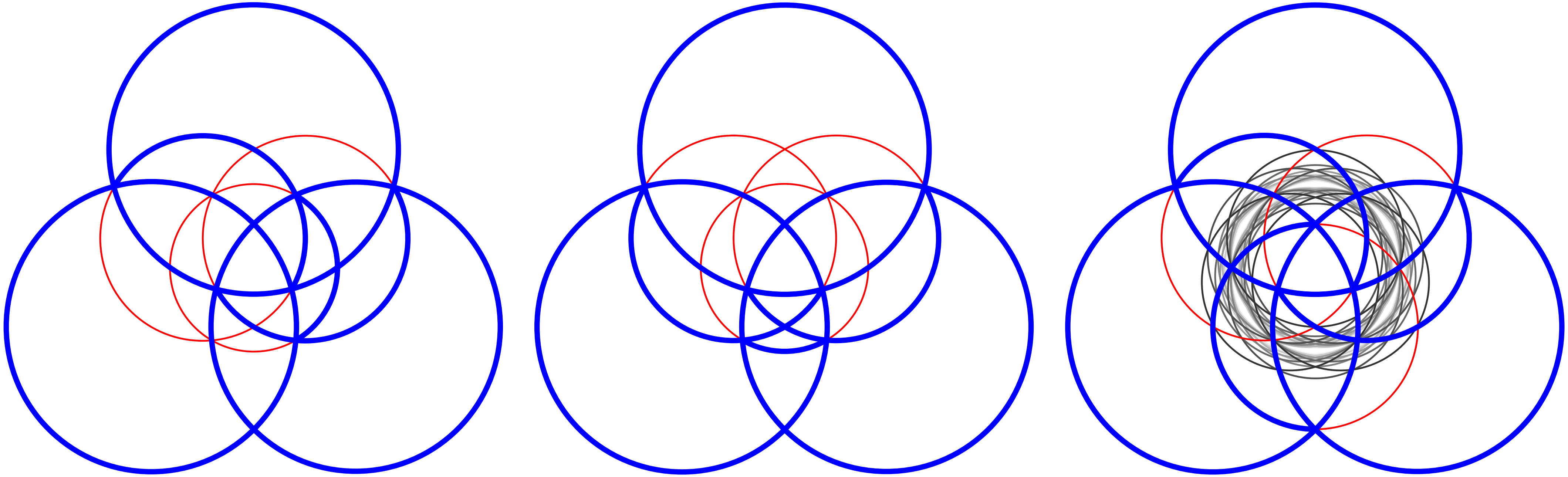
III. UNIVERSAL ASSOCIAHEDRON & NON-KISSING ASSOCIAHEDRON



V. PILAUD (Universitat de Barcelona)

Osnabrück, Tuesday February 25th, 2025

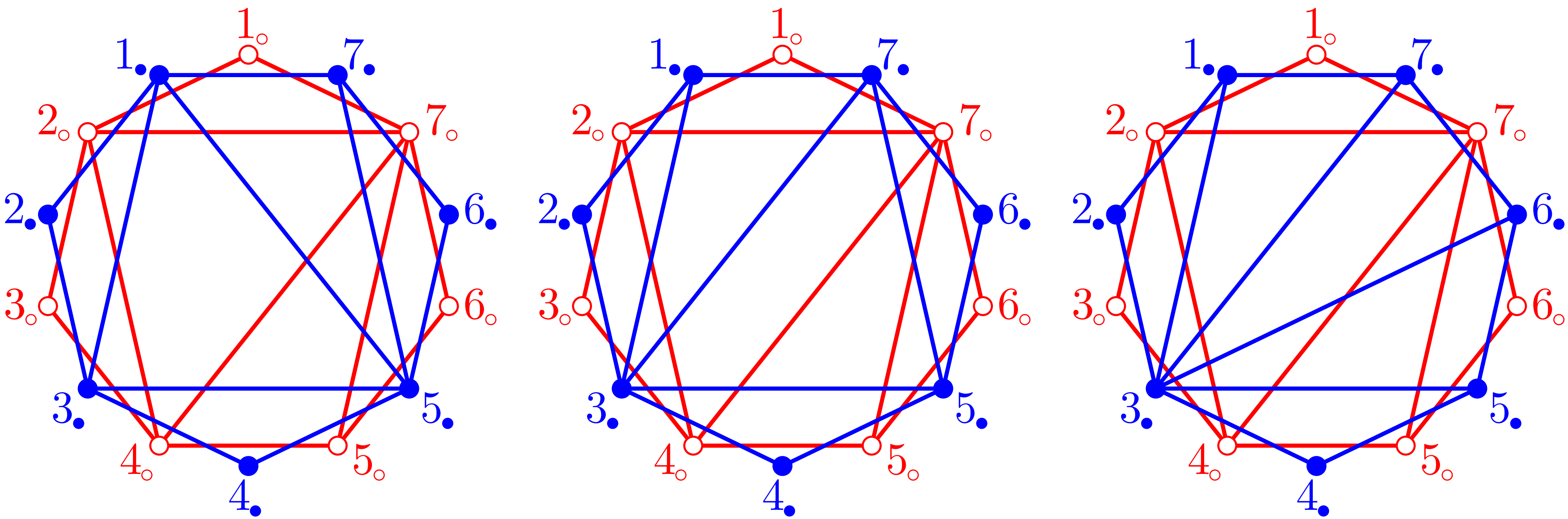
slides available at: https://www.ub.edu/comb/vincentpilaud/documents/presentations/Osnabruck/3.pdf

g- AND c-VECTORS

TWO POLYGONS

Consider simultaneously two n-gons:

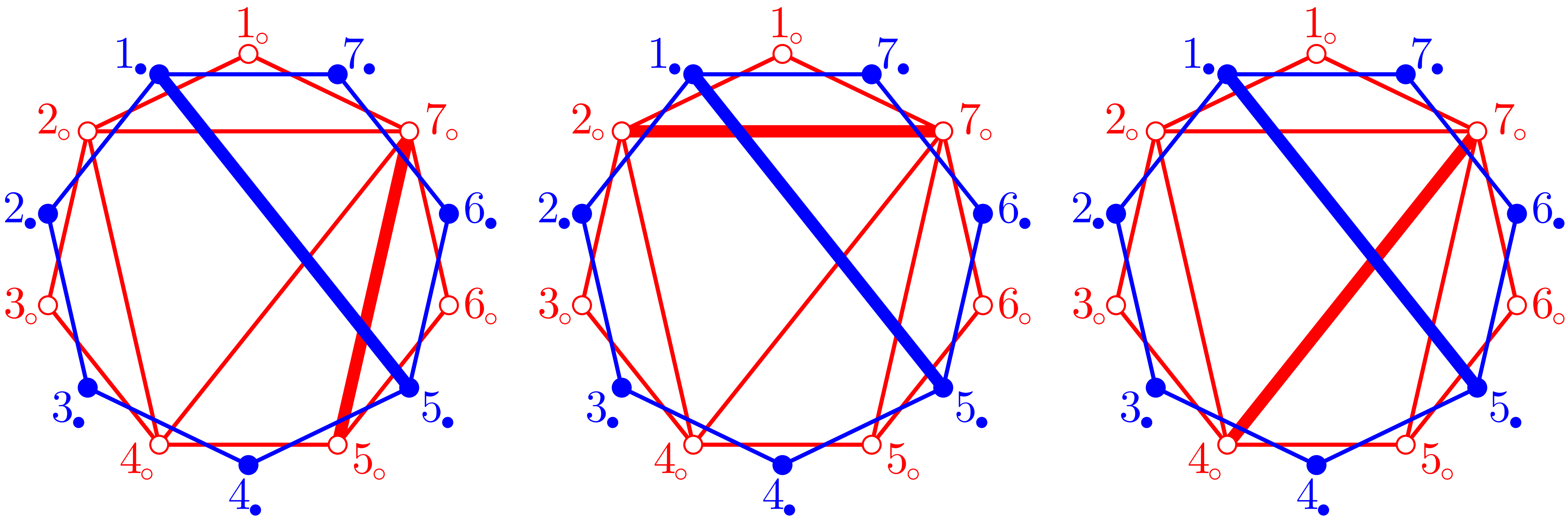
- the red polygon supports a reference triangulation,
- the blue polygon is the ground set.



G-VECTORS

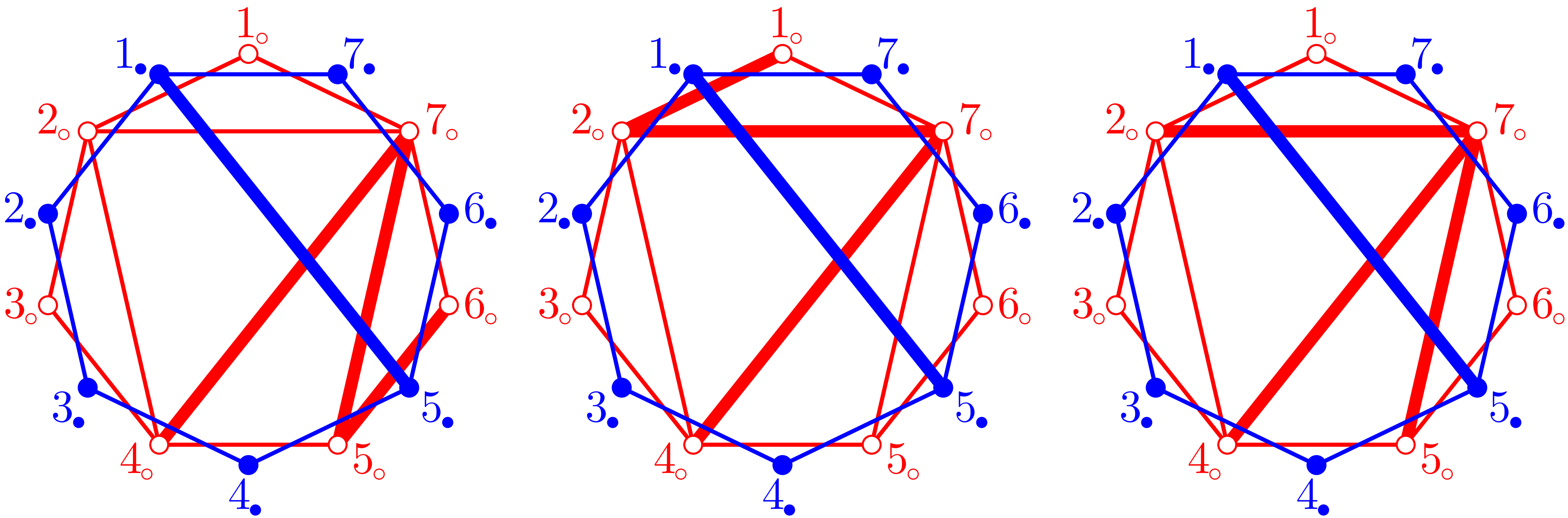
For $\mathrm{T}_\circ$ red triangulation, $\delta_\circ \in \mathrm{T}_\circ$ and $\delta_\bullet$ a blue diagonal, let

$$\varepsilon_\circ(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as an Ƨ} \\ 0 & \text{otherwise} \end{cases}$$



For $\mathrm{T}_\circ$ red triangulation, $\delta_\circ \in \mathrm{T}_\circ$ and $\delta_\bullet$ a blue diagonal, let

$$\varepsilon_\circ(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as an } \Sigma \\ 0 & \text{otherwise} \end{cases}$$



Z $= 1$ $\Sigma = -1$ $\forall = 0$

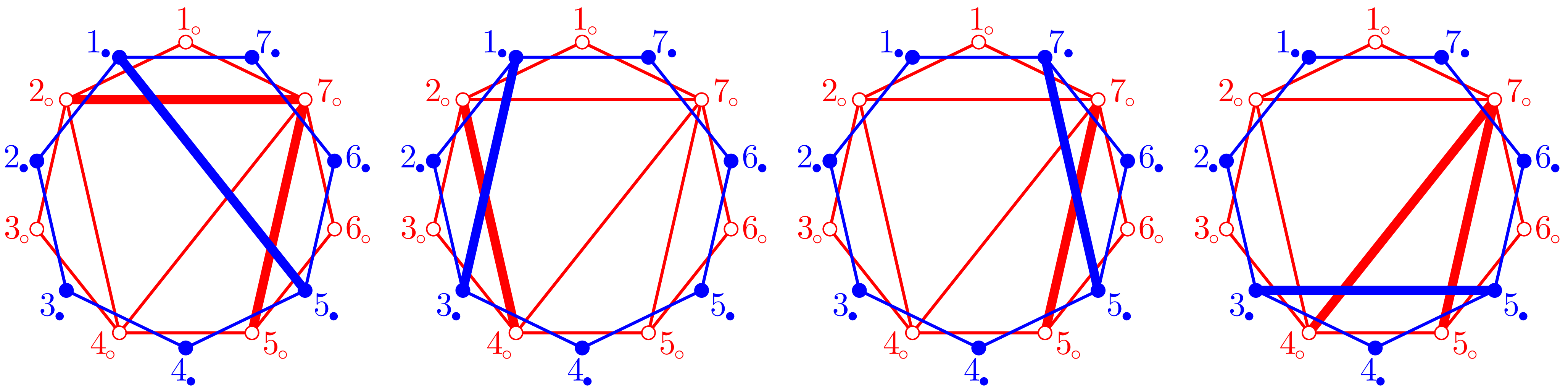
$\boldsymbol{G}$-VECTORS

For $\mathrm{T}_\circ$ red triangulation, $\delta_\circ \in \mathrm{T}_\circ$ and $\delta_\bullet$ a blue diagonal, let

$$\varepsilon_\circ\big(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet\big) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as an } \Sigma \\ 0 & \text{otherwise} \end{cases}$$

$$\boldsymbol{g}(\mathrm{T}_\circ, \delta_\bullet) = \underline{\boldsymbol{g}\text{-vector}} \text{ of } \delta_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet\big)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$

$$= \text{alternating } \pm 1 \text{ along the zigzag crossed by } \delta_\bullet \text{ in } \mathrm{T}_\circ$$



$\boldsymbol{g}(\mathrm{T}_\circ, (1_\bullet, 5_\bullet)) = \boldsymbol{e}_{5_\circ 7_\circ} - \boldsymbol{e}_{2_\circ 7_\circ}$

$\boldsymbol{g}(\mathrm{T}_\circ, (1_\bullet, 3_\bullet)) = -\boldsymbol{e}_{2_\circ 4_\circ}$

$\boldsymbol{g}(\mathrm{T}_\circ, (5_\bullet, 7_\bullet)) = \boldsymbol{e}_{5_\circ 7_\circ}$

$\boldsymbol{g}(\mathrm{T}_\circ, (3_\bullet, 5_\bullet)) = \boldsymbol{e}_{5_\circ 7_\circ} - \boldsymbol{e}_{4_\circ 7_\circ}$

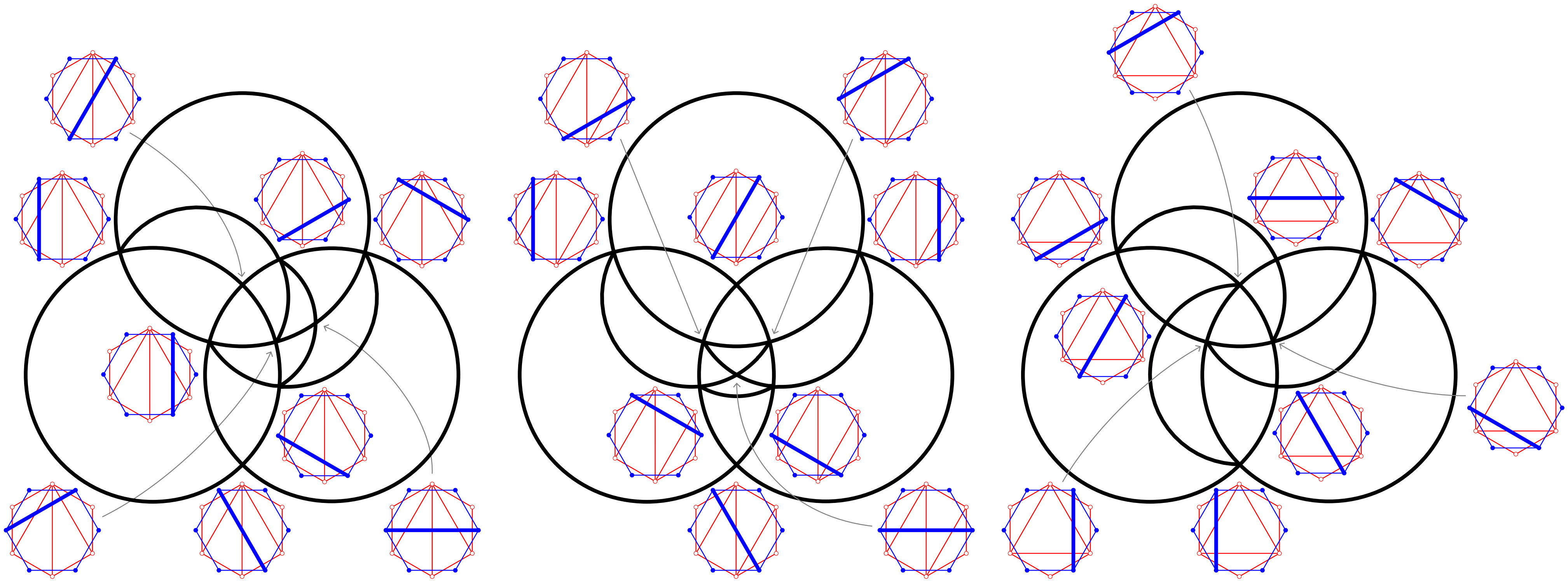
$\boldsymbol{G}$-VECTOR FAN

$\boldsymbol{g}(\mathrm{T}_\circ, \delta_\bullet)$ = $\boldsymbol{g}$-vector of $\delta_\bullet$ with respect to $\mathrm{T}_\circ$ = $\Big[\varepsilon_\circ(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet) \Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$

THM. For any red triangulation $\mathrm{T}_\circ$, the collection of cones

$$\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ) := \big\{ \mathbb{R}_{\geq 0}\boldsymbol{g}(\mathrm{T}_\circ, \mathrm{D}_\bullet) \;\big|\; \mathrm{D}_\bullet \text{ any blue dissection} \big\}$$

forms a complete simplicial fan, called $\boldsymbol{g}$-vector fan of $\mathrm{T}_\circ$.



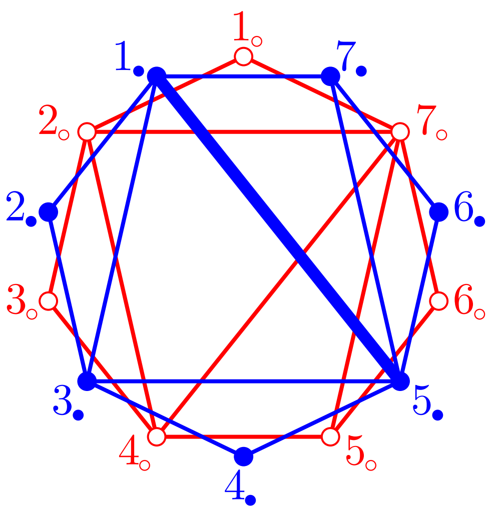
Loday Hohlweg-Lange / Reading Hohlweg-P.-Stella

$\boldsymbol{C}$-VECTORS

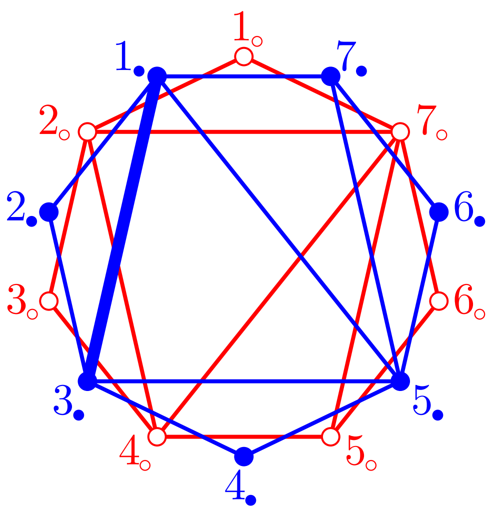
For $\mathrm{T}_\circ$ red triangulation and $\mathrm{T}_\bullet$ blue triangulation
and two diagonals $\delta_\circ \in \mathrm{T}_\circ$ and $\delta_\bullet \in \mathrm{T}_\bullet$, let

$$\varepsilon_\bullet(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \begin{cases} 1 & \text{if } \delta_\circ \text{ slaloms on } \delta_\bullet \in \mathrm{T}_\bullet \text{ as a } \Sigma \\ -1 & \text{if } \delta_\circ \text{ slaloms on } \delta_\bullet \in \mathrm{T}_\bullet \text{ as an Z} \\ 0 & \text{otherwise} \end{cases}$$

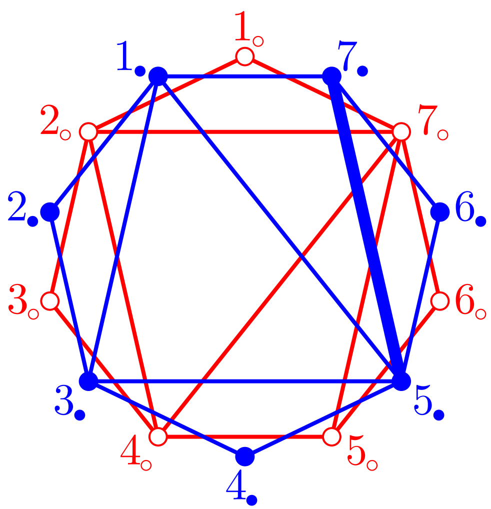
$$\boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \underline{\boldsymbol{c}\text{-vector}} \text{ of } \delta_\bullet \text{ in } \mathrm{T}_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\bullet(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet) \Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$
$$= \pm \text{ charac. vector of diagonals of } \mathrm{T}_\circ \text{ crossed by opposite neighbors of } \delta_\bullet$$



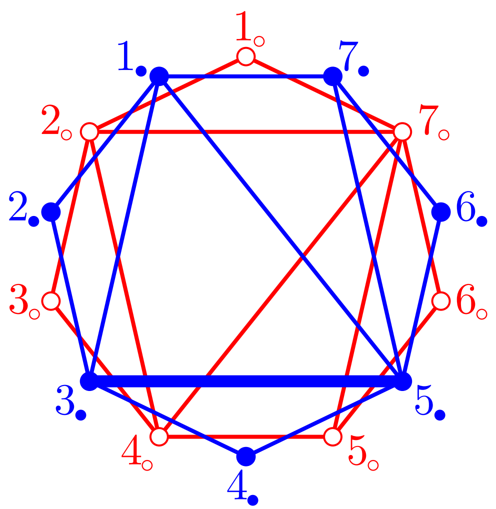
$\boldsymbol{c}(\mathrm{T}_\circ, (1_\bullet, 5_\bullet) \in \mathrm{T}_\bullet) = -\boldsymbol{e}_{2_\circ 7_\circ}$



$\boldsymbol{c}(\mathrm{T}_\circ, (1_\bullet, 3_\bullet) \in \mathrm{T}_\bullet) = -\boldsymbol{e}_{2_\circ 4_\circ}$



$\boldsymbol{c}(\mathrm{T}_\circ, (5_\bullet, 7_\bullet) \in \mathrm{T}_\bullet) = \boldsymbol{e}_{2_\circ 7_\circ} + \boldsymbol{e}_{4_\circ 7_\circ} + \boldsymbol{e}_{5_\circ 7_\circ}$



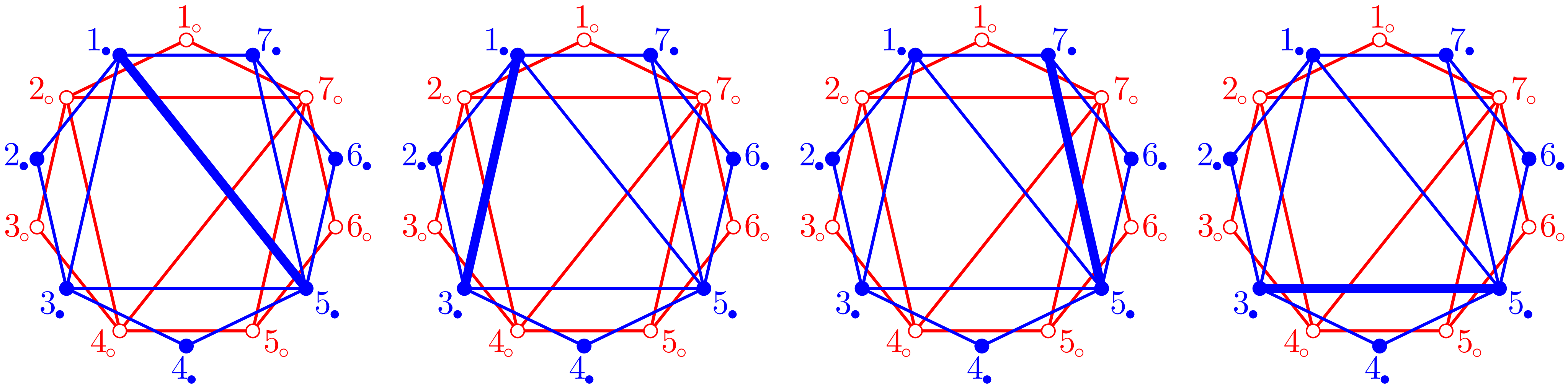
$\boldsymbol{c}(\mathrm{T}_\circ, (5_\bullet, 7_\bullet) \in \mathrm{T}_\bullet) = -\boldsymbol{e}_{4_\circ 7_\circ}$

$\boldsymbol{G}$- AND $\boldsymbol{C}$-VECTORS

For $\mathrm{T}_\circ$ red triangulation and $\mathrm{T}_\bullet$ blue triangulation

$$\boldsymbol{g}(\mathrm{T}_\circ, \delta_\bullet) = \underline{\boldsymbol{g}\text{-vector}} \text{ of } \delta_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet\big)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$

$$\boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \underline{\boldsymbol{c}\text{-vector}} \text{ of } \delta_\bullet \text{ in } \mathrm{T}_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$



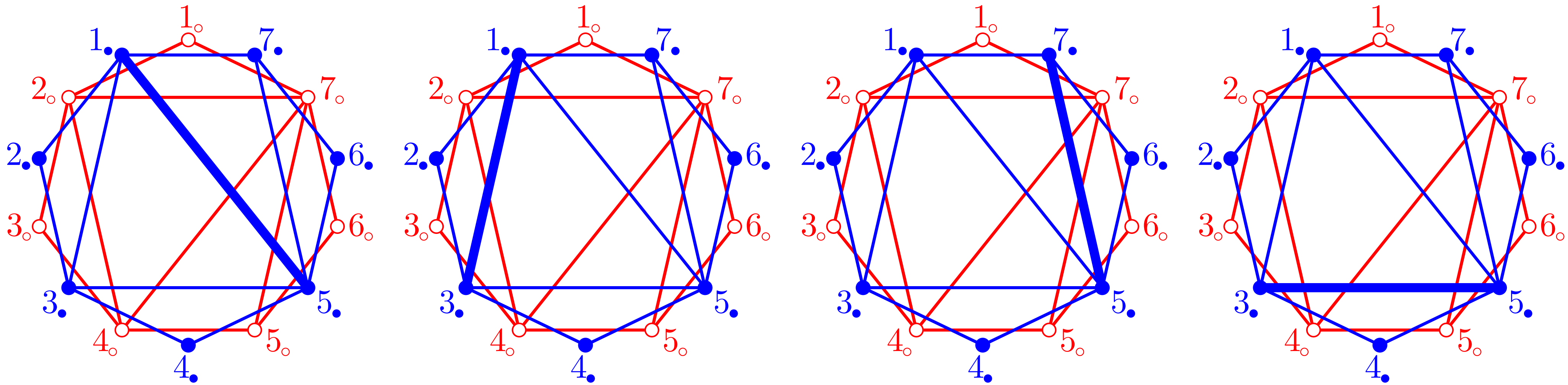
$\boldsymbol{g}$	$\boldsymbol{e}_{5_\circ 7_\circ} - \boldsymbol{e}_{2_\circ 7_\circ}$	$-\boldsymbol{e}_{2_\circ 4_\circ}$	$\boldsymbol{e}_{5_\circ 7_\circ}$	$\boldsymbol{e}_{5_\circ 7_\circ} - \boldsymbol{e}_{4_\circ 7_\circ}$
$\boldsymbol{c}$	$-\boldsymbol{e}_{2_\circ 7_\circ}$	$-\boldsymbol{e}_{2_\circ 4_\circ}$	$\boldsymbol{e}_{2_\circ 7_\circ} + \boldsymbol{e}_{4_\circ 7_\circ} + \boldsymbol{e}_{5_\circ 7_\circ}$	$-\boldsymbol{e}_{4_\circ 7_\circ}$

$\boldsymbol{G}$- AND $\boldsymbol{C}$-VECTORS

For $\mathrm{T}_\circ$ red triangulation and $\mathrm{T}_\bullet$ blue triangulation

$$\boldsymbol{g}(\mathrm{T}_\circ, \delta_\bullet) = \underline{\boldsymbol{g}\text{-vector}} \text{ of } \delta_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet\big) \Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$

$$\boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \underline{\boldsymbol{c}\text{-vector}} \text{ of } \delta_\bullet \text{ in } \mathrm{T}_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$



$\boldsymbol{g}$	$\boldsymbol{e}_{5_\circ 7_\circ} - \boldsymbol{e}_{2_\circ 7_\circ}$	$-\boldsymbol{e}_{2_\circ 4_\circ}$	$\boldsymbol{e}_{5_\circ 7_\circ}$	$\boldsymbol{e}_{5_\circ 7_\circ} - \boldsymbol{e}_{4_\circ 7_\circ}$
$\boldsymbol{c}$	$-\boldsymbol{e}_{2_\circ 7_\circ}$	$-\boldsymbol{e}_{2_\circ 4_\circ}$	$\boldsymbol{e}_{2_\circ 7_\circ} + \boldsymbol{e}_{4_\circ 7_\circ} + \boldsymbol{e}_{5_\circ 7_\circ}$	$-\boldsymbol{e}_{4_\circ 7_\circ}$

PROP. The $\boldsymbol{g}$-vectors $\boldsymbol{g}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$ and the $\boldsymbol{c}$-vectors $\boldsymbol{c}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$ form dual bases.

