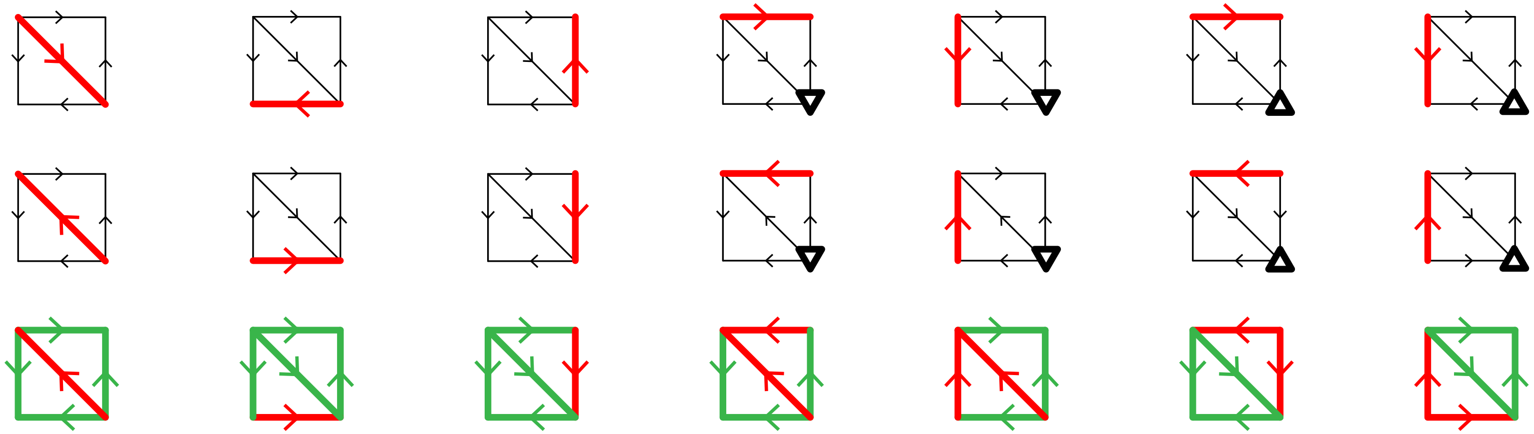
ropes

join irreducibles



THM. join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ ropes of D

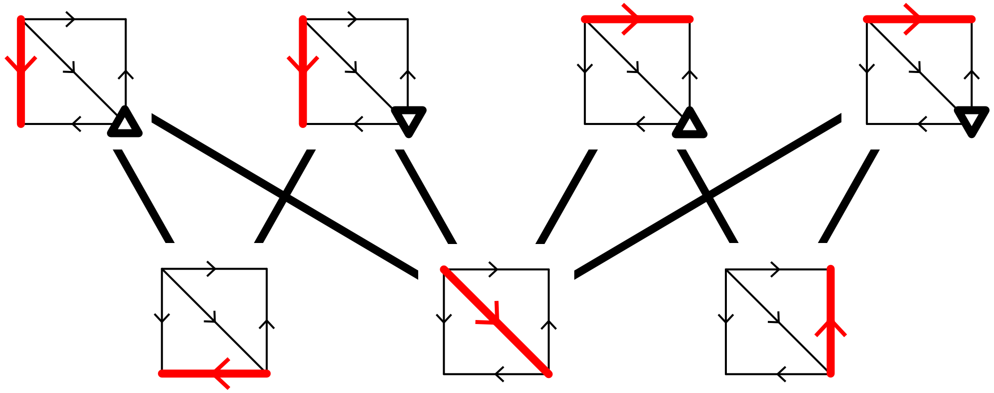
canonical join representations of $\mathcal{AR}_D$ $\longleftrightarrow$ non-crossing rope diagrams of $\mathcal{AR}_D$

$(u, v, \bigtriangledown, \bigtriangleup)$ and $(u', v', \bigtriangledown', \bigtriangleup')$ are crossing if there are $w \neq w'$ such that $w \in (\bigtriangledown \cup \{u, v\}) \cap (\bigtriangleup' \cup \{u', v'\})$ and $w' \in (\bigtriangleup \cup \{u, v\}) \cap (\bigtriangledown' \cup \{u', v'\})$

CONGRUENCES & QUOTIENTS

SUBROPES ORDER

$(u, v, \bigtriangledown, \bigtriangleup)$ subrope of $(u', v', \bigtriangledown', \bigtriangleup')$ if $u, v \in \{u', v'\} \cup \bigtriangledown' \cup \bigtriangleup'$ and $\bigtriangledown \subseteq \bigtriangledown'$ and $\bigtriangleup \subseteq \bigtriangleup'$



PROP. congruence lattice of $\mathcal{AR}_D \simeq$ lower ideal lattice of subrope order

CORO. $\equiv$ lattice congruence of $\mathcal{AR}_D$

- E minimal in its $\equiv$-class $\iff$ $\delta(E) \subseteq \mathcal{R}_\equiv$
- quotient $\mathcal{AR}_D/\equiv$ $\simeq$ subposet of $\mathcal{AR}_D$ induced by $\{E \in \mathcal{AR}_D \mid \delta(E) \subseteq \mathcal{R}_\equiv\}$

COHERENT CONGRUENCES

$(\mho, \Omega)$ = two of arbitrary subsets of V

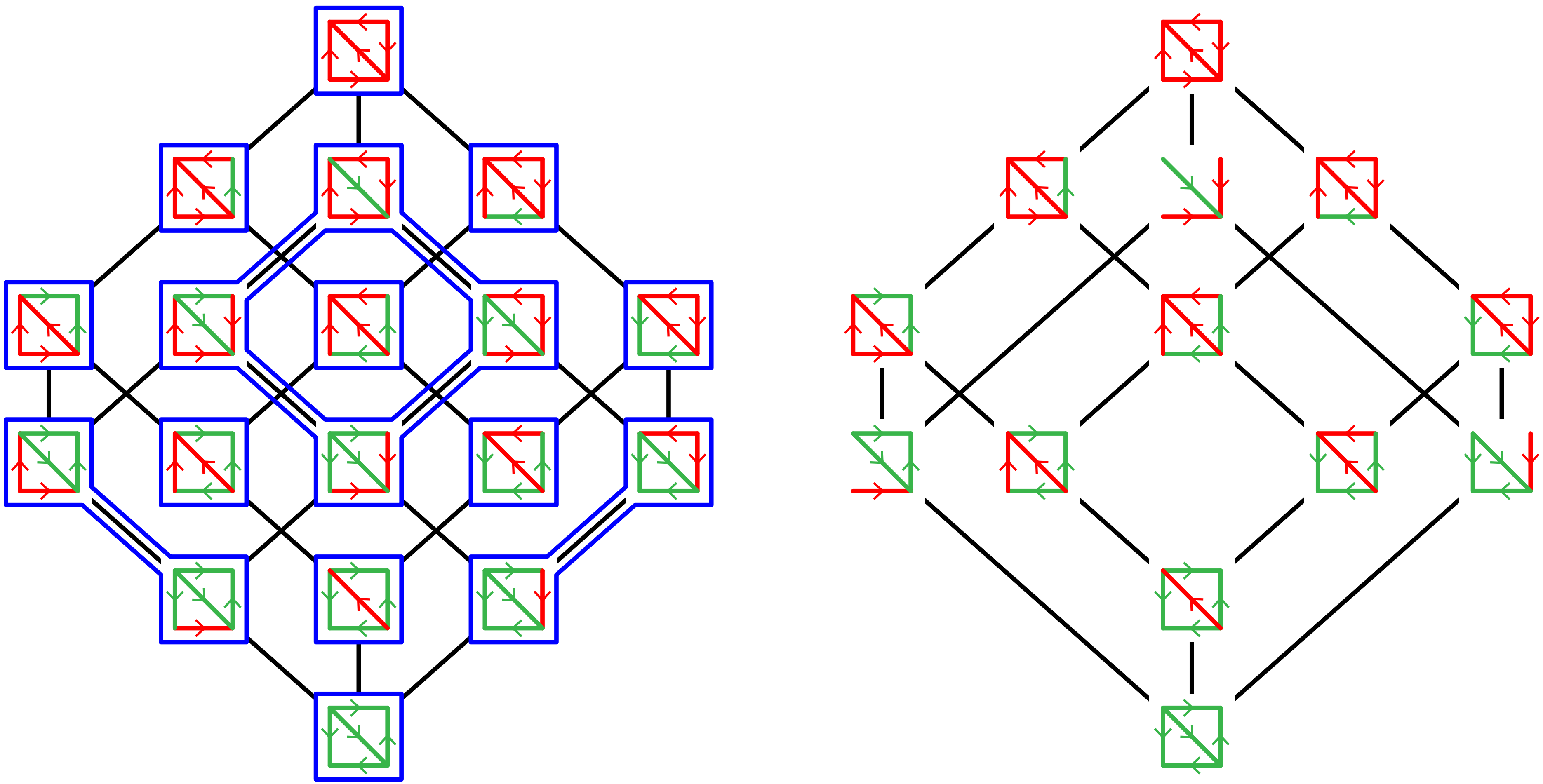
$\mathcal{R}_{(\mho,\Omega)}$ = lower ideal of ropes $(u, v, \bigtriangledown, \bigtriangleup)$ of D such that $\bigtriangledown \subseteq \mho$ and $\bigtriangleup \subseteq \Omega$

