Polytopes and Hopf algebras from lattice quotients of the weak order

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Noncrossing arc diagrams

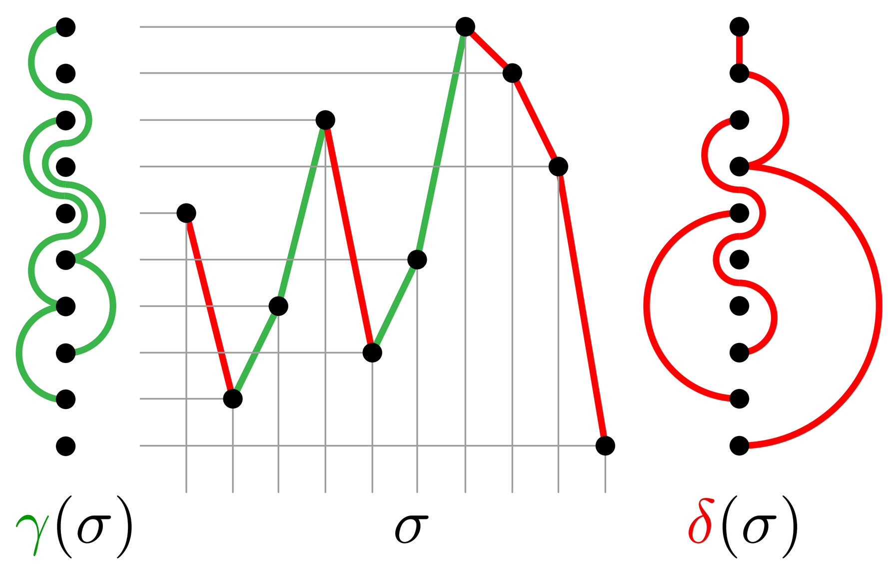
noncrossing arc diagram = set $\mathcal{D}$ of arcs s.t. $\forall\ \alpha, \beta \in \mathcal{D}$

- $t(\alpha) \ne t(\beta)$ and $b(\alpha) \ne b(\beta)$
- α and β are not crossing

THM. bij. $\mathfrak{S}_n \xleftrightarrow{\gamma \text{ or } \delta}$ NCAD

$\gamma(\sigma)$ = projection of ascents

$\delta(\sigma)$ = projection of descents



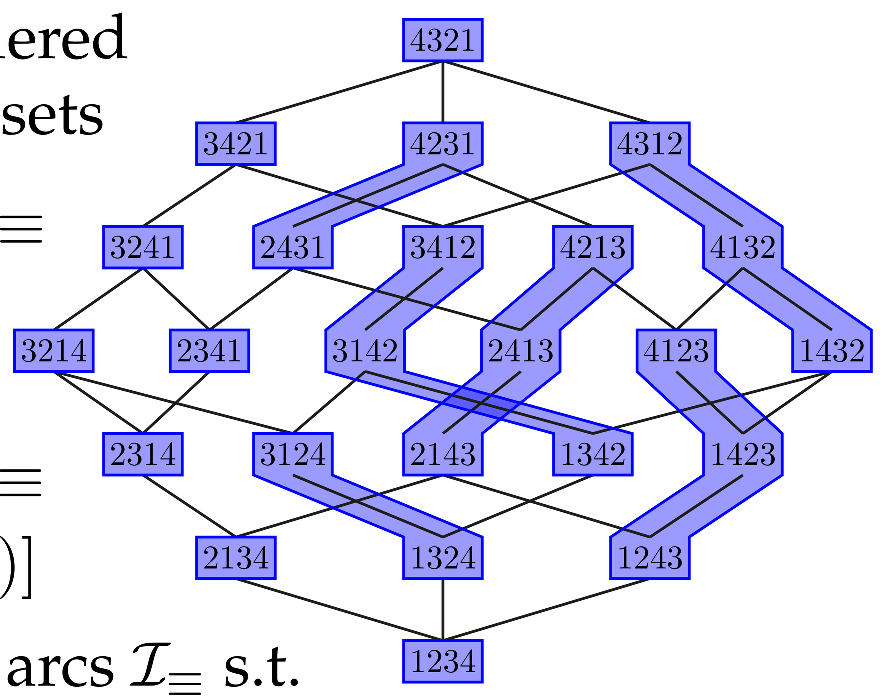
N. Reading. Noncrossing arc diagrams and canonical join representations ('15)

