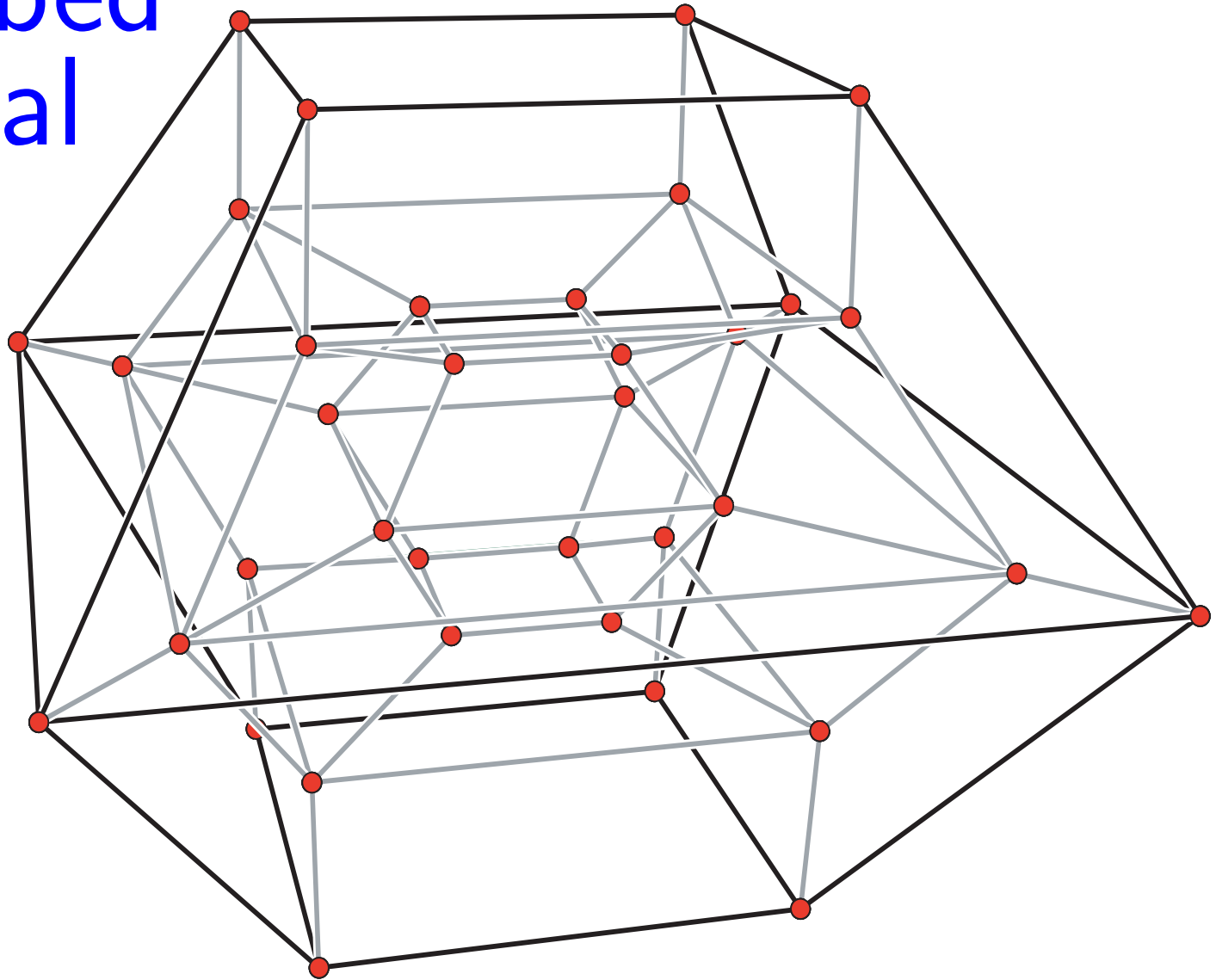


Multitriangulations, pseudotriangulations and some problems of realization of polytopes

Vincent PILAUD



Introduction: Polytopes and spheres with prescribed combinatorial structure



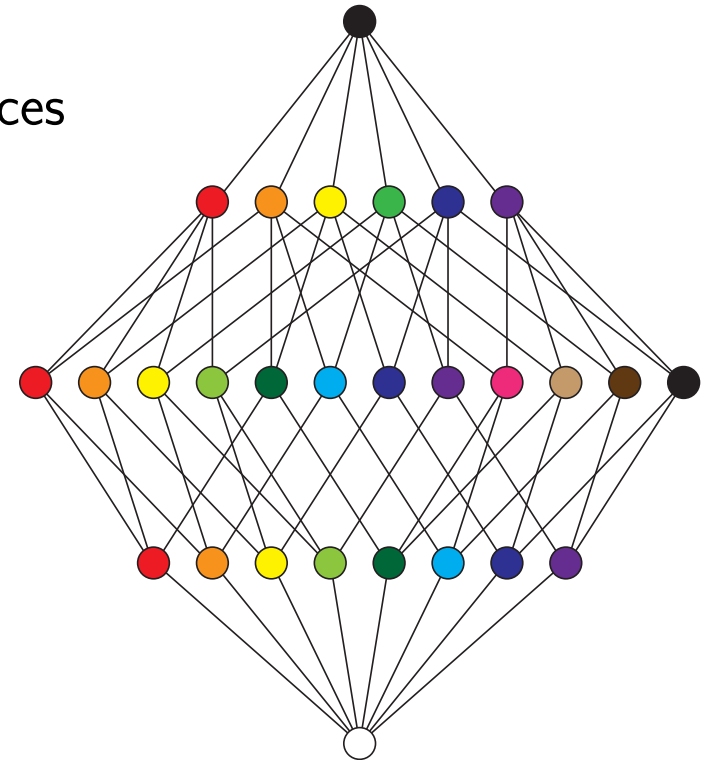
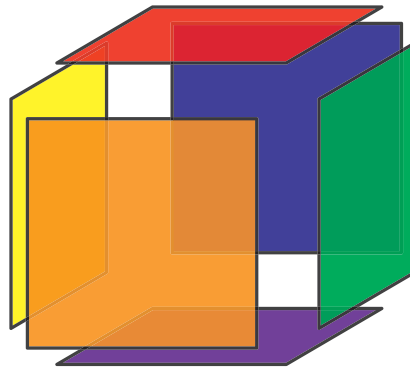
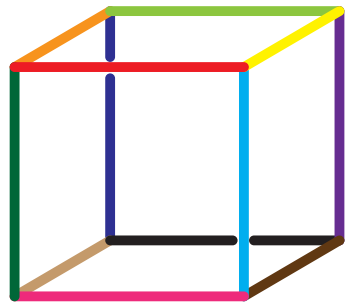
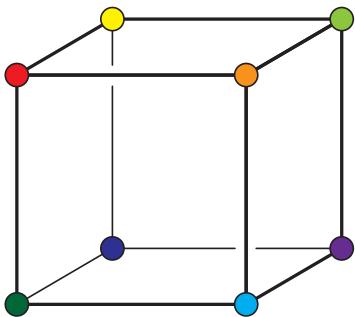
COMBINATORICS OF POLYTOPES

POLYTOPES FROM COMBINATORICS

polytope = convex hull of a finite set of \mathbb{R}^d
= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations



Given a set of points, determine the face lattice of its convex hull.

Given (part of) a face lattice, is there a **polytope which realizes it**?
In **which dimension(s)**?

POLYTOPES WITH PRESCRIBED COMBINATORICS

Given (part of) a face lattice, is there a polytope which realizes it?

For example, which graphs are polytopal?

THEOREM. (Steinitz) Graphs of 3-polytopes = planar and 3-connected graphs.

Realizability questions are interesting for two kinds of structures:

1. lattices coming from combinatorial structures: for example, transformation graphs on combinatorial objects (permutohedron, associahedron, ...).
⇒ understanding of the combinatorial objects.
2. lattices derived from operations on other lattices: Cartesian product, ΔY , ...
⇒ understanding of polytopes.

cell complex \longrightarrow topological sphere \longrightarrow matroid polytope \longrightarrow polytope

CONTENTS

MULTITRIANGULATIONS

1. Introduction
2. Stars in multitriangulations
3. Multipseudotriangulations
4. Three open problems: bijective counting, rigidity, multiassociahedron
- A. Two enumeration algorithms

POLYTOPALITY OF PRODUCTS

5. Introduction
6. Cartesian products of non-polytopal graphs
7. Prodsimplicial neighborly polytopes

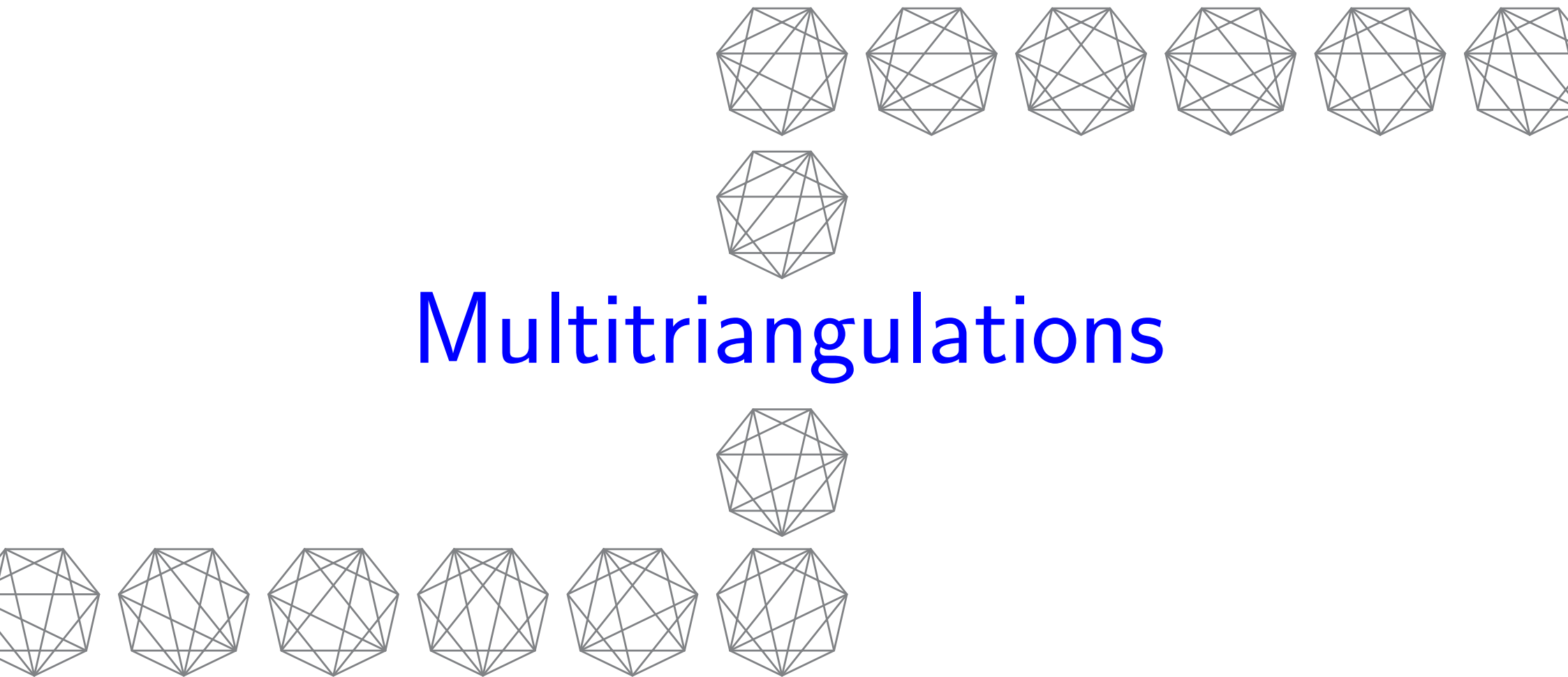
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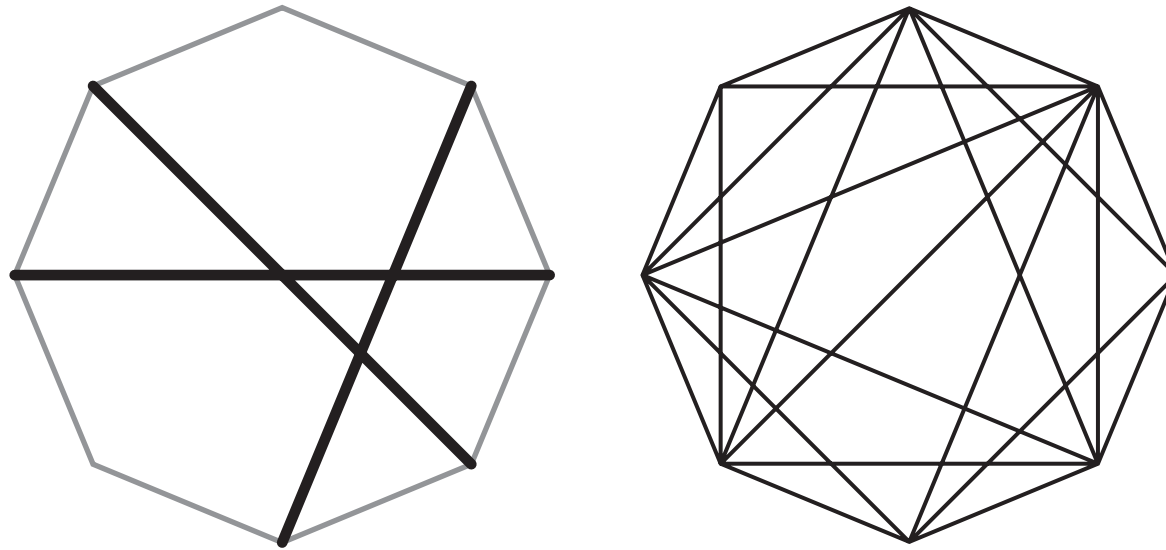
Multitriangulations

DEFINITION

$k \geq 1$ and $n \geq 2k + 1$ two fixed integers.

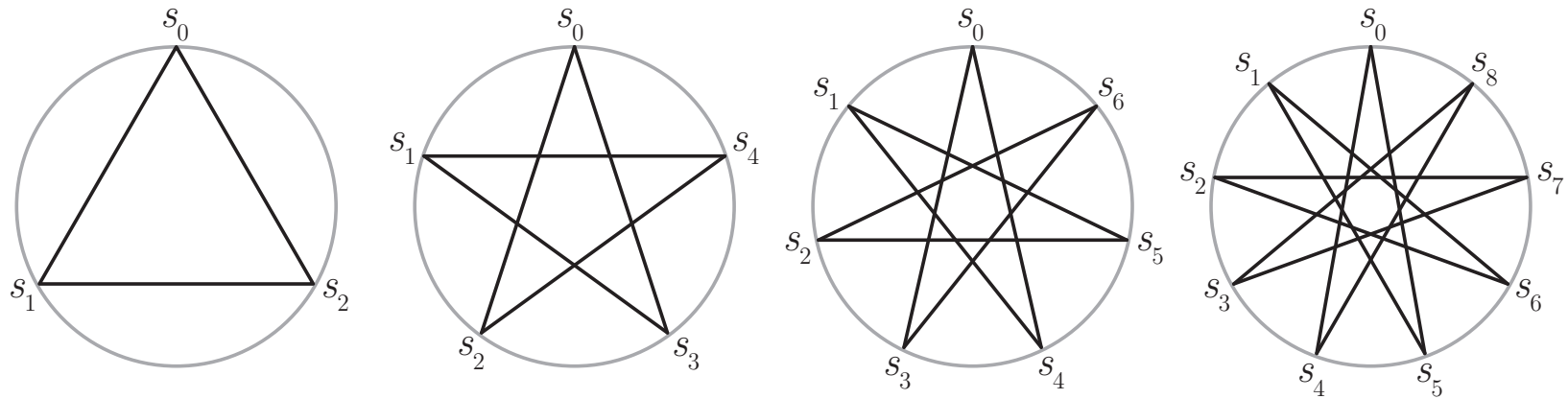
ℓ -crossing = set of ℓ mutually crossing diagonals of the convex n -gon.

k -triangulation = maximal $(k + 1)$ -crossing-free set of diagonals of the n -gon.

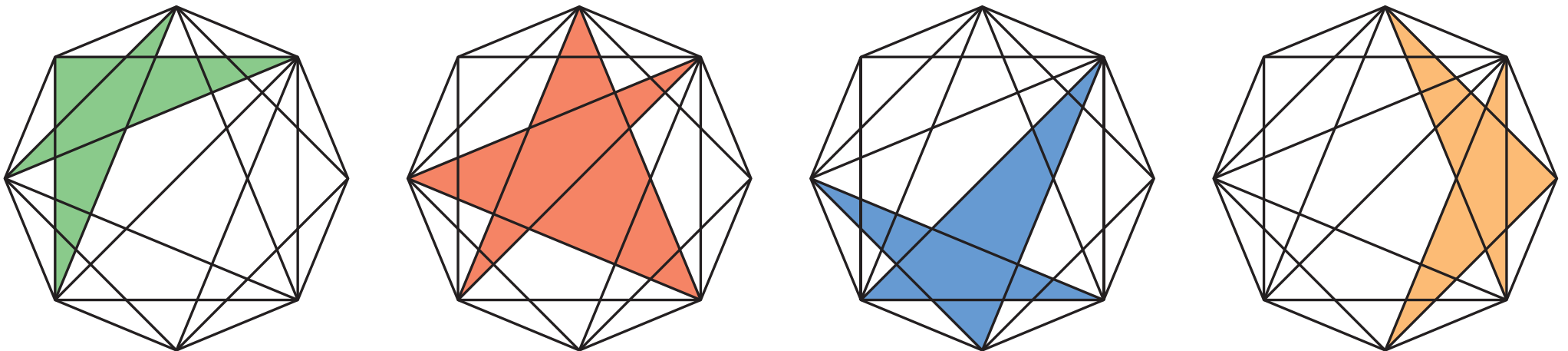


- V. Capoleas & J. Pach, A Turán-type theorem on chords of a convex polygon, 1992.
- T. Nakamigawa, A generalization of diagonal flips in a convex polygon, 2000.
- A. Dress, J. Koolen & V. Moulton, On line arrangements in the hyperbolic plane, 2002.
- J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

STARS IN MULTITRIANGULATIONS



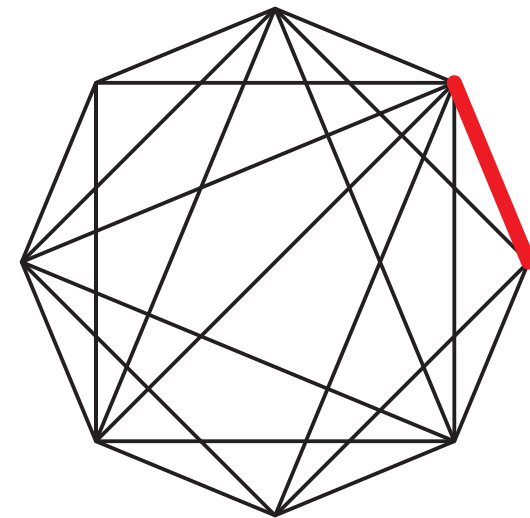
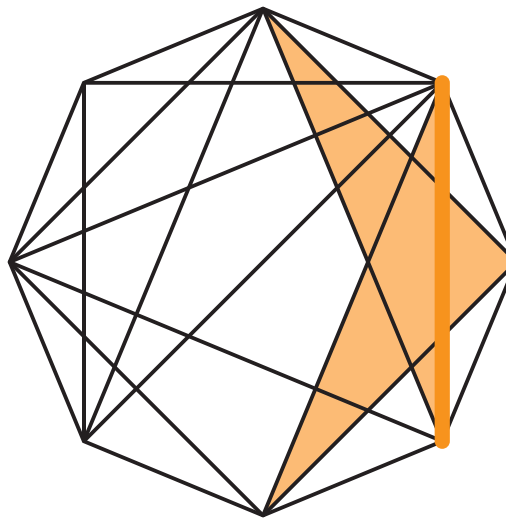
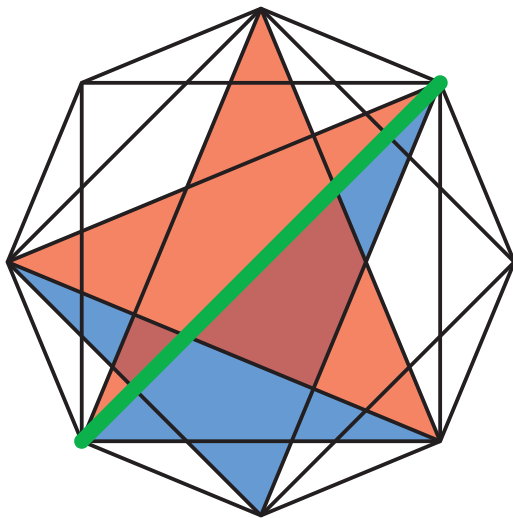
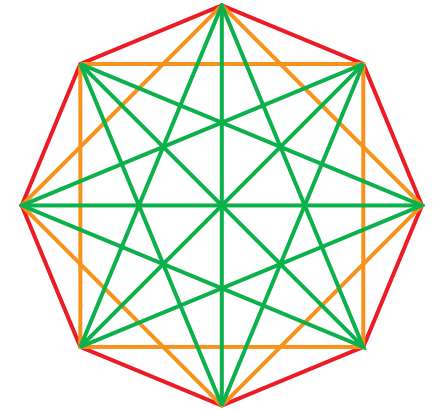
k -star = star polygon with vertices s_0, s_1, \dots, s_{2k} cyclically ordered
and edges $[s_0, s_k], [s_1, s_{1+k}], \dots, [s_k, s_{2k}], [s_{k+1}, s_0], \dots, [s_{2k}, s_{k-1}]$.