PROP. Duality: $\boldsymbol{g}(\mathrm{T}_\circ, \mathrm{T}_\bullet) = -\boldsymbol{c}(\mathrm{T}_\bullet, \mathrm{T}_\circ)^t$ and $\boldsymbol{c}(\mathrm{T}_\circ, \mathrm{T}_\bullet) = -\boldsymbol{g}(\mathrm{T}_\bullet, \mathrm{T}_\circ)^t$

ASSOCIAHEDRA FOR g-VECTOR FANS

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('18)

$\mathrm{T}_\circ$-ZONOTOPE

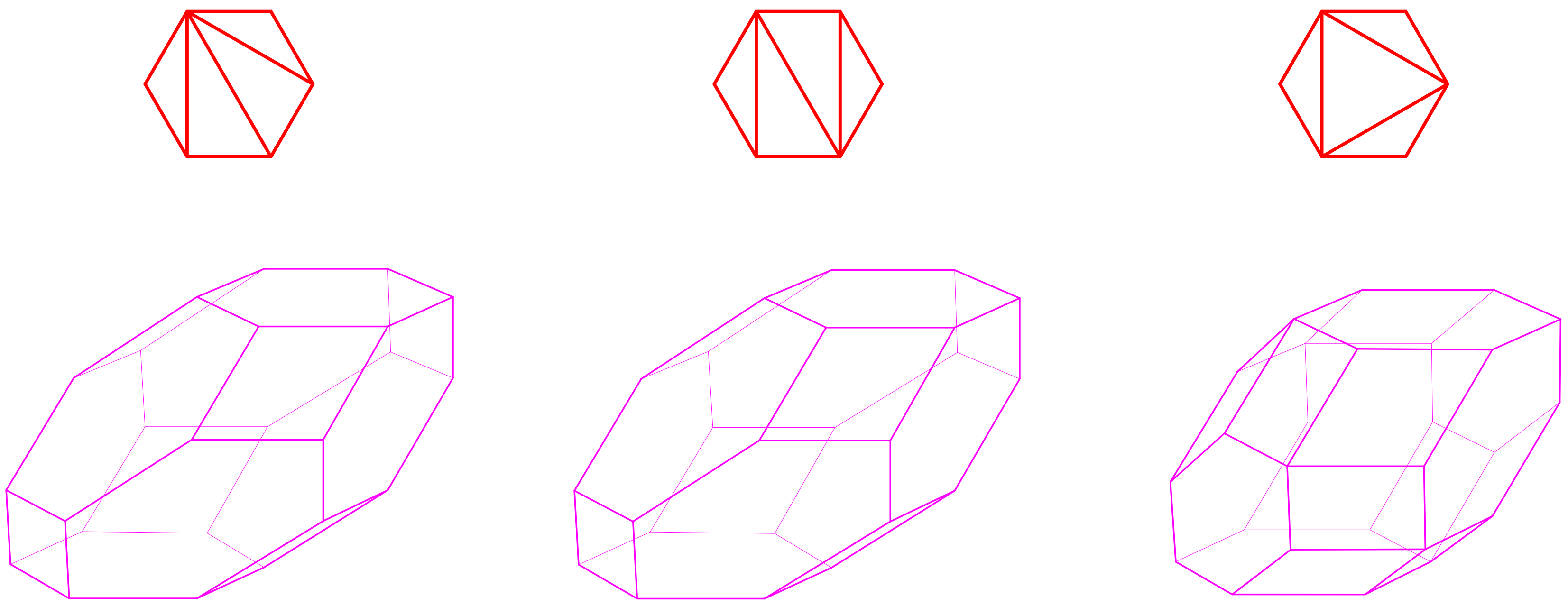
$\underline{\mathrm{T}_\circ\text{-zonotope}} = \mathbb{Z}\mathsf{ono}(\mathrm{T}_\circ) =$ Minkowski sum of all $\boldsymbol{c}$-vectors $\boldsymbol{C}(\mathrm{T}_\circ) = \bigcup_{\mathrm{T}_\bullet} \boldsymbol{c}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$

$$\mathbb{Z}\mathsf{ono}(\mathrm{T}_\circ) = \sum_{\boldsymbol{c} \in \boldsymbol{C}(\mathrm{T}_\circ)} \boldsymbol{c}.$$

PROP. For any diagonal $\gamma_\bullet$, $\mathbb{Z}\mathsf{ono}(\mathrm{T}_\circ)$ has a facet defined by the inequality

$$\langle\, \boldsymbol{g}(\mathrm{T}_\circ, \gamma_\bullet) \mid \boldsymbol{x} \,\rangle \leq \omega(\gamma_\bullet)$$

where $\omega(\gamma_\bullet) =$ number of red diagonals that cross $\gamma_\bullet$.



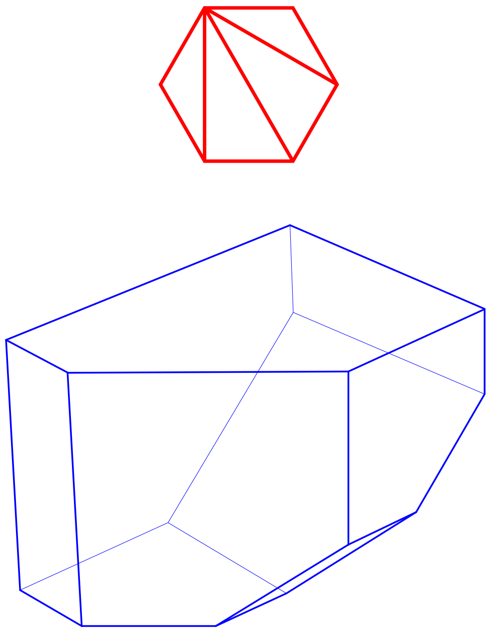
$\mathrm{T}_\circ$-ASSOCIAHEDRON

Define $$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet)$$

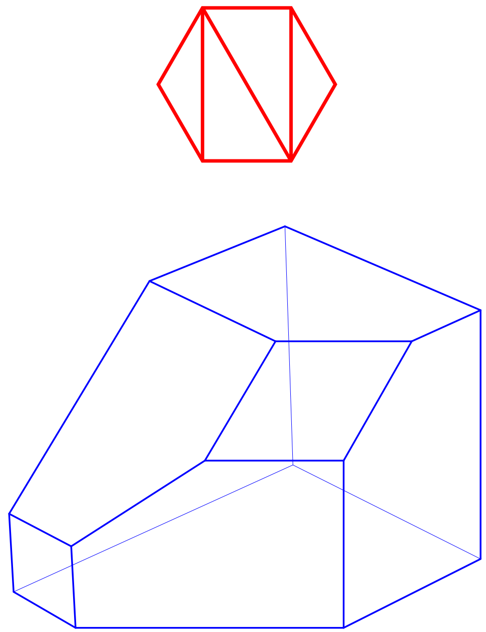
THM. For any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\begin{aligned}\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ) &= \mathrm{conv}\left\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation}\right\} \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^{\mathrm{T}_\circ} \;\middle|\; \langle\, \boldsymbol{g}(\mathrm{T}_\circ, \delta_\bullet) \mid \boldsymbol{x} \,\rangle \le \omega(\delta_\bullet) \text{ for any blue diagonal } \delta_\bullet\right\}.\end{aligned}$$

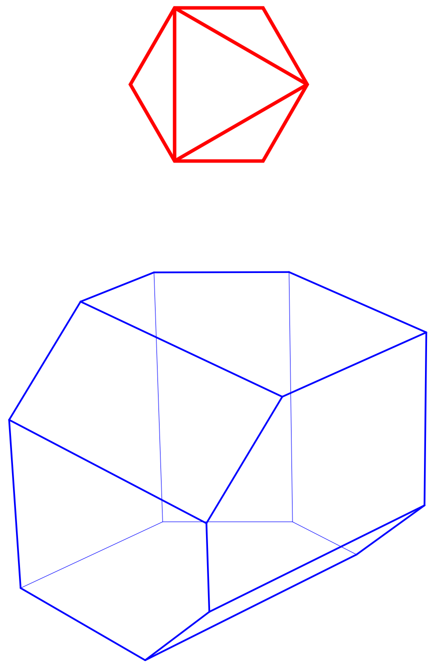
Hohlweg-P.-Stella, ('18)



Loday



Hohlweg-Lange



Hohlweg-P.-Stella

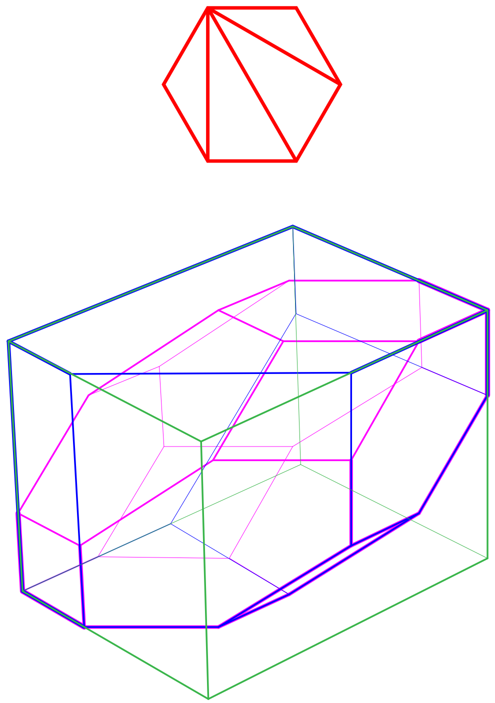
$\mathrm{T}_\circ$-ASSOCIAHEDRON

Define $$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet)$$

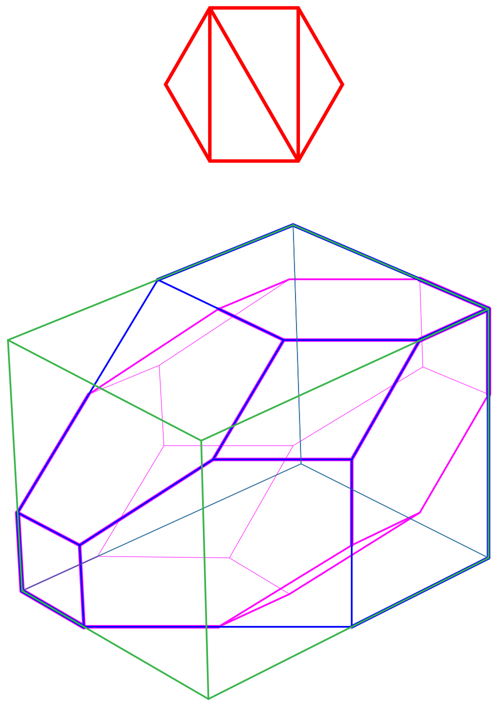
THM. For any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\mathbb{A}\mathrm{sso}(\mathrm{T}_\circ) = \operatorname{conv}\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation}\}$$
$$= \left\{\boldsymbol{x} \in \mathbb{R}^{\mathrm{T}_\circ} \;\middle|\; \langle\, \boldsymbol{g}(\mathrm{T}_\circ, \delta_\bullet) \mid \boldsymbol{x} \,\rangle \le \omega(\delta_\bullet) \text{ for any blue diagonal } \delta_\bullet \right\}.$$

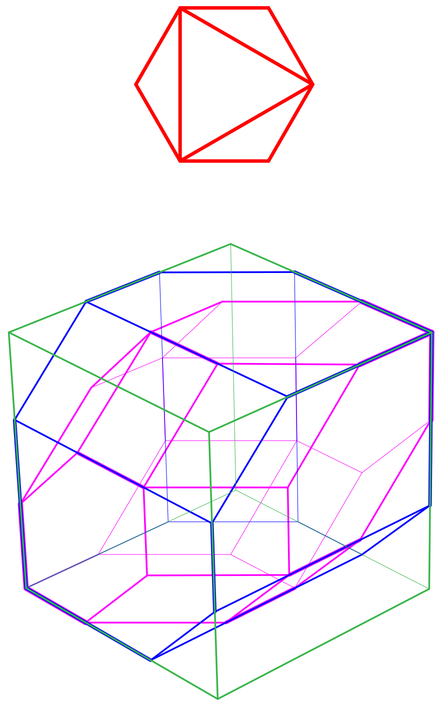
Hohlweg-P.-Stella, ('18)



Loday



Hohlweg-Lange



Hohlweg-P.-Stella

UNIVERSAL ASSOCIAHEDRON

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('18)

UNIVERSAL ASSOCIAHEDRON

THM. For any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ) = \mathrm{conv}\left\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation}\right\}$$

where

$$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \sum_{\delta_\circ \in \mathrm{T}_\circ} \bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big)\bigg) \, \boldsymbol{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}.$$

Hohlweg-P.-Stella ('18)

$\Longrightarrow$ the $\delta_\circ$-coordinate of $\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$ does not really depends on $\mathrm{T}_\circ$

UNIVERSAL ASSOCIAHEDRON

THM. For any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ) = \mathrm{conv}\left\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation}\right\}$$

where

$$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \boldsymbol{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \sum_{\delta_\circ \in \mathrm{T}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \boldsymbol{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}.$$

Hohlweg-P.-Stella ('18)

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \boldsymbol{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella ('18)

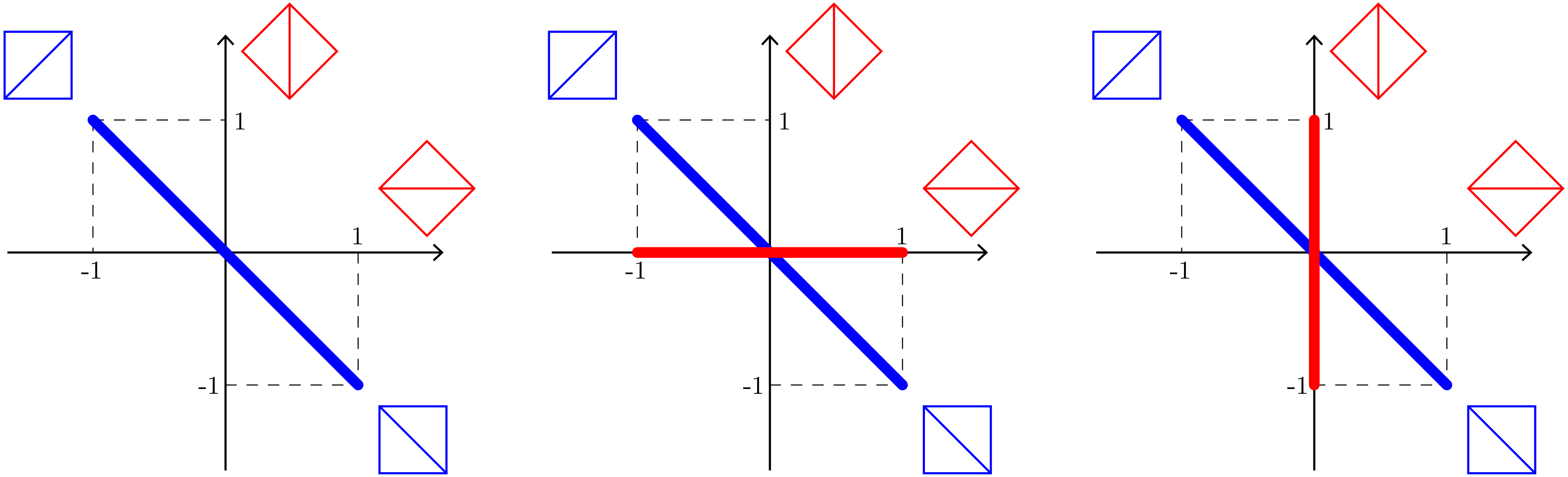
UNIVERSAL ASSOCIAHEDRON

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \boldsymbol{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella ('18)



UNIVERSAL ASSOCIAHEDRON

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Big(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big)\Big) \, \boldsymbol{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathbb{A}\mathrm{sso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella ('18)

n	dimension of ambient space	dimension	# vertices	# facets	# vertices / facet	# facets / vertex
1	2	1	2	2	1	1
2	5	4	5	5	4	4
3	9	8	14	60	$9 \le \cdot \le 10$	$30 \le \cdot \le 42$
4	14	13	42	8960	$14 \le \cdot \le 28$	$3463 \le \cdot \le 4244$

UNIVERSAL ASSOCIAHEDRON

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Big(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Big) \, \boldsymbol{e}_{\delta_\circ} \; \in \; \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella ('18)

THM. The origin is the vertex barycenter of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$.

Hohlweg-P.-Stella ('18)

CORO. For any red triangulation $\mathrm{T}_\circ$, the origin is the vertex barycenter of the $\mathrm{T}_\circ$-associahedron $\mathbb{A}\mathsf{sso}(\mathrm{T}_\circ)$.

Hohlweg-P.-Stella ('18)

SECTIONS AND PROJECTIONS

Manneville-P., *Geometric realizations of the accordion complex* ('19)

SECTIONS AND PROJECTIONS

THM. For any red triangulation $\mathrm{T}_\circ$, the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathbb{A}\mathrm{sso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

What happens if we project on other coordinate planes?
No clue in general, but...

For a red dissection $\mathrm{D}_\circ$, define

$$\mathbb{A}\mathrm{sso}(\mathrm{D}_\circ) = \text{projection of } \mathrm{Asso}_{\mathrm{un}}(n) \text{ on the coordinate plane } \mathbb{R}^{\mathrm{D}_\circ}$$

Since normal fan of projections are sections of normal fans,

$$\begin{aligned}\text{normal fan of } \mathbb{A}\mathrm{sso}(\mathrm{D}_\circ) &= \text{section of the normal fan of } \mathrm{Asso}_{\mathrm{un}}(n) \text{ by the plane } \mathbb{R}^{\mathrm{D}_\circ} \\ &= \text{subfan of the normal fan of } \mathrm{Asso}_{\mathrm{un}}(n) \text{ induced by the rays in } \mathbb{R}^{\mathrm{D}_\circ} \\ &= \text{subfan of the normal fan of } \mathbb{A}\mathrm{sso}(\mathrm{T}_\circ) \text{ induced by the rays in } \mathbb{R}^{\mathrm{D}_\circ}\end{aligned}$$

for a triangulation $\mathrm{T}_\circ$ containing $\mathrm{D}_\circ$

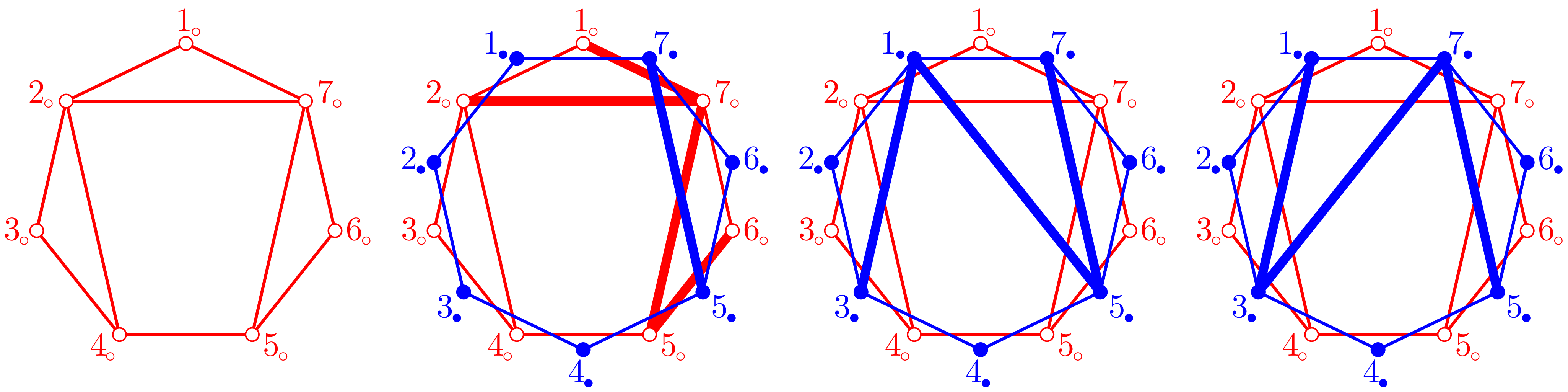
ACCORDION COMPLEX

LEM. For a red dissection $D_\circ$ contained in a red triangulation $T_\circ$, and a blue diagonal $\delta_\bullet$, $\boldsymbol{g}(T_\circ, \delta_\bullet) \in \mathbb{R}^{D_\circ} \iff \delta_\bullet$ never crosses a cell of $D_\circ$ through two non-consecutive edges

$D_\circ$-accordion diagonal = diagonal of the blue solid polygon that crosses an accordion of $D_\circ$

$D_\circ$-accordion dissection = set of non-crossing $D_\circ$-accordion diagonals

$D_\circ$-accordion complex = simplicial complex of $D_\circ$-accordion dissections

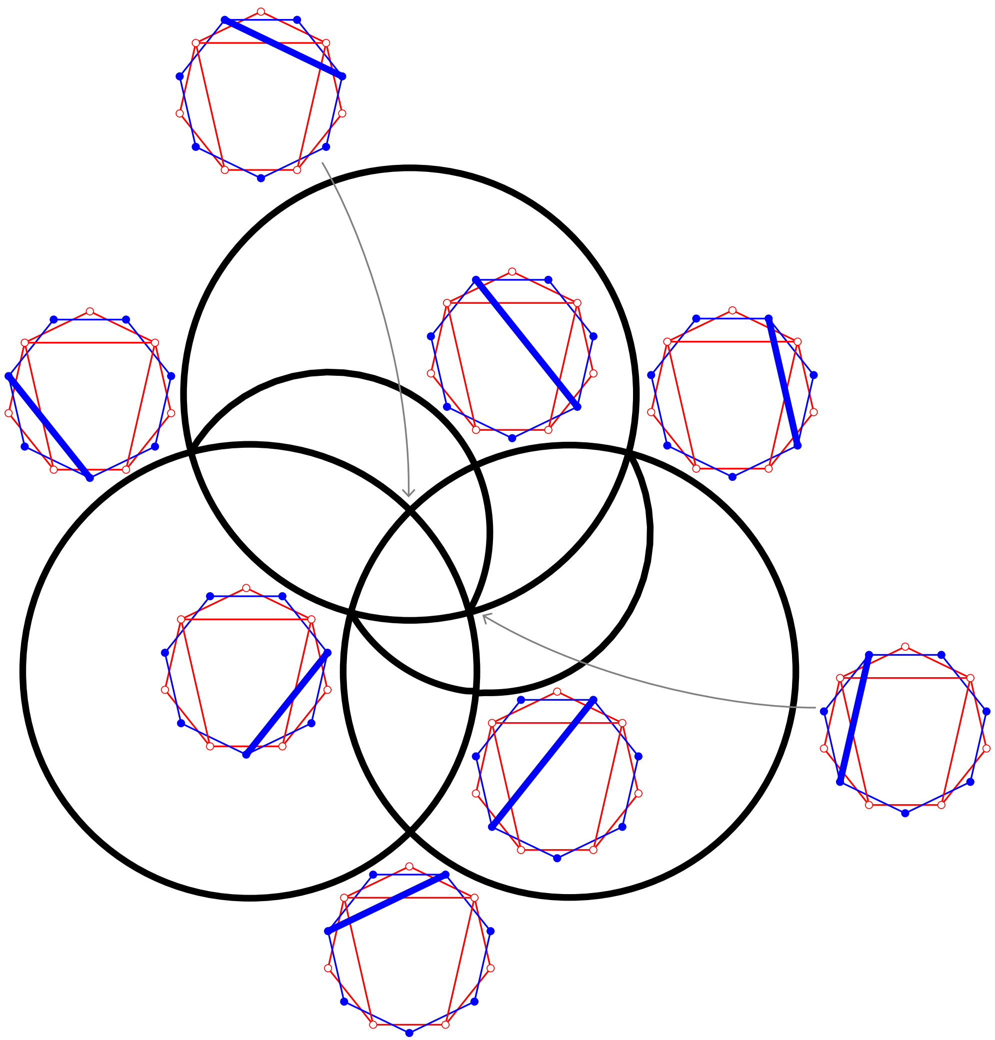


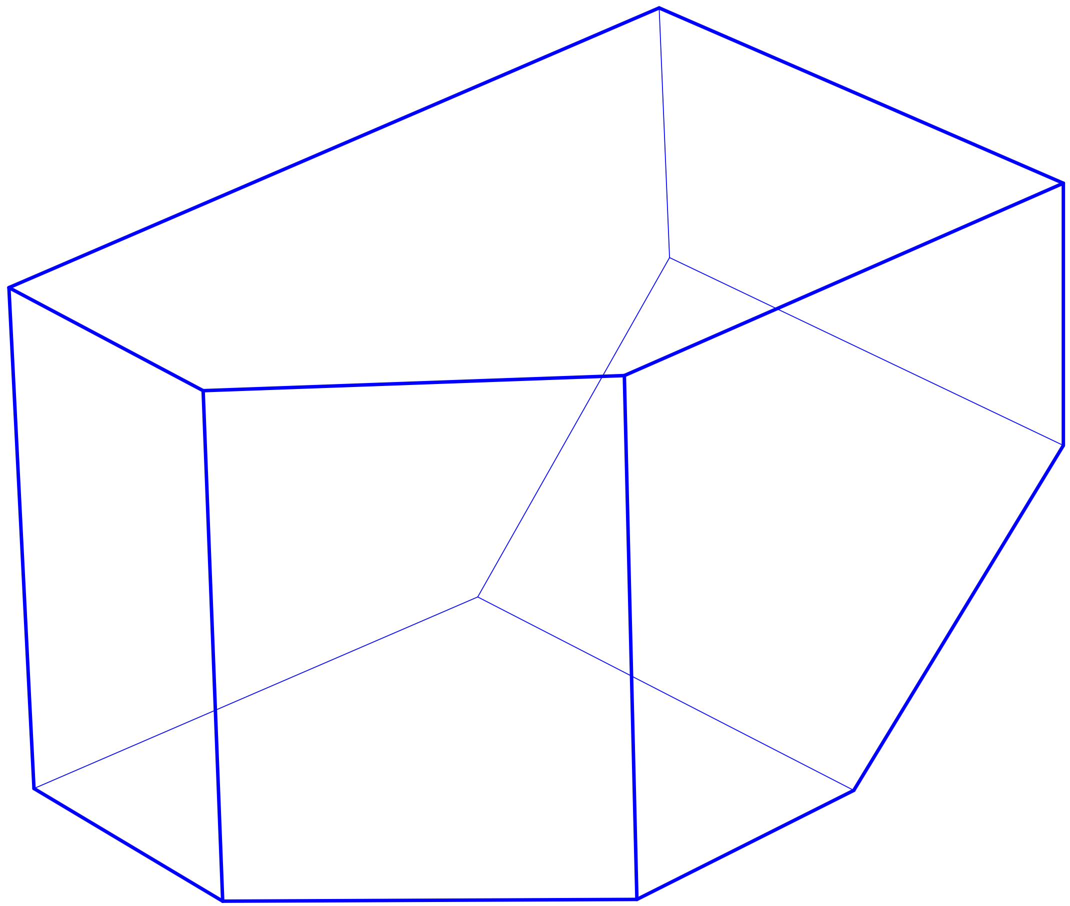
dissection $D_\circ$ $D_\circ$-accordion diagonal two maximal $D_\circ$-accordion dissections

ACCORDIOHEDRON

THM. For any red dissection $D_\circ$, the projection $\mathbb{A}\mathrm{sso}(D_\circ)$ of the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{D_\circ}$ realizes the $D_\circ$-accordion complex.

Manneville-P., ('19)

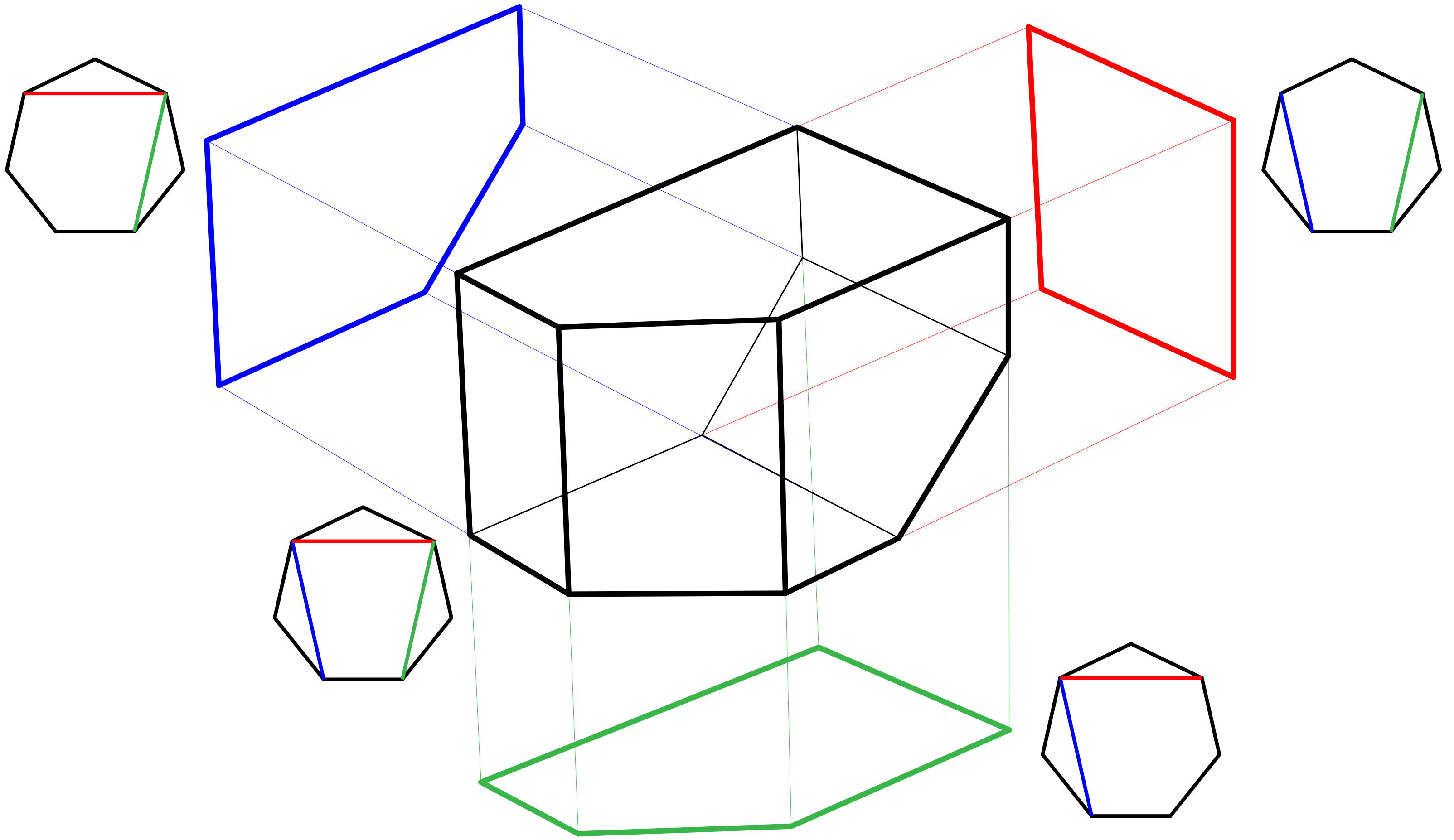




PROJECTIONS OF PROJECTIONS

PROP. If $\mathrm{D}_\circ \subseteq \mathrm{D}'_\circ$, then

- $\mathcal{F}^{\boldsymbol{g}}(\mathrm{D}_\circ)$ is the section of $\mathcal{F}^{\boldsymbol{g}}(\mathrm{D}'_\circ)$ with the coordinate plane $\langle \boldsymbol{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$,
- therefore, $\mathcal{F}^{\boldsymbol{g}}(\mathrm{D}_\circ)$ is also realized by the projection of $\mathbb{A}\mathrm{sso}(\mathrm{D}_\circ)$ on $\langle \boldsymbol{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$.



