Algebraic structures on integer posets

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We define a lattice structure and a Hopf algebra on integer posets and use them to recover relevant structures on the elements, the intervals and the faces in the permutahedron, the associahedron, the cube and more generally all permutreehedra [2, 5, 6].

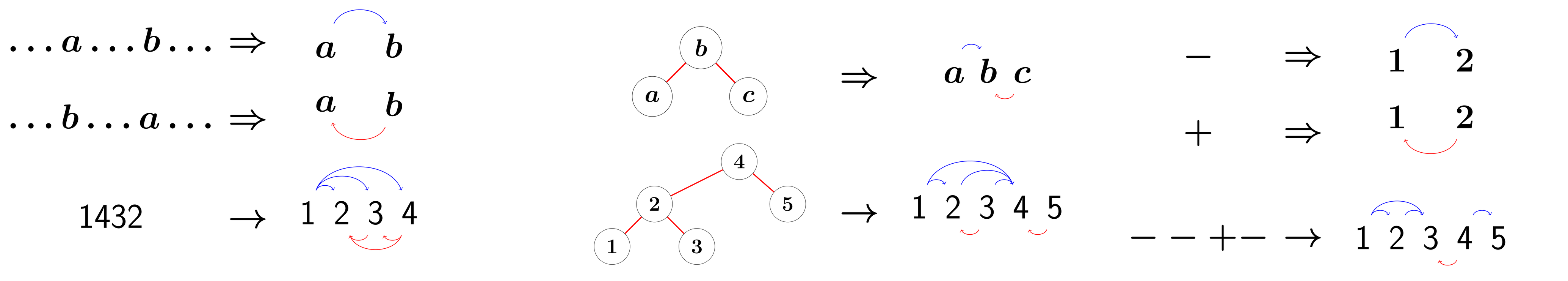
Combinatorial objects as integer posets

Many combinatorial objects can be interpreted as **binary relations.**

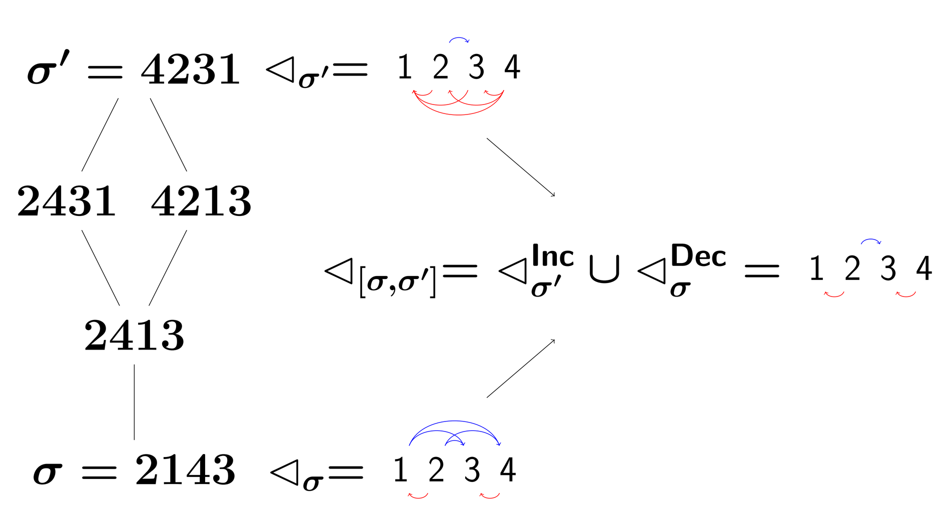
Permutations

Binary trees