coherent congruence $\equiv_{(\mho,\Omega)}$ = congruence with subrope ideal $\mathcal{R}_{(\mho,\Omega)}$

P.–Pons ('18)

examples:

- sylvester congruence = subrope ideal contains only ropes $(u, v, \bigtriangledown, \varnothing)$



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P.–Pons ('18)

examples:

- sylvester congruence = subrope ideal contains only ropes $(u, v, \triangledown, \varnothing)$
- Cambrian congruences = when $\mho \sqcup \Omega = V$

Reading ('06)

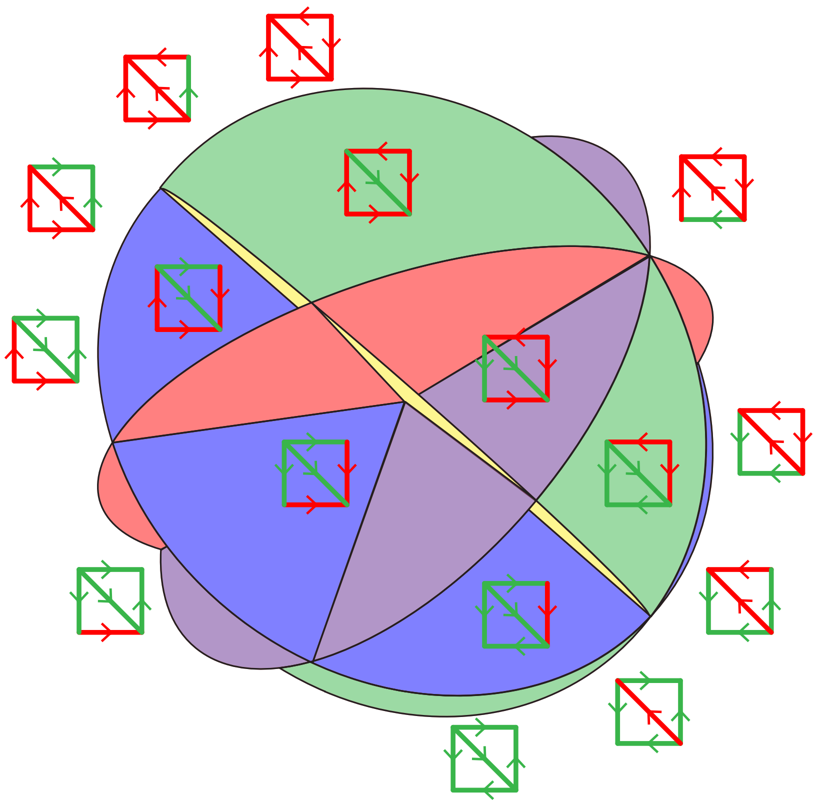
QUOTIENT FANS & QUOTIENTOPES

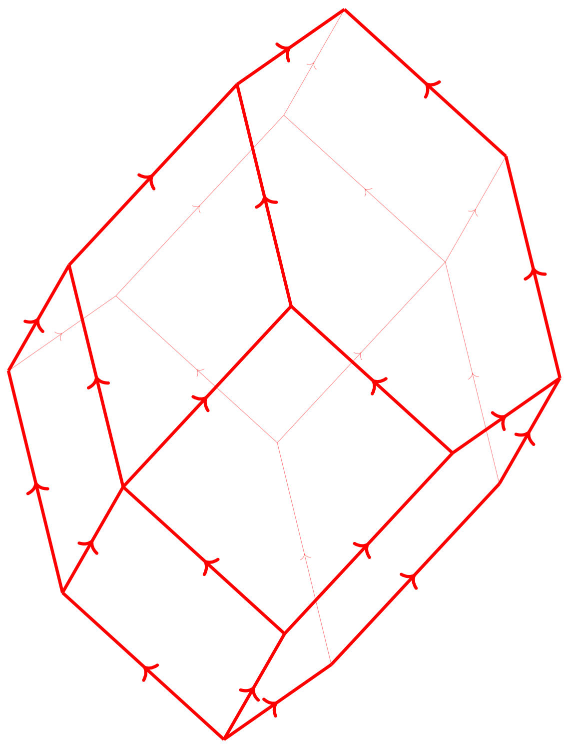
GRAPHICAL ARRANGEMENT & GRAPHICAL ZONOTOPE

D directed acyclic graph

graphical arrangement $\mathcal{H}_D$ = arrangement of hyperplanes $x_u = x_v$ for all arcs $(u, v) \in D$

graphical zonotope $\mathcal{Z}_D$ = Minkowski sum of $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all arcs $(u, v) \in D$