Lattice quotients and arc ideals

weak order = permutations ordered by inclusion of their inversions sets

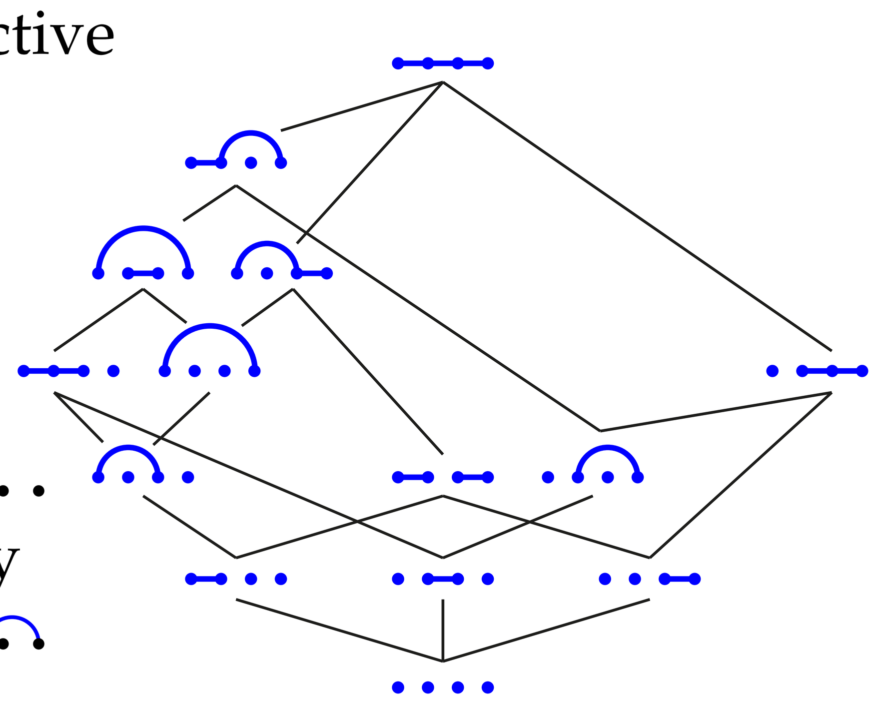
lattice congruence = equiv. rel. $\equiv$ s.t. $\begin{cases} x \equiv x' \\ y \equiv y' \end{cases} \Rightarrow \begin{cases} x \vee y \equiv x' \vee y' \\ x \wedge y \equiv x' \wedge y' \end{cases}$

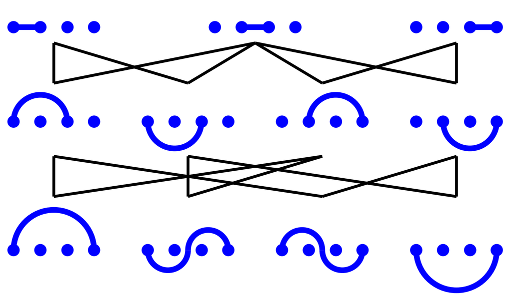
$\Rightarrow$ each congruence class X of $\equiv$ is an interval $[\pi_\downarrow^{\equiv}(X), \pi^\uparrow_{\equiv}(X)]$



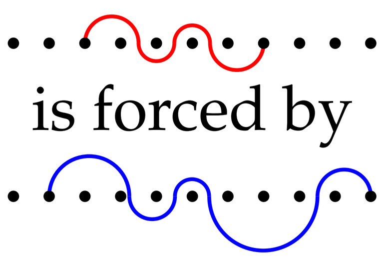
THM. $\forall$ lattice congr. $\equiv$, $\exists$ set of arcs $\mathcal{I}_\equiv$ s.t. $\mathfrak{S}_n/\!\equiv\ \to \{\text{NCAD in } \mathcal{I}_\equiv\}$ is bijective
$X \mapsto \delta(\pi_\downarrow^{\equiv}(X))$

