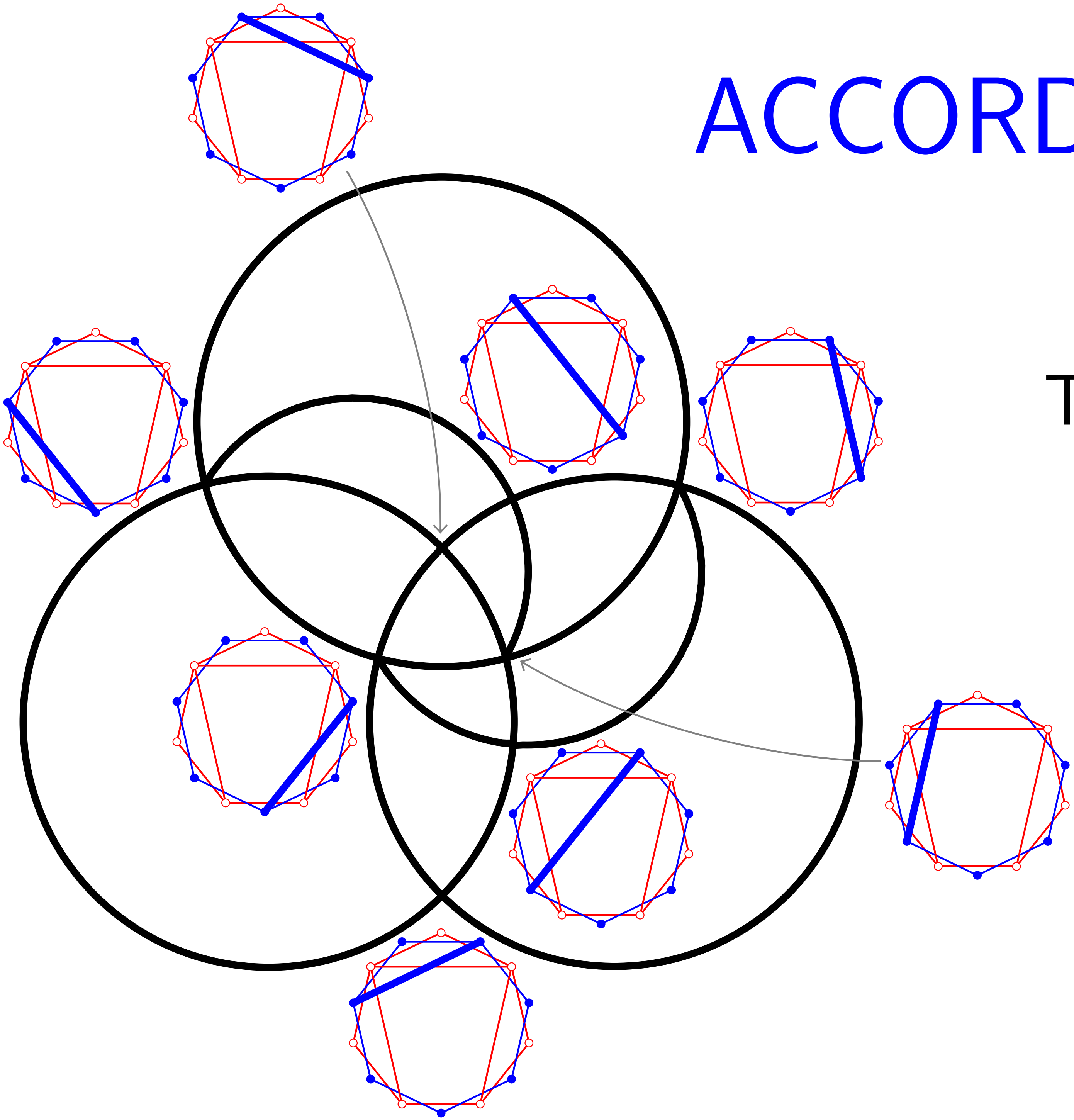


ACCORDIOHEDRA

C. HOHLWEG

LaCIM, UQAM

T. MANNEVILLE

LIX, École Poytechnique

V. PILAUD

CNRS & LIX,

École Poytechnique

S. STELLA

Univ. Roma

Séminaire

CALIN LIPN

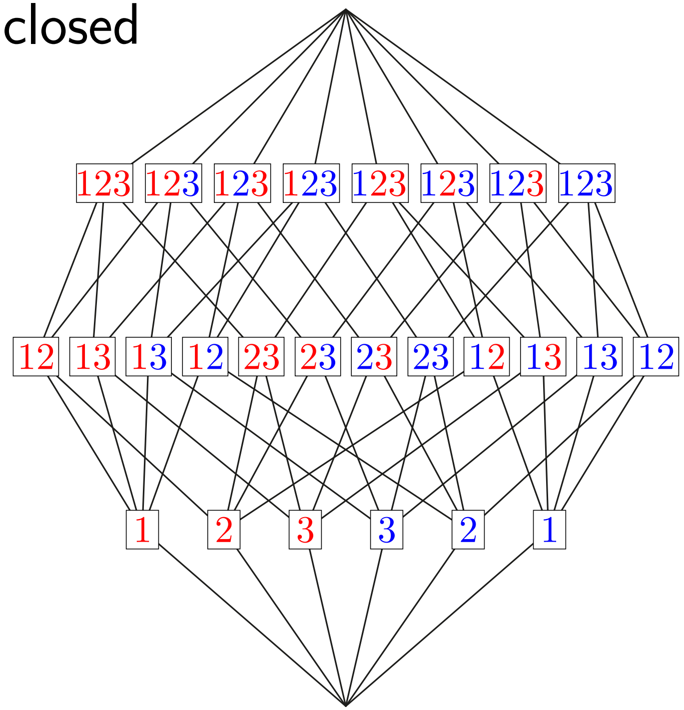
October 25, 2016

FANS & POLYTOPES

Ziegler, *Lectures on polytopes* ('95)
Matoušek, *Lectures on Discrete Geometry* ('02)

SIMPLICIAL COMPLEX

simplicial complex = collection of subsets of X downward closed



exm:

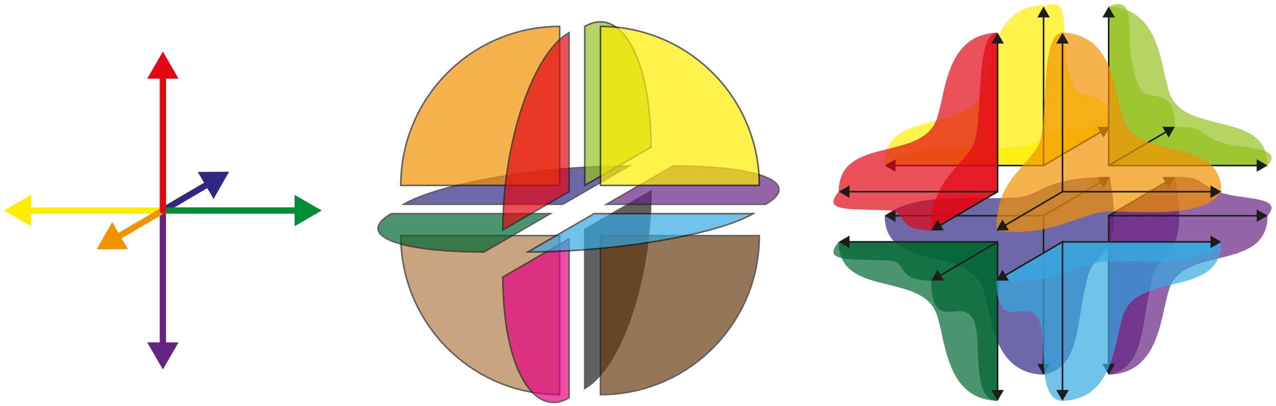
$$X = [n] \cup [n]$$

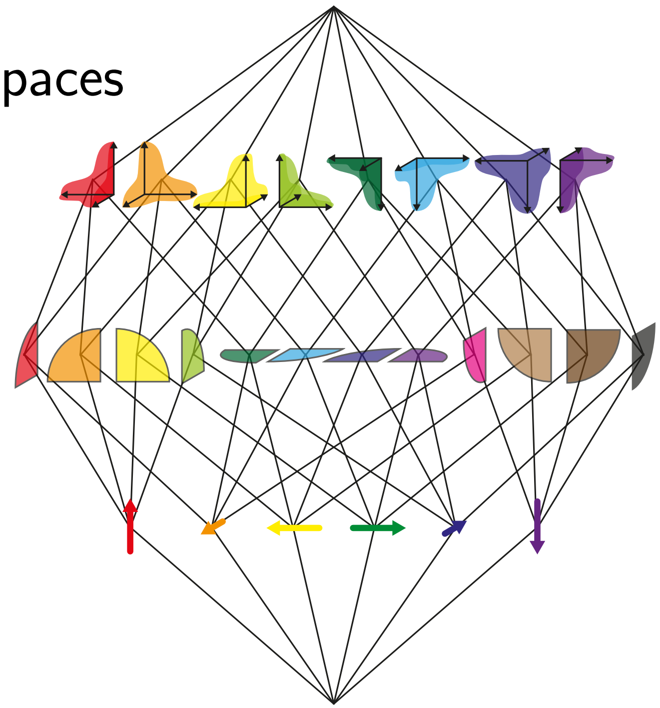
$$\Delta = \{ I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I \}$$

FANS

polyhedral cone = positive span of a finite set of $\mathbb{R}^d$
= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face





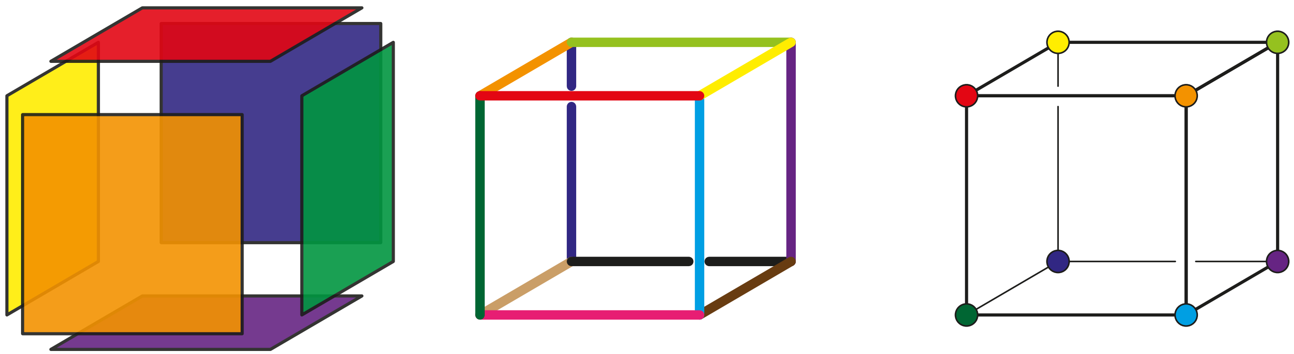
simplicial fan = maximal cones generated by d rays

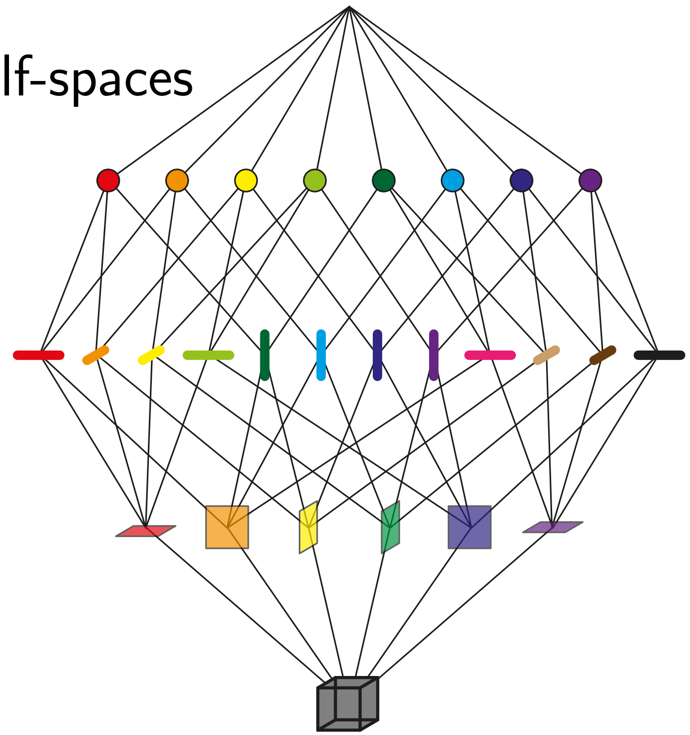
POLYTOPES

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

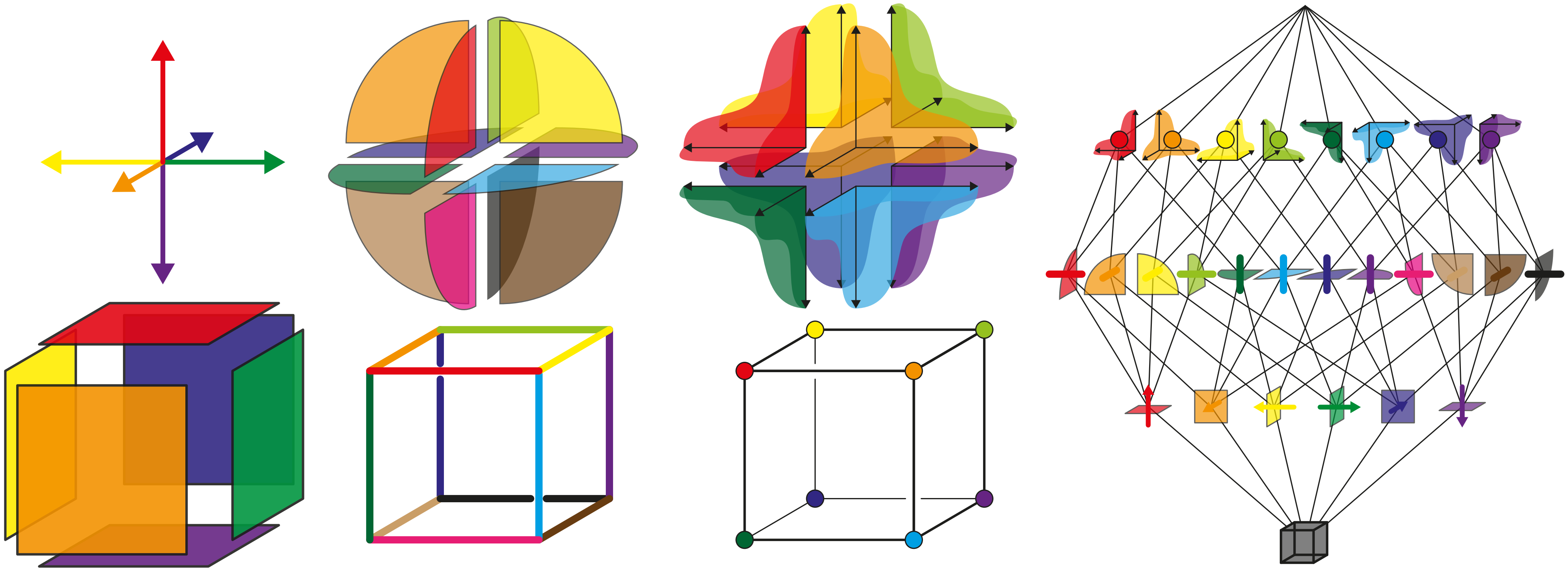
face lattice = all the faces with their inclusion relations





simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

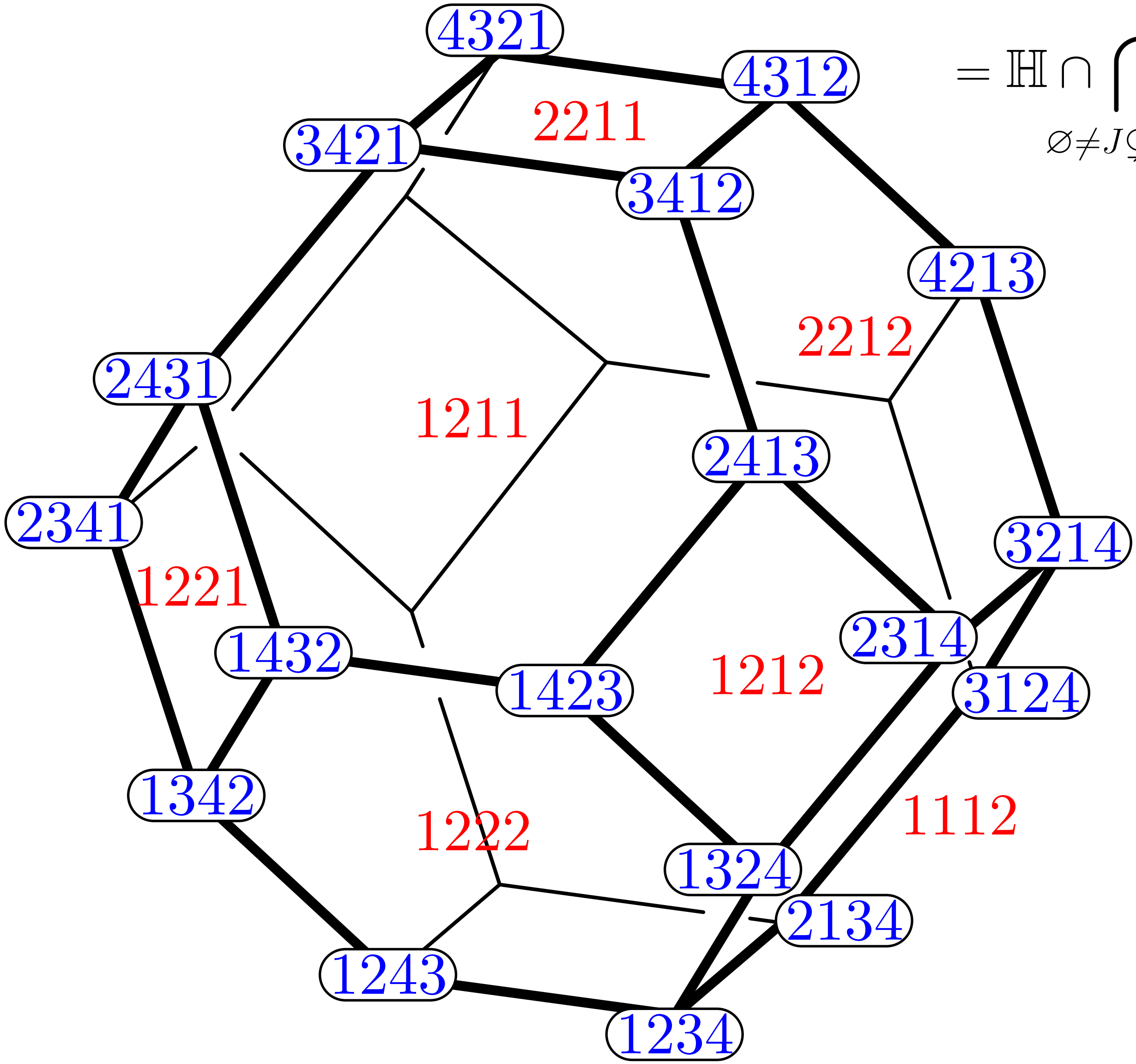
normal fan of $P = \{$ normal cone of $F \mid F$ face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

EXAMPLE: PERMUTAHEDRON

Ziegler, *Lectures on polytopes* ('95)
Hohlweg, *Permutahedra and associahedra* ('12)

PERMUTAHEDRON

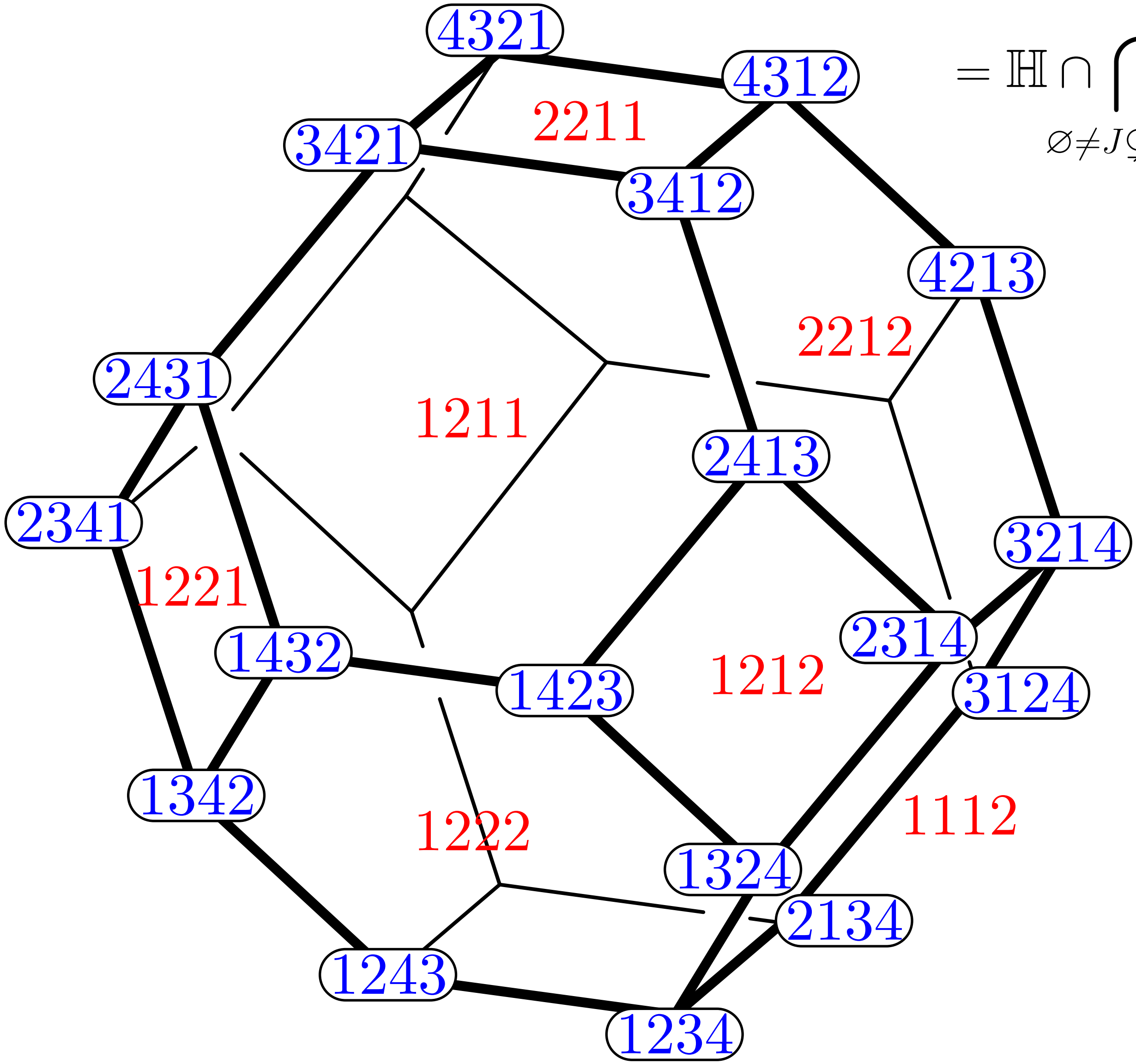


Permutahedron $\text{Perm}(n)$

$$= \text{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

PERMUTAHEDRON



Permutahedron Perm(n)