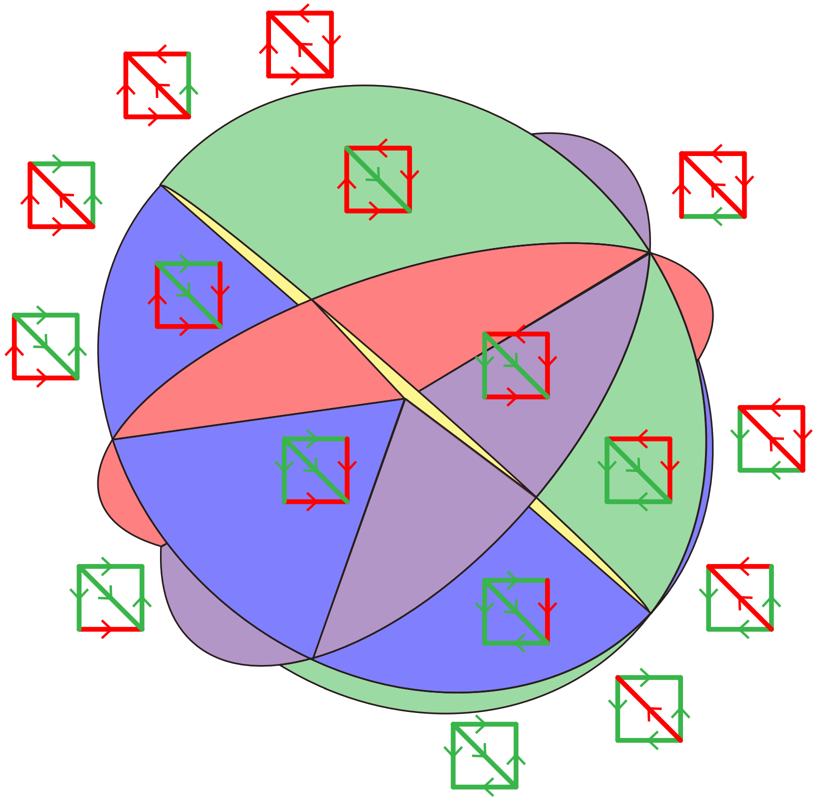


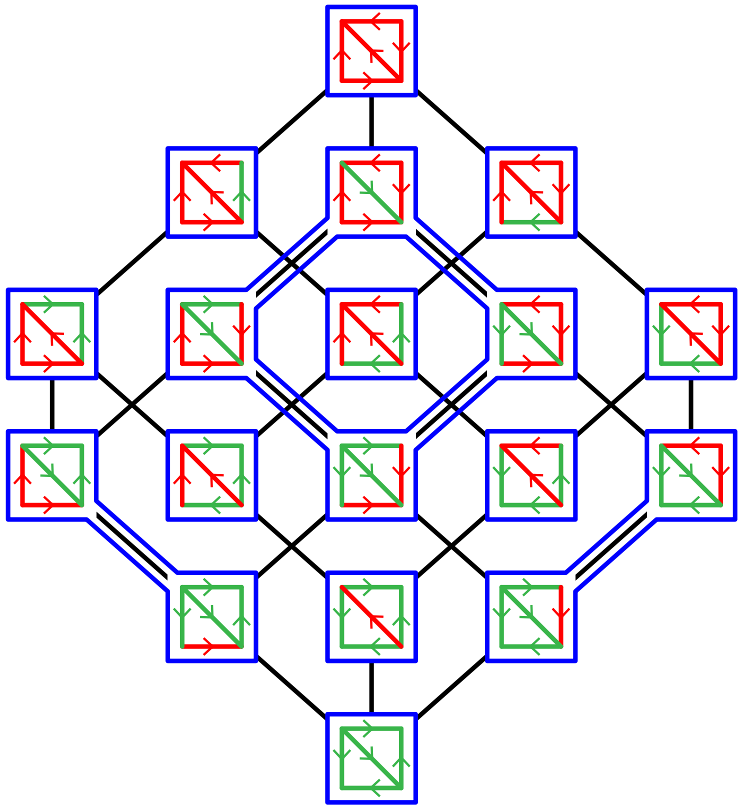


hyperplanes of $\mathcal{H}_D$	$\longleftrightarrow$	summands of $\mathcal{Z}_D$	$\longleftrightarrow$	arcs of D
regions of $\mathcal{H}_D$	$\longleftrightarrow$	vertices of $\mathcal{Z}_D$	$\longleftrightarrow$	acyclic reorientations of D
poset of regions of $\mathcal{H}_D$	$\longleftrightarrow$	oriented graph of $\mathcal{Z}_D$	$\longleftrightarrow$	acyclic reorientation poset of D

QUOTIENT FAN

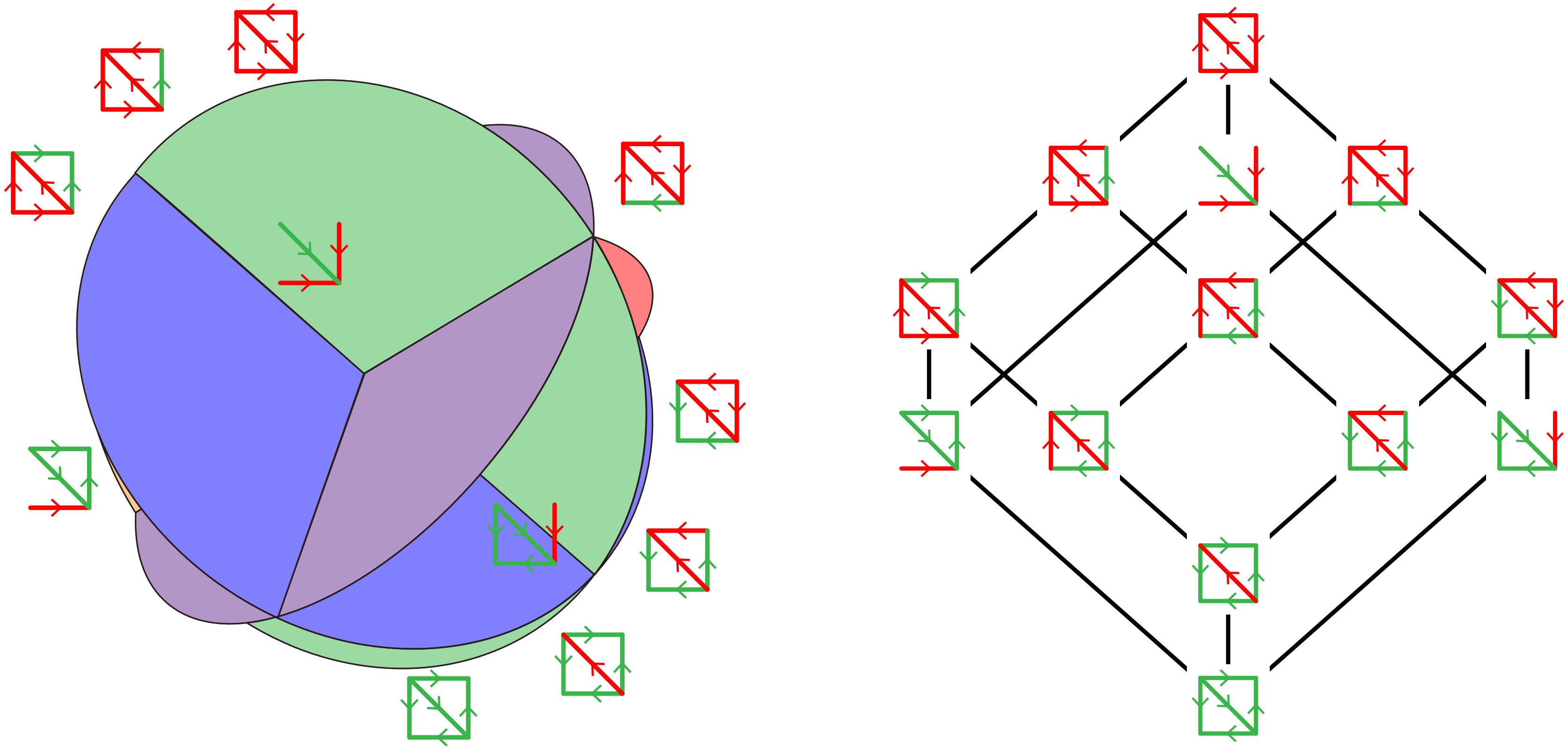
THM. A lattice congruence $\equiv$ of $\mathcal{AR}_D$ defines a quotient fan $\mathcal{F}_\equiv$ where the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of $\mathcal{H}_D$ corresponding to acyclic reorientations in the same equivalence class of $\equiv$





QUOTIENT FAN

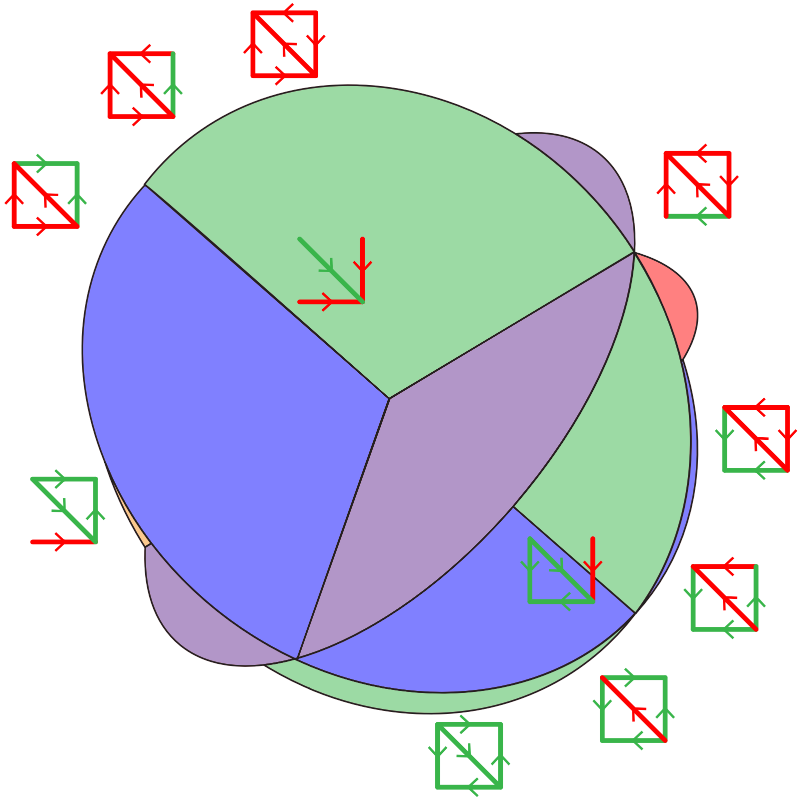
THM. A lattice congruence $\equiv$ of $\mathcal{AR}_D$ defines a quotient fan $\mathcal{F}_\equiv$ where the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of $\mathcal{H}_D$ corresponding to acyclic reorientations in the same equivalence class of $\equiv$