THM. $\equiv\ \mapsto \mathcal{I}_\equiv$ is a bijection from {lattice congr. on $\mathfrak{S}_n$} to {upper ideals of forcing}





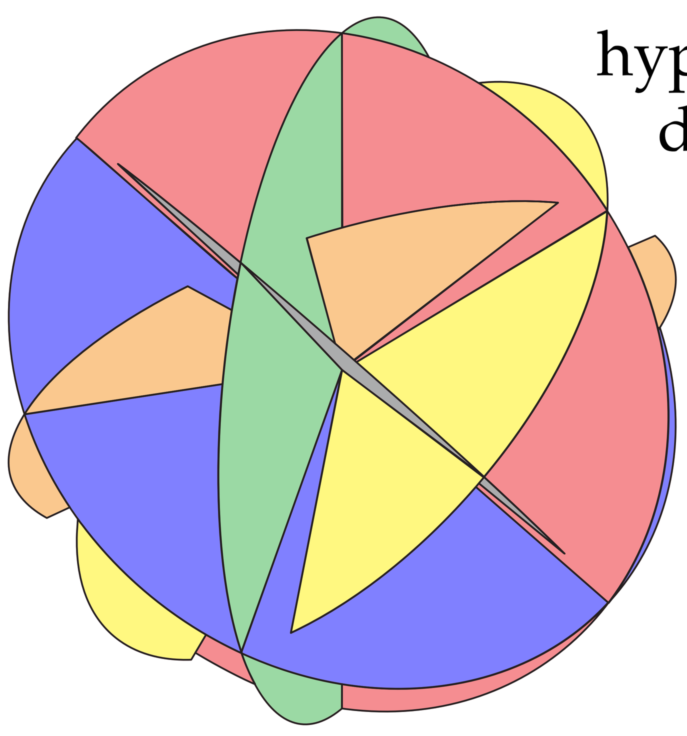
is forced by



N. Reading. Noncrossing arc diagrams and canonical join representations ('15)

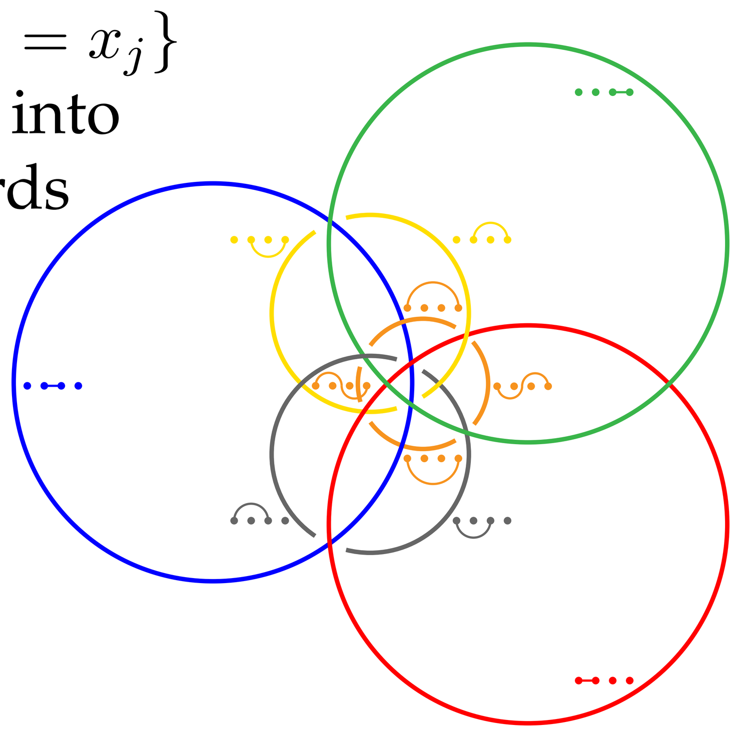
Shards

shard $\mathbf{S}(\ldots) = \{\mathbf{x} \in \mathbb{R}^n \mid x_{\text{below}} \le x_i = x_j \le x_{\text{above}}\}$