Binary sequences



Intervals of permutations

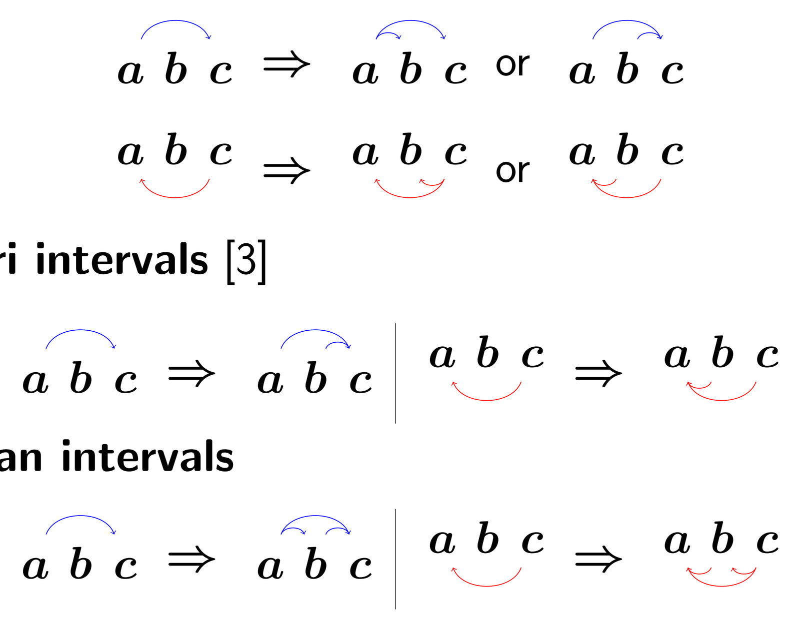


Characterization

Permutation intervals

Tamari intervals [3]

Boolean intervals

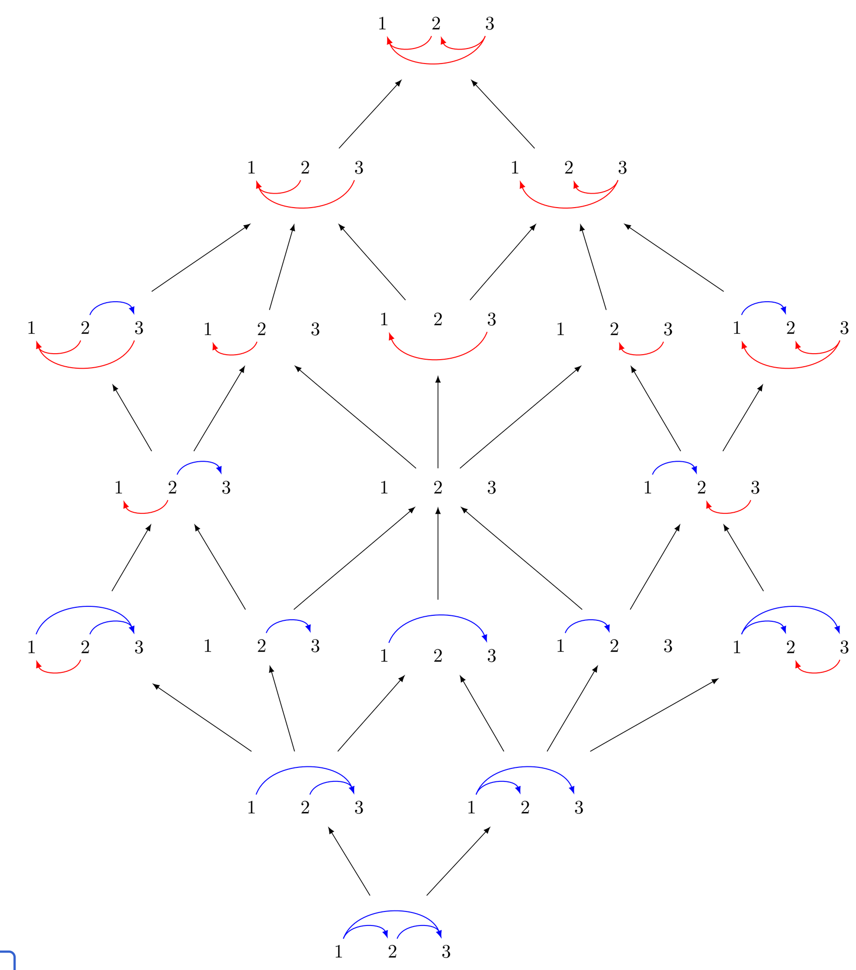


Weak order

We define the **weak order on binary relations** by inclusion of *increasing relations* and inverse inclusion of *decreasing relations*.

$$\mathbf{R} \preccurlyeq \mathbf{S} \Leftrightarrow \mathbf{R}^{\mathsf{Inc}} \supseteq \mathbf{S}^{\mathsf{Inc}} \text{ and } \mathbf{R}^{\mathsf{Dec}} \subseteq \mathbf{S}^{\mathsf{Dec}}.$$