EXTENSIONS TO CLUSTER ALGEBRAS

Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02 – '07)

CLUSTER ALGEBRAS

cluster algebra $=$ commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process :

cluster seed $=$ algebraic data $\{x_1, \dots, x_n\}$, combinatorical data B (matrix or quiver)

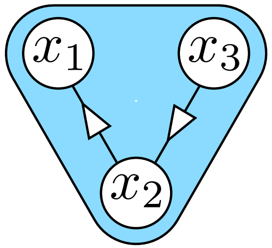
cluster mutation $=$ $\big(\{x_1, \dots, x_k, \dots, x_n\}, B\big) \overset{\mu_k}{\longleftrightarrow} \big(\{x_1, \dots, x'_k, \dots, x_n\}, \mu_k(B)\big)$

$$x_k \cdot x'_k = \prod_{\{i \mid b_{ik}>0\}} x_i^{b_{ik}} + \prod_{\{i \mid b_{ik}<0\}} x_i^{-b_{ik}}$$

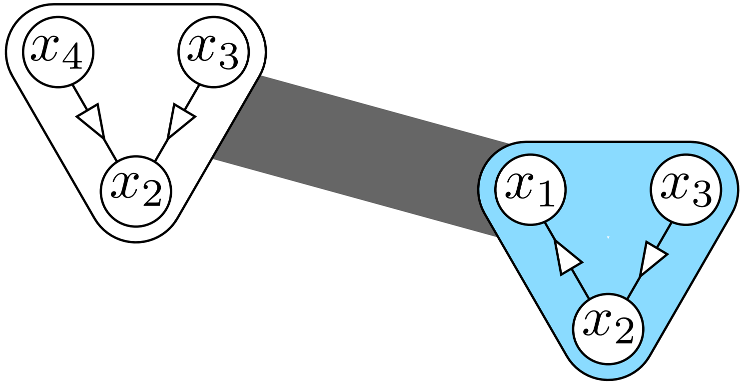
$$\big(\mu_k(B)\big)_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i,j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

cluster complex $=$ simplicial complex w/ vertices $=$ cluster variables & facets $=$ clusters

Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02 – '07)


x_1
x_3
x_2

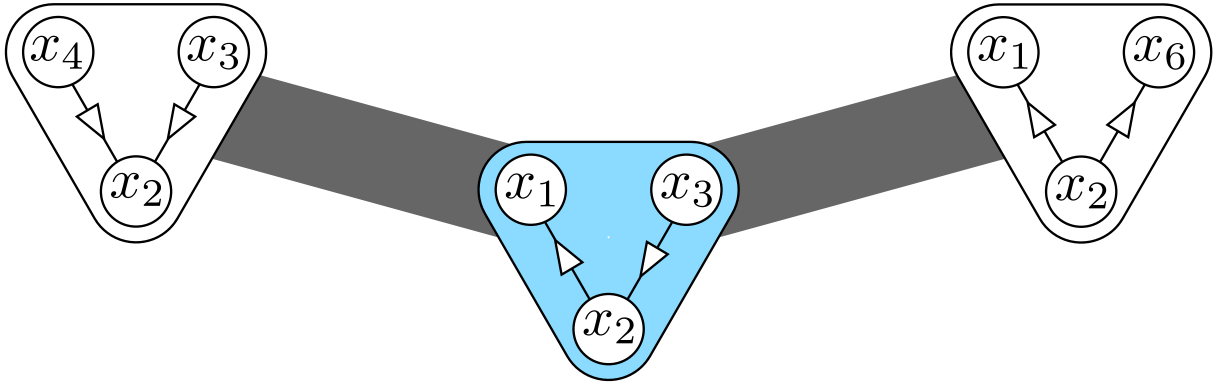
$$x_4 = \frac{1 + x_2}{x_1}$$



CLUSTER MUTATION

$$x_4 = \frac{1 + x_2}{x_1}$$

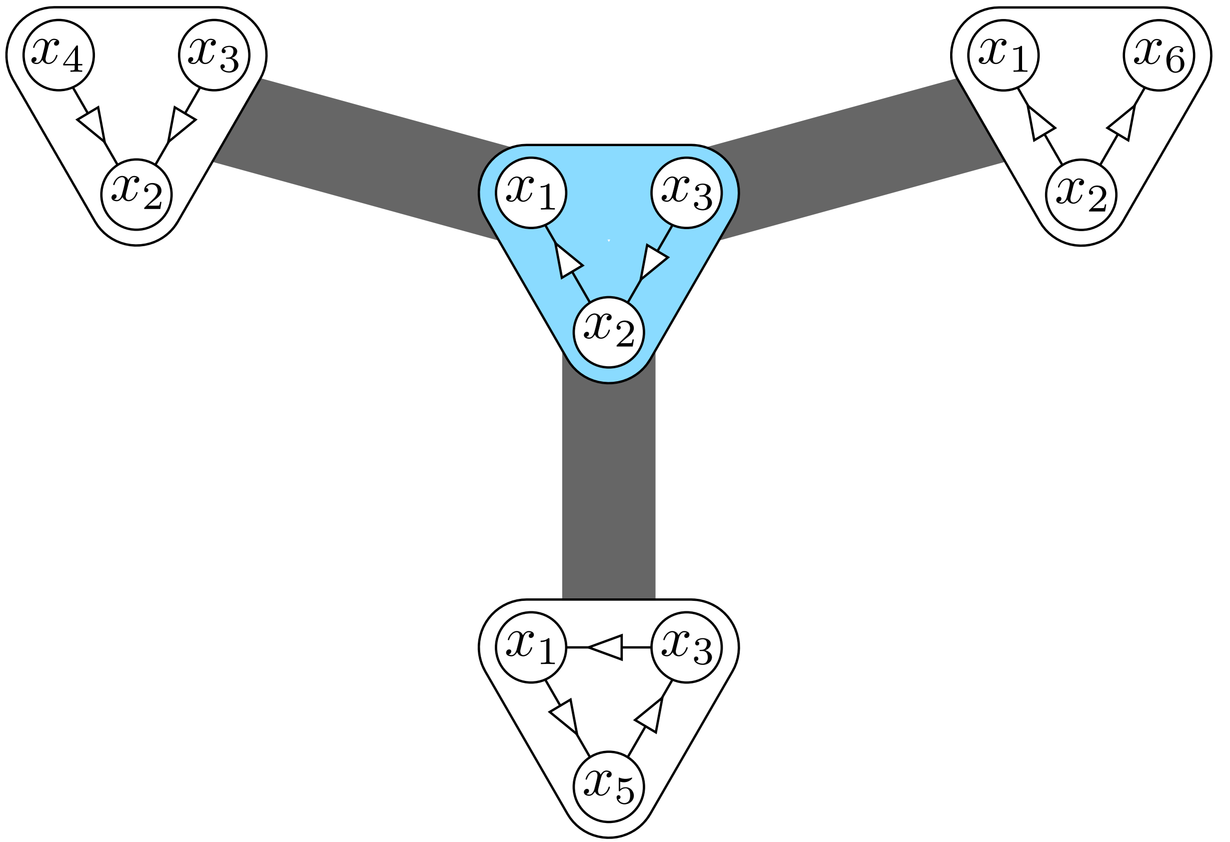
$$x_6 = \frac{1 + x_2}{x_3}$$



CLUSTER MUTATION

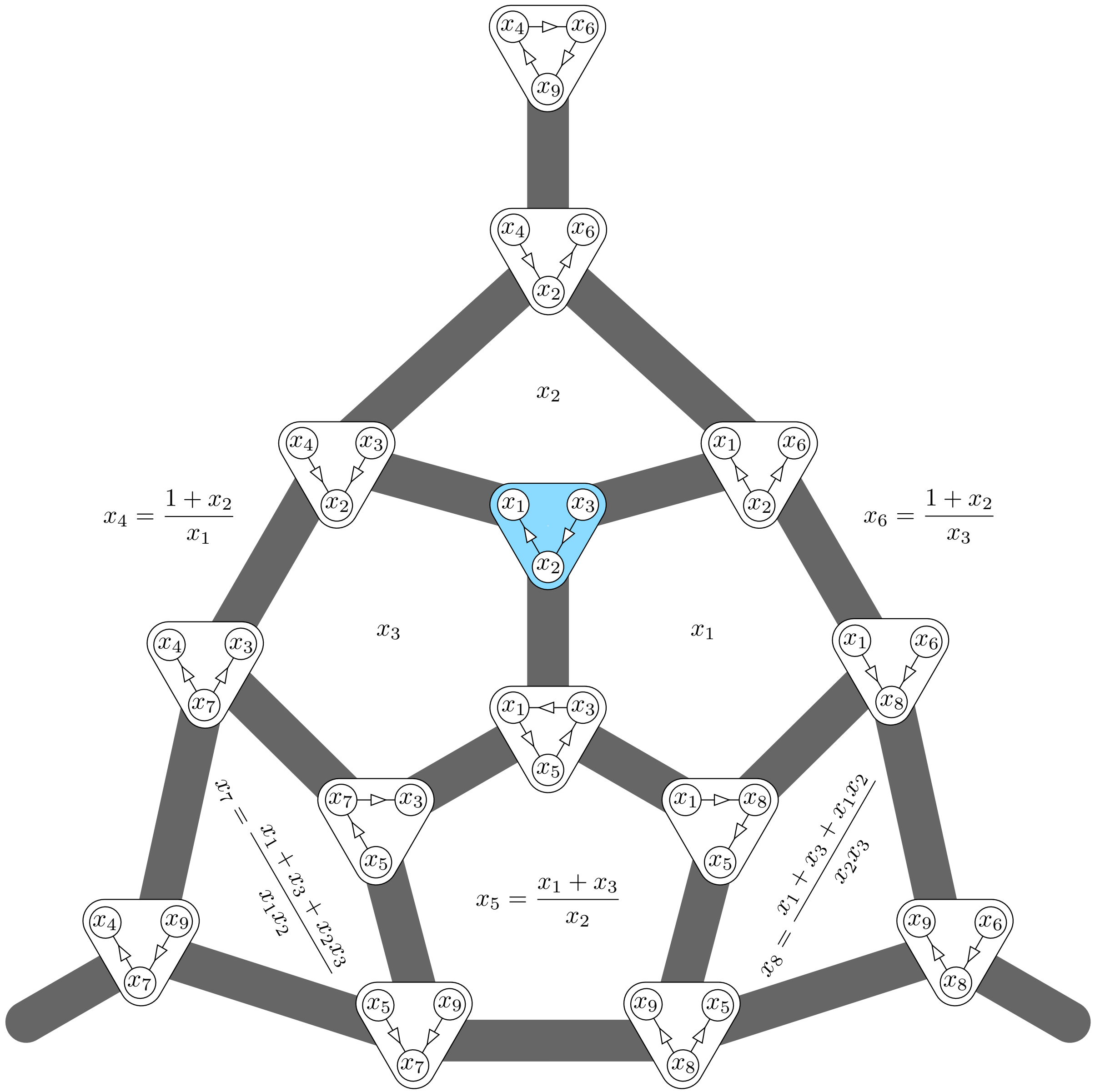
$$x_4 = \frac{1 + x_2}{x_1}$$

$$x_6 = \frac{1 + x_2}{x_3}$$



$$x_5 = \frac{x_1 + x_3}{x_2}$$

CLUSTER MUTATION GRAPH

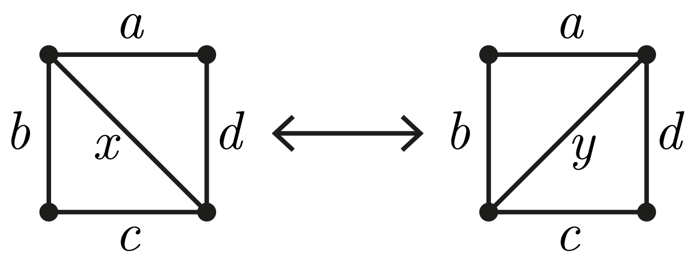


$$x_9 = \frac{(x_1+x_3)(1+x_2)}{x_1x_2x_3}$$

CLUSTER ALGEBRA FROM TRIANGULATIONS

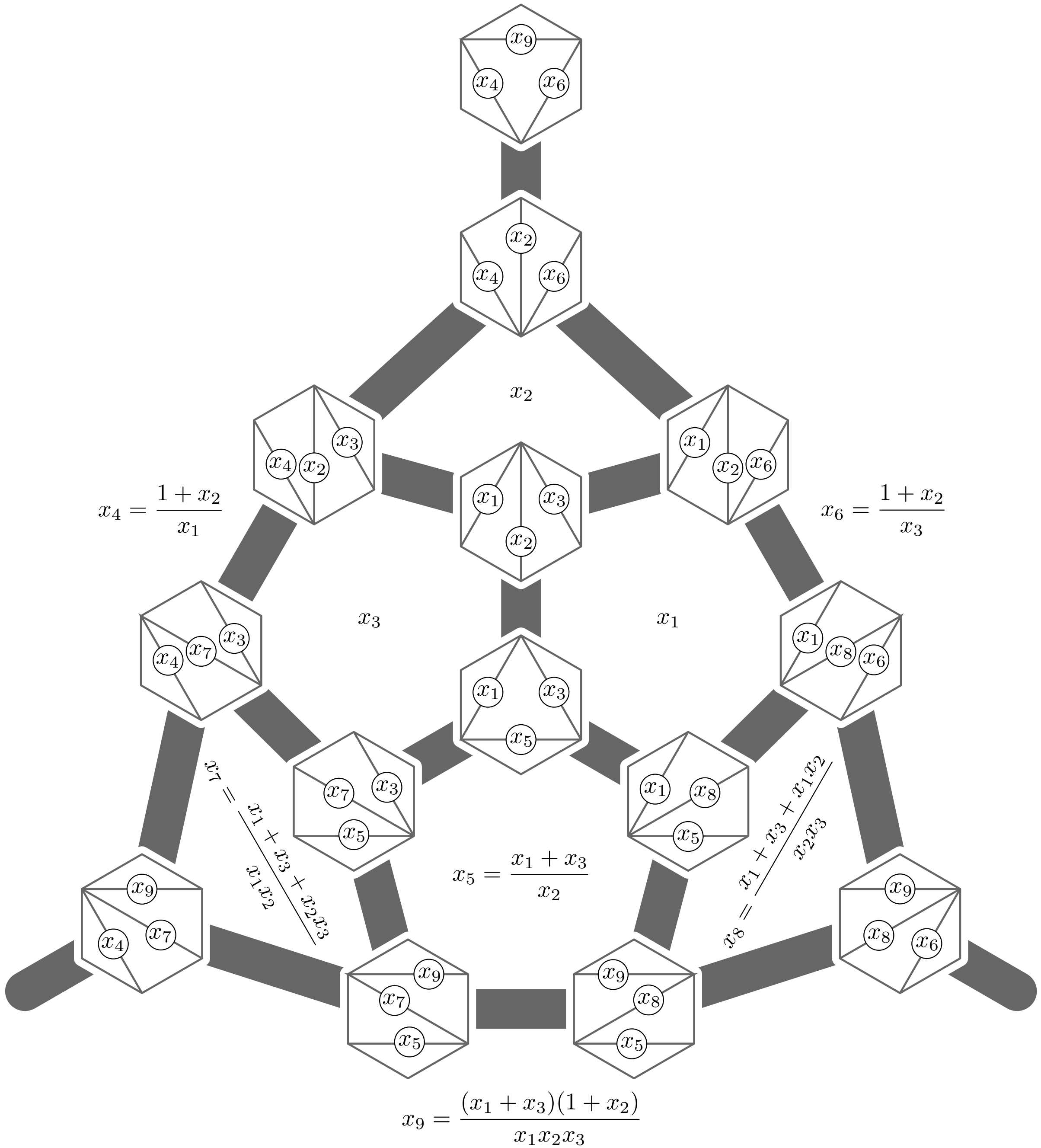
One constructs a cluster algebra from the triangulations of a polygon:

diagonals	$\longleftrightarrow$	cluster variables
triangulations	$\longleftrightarrow$	clusters
flip	$\longleftrightarrow$	mutation



$\longleftrightarrow$ $\quad xy = ac + bd$

CLUSTER MUTATION GRAPH



CLUSTER ALGEBRAS

THM. (Laurent phenomenon) — Fomin-Zelevinsky ('02)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.

THM. (Classification) — Fomin-Zelevinsky ('03)
Finite type cluster algebras are classified by the Cartan-Killing classification for finite type crystallographic root systems.

for a root system Φ, and an acyclic initial cluster $X = \{x_1, \ldots, x_n\}$, there is a bijection

cluster variables of $\mathcal{A}_\Phi$	$\xleftrightarrow{\theta_X}$	$\Phi_{\geq -1} = \Phi^+ \cup -\Delta$
$y = \dfrac{F(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$	$\xleftrightarrow{\theta_X}$	$\beta = d_1\alpha_1 + \cdots + d_n\alpha_n$
cluster of $\mathcal{A}_\Phi$	$\xleftrightarrow{\theta_X}$	X-cluster in $\Phi_{\geq -1}$
cluster complex of $\mathcal{A}_\Phi$	$\xleftrightarrow{\theta_X}$	X-cluster complex in $\Phi_{\geq -1}$

COXETER UNIVERSAL ASSOCIAHEDRON

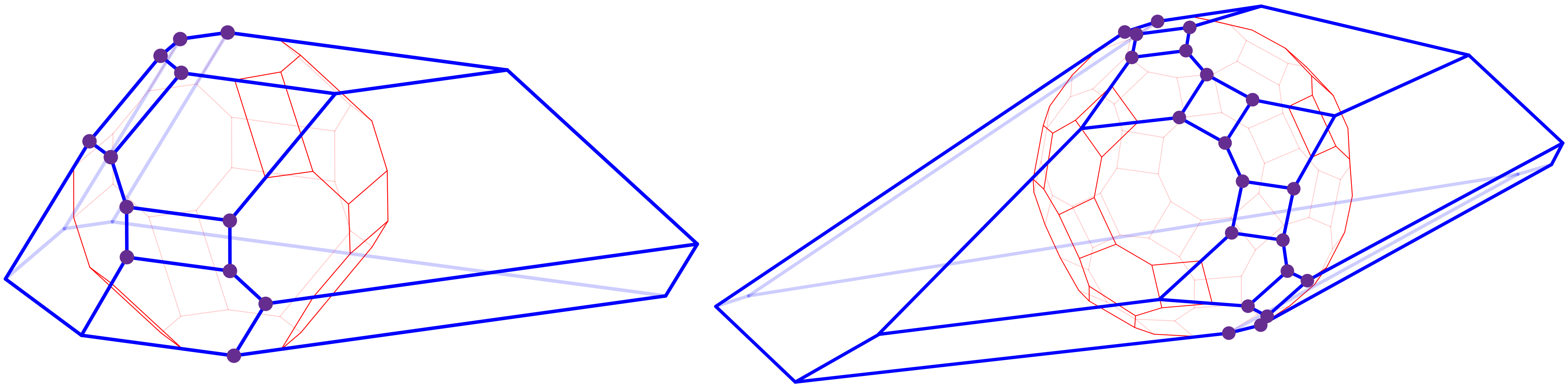
$\boldsymbol{g}$- and $\boldsymbol{c}$-vectors of cluster variables are defined using principal coefficients
universal $\boldsymbol{c}$-vectors are defined using universal coefficients

THM. Γ finite type Dynkin diagram and h : cluster vars $\to \mathbb{R}$ exchange submodular.
Define the universal Γ-associahedron $\mathsf{Asso}_{\mathrm{un}}(\Gamma)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\Sigma) := \sum_{x \in \Sigma} h(x) \cdot \mathbf{c}_{\mathrm{un}}(x \in \Sigma)$$

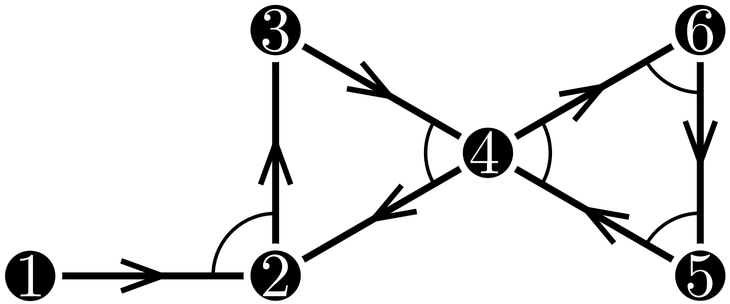
for all seeds Σ in the cluster algebra of type Γ.
Then for any initial seed $\Sigma_\circ$, the $\boldsymbol{g}$-vetor fan $\mathcal{F}^{\boldsymbol{g}}(\Sigma_\circ)$ is the normal fan of the projection $\mathbb{A}\mathsf{sso}(\Sigma_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(\Gamma)$ on the coordinate plane $\mathbb{R}^\Gamma$.

Hohlweg-P.-Stella ('18)



NON-KISSING COMPLEX

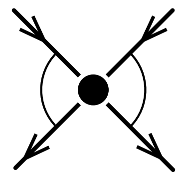
Brüstle–Douville–Mousavand–Thomas–Yıldırım, *Combinatorics of gentle algebras* ('20)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)



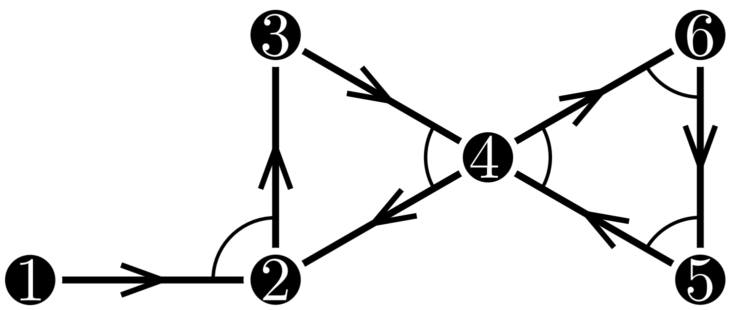
gentle quiver $\bar{Q} =$

- quiver $Q =$ oriented graph (Q_0, Q_1, s, t)
- relations $I =$ forbid certain paths

where

- forbidden paths all of length 2
- locally at each vertex, subgraph of 

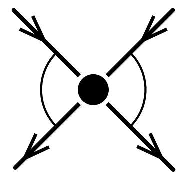
GENTLE QUIVERS AND STRINGS

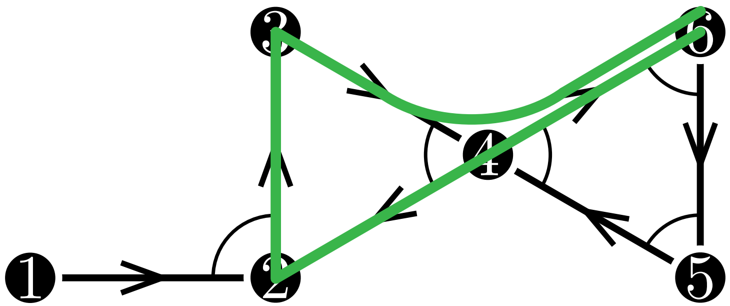


gentle quiver $\bar{Q}$ =

- quiver Q = oriented graph (Q_0, Q_1, s, t)
- relations I = forbid certain paths

where

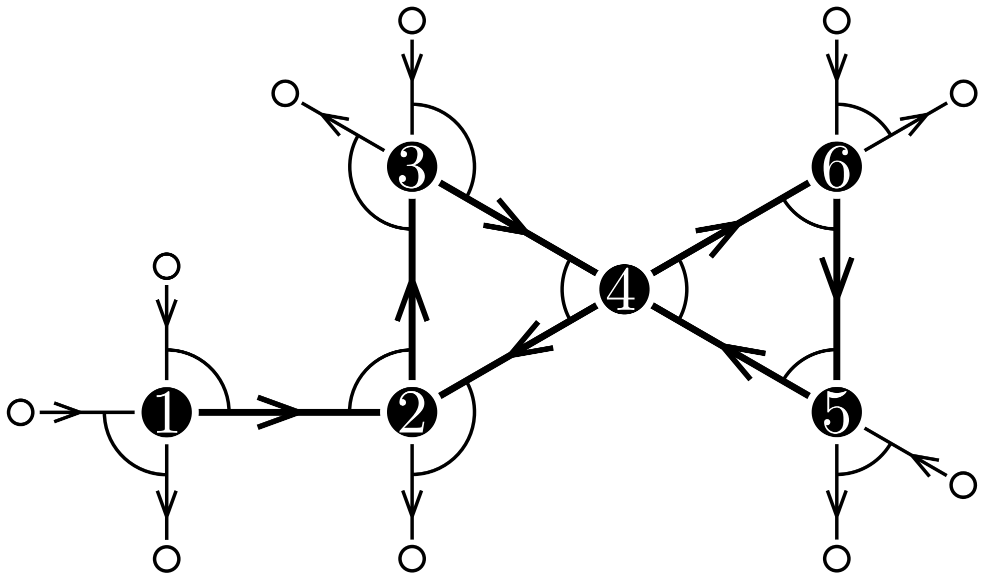
- forbidden paths all of length 2
- locally at each vertex, subgraph of 

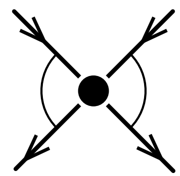


string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$ with $\alpha_k \in Q_1$, $\varepsilon_k \in \{-1, 1\}$ such that

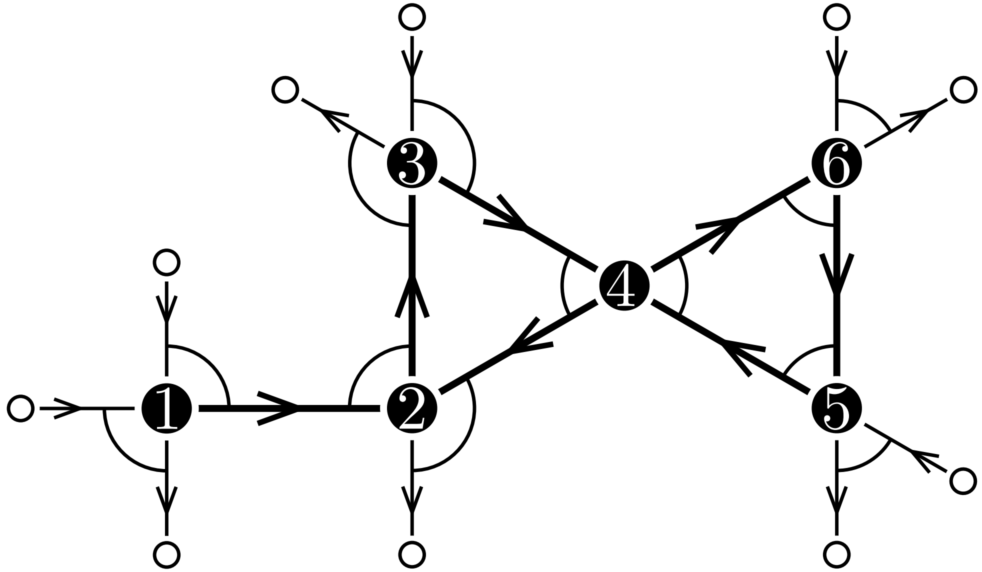
- $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$
- contains no factor π or π^{-1} for any path $\pi \in I$
- contains no $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for any arrow $\alpha \in Q_1$

BLOSSOMING QUIVERS AND WALKS



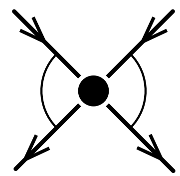
blossoming quiver $\bar{Q}^{❀} =$
add blossoms to complete each vertex to 

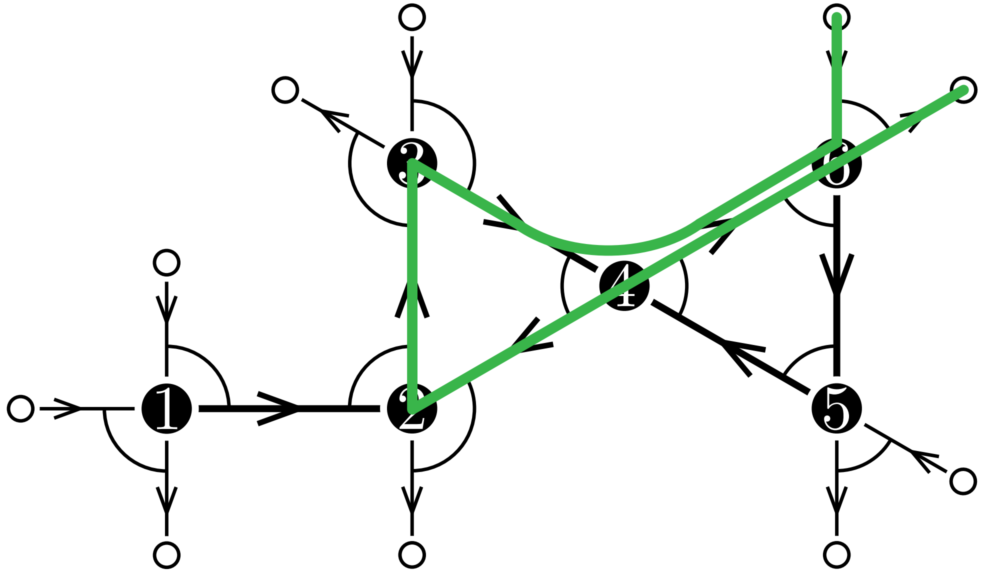
BLOSSOMING QUIVERS AND WALKS



blossoming quiver $\bar{Q}^{❀} =$

add blossoms to complete each vertex to

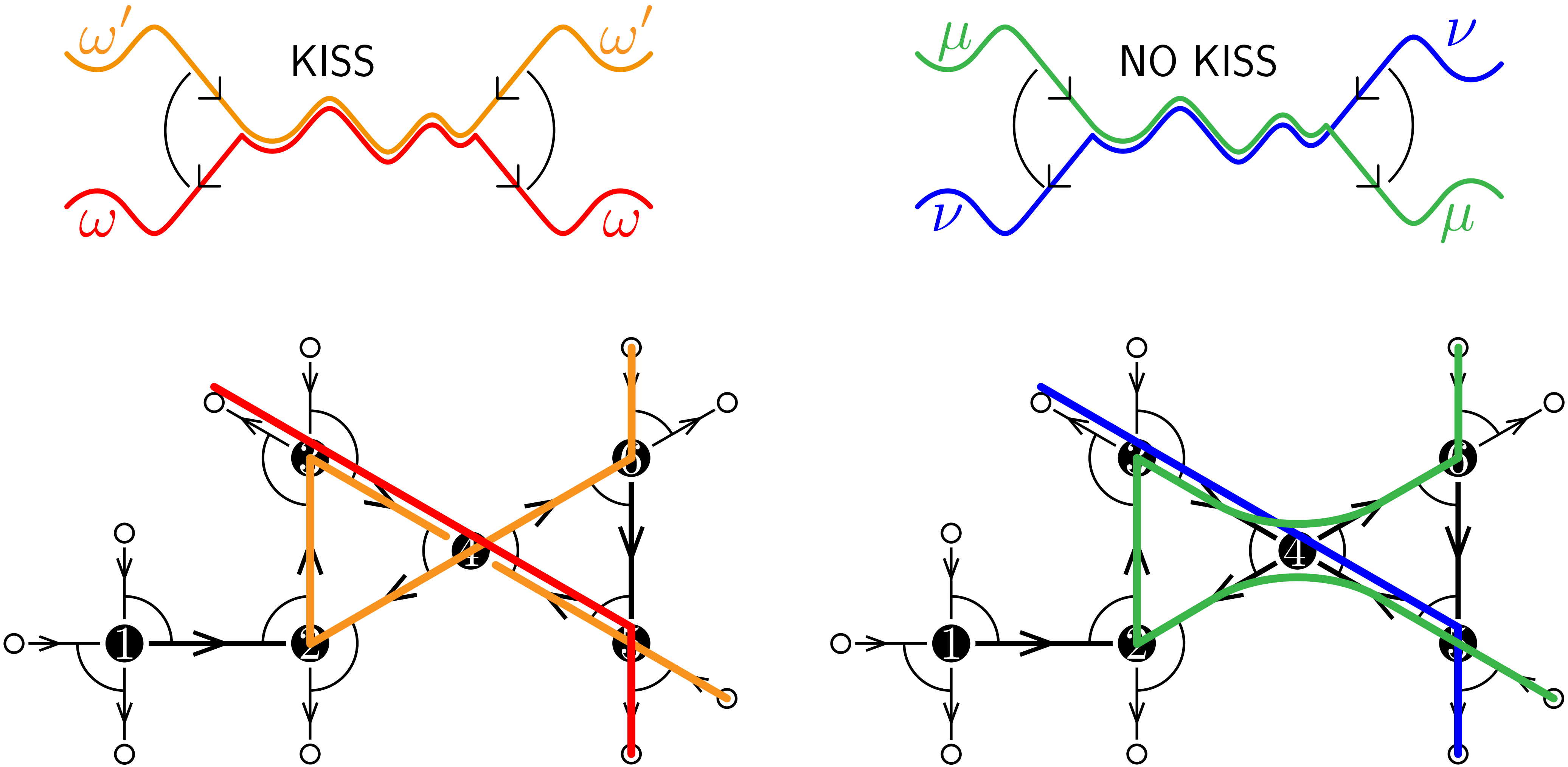




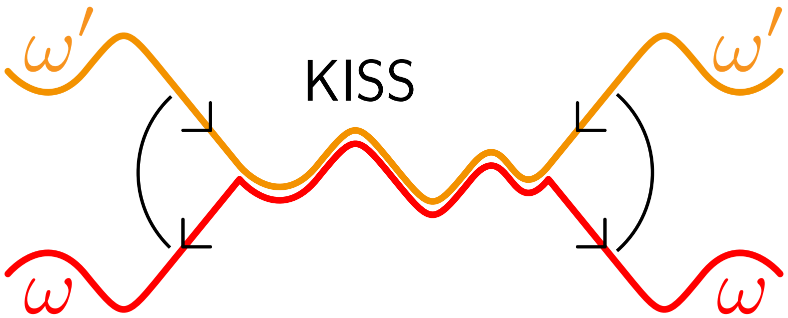
walk $\omega =$ maximal string in $\bar{Q}^{❀}$