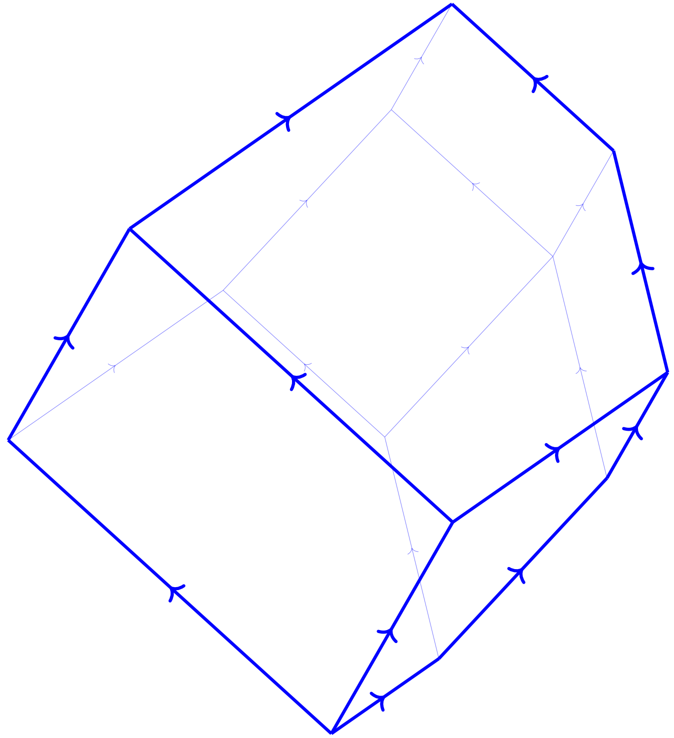


QUOTIENTOPES

THM. The quotient fan $\mathcal{F}_{\equiv}$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of

- a Minkowski sum of associahedra of Hohlweg – Lange, and
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter





ρ-alternating matching $=$ pair $(M_\triangledown, M_\vartriangle)$ with $M_\triangledown \subseteq \{u\} \cup \triangledown$ and $M_\vartriangle \subseteq \vartriangle \cup \{v\}$ s.t. $M_\triangledown$ and $M_\vartriangle$ are alternating along the transitive reduction of D

shard polytope of ρ $=$ convex hull of signed charact. vectors of ρ-alternating matchings

QUOTIENTOPES

THM. The quotient fan $\mathcal{F}_{\equiv}$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of

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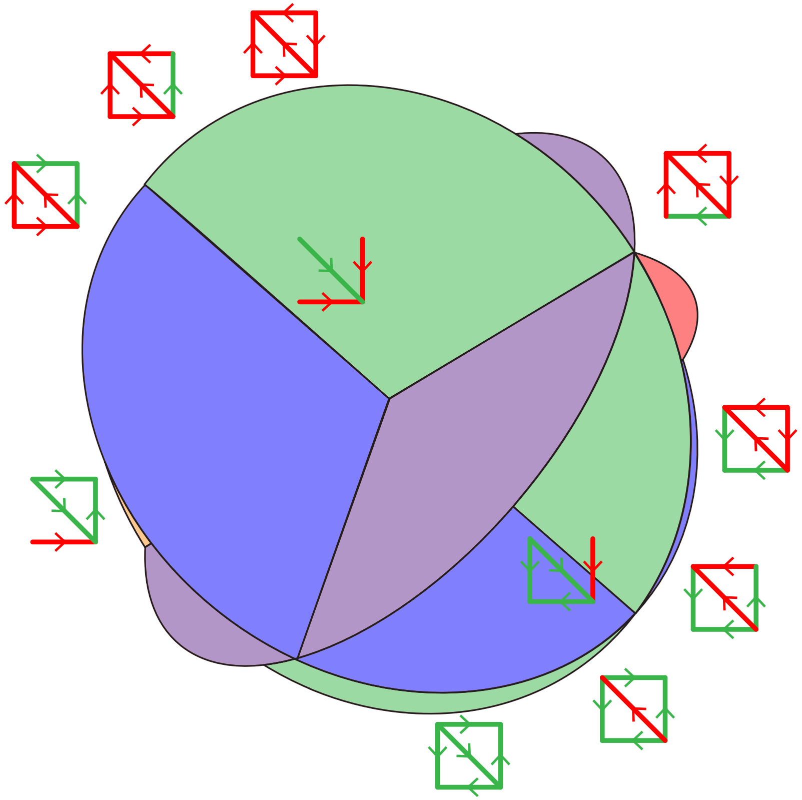
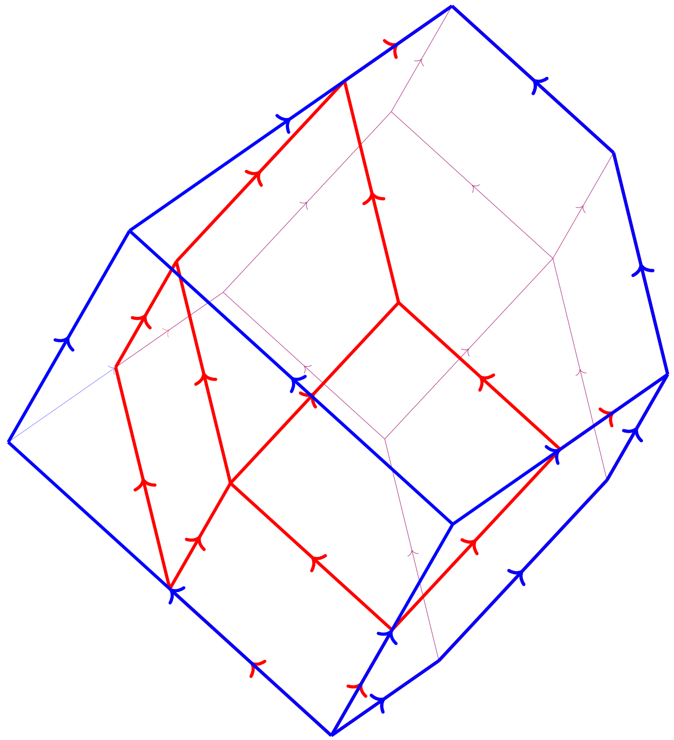
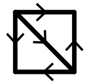
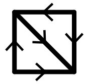




PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of D are facet defining inequalities of the graphical zonotope of D