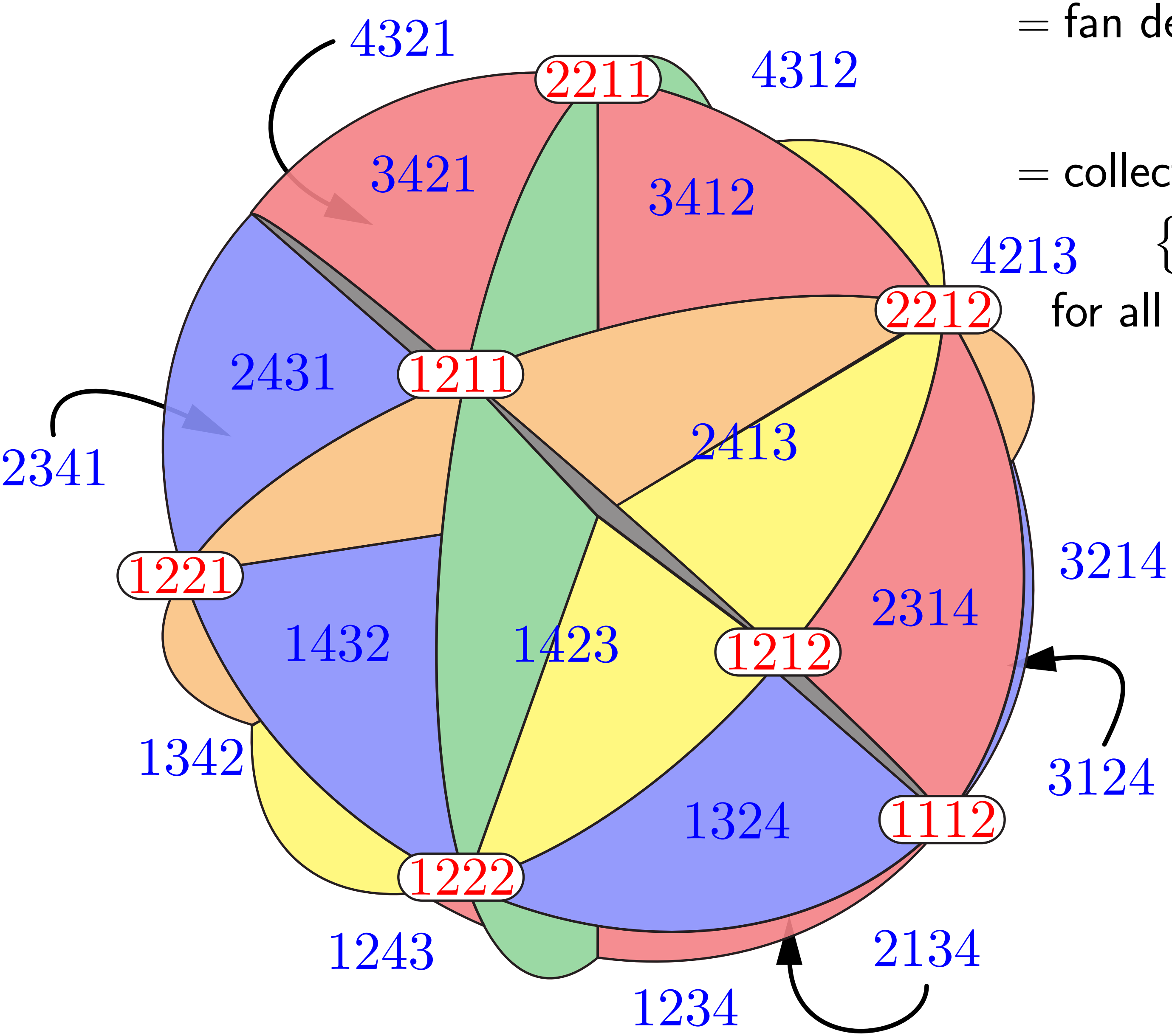
$$= \text{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

COXETER ARRANGEMENT



Coxeter fan

= fan defined by the hyperplane arrangement

$$\left\{\mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i = x_j\right\}_{1 \le i < j \le n+1}$$

= collection of all cones

$$\left\{\mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i < x_j \text{ if } \pi(i) < \pi(j)\right\}$$

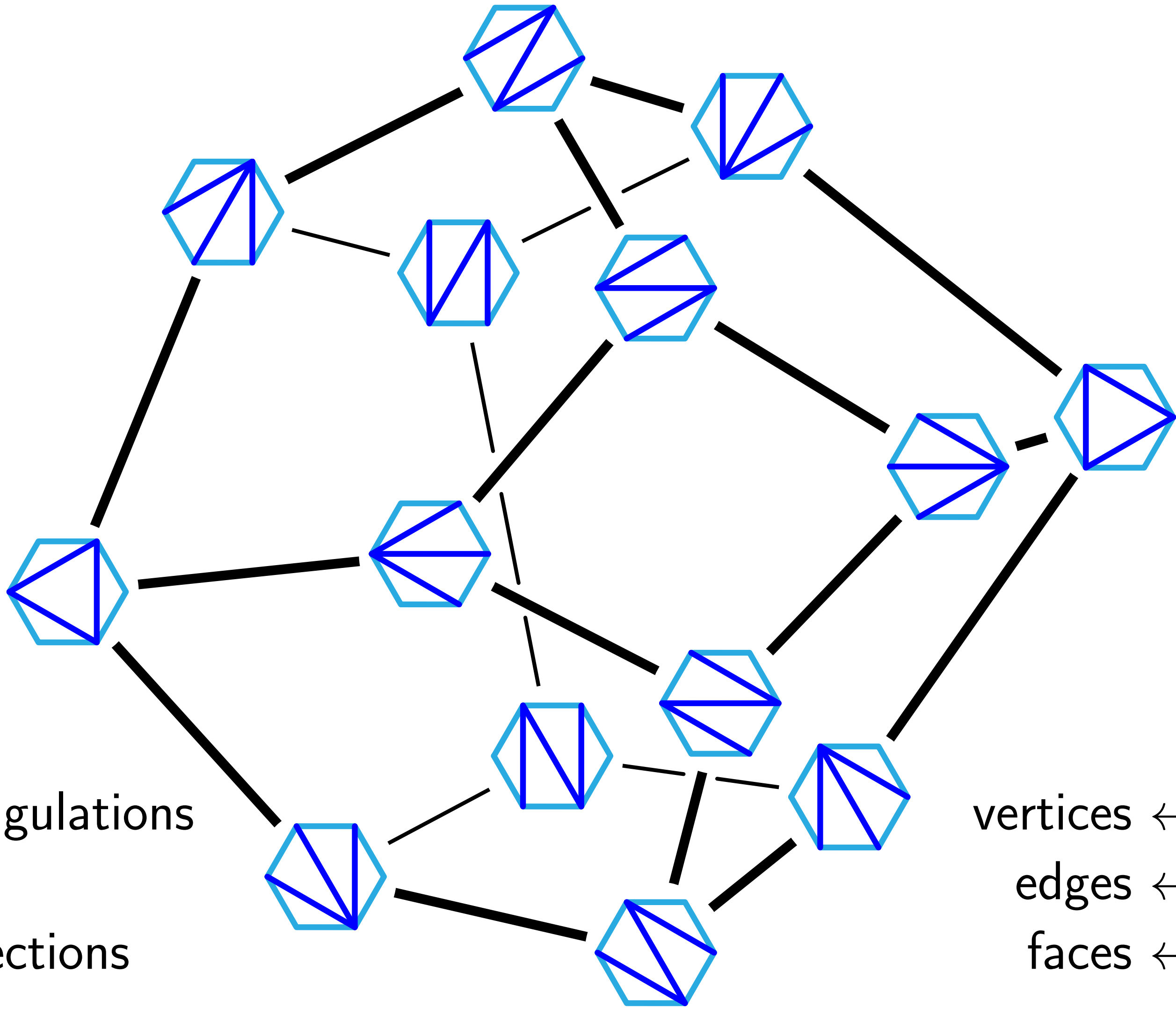
for all surjections $\pi : [n+1] \to [n+1-k]$

ASSOCIAHEDRA

Ceballos-Santos-Ziegler,
Many non-equivalent realizations of the associahedron ('11)

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the reverse-inclusion lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon

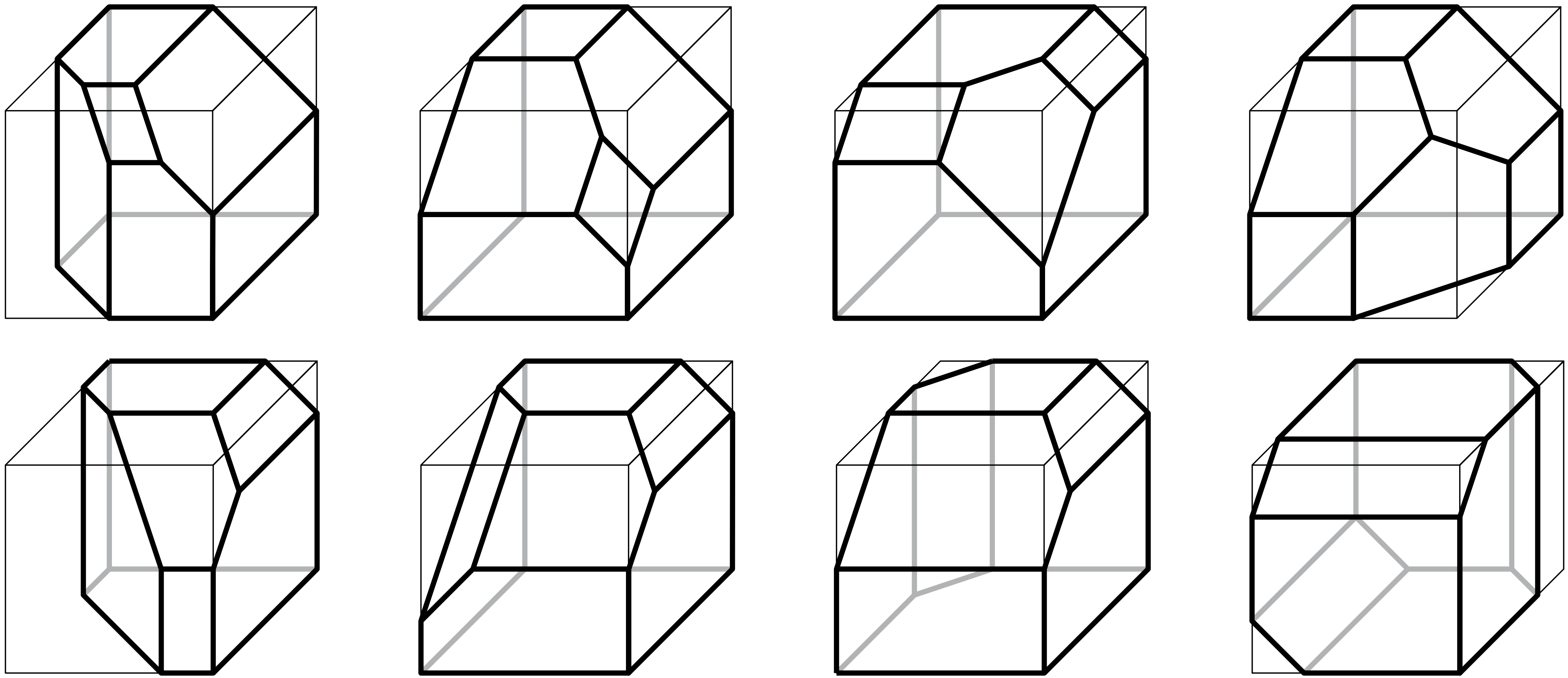


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the reverse-inclusion lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon



(Pictures by Ceballos-Santos-Ziegler)

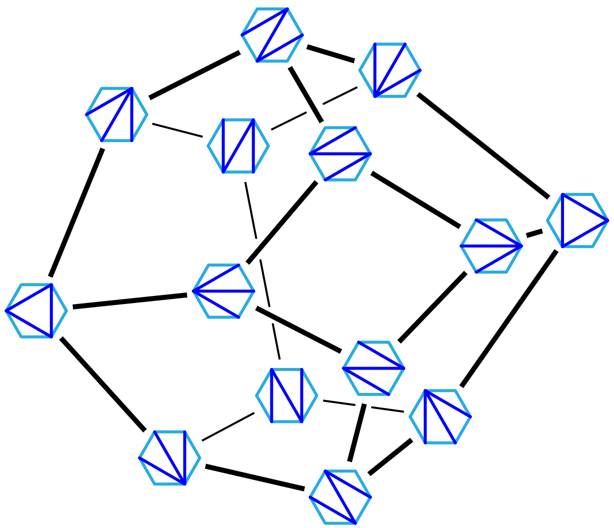
Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

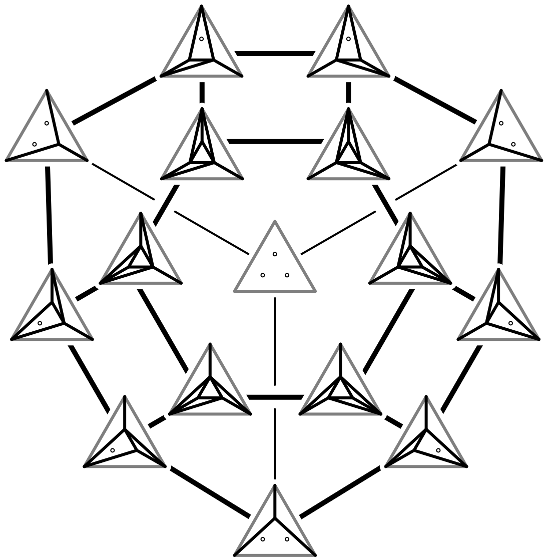
— Ceballos-Santos-Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

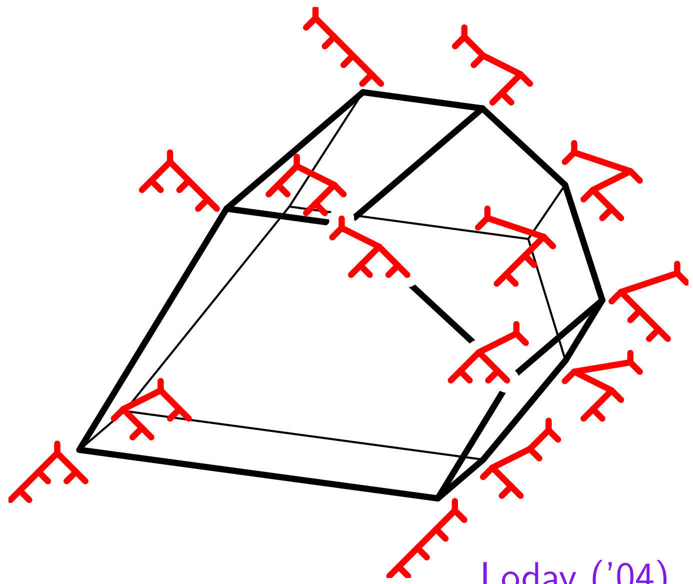
SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)



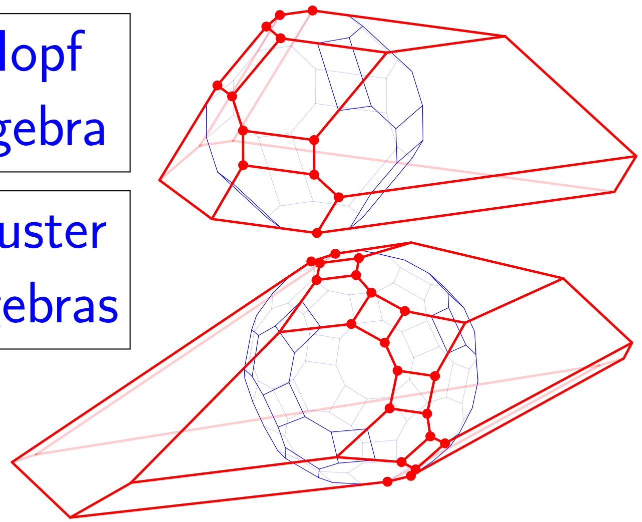
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

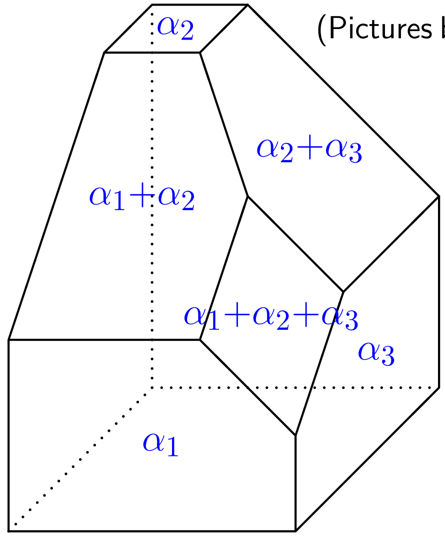
Hopf algebra

Cluster algebras



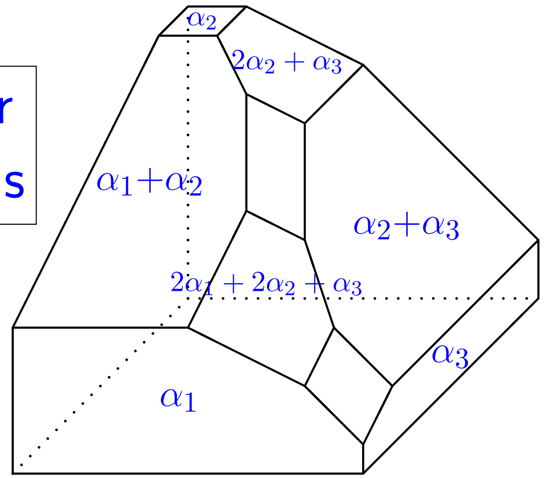
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



g-VECTOR FAN

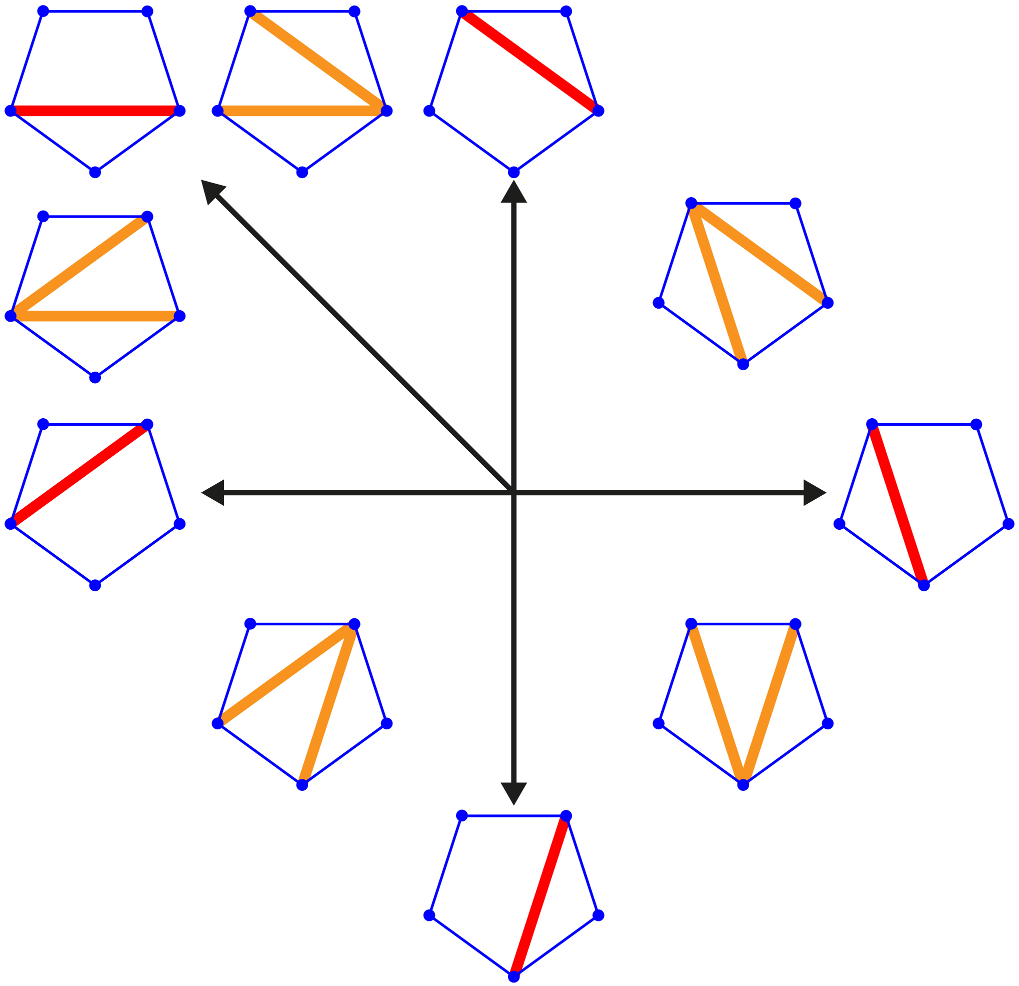
δ internal diagonal

<u>g-vector</u> of δ

$\mathbf{g}(\delta)$ = characteristic vector of points separated by δ from the top boundary edge