hyperplane $\{x_i = x_j\}$ decomposed into 2^{j-i-1} shards

stereographic projection



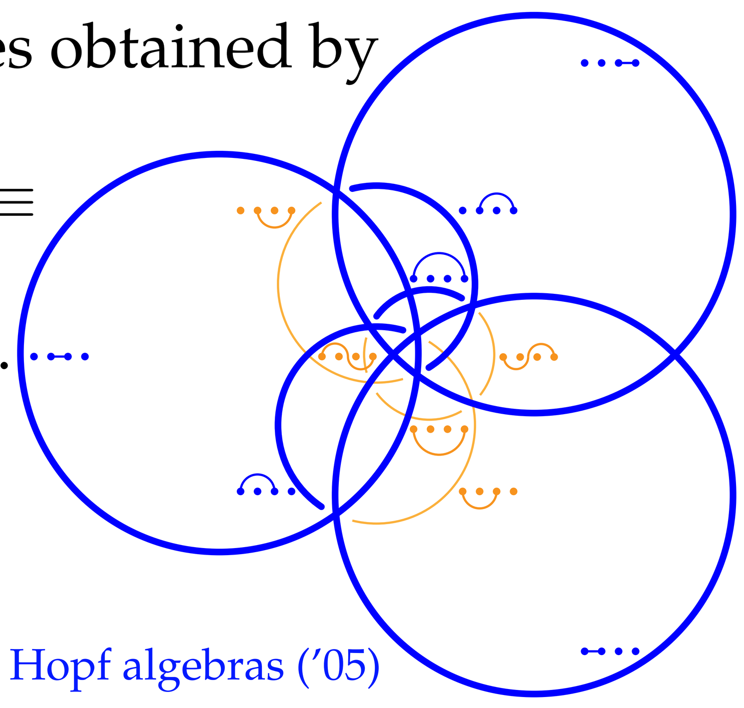
N. Reading, Lattice theory of the poset of regions ('16)

Quotient fans

THM. For a lattice congr. $\equiv$, the cones obtained by

- either glueing the regions of the perm. in the same congr. class of $\equiv$
- or as the connected components of the union of the shards corresp. to the arcs of the upper ideal $\mathcal{I}_\equiv$

form a fan $\mathcal{F}_\equiv$ whose dual graph realizes the lattice quotient $\mathfrak{S}_n/\!\equiv$