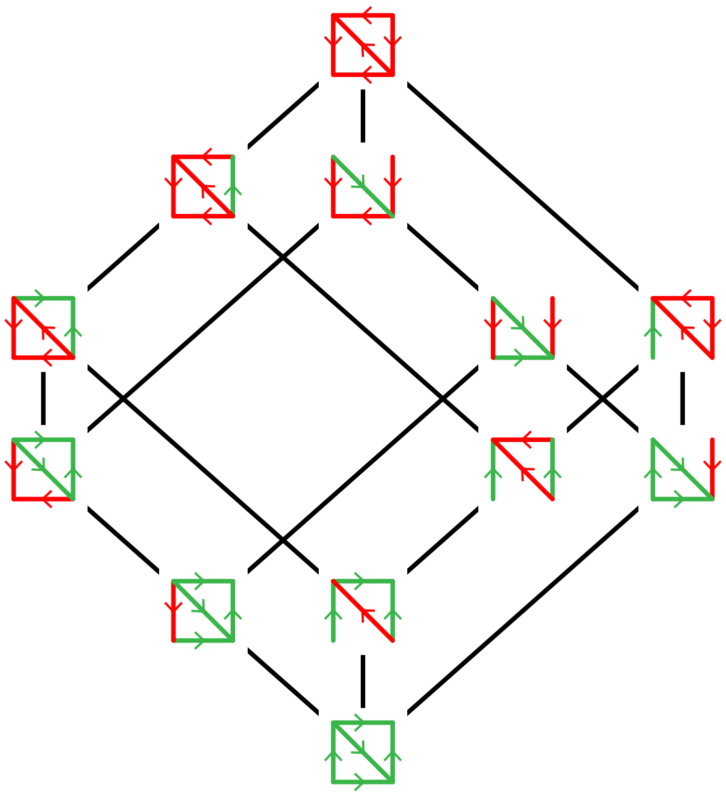
SOME OPEN PROBLEMS

SIMPLE ASSOCIAHEDRA

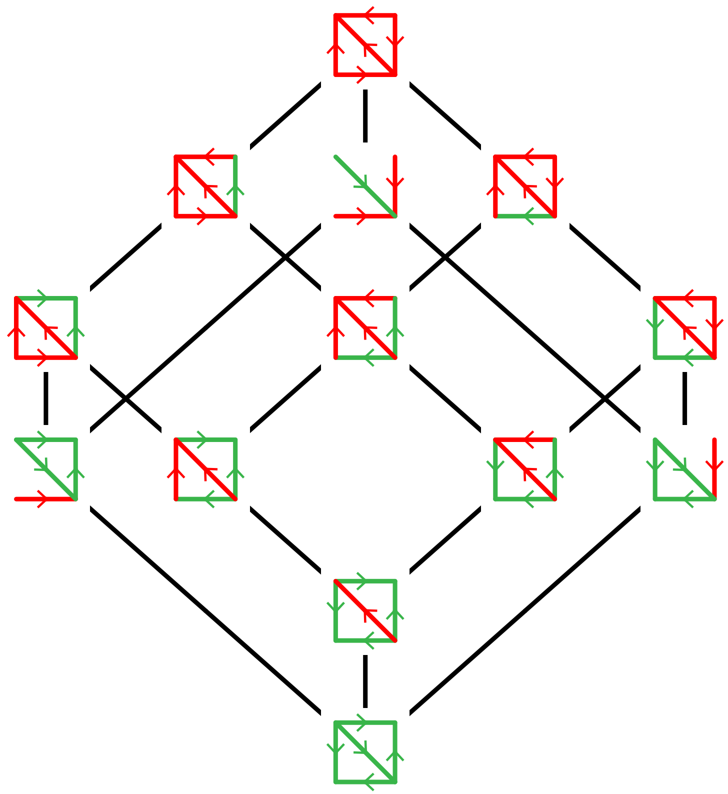
CONJ. D has no induced subgraph isomorphic to  or 

$\Longleftrightarrow$ the Hasse diagram of the D-Tamari lattice is regular

$\Longleftrightarrow$ the D-associahedron is a simple polytope

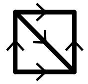


regular



non regular

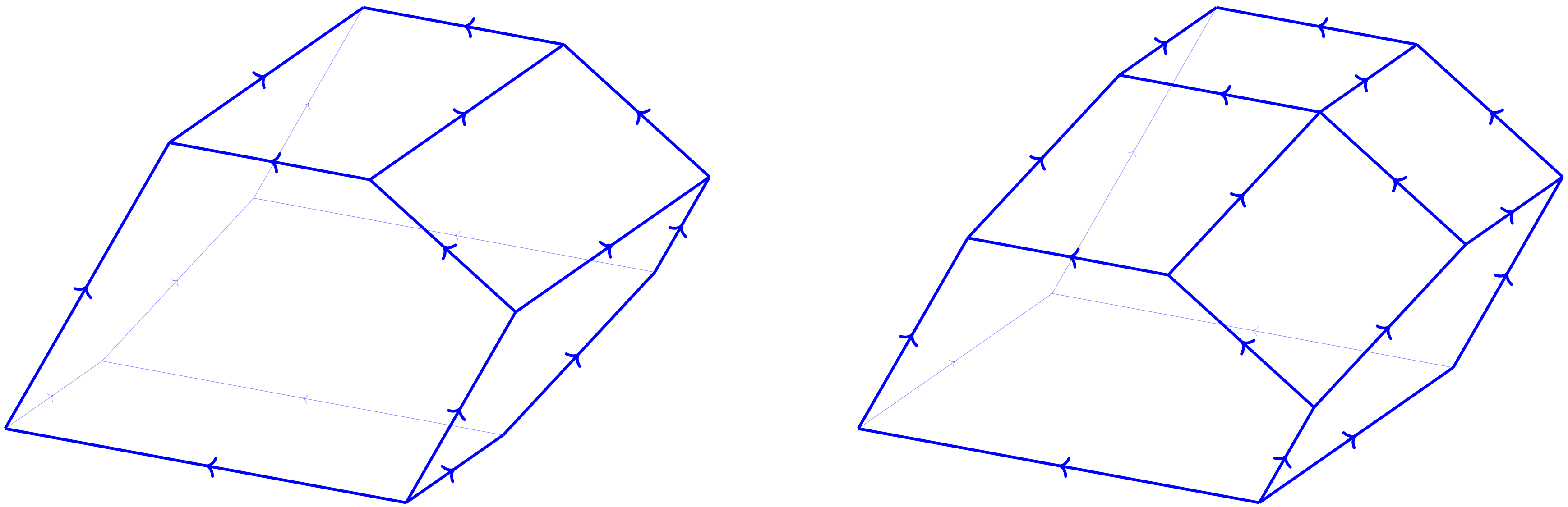
ISOMORPHIC CAMBRIAN ASSOCIAHEDRA

CONJ. D has no induced subgraph isomorphic to 

$\iff$ all Cambrian associahedra of D have the same number of vertices

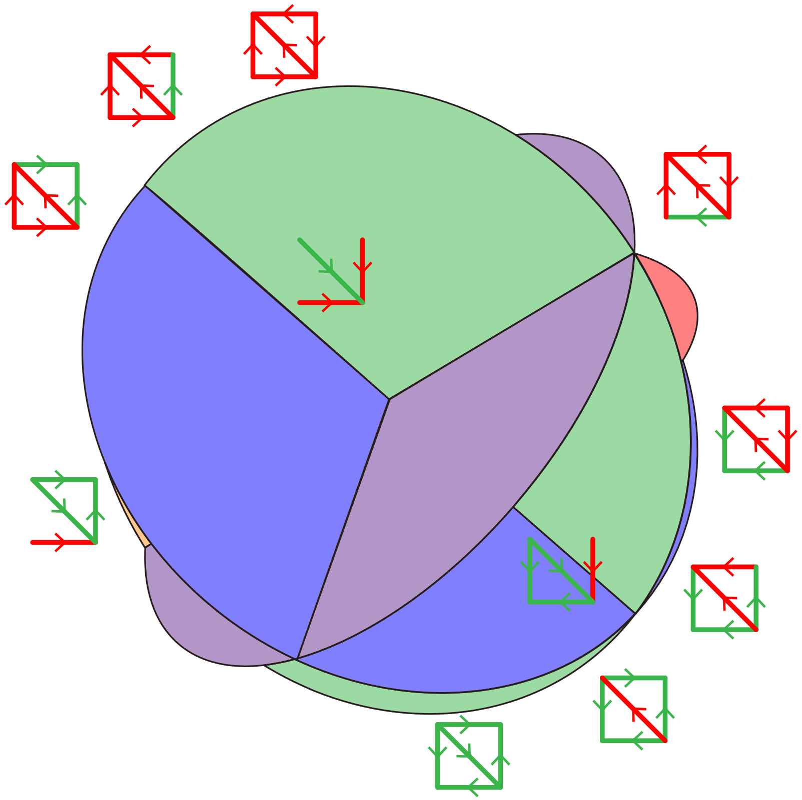
$\iff$ all Cambrian associahedra of D have isomorphic 1-skeleta