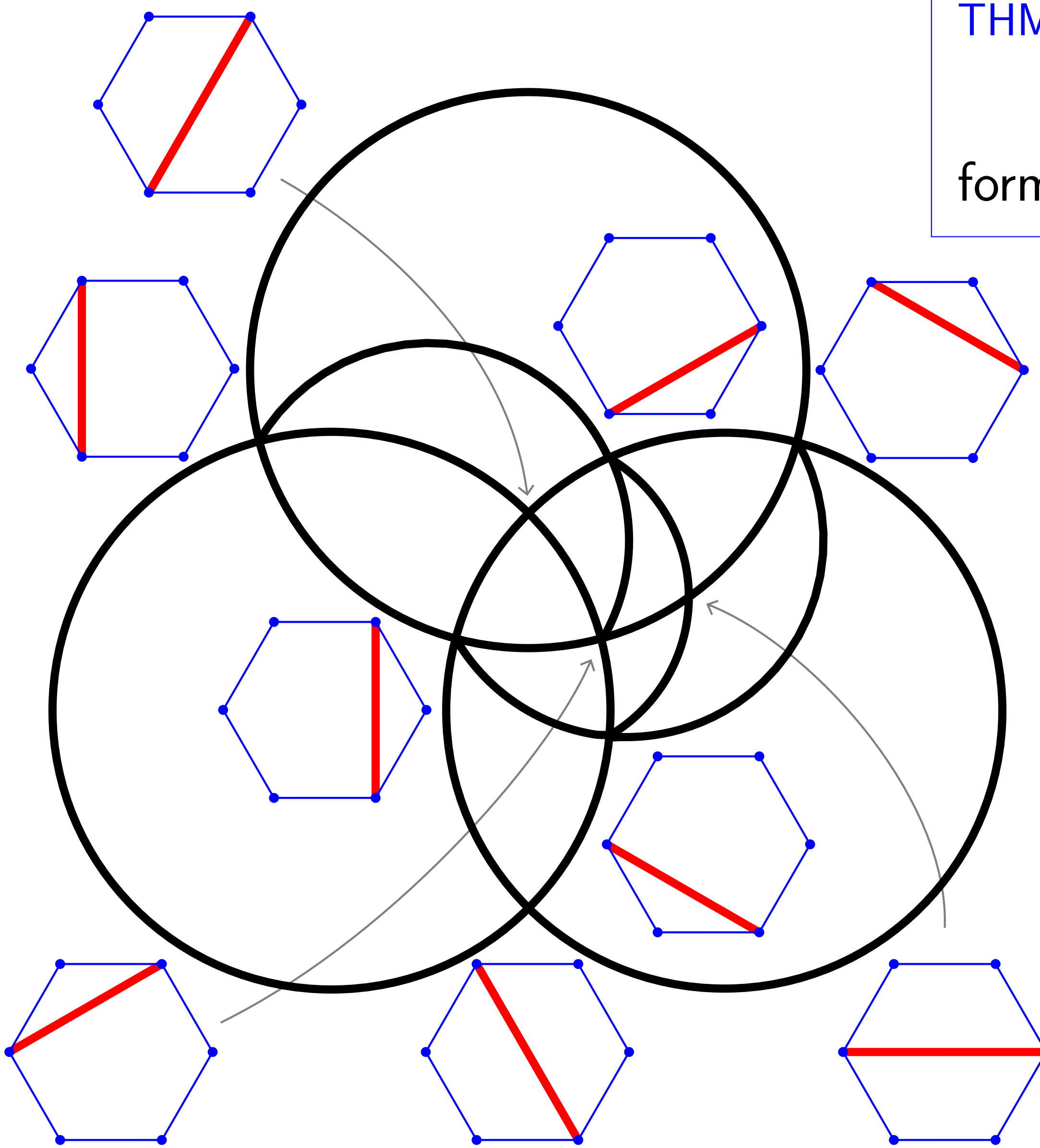
<u>g-vector fan</u>:

$$\mathcal{F}^{\mathbf{g}} = \{\mathbb{R}_{\geq 0}\,\mathbf{g}(\mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$$



g-VECTOR FAN

$\mathbf{g}(\delta) =$ g-vector of $\delta =$ characteristic vector of points separated by δ from the top diagonal

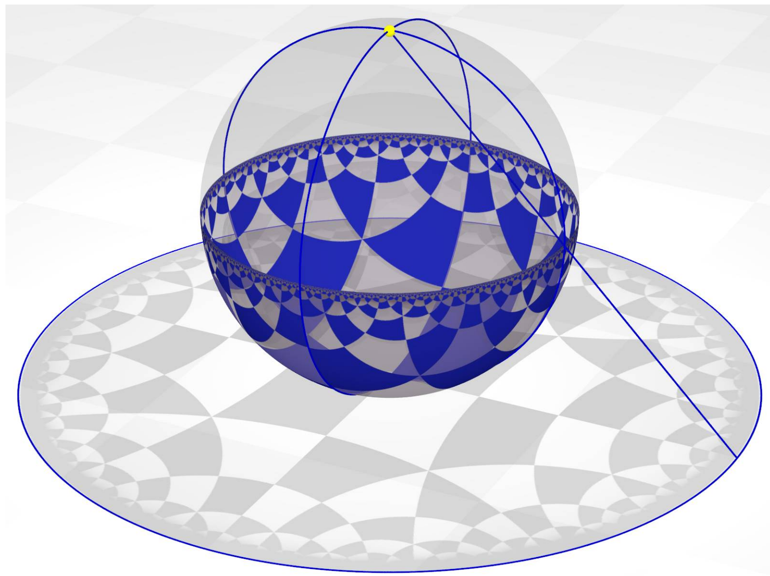


THM. The collection of cones

$$\mathcal{F}^{\mathbf{g}} := \left\{ \mathbb{R}_{\geq 0}\mathbf{g}(\mathrm{D}) \;\middle|\; \mathrm{D} \text{ dissection} \right\}$$

forms a compl. simpl. fan, called g-vector fan.

stereographic projection from $(1, 2, 3)$

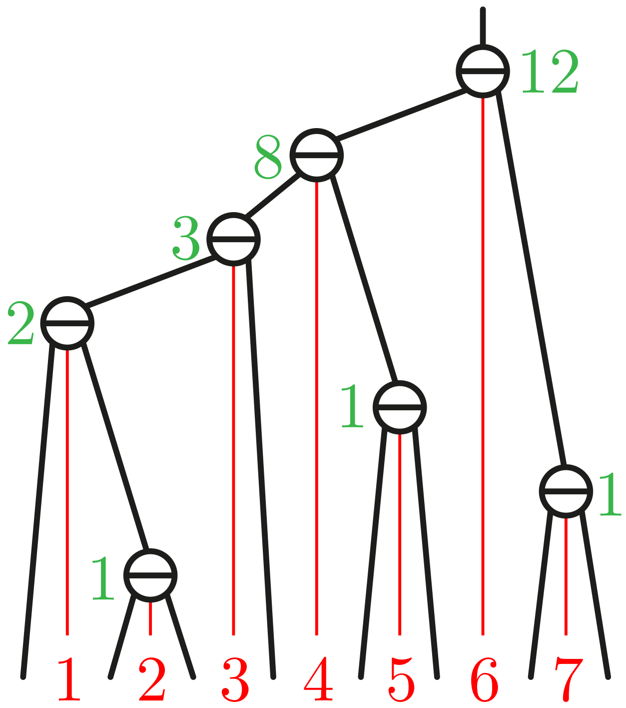


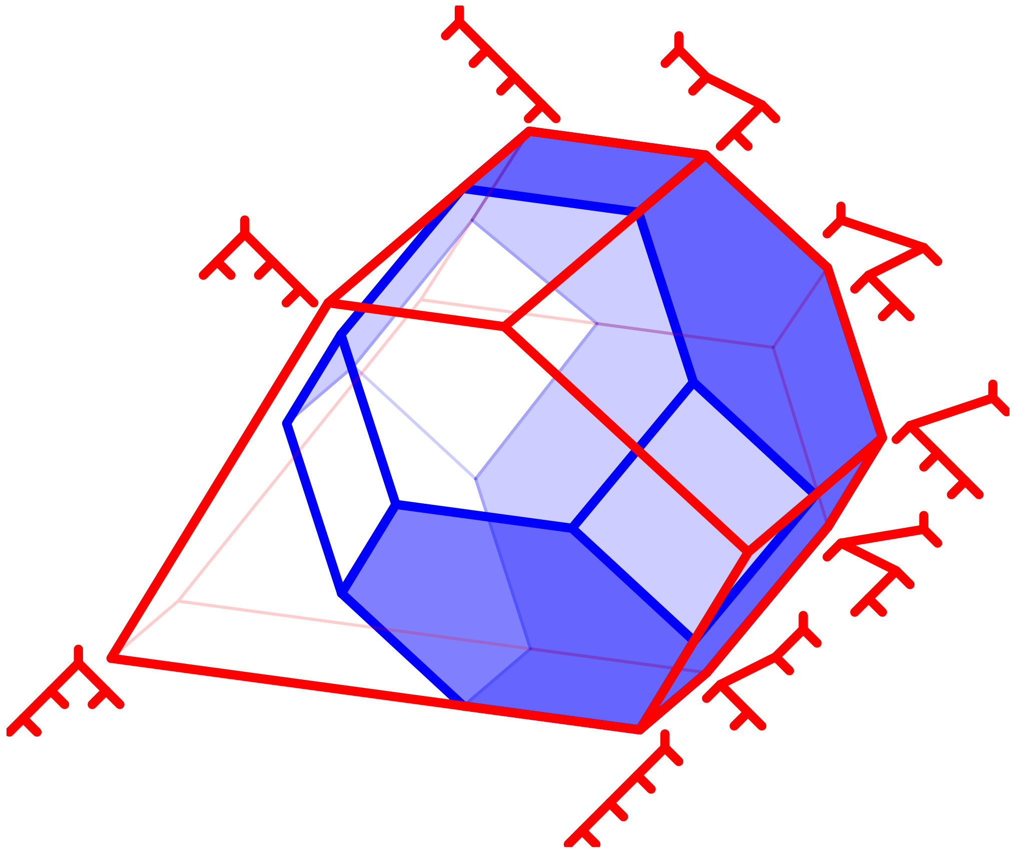
LODAY'S ASSOCIAHEDRON

$$\mathsf{Asso}(n) = \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i) \cdot r(\mathrm{T},i)\big]_{i \in [n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Loday, *Realization of the Stasheff polytope* ('04)





COMPATIBILITY FANS FOR ASSOCIAHEDRA

$\mathrm{T}_\circ$ an initial triangulation

δ, δ' two internal diagonals

compatibility degree between δ and δ':

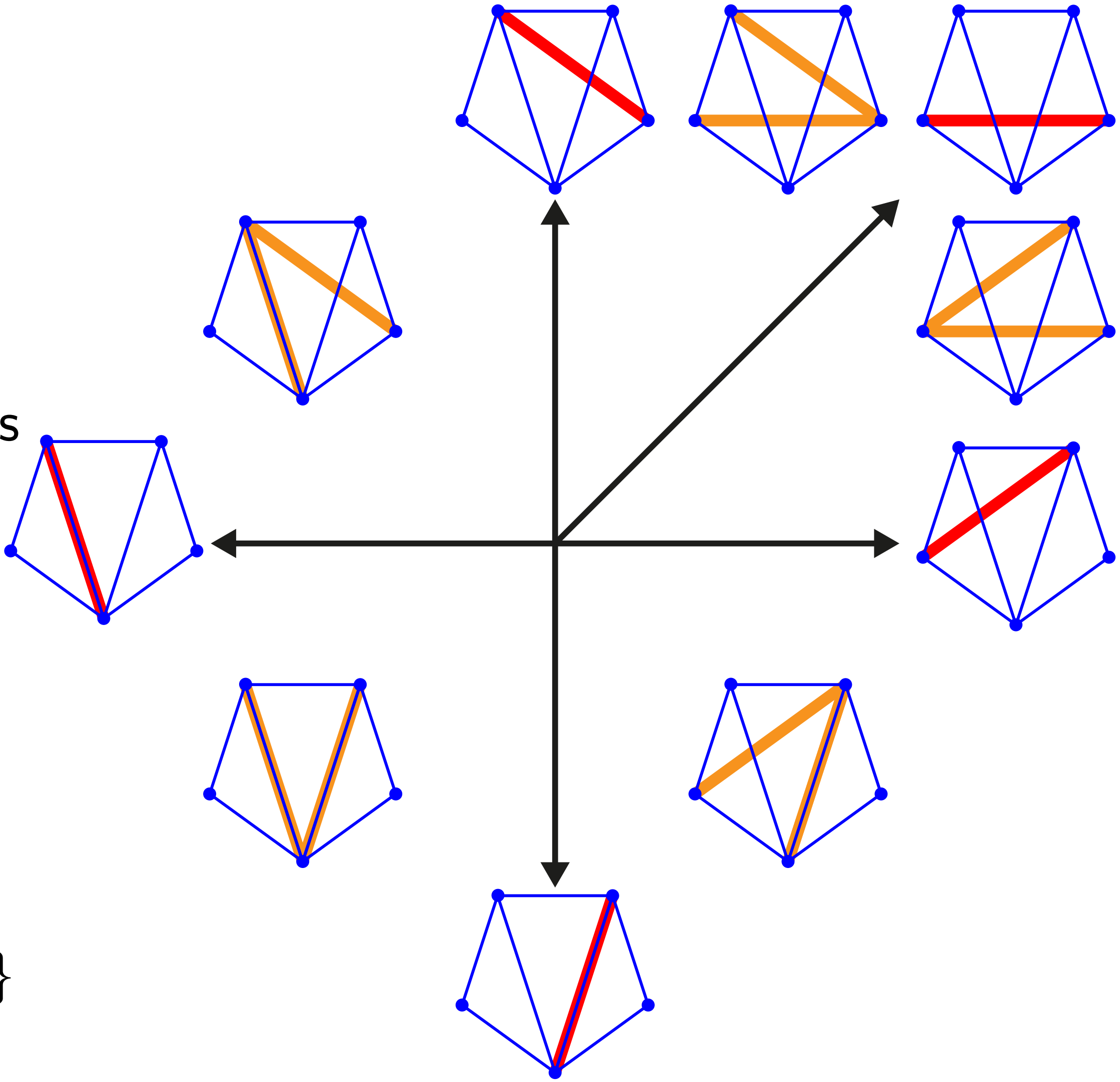
$$(\delta \parallel \delta') = \begin{cases} -1 & \text{if } \delta = \delta' \\ 0 & \text{if } \delta \text{ and } \delta' \text{ do not cross} \\ 1 & \text{if } \delta \text{ and } \delta' \text{ cross} \end{cases}$$

compatibility vector of δ wrt $\mathrm{T}_\circ$:

$$\mathbf{d}(\mathrm{T}_\circ, \delta) = \big[(\delta_\circ \parallel \delta)\big]_{\delta_\circ \in \mathrm{T}_\circ}$$

compatibility fan wrt $\mathrm{T}_\circ$:

$$\mathcal{F}^{\mathbf{d}}(\mathrm{T}_\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}_\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$$



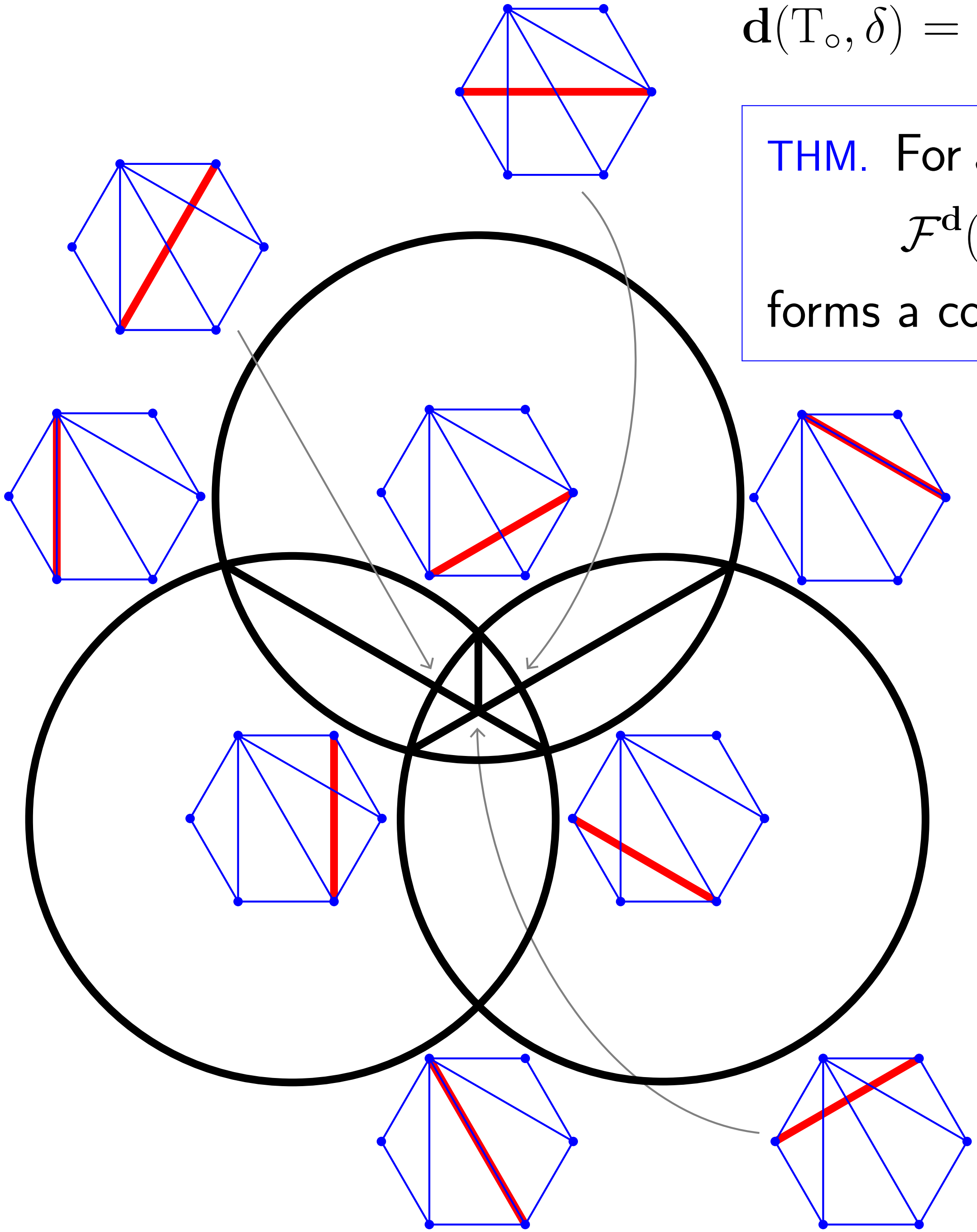
Fomin-Zelevinsky, *Y-Systems and generalized associahedra* ('03)

Fomin-Zelevinsky, *Cluster algebras II: Finite type classification* ('03)

Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized associahedra* ('02)

Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)

d-VECTOR FAN



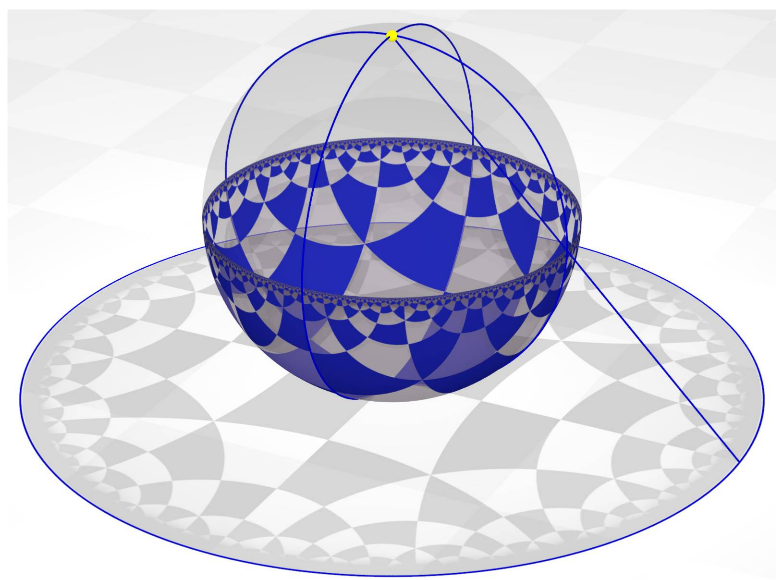
$\mathbf{d}(\mathrm{T}_\circ, \delta) = \mathbf{d}$-vector of δ wrt $\mathrm{T}_\circ = \big[(\delta_\circ \,\|\, \delta)\big]_{\delta_\circ \in \mathrm{T}_\circ}$

THM. For any triangulation $\mathrm{T}_\circ$, the collection of cones

$$\mathcal{F}^{\mathbf{d}}(\mathrm{T}_\circ) := \big\{ \mathbb{R}_{\geq 0}\mathbf{d}(\mathrm{T}_\circ, \mathrm{D}) \;\big|\; \mathrm{D} \text{ dissection} \big\}$$

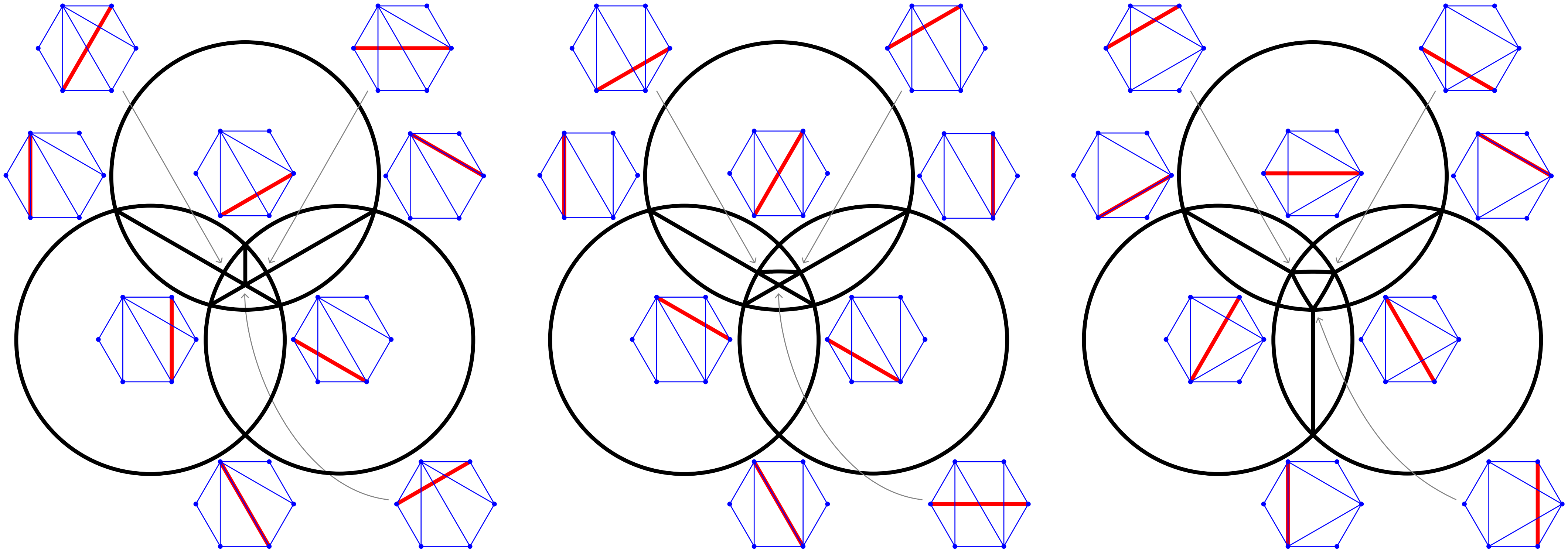
forms a compl. simpl. fan, called $\mathbf{d}$-vector fan of $\mathrm{T}_\circ$.

stereographic projection
from $(-1, -1, -1)$



COMPATIBILITY FANS FOR ASSOCIAHEDRA

Different initial triangulations $\mathrm{T}_\circ$ yield different realizations



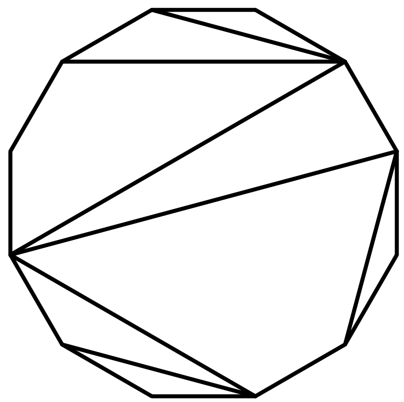
THM. For any initial triangulation $\mathrm{T}_\circ$, the cones $\{\mathbb{R}_{\geq 0}\,\mathbf{d}(\mathrm{T}_\circ, D) \mid D \text{ dissection}\}$ form a complete simplicial fan. Moreover, this fan is always polytopal.

Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)

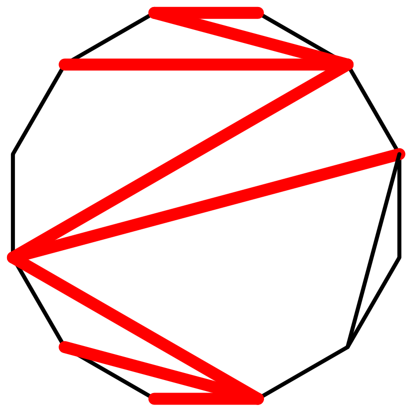
ACCORDION COMPLEX

Garver-McConville,
Oriented flip graphs and noncrossing tree partitions ('16^+)

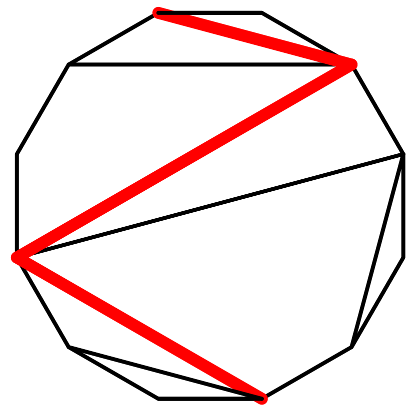
ACCORDIONS AND ZIGZAGS



dissection

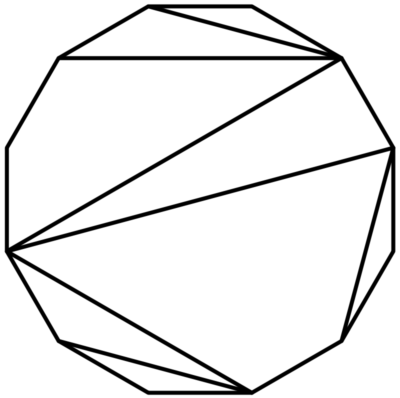


accordion

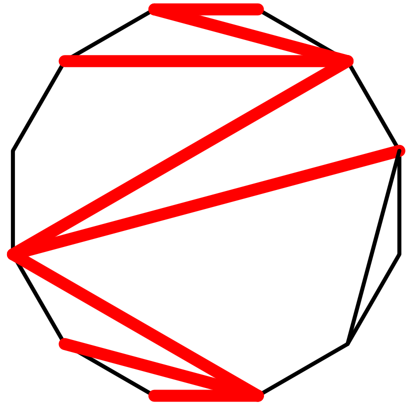


zigzag

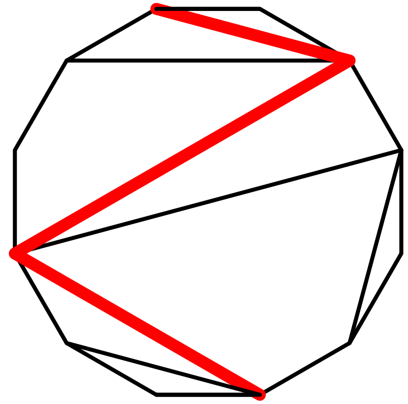
ACCORDIONS AND ZIGZAGS



dissection



accordion



zigzag

$D_\circ$-ACCORDION COMPLEX

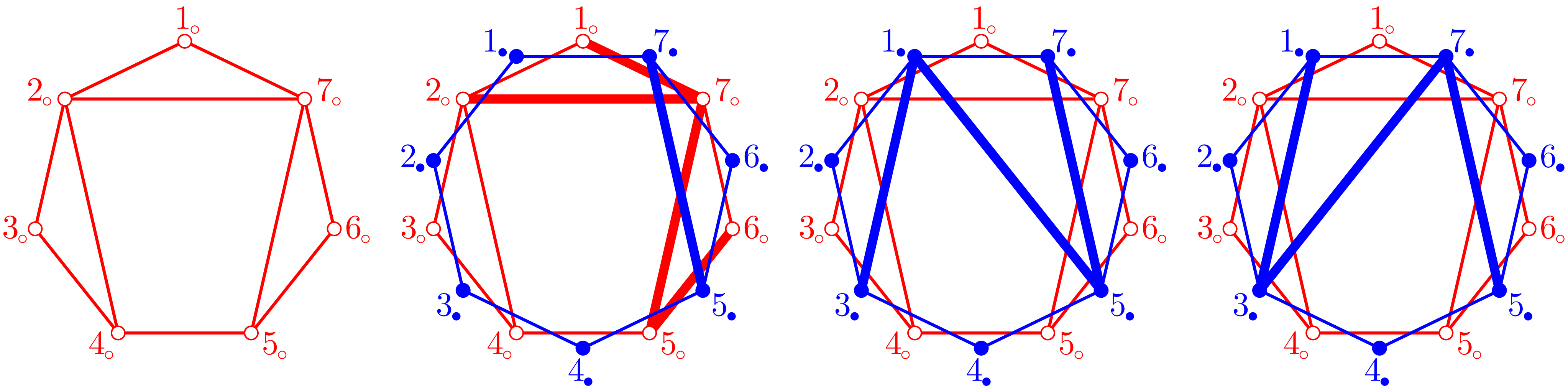
$2n$ points of the unit circle labeled counterclockwise by $1_\circ, 1_\bullet, 2_\circ, 2_\bullet, \dots, n_\circ, n_\bullet$

Fix a dissection $D_\circ$ of the red hollow polygon

$D_\circ$-accordion diagonal $=$ diagonal of the blue solid polygon that crosses an accordion of $D_\circ$

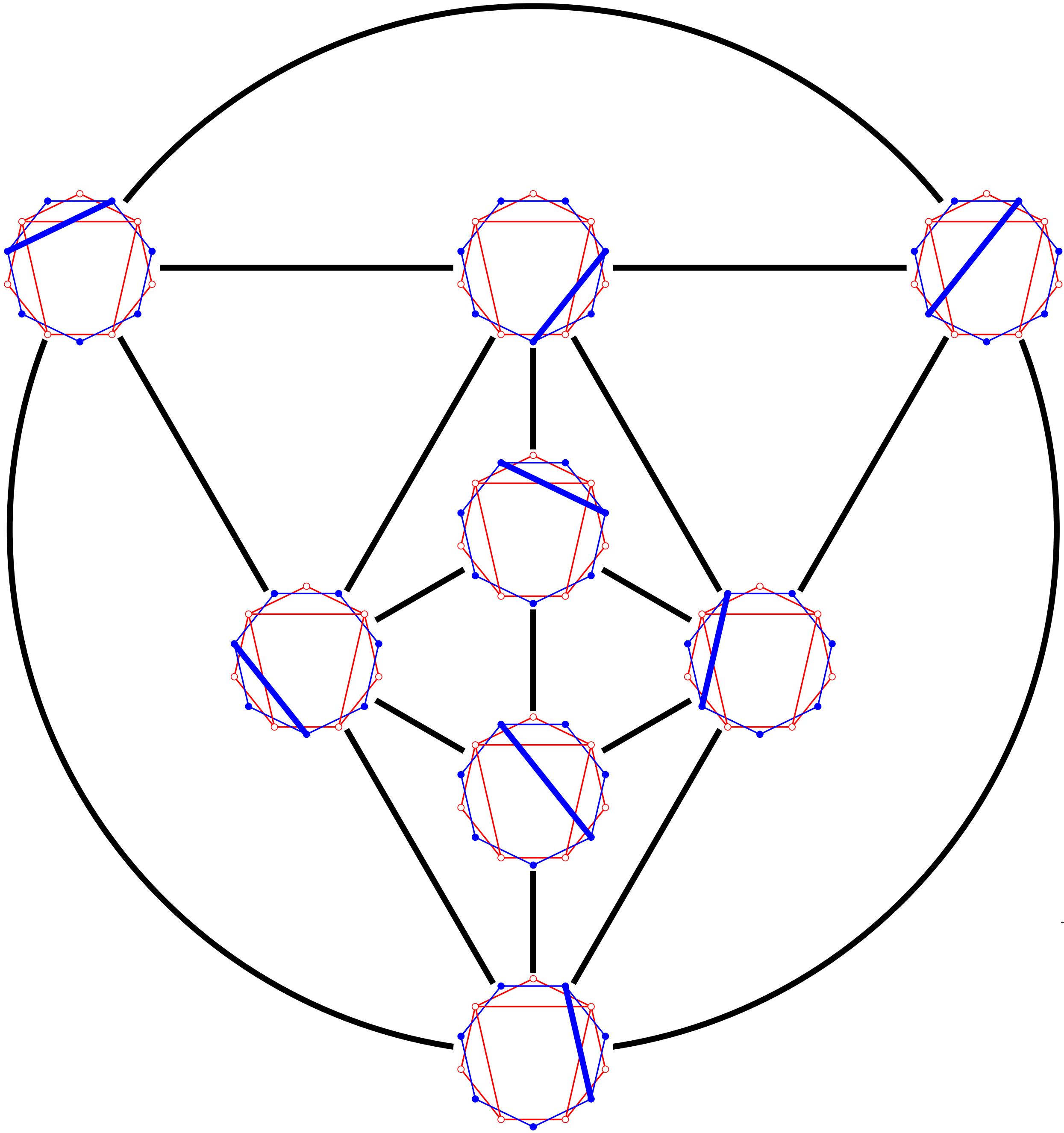
$D_\circ$-accordion dissection $=$ set of non-crossing $D_\circ$-accordion diagonals

$D_\circ$-accordion complex $=$ simplicial complex of $D_\circ$-accordion dissections



dissection $D_\circ$ $\quad$ $D_\circ$-accordion diagonal $\quad$ two maximal $D_\circ$-accordion dissections

D∘-ACCORDION COMPLEX



D∘-accordion complex = simplicial complex of D∘-accordion dissections

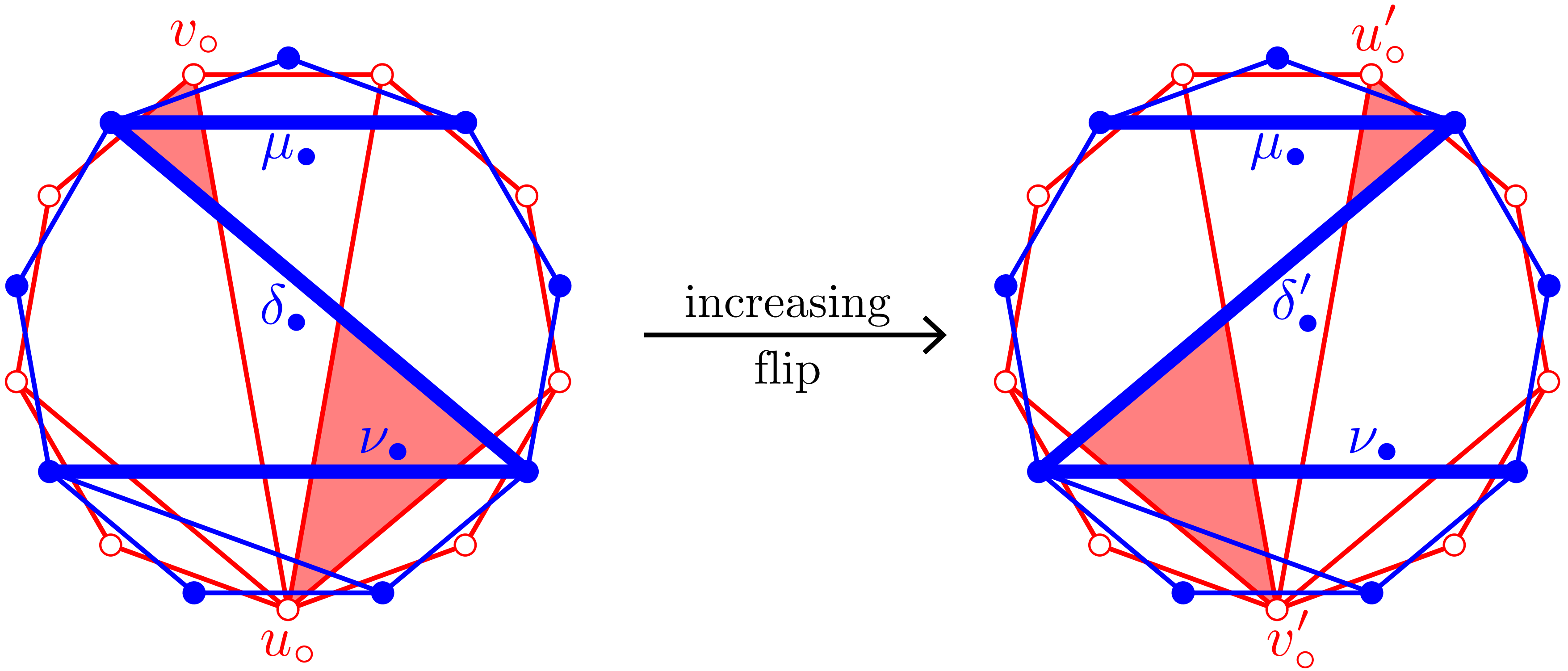
Exm: for a triangulation T∘, the T∘-accordion complex is a simplicial associahedron

FLIPS

PROP. The $D_\circ$-accordion complex is a pseudomanifold:

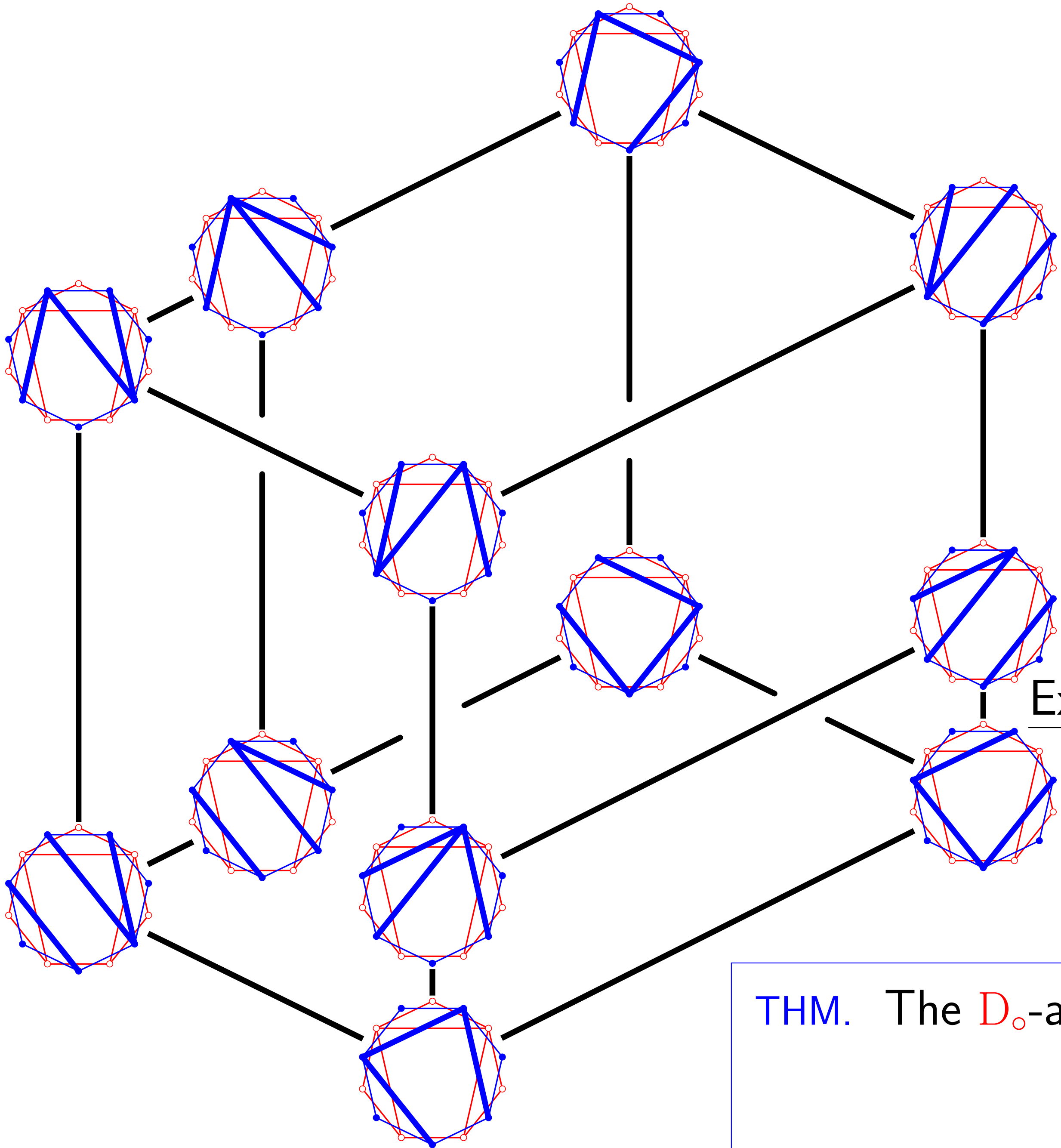
- pure: any maximal $D_\circ$-accordion dissection has $|D_\circ|$ diagonals
- thin: for any maximal $D_\circ$-accordion dissection $D_\bullet$ and any $\delta_\bullet \in D_\bullet$, there is a unique $\delta'_\circ \neq \delta_\circ$ such that $D_\circ \triangle \{\delta_\circ, \delta'_\circ\}$ is again a $D_\circ$-accordion dissection

Garver-McConville, *Oriented flip graphs and noncrossing tree partitions* ('16$^+$)



increasing flip = flip that changes a Σ to a Z

$\mathrm{D}_\circ$-ACCORDION LATTICE



increasing flip =
flip that changes a Σ to a Z

$\mathrm{D}_\circ$-accordion poset =
increasing flip poset on maximal $\mathrm{D}_\circ$-accordion dissections

Exm: for a comb triangulation $\mathrm{T}_\circ$,
the $\mathrm{T}_\circ$-accordion poset is the Tamari lattice

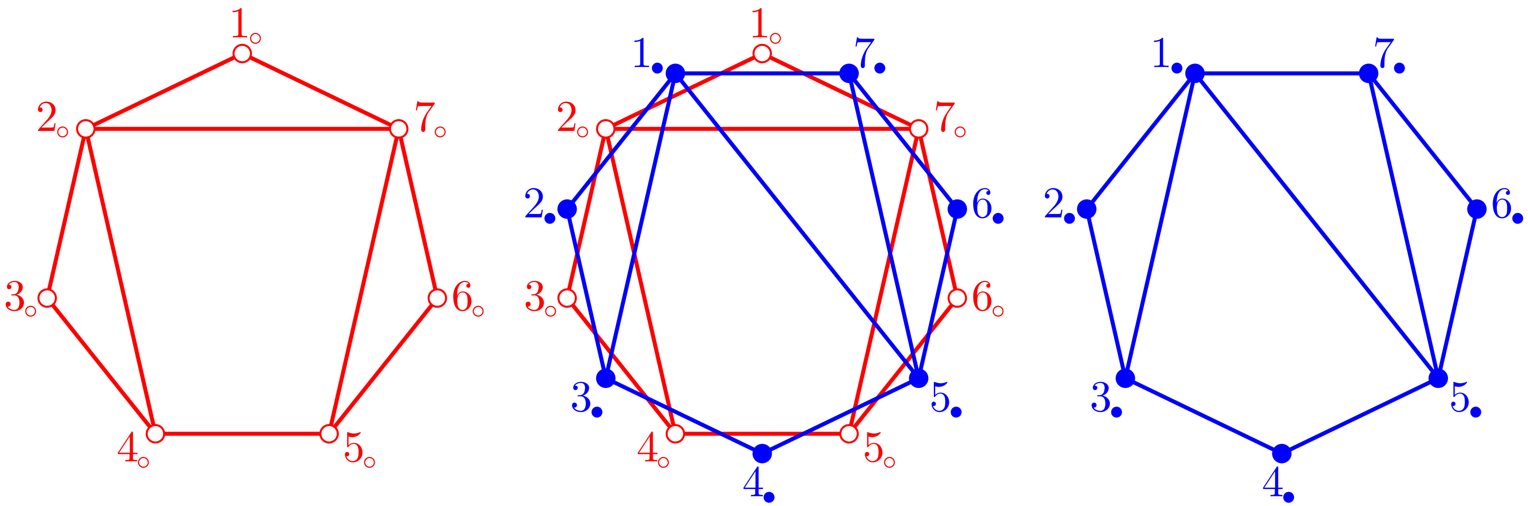
THM. The $\mathrm{D}_\circ$-accordion poset is a lattice

Garver-McConville, *Oriented flip graphs and noncrossing tree partitions* ('16$^+$)

DUALITY

PROP. $\mathrm{D}_\circ$ red hollow dissection & $\mathrm{D}_\bullet$ blue solid dissection

$\mathrm{D}_\bullet$ is a maximal $\mathrm{D}_\circ$-accordion dissection $\iff$ $\mathrm{D}_\circ$ is a maximal $\mathrm{D}_\bullet$-accordion dissection



"Look from the other side of the board..."