COMPLEXES OF STARS

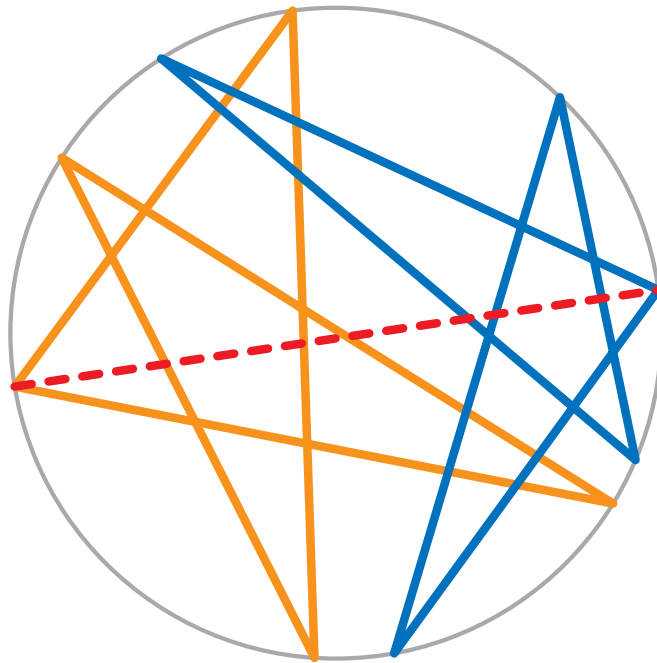
THEOREM. In a k -triangulation T ,

- (i) a k -relevant diagonal belongs to exactly two k -stars of T ,
- (ii) a k -boundary diagonal belongs to exactly one k -star of T ,
- (iii) a k -irrelevant diagonal does not belong to any k -star of T .



COMMON BISECTORS

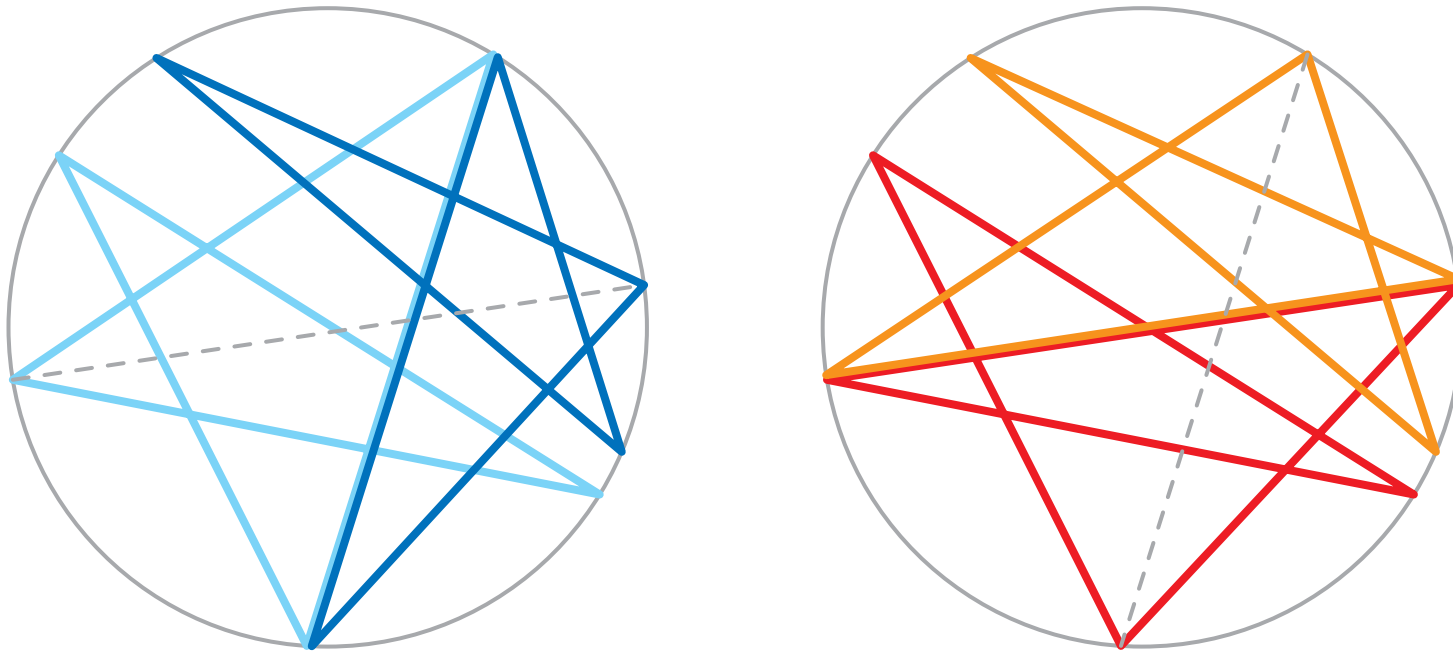
THEOREM. T a k -triangulation of the n -gon. Every pair of k -stars of T have a **unique common bisector**. Reciprocally, any diagonal not in T is the common bisector of a unique pair of k -stars of T .



COROLLARY. Any k -triangulation of the n -gon contains exactly
 $n - 2k$ k -stars and $k(2n - 2k - 1)$ diagonals.

THE GRAPH OF FLIPS

THEOREM. Let e be a k -relevant diagonal of a k -triangulation T , let R and S be the two k -stars of T containing e , and let f be the common bisector of R and S . Then $T \Delta \{e, f\}$ is the only k -triangulation other than T containing $T \setminus \{e\}$.



THEOREM. The graph of flips is connected, regular of degree $k(n - 2k - 1)$, and its diameter is at most $2k(n - 2k - 1)$.

THE (POLYTOPAL?) SIMPLICIAL COMPLEX $\Delta_{n,k}$

$k \geq 1$ and $n \geq 2k + 1$ two fixed integers.

ℓ -crossing = set of ℓ mutually crossing diagonals of the convex n -gon.

k -relevant diagonal = at least k vertices on each side
= diagonals which may appear in a $(k + 1)$ -crossing.

$\Delta_{n,k}$ = simplicial complex of $(k + 1)$ -crossing-free sets
of k -relevant diagonals of the convex n -gon.

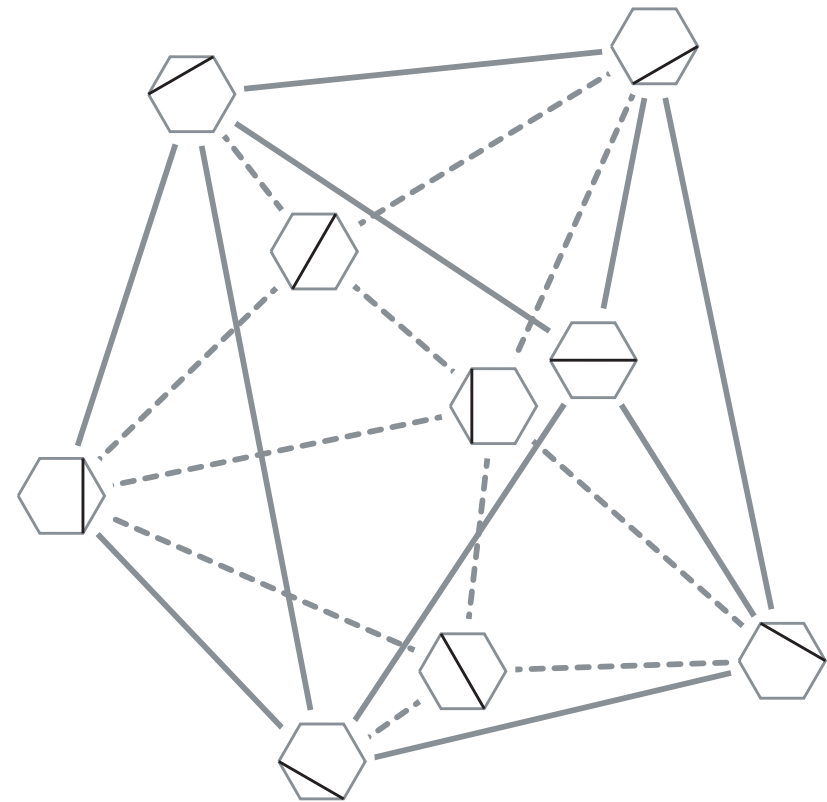
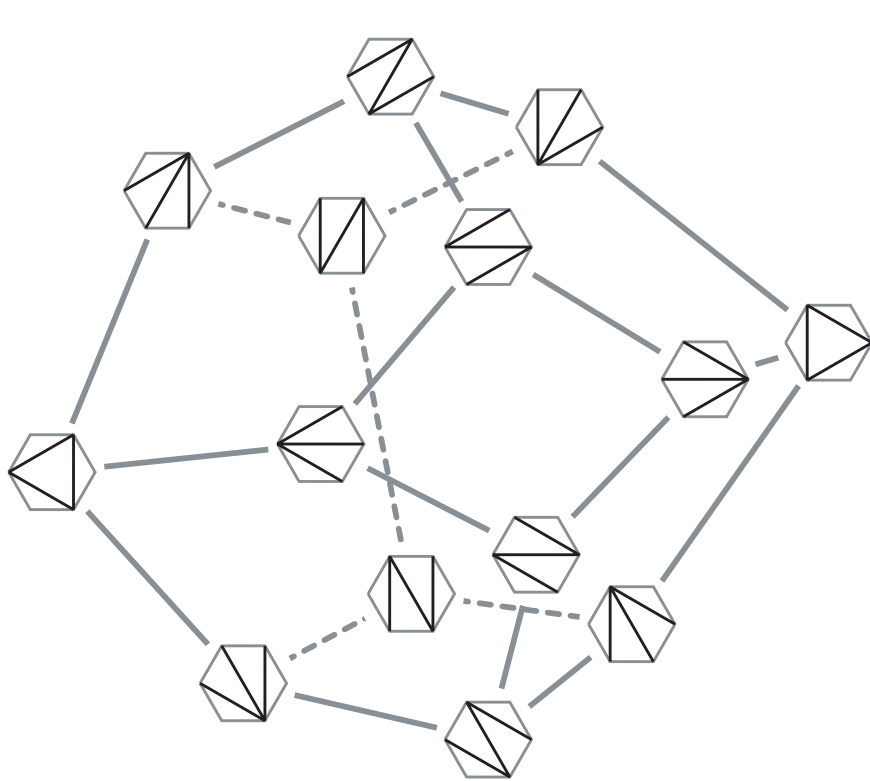
THEOREM. $\Delta_{n,k}$ is a **topological sphere** of dimension $k(n - 2k - 1) - 1$.

J. Jonsson, *Generalized triangulations of the n -gon*, 2003.

QUESTION. Is $\Delta_{n,k}$ the **boundary complex** of a **simplicial $k(n - 2k - 1)$ -polytope**?

ASSOCIAHEDRON

- $k = 1$ Maximal elements of $\Delta_{n,1} =$ triangulations of the n -gon.
 $\Delta_{n,1} =$ boundary complex of the dual of the $(n - 3)$ -dimensional **associahedron**.



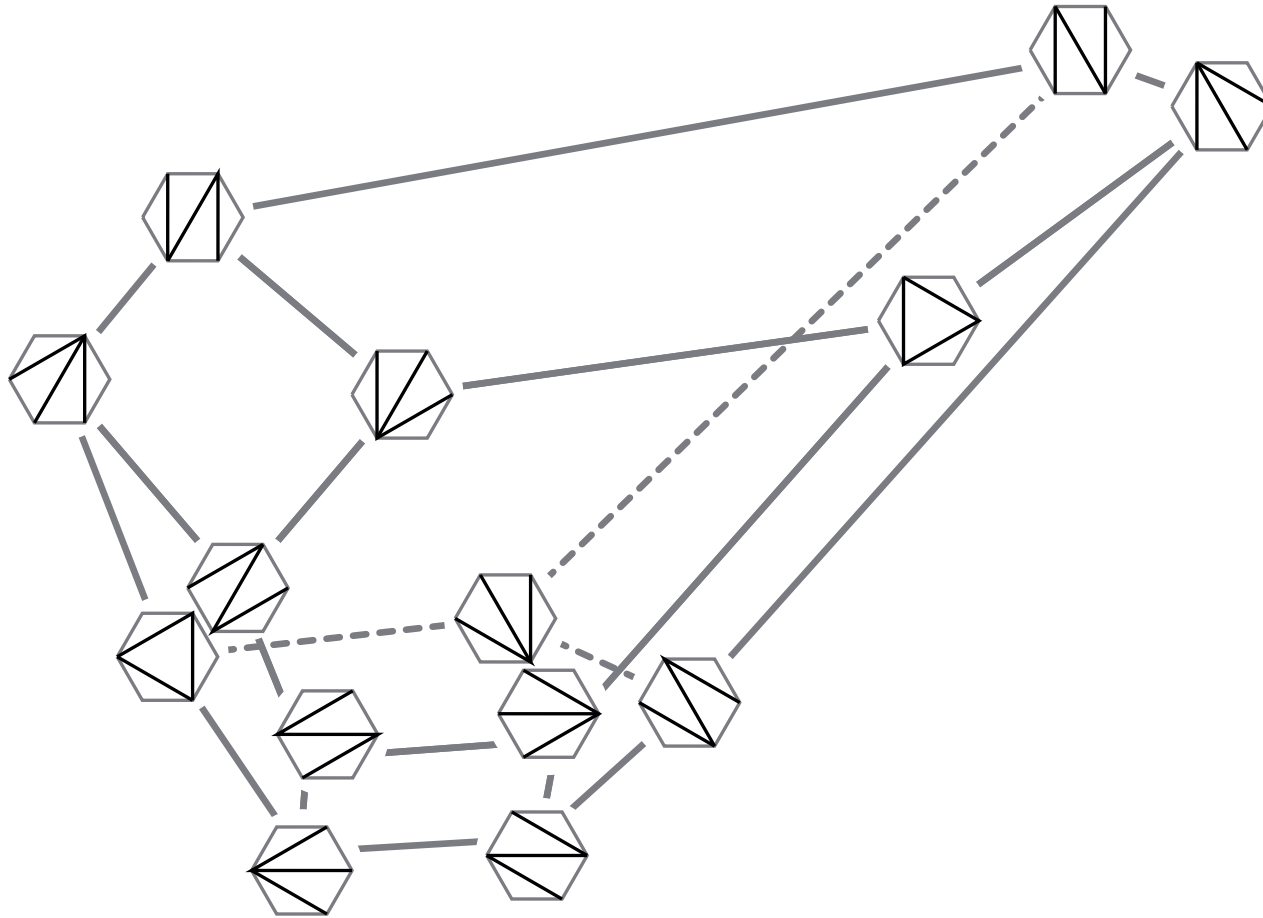
L.J. Billera, P. Filliman & B. Sturmfels,
Constructions and complexity of secondary polytopes, 1990.

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OTHER EXAMPLES

$k = 1$ Maximal elements of $\Delta_{n,1}$ = triangulations of the n -gon.
 $\Delta_{n,1}$ = boundary complex of the dual of the $(n - 3)$ -dimensional [associahedron](#).

$n = 2k + 1$ $\Delta_{2k+1,k}$ = single k -triangulation.

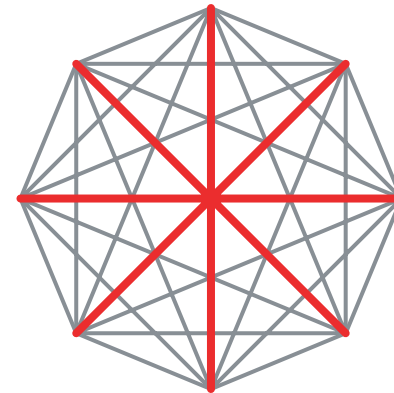
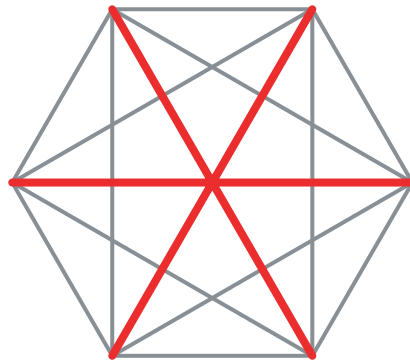
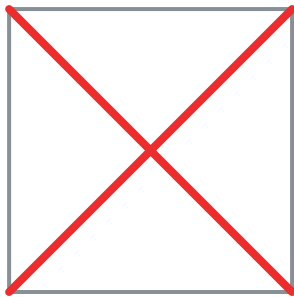
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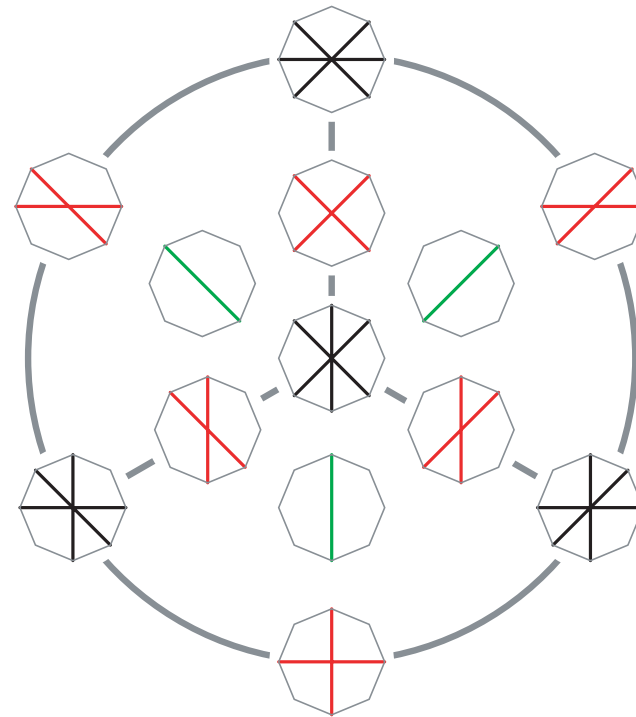
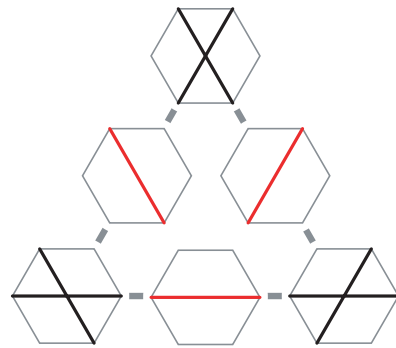
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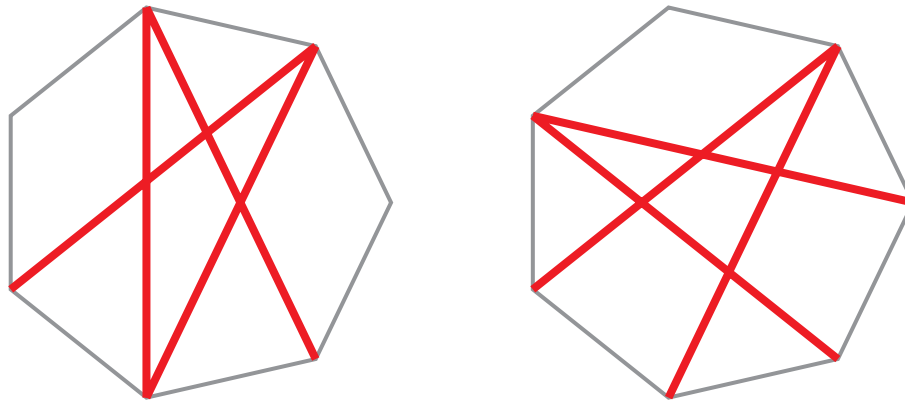
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$n = 2k + 3$ $\Delta_{2k+3,k} =$ boundary complex of the **cyclic polytope**
of dimension $2k$ with $2k + 3$ vertices.



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of dimension $2k$ with $2k + 3$ vertices.

$n = 8$ & $k = 2$

f -vector of $\Delta_{8,2} = (12, 66, 192, 306, 252, 84)$

THEOREM. The space of symmetric realizations of $\Delta_{8,2}$ has dimension 4.

J. Bokowski & V. P., On symmetric realizations of the simplicial complex of 3-crossing-free sets of diagonals of the octagon, 2009.



Flip graphs on pseudoline arrangements

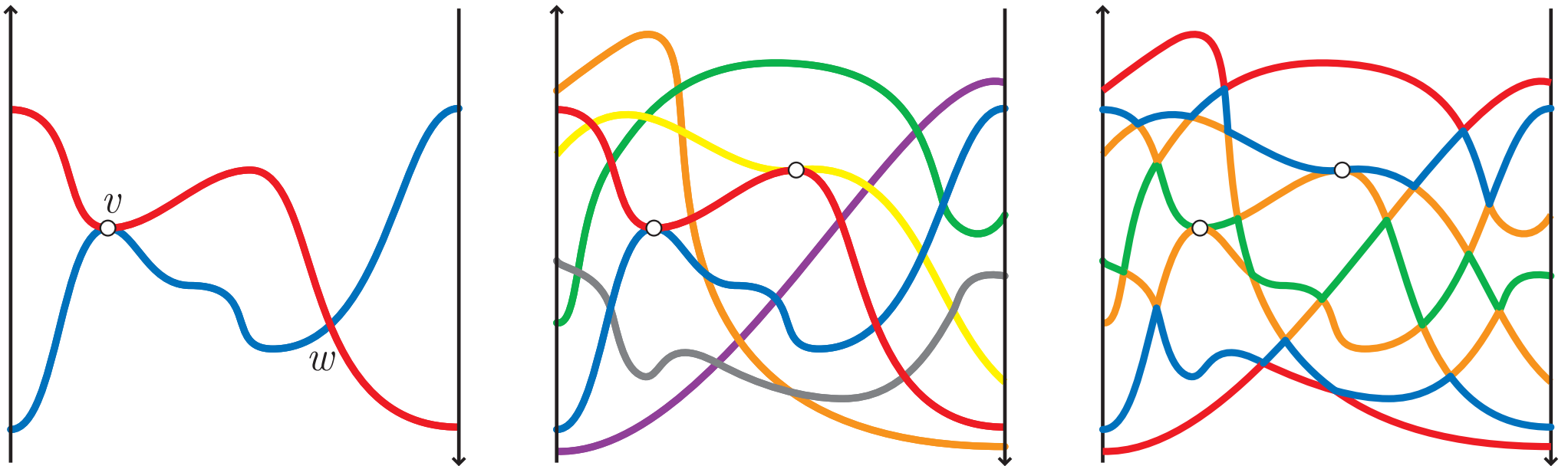
V. P. & M. Pocchiola, Multipseudotriangulations, 2010.