N. Reading. Lattice congr., fans and Hopf algebras ('05)

Quotientopes

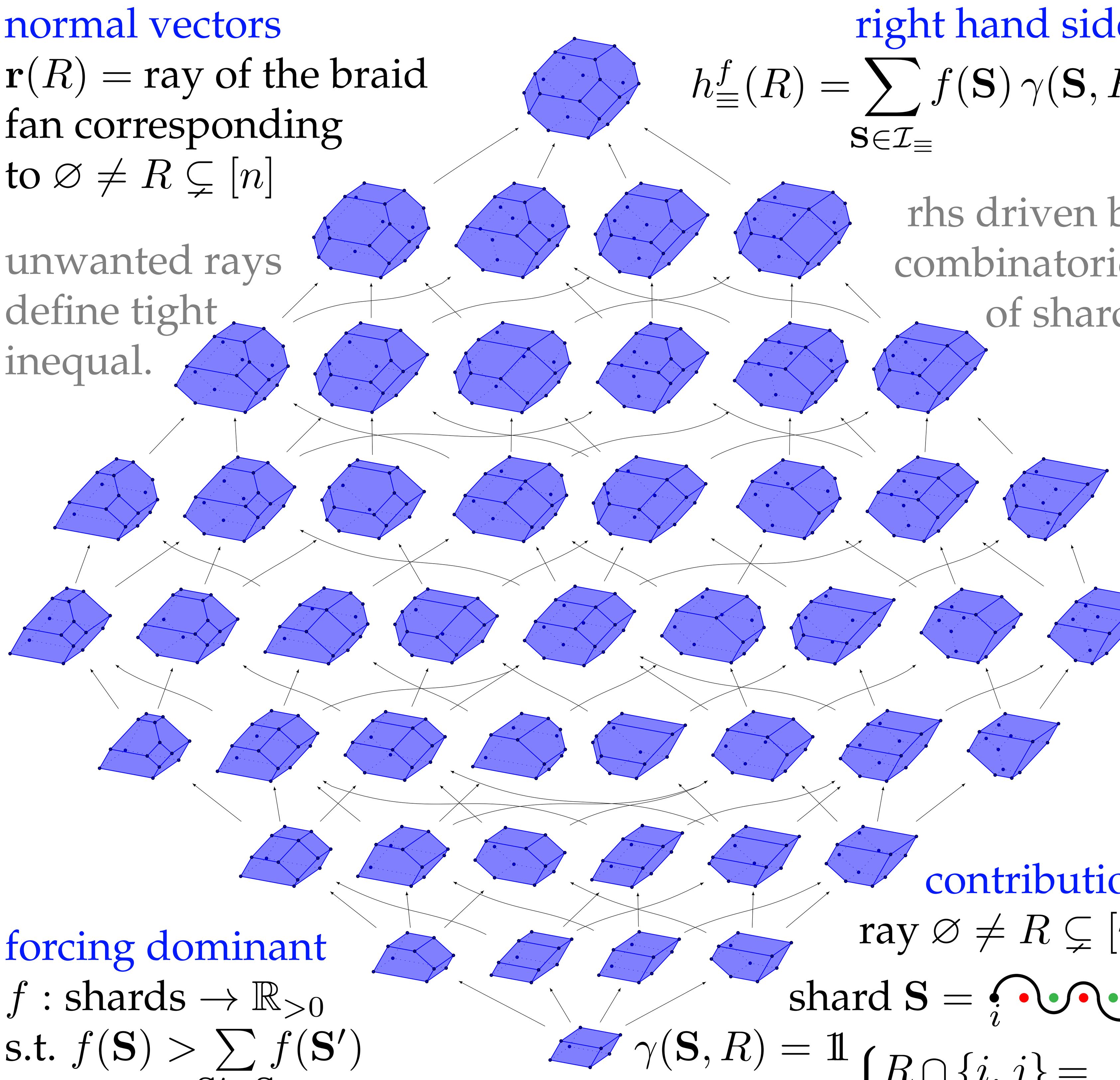
THM. The quotient fan $\mathcal{F}_\equiv$ is the normal fan of a quotientope
$P_\equiv = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{r}(R) \mid \mathbf{x} \rangle \le h^f_\equiv(R) \text{ for all } \varnothing \ne R \subsetneq [n]\}$

normal vectors
$\mathbf{r}(R)$ = ray of the braid fan corresponding to $\varnothing \ne R \subsetneq [n]$

right hand sides
$h^f_\equiv(R) = \sum_{\mathbf{S} \in \mathcal{I}_\equiv} f(\mathbf{S})\, \gamma(\mathbf{S}, R)$

unwanted rays define tight inequal.

rhs driven by combinatorics of shards



forcing dominant
f : shards $\to \mathbb{R}_{>0}$
s.t. $f(\mathbf{S}) > \sum_{\mathbf{S}' \succ \mathbf{S}} f(\mathbf{S}')$

contribution
ray $\varnothing \ne R \subsetneq [n]$
shard $\mathbf{S} = \ldots$
$\gamma(\mathbf{S}, R) = \mathbb{1} \begin{cases} R \cap \{i, j\} = 1 \\ R \cap\,]i, j[\ = \{\bullet\} \end{cases}$

VP & F. Santos, Quotientopes ('19)

Hopf algebras

GOAL. Extend the Malvenuto–Reutenauer Hopf algebra on permutations and the Loday–Ronco algebra on binary trees

THM. There is a Hopf algebra structure on all classes of all lattice congruences of the weak order on $\mathfrak{S}_n$ for $n \in \mathbb{N}$

decoration set $\mathfrak{X} = \bigsqcup \mathfrak{X}_n$ with a selection sel : $\mathfrak{X}_m \times \binom{[m]}{k} \to \mathfrak{X}_k$ and a concatenation cc : $\mathfrak{X}_m \times \mathfrak{X}_n \to \mathfrak{X}_{m+n}$ s.t.