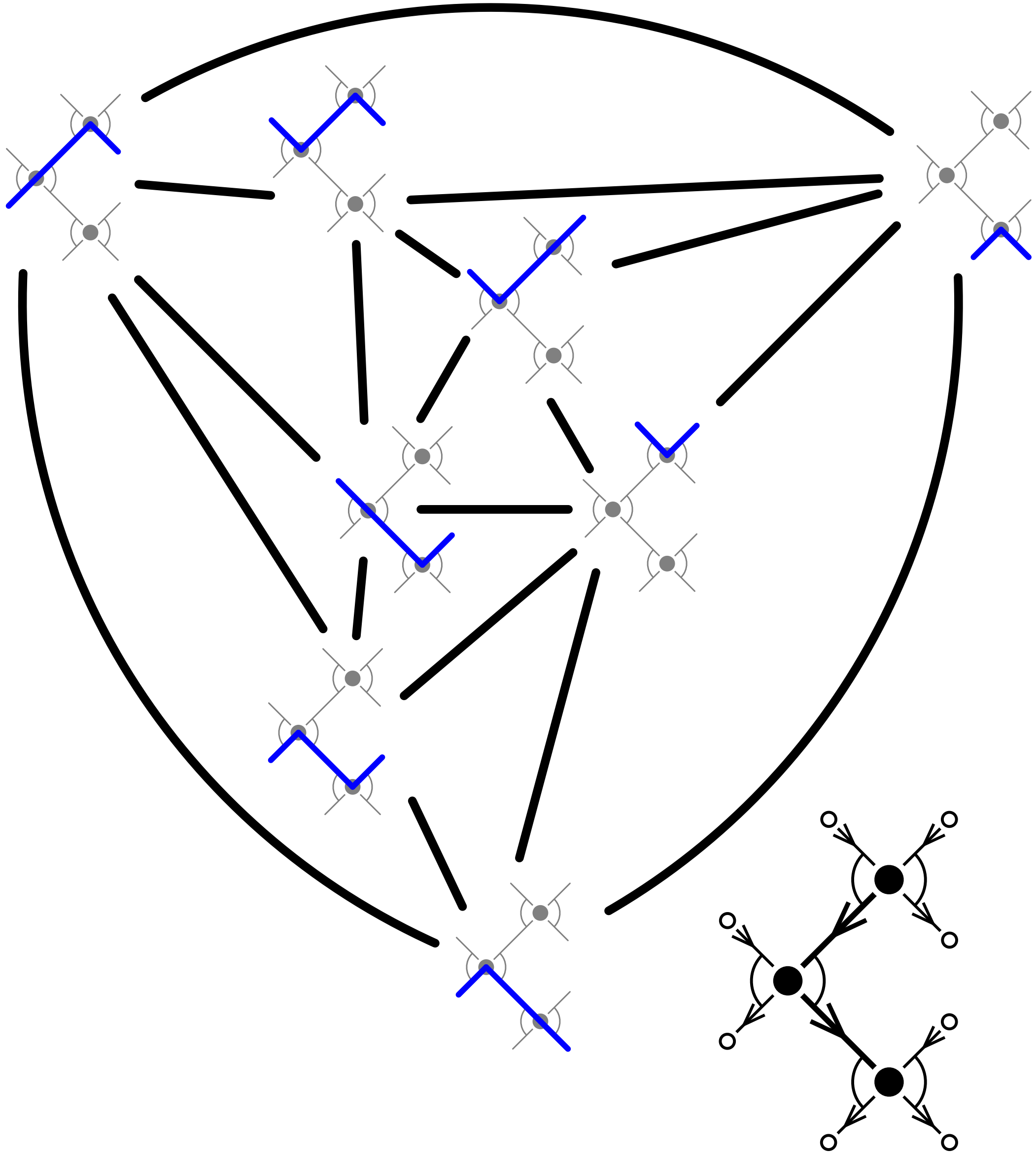
from blossoms to blossoms

KISSING



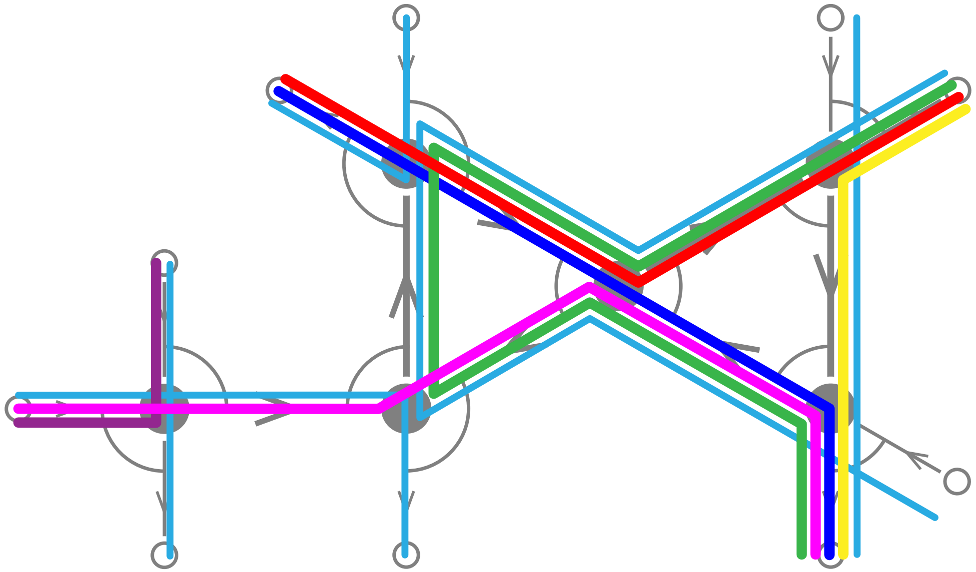
NON-KISSING COMPLEX





[reduced] non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q}) =$

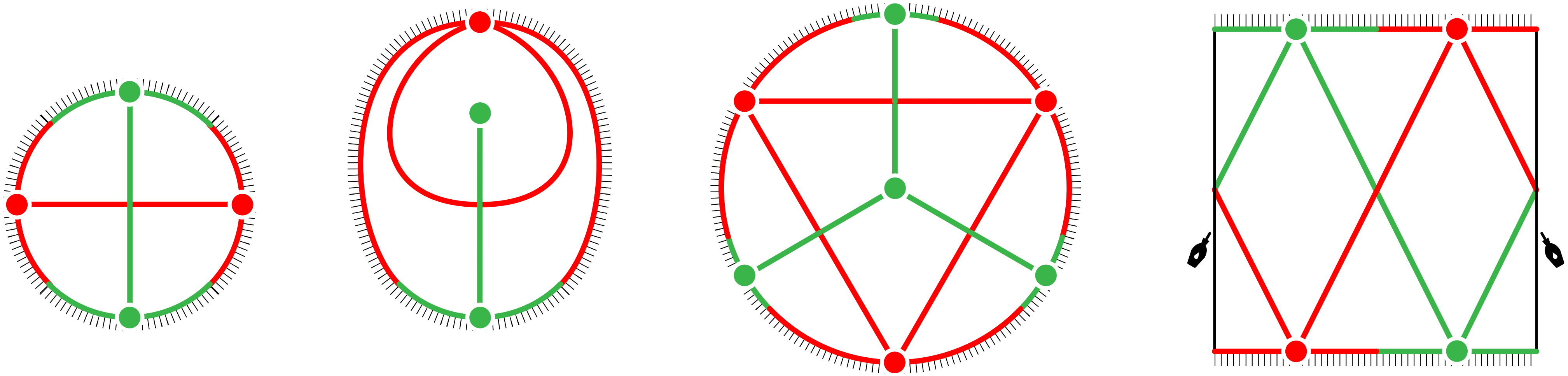
- vertices = [bending] walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing [bending] walks in $\bar{Q}^{❀}$



NON-KISSING VS NON-CROSSING

Palu–P.–Plamondon,
Non-kissing and non-crossing complexes for locally gentle algebras ('19)

DUAL DISSECTIONS

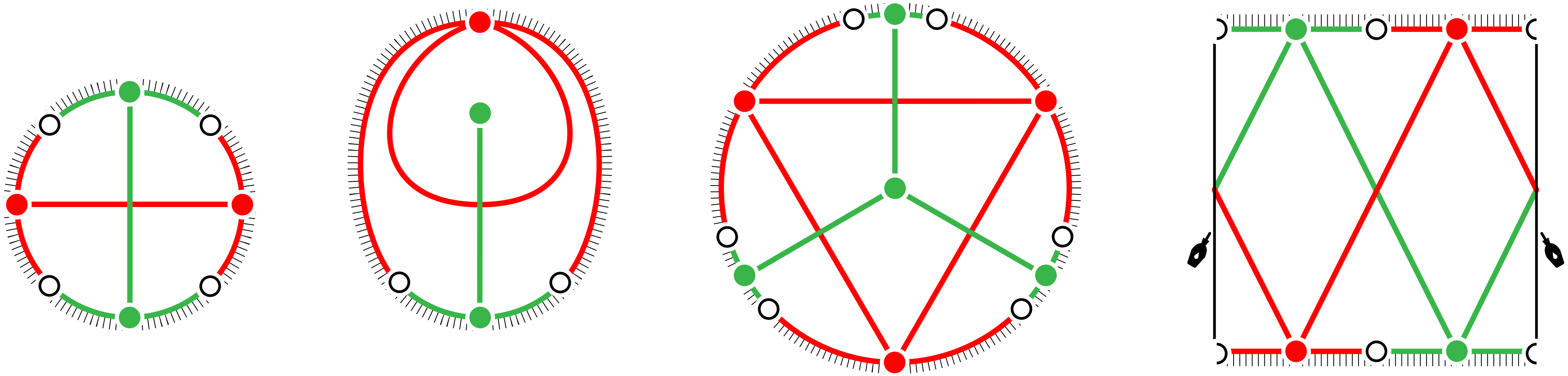


$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

DUAL DISSECTIONS



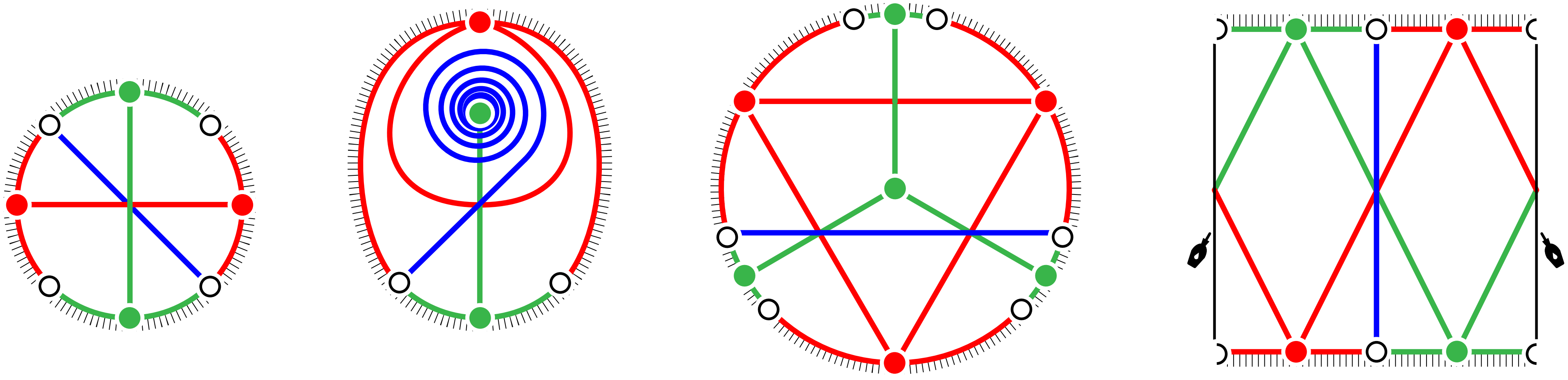
$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

blossom vertices = white vertices, alternating with $\mathrm{V} \cup \mathrm{V}^*$ along the boundary of $\mathcal{S}$

DUAL DISSECTIONS



$\mathcal{S}$ = orientable surface with or without boundaries

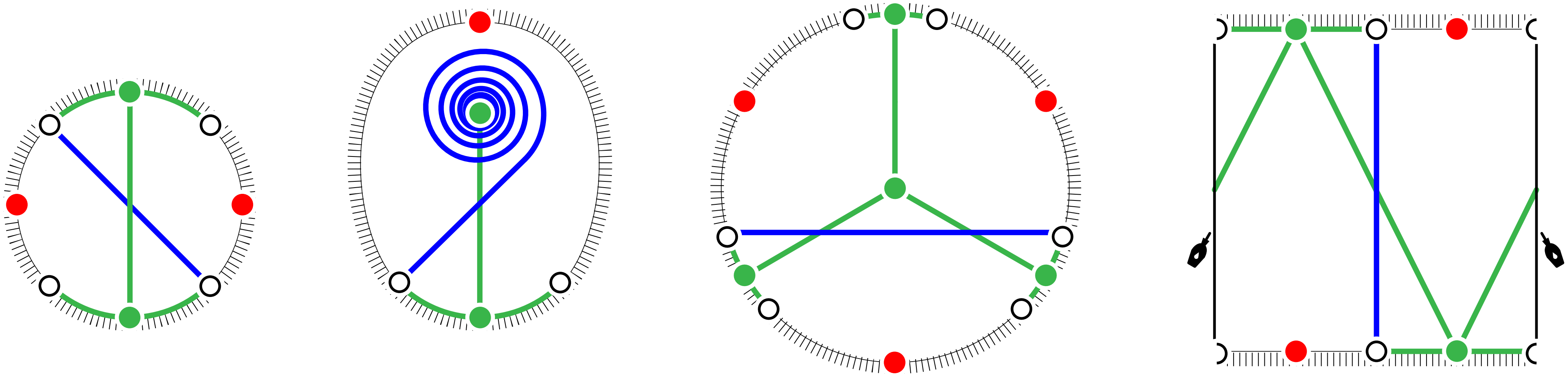
V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

blossom vertices = white vertices, alternating with $\mathrm{V} \cup \mathrm{V}^*$ along the boundary of $\mathcal{S}$

B-curve = curve which at each endpoint either reaches a blossom point or infinitely circles around a puncture of $\mathcal{S}$

ACCORDIONS



D-accordion = B-curve α such that whenever α meets a face f of D,

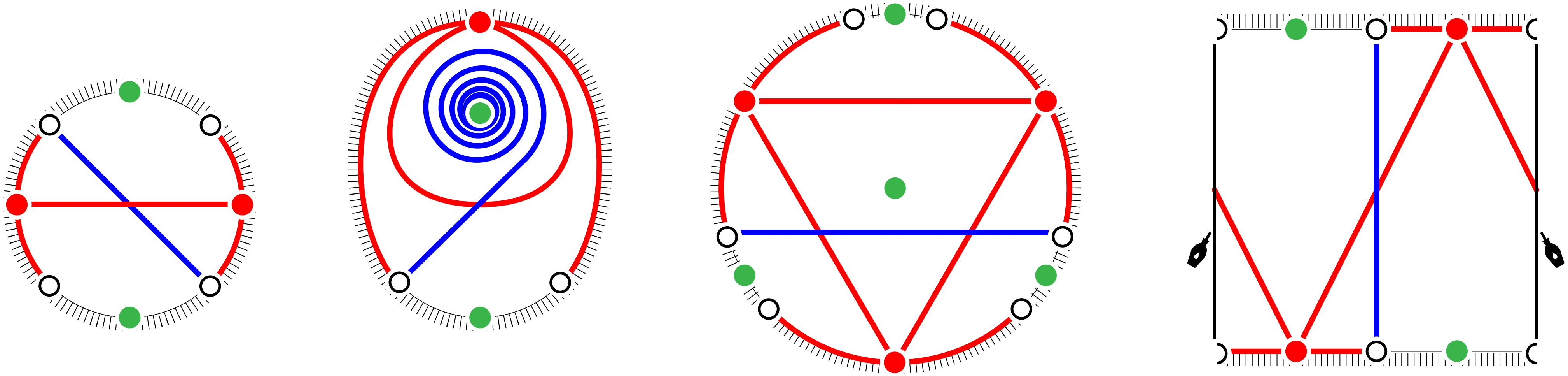
(i) it enters crossing an edge a of f and leaves crossing an edge b of f

(ii) the two edges a and b of f crossed by α are consecutive along the boundary of f,

(iii) α, a and b bound a disk inside f that does not contain f^*.

D-accordion complex = simplicial complex of pairwise non-crossing sets of D-accordions

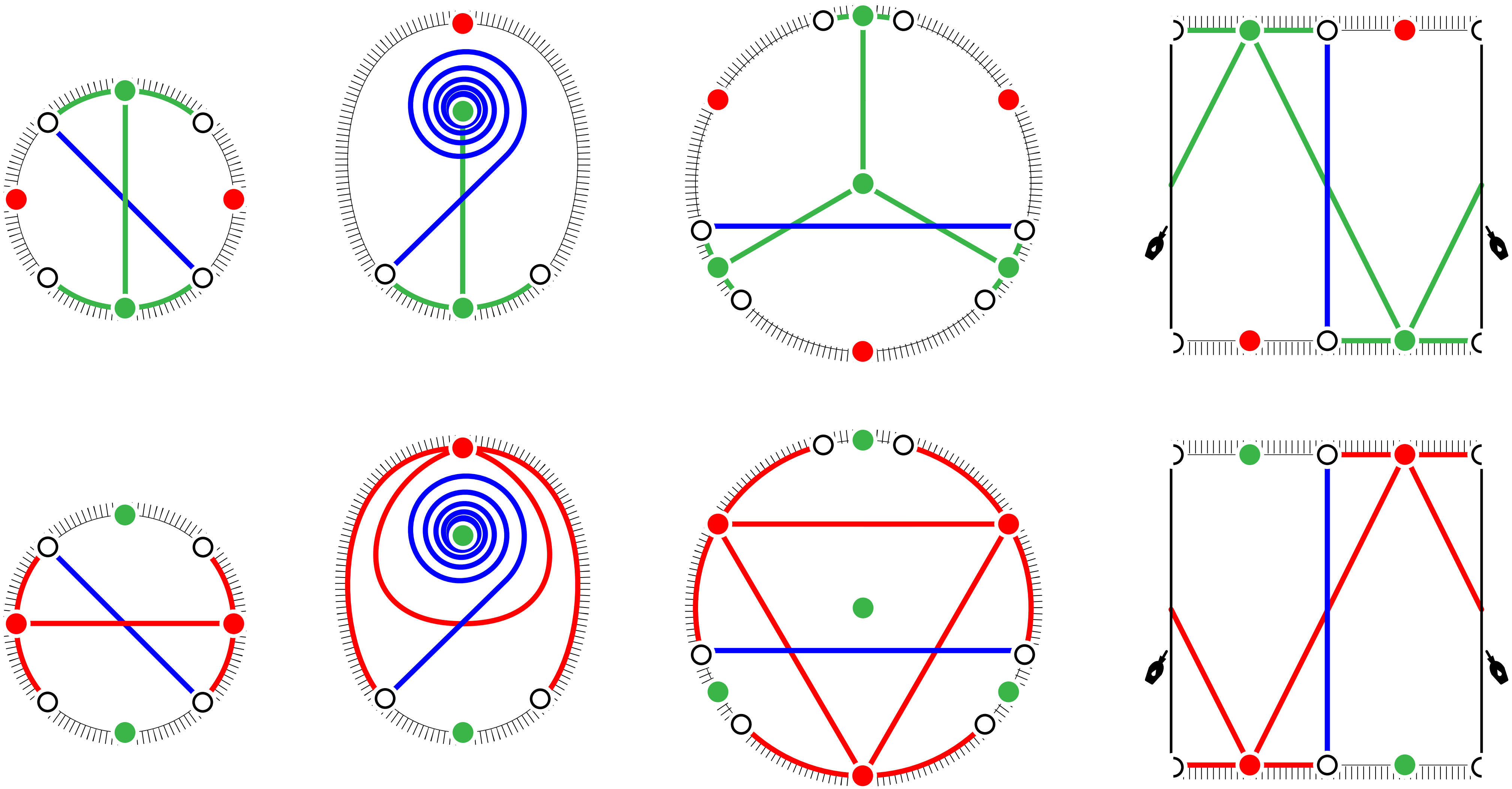
SLALOMS



D*-slalom = B-curve α of $\bar{S}$ such that, whenever α crosses an edge a^* of D* contained in two faces f^*, g^* of D*, the marked points f and g lie on opposite sides of α in the union of f^* and g^* glued along a^*.

D*-slalom complex = simplicial complex of pairwise non-crossing sets of D*-slaloms

D-ACCORDIONS = D*-SLALOMS

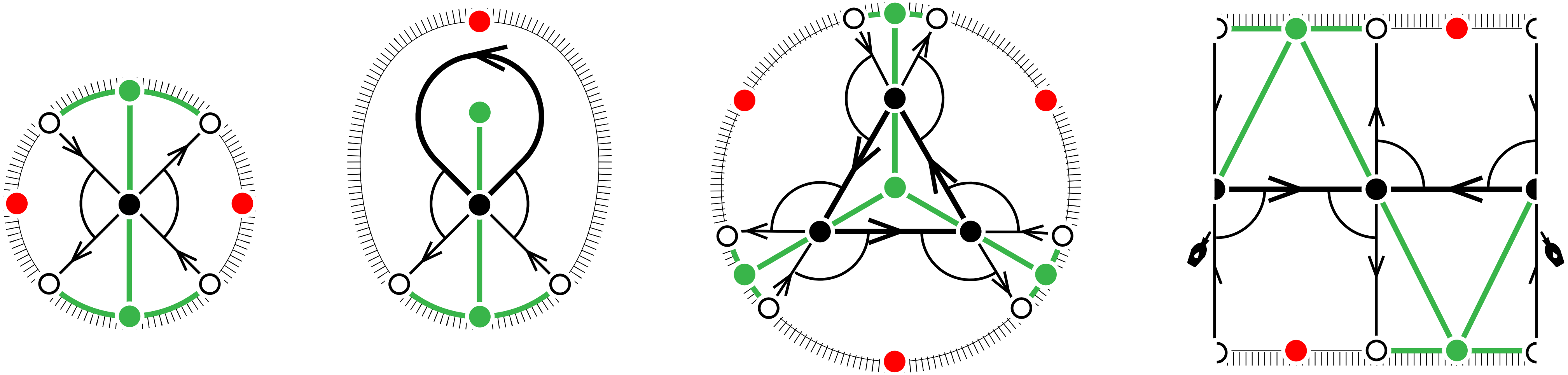


(D, D*)-non-crossing complex = D-accordion complex = D*-slalom complex

QUIVER OF A DISSECTION

quiver $\bar{Q}_{\mathrm{D}}$ of a dissection =

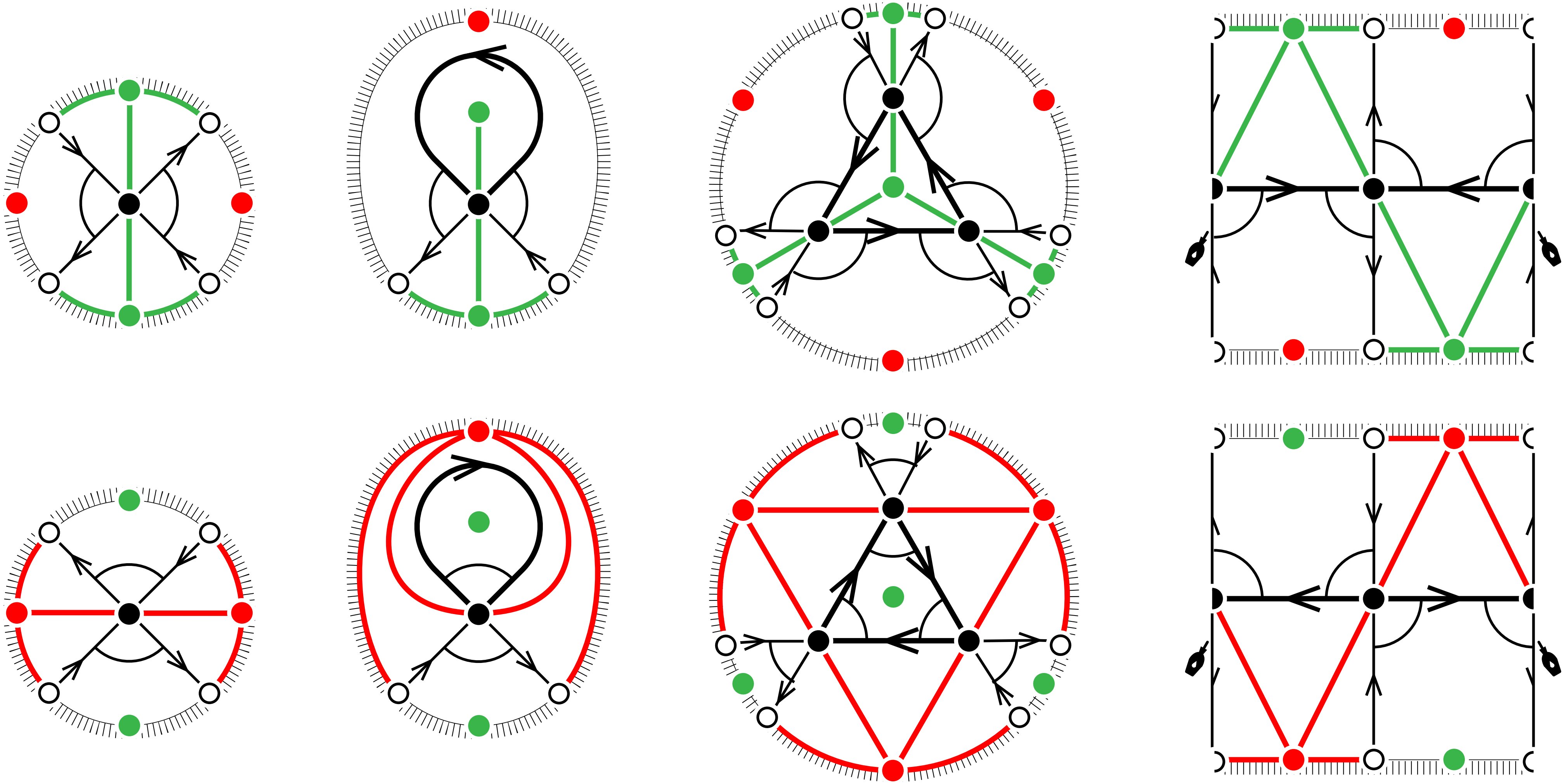
- vertices = edges of D (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of D
- relations = three consecutive edges around a face of D



QUIVER OF A DISSECTION

quiver $\bar{Q}_{\mathrm{D}}$ of a dissection =

- vertices = edges of D (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of D
- relations = three consecutive edges around a face of D



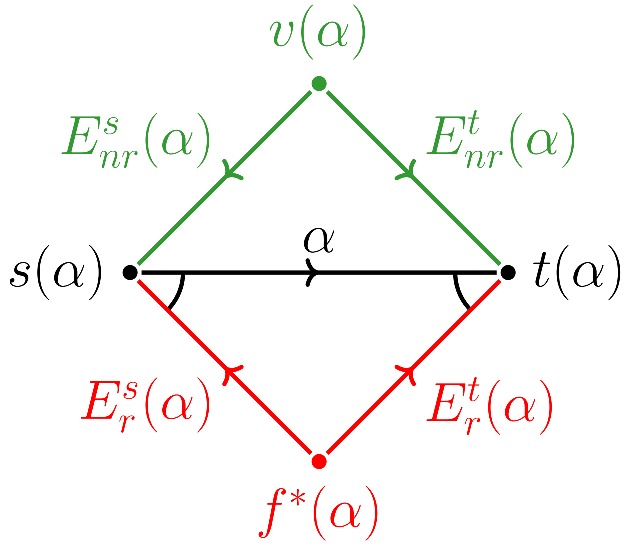
SURFACE OF A GENTLE QUIVER

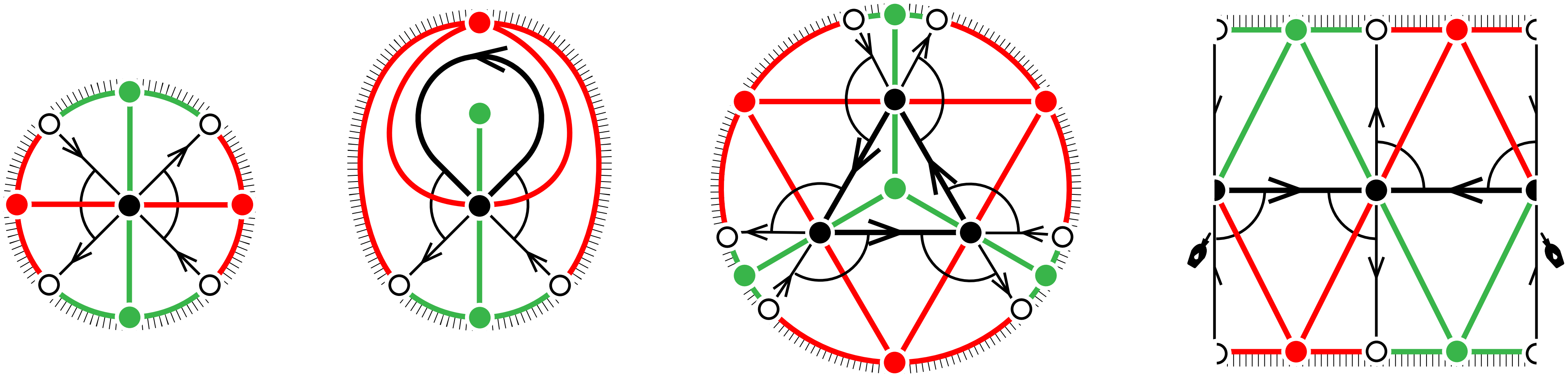
surface $\mathcal{S}_{\bar{Q}}$ of quiver $\bar{Q}$ = surface obtained from the blossoming quiver $\bar{Q}^{\text{❀}}$ as follows:

(i) for each arrow $\alpha \in Q_1^{\text{❀}}$, consider a lozenge

(ii) for any $\alpha, \beta \in Q_1^{\text{❀}}$ with $t(\alpha) = s(\beta)$, proceed to the following identifications:

- if $\alpha\beta \in I$, then glue $E_r^t(\alpha)$ with $E_r^s(\beta)$,
- if $\alpha\beta \notin I$, then glue $E_{nr}^t(\alpha)$ with $E_{nr}^s(\beta)$.