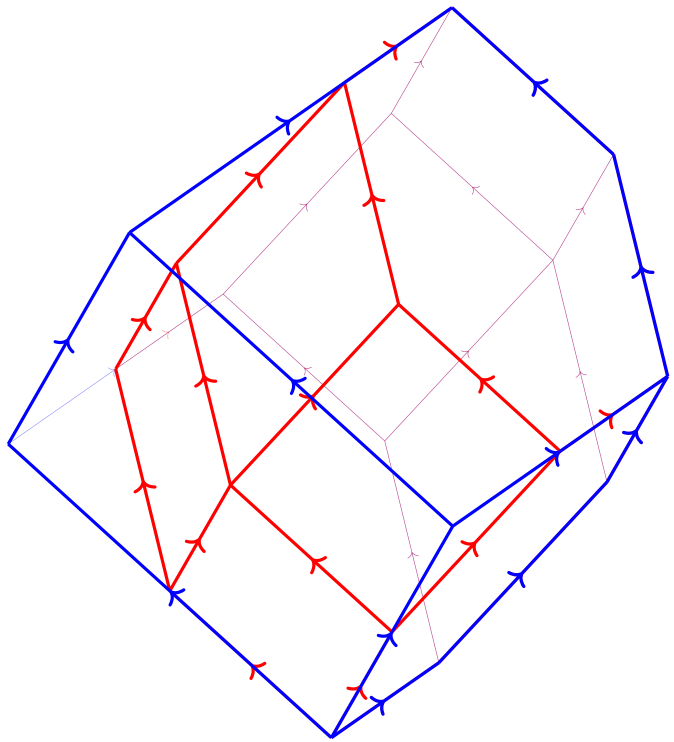
$\iff$ all Cambrian associahedra of D have isomorphic face lattices



REMOVAHEDRA

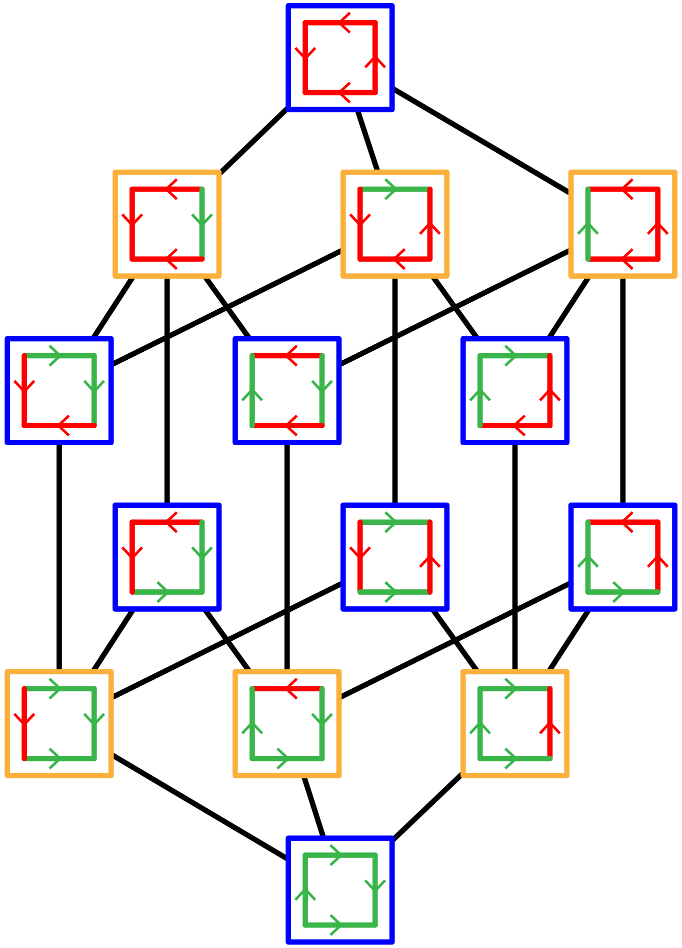
PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of D are facet defining inequalities of the graphical zonotope of D

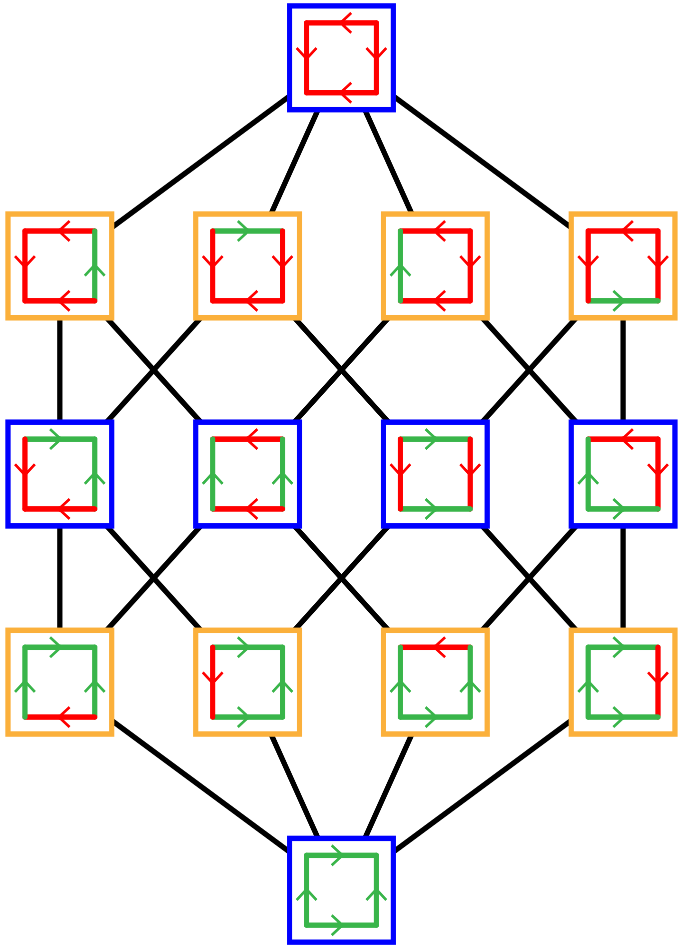




CONJ. For any $\mho, \Omega \subseteq V$, the quotient fan $\mathcal{F}_{(\mho,\Omega)}$ is the normal fan of the polytope obtained by deleting inequalities of the graphical zonotope of D