- $\mathsf{sel}(\mathsf{sel}(\mathcal{X}, R), S) = \mathsf{sel}(\mathcal{X}, \{r_s \mid s \in S\})$
- $\mathsf{cc}(\mathcal{X}, \mathsf{cc}(\mathcal{Y}, \mathcal{Z})) = \mathsf{cc}(\mathsf{cc}(\mathcal{X}, \mathcal{Y}), \mathcal{Z})$
- $\mathsf{cc}(\mathsf{sel}(\mathcal{X}, R), \mathsf{sel}(\mathcal{Y}, S)) = \mathsf{sel}(\mathsf{cc}(\mathcal{X}, \mathcal{Y}), R \cup S^{\to m})$

decorated permutation = pair $(\sigma, \mathcal{X})$ with $\sigma \in \mathfrak{S}_n$ and $\mathcal{X} \in \mathfrak{X}_n$

standardization $\mathsf{std}((\rho, \mathcal{Z}), R) := (\mathsf{std}(\rho, R), \mathsf{sel}(\mathcal{Z}, \rho^{-1}(R)))$

THM. Hopf algebra on decorated permutations:

- product $\mathbb{F}_{(\sigma,\mathcal{X})} \cdot \mathbb{F}_{(\tau,\mathcal{Y})} = \sum_{\rho \in \sigma \,\bar{\sqcup}\, \tau} \mathbb{F}_{(\rho, \mathsf{cc}(\mathcal{X}, \mathcal{Y}))}$
- coproduct $\triangle \mathbb{F}_{(\rho,\mathcal{Z})} = \sum_{k=0}^{p} \mathbb{F}_{\mathsf{std}((\rho,\mathcal{Z}),[k])} \otimes \mathbb{F}_{\mathsf{std}((\rho,\mathcal{Z}),[p] \smallsetminus [k])}$

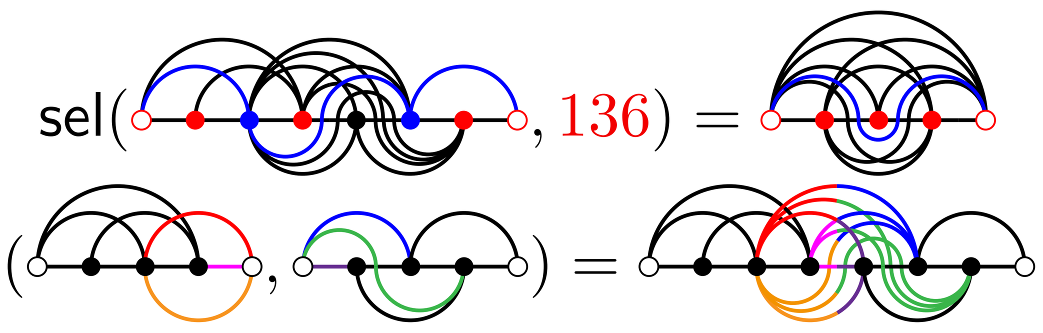
Ψ : {decorations} $\to$ {arc ideals} compatible with sel and cc

decorated noncrossing arc diagram = pair $(\mathcal{D}, \mathcal{X})$ s.t. $\mathcal{D} \subseteq \Psi(\mathcal{X})$

Define $\mathbb{P}_{(\mathcal{D},\mathcal{X})} = \sum \mathbb{F}_{(\sigma,\mathcal{X})}$ over all σ s.t. $\delta(\pi_\downarrow^{\Psi(\mathcal{X})}(\sigma)) = \mathcal{D}$

THM. The subspace generated by $\mathbb{P}_{(\mathcal{D},\mathcal{X})}$ is a Hopf subalgebra

EXM. $\mathfrak{X}$ = ext. arc ideals
$\Psi(\mathcal{X})$ = strict arcs in $\mathcal{X}$
$\Rightarrow$ contains the permutree algebra



VP, Hopf algebras on decorated noncrossing arc diagrams ('19)