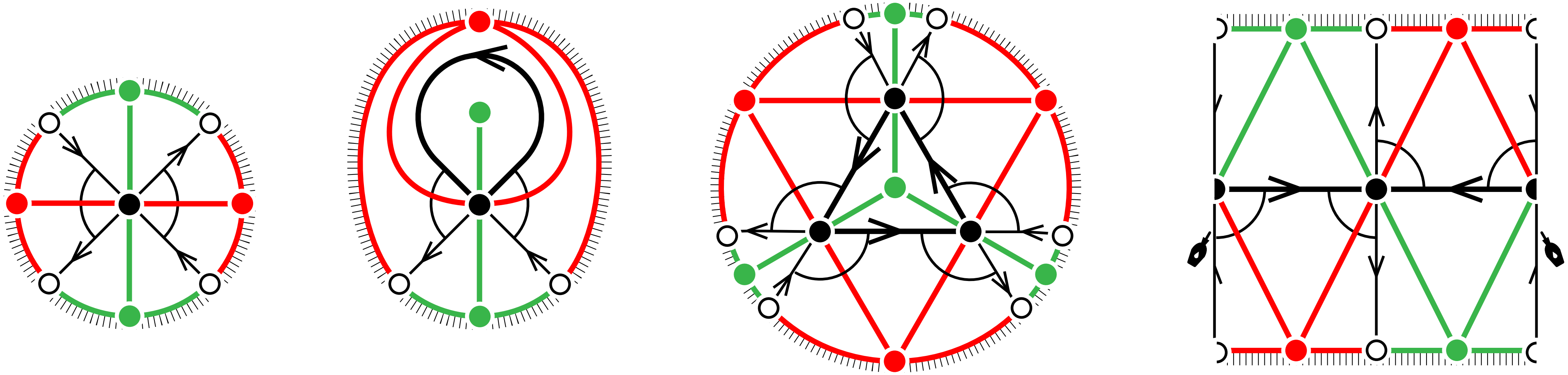




NON-CROSSING VS NON-KISSING

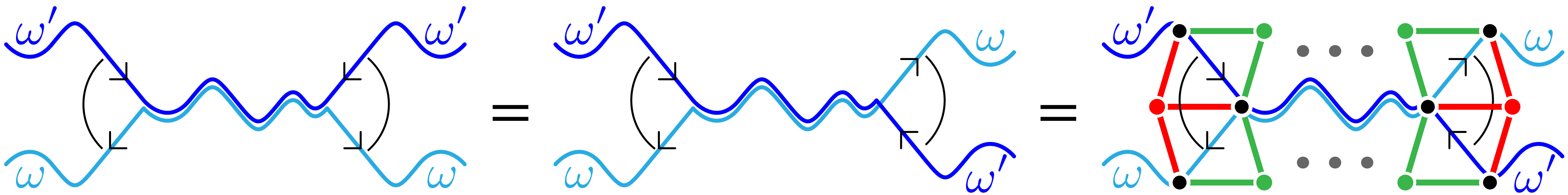
PROP. The two previous constructions are inverse to each other and define bijections:

pairs of dual dissections on a surface $\longleftrightarrow$ gentle quivers



PROP. It defines isomorphisms between:

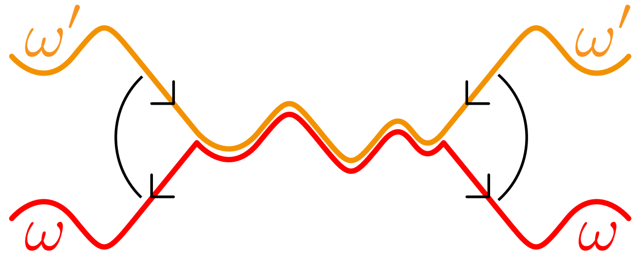
non-crossing complex of dissections $\longleftrightarrow$ non-kissing complex of gentle quiver



Palu–P.–Plamondon ('19)

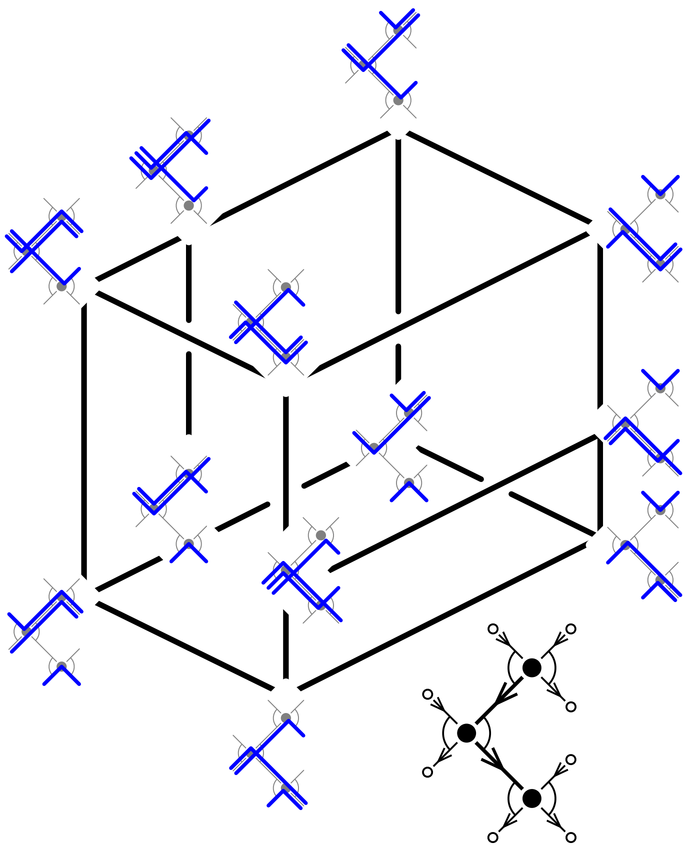
END OF THE TALK

non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q}) =$
- vertices $=$ walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces $=$ collections of pairwise non-kissing walks in $\bar{Q}^{❀}$

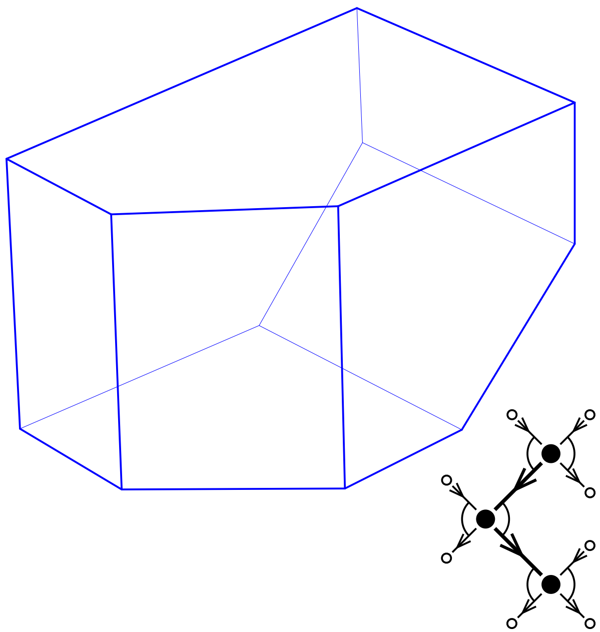


... generalizing the associahedron

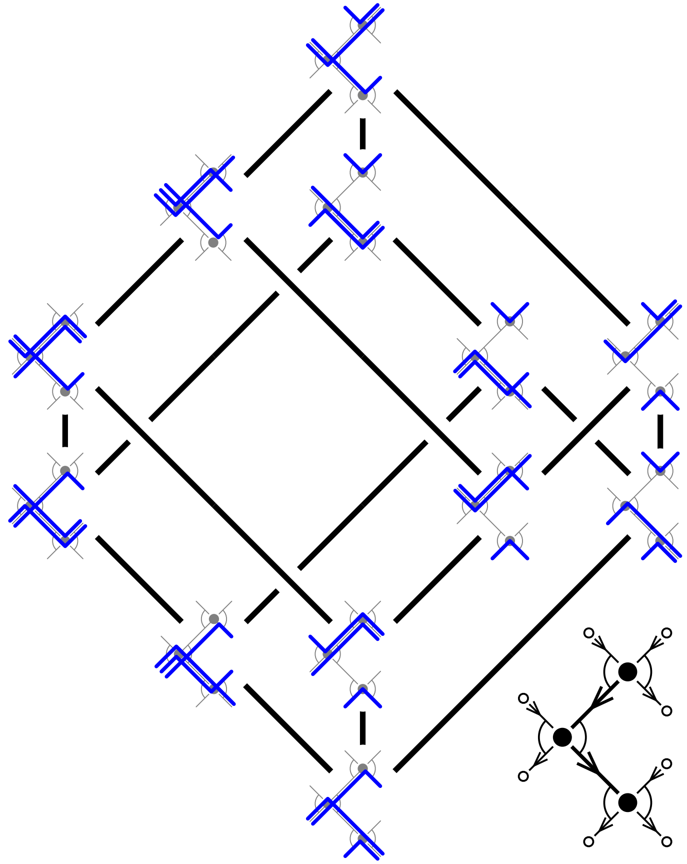
Flip graph



Associahedron



Tamari lattice

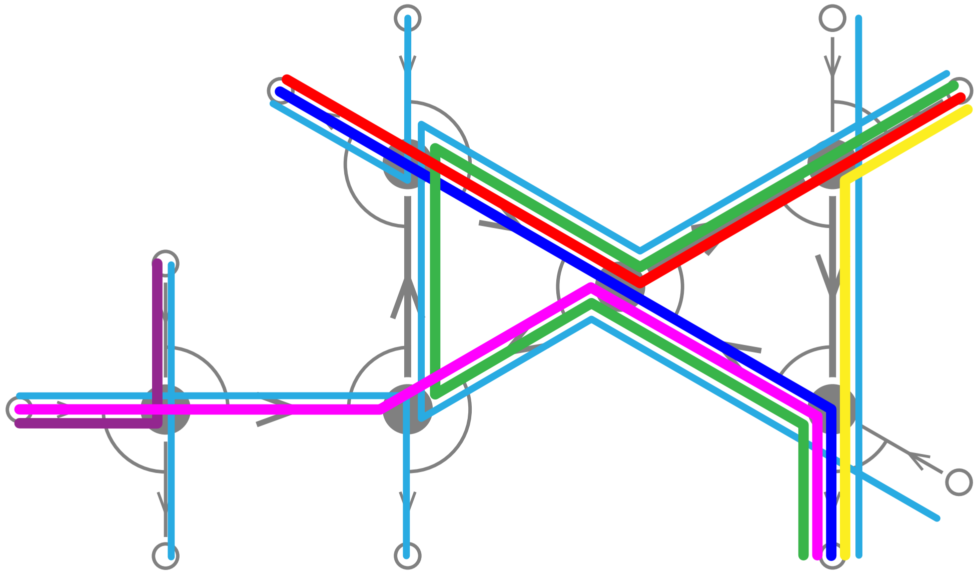


DISTINGUISHED ARROWS AND FLIPS

McConville, *Lattice structures of grid Tamari orders* ('17)

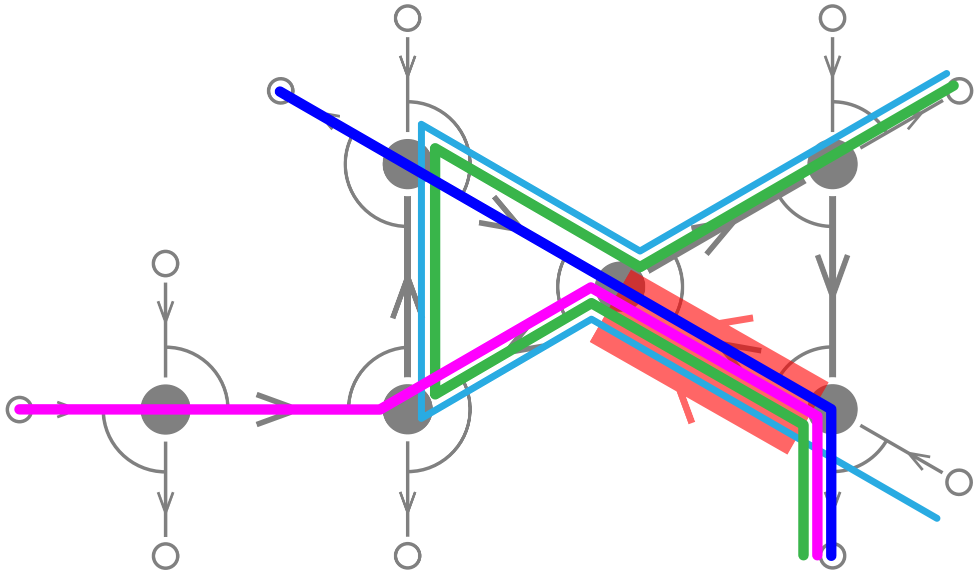
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

DISTINGUISHED WALKS, ARROWS AND STRINGS

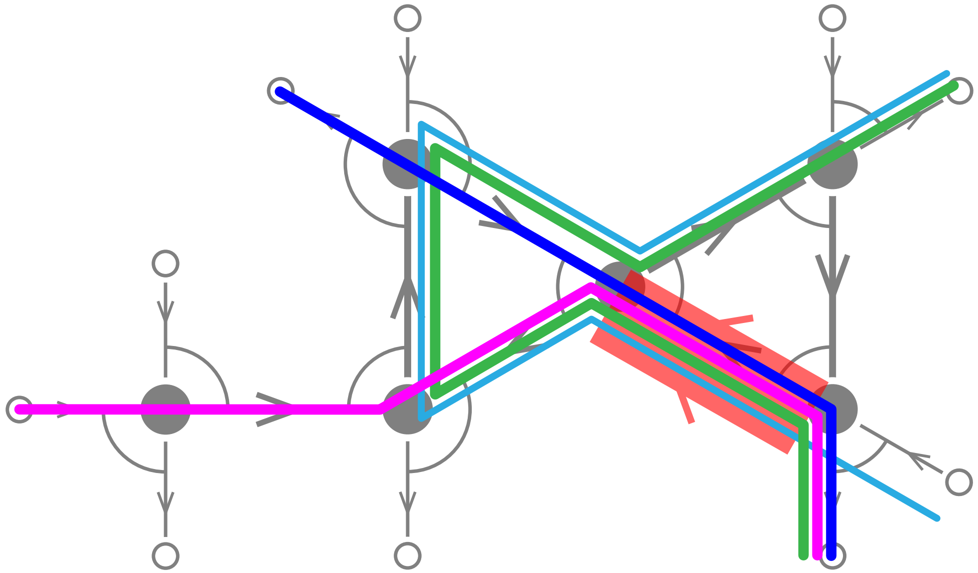


F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

$\alpha \in Q_1$

$F_{\alpha} = \{\omega \in F \mid \alpha \in \omega\}$

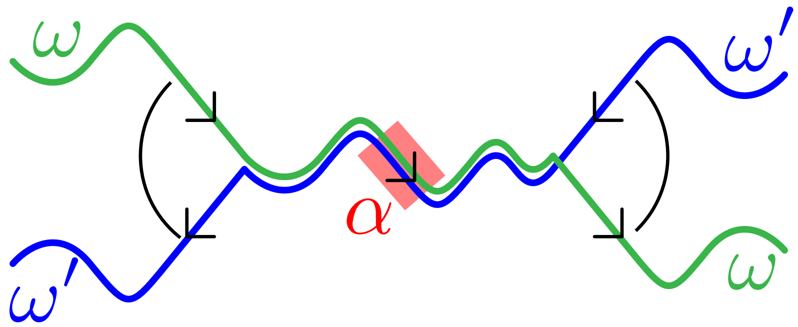
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

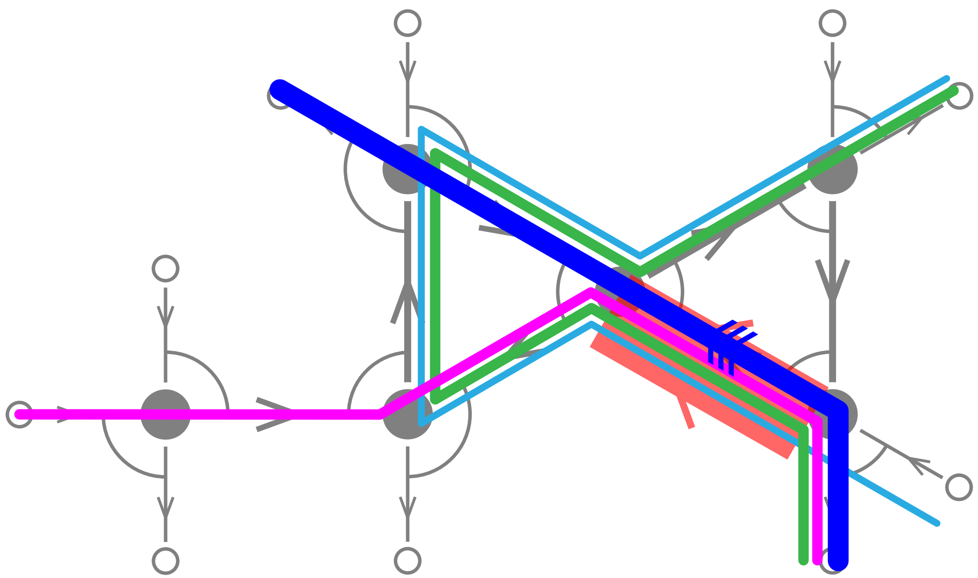
$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

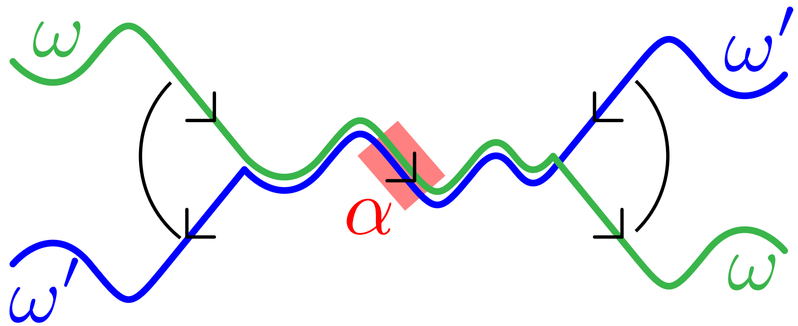
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$

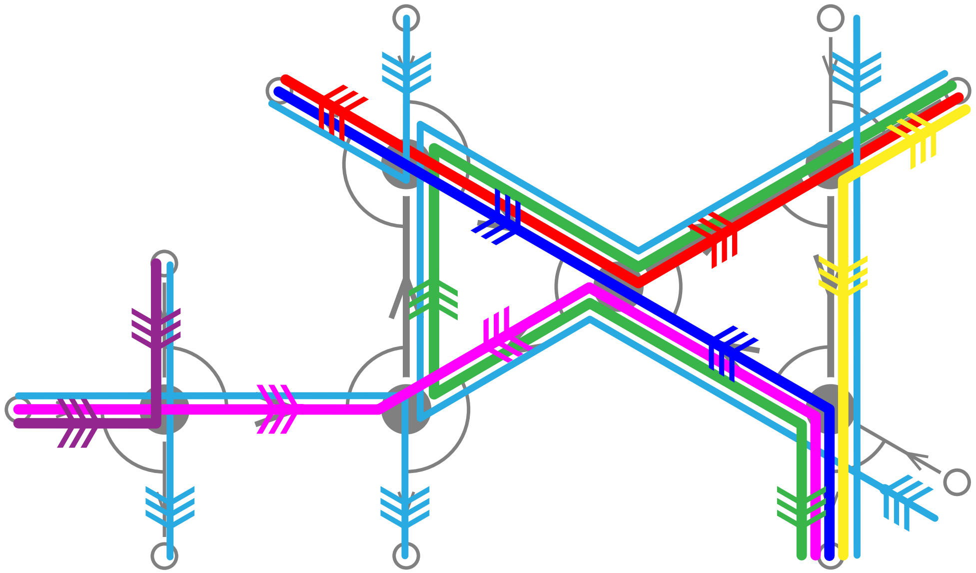


$\omega \prec_\alpha \omega'$ countercurrent order at α

distinguished walk at α in F = dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in F = da$(\omega, F) = \{\alpha \in Q_1 \mid \omega =$ dw$(\alpha, F)\}$

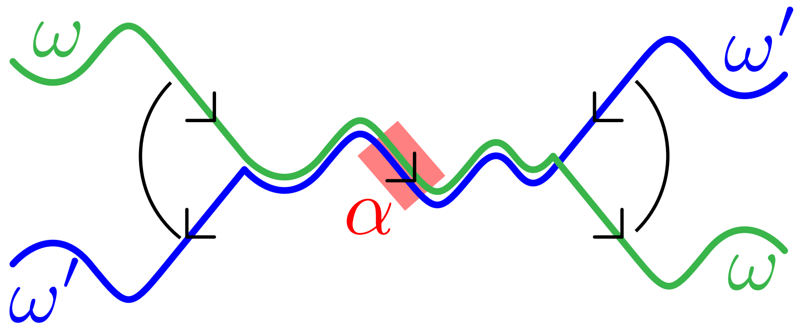
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

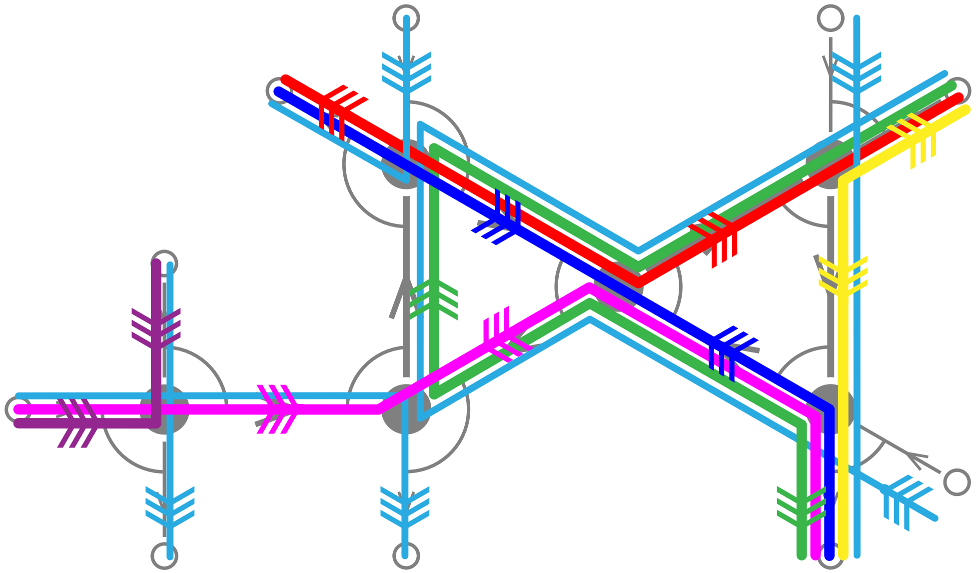
distinguished walk at α in F = dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in F = da$(\omega, F) = \{\alpha \in Q_1 \mid \omega =$ dw$(\alpha, F)\}$

PROP. For any facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$,

- each bending walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

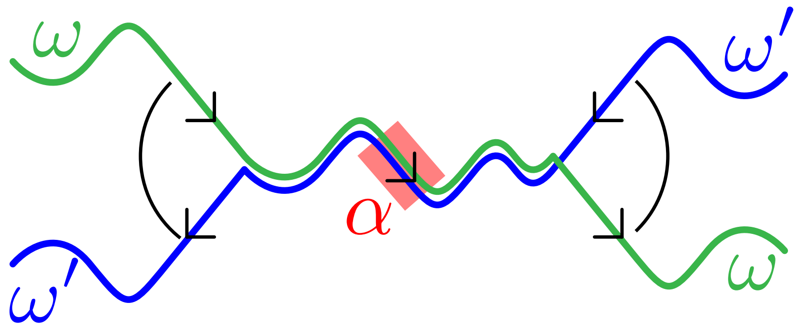
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

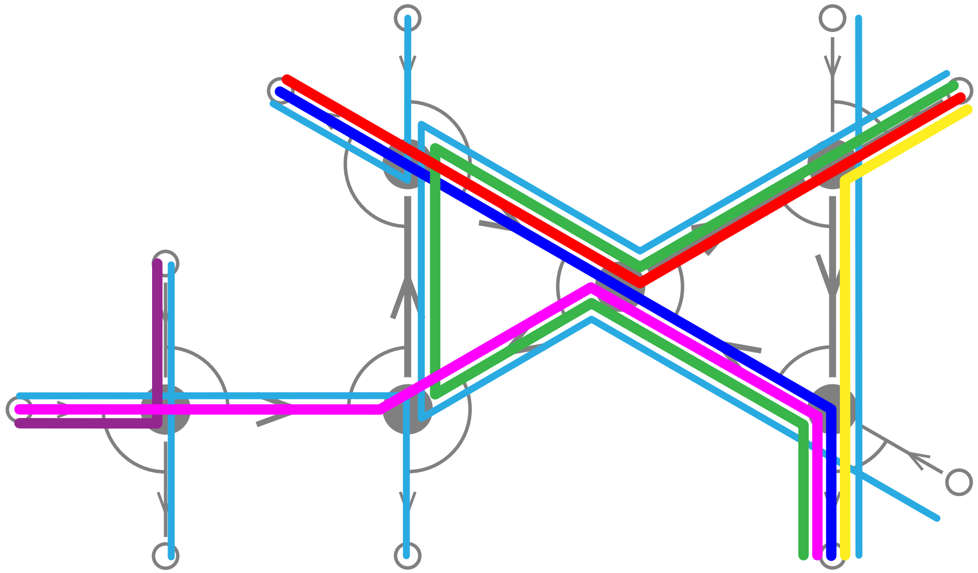
distinguished walk at α in F = dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in F = da$(\omega, F) = \{\alpha \in Q_1 \mid \omega =$ dw$(\alpha, F)\}$

PROP. For any facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$,

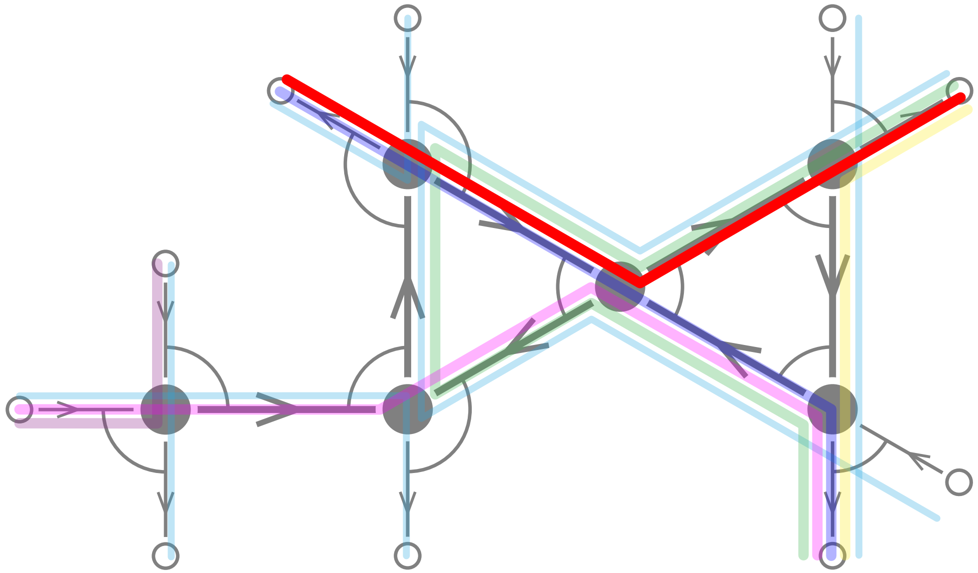
- each bending walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

CORO. $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is pure of dimension $|Q_0|$.

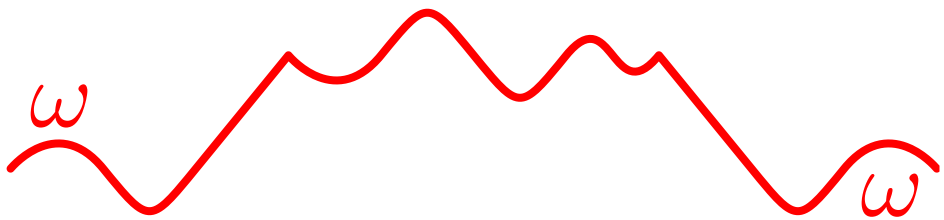


F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

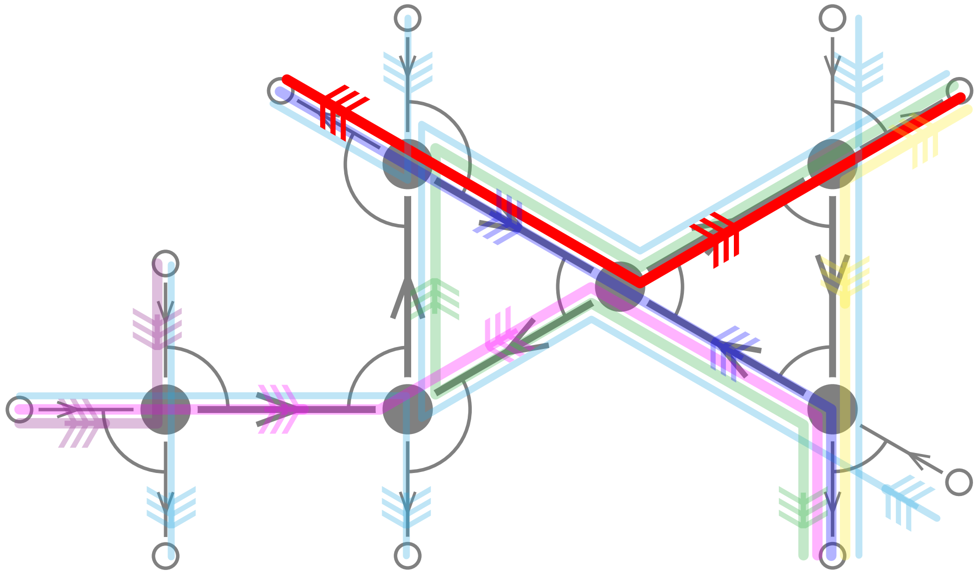
FLIPS



F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$\omega \in F$ we want to "flip"



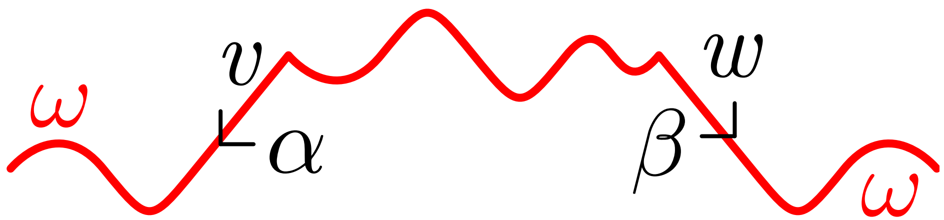
FLIPS



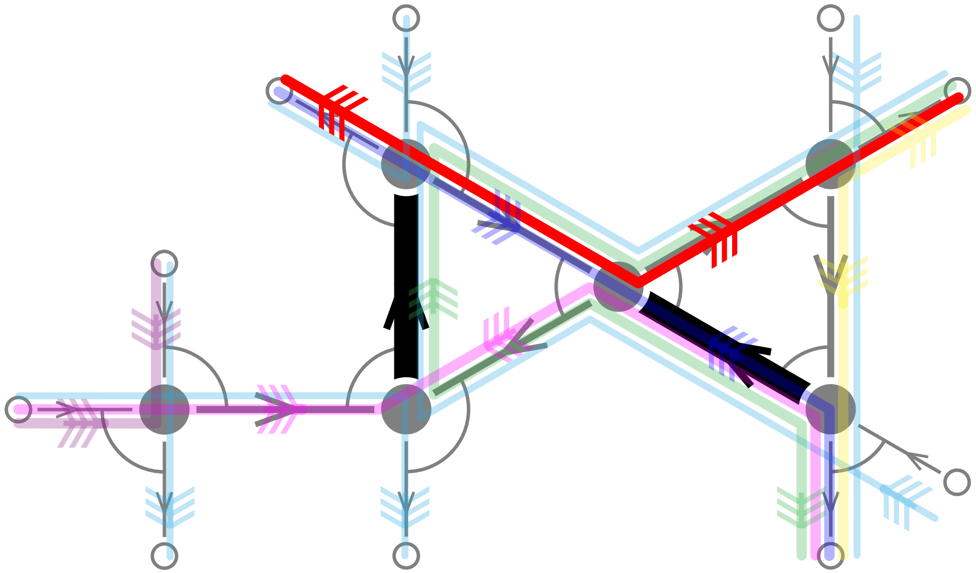
F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$