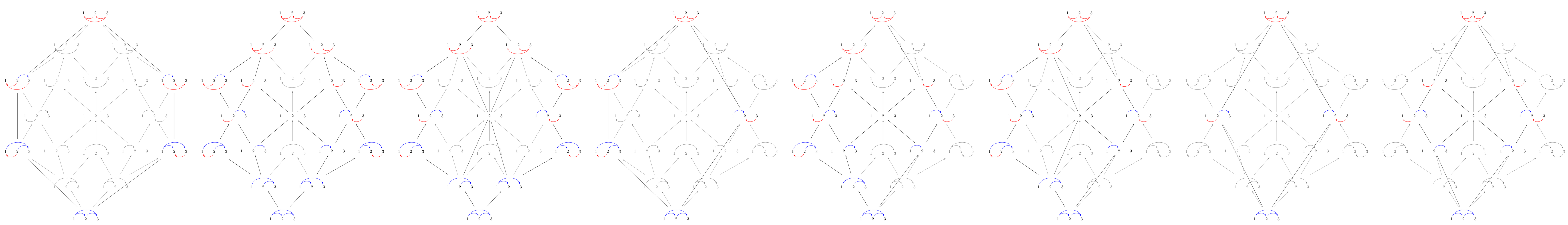
We then restrict this order to relations which are **transitive** and **antisymmetric**, *i.e.*, integer posets.



Theorem

The weak order on integer posets is a lattice.

Many well known lattices can be induced from the lattice of integer posets by selecting elements satisfying specific local conditions.



permutations | perm. intervals | perm. faces | Tamari lattice | Tamari intervals | Assoc. faces | Boolean lattice | Boolean intervals

Hopf algebras

Hopf algebras of integer relations and posets

We define a **product** of integer relations by $\mathbf{R} \cdot \mathbf{S} := \sum \mathbf{T}$ where $\mathbf{T}$ ranges over all integer relations $\mathbf{T} \in \mathcal{R}_{m+n}$ with $\mathbf{T}_{[m]} = \mathbf{R}$ and $\mathbf{T}_{[n+m] \smallsetminus [m]} = \mathbf{S}$.

$$1\,2 \cdot 1 = 1\,2\,3 + 1\,2\,3 + 1\,2\,3 + \cdots + 1\,2\,3 + \cdots + 1\,2\,3$$

We define the **coproduct** of an integer relation by $\triangle(\mathbf{T}) := \sum \mathbf{T}_X \otimes \mathbf{T}_Y$ where the sum ranges over all partitions $X \sqcup Y \subseteq [n]$ such that $x\,\mathbf{T}\,y$ and $y \not\mathbf{T}\, x$ for all $(x, y) \in X \times Y$.

$$\triangle(1\,2\,3) = 1\,2\,3 \otimes \varnothing + 1 \otimes 1\,2 + 1\,2 \otimes 1 + \varnothing \otimes 1\,2\,3$$

Theorem

The vector space indexed by integer relations endowed with the product and coproduct operations forms a **Hopf algebra**. In other words, $\triangle(\mathbf{R} \cdot \mathbf{S}) = \triangle(\mathbf{R}) \cdot \triangle(\mathbf{S})$.

We obtain a **Hopf algebra on integer posets** as a **quotient** of the integer relations Hopf algebra: **we identify all relations which are not posets to 0.** Here is an example of this product in the quotient:

$$1\,2 \cdot 1 = 1\,2\,3 + 1\,2\,3 + 1\,2\,3 + 1\,2\,3 + 1\,2\,3 + 1\,2\,3$$

As an integer poset cannot be cut by the coproduct into two non-posets: **the coproduct stays the same as for integer relations.**

Hopf algebras on other objects

By a similar quotient operation, we recover the **Malvenuto–Reutenauer Hopf algebra on permutations** [4] as well as the **Chapoton Hopf algebra on ordered partitions** [1]. We also define a new Hopf algebra on **permutation intervals**. As an example, here is a product of permutations in a quotient Hopf algebra

$$1\,2 \cdot 1 = 1\,2\,3 + 1\,2\,3 + 1\,2\,3.$$

Hopf algebras on binary trees, Tamari intervals and Associahedron faces can be obtained as **subalgebras** of the integer poset Hopf algebra.

References

[1] F. Chapoton, *Algèbres de Hopf des permutahèdres, associahèdres et hypercubes*, Adv. Math. **150** (2000), no. 2, 264–275.

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[3] G. Châtel and V. Pons, *Counting smaller elements in the Tamari and m-Tamari lattices*, J. Combin. Theory Ser. A **134** (2015), 58–97.

[4] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982.

[5] V. Pilaud and V. Pons, *Hopf algebra on integer posets*, 2017, In preparation.

[6] V. Pilaud and V. Pons, *Permutrees*, Algebraic Combinatorics **1** (2018), no. 2, 173–224.