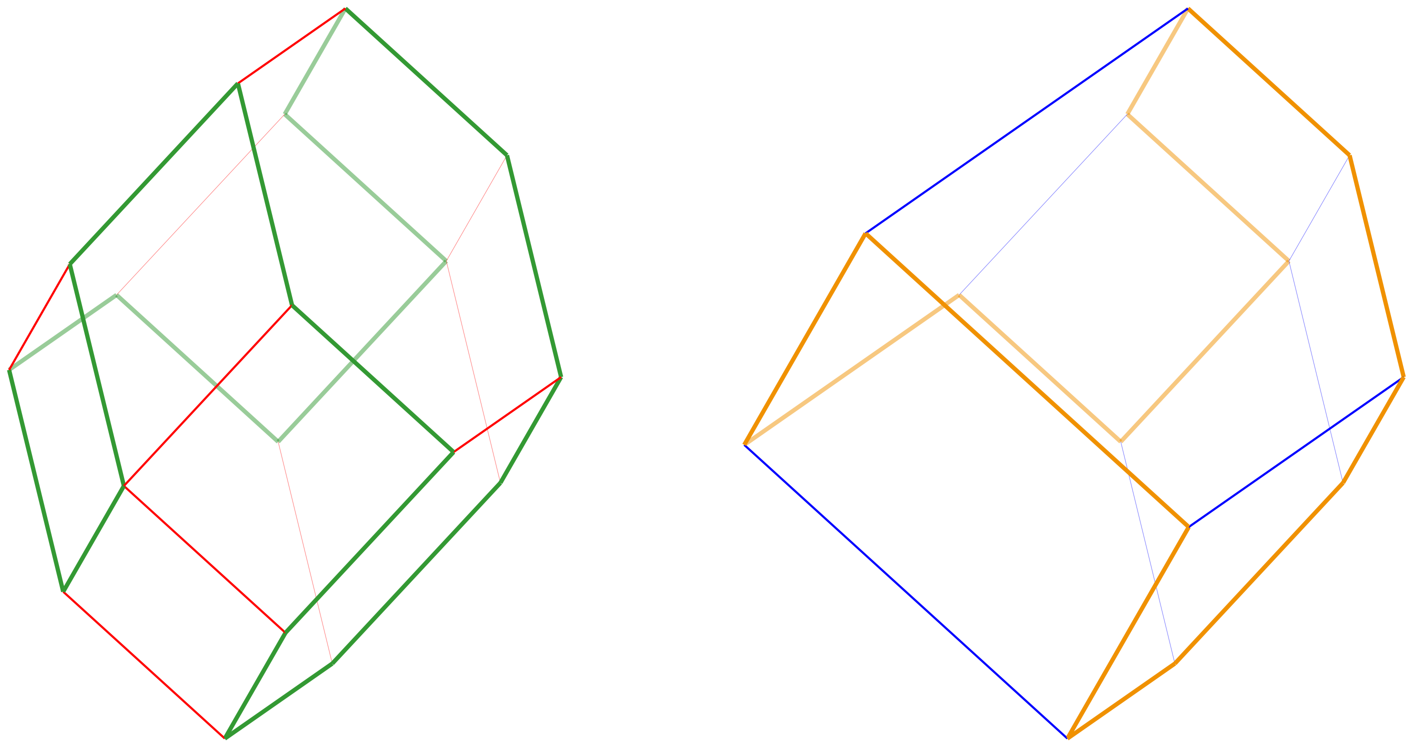
Not all acyclic reorientation flip graphs admit a Hamiltonian cycle





HAMILTONIAN CYCLES

THM [SSW '93]. For D chordal, the acyclic reorientation flip graph is Hamiltonian



CONJ. When D is skeletal, all quotientopes admit a Hamiltonian cycle

... checked for all quotients, for all skeletal acyclic directed graphs up to 5 vertices ...

LATTICE OF REGIONS OF HYPERPLANE ARRANGEMENTS

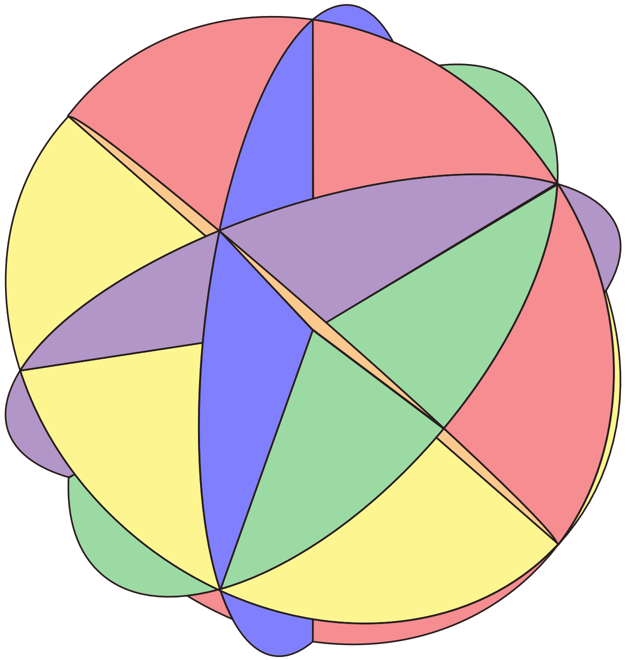
$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$

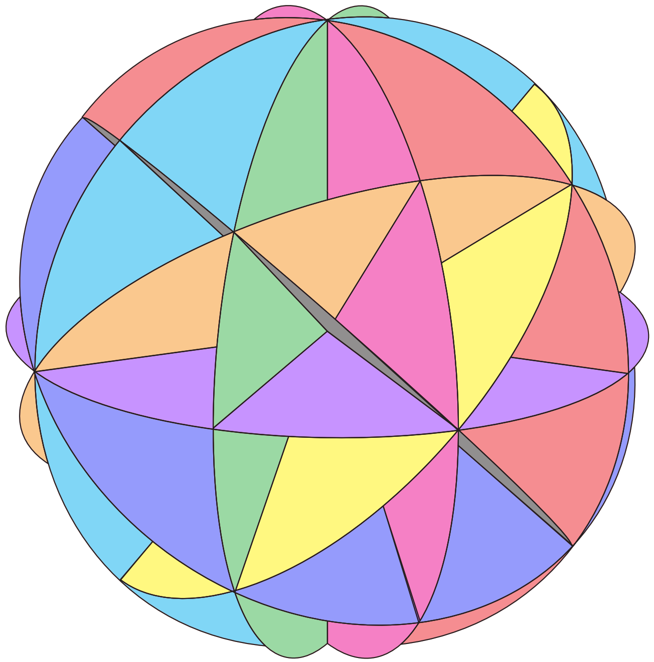
base region B = distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$

inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C

poset of regions $\mathrm{PR}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

QU. For which $(\mathcal{H}, B)$ is the poset of regions PR a lattice?





LATTICE OF REGIONS OF HYPERPLANE ARRANGEMENTS

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QU. For which $(\mathcal{H}, B)$ is the poset of regions PR a lattice?

THM. The poset of regions $\mathrm{PR}(\mathcal{H}, B)$ — Björner–Edelman–Ziegler ('90)

- is never a lattice when B is not a simplicial region
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement

THM. The poset of regions $\mathrm{PR}(\mathcal{H}, B)$ is a semidistributive lattice
$\iff$ $\mathcal{H}$ is tight with respect to B — Reading ('16)

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

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poset of regions $\mathrm{PR}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