FLIPS

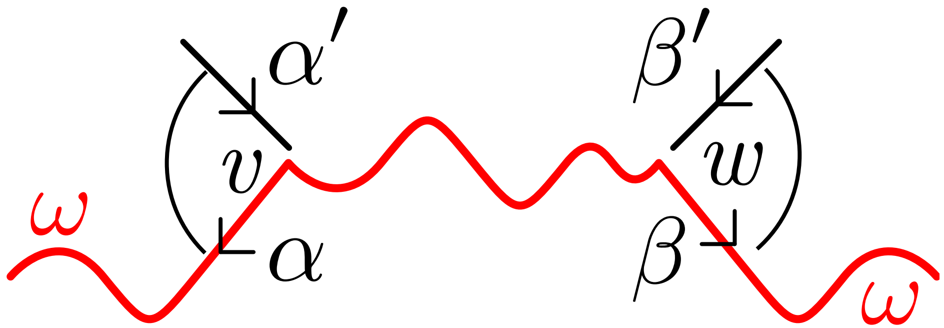


F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

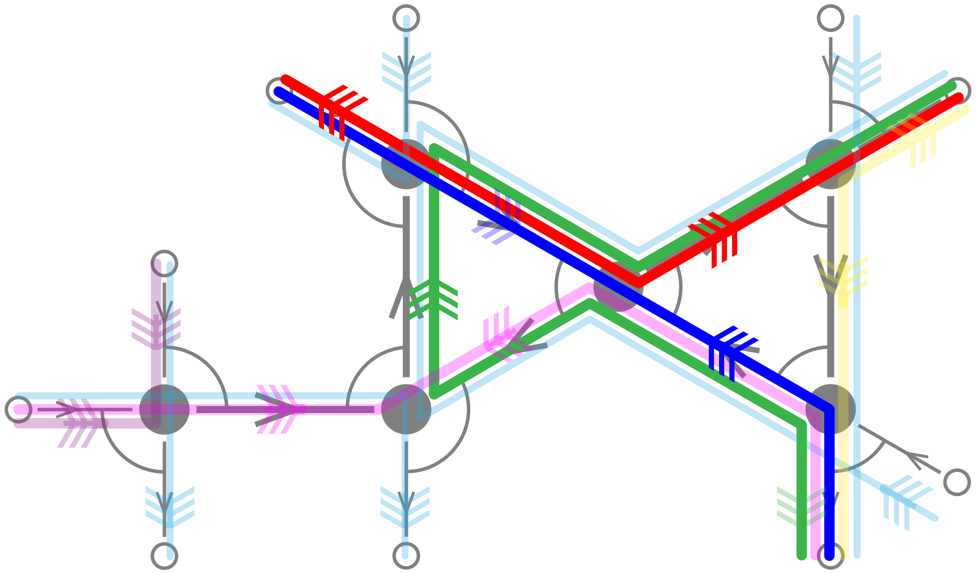
$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$



FLIPS



F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

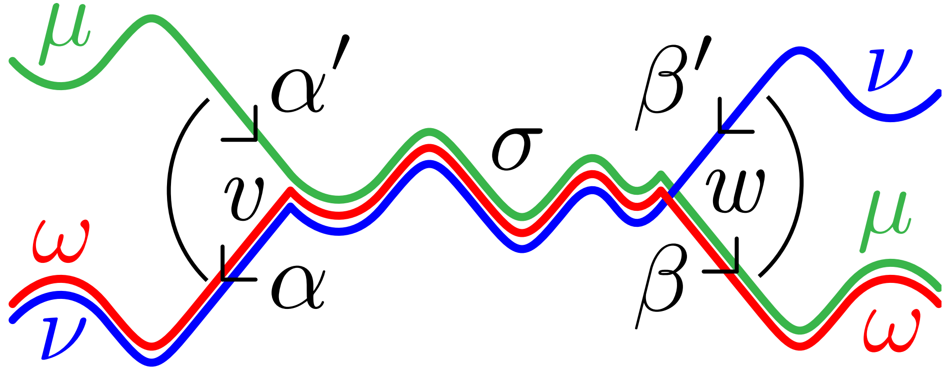
$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

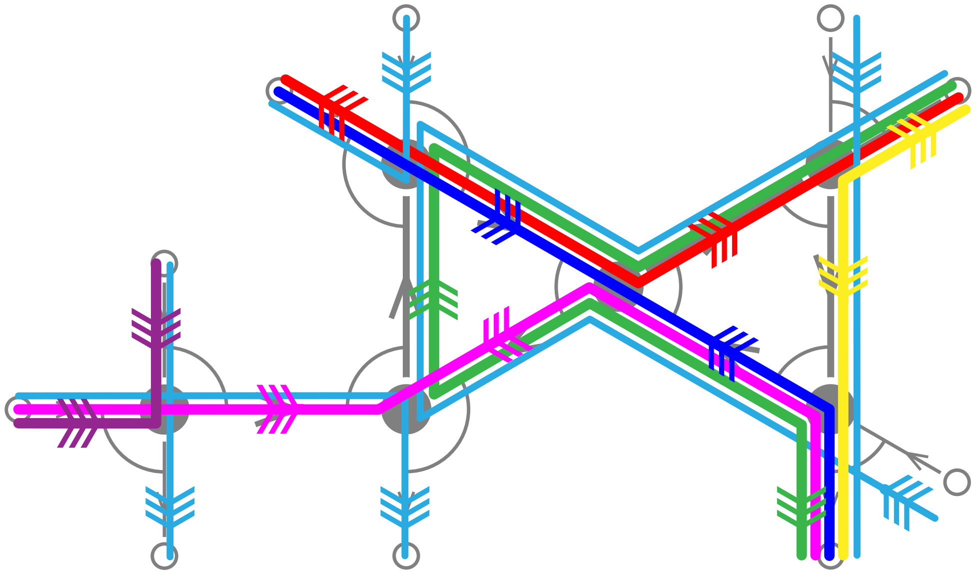
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

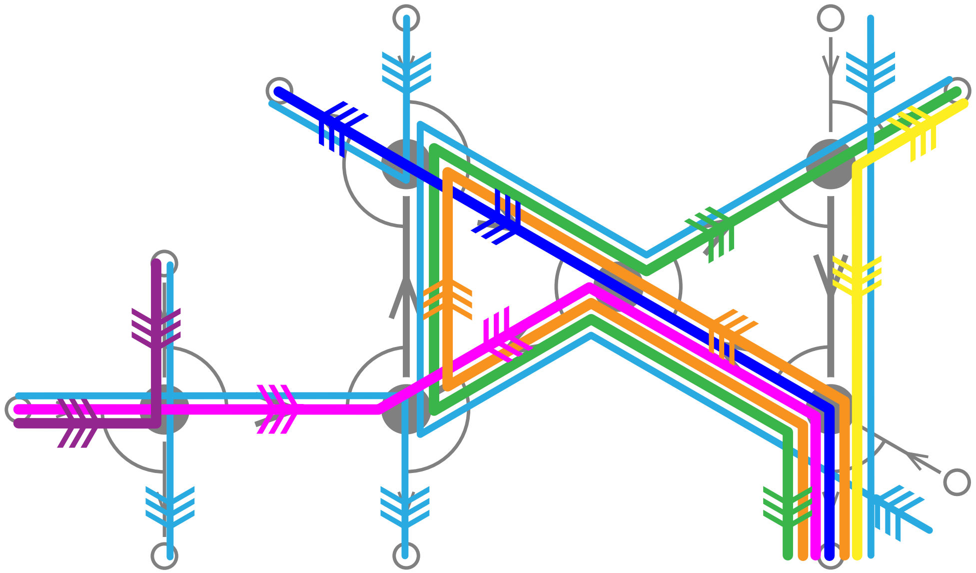
$\mu = \mathsf{dw}(\alpha', F)$ and $\nu = \mathsf{dw}(\beta', F)$

$\omega = \nu[\cdot, v]\, \sigma\, \mu[w, \cdot]$



FLIPS





F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

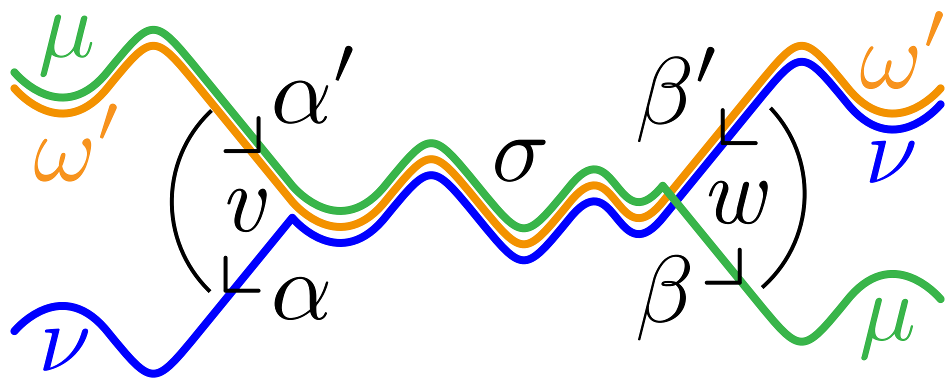
$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

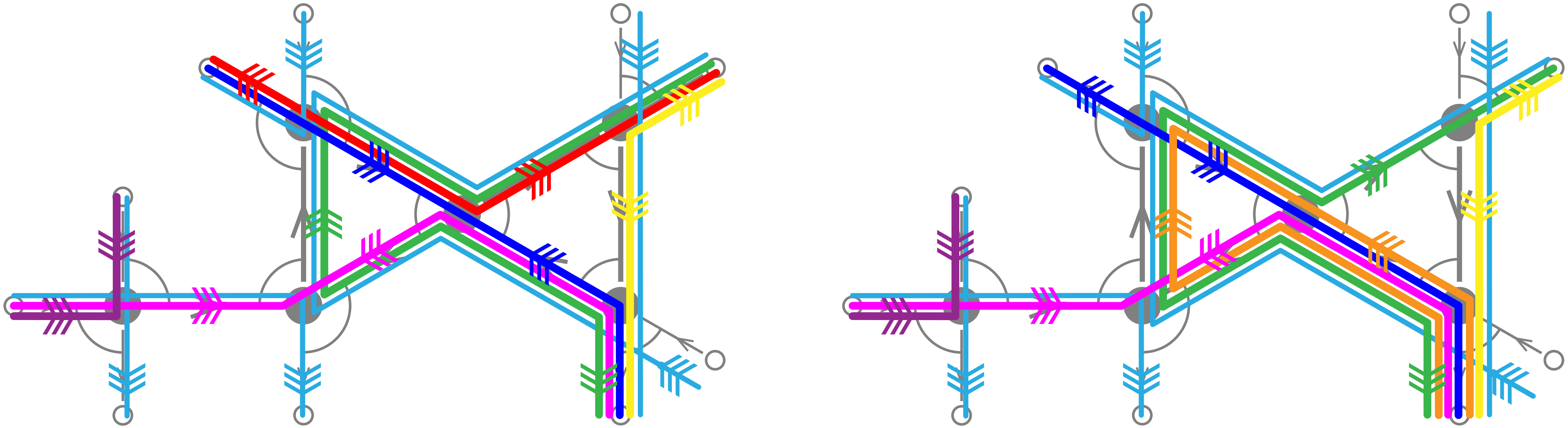
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = \mathsf{dw}(\alpha', F)$ and $\nu = \mathsf{dw}(\beta', F)$

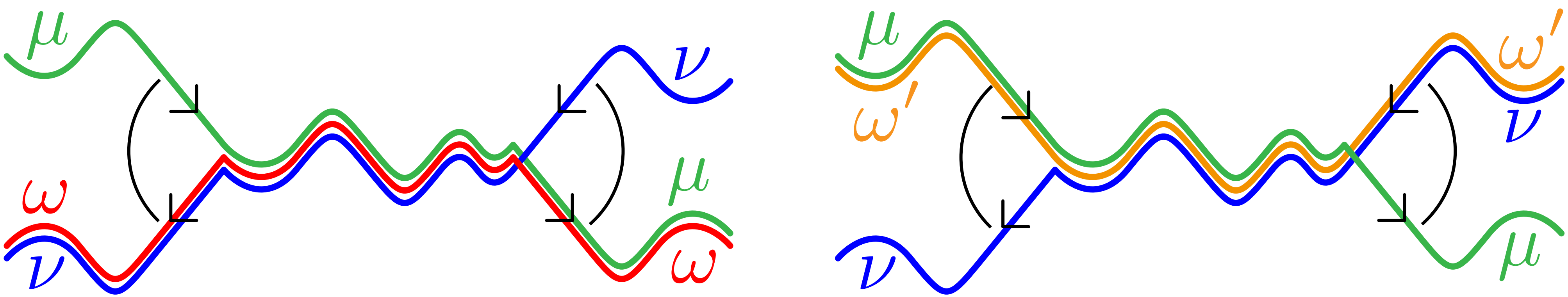
$\omega = \nu[\cdot, v]\,\sigma\,\mu[w, \cdot]$

$\omega' = \mu[\cdot, v]\,\sigma\,\nu[w, \cdot]$





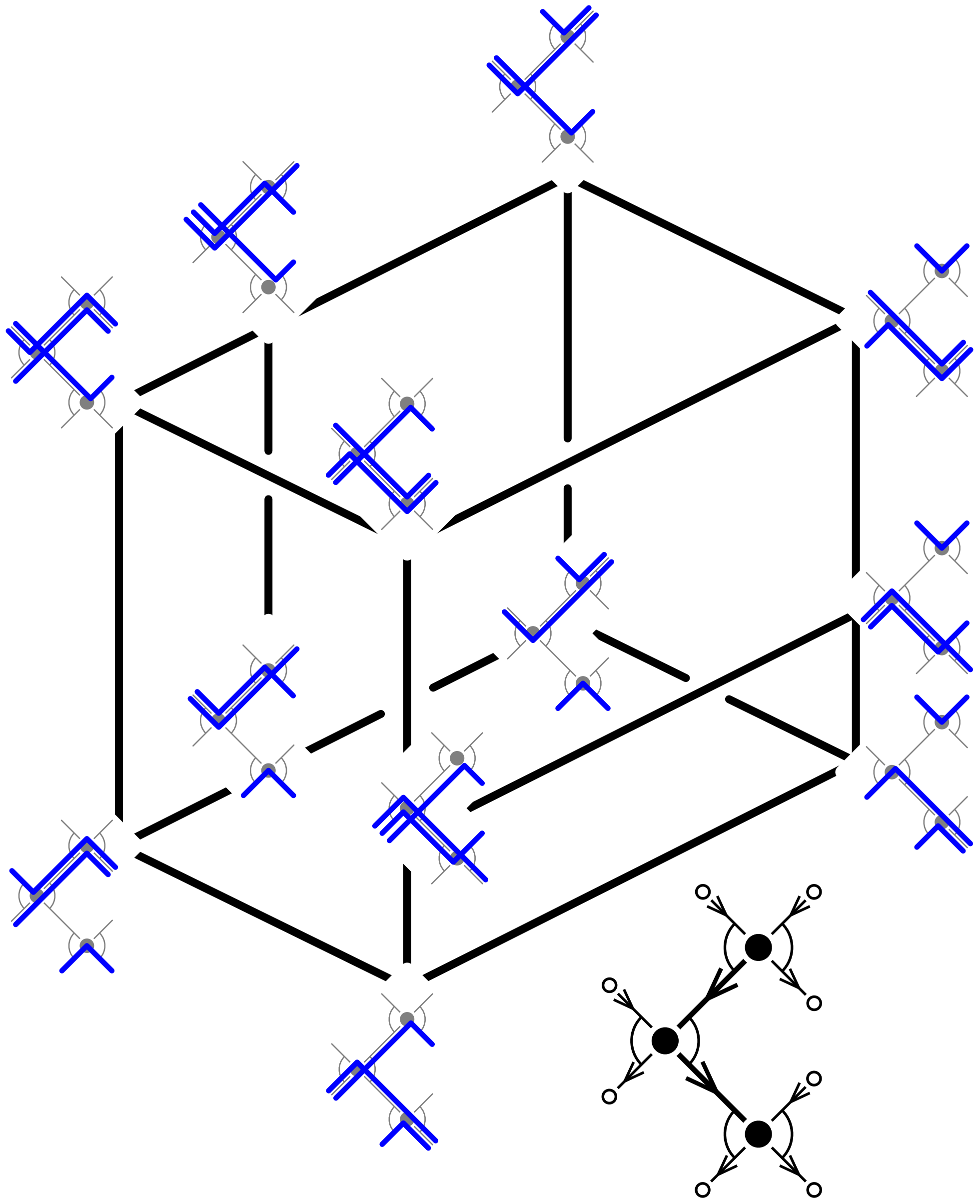
PROP. ω' kisses ω but no other walk of F. Moreover, ω' is the only such walk.



FLIPS

flip graph =

- vertices = non-kissing facets
- edges = flips



GENTLE ASSOCIAHEDRA

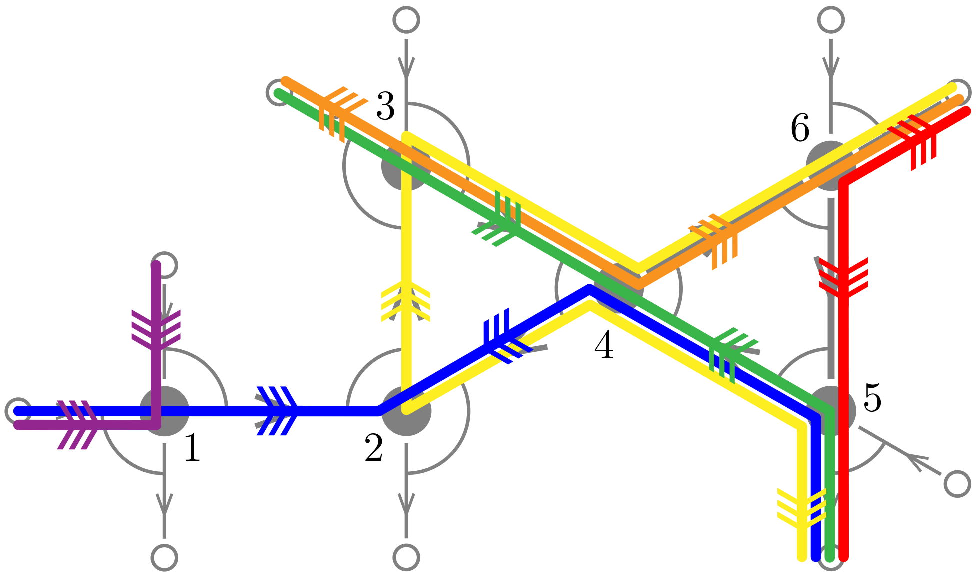
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

$\boldsymbol{G}$-VECTORS & $\boldsymbol{C}$-VECTORS

<u>multiplicity vector</u> $\boldsymbol{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of Q_0 $\quad = \quad \sum_{i \in [m]} \boldsymbol{e}_{v_i} \in \mathbb{R}^{Q_0}$

<u>$\boldsymbol{g}$-vector</u> $\boldsymbol{g}(\omega)$ of a walk ω $\quad = \quad \boldsymbol{m}_{\mathsf{peaks}(\omega)} - \boldsymbol{m}_{\mathsf{deeps}(\omega)}$

<u>$\boldsymbol{c}$-vector</u> $\boldsymbol{c}(\omega \in F)$ of a walk ω in a non-kissing facet F $\quad = \quad \varepsilon(\omega, F)\, \boldsymbol{m}_{\mathsf{ds}(\omega, F)}$

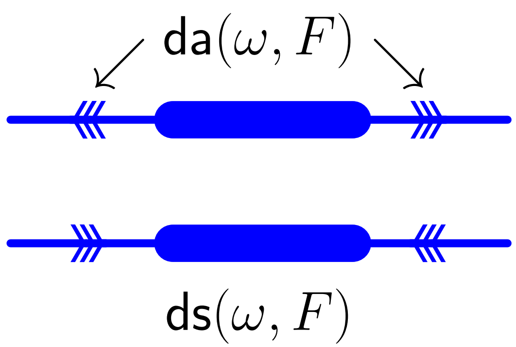


$$\begin{matrix}1\\2\\3\\4\\5\\6\end{matrix}\begin{pmatrix}0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$\boldsymbol{g}(F)$

$$\begin{matrix}1\\2\\3\\4\\5\\6\end{matrix}\begin{pmatrix}0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$\boldsymbol{c}(F)$

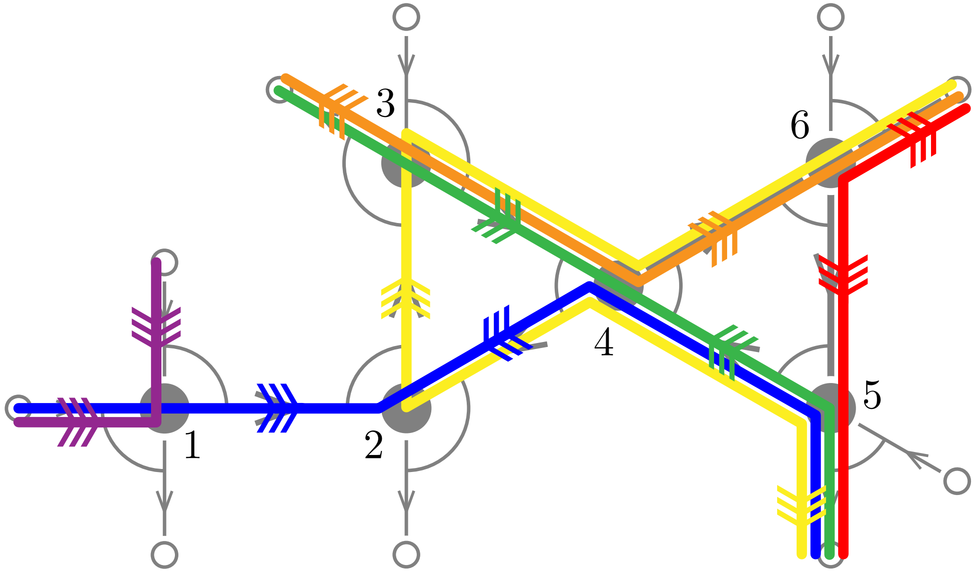


$\boldsymbol{G}$-VECTORS & $\boldsymbol{C}$-VECTORS

multiplicity vector $\boldsymbol{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of $Q_0 \;=\; \sum_{i \in [m]} \boldsymbol{e}_{v_i} \in \mathbb{R}^{Q_0}$

$\boldsymbol{g}$-vector $\boldsymbol{g}(\omega)$ of a walk $\omega \;=\; \boldsymbol{m}_{\text{peaks}(\omega)} - \boldsymbol{m}_{\text{deeps}(\omega)}$

$\boldsymbol{c}$-vector $\boldsymbol{c}(\omega \in F)$ of a walk ω in a non-kissing facet $F \;=\; \varepsilon(\omega, F)\, \boldsymbol{m}_{\text{ds}(\omega,F)}$



$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{g}(F)$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{c}(F)$

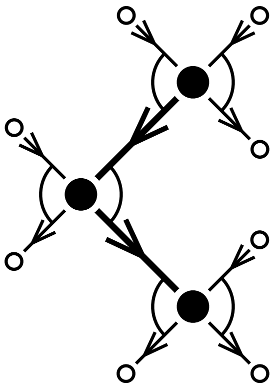
PROP. For any non-kissing facet F, the sets of vectors

$$\boldsymbol{g}(F) := \{\boldsymbol{g}(\omega) \mid \omega \in F\} \qquad \text{and} \qquad \boldsymbol{c}(F) := \{\boldsymbol{c}(\omega \in F) \mid \omega \in F\}$$

form dual bases.

Palu–P.–Plamondon ('21)

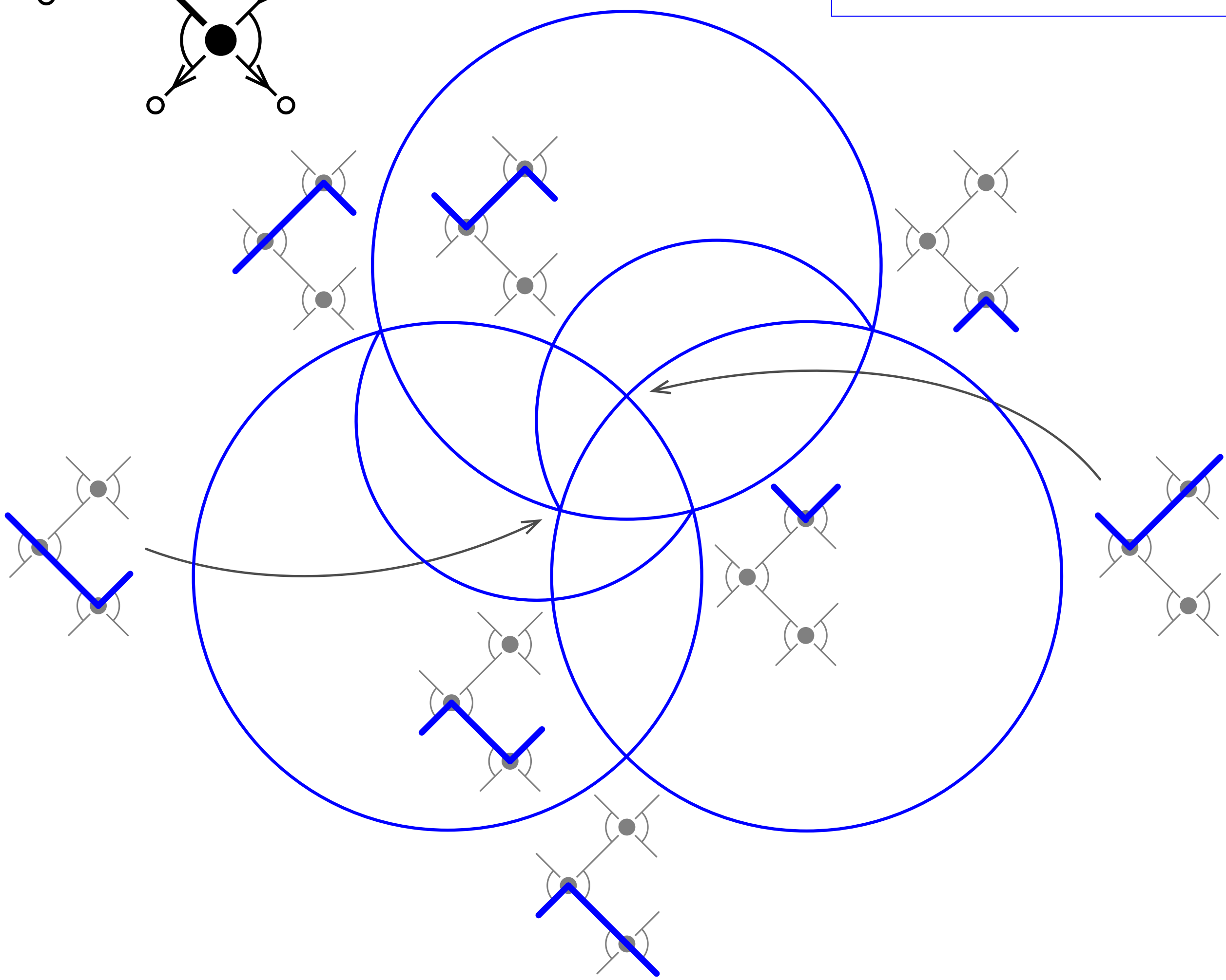
$\boldsymbol{G}$-VECTOR FAN



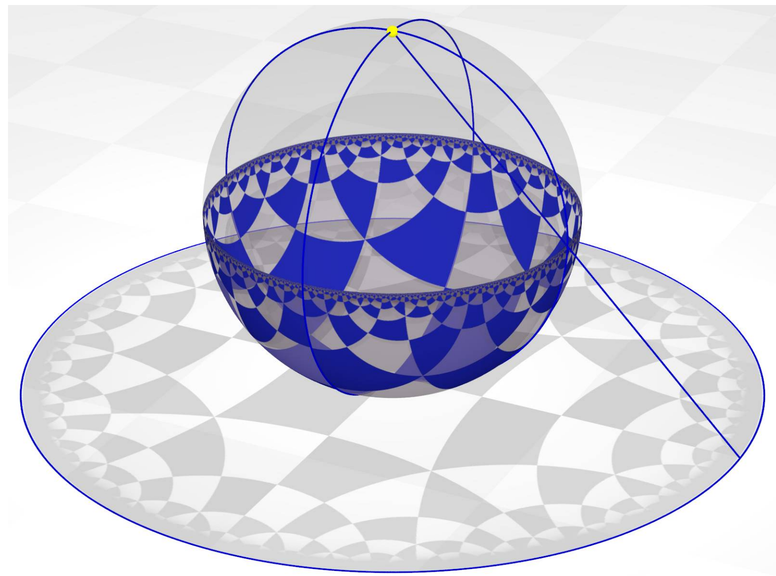
THM. For any gentle quiver $\bar{Q}$, the collection of cones

$$\mathcal{F}^{\boldsymbol{g}}(\bar{Q}) := \left\{ \mathbb{R}_{\ge 0}\,\boldsymbol{g}(F) \;\middle|\; F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q}) \right\}$$

forms a compl. simpl. fan, called $\boldsymbol{g}$-vector fan of $\bar{Q}$.



stereographic projection from $(1,1,1)$



NON-KISSING ASSOCIAHEDRON

kissing number $\mathsf{kn}(\omega) = \sum_{\omega'}$ number of times ω and ω' kiss

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$, the two sets of $\mathbb{R}^{Q_0}$ given by

(i) the convex hull of the points

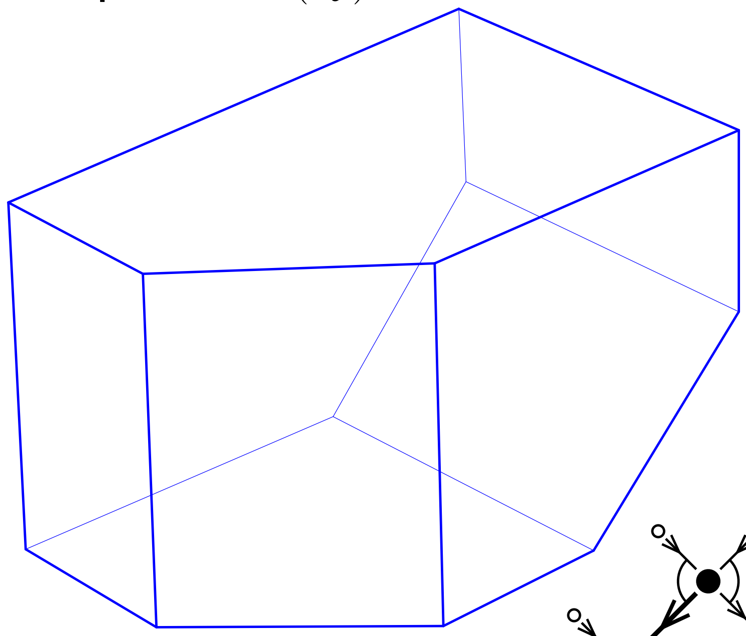
$$\mathbf{p}(F) := \sum_{\omega \in F} \mathsf{kn}(\omega)\, \boldsymbol{c}(\omega \in F),$$

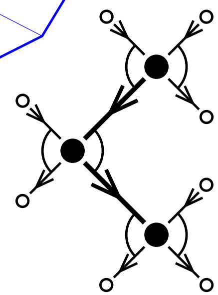
for all non-kissing facets $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$,

(ii) the intersection of the halfspaces

$$\boldsymbol{H}^{\geq}(\omega) := \left\{ \boldsymbol{x} \in \mathbb{R}^{Q_0} \;\middle|\; \langle\, \boldsymbol{g}(\omega) \mid \boldsymbol{x} \,\rangle \leq \mathsf{kn}(\omega) \right\}.$$

for all walks ω of $\bar{Q}$,





define the same polytope, whose normal fan is the $\boldsymbol{g}$-vector fan $\mathcal{F}^{\boldsymbol{g}}$. We call it the $\bar{Q}$-associahedron and denote it by $\mathbb{A}$sso.