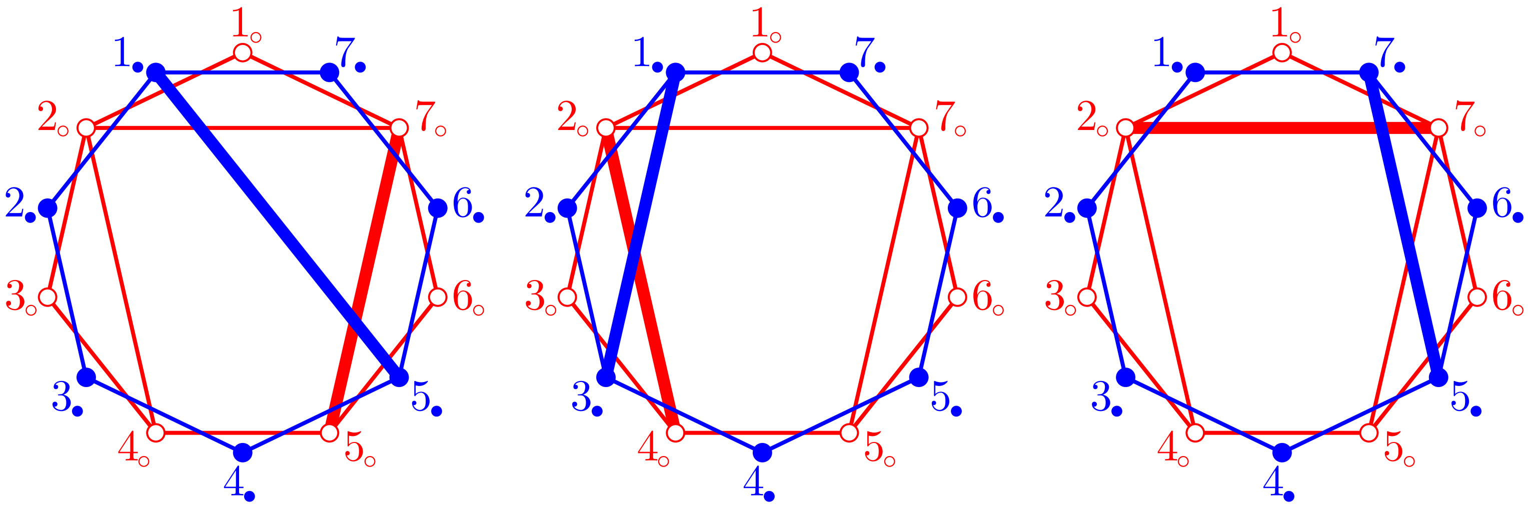
G-VECTOR FAN

Manneville-P.,
Geometric realizations of the accordion complex of a dissection ('16$^+$)

g-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection, $\delta_\circ \in \mathrm{D}_\circ$ and $\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal, let

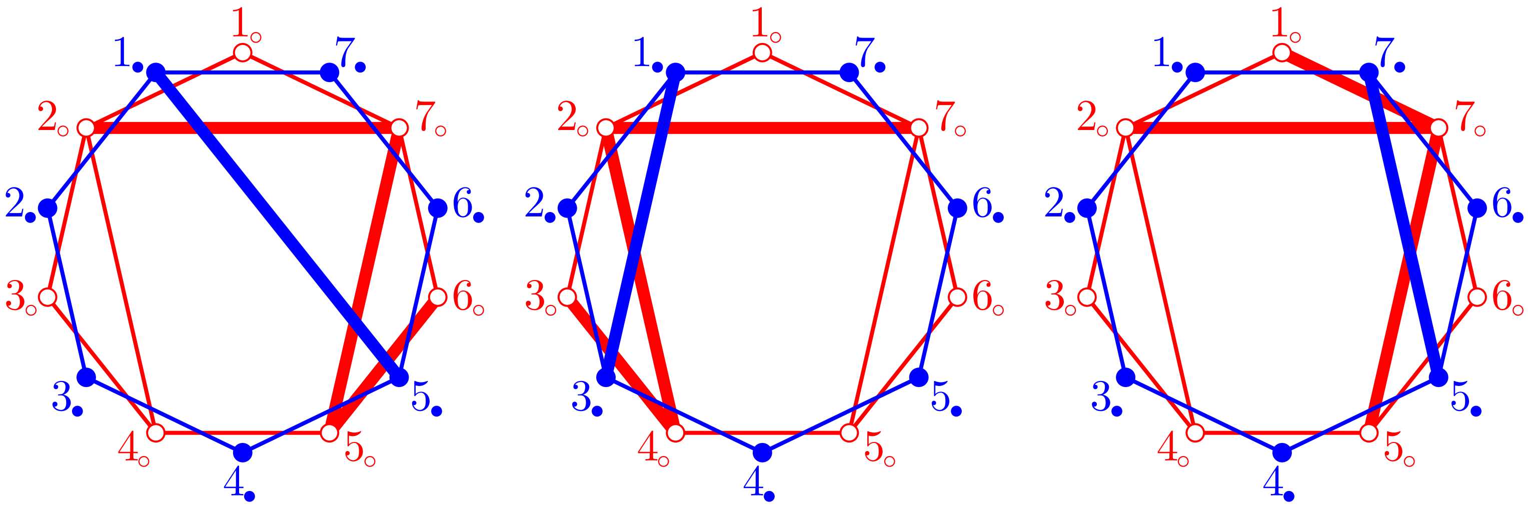
$$\varepsilon_\circ(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{D}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{D}_\circ \text{ as an } \Sigma \\ 0 & \text{otherwise} \end{cases}$$



g-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection, $\delta_\circ \in \mathrm{D}_\circ$ and $\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal, let

$$\varepsilon_\circ(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{D}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{D}_\circ \text{ as an Σ} \\ 0 & \text{otherwise} \end{cases}$$

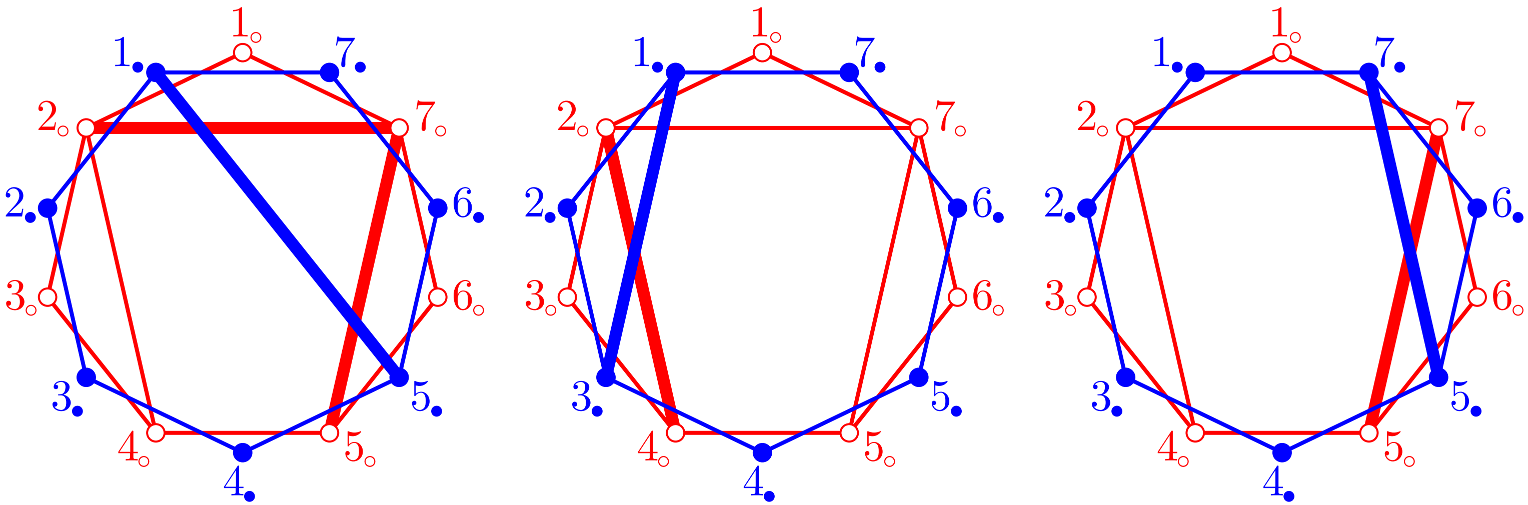


Z $= 1$ Σ $= -1$ Ψ $= 0$

g-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection, $\delta_\circ \in \mathrm{D}_\circ$ and $\delta_\bullet$ a $\mathrm{D}_\circ$-accordion diagonal, let

$$\varepsilon_\circ\big(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet\big) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{D}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{D}_\circ \text{ as an } \Sigma \\ 0 & \text{otherwise} \end{cases}$$



$\mathbf{g}(\mathrm{D}_\circ, (1_\bullet, 5_\bullet)) = \mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{2_\circ 7_\circ}$ $\qquad$ $\mathbf{g}(\mathrm{D}_\circ, (1_\bullet, 3_\bullet)) = -\mathbf{e}_{2_\circ 4_\circ}$ $\qquad$ $\mathbf{g}(\mathrm{D}_\circ, (5_\bullet, 7_\bullet)) = \mathbf{e}_{5_\circ 7_\circ}$

$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet)$ = g-vector of $\delta_\bullet$ with respect to $\mathrm{D}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet\big)\Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$

= alternating ± 1 along the zigzag crossed by $\delta_\bullet$ in $\mathrm{D}_\circ$

g-VECTOR FAN

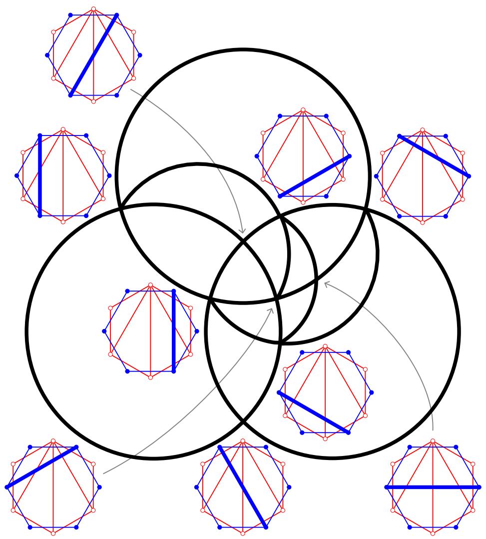
$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet)$ = g-vector of $\delta_\bullet$ with respect to $\mathrm{D}_\circ = \left[\varepsilon_\circ(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet) \right]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$

THM. For any dissection $\mathrm{D}_\circ$, the collection of cones

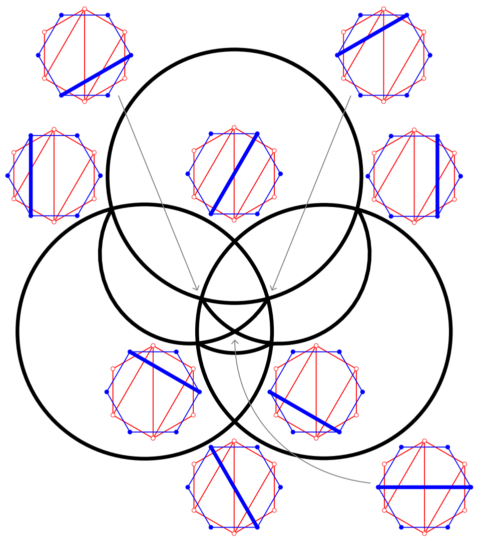
$$\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ) := \left\{ \mathbb{R}_{\geq 0}\mathbf{g}(\mathrm{D}_\circ, \mathrm{D}_\bullet) \mid \mathrm{D}_\bullet \text{ any } \mathrm{D}_\circ\text{-accordion dissection} \right\}$$

forms a complete simplicial fan, called g-vector fan of $\mathrm{D}_\circ$.

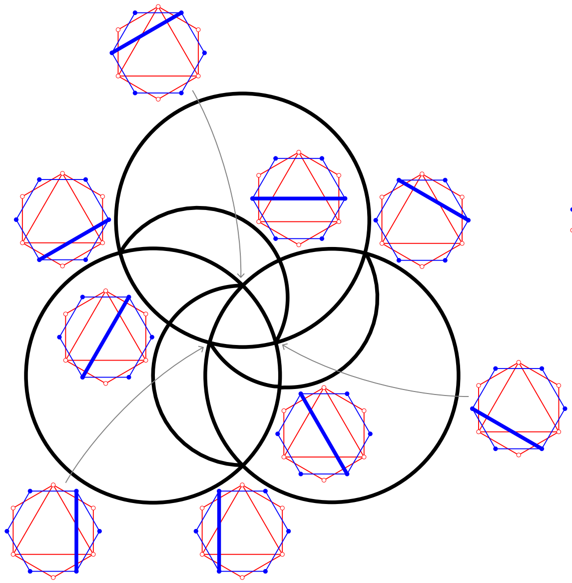
Manneville-P., *Geometric realizations of the accordion complex of a dissection* ('16$^+$)



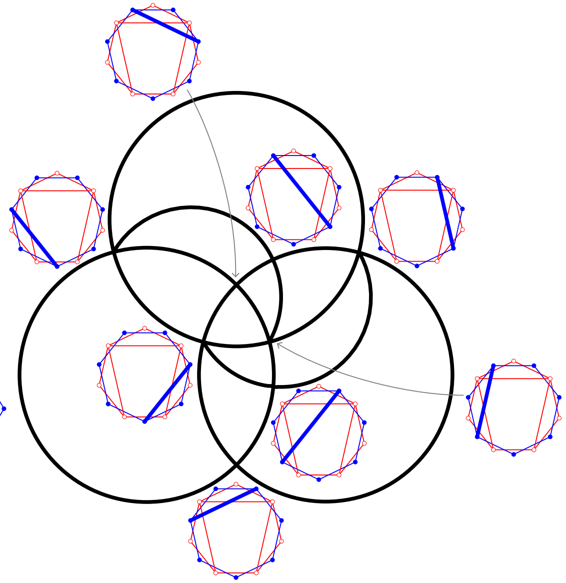
Loday



Hohlweg-Lange
Reading

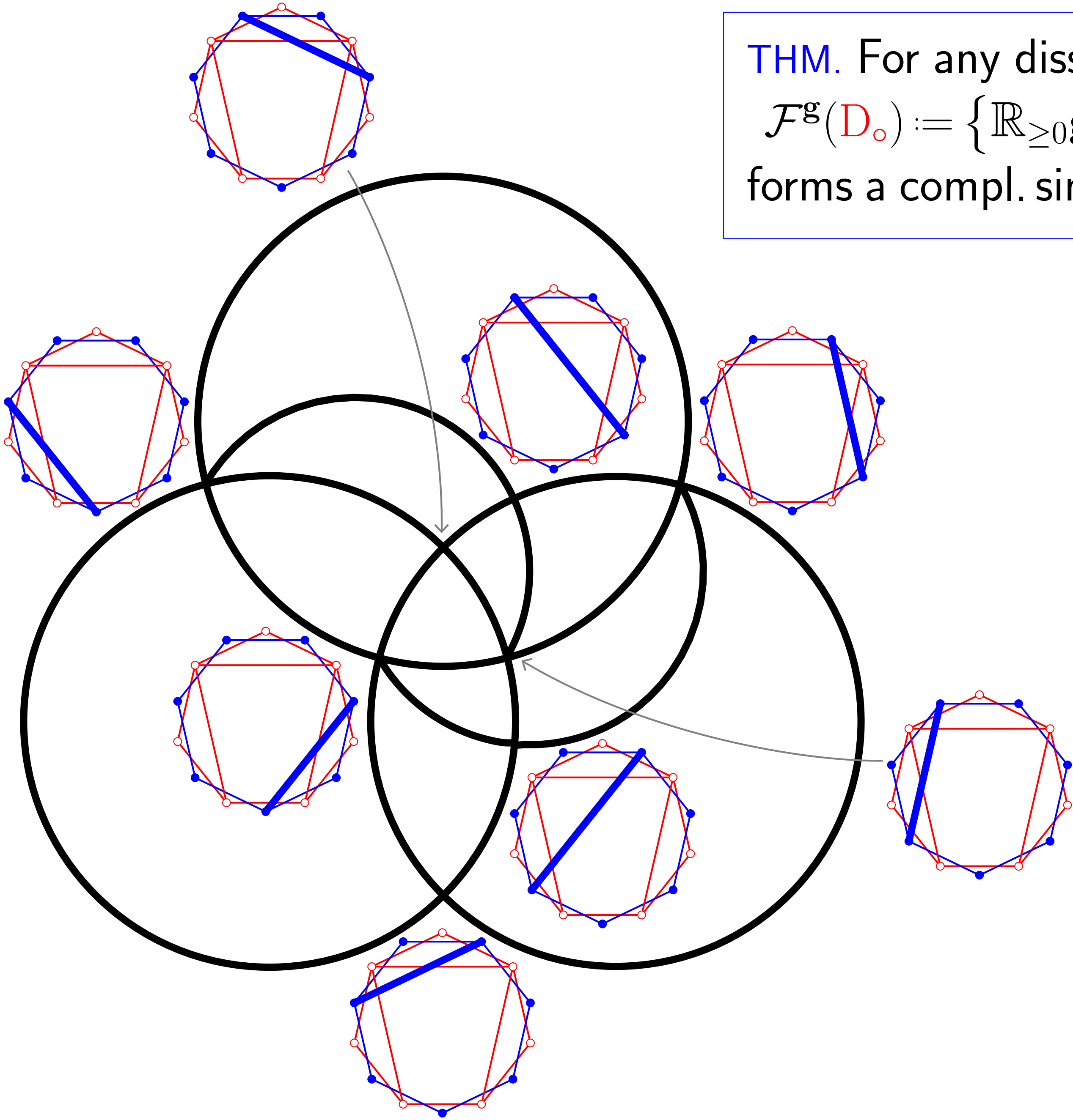


Hohlweg-P.-Stella



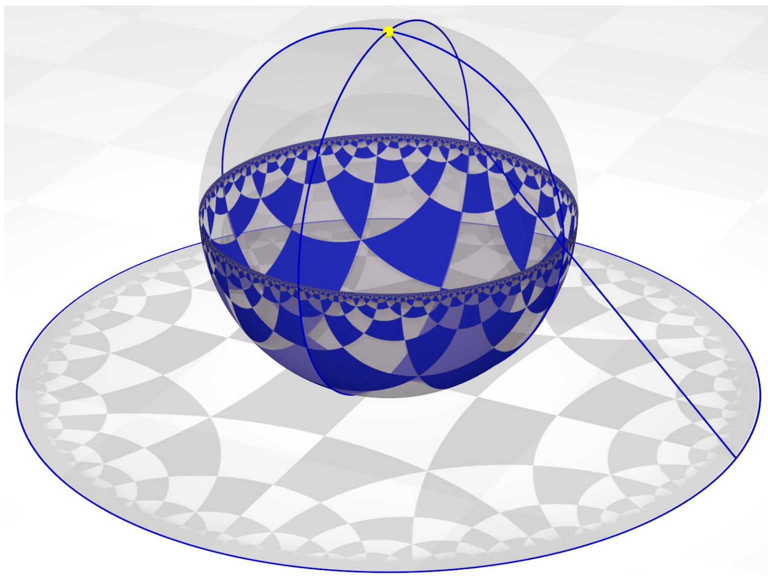
Manneville-P.

g-VECTOR FAN



THM. For any dissection $D_\circ$, the collection of cones
$$\mathcal{F}^{\mathbf{g}}(D_\circ) := \{\mathbb{R}_{\geq 0}\mathbf{g}(D_\circ, D_\bullet) \mid D_\bullet \text{ any } D_\circ\text{-acc. diss.}\}$$
forms a compl. simpl. fan, called g-vector fan of $D_\circ$.

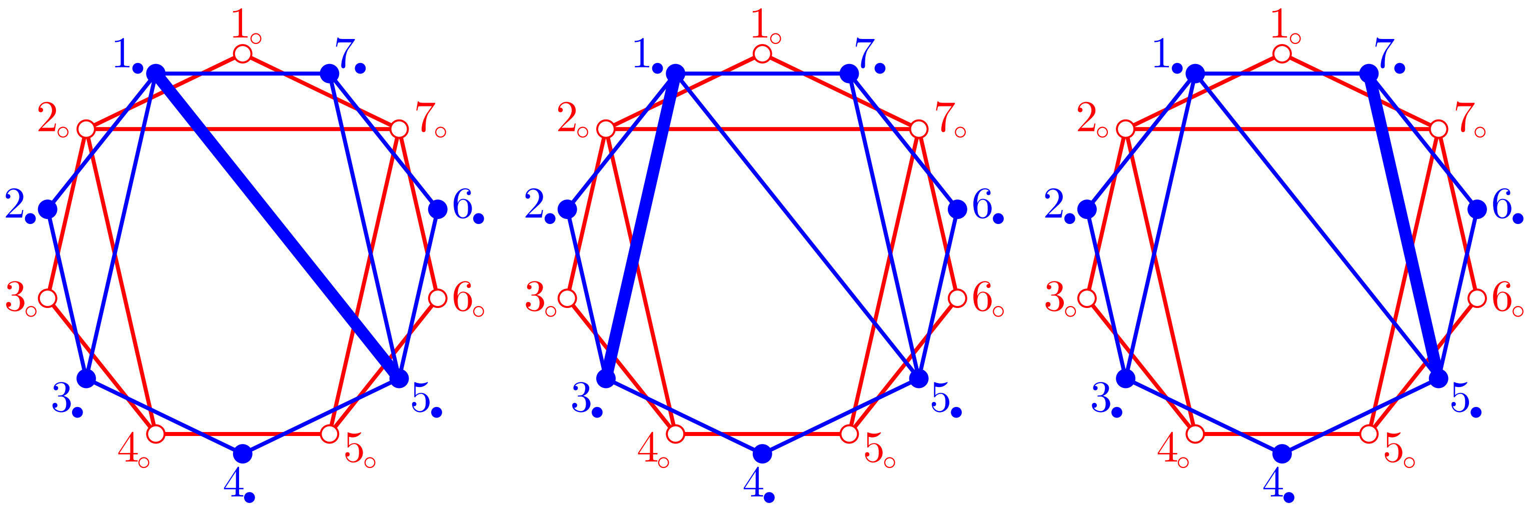
stereographic projection
from $(1,1,1)$



c-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection & $\mathrm{D}_\bullet$ blue solid dissection, accordion dissections of eachother and two diagonals $\delta_\circ \in \mathrm{D}_\circ$ and $\delta_\bullet \in \mathrm{D}_\bullet$, let

$$\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{D}_\bullet\big) = \begin{cases} 1 & \text{if } \delta_\circ \text{ slaloms on } \delta_\bullet \in \mathrm{D}_\bullet \text{ as a } \Sigma \\ -1 & \text{if } \delta_\circ \text{ slaloms on } \delta_\bullet \in \mathrm{D}_\bullet \text{ as an } Z \\ 0 & \text{otherwise} \end{cases}$$



$\mathbf{c}(\mathrm{D}_\circ, (1_\bullet, 5_\bullet) \in \mathrm{D}_\bullet) = -\mathbf{e}_{2_\circ 7_\circ}$

$\mathbf{c}(\mathrm{D}_\circ, (1_\bullet, 3_\bullet) \in \mathrm{D}_\bullet) = -\mathbf{e}_{2_\circ 4_\circ}$

$\mathbf{c}(\mathrm{D}_\circ, (5_\bullet, 7_\bullet) \in \mathrm{D}_\bullet) = \mathbf{e}_{2_\circ 7_\circ} + \mathbf{e}_{5_\circ 7_\circ}$

$\mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet) =$ <u>$\mathbf{c}$-vector</u> of $\delta_\bullet$ in $\mathrm{D}_\bullet$ with respect to $\mathrm{D}_\circ$ $= \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{D}_\bullet\big)\Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$

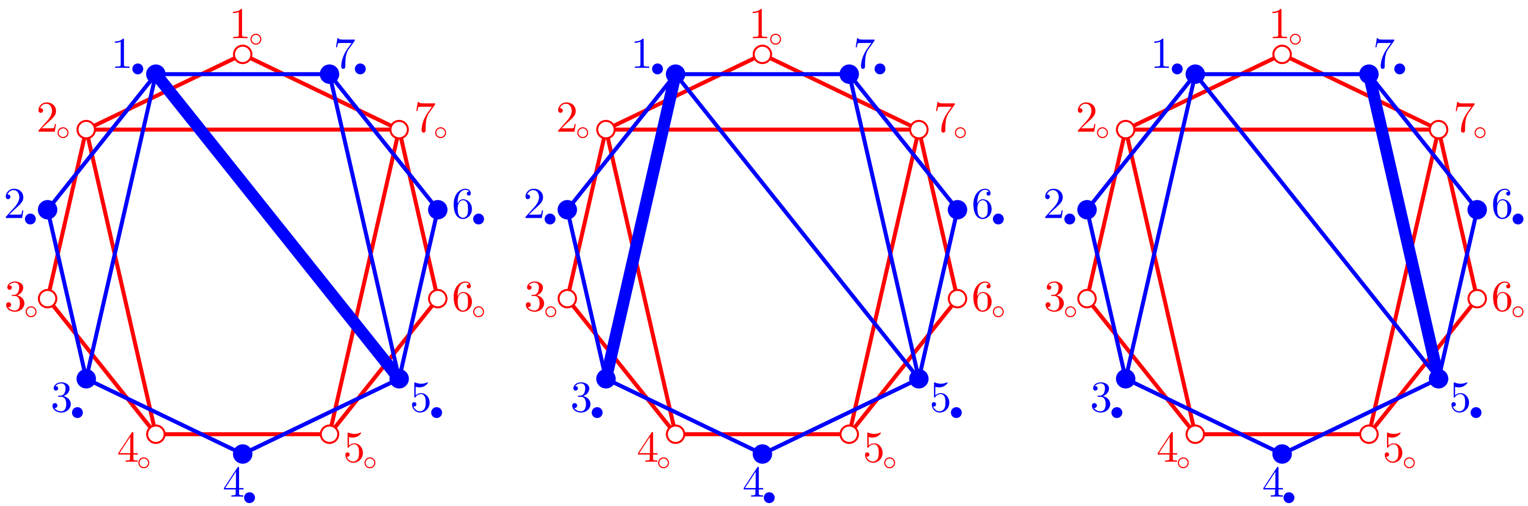
$=\pm$ charac. vector of diagonals of $\mathrm{D}_\circ$ crossed by opposite neighbors of $\delta_\bullet$

g- AND c-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection & $\mathrm{D}_\bullet$ blue solid dissection, accordion dissections of eachother,

$$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) = \underline{\text{g-vector}} \text{ of } \delta_\bullet \text{ with respect to } \mathrm{D}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet\big) \Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$$

$$\mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet) = \underline{\text{c-vector}} \text{ of } \delta_\bullet \text{ in } \mathrm{D}_\bullet \text{ with respect to } \mathrm{D}_\circ = \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{D}_\bullet\big) \Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$$