PSEUDOLINE ARRANGEMENTS

Möbius strip = $\mathbb{R}^2 / (x, y) \sim (x + \pi, -y)$.

pseudoline = non-separating simple closed curve in the Möbius strip.

pseudoline arrangement = finite set of pseudolines such that any two of them have exactly one crossing point and possibly some contact points.

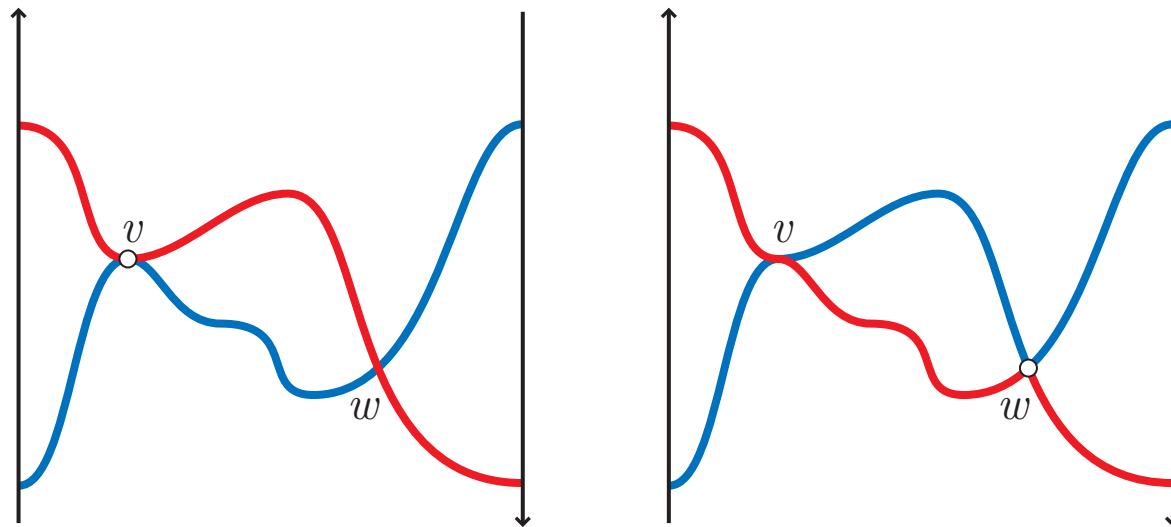


support = union of pseudolines

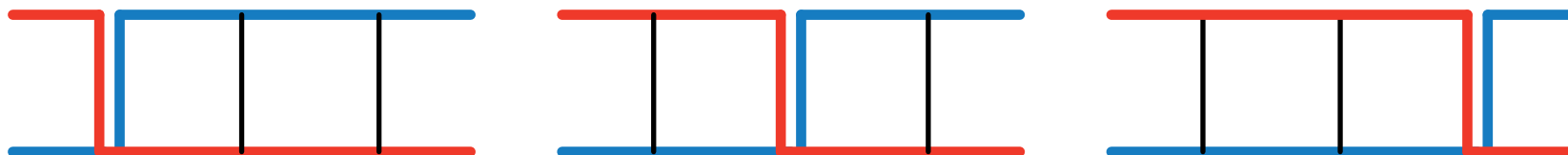
levels = layers of the arrangement

FLIP GRAPHS

Flip = exchange a contact point between two pseudolines with their crossing point.
 $G(\mathcal{S})$ = the **flip graph** on all pseudoline arrangements supported by a given support \mathcal{S} .

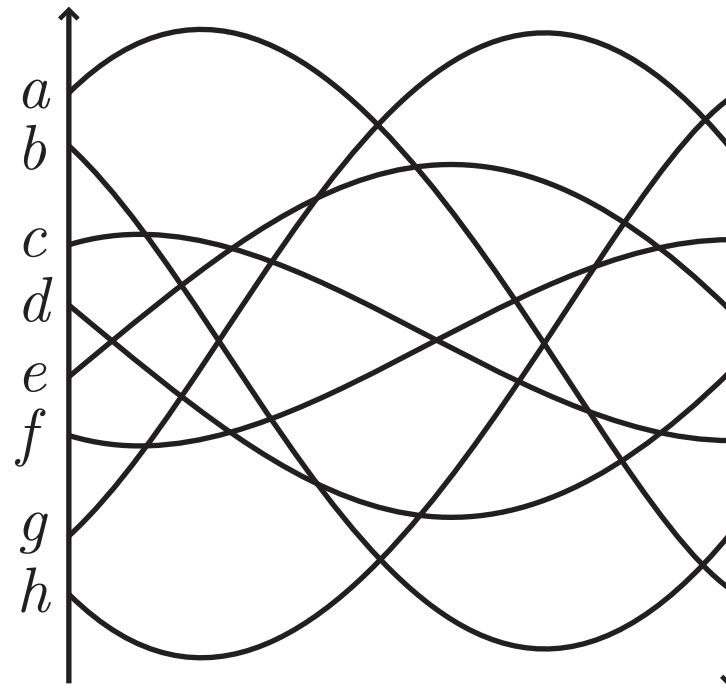
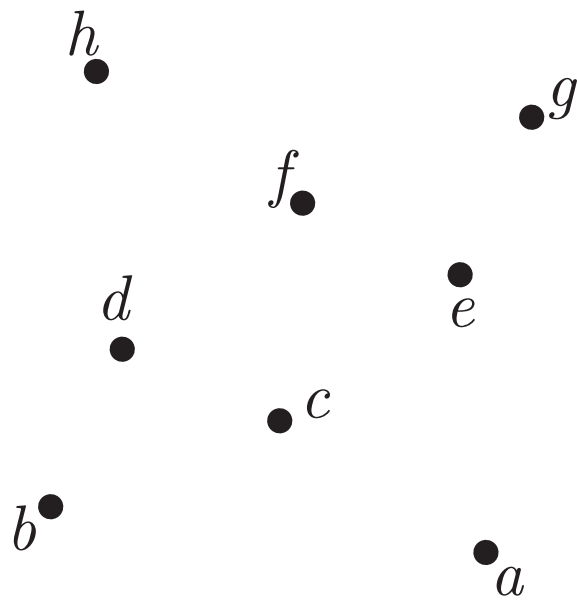


EXAMPLE. \mathcal{S} = support with 2 levels and p intersection points.
Then $G(\mathcal{S})$ = complete graph K_p .



DUALITY

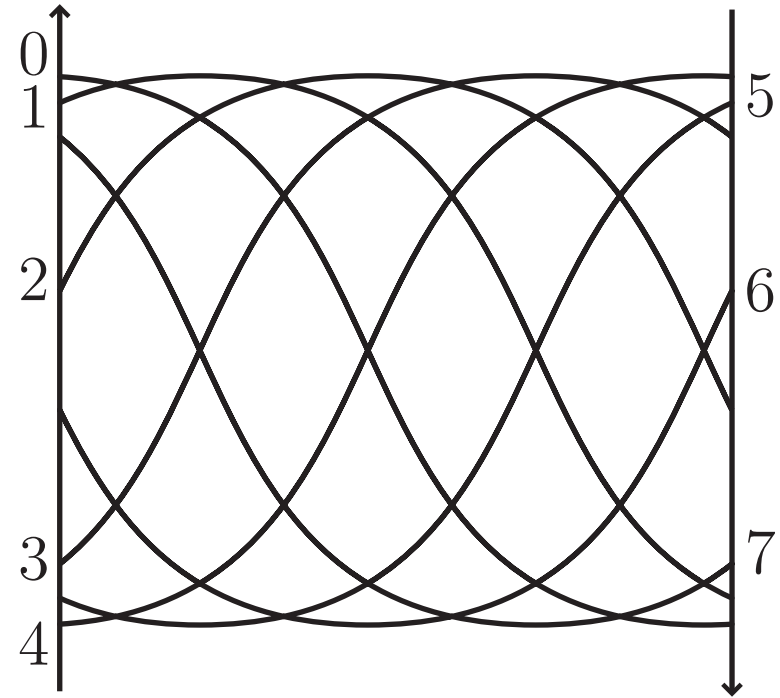
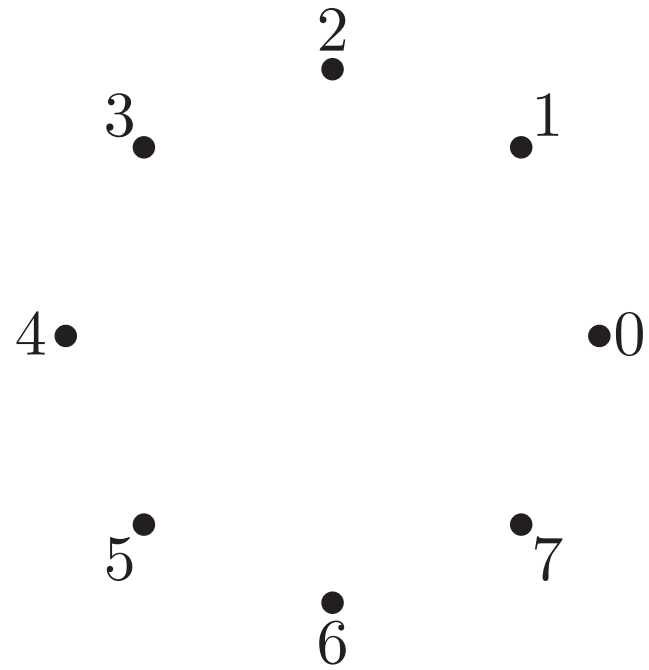
line space of the Euclidean plane = $\mathbb{R}^2/(\theta, d) \sim (\theta + \pi, -d) = \text{Möbius strip}$.



p point of the plane
 P point set in general position

$p^* = \{\text{lines passing through } p\}$ dual pseudoline
 $P^* = \{p^* \mid p \in P\}$ dual pseudoline arrangement

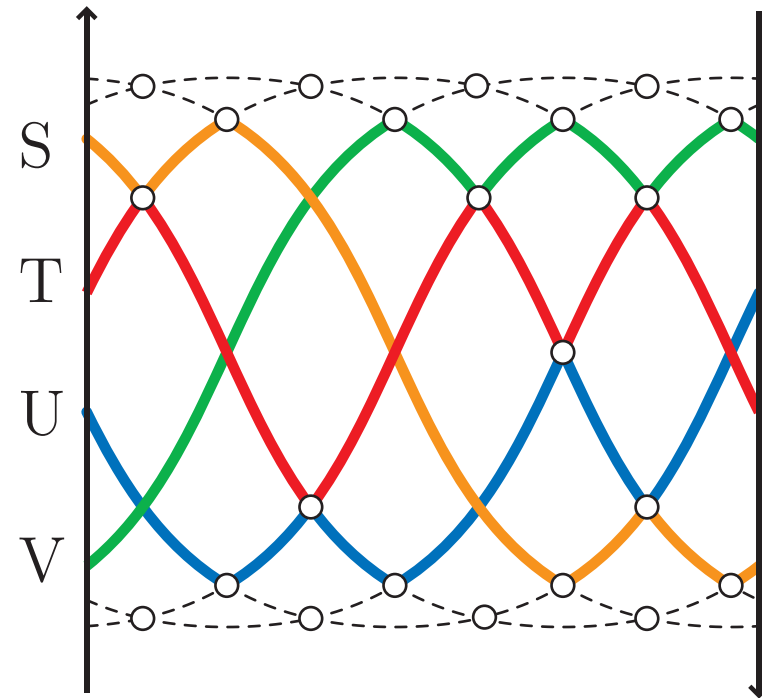
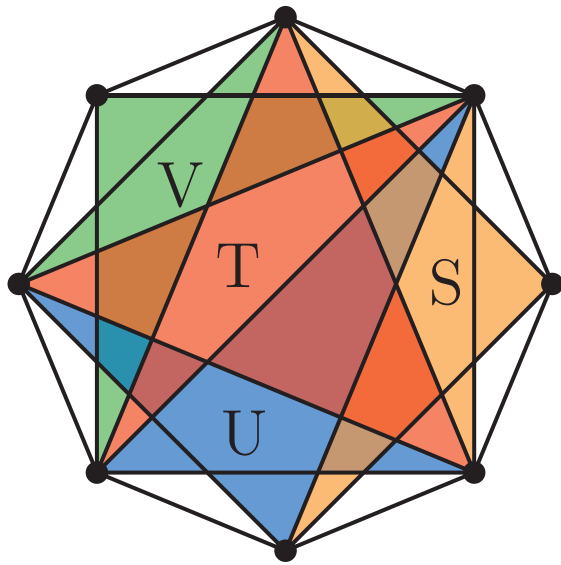
DUALITY AND MULTITRIANGULATIONS



V_n vertices of the convex n -gon

V_n^* dual pseudoline arrangement of V_n

DUALITY AND MULTITRIANGULATIONS



V_n vertices of the convex n -gon

S k -star of a k -triangulation T

T k -triangulation of V_n

V_n^* dual pseudoline arrangement of V_n

$S^* = \{\text{bisectors of } S\}$ dual pseudoline of S

$T^* = \{S^* \mid S \text{ } k\text{-star of } T\}$

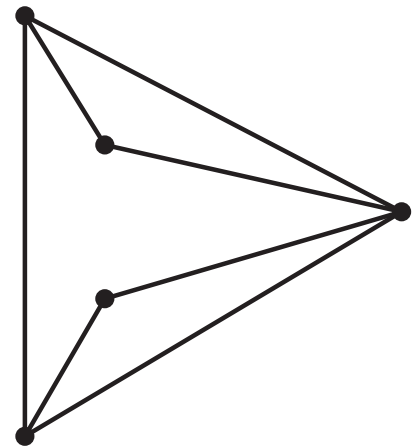
dual pseudoline arrangement of T

THEOREM. $T \subset \binom{V_n}{2}$ k -triangulation of $V_n \Leftrightarrow T^*$ covers V_n^* minus its first k levels.

PSEUDOTRIANGULATIONS

A **pseudotriangulation** of a finite point set P is:

- (i) a maximal **crossing-free pointed** subset of $\binom{P}{2}$,
- (ii) a pointed subset of $\binom{P}{2}$ that decomposes the convex hull of P into **pseudotriangles**.



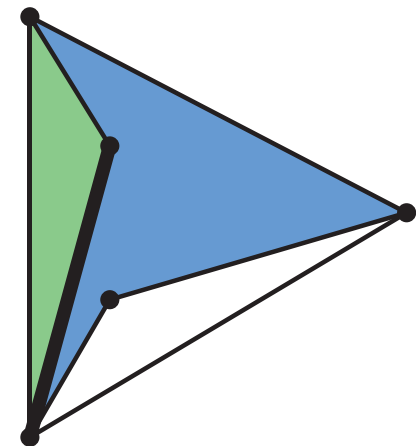
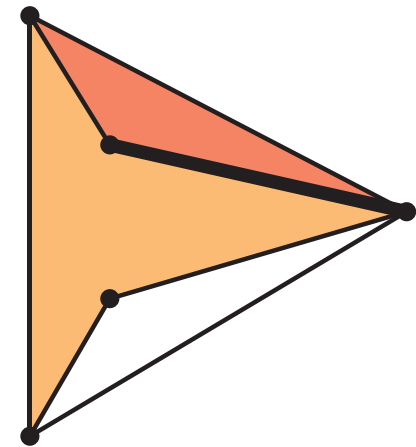
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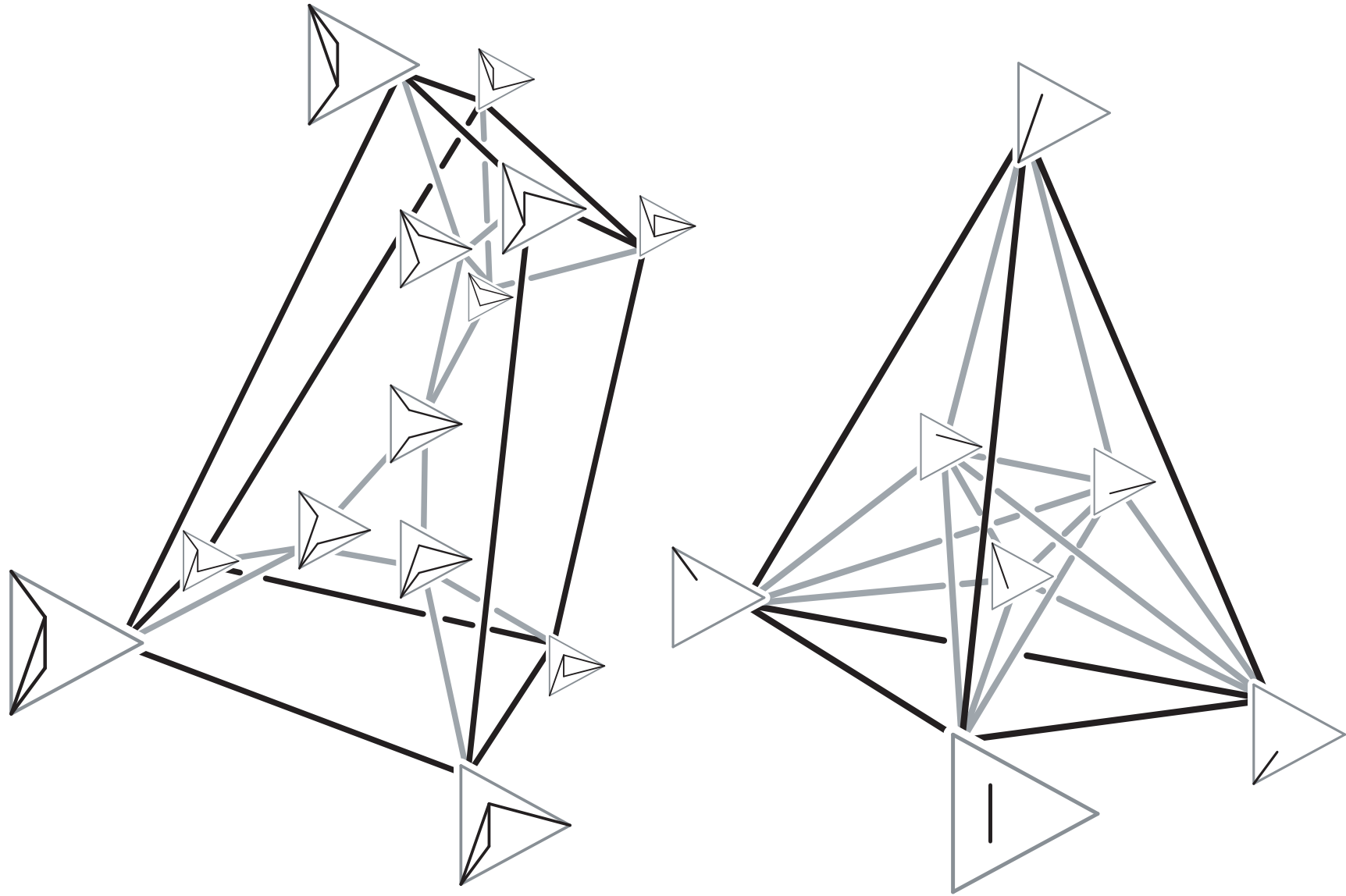
- (i) a maximal **crossing-free pointed** subset of $\binom{P}{2}$,
- (ii) a pointed subset of $\binom{P}{2}$ that decomposes the convex hull of P into **pseudotriangles**.

PROPERTIES.

- (i) A pseudotriangulation of P has exactly $2|P| - 3$ edges.
- (ii) Two pseudotriangles have a unique **common tangent**.
- (iii) T pseudotriangulation of P ; e **internal** edge of T ;
 f common tangent between the two pseudotriangles of T containing $e \Rightarrow T \Delta \{e, f\}$ pseudotriangulation of P .
- (iv) The graph of flips is **polytopal**.