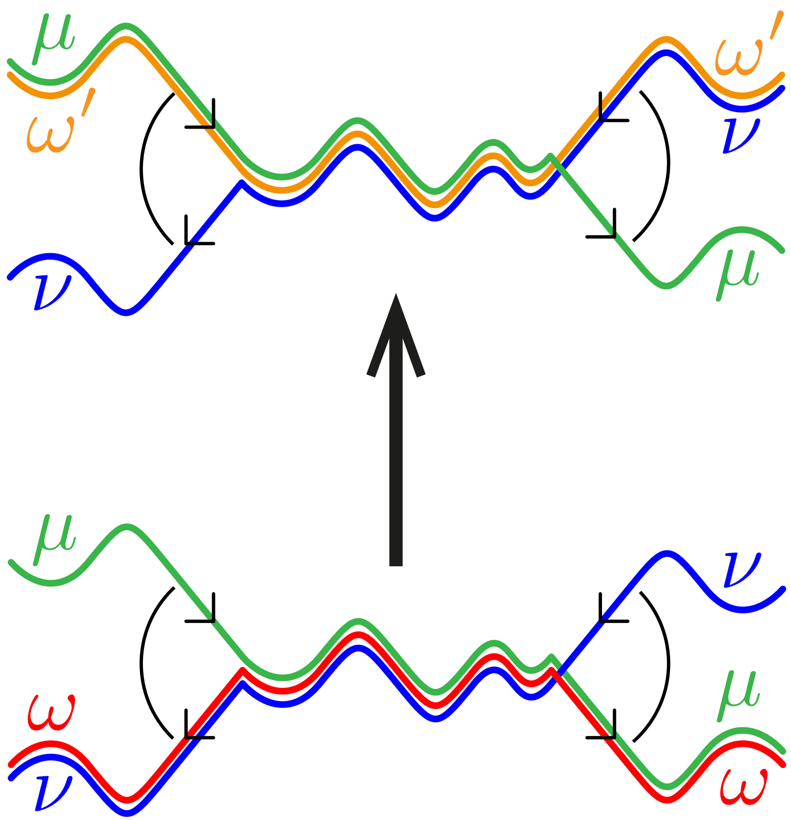
Palu–P.–Plamondon ('21)

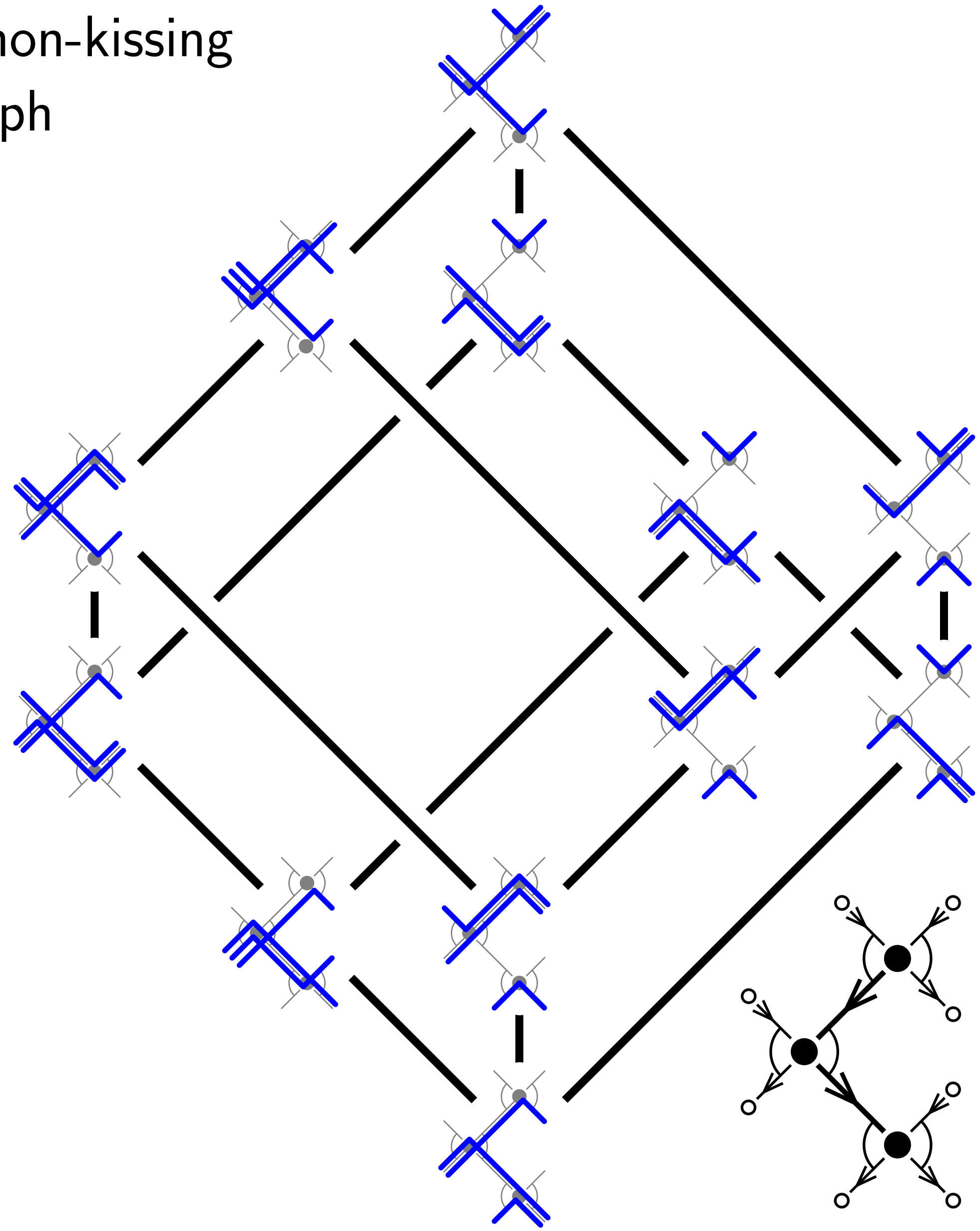
NON-KISSING LATTICE

McConville, *Lattice structures of grid Tamari orders* ('17)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

NON-KISSING LATTICE

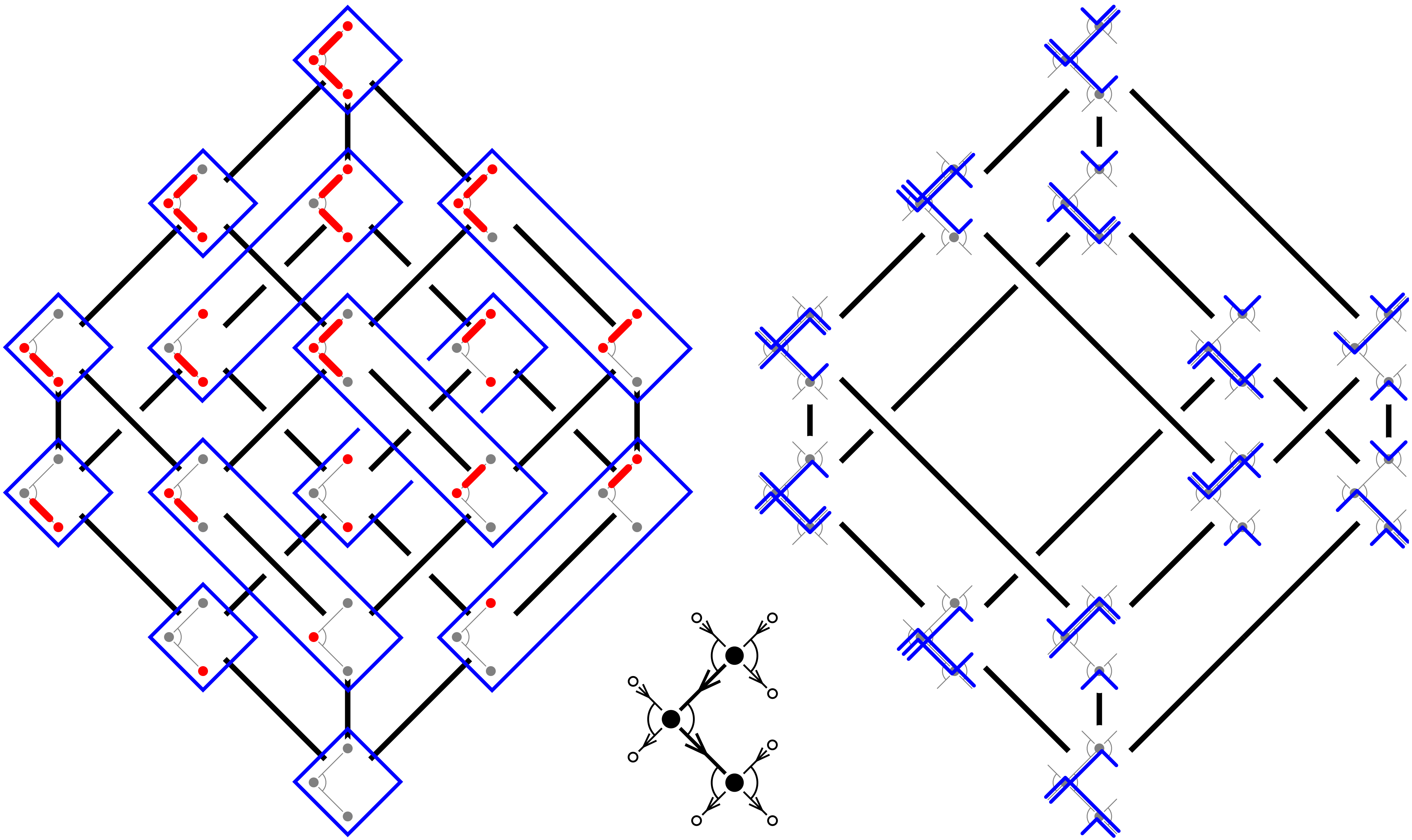
THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.





Palu–P.–Plamondon ('21)

NON-KISSING LATTICE



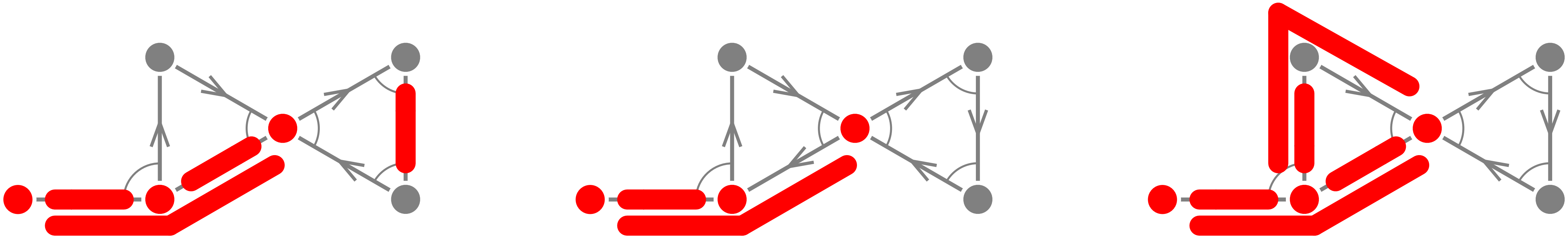
BICLOSED SETS OF STRINGS

σ, τ oriented strings

concatenation $\sigma \circ \tau = \big\{ \sigma \alpha \tau \;\big|\; \alpha \in Q_1$ and $\sigma\alpha\tau$ string of $\bar{Q} \big\}$

closure $S^{\mathrm{cl}} = \bigcup_{\substack{\ell \in \mathbb{N} \\ \sigma_1, \dots, \sigma_\ell \in S}} \sigma_1 \circ \cdots \circ \sigma_\ell =$ all strings obtained by concatenation of some strings of S

closed $\iff S^{\mathrm{cl}} = S$ coclosed $\iff \bar{S}^{\mathrm{cl}} = \bar{S}$ biclosed $=$ closed and coclosed

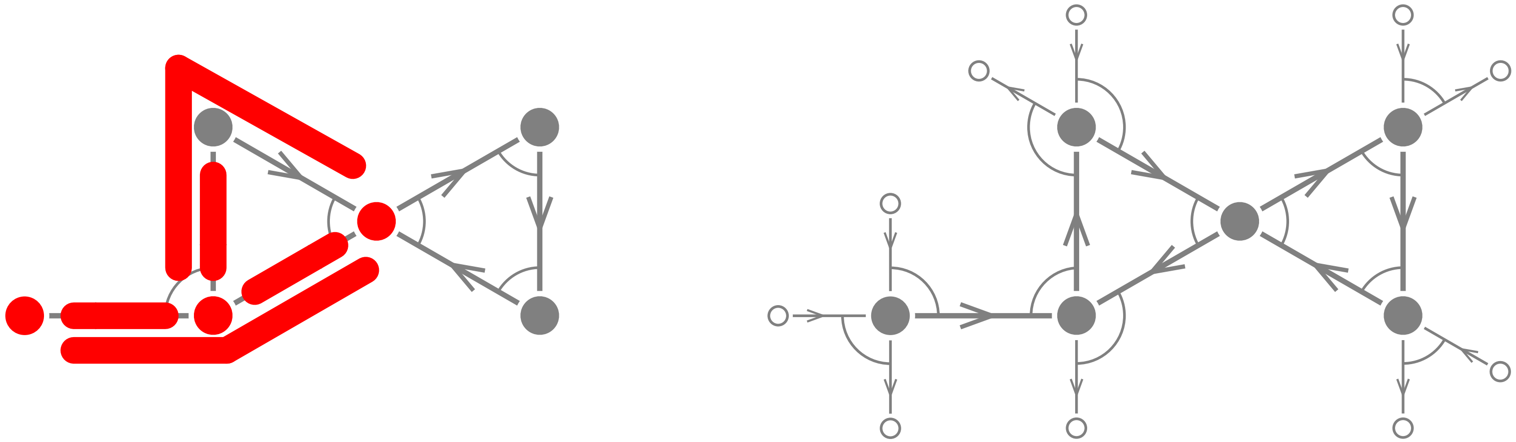


THM. For any gentle quiver $\bar{Q}$ such that $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is finite, the inclusion poset on biclosed sets of strings of $\bar{Q}$ is a congruence-uniform lattice.

McConville ('17) Palu–P.–Plamondon ('21)

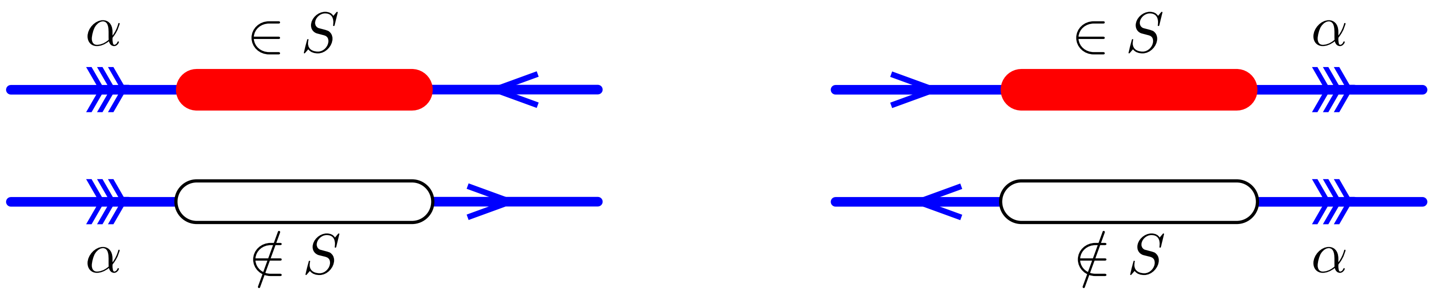
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

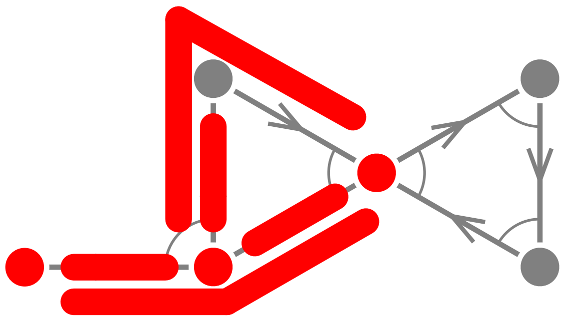
$\omega(\alpha, S) =$ walk constructed with the local rules:

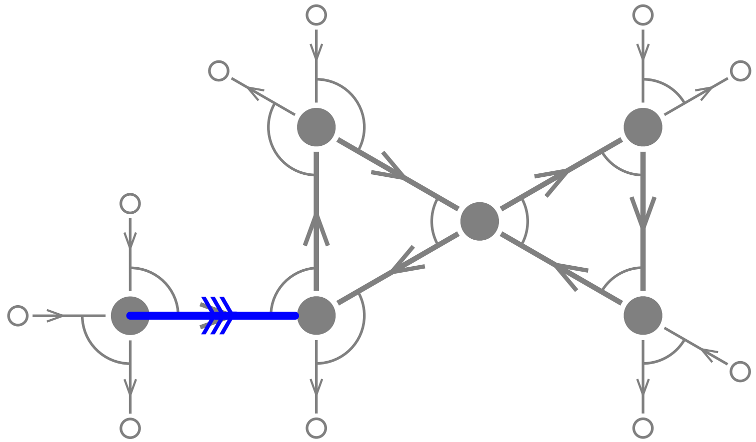


McConville ('17)

NON-KISSING INSERTION

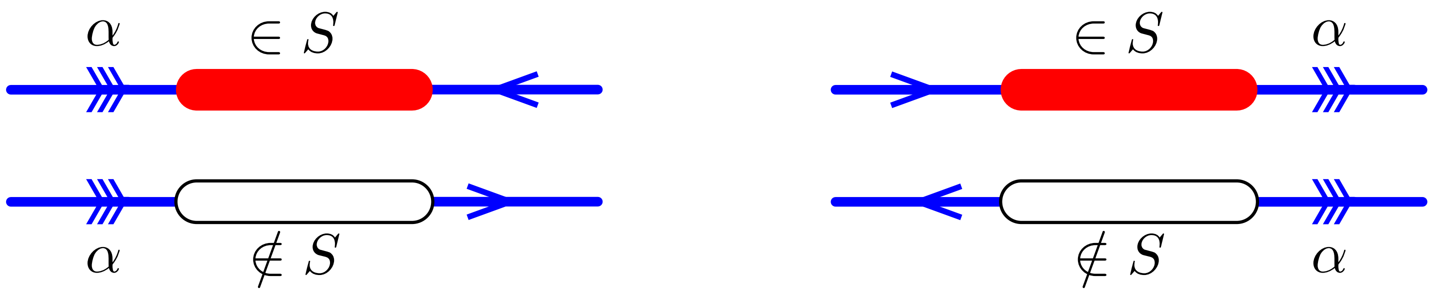
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

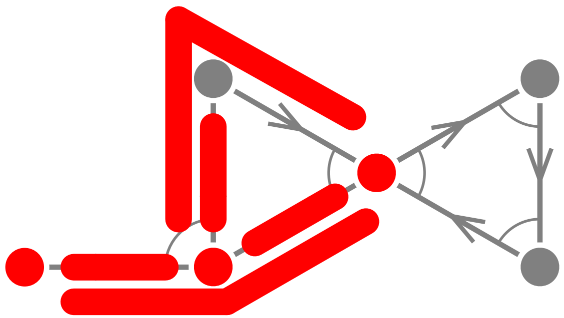
$\omega(\alpha, S) =$ walk constructed with the local rules:

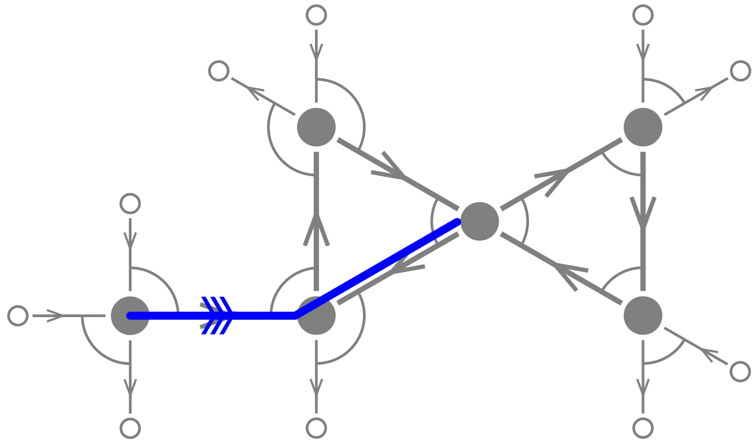


McConville ('17)

NON-KISSING INSERTION

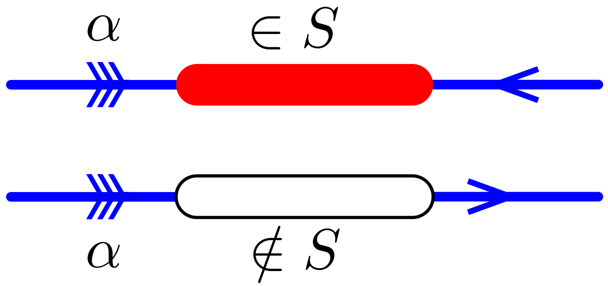
Surjection from biclosed sets of strings to non-kissing facets

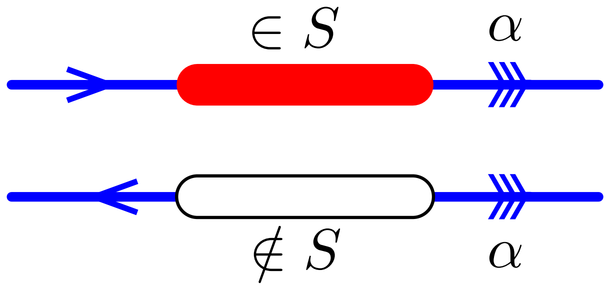




S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:

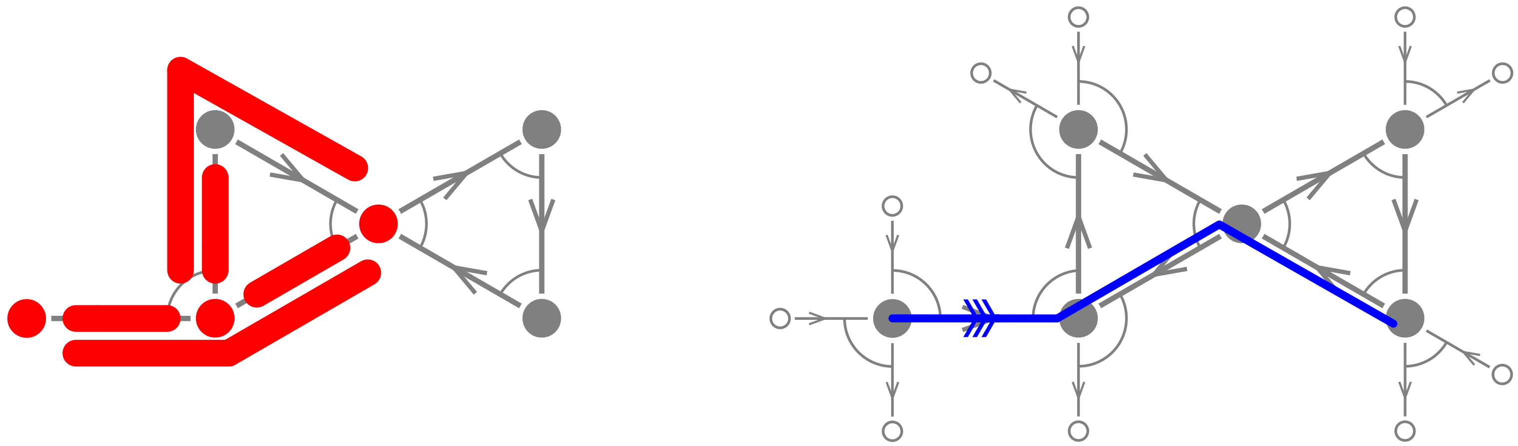




McConville ('17)

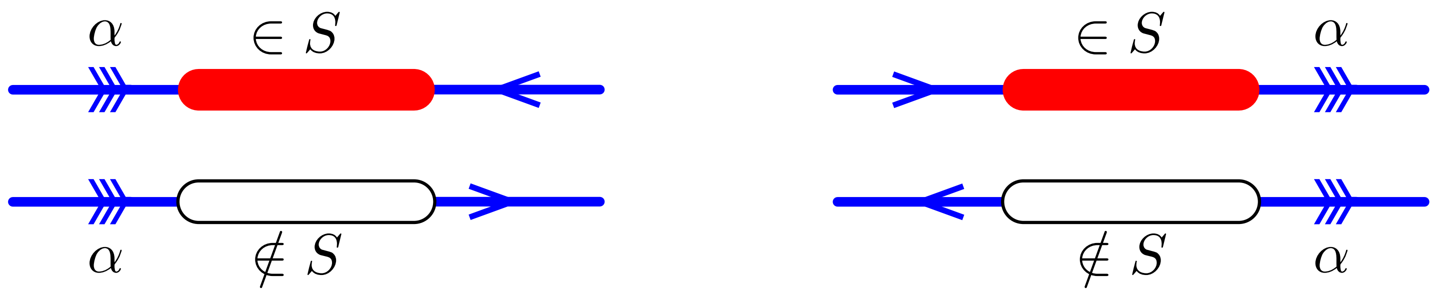
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

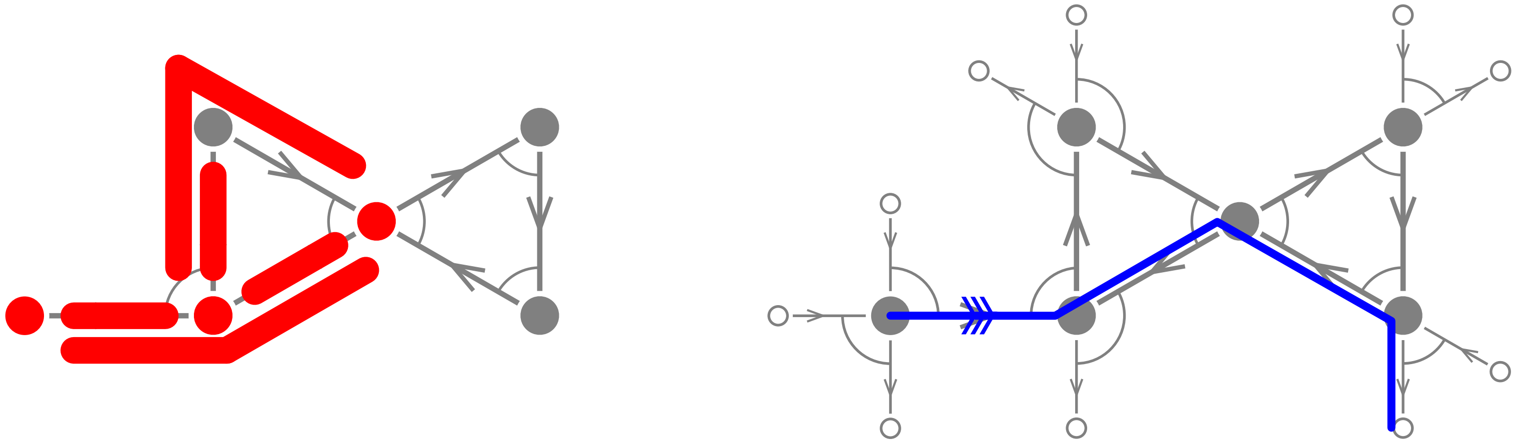
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville ('17)

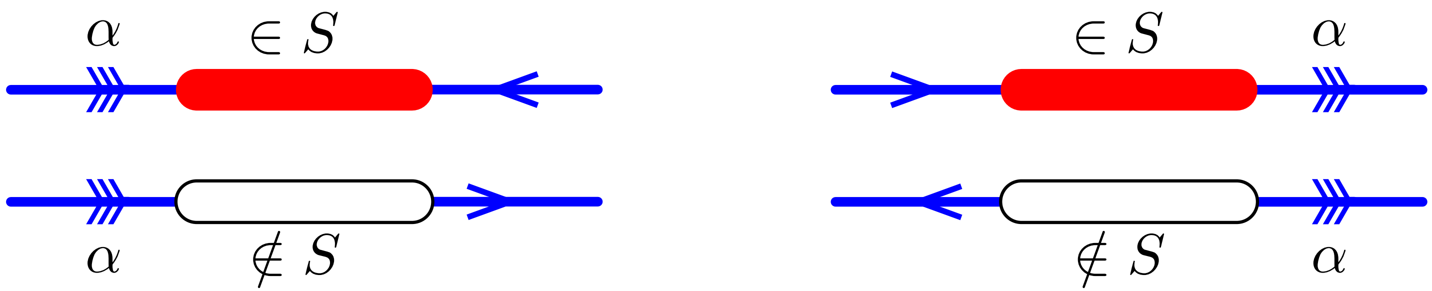
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

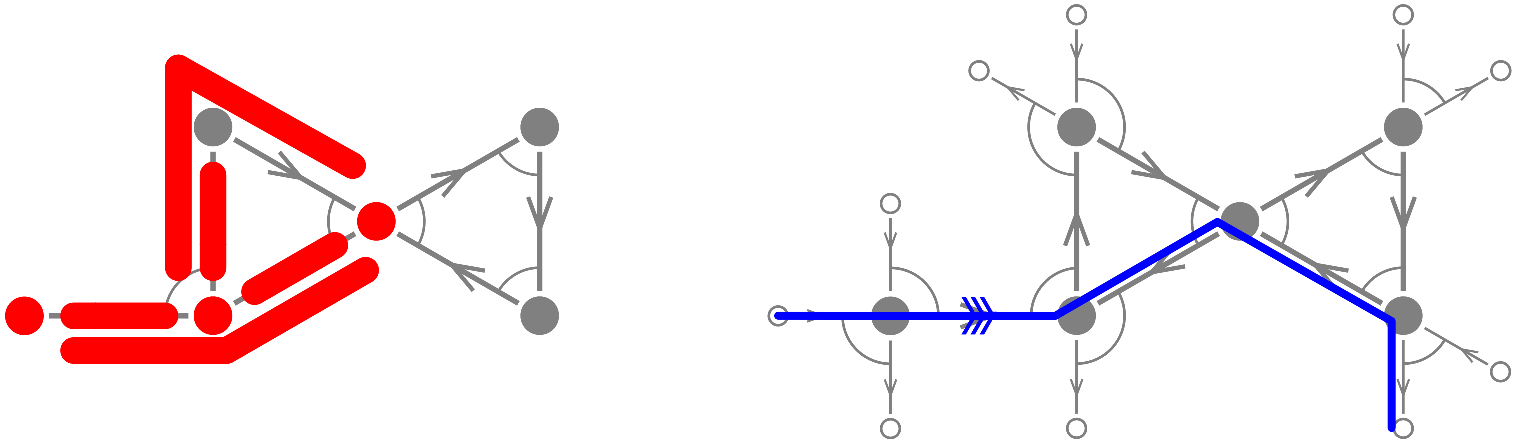
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville ('17)

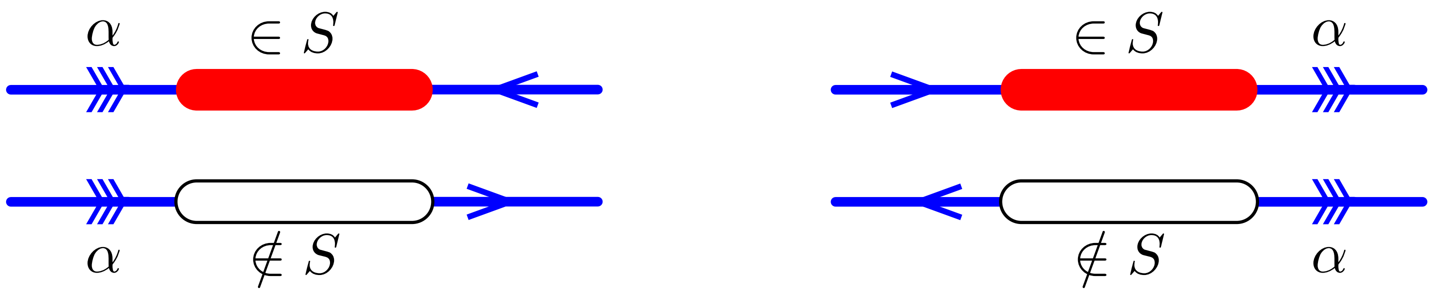
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