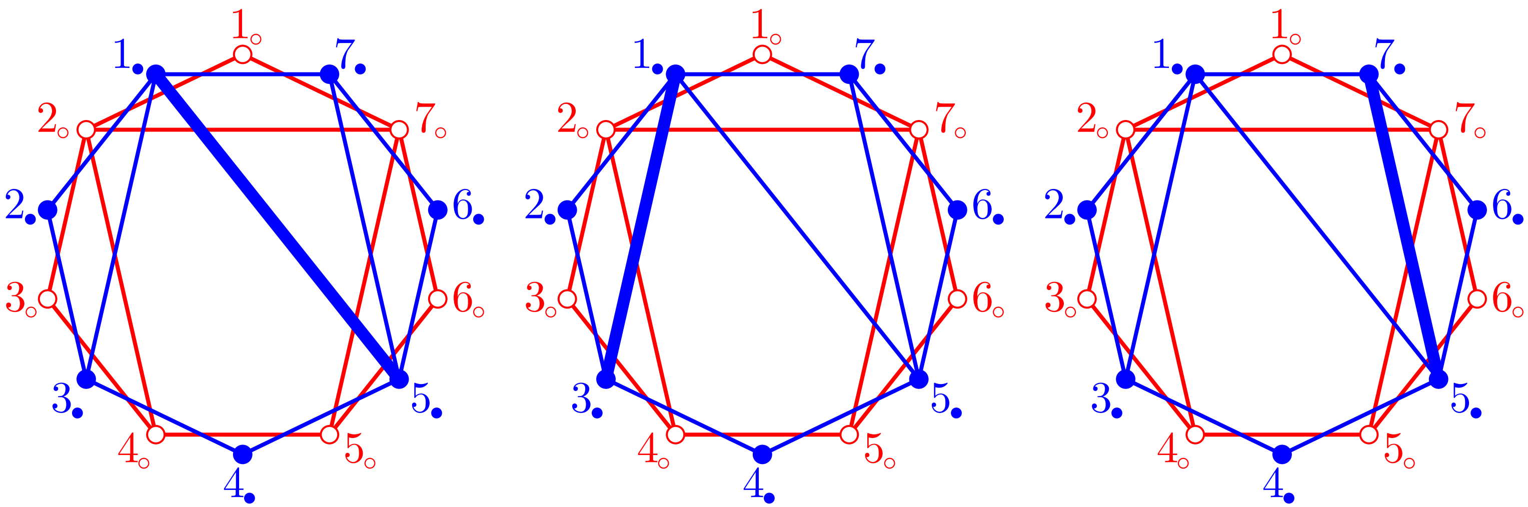
$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) =$	$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{5_\circ 7_\circ}$
$\mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet) =$	$-\mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{2_\circ 7_\circ} + \mathbf{e}_{5_\circ 7_\circ}$

g- AND c-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection & $\mathrm{D}_\bullet$ blue solid dissection, accordion dissections of eachother,

$$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) = \underline{\text{g-vector}} \text{ of } \delta_\bullet \text{ with respect to } \mathrm{D}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{D}_\circ, \delta_\bullet\big) \Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$$

$$\mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet) = \underline{\text{c-vector}} \text{ of } \delta_\bullet \text{ in } \mathrm{D}_\bullet \text{ with respect to } \mathrm{D}_\circ = \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{D}_\bullet\big) \Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$$



$\mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) =$	$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{5_\circ 7_\circ}$
$\mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet) =$	$-\mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{2_\circ 7_\circ} + \mathbf{e}_{5_\circ 7_\circ}$

PROP. The g-vectors $\mathbf{g}(\mathrm{D}_\circ, \mathrm{D}_\bullet)$ and the c-vectors $\mathbf{c}(\mathrm{D}_\circ, \mathrm{D}_\bullet)$ form dual bases.

PROP. Duality: $\mathbf{g}(\mathrm{D}_\circ, \mathrm{D}_\bullet) = -\mathbf{c}(\mathrm{D}_\bullet, \mathrm{D}_\circ)^t$ and $\mathbf{c}(\mathrm{D}_\circ, \mathrm{D}_\bullet) = -\mathbf{g}(\mathrm{D}_\bullet, \mathrm{D}_\circ)^t$

$\mathrm{D}_\circ$-ZONOTOPE

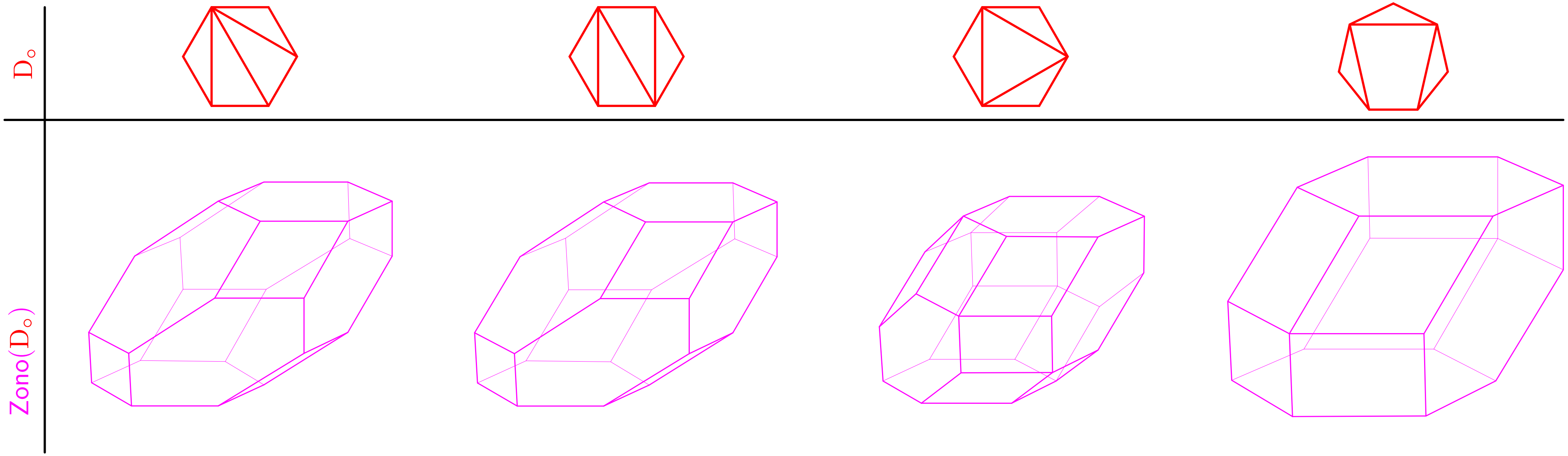
$\mathrm{D}_\circ$-zonotope $=$ Zono$(\mathrm{D}_\circ)$ $=$ Minkowski sum of all $\mathbf{c}$-vectors $\mathbf{C}(\mathrm{D}_\circ) = \bigcup_{\mathrm{D}_\bullet} \mathbf{c}(\mathrm{D}_\circ, \mathrm{D}_\bullet)$

$$\text{Zono}(\mathrm{D}_\circ) = \sum_{\mathbf{c} \in \mathbf{C}(\mathrm{D}_\circ)} \mathbf{c}.$$

PROP. For any $\mathrm{D}_\circ$-accordion diagonal $\gamma_\bullet$, Zono$(\mathrm{D}_\circ)$ has a facet defined by the inequality

$$\langle\, \mathbf{g}(\mathrm{D}_\circ, \gamma_\bullet) \mid \mathbf{x} \,\rangle \leq \omega(\mathrm{D}_\circ, \gamma_\bullet),$$

where $\omega(\mathrm{D}_\circ, \gamma_\bullet)$ $=$ $\mathrm{D}_\circ$-height of $\gamma_\bullet$ $=$ number of $\mathrm{D}_\circ$-accordion diagonals that cross $\gamma_\bullet$.



$\mathrm{D}_\circ$-ACCORDIOHEDRON

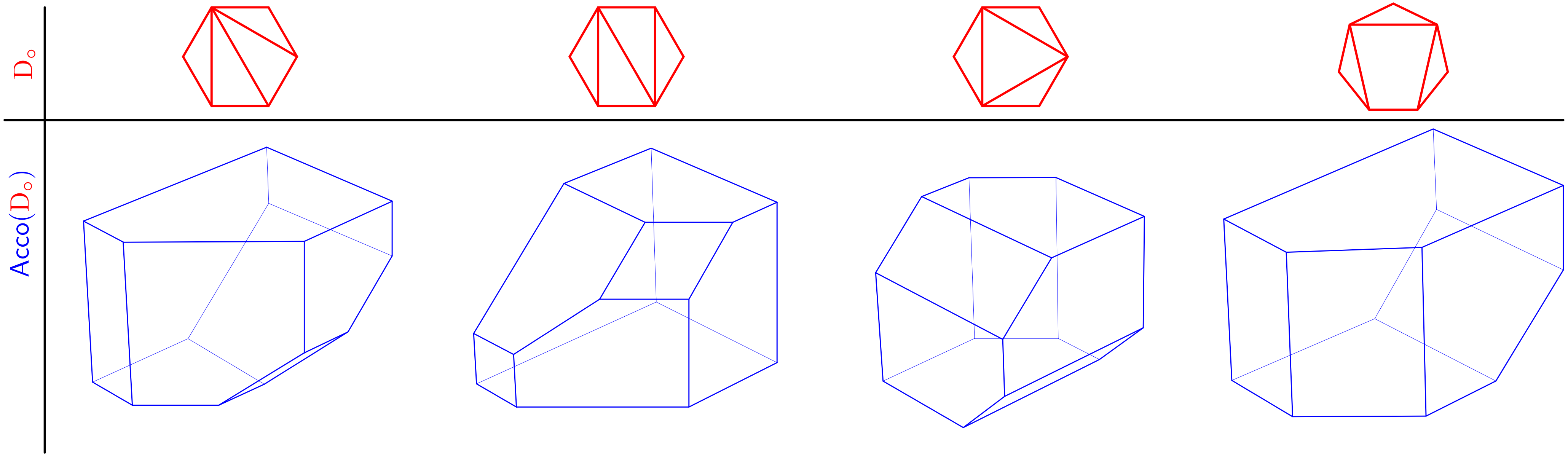
Define $\mathbf{p}(\mathrm{D}_\circ, \mathrm{D}_\bullet) := \sum_{\delta_\bullet \in \mathrm{D}_\bullet} \omega(\mathrm{D}_\circ, \delta_\bullet) \cdot \mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet)$ and

$\omega(\mathrm{D}_\circ, \gamma_\bullet) =$ number of $\mathrm{D}_\circ$-accordion diagonals that cross $\gamma_\bullet$

THM. The $\mathrm{D}_\circ$-accordiohedron

$$\mathrm{Acco}(\mathrm{D}_\circ) = \mathrm{conv}\left\{\mathbf{p}(\mathrm{D}_\circ, \mathrm{D}_\bullet) \mid \mathrm{D}_\bullet \text{ maximal } \mathrm{D}_\circ\text{-accordion dissection}\right\}$$
$$= \left\{\mathbf{x} \in \mathbb{R}^{\mathrm{D}_\circ} \;\middle|\; \langle\, \mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) \mid \mathbf{x} \,\rangle \le \omega(\mathrm{D}_\circ, \delta_\bullet) \text{ for any } \mathrm{D}_\circ\text{-accordion diagonal } \delta_\bullet\right\}$$

has for normal fan the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$, and thus realizes the $\mathrm{D}_\circ$-accordion complex.



Loday | Hohlweg-Lange | Hohlweg-P.-Stella | Manneville-P.

$\mathrm{D}_\circ$-ACCORDIOHEDRON

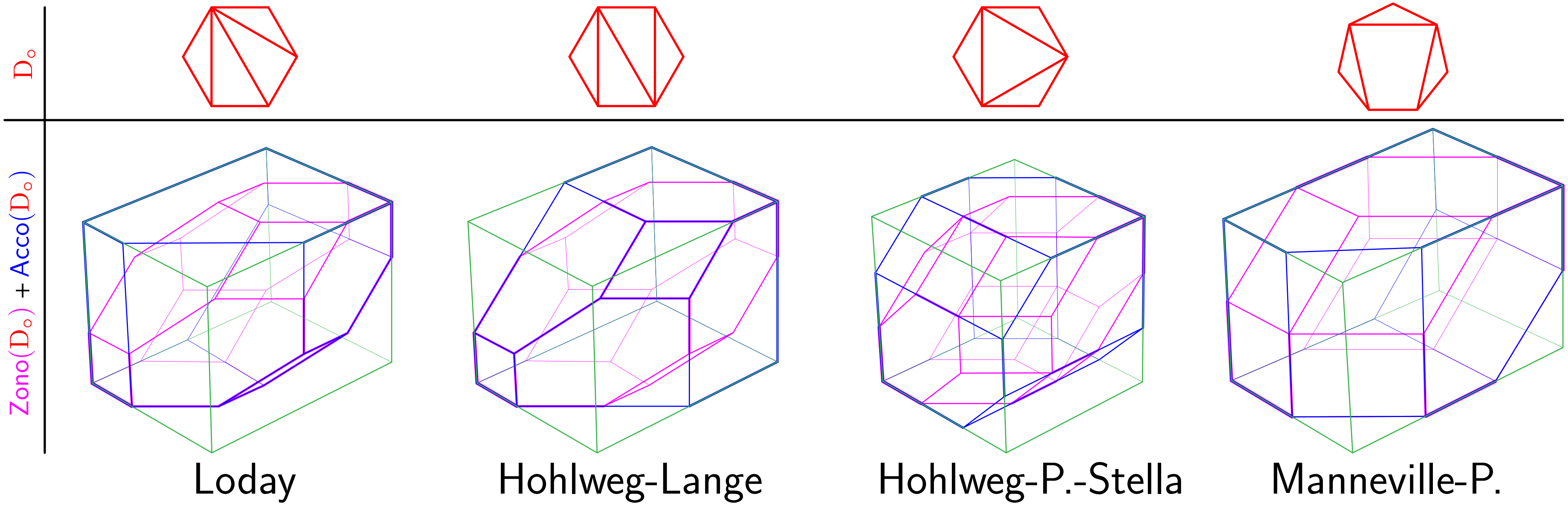
Define $\mathbf{p}(\mathrm{D}_\circ, \mathrm{D}_\bullet) := \sum_{\delta_\bullet \in \mathrm{D}_\bullet} \omega(\mathrm{D}_\circ, \delta_\bullet) \cdot \mathbf{c}(\mathrm{D}_\circ, \delta_\bullet \in \mathrm{D}_\bullet)$ and

$\omega(\mathrm{D}_\circ, \gamma_\bullet)$ = number of $\mathrm{D}_\circ$-accordion diagonals that cross $\gamma_\bullet$

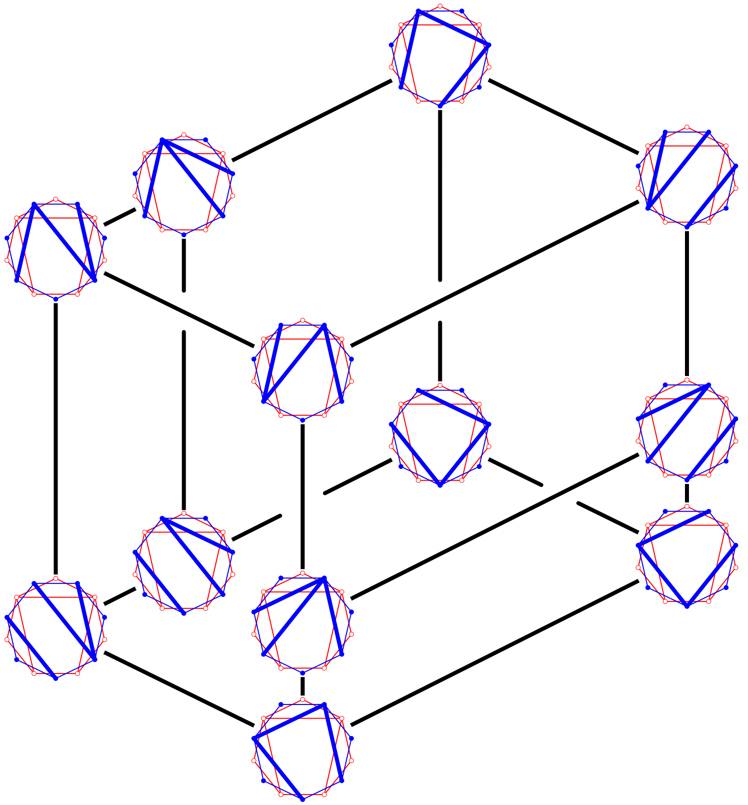
THM. The $\mathrm{D}_\circ$-accordiohedron

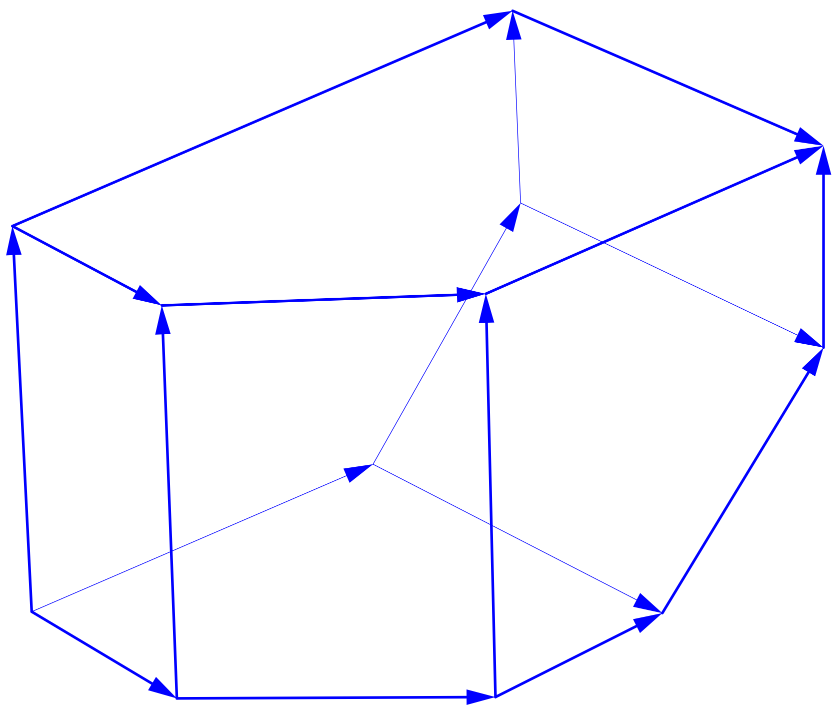
$$\mathrm{Acco}(\mathrm{D}_\circ) = \mathrm{conv}\left\{\mathbf{p}(\mathrm{D}_\circ, \mathrm{D}_\bullet) \;\middle|\; \mathrm{D}_\bullet \text{ maximal } \mathrm{D}_\circ\text{-accordion dissection}\right\}$$
$$= \left\{\mathbf{x} \in \mathbb{R}^{\mathrm{D}_\circ} \;\middle|\; \langle\, \mathbf{g}(\mathrm{D}_\circ, \delta_\bullet) \mid \mathbf{x} \,\rangle \le \omega(\mathrm{D}_\circ, \delta_\bullet) \text{ for any } \mathrm{D}_\circ\text{-accordion diagonal } \delta_\bullet\right\}$$

has for normal fan the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$, and thus realizes the $\mathrm{D}_\circ$-accordion complex.



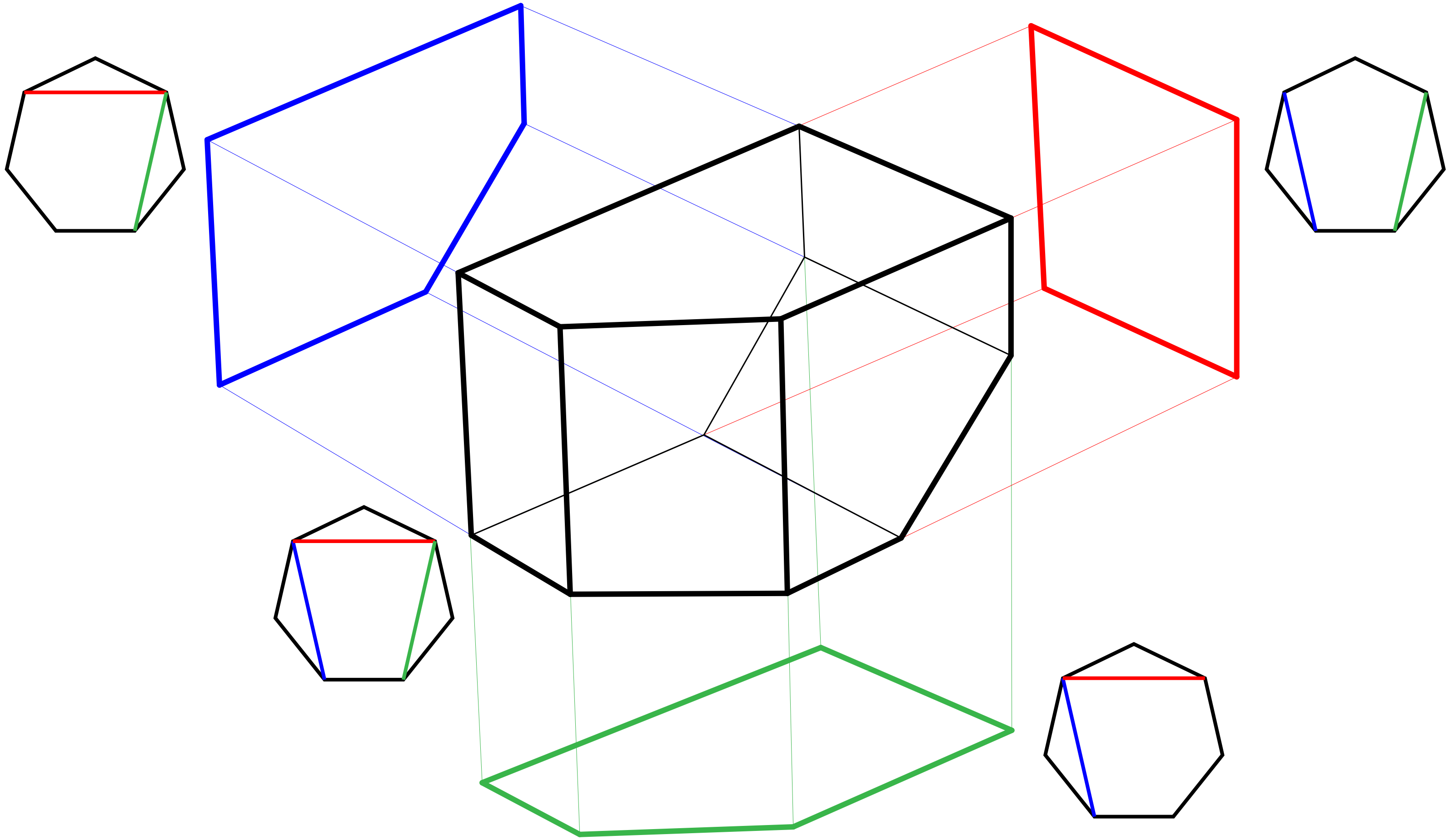
PROP. The graph of the $\mathrm{D}_\circ$-accordiohedron $\mathrm{Acco}(\mathrm{D}_\circ)$ linearly oriented in the direction $\mathbb{1} := \sum_{i\in[n]} \mathbf{e}_i$ is the Hasse diagram of the $\mathrm{D}_\circ$-accordion lattice.