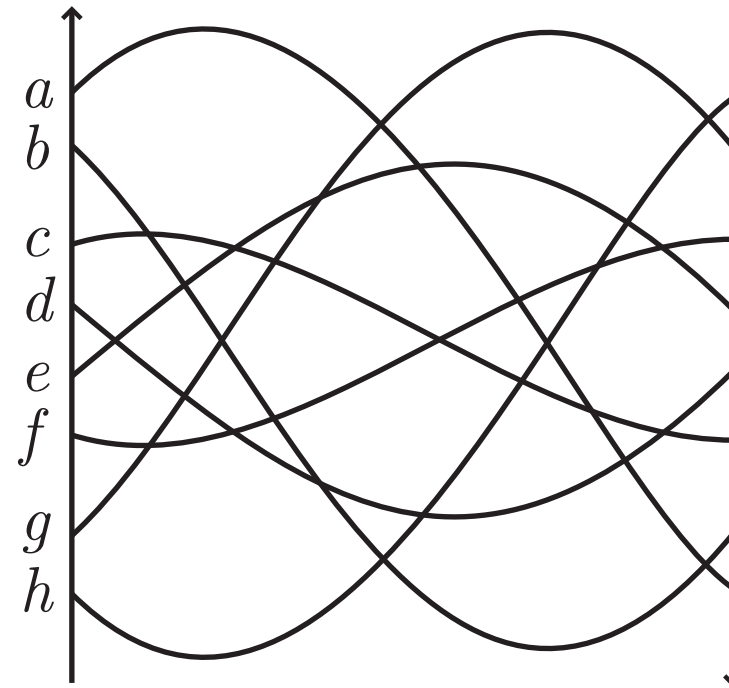
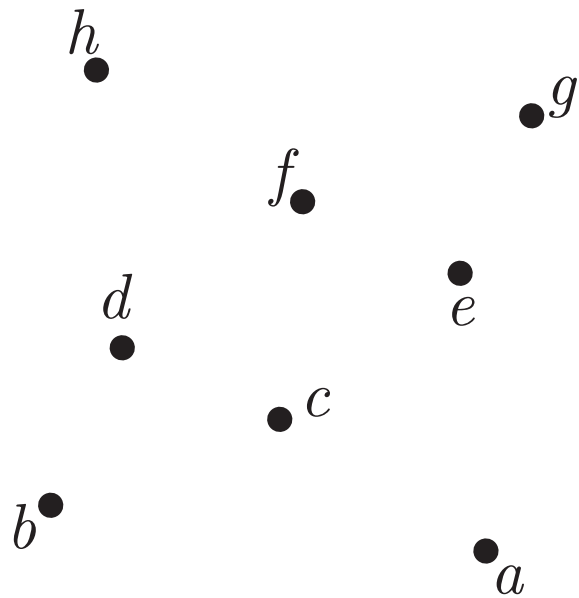


PSEUDOTRIANGULATIONS



G. Rote, F. Santos, & I. Streinu,
Expansive motions and the polytope of pointed pseudo-triangulations, 2003.

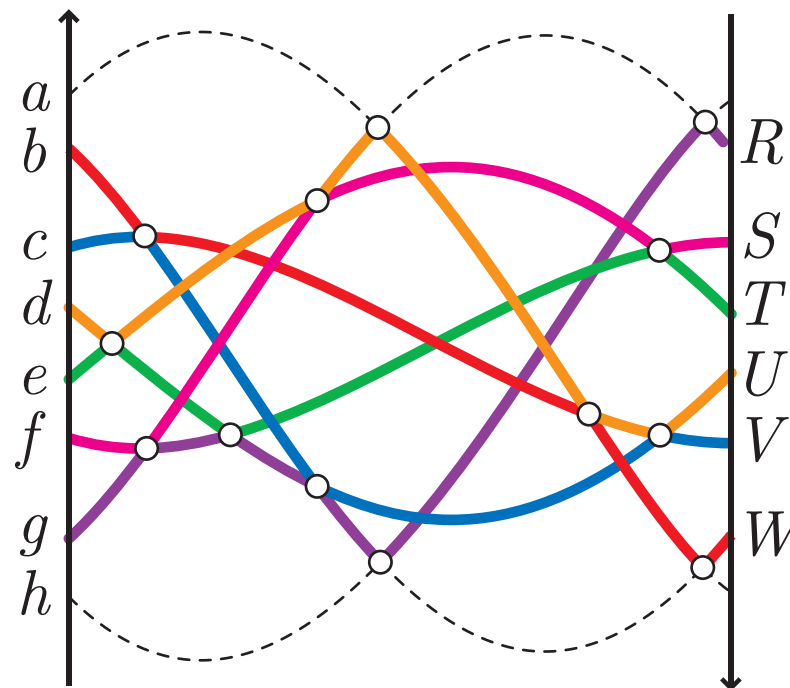
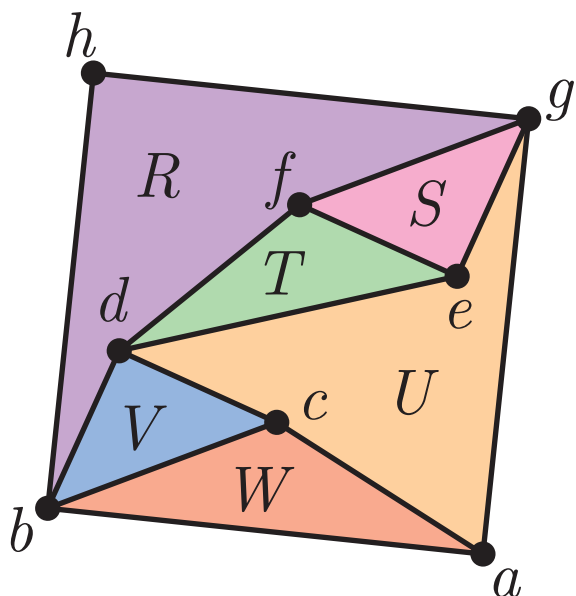
DUALITY AND PSEUDOTRIANGULATIONS



P point set in general position

P^* dual pseudoline arrangement of P

DUALITY AND PSEUDOTRIANGULATIONS



P point set in general position

Δ pseudotriangle

T pseudotriangulation of P

P^* dual pseudoline arrangement of P

$\Delta^* = \{\text{tangent to } \Delta\}$ dual pseudoline of Δ

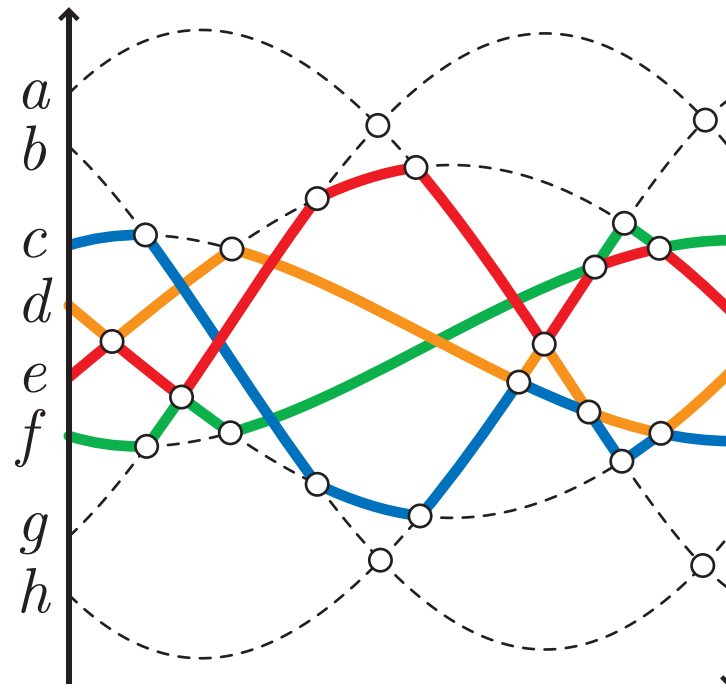
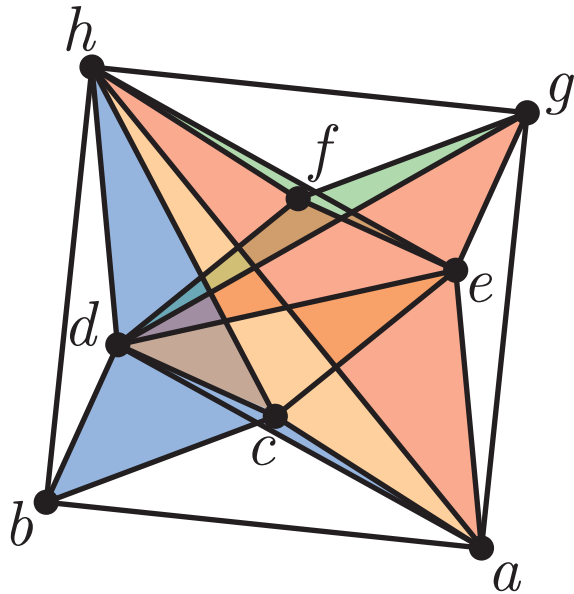
$T^* = \{\Delta^* \mid \Delta \text{ pseudotriangle of } T\}$

dual pseudoline arrangement of T

THEOREM. $T \subset \binom{P}{2}$ pseudotriangulation of $P \Leftrightarrow T^*$ covers P^* minus its first level.

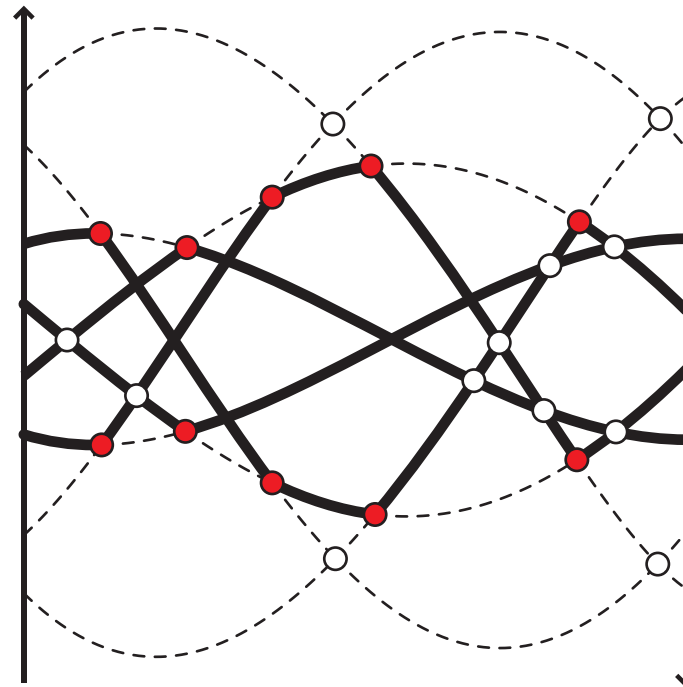
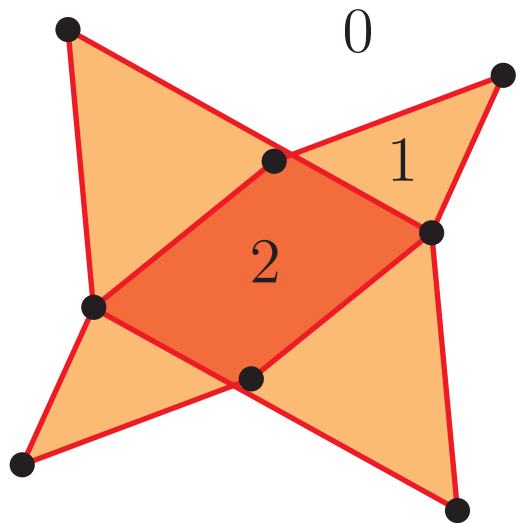
MULTIPSEUDOTRIANGULATIONS

k -pseudotriangulation of a point set P in general position in the plane = set T of edges of $\binom{P}{2}$ which corresponds via duality to the contact points of a pseudoline arrangement T^* supported by P^* minus its first k levels.



MULTIPSEUDOTRIANGULATIONS

k -pseudotriangulation of a point set P in general position in the plane = set T of edges of $\binom{P}{2}$ which corresponds via duality to the contact points of a pseudoline arrangement T^* supported by P^* minus its first k levels.

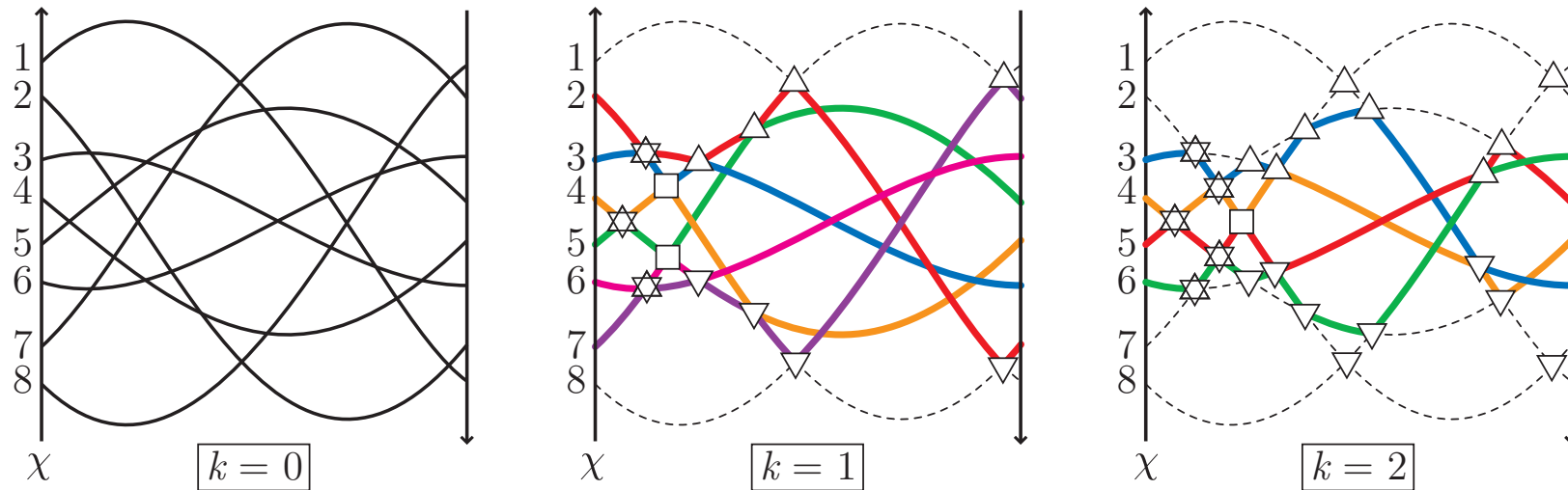


PROPOSITION. $P \cup \{q\}$ point set in general position. T a k -pseudotriangulation of P .

$$k\text{-depth of } q \text{ in } P = \sum_{\lambda \in T^*} \text{winding number of } S(\lambda).$$

EXAMPLES OF APPLICATIONS

ENUMERATION ... greedy pseudoline arrangements



\Rightarrow enumeration algorithm for pseudoline arrangements covering a given support.

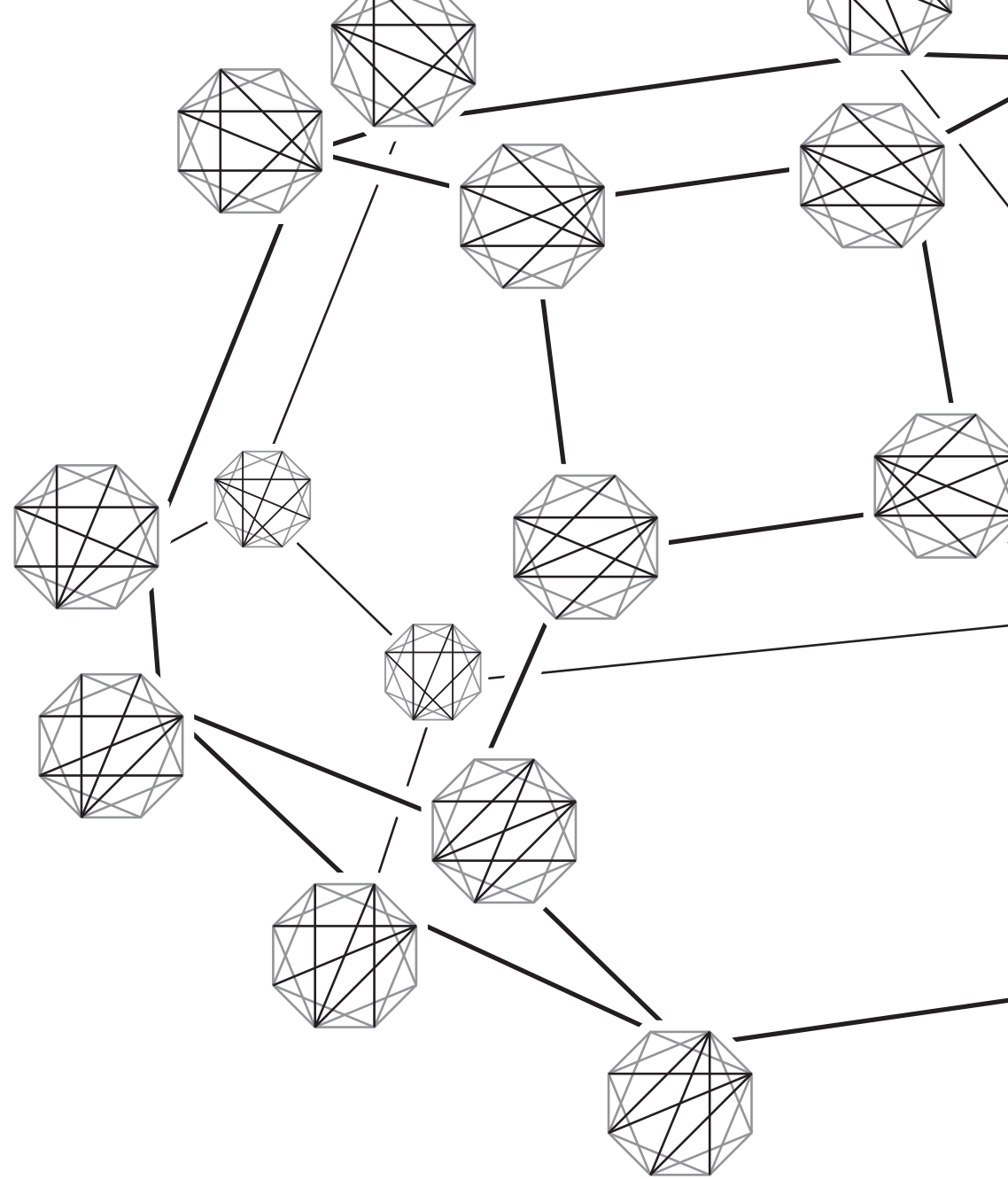
CHARACTERIZATION THEOREM

THEOREM. A set Σ of k -stars of the n -gon such that:

- (i) any k -relevant edge is contained in zero or two k -stars of Σ , one on each side, and
 - (ii) any k -boundary edge is contained in exactly one k -star of Σ ,
- is the set of k -stars of a k -triangulation of the n -gon.

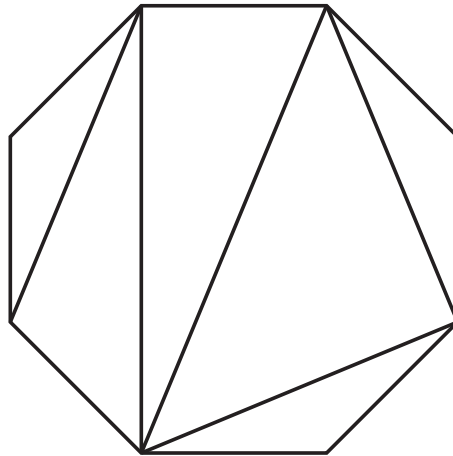
LOWER BOUND THEOREM ... for d -polytopes with $d + 3$ vertices.

The brick polytope



LODAY'S ASSOCIAHEDRON REVISITED

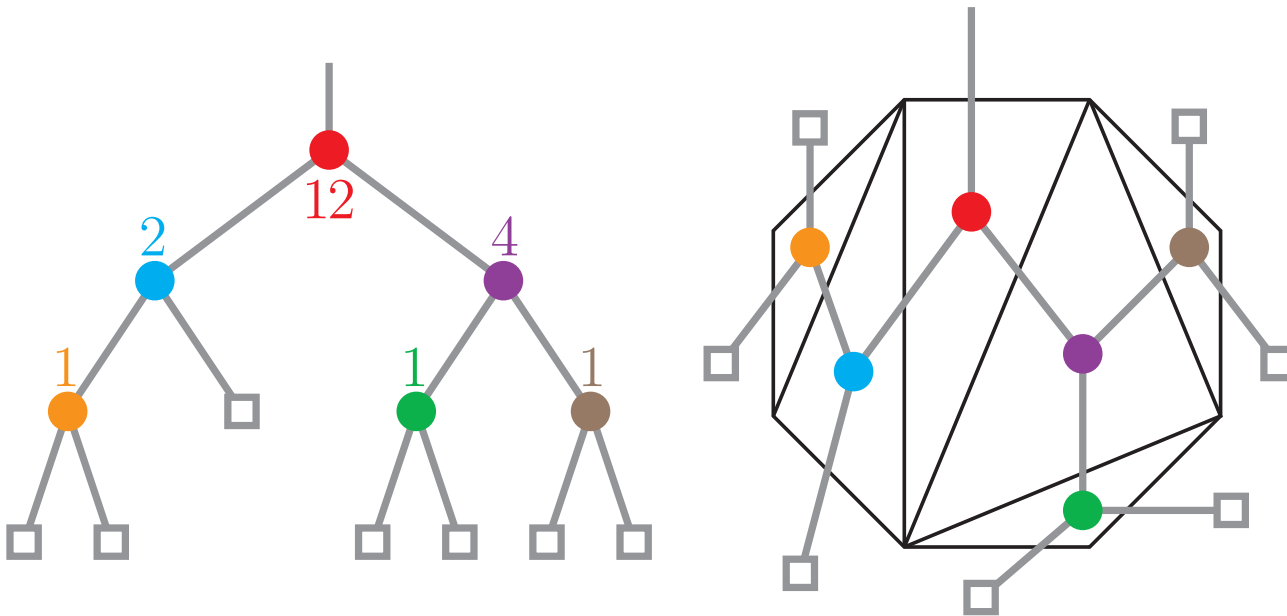
T triangulation of the n -gon \mapsto vector $\omega(T) \in \mathbb{R}^{n-2}$.
Loday's associahedron $\Omega(n) = \text{conv}\{\omega(T) \mid T \text{ triangulation of the } n\text{-gon}\}$.



LODAY'S ASSOCIAHEDRON REVISITED

T triangulation of the n -gon \mapsto vector $\omega(T) \in \mathbb{R}^{n-2}$.