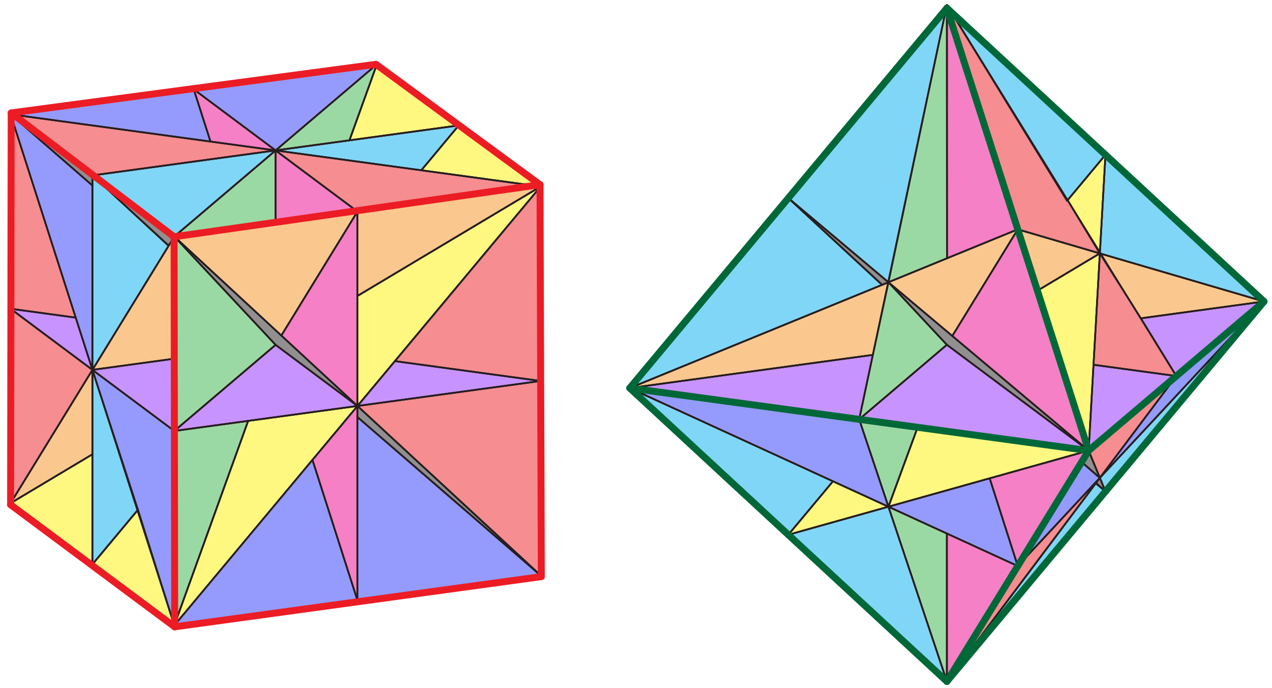
THM. If $\mathrm{PR}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a congruence of $\mathrm{PR}(\mathcal{H}, B)$, the cones obtained by glueing the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan $\mathcal{F}_\equiv$

Reading ('05)

QU. Is the quotient fan $\mathcal{F}_\equiv$ always polytopal?

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

hyperoctahedral group = isometry group of the hypercube (or of its dual cross-polytope)

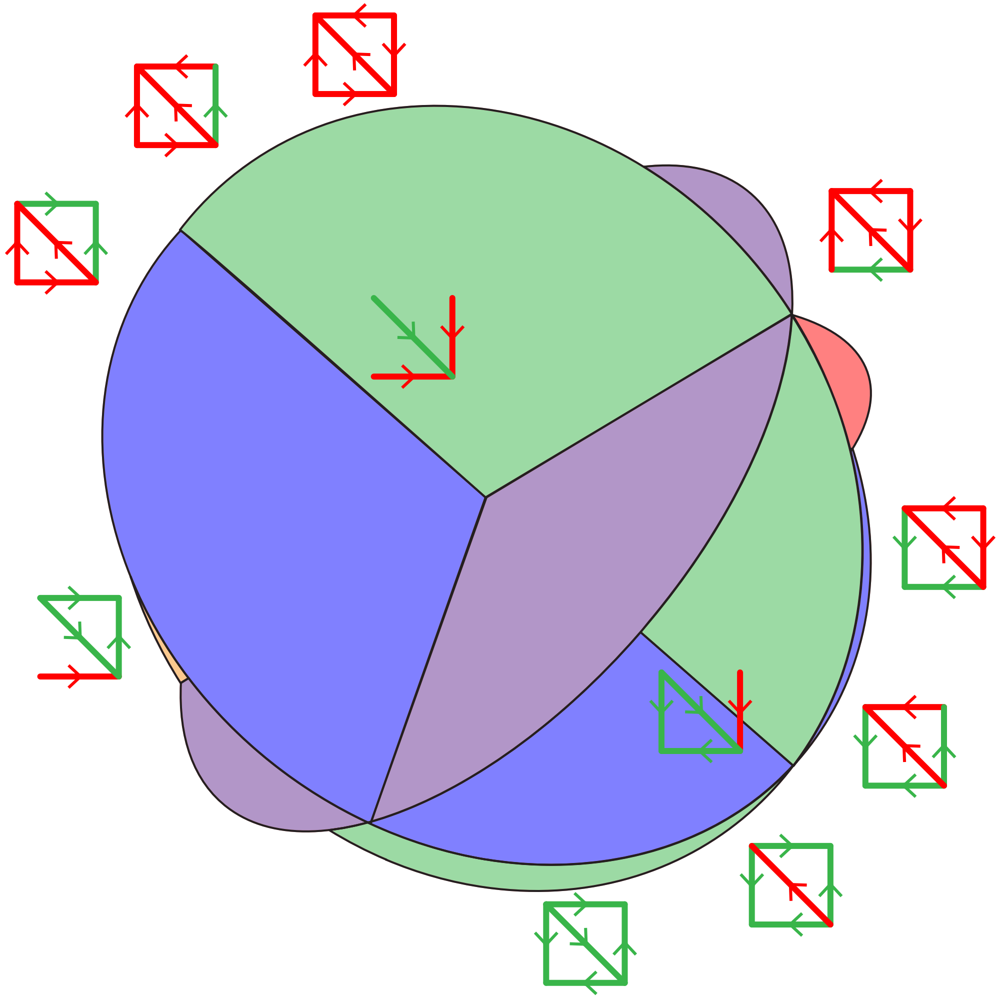


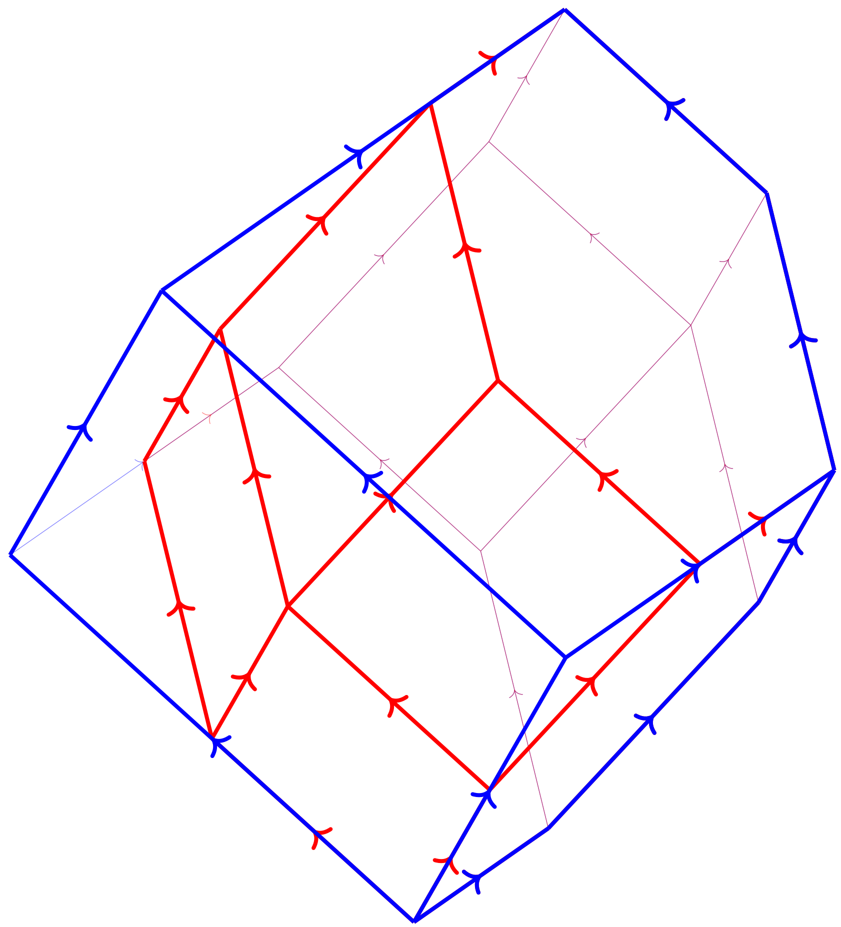
THM. The quotient fan of any lattice congruence of the type B weak order is polytopal

Padrol–P.–Ritter ('20$^+$)

Type B quotientopes are obtained

- not as removahedra,
- not as Minkowski sum of cyclohedra,
- but as Minkowski sum of shard polytopes (but this is another story...)





THANK YOU