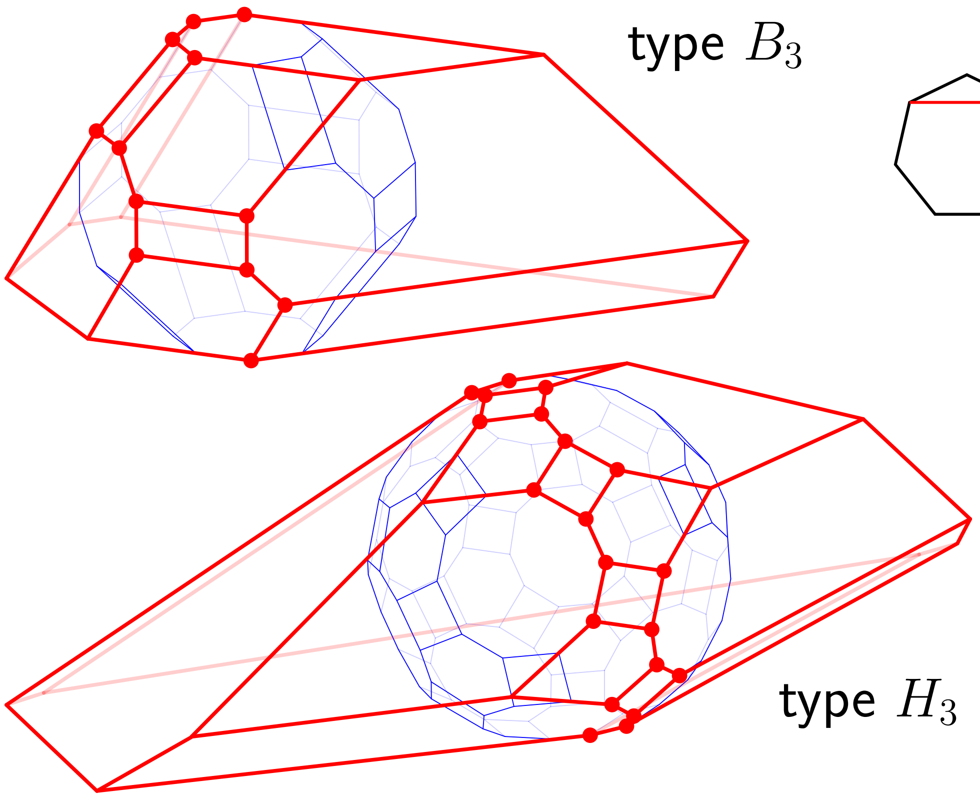
PROP. If $\mathrm{D}_\circ \subseteq \mathrm{D}'_\circ$, then

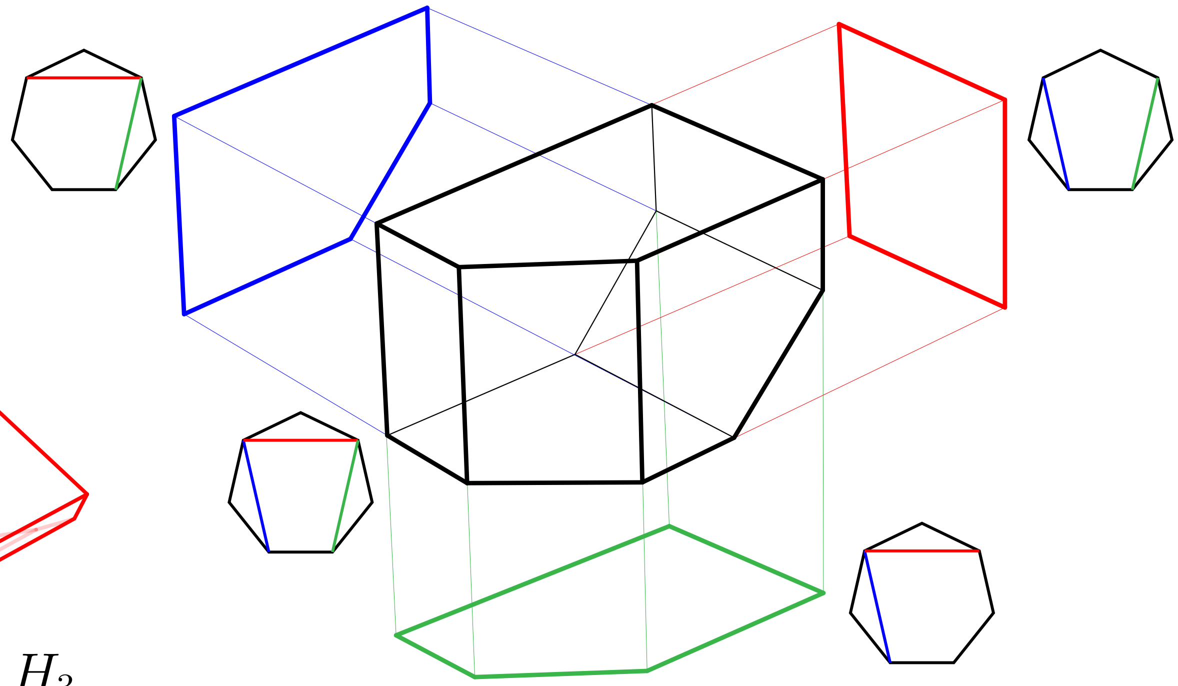
- $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ is the section of $\mathcal{F}^{\mathbf{g}}(\mathrm{D}'_\circ)$ with the coordinate plane $\langle \mathbf{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$,
- therefore, $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ is also realized by the projection of $\mathrm{Asso}(\mathrm{D}_\circ)$ on $\langle \mathbf{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$.



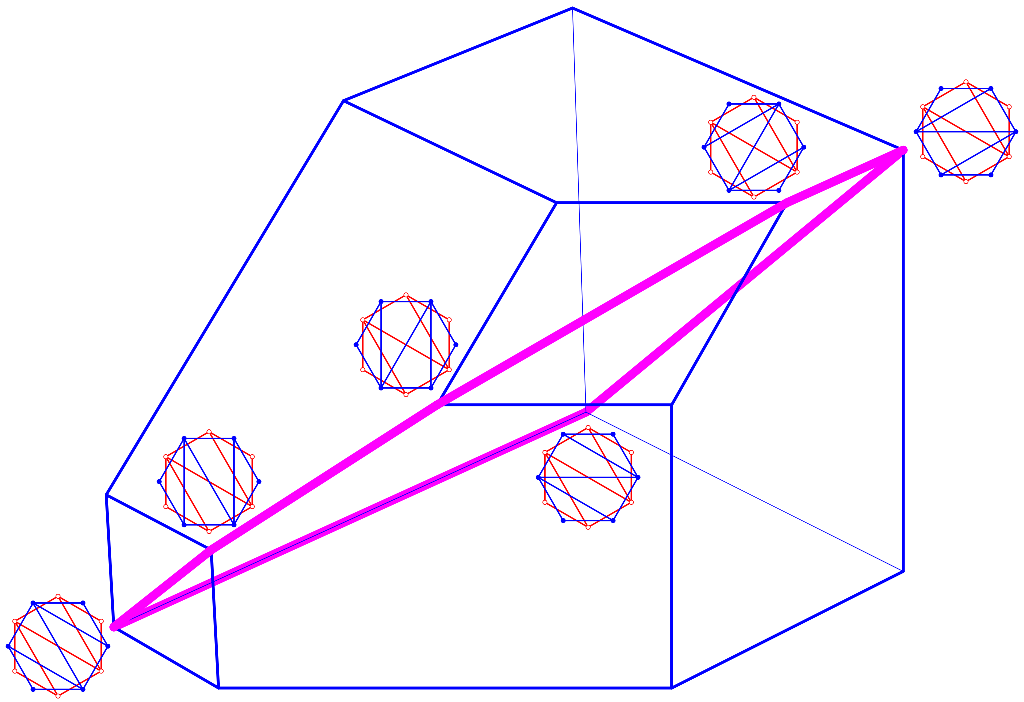
FURTHER TOPICS ON ACCORDIOHEDRA

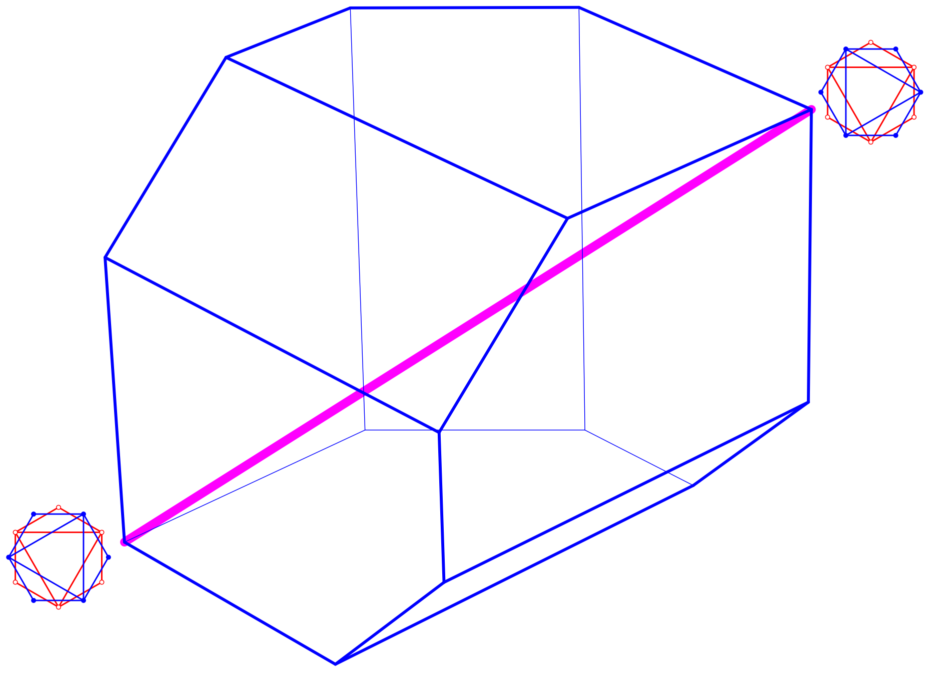
CONJ. For any Coxeter group W and any Coxeter element c of W, the transitive closure of the oriented graph of any projection of the c-associahedron on a coordinate plane is a lattice.





OBS. Symmetries in $D_\circ$ induce symmetries in $\mathrm{Acco}(D_\circ)$





D-VECTOR FAN

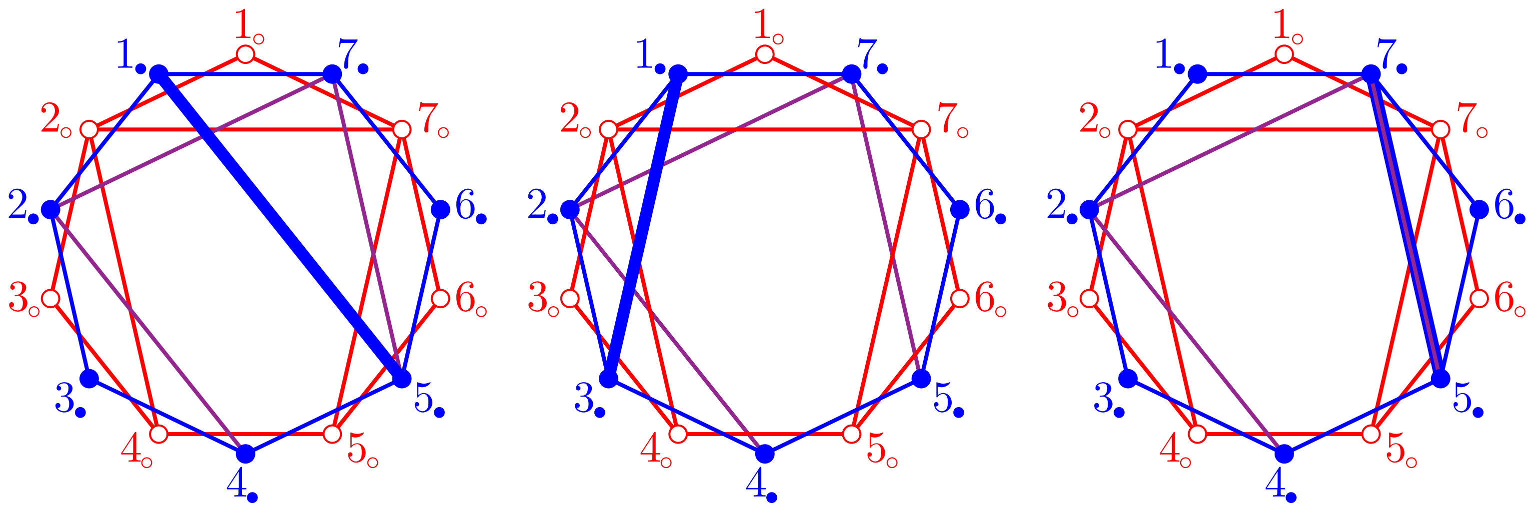
Manneville-P.,
Geometric realizations of the accordion complex of a dissection ('16$^+$)

d-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection, $\delta_\circ = i_\circ j_\circ$ red hollow diagonal and $\delta_\bullet$ blue solid diagonal, let

$$(\delta_\circ, \delta_\bullet) := \begin{cases} -1 & \text{if } \delta_\bullet = i_\bullet j_\bullet, \\ 0 & \text{if } \delta_\bullet \text{ and } i_\bullet j_\bullet \text{ do not cross}, \\ 1 & \text{if } \delta_\bullet \text{ and } i_\bullet j_\bullet \text{ cross}. \end{cases}$$

$\mathbf{d}(\mathrm{D}_\circ, \delta_\bullet)$ = $\underline{\mathbf{d}\text{-vector}}$ of $\delta_\bullet$ with respect to $\mathrm{D}_\circ = \Big[(\delta_\circ, \delta_\bullet)\Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$



$\mathbf{d}(\mathrm{D}_\circ, (1_\bullet, 5_\bullet)) = \mathbf{e}_{2_\circ 7_\circ}$

$\mathbf{d}(\mathrm{D}_\circ, (1_\bullet, 3_\bullet)) = \mathbf{e}_{2_\circ 4_\circ} + \mathbf{e}_{2_\circ 7_\circ}$

$\mathbf{d}(\mathrm{D}_\circ, (5_\bullet, 7_\bullet)) = -\mathbf{e}_{5_\circ 7_\circ}$

d-VECTORS

For $\mathrm{D}_\circ$ red hollow dissection, $\delta_\circ = i_\circ j_\circ$ red hollow diagonal and $\delta_\bullet$ blue solid diagonal, let

$$(\delta_\circ, \delta_\bullet) := \begin{cases} -1 & \text{if } \delta_\bullet = i_\bullet j_\bullet, \\ 0 & \text{if } \delta_\bullet \text{ and } i_\bullet j_\bullet \text{ do not cross,} \\ 1 & \text{if } \delta_\bullet \text{ and } i_\bullet j_\bullet \text{ cross.} \end{cases}$$

$\mathbf{d}(\mathrm{D}_\circ, \delta_\bullet)$ = $\mathbf{d}$-vector of $\delta_\bullet$ with respect to $\mathrm{D}_\circ = \left[(\delta_\circ, \delta_\bullet) \right]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$

QU. Is the collection of cones

$$\mathcal{F}^{\mathbf{d}}(\mathrm{D}_\circ) := \left\{ \mathbb{R}_{\geq 0} \mathbf{d}(\mathrm{D}_\circ, \mathrm{D}_\bullet) \;\middle|\; \mathrm{D}_\bullet \text{ any } \mathrm{D}_\circ\text{-accordion dissection} \right\}$$

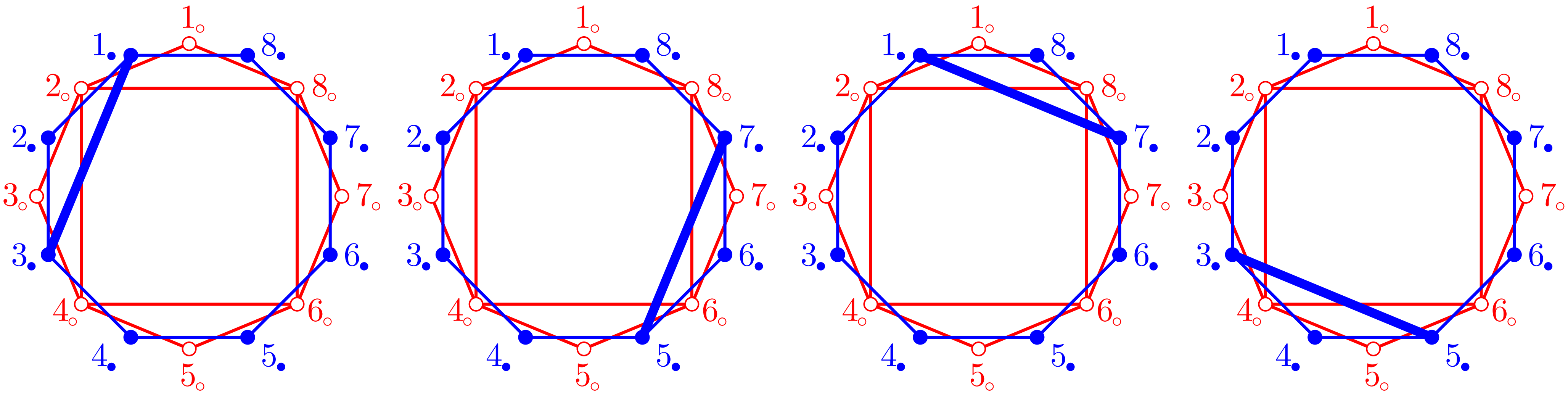
a complete simplicial fan?

OBSTRUCTION: EVEN INTERIOR CELLS

Assume $\mathrm{D}_\circ$ contains an even interior cell with edges $\delta_\circ^k = i_\circ^k i_\circ^{k+1}$ for $k \in [2p]$

Then the $\mathbf{d}$-vectors of the edges $\delta_\bullet^k = (i^k - 1)_\bullet (i^{k+1} - 1)_\bullet$ satisfy

$$\sum_{\substack{k \in [2p] \\ k \text{ even}}} \mathbf{d}(\mathrm{D}_\circ, \delta_\bullet^k) = \sum_{\substack{k \in [2p] \\ k \text{ odd}}} \mathbf{d}(\mathrm{D}_\circ, \delta_\bullet^k).$$



$$\mathbf{d}(\mathrm{D}_\circ, (1_\bullet, 3_\bullet)) \quad + \quad \mathbf{d}(\mathrm{D}_\circ, (5_\bullet, 7_\bullet)) \quad = \quad \mathbf{d}(\mathrm{D}_\circ, (1_\bullet, 7_\bullet)) \quad + \quad \mathbf{d}(\mathrm{D}_\circ, (3_\bullet, 5_\bullet))$$

$$\mathbf{e}_{2_\circ 4_\circ} + \mathbf{e}_{2_\circ 8_\circ} \quad + \quad \mathbf{e}_{4_\circ 6_\circ} + \mathbf{e}_{6_\circ 8_\circ} \quad = \quad \mathbf{e}_{2_\circ 8_\circ} + \mathbf{e}_{6_\circ 8_\circ} \quad + \quad \mathbf{e}_{2_\circ 4_\circ} + \mathbf{e}_{4_\circ 6_\circ}$$

even interior cells $\Longrightarrow$ obstruction for $\mathbf{d}$-vector fans

d-VECTOR FAN

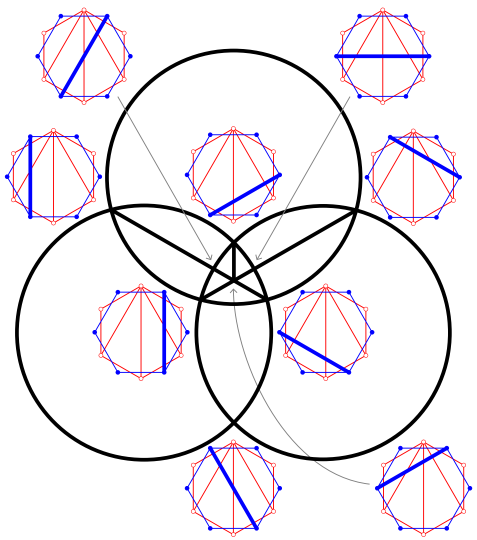
$\mathbf{d}(\mathrm{D}_\circ, \delta_\bullet) =$ $\mathbf{d}$-vector of $\delta_\bullet$ with respect to $\mathrm{D}_\circ = \Big[(\delta_\circ, \delta_\bullet) \Big]_{\delta_\circ \in \mathrm{D}_\circ} \in \mathbb{R}^{\mathrm{D}_\circ}$

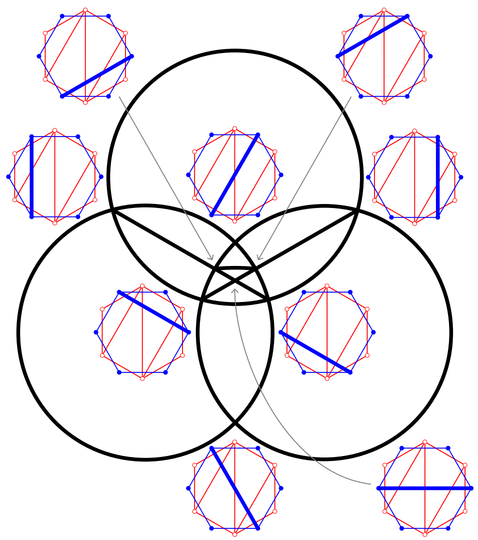
THM. The collection of cones

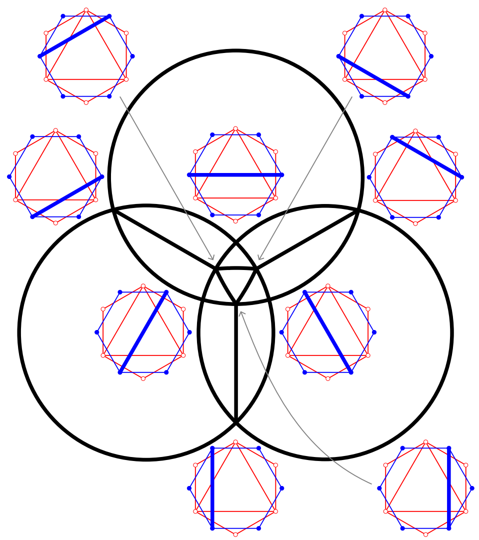
$$\mathcal{F}^{\mathbf{d}}(\mathrm{D}_\circ) := \left\{ \mathbb{R}_{\geq 0}\mathbf{d}(\mathrm{D}_\circ, \mathrm{D}_\bullet) \;\middle|\; \mathrm{D}_\bullet \text{ any } \mathrm{D}_\circ\text{-accordion dissection} \right\}$$

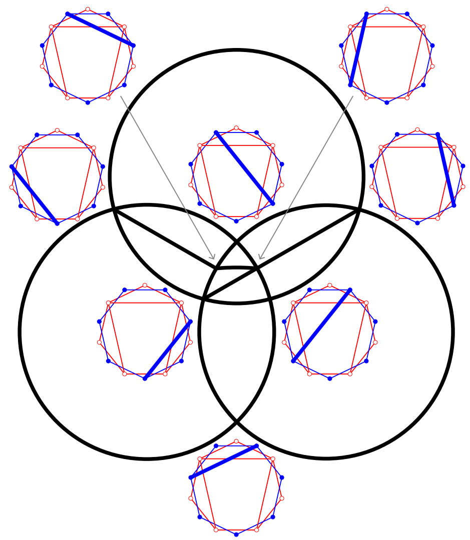
forms a complete simplicial fan, called $\mathbf{d}$-vector fan of $\mathrm{D}_\circ$, if and only if $\mathrm{D}_\circ$ contains no even interior cell.