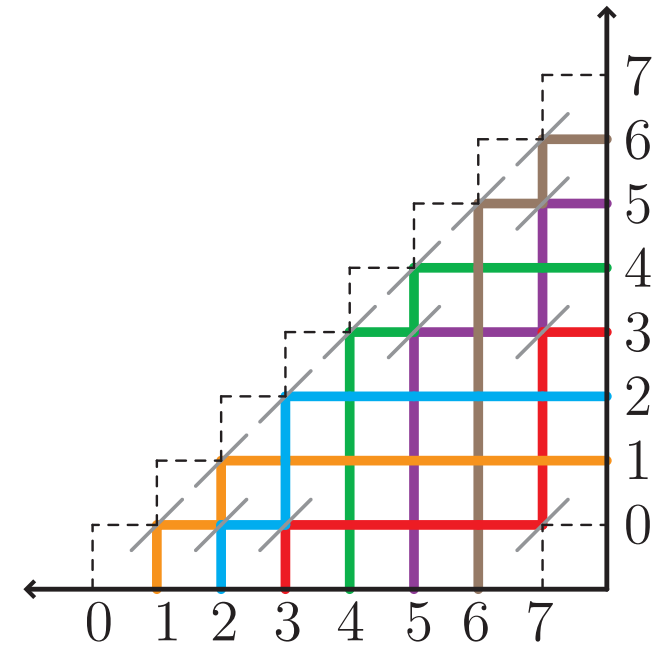
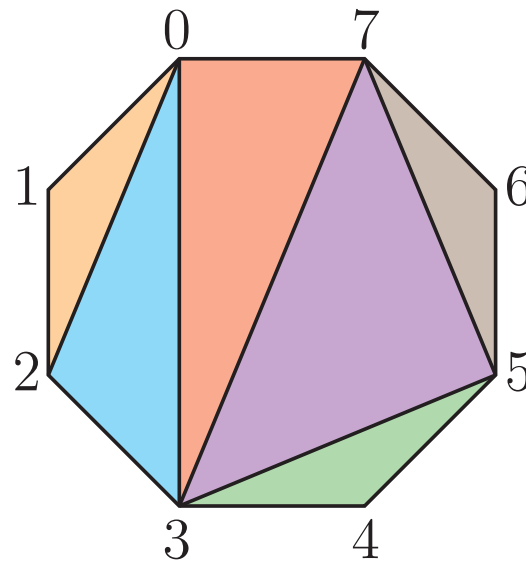
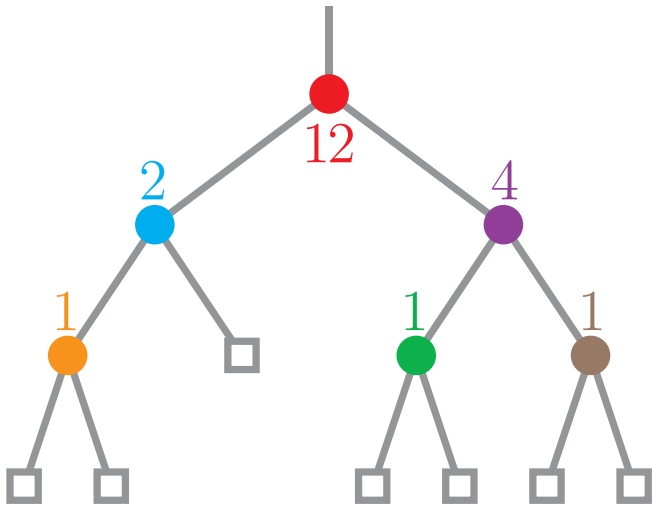
Loday's associahedron $\Omega(n) = \text{conv}\{\omega(T) \mid T \text{ triangulation of the } n\text{-gon}\}$.



LODAY'S ASSOCIAHEDRON REVISITED

T triangulation of the n -gon \mapsto vector $\omega(T) \in \mathbb{R}^{n-2}$.

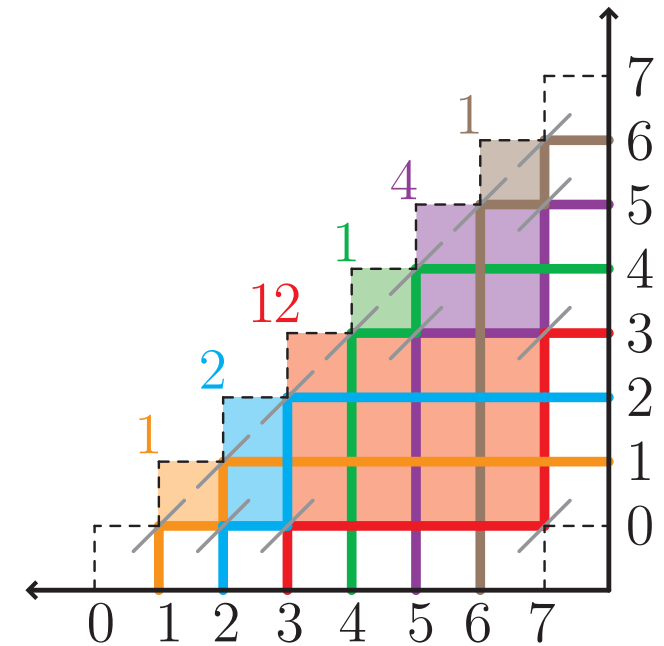
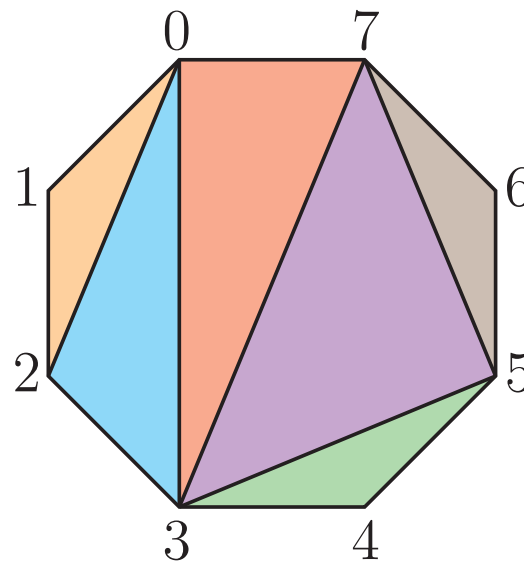
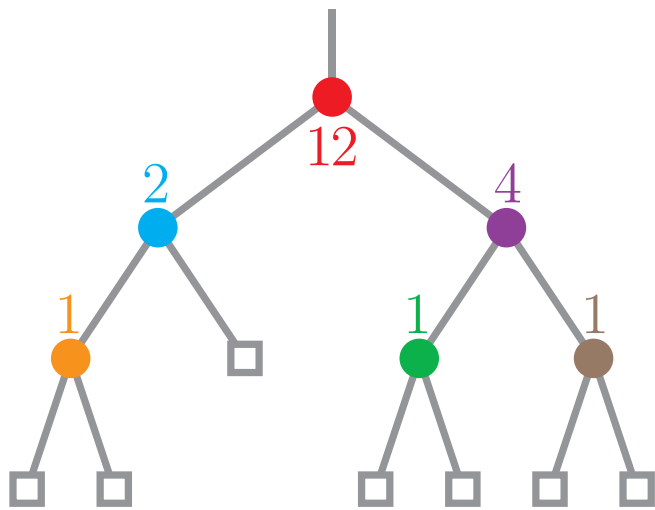
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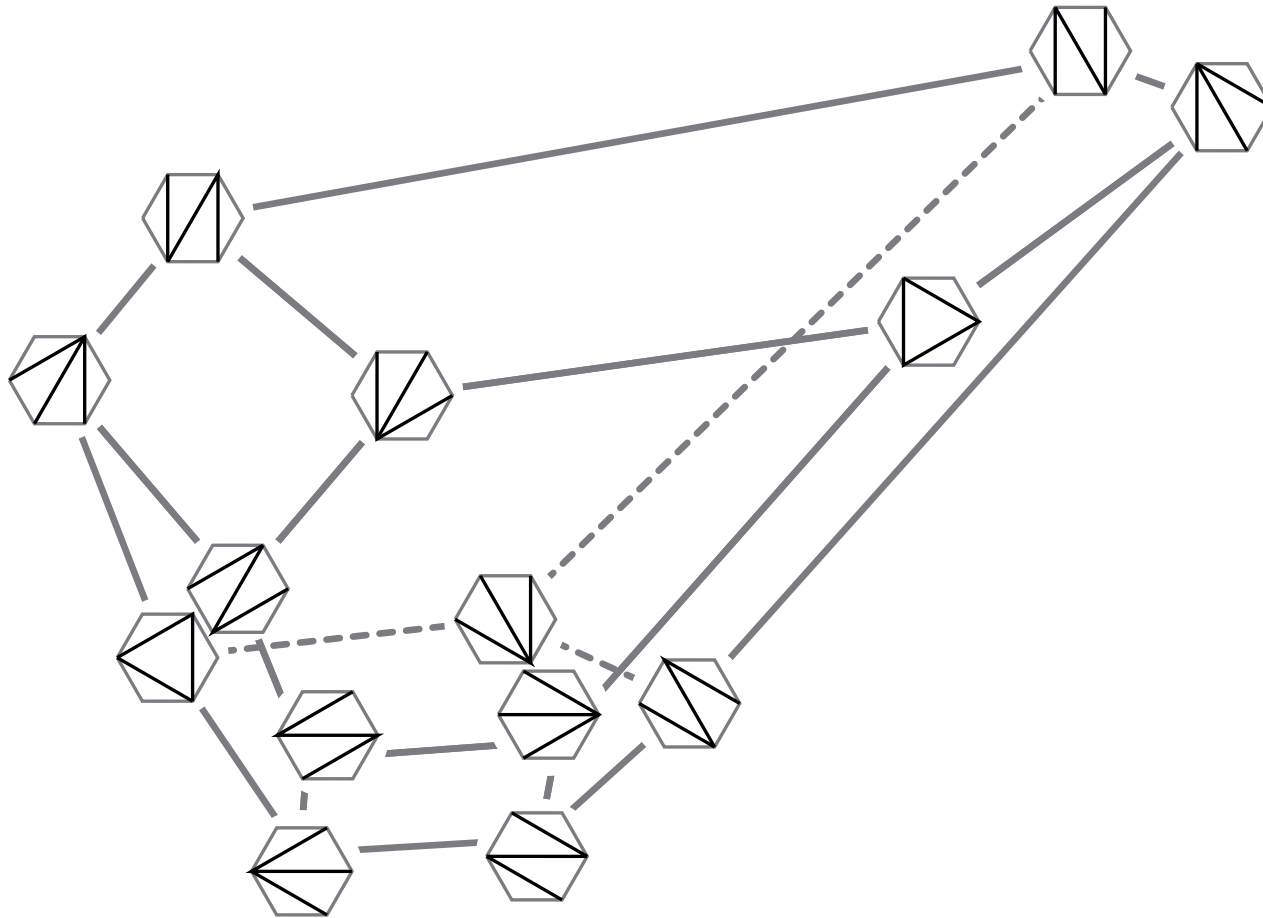
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LODAY'S ASSOCIAHEDRON REVISITED

THEOREM. $\Omega(n)$ is a realization of the $(n - 3)$ -dimensional associahedron.

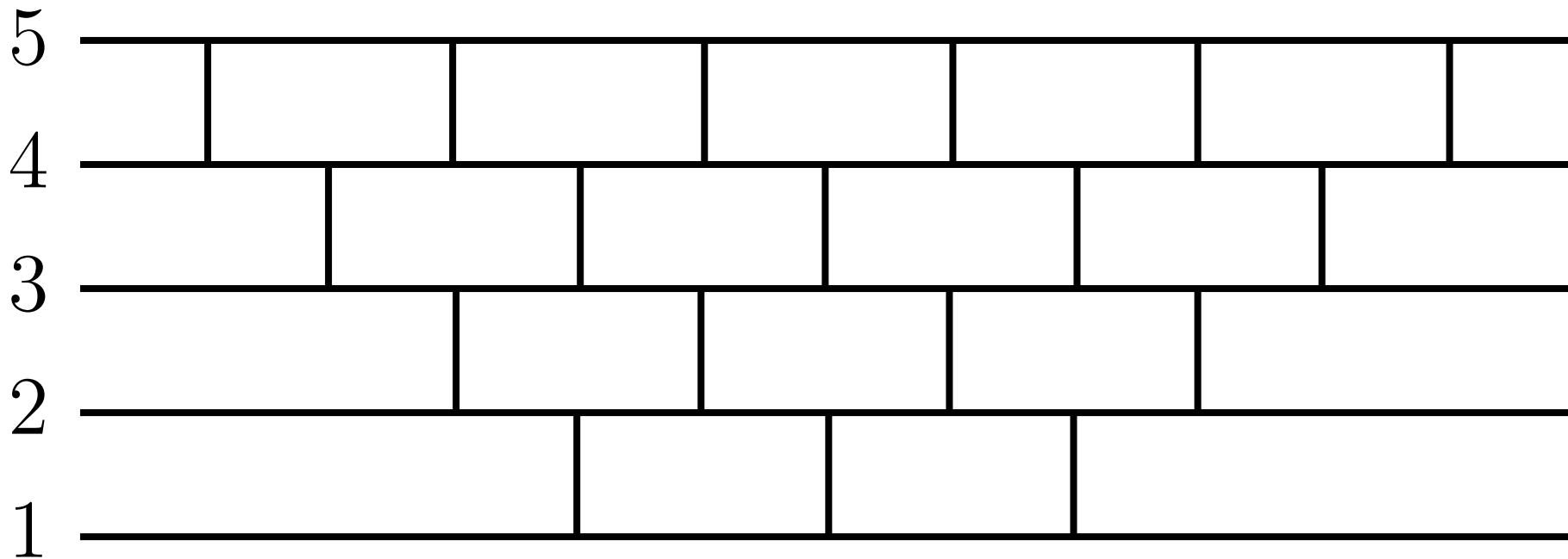


THE BRICK POLYTOPE

\mathcal{S} = sorting network = support of pseudoline arrangements.

Λ pseudoline arrangement supported by \mathcal{S} \mapsto vector $\omega(\Lambda) \in \mathbb{R}^m$.

Brick polytope $\Omega(\mathcal{S}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{S}\}$.

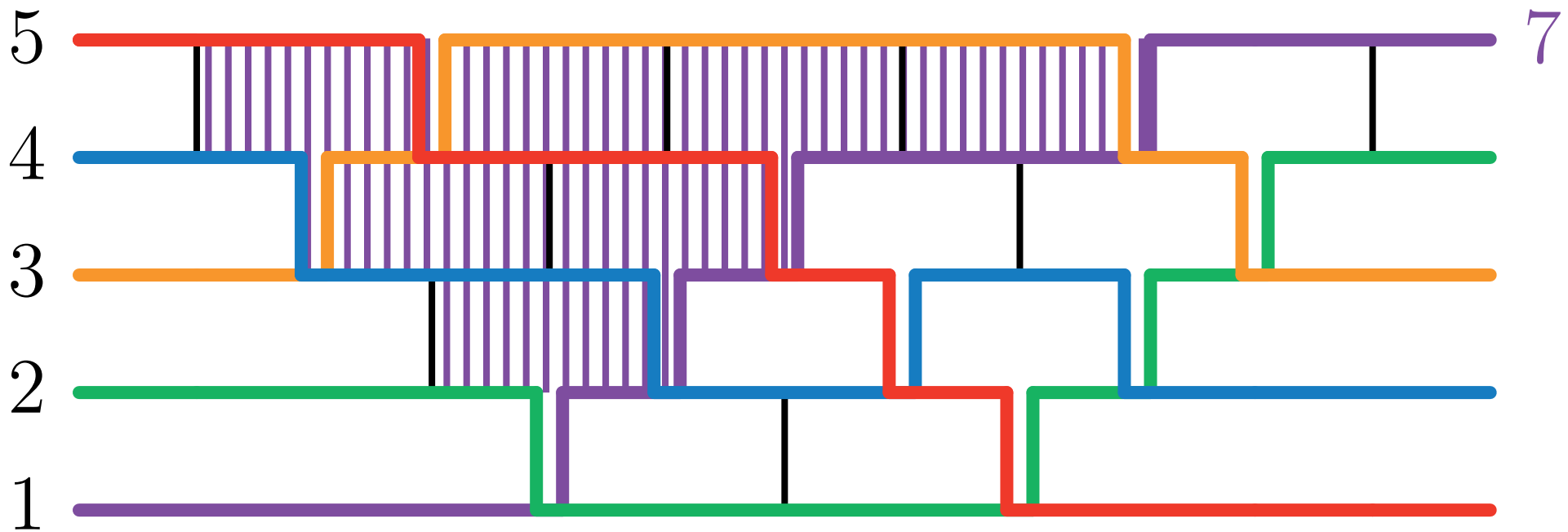


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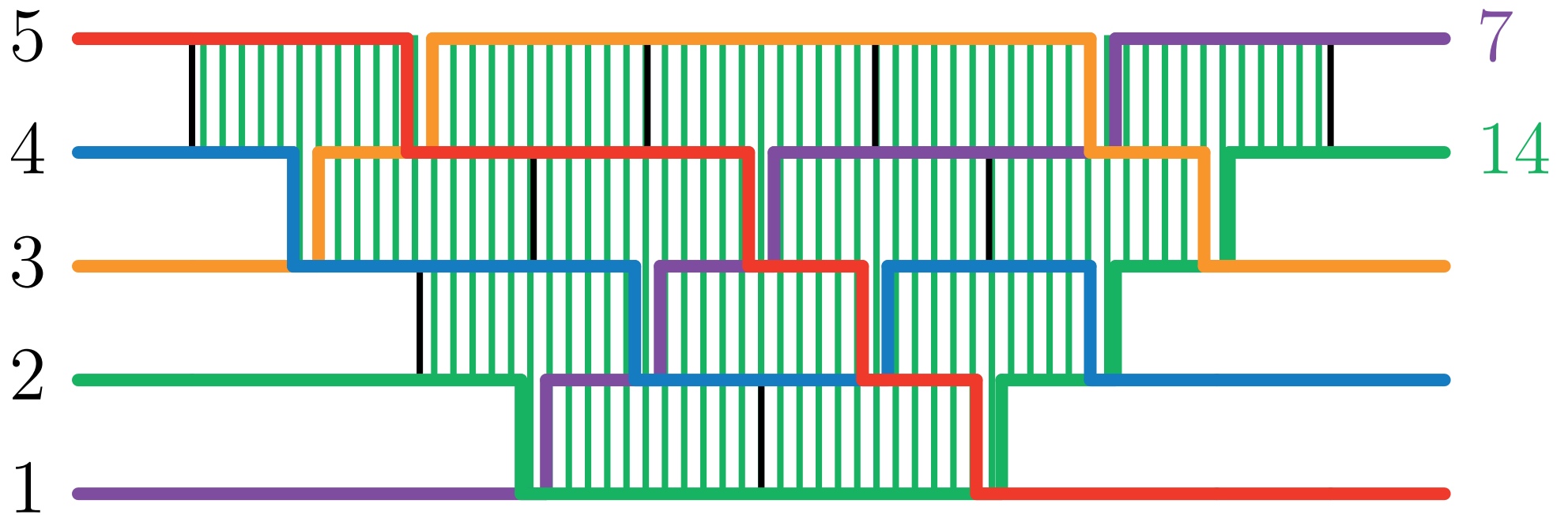
$\omega(\Lambda)_j =$ number of bricks above the j th pseudoline of Λ .

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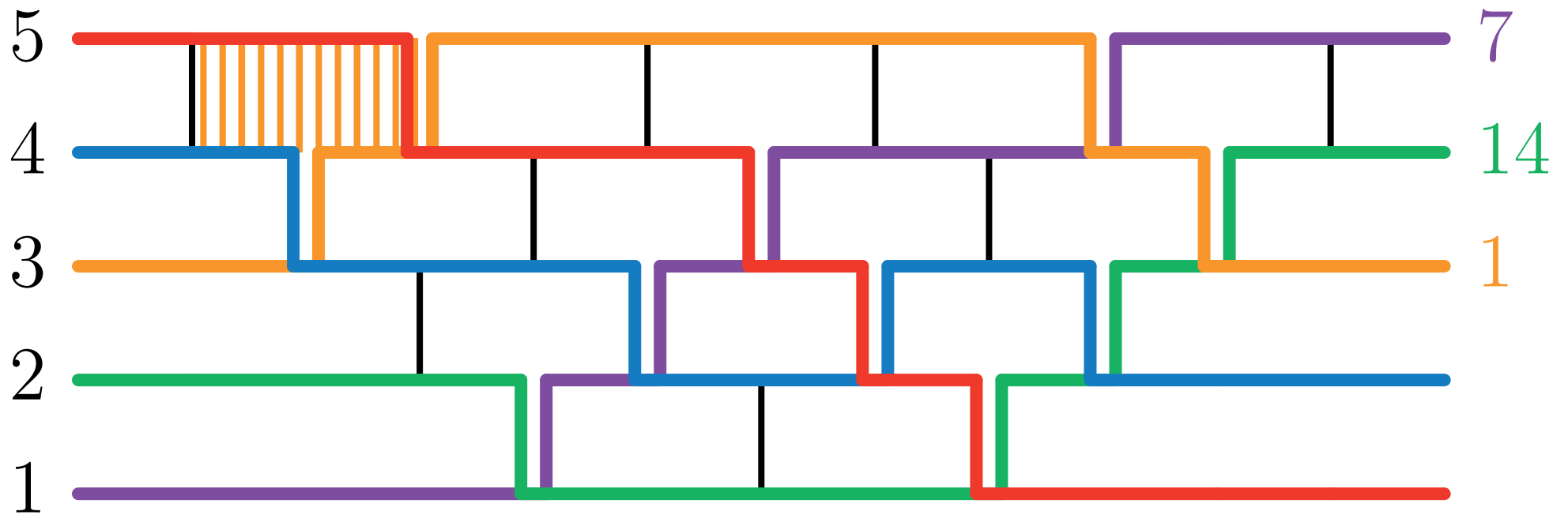
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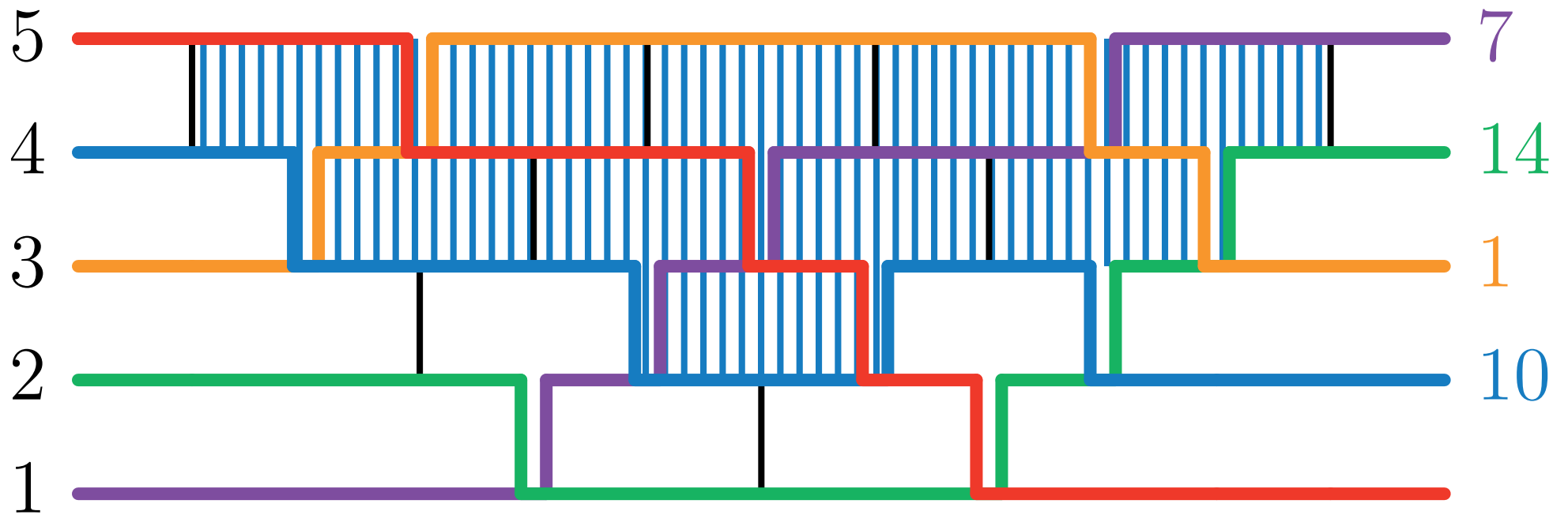
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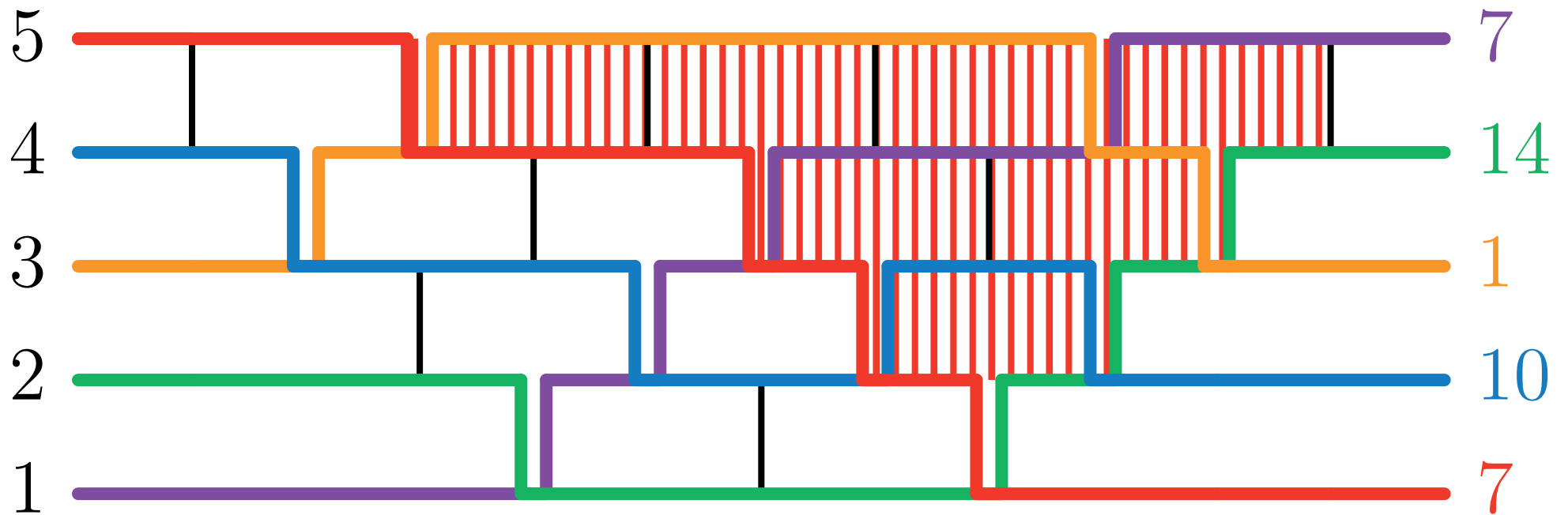
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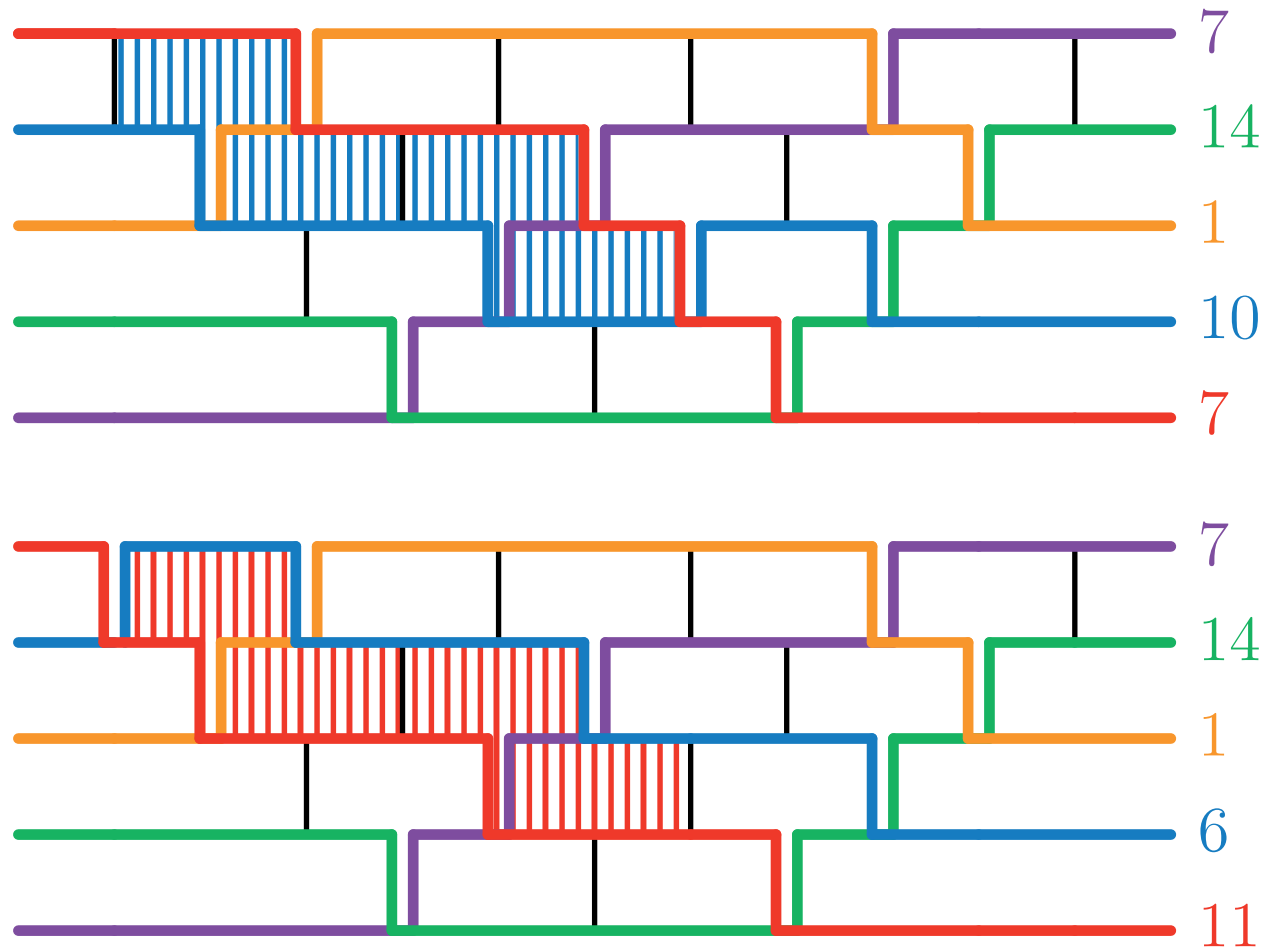
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BRICK VECTORS AND FLIPS



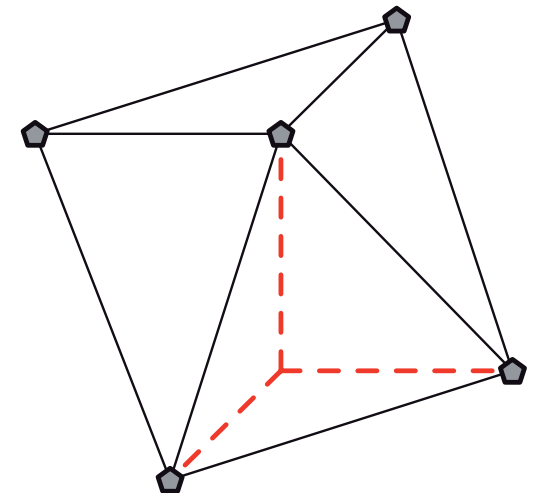
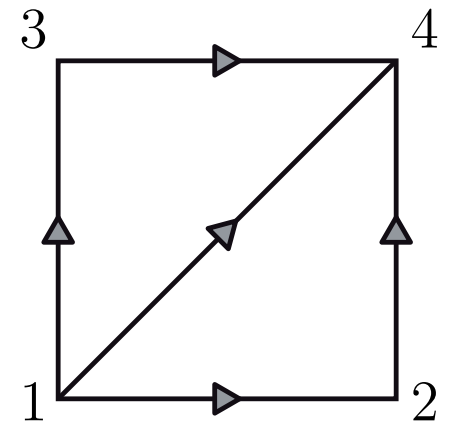
LEMMA. Λ and Λ' related by a flip between their i th and j th pseudolines
 $\Rightarrow \omega(\Lambda) - \omega(\Lambda') = \lambda(e_i - e_j).$

THE INCIDENCE CONE OF A MULTIGRAPH

G oriented (multi)graph \mapsto Incidence configuration $I(G) = \{e_i - e_j \mid (i, j) \in G\}$,
 \mapsto Incidence cone $C(G) = \text{cone generated by } I(G)$.

REMARK. circuits in $I(G) \longleftrightarrow$ simple cycles in G ,
cocircuits in $I(G) \longleftrightarrow$ minimal cuts in G ,
(and signs correspond to the orientations of the edges).

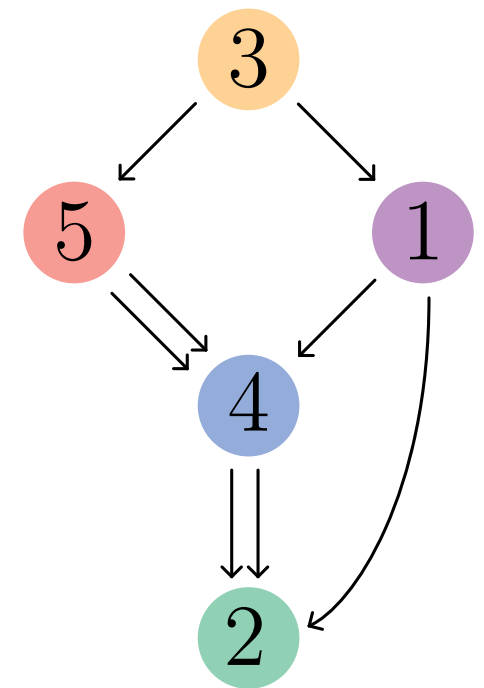
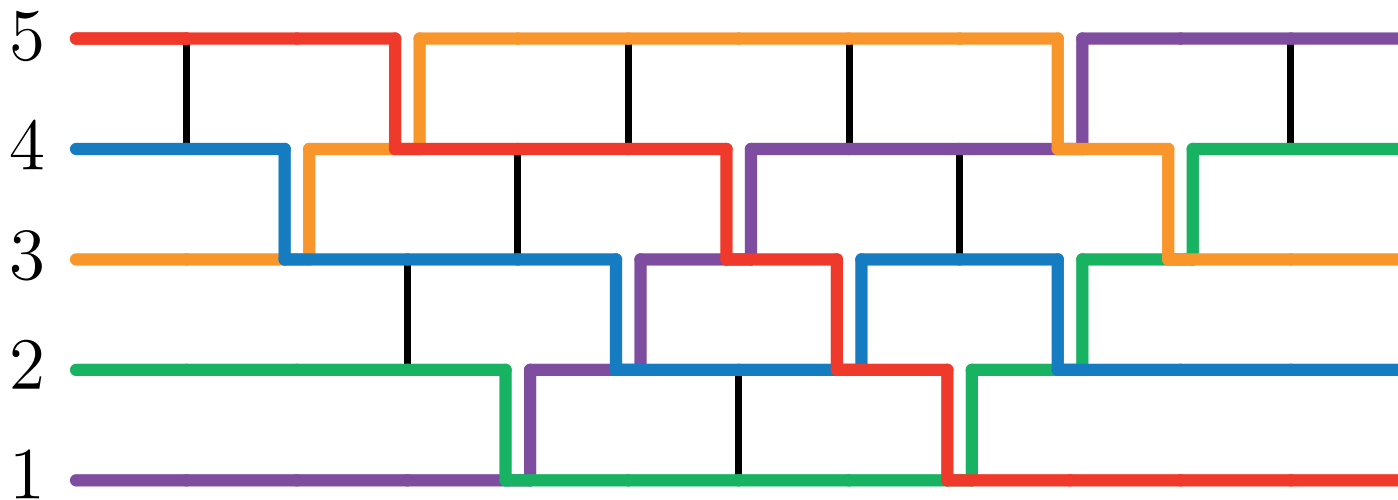
REMARK. $C(G)$ is pointed $\longleftrightarrow G$ is acyclic.
facets of $C(G) \longleftrightarrow$ complements of the
minimal directed cuts of G .



CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

Contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ , and
- an arc for each contact point of Λ oriented from top to bottom.



VERTEX CHARACTERIZATION AND FACET DESCRIPTION

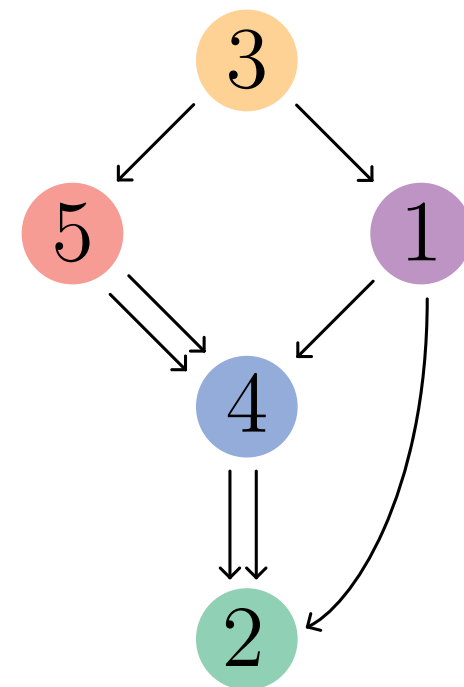
Contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$

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THEOREM. Cone of $\Omega(\mathcal{S})$ at $\omega(\Lambda) =$ incidence cone $C(\Lambda^\#)$.

COROLLARY. $\omega(\Lambda)$ vertex of $\Omega(\mathcal{S}) \iff \Lambda^\#$ acyclic.

COROLLARY. Normal vectors of $\Omega(\mathcal{S}) =$ characteristic vectors of sinks of directed cuts of acyclic contact graphs of pseudoline arrangements supported by \mathcal{S} .



TRIANGULATIONS AND MULTITRIANGULATIONS

T k -triangulation of the n -gon. Then:

$$\begin{aligned}(T^*)^\# &= \text{contact graph of the dual pseudoline arrangement of } T \\ &= \text{dual graph of } T \text{ as complex of } k\text{-stars.}\end{aligned}$$

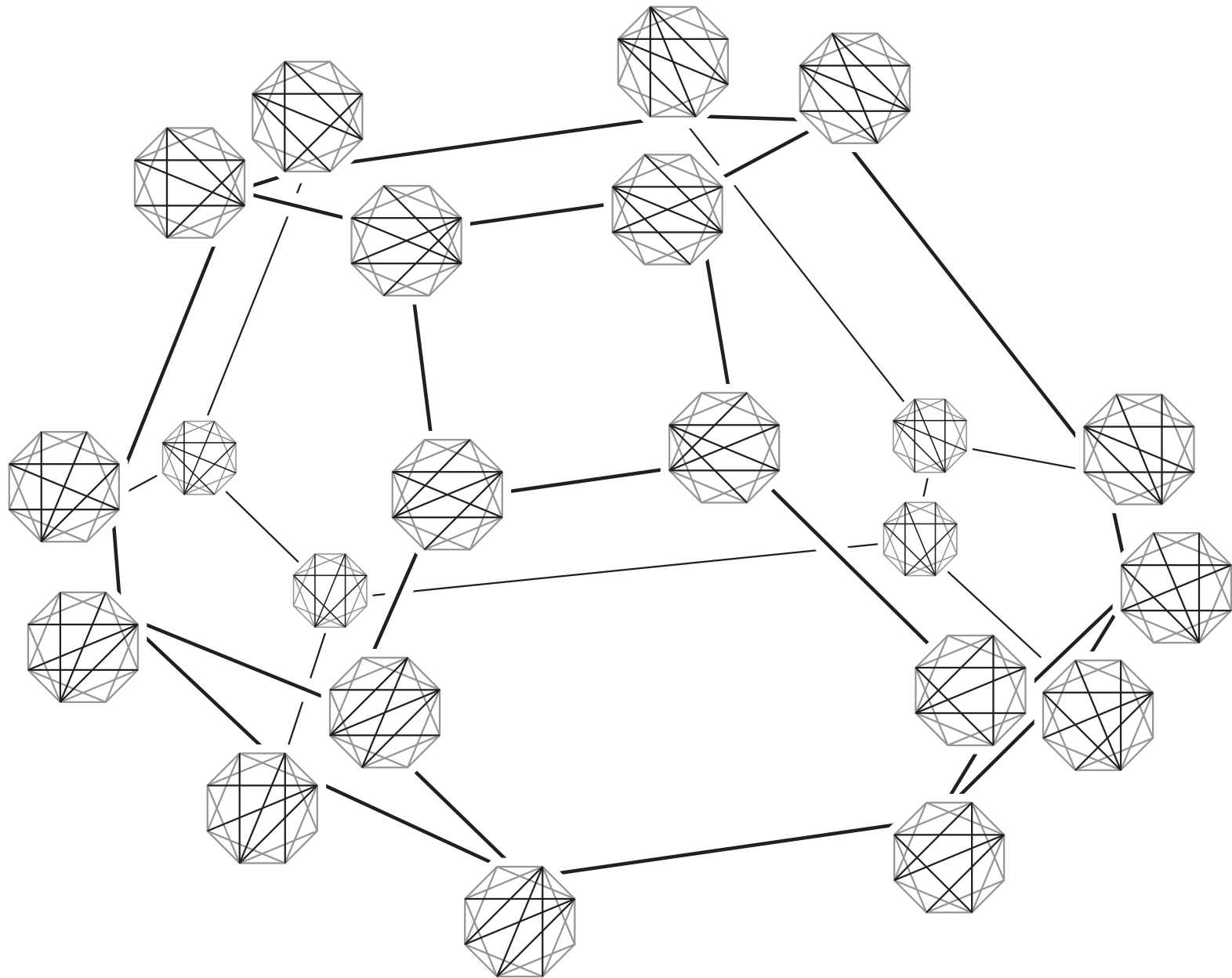
TRIANGULATIONS

1. Up to translation, **Loday's associahedron** = **Brick polytope** of V_n^* minus its first level.
2. Contact graph = dual tree \Rightarrow each triangulation appears as a simple vertex.
3. Normal vectors of facets of $\Omega(n) = \{0^{i-1}1^{j-i-1}0^{n-j} \mid [i, j] \text{ internal diagonal}\}$.

MULTITRIANGULATIONS

1. Not all k -triangulations appear as vertices of $\Omega(V_n^{*k})$, and not all vertices are simple.
2. Normal vectors of facets of $\Omega(V_n^{*k}) =$ 0/1-sequences of length $n - 2k$,
distinct from 0^{n-2k} and 1^{n-2k}
and not containing 10^r1 , for $r \geq k$.

TRIANGULATIONS AND MULTITRIANGULATIONS





Open problems and perspectives

OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths

THEOREM. The number of k -triangulations of the n -gon is

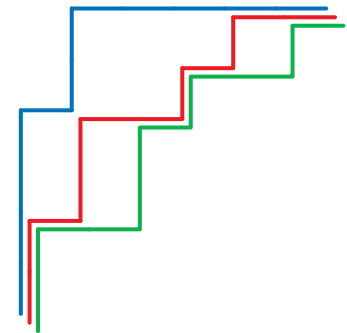
$$\det(C_{n-i-j})_{1 \leq i, j \leq k} = \left| \begin{pmatrix} C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k} \end{pmatrix} \right|,$$

where $C_m = \frac{1}{m+1} \binom{2m}{m}$ is the m th **Catalan number**.

J. Jonsson, *Generalized triangulations and diagonal-free subsets of stack polyominoes*, 2005.

PROBLEM. Find an explicit bijection

k -triangulations \longleftrightarrow k -Dyck paths.



S. Elizalde, *A bijection between 2-triangulations and pairs of non-crossing Dyck paths*, 2006.

C. Nicolas, *Another bijection between 2-triangulations and pairs of non-crossing Dyck paths*, 2009.

OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths
2. Pseudotriangulations and multipseudotriangulations in higher dimension

multipseudotriangulations of 2-dimensional point sets

→ Positivity of the j -depth for all j

→ Lower Bound Theorem for d -polytopes with $d + 3$ vertices

E. Welzl, *Entering and leaving j -facets*, 2001.

PROBLEM. Define (multi)pseudotriangulations in higher dimension.

OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths
2. Pseudotriangulations and multipseudotriangulations in higher dimension
3. Polytopality of flip graphs

\mathcal{S} support of pseudoline arrangements.

$\Delta(\mathcal{S}) =$ simplicial complex whose maximal simplices are the sets of contact points of pseudoline arrangements supported by \mathcal{S} .

PROBLEM. Is $\Delta(\mathcal{S})$ the boundary complex of a polytope?