Manneville-P., *Geometric realizations of the accordion complex of a dissection* ('16$^+$)

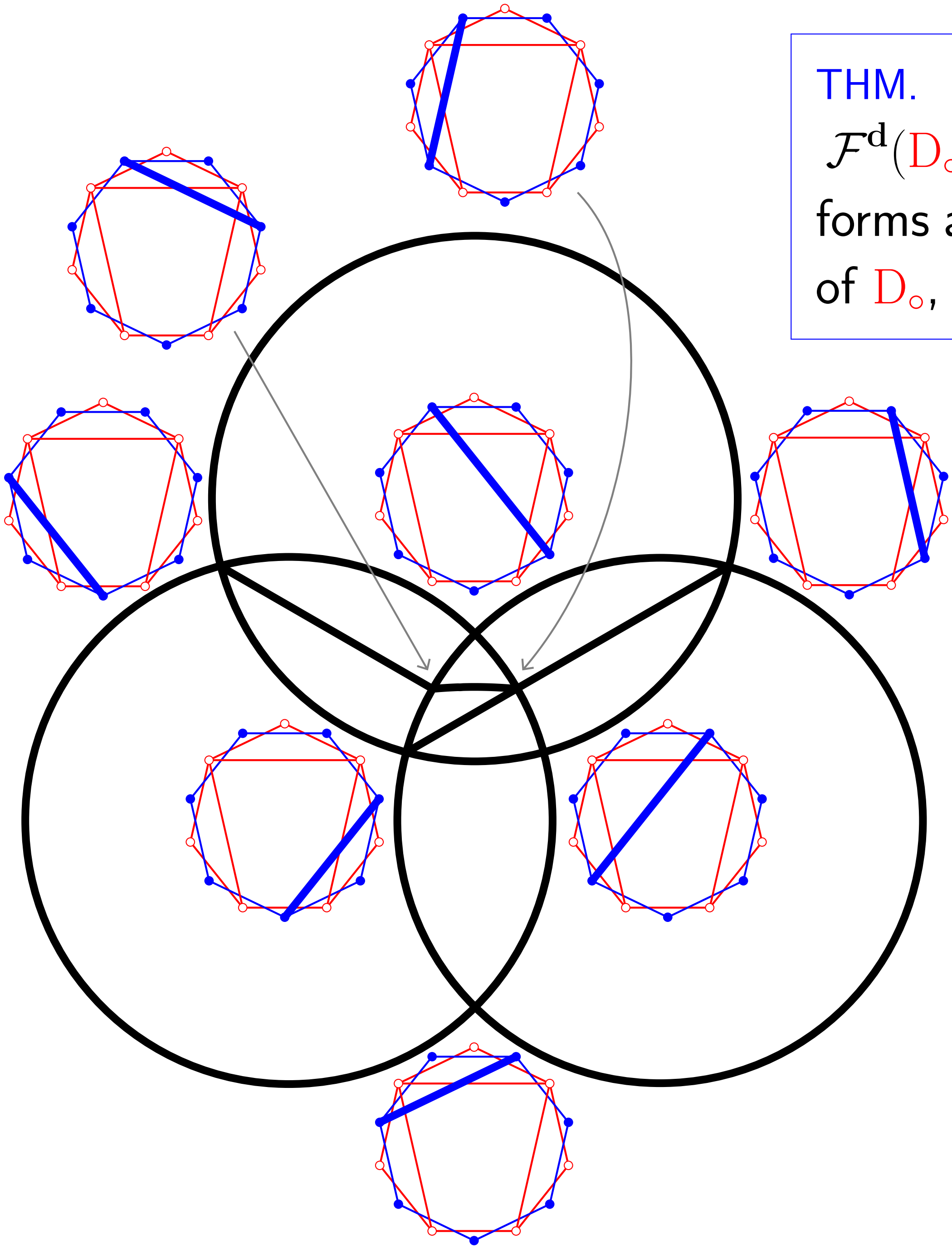

Fomin-Zelevinsky


Stella


Santos

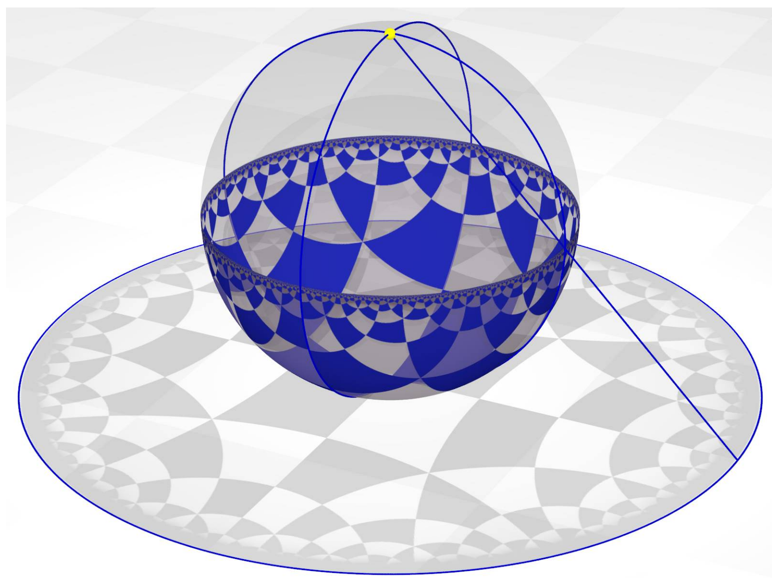

Manneville-P.

d-VECTOR FAN



THM. The collection of cones
$\mathcal{F}^{\mathbf{d}}(\mathrm{D}_\circ) := \big\{ \mathbb{R}_{\geq 0}\mathbf{d}(\mathrm{D}_\circ, \mathrm{D}_\bullet) \;\big|\; \mathrm{D}_\bullet \text{ any } \mathrm{D}_\circ\text{-acc. diss.} \big\}$
forms a complete simplicial fan, called $\mathbf{d}$-vector fan of $\mathrm{D}_\circ$, iff $\mathrm{D}_\circ$ contains no even interior cell.

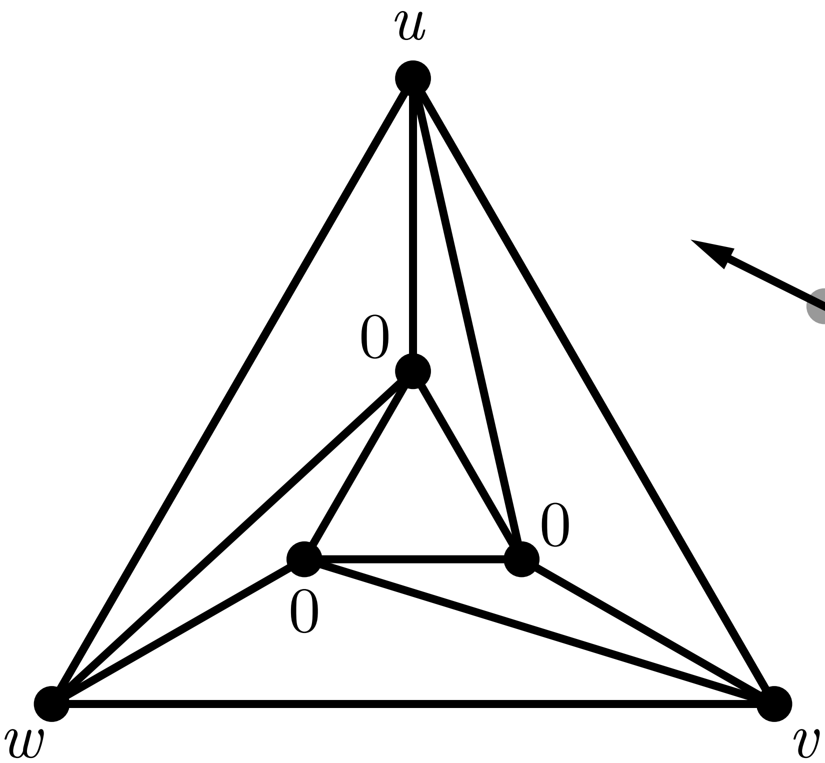
stereographic projection
from $(-1, -1, -1)$

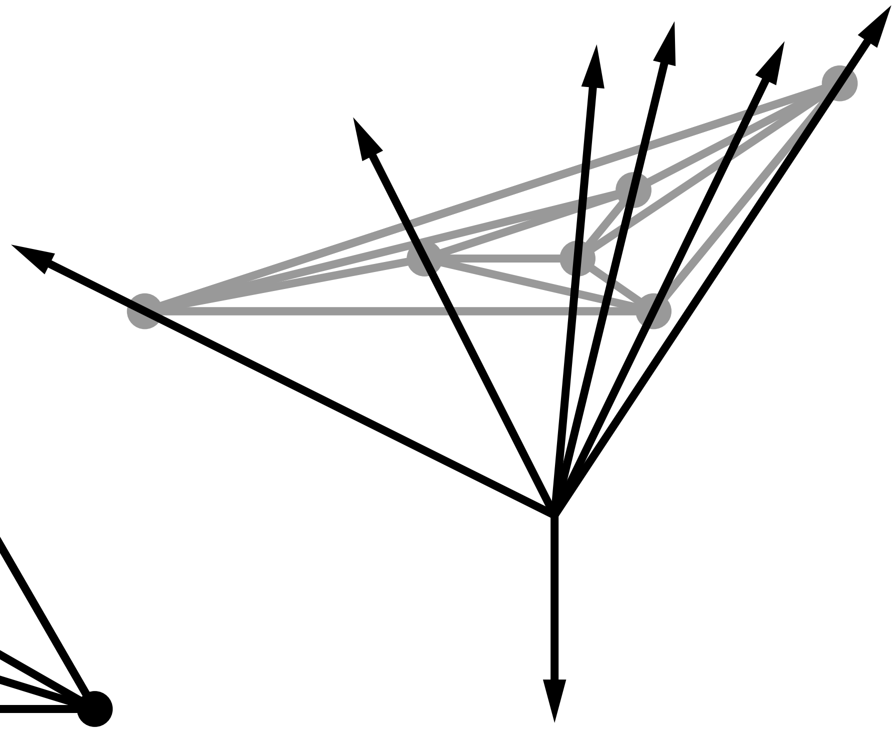


POLYTOPALITY?

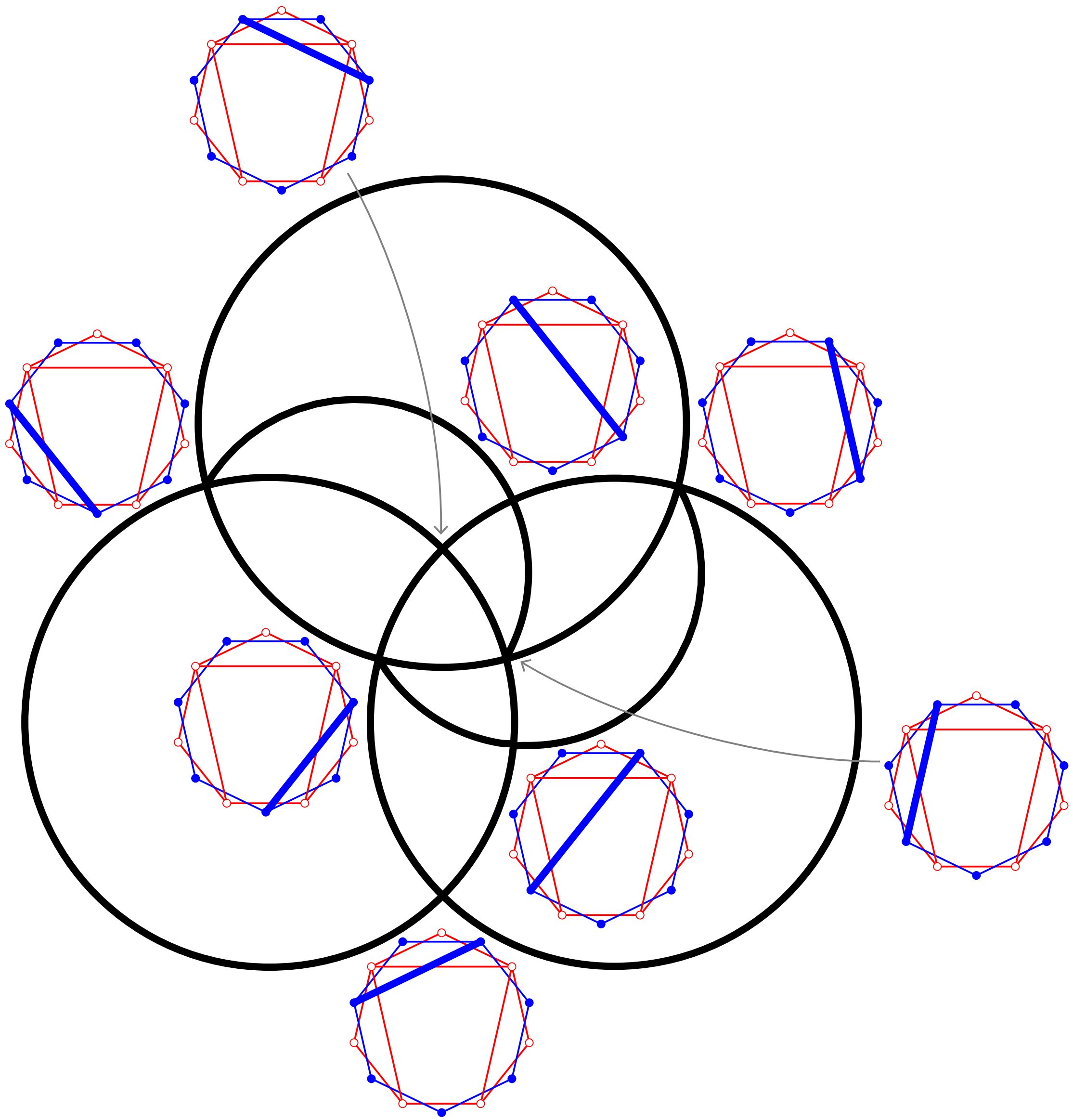
QU. Are all $\mathbf{d}$-vector fans polytopal?

Not all complete simplicial fans are polytopal... Escher always falling water:

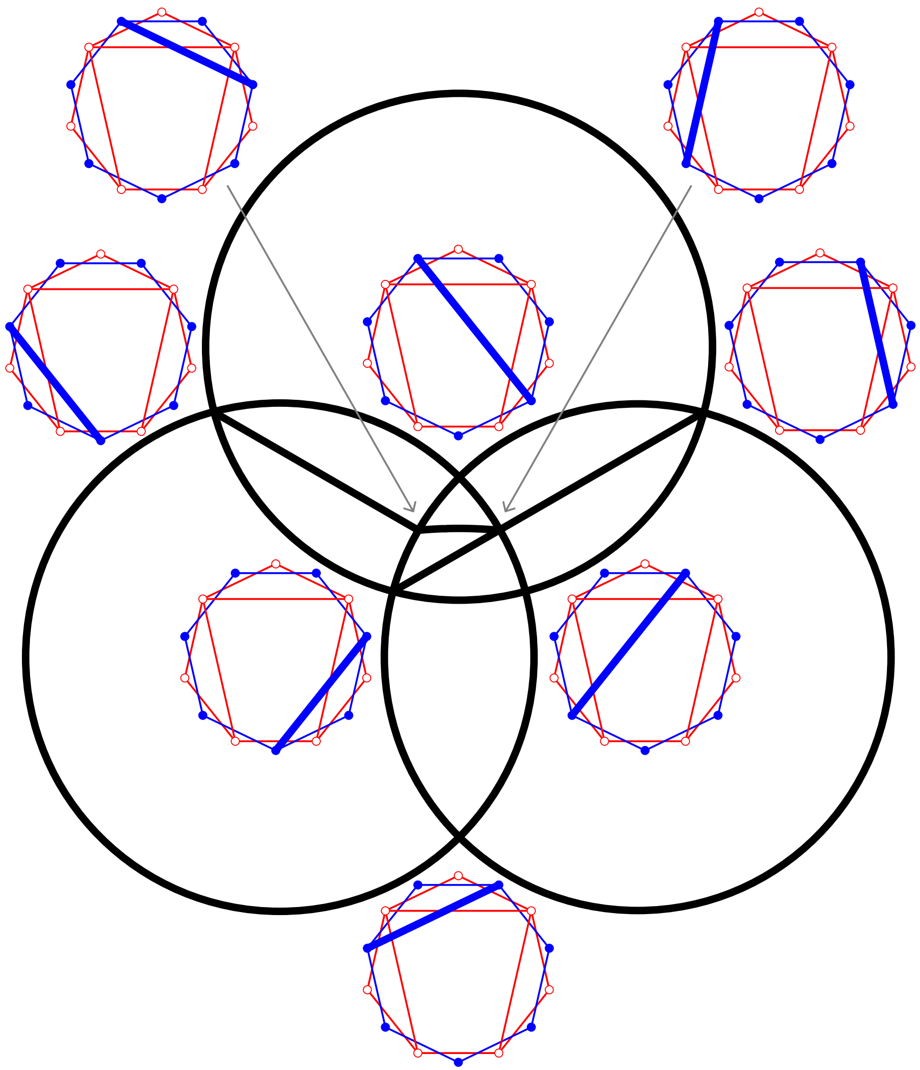




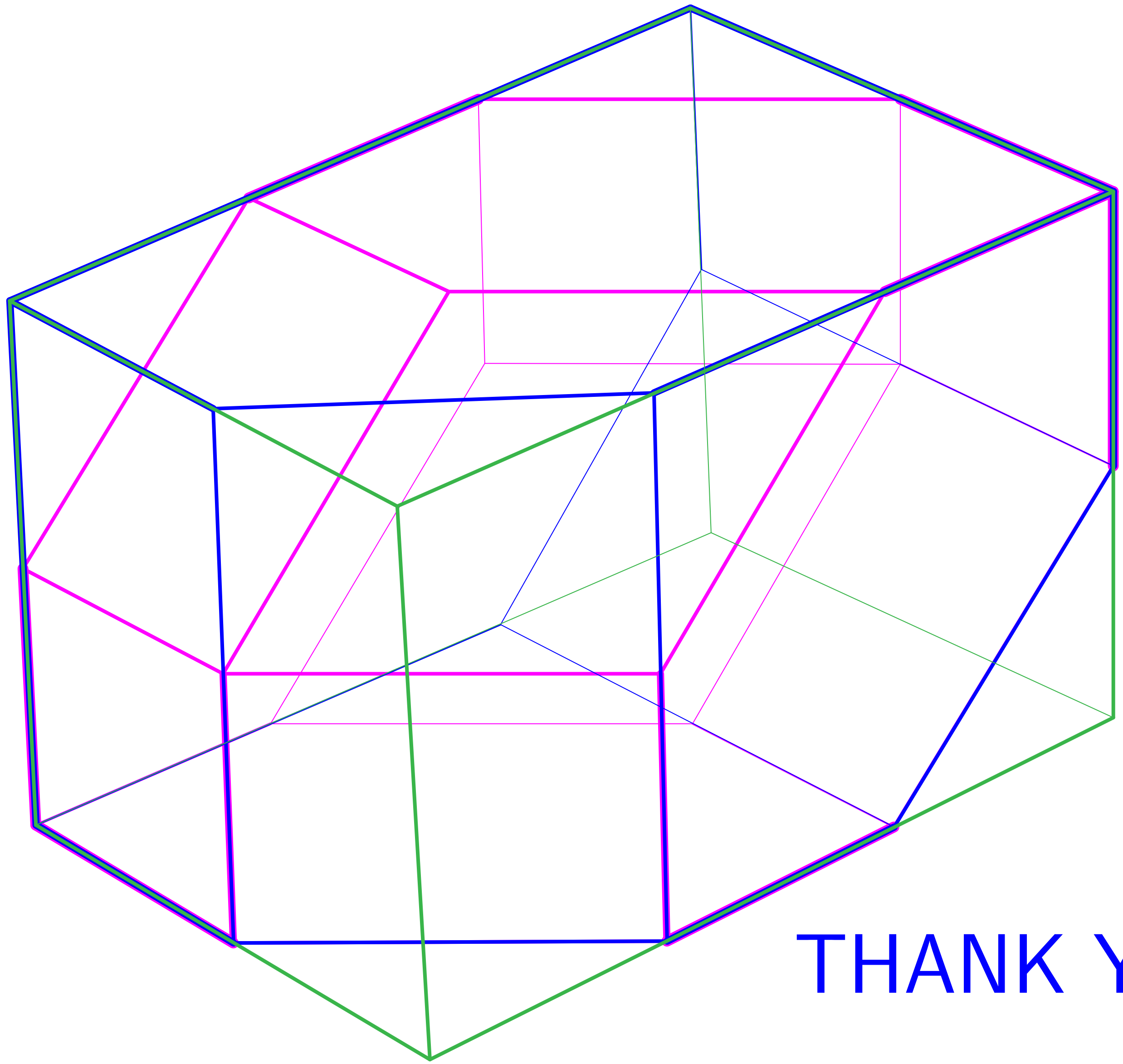
TWO FAN REALIZATIONS OF THE $D_\circ$-ACCORDION COMPLEX



g-vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$
realized by the $D_\circ$-accordiohedron $\mathrm{Acco}(D_\circ)$



d-vector fan $\mathcal{F}^{\mathbf{d}}(D_\circ)$
Polytopal ?



THANK YOU