Remark:

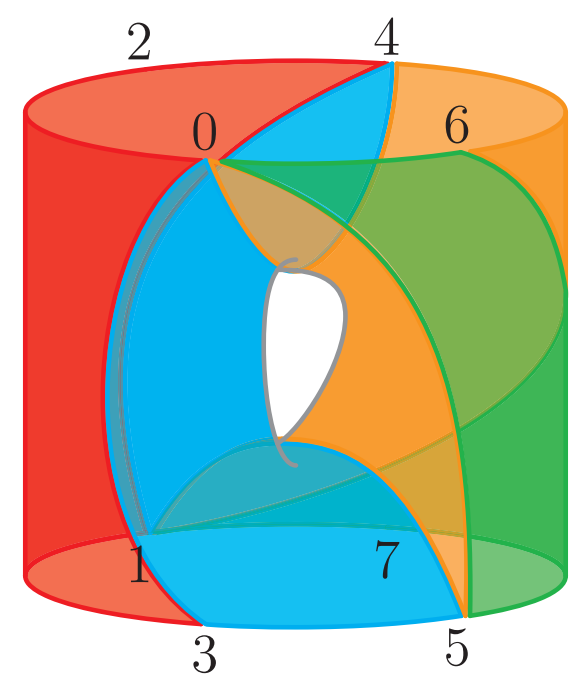
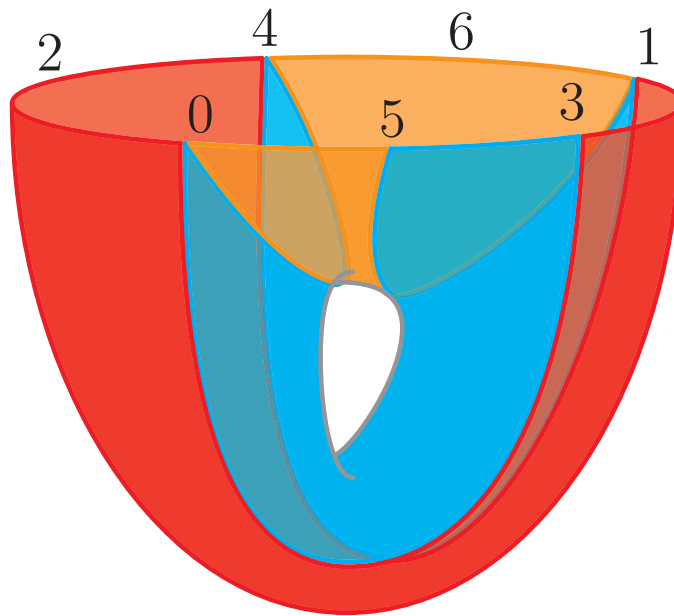
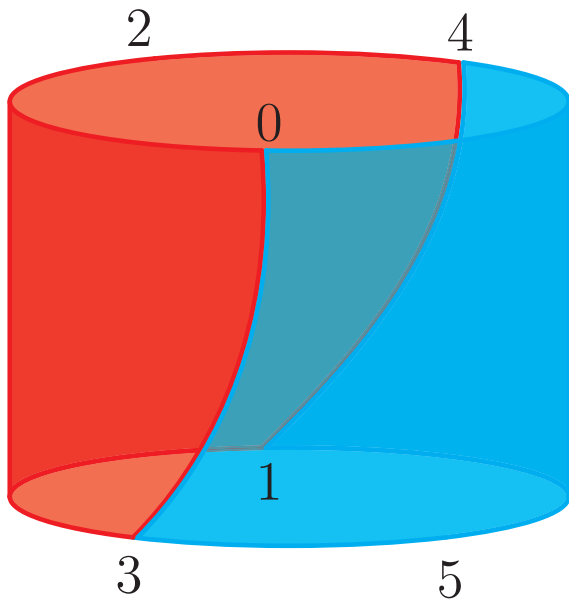
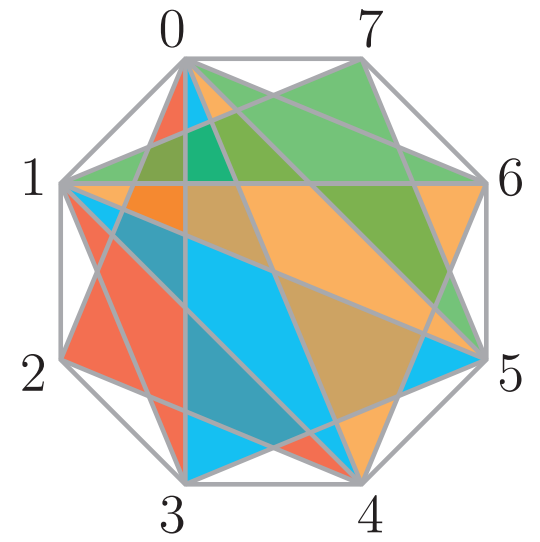
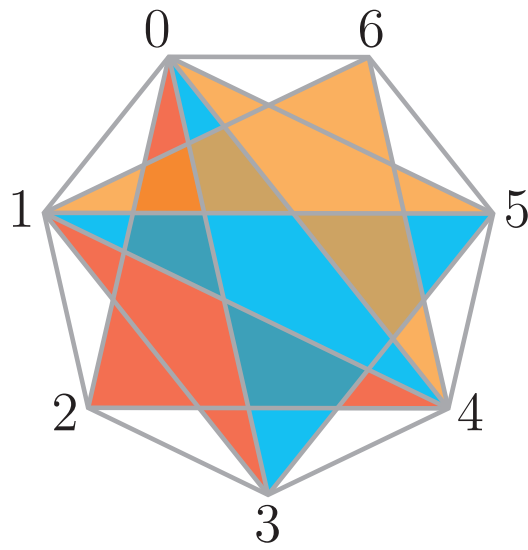
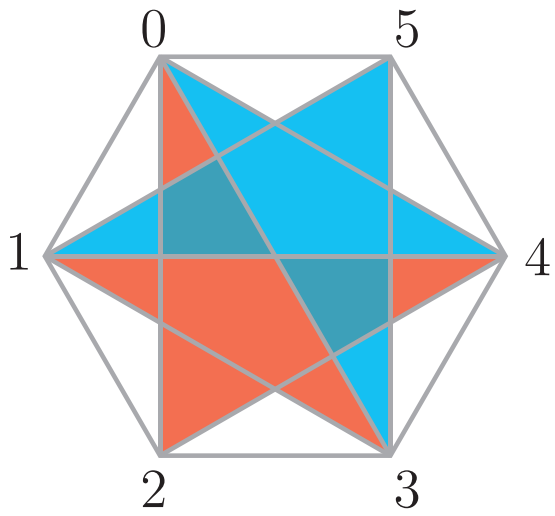
- Multitriangulations are **universal**.
- First open case: pseudotriangulations of non-realizable pseudoline arrangements.

Thank you

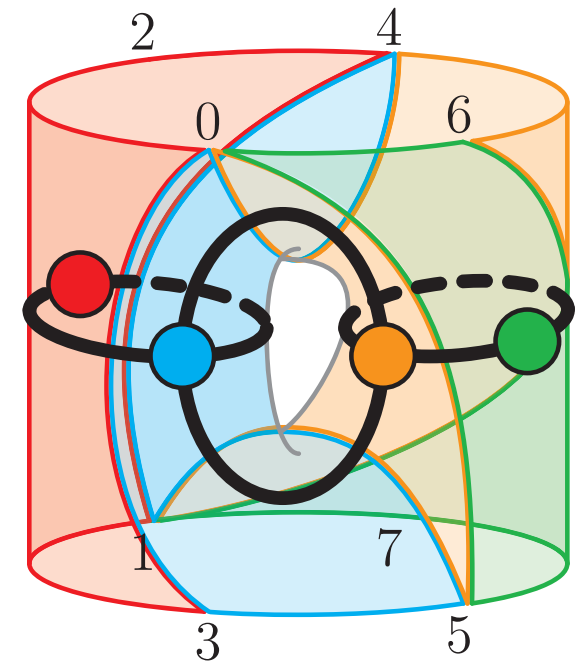
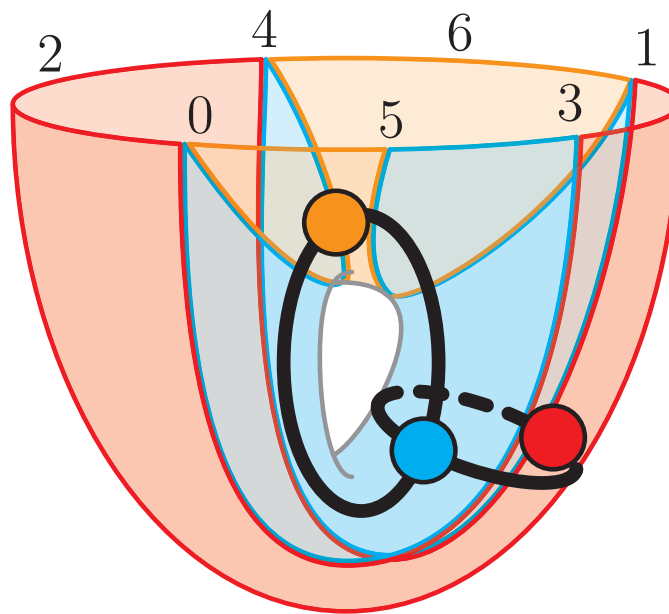
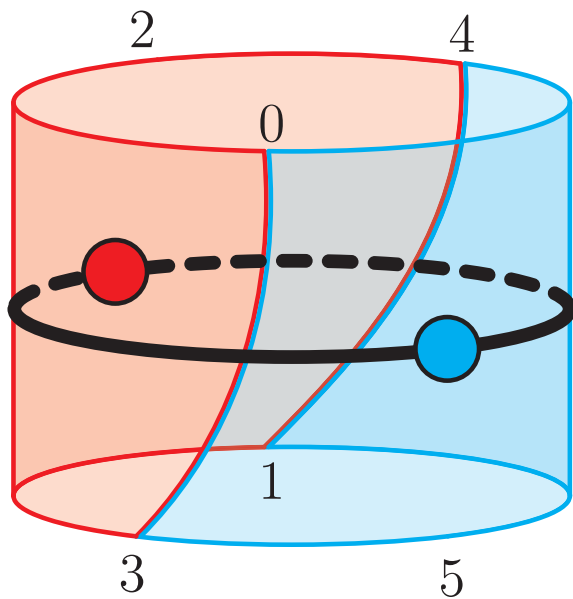
Questions

You will ask about that, right?

MAPS ON SURFACES



MAPS ON SURFACES

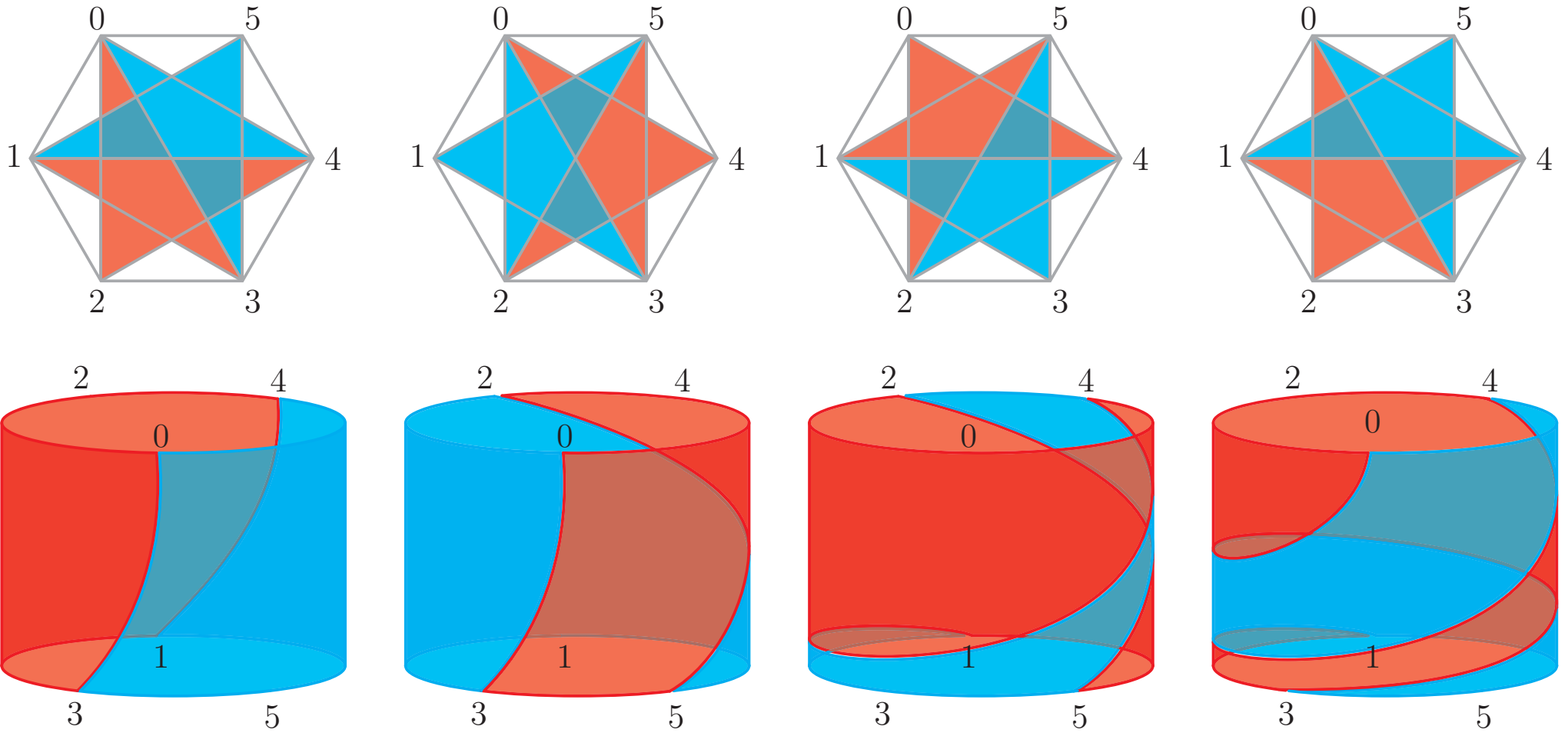


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MAPS ON SURFACES



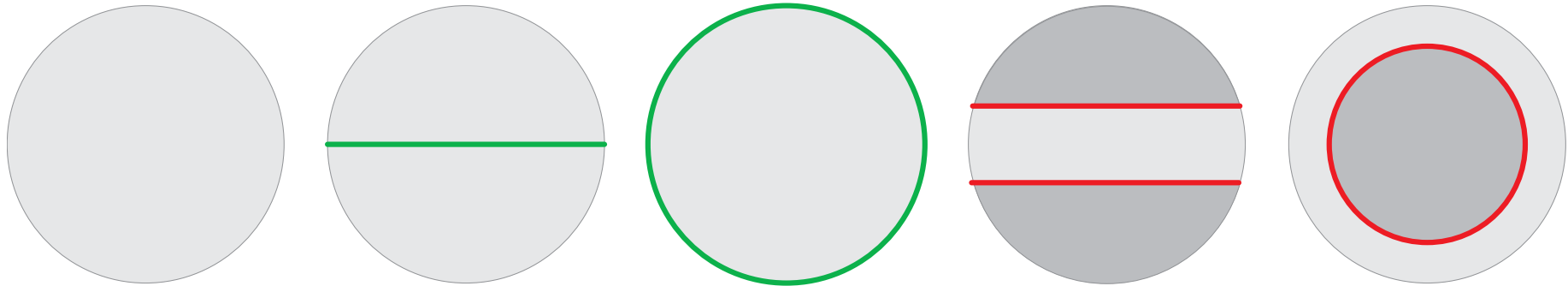
fundamental group of the flip graph $G_{n,k} \longmapsto$ mapping class group of the surface $\mathcal{S}_{n,k}$

ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS

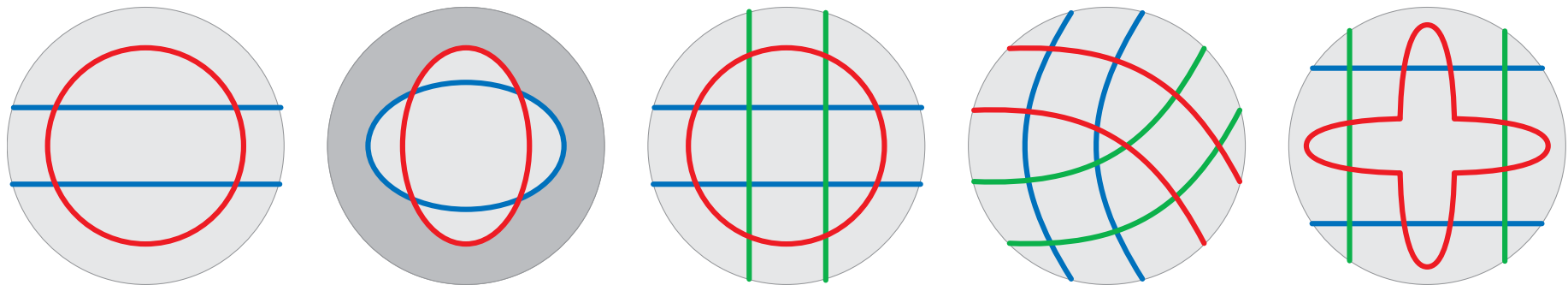
\mathcal{P} = projective plane = disk with antipodal boundary points identified.

pseudoline = non-separating simple closed curve

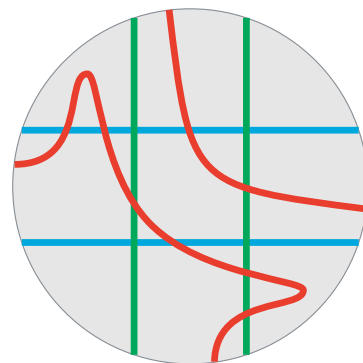
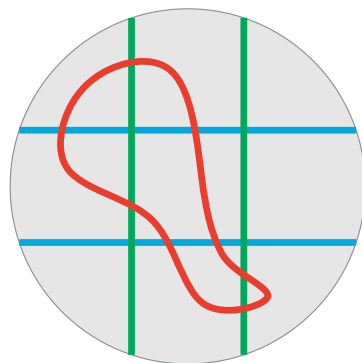
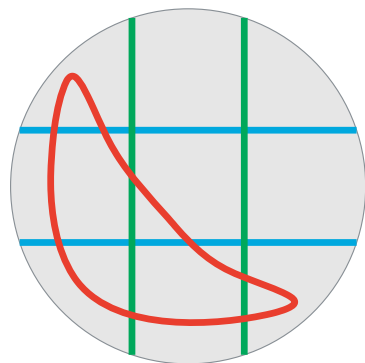
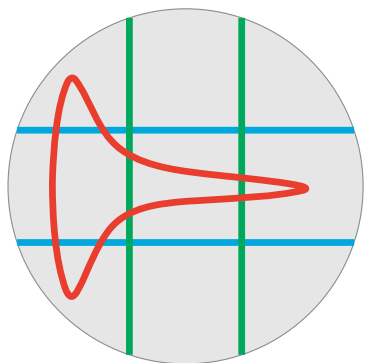
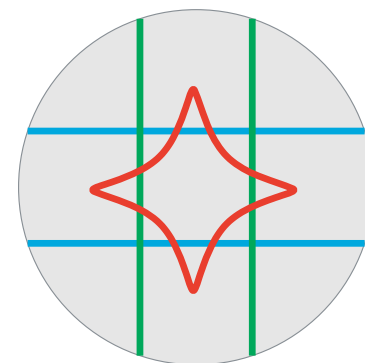
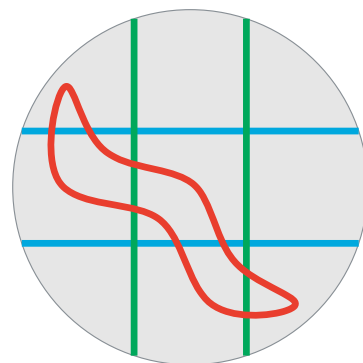
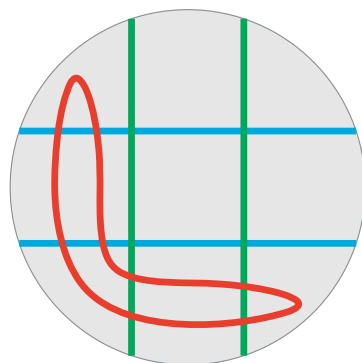
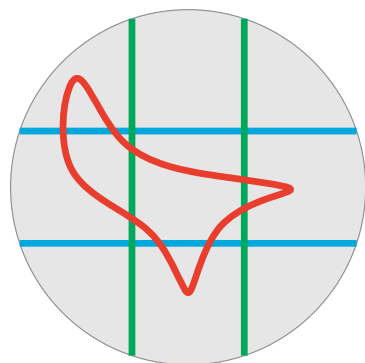
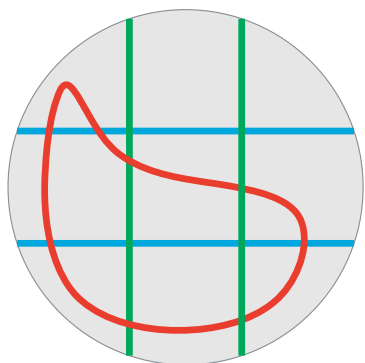
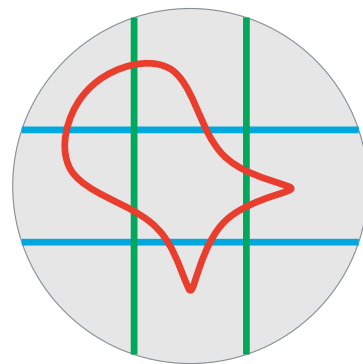
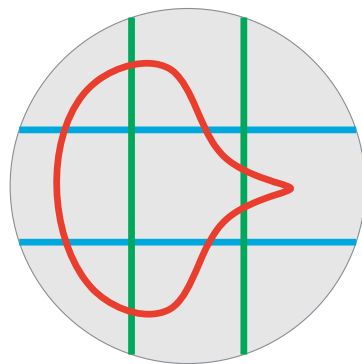
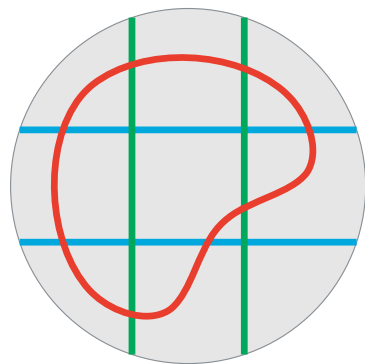
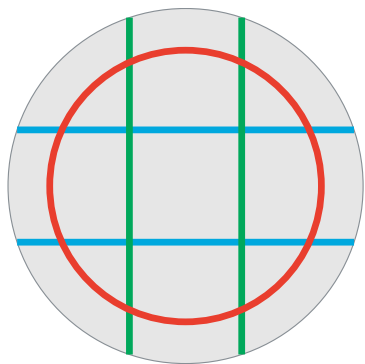
double pseudoline = separating simple closed curve



double pseudoline arrangement = finite set of double pseudolines such that any two of them cross in exactly four points, transversally at these points and induce a cell decomposition of \mathcal{P} .



ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS



ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS

Number of arrangements with n pseudolines and m double pseudolines:

	0	1	2	3	4	5
0			1	13	6 570	181 403 533
1		1	4	626	4 822 394	
2	1	2	48	86 715		
3	1	5	1 329			
4	1	25	80 253			
5	1	302				
6	4	9 194				
7	11	556 298				
8	135					
9	4 382					
10	312 356					

ENUMERATION OF SYMMETRIC REALIZATIONS OF $\Delta_{8,2}$

USE SYMMETRY

\mathbb{D}_n = dihedral group = isometries of the regular n -gon

Natural action of \mathbb{D}_n on $\Delta_{n,k}$:
$$\begin{array}{ccc} \mathbb{D}_n \times \Delta_{n,k} & \longrightarrow & \Delta_{n,k} \\ (\rho, E) & \longmapsto & \rho E = \{\rho e \mid e \in E\} \end{array}$$

DECOMPOSE INTO TWO STEPS

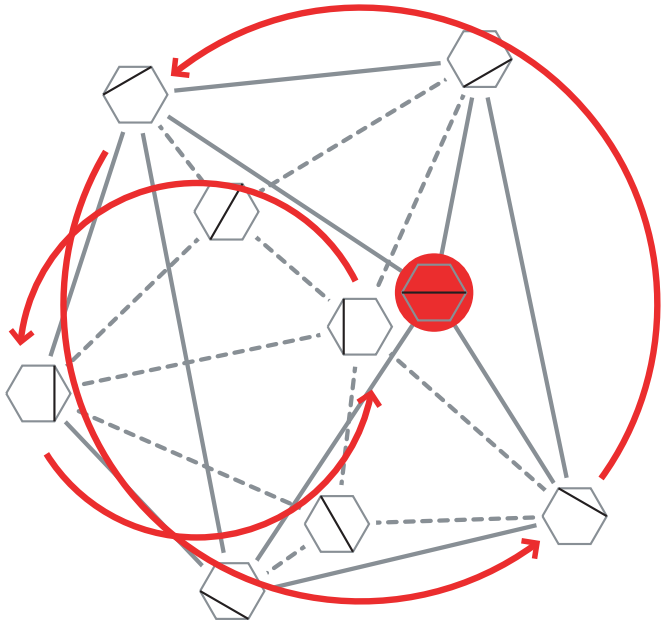
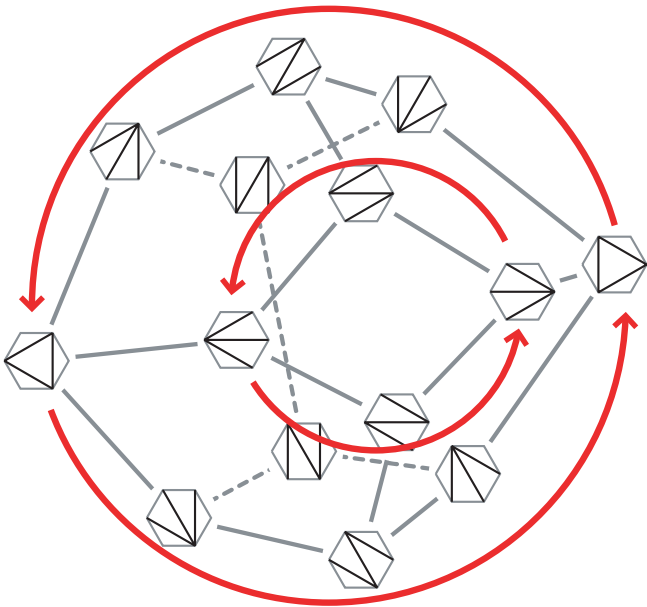
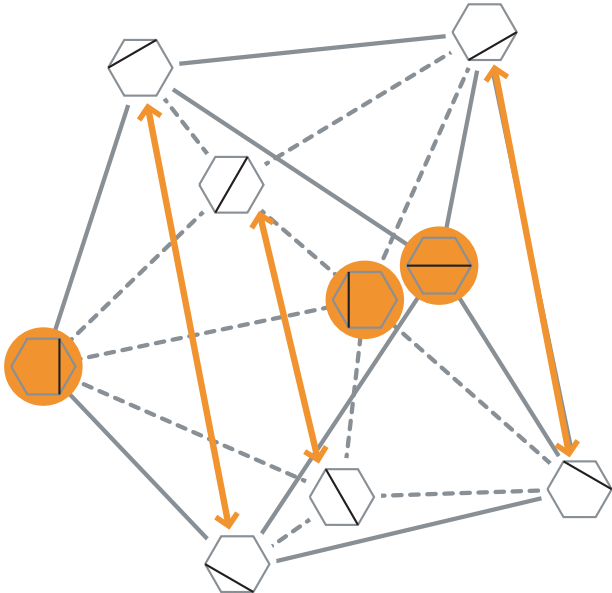
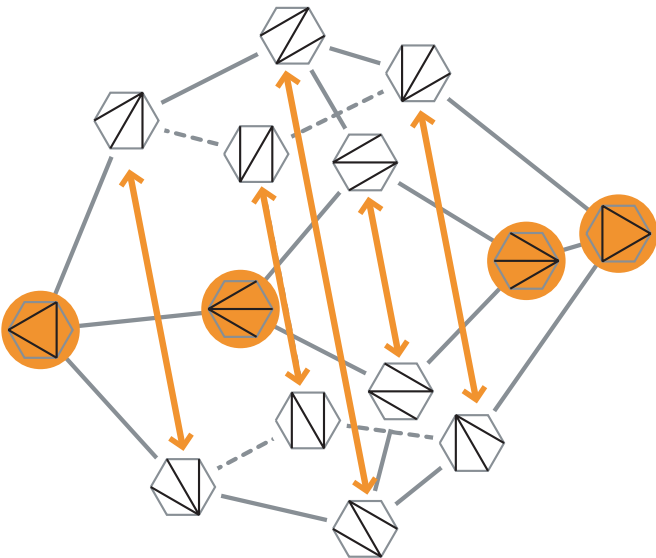
1. From face lattice to oriented matroids

Find all possible symmetric oriented matroids realizing $\Delta_{n,k}$

2. From oriented matroids to polytopes

Deduce the space of symmetric realizations of $\Delta_{n,k}$

SYMMETRY



FROM FACE LATTICE TO ORIENTED MATROIDS

Δ a simplicial complex with an action of a group G

$P \subset \mathbb{R}^d$ a realization of Δ symmetric under G , and V its vertex set

$$\sigma : \left[\begin{array}{ll} V^{d+1} & \longrightarrow \{-1, 0, +1\} \\ (v_0, v_1, \dots, v_d) & \longmapsto \begin{array}{l} \text{orientation of the simplex} \\ \text{spanned by } v_0, v_1, \dots, v_d \end{array} \end{array} \right] = \text{sign det} \begin{pmatrix} v_0 & v_1 & \dots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

FROM FACE LATTICE TO ORIENTED MATROIDS

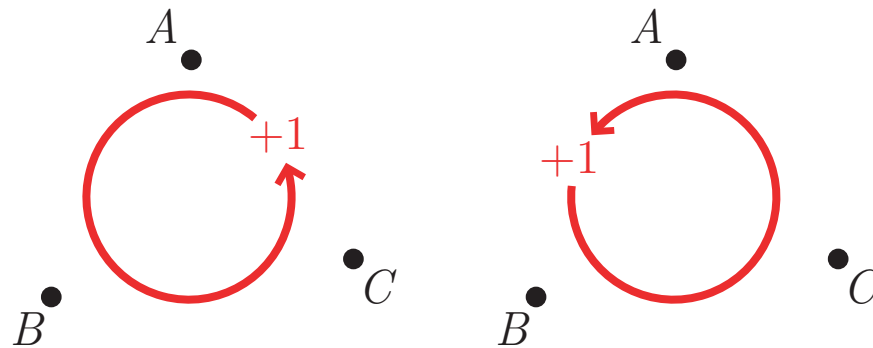
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satisfies the relations:

(i) **Alternating relations**



FROM FACE LATTICE TO ORIENTED MATROIDS

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satisfies the relations:

- (i) Alternating relations
- (ii) Grassmann-Plucker relations

$$\text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} = 0$$

FROM FACE LATTICE TO ORIENTED MATROIDS

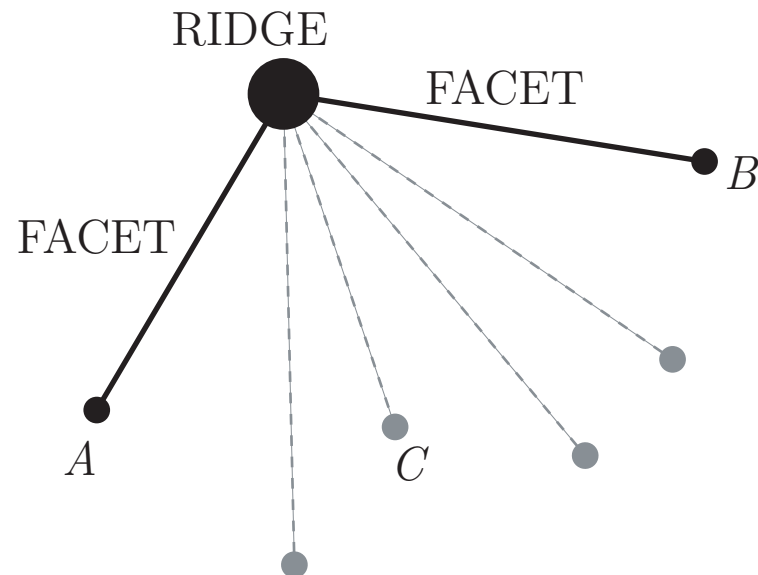
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$P \subset \mathbb{R}^d$ a realization of Δ symmetric under G , and V its vertex set

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satisfies the relations:

- (i) Alternating relations
- (ii) Grassmann-Plucker relations
- (iii) Necessary simplex orientations



FROM FACE LATTICE TO ORIENTED MATROIDS

Δ a simplicial complex with an action of a group G

$P \subset \mathbb{R}^d$ a realization of Δ symmetric under G , and V its vertex set

$$\sigma : \begin{cases} V^{d+1} & \longrightarrow \{-1, 0, +1\} \\ (v_0, v_1, \dots, v_d) & \longmapsto \begin{array}{l} \text{orientation of the simplex} \\ \text{spanned by } v_0, v_1, \dots, v_d \end{array} \end{cases} = \text{sign det} \begin{pmatrix} v_0 & v_1 & \dots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

satisfies the relations:

- (i) Alternating relations
- (ii) Grassmann-Plucker relations
- (iii) Necessary simplex orientations
- (iv) Symmetry

FROM ORIENTED MATROIDS TO POLYTOPES

Problem. For a given oriented matroid, find a matrix representing it or a proof that such a matrix is impossible to find.

“On the one hand, there is a general algorithm to solve this problem. On the other hand, it is known that this algorithm from real algebraic geometry is far from applicable for our cases in the theory of oriented matroids.”

J. Bokowski, Computational Oriented Matroids, 2006

⇒ USE HEURISTICAL METHODS

Our heuristic is symmetry

THE SPACE OF SYMMETRIC REALIZATIONS OF $\Delta_{8,2}$

Proposition. The space of symmetric realizations of $\Delta_{8,2}$ has dimension 4.

Example. With some arbitrary values of the 4 parameters, we obtain a particular symmetric realization of $\Delta_{8,2}$:

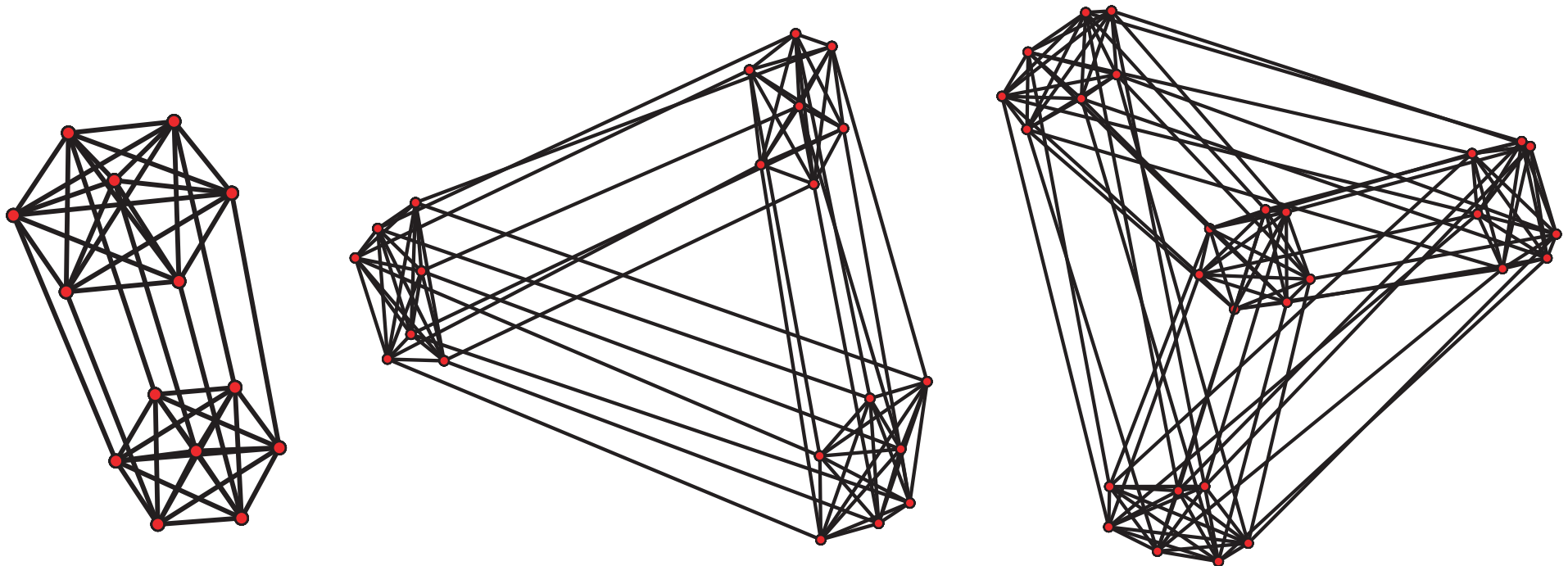
$$M \simeq \begin{pmatrix} 0.21 & 0 & 0 & -0.21 & 0.52 & -0.74 & 0.74 & -0.52 & 0 & 0 & 0 & 0 \\ -0.95 & 0.66 & 0 & -0.63 & 0.8 & -0.4 & -0.3 & 0.68 & 0 & 0 & 0 & 0 \\ -0.17 & 0.75 & -1 & 0.77 & -0.21 & -0.4 & 0.60 & -0.34 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0.55 & 0.55 & -0.55 & -0.55 & 0.5 & 0.7 & -0.7 & -0.4 & 1 & 0 & -1 & 0 \\ -0.55 & 0.55 & 0.55 & -0.4 & -0.7 & 0.7 & 0.5 & -0.55 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

```
> polymake multiassociahedron82 F_VECTOR
F_VECTOR
12 66 192 306 252 84
```

POLYTOPALITY OF PRODUCTS OF NON-POLYTOPAL GRAPHS

Cartesian product of polytopes: $P \times Q := \{(p, q) \mid p \in P, q \in Q\}$.

Cartesian product of graphs:
$$\begin{cases} V(G \times H) := V(G) \times V(H), \\ E(G \times H) := (V(G) \times E(H)) \cup (E(G) \times V(H)). \end{cases}$$



REMARK. graph of $P \times Q = (\text{graph of } P) \times (\text{graph of } Q)$.

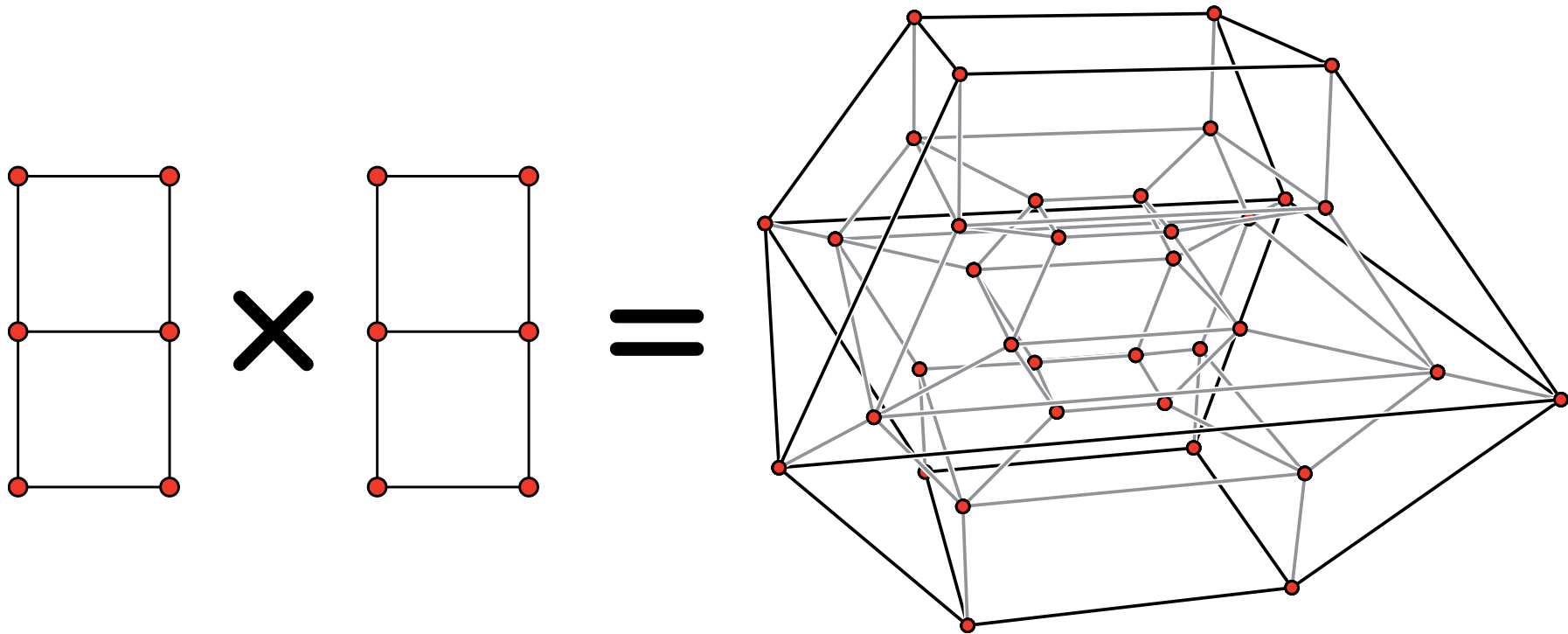
PROBLEM. Does the polytopality of $P \times Q$ imply that of P and Q ?

POLYTOPALITY OF PRODUCTS OF NON-POLYTOPAL GRAPHS

PROBLEM. Does the polytopality of $P \times Q$ imply that of P and Q ?

THEOREM. $G \times H$ simple polytopal $\iff G$ and H simply polytopal.

THEOREM. The product of a d -polytopal graph by the graph of a regular subdivision of an e -polytope is $(d + e)$ -polytopal.



PRODSIMPLICIAL NEIGHBORLY POLYTOPES

$k \geq 0$ and $\underline{n} := (n_1, \dots, n_r)$.

A polytope is (k, \underline{n}) -prodsimplicial-neighborly if its k -skeleton is combinatorially equivalent to that of the product of simplices $\Delta_{\underline{n}} := \Delta_{n_1} \times \dots \times \Delta_{n_r}$.

EXAMPLE.

(i) neighborly polytopes arise when $r = 1$.

For example, the cyclic polytope $C_{2k+2}(n+1)$ is (k, n) -PSN.

(ii) neighborly cubical polytopes arise when $\underline{n} = (1, 1, \dots, 1)$.

M. Joswig and G. Ziegler, Neighborly cubical polytopes, 2000.

PROBLEM. What is the minimal dimension of a (k, n) -PSN polytope?

PRODSIMPLICIAL NEIGHBORLY POLYTOPES

CONSTRUCTIONS

- (i) products of cyclic polytopes.
- (ii) reflections of cyclic polytopes.
- (iii) Minkowski sums of cyclic polytopes.
- (iv) projections of deformed products of polytopes.

OBSTRUCTIONS

A (k, \underline{n}) -PSN polytope is (k, \underline{n}) -projected-prodsimplicial-neighborly if it is a projection of a polytope combinatorially equivalent to $\Delta_{\underline{n}}$.

Sanyal's topological obstruction method:

Projection preserving the k -skeleton of $\Delta_{\underline{n}}$

- ⟶ simplicial complex embeddable in a certain dimension (Gale duality)
- ⟶ topological obstruction (Sarkaria's criterion).

B. Matschke, J. Pfeifle, and V. P., Prodsimplicial neighborly polytopes, 2010.

PROGRAM

2pm. Room 0C05. Thesis defense.

3pm. Room 0C08. Pot de thèse.

Be careful, scientific program is not over yet...

6pm. Room 0C05. FRANCISCO SANTOS disproves the Hirsch Conjecture !!!!!

7pm. Room 0C08. Back to the pot.

The scientific program is over now...