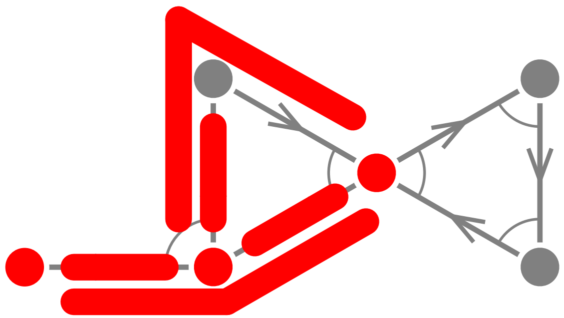
$\omega(\alpha, S)$ = walk constructed with the local rules:

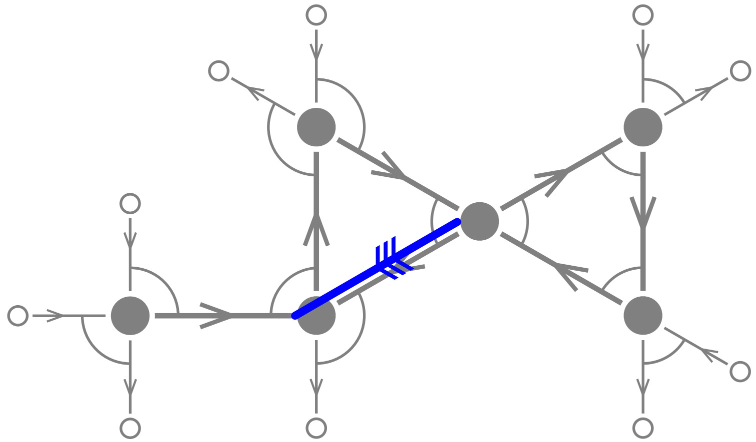


McConville ('17)

NON-KISSING INSERTION

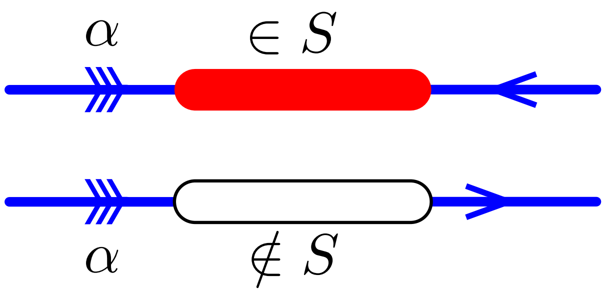
Surjection from biclosed sets of strings to non-kissing facets

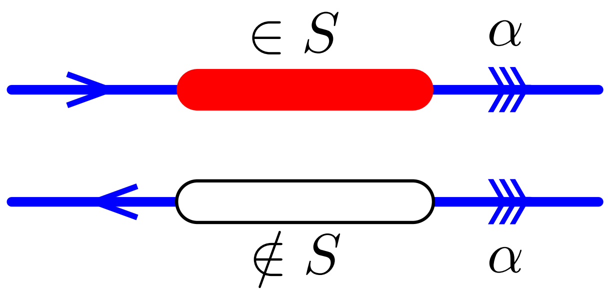




S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S) =$ walk constructed with the local rules:

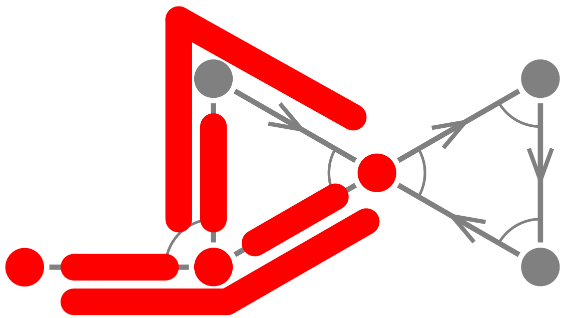


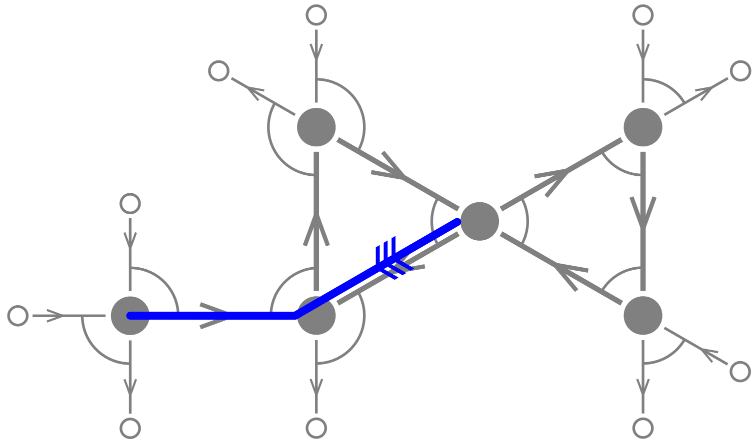


McConville ('17)

NON-KISSING INSERTION

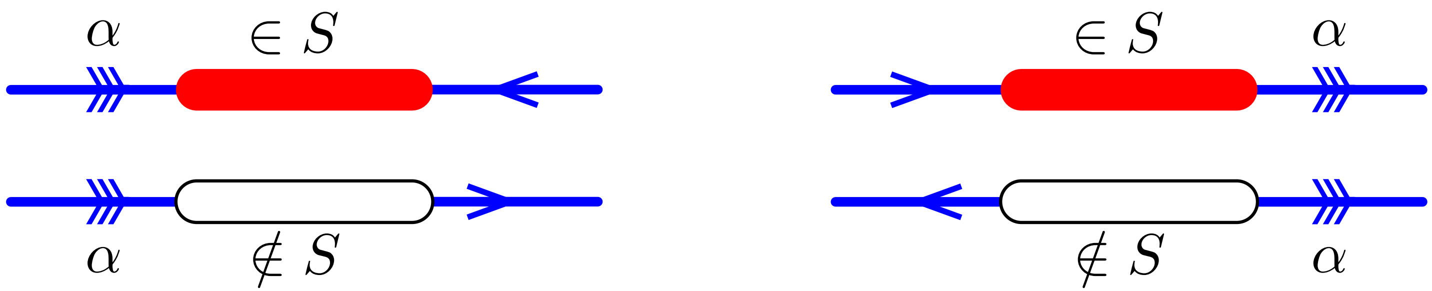
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

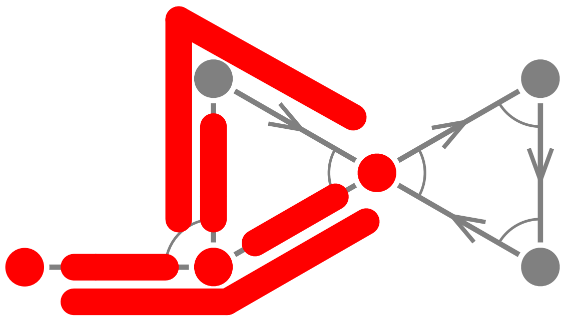
$\omega(\alpha, S) =$ walk constructed with the local rules:

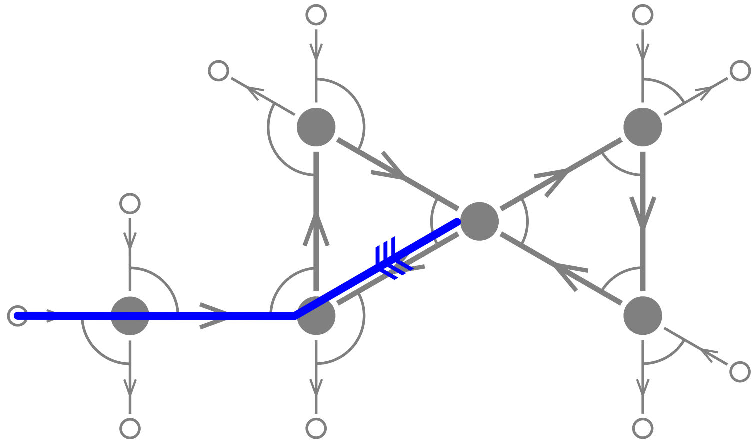


McConville ('17)

NON-KISSING INSERTION

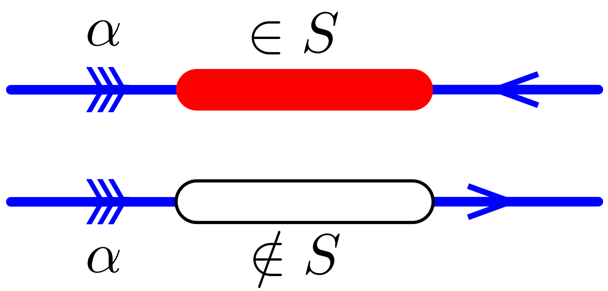
Surjection from biclosed sets of strings to non-kissing facets

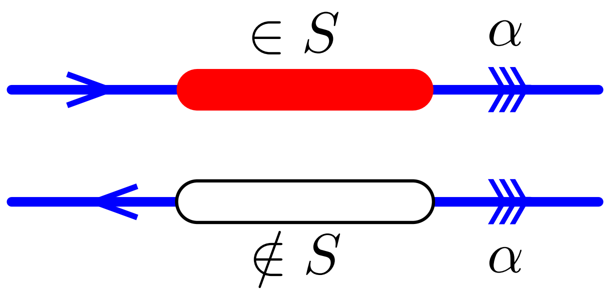




S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:

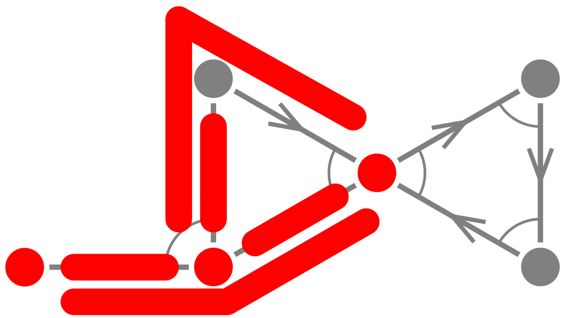


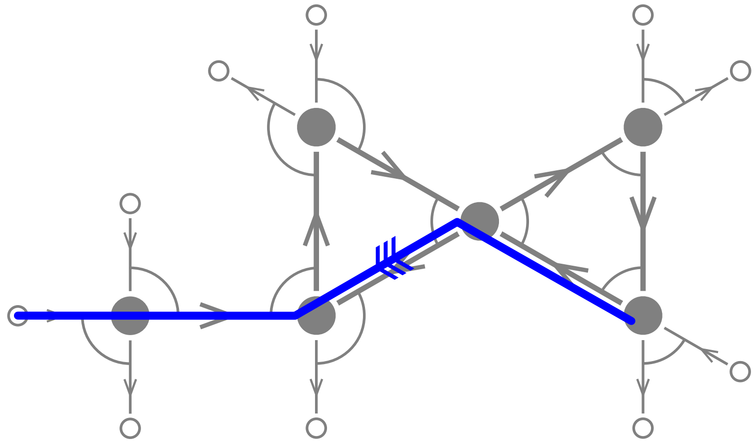


McConville ('17)

NON-KISSING INSERTION

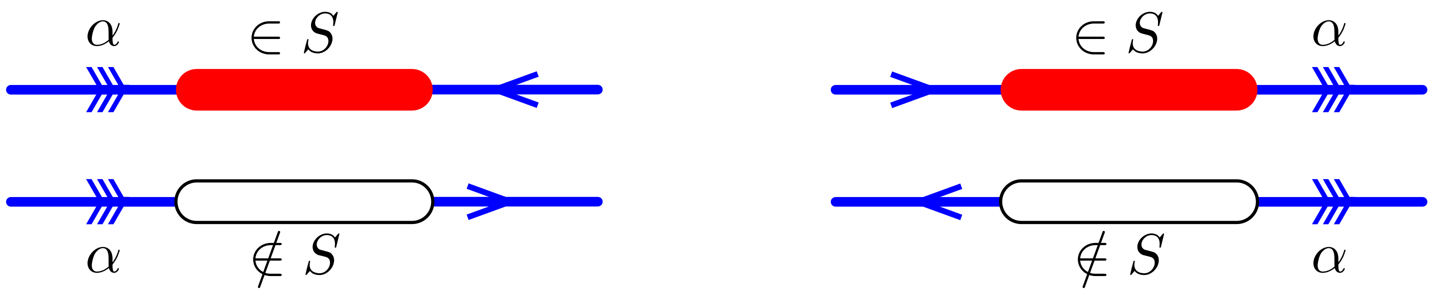
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

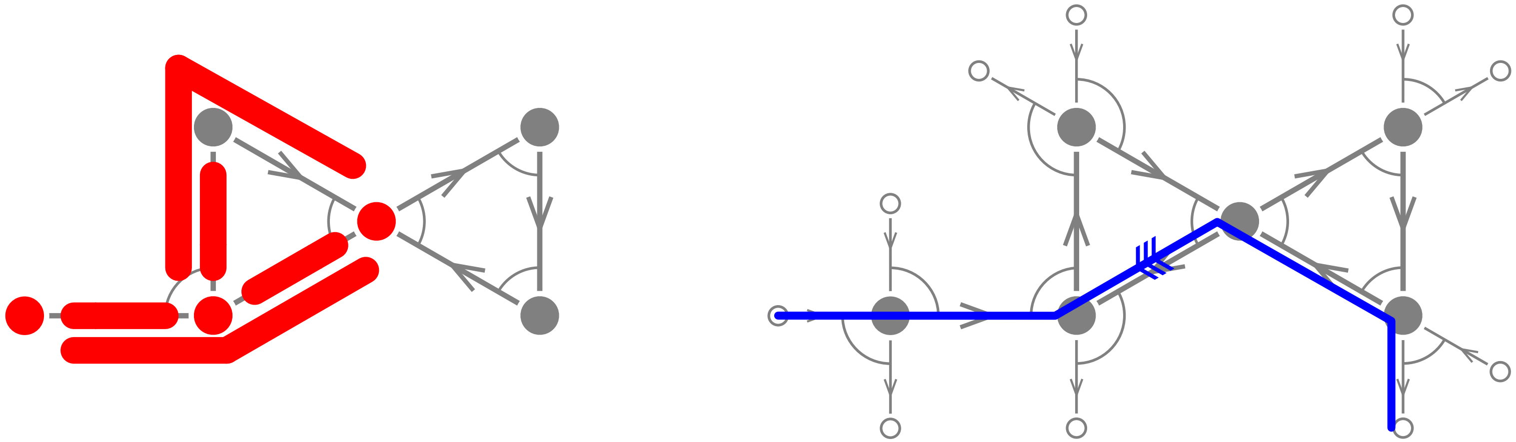
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville ('17)

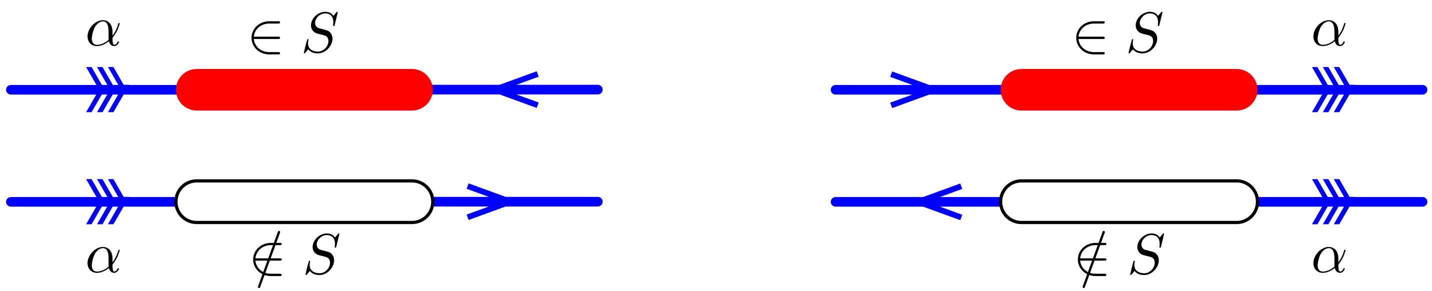
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

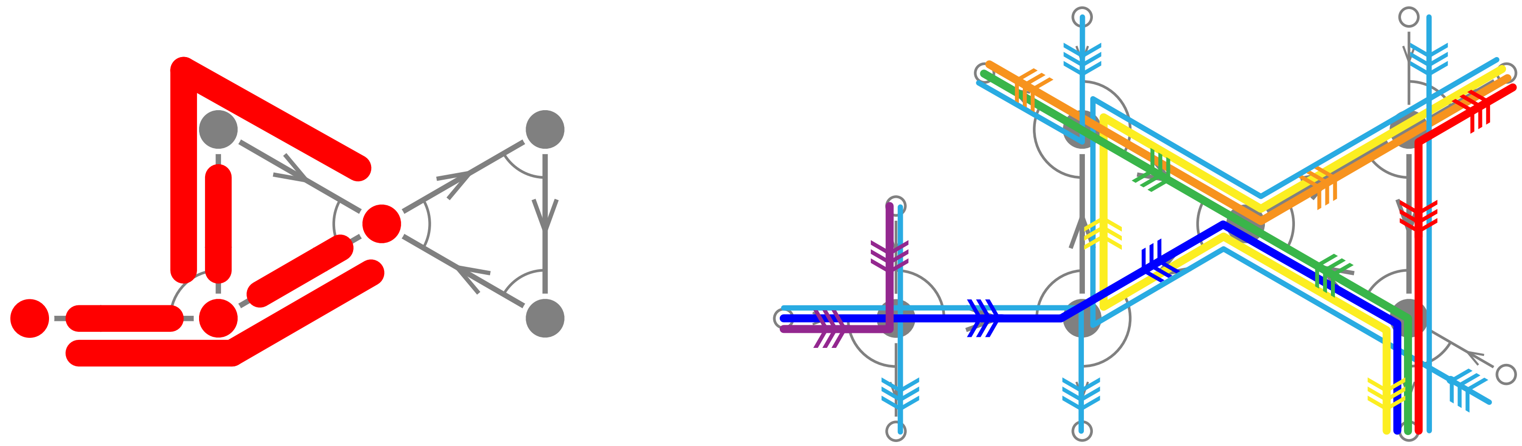
$\omega(\alpha, S)$ = walk constructed with the local rules:



McConville ('17)

NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets

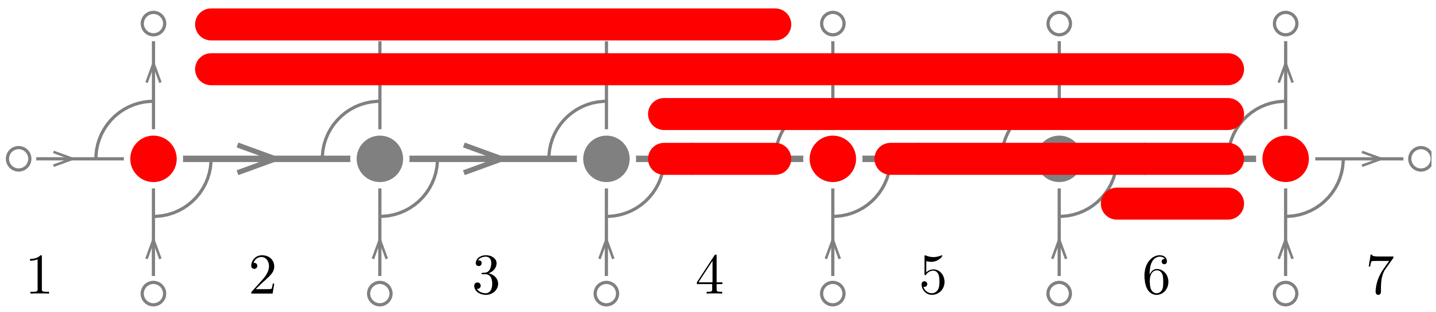


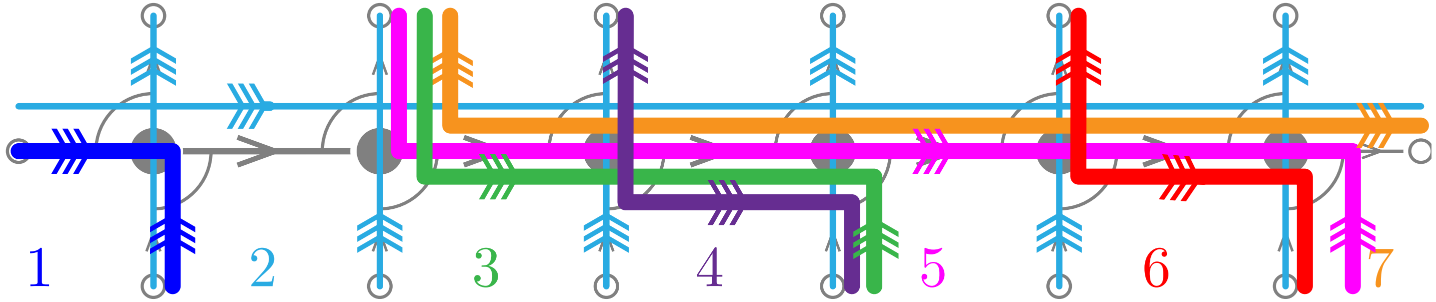
PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

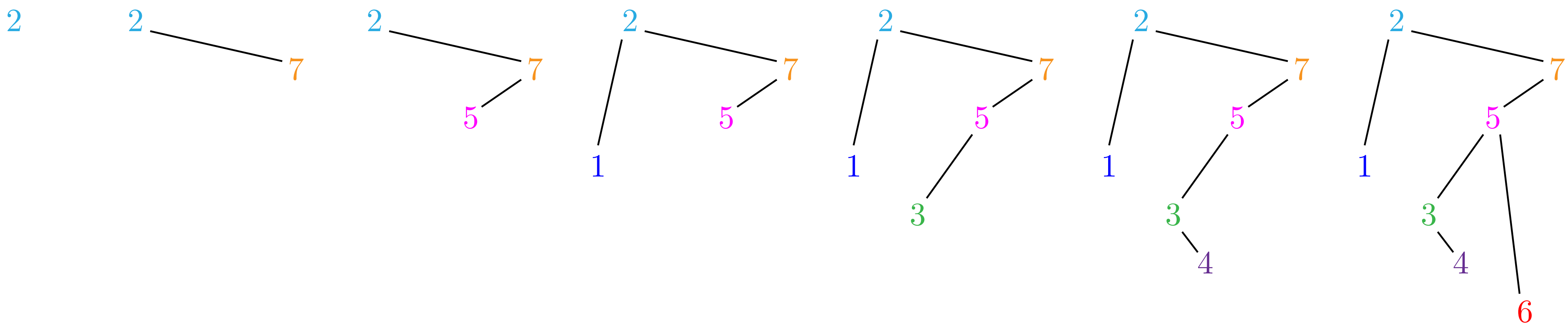
McConville ('17)

EXM: BINARY SEARCH TREE INSERTION AGAIN

inversion set of 2751346

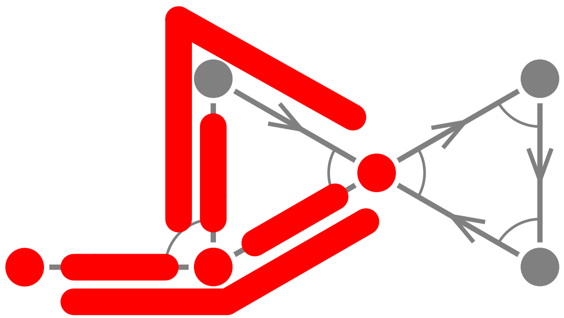


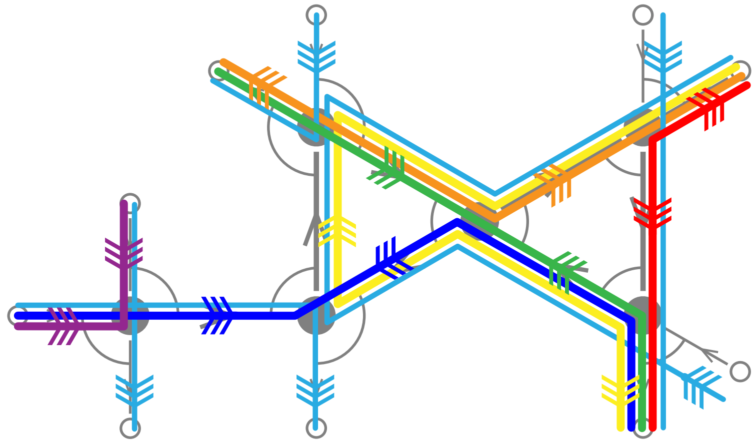




NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



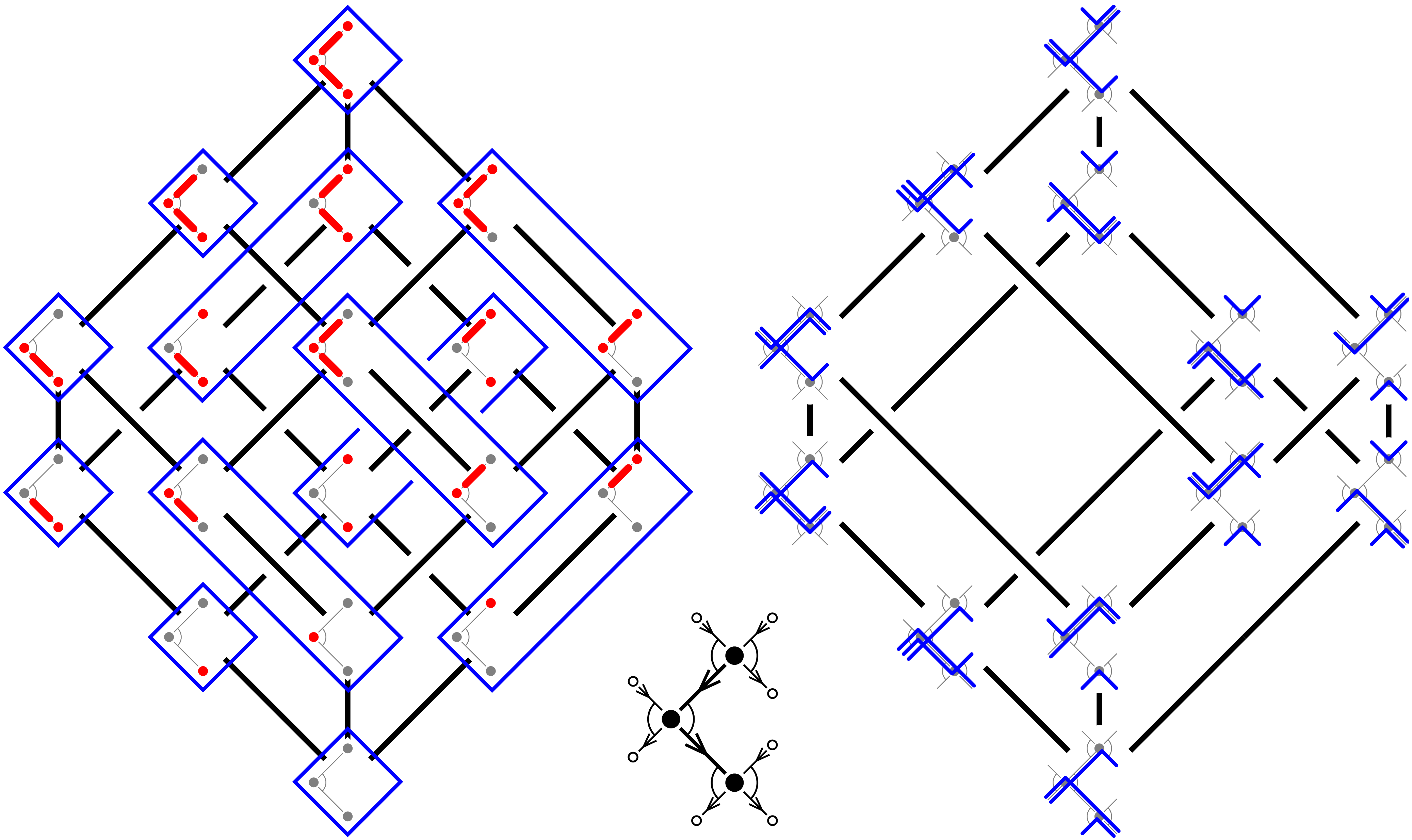


PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

THM. The map η defines a lattice morphism from biclosed sets to non-kissing facets.

McConville ('17)

NON-KISSING LATTICE

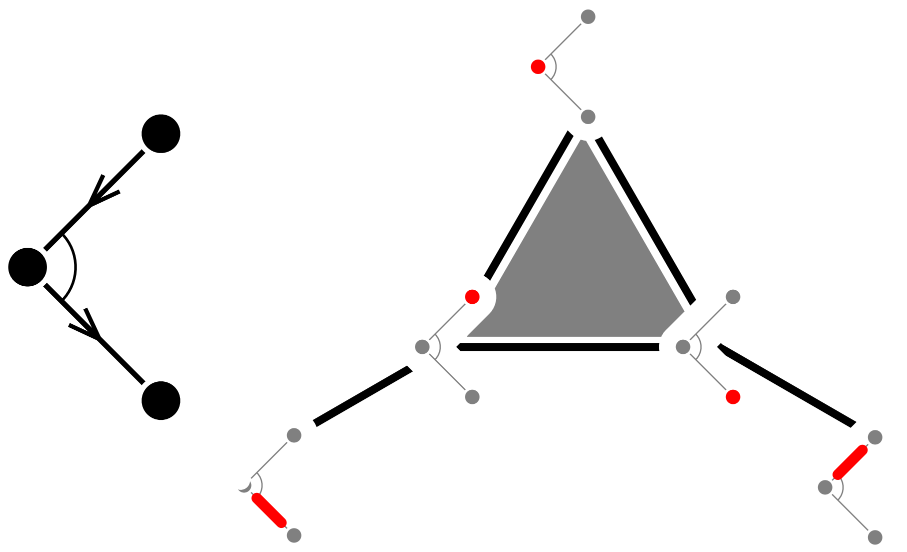


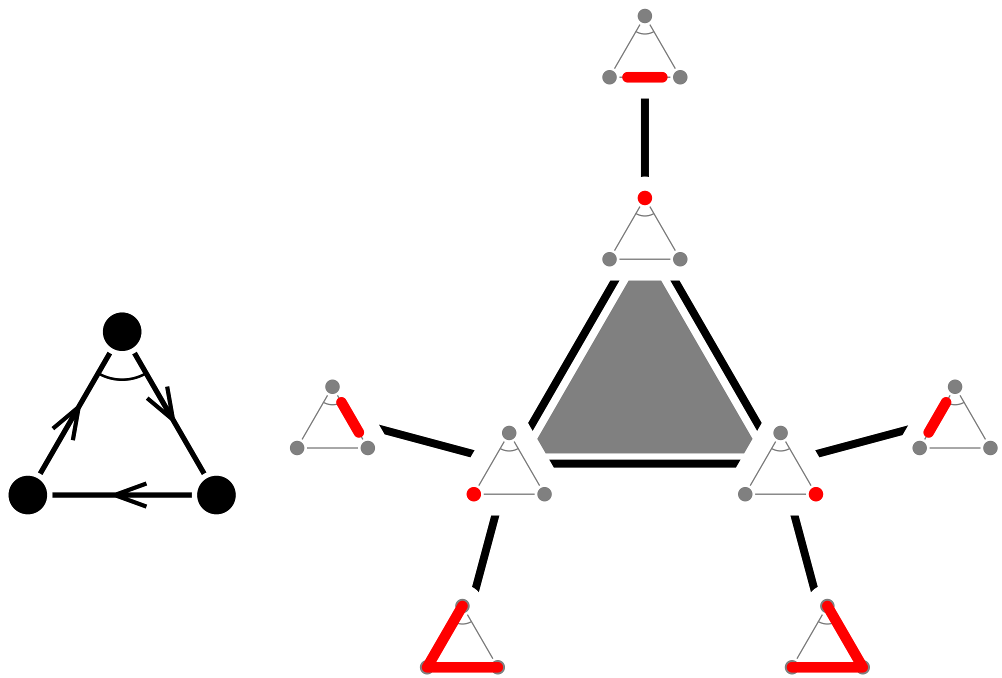
NON-KISSING LATTICE

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice. Palu–P.–Plamondon ('21)

Much more nice combinatorics:

- join-irreducible elements of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ are in bijection with distinguishable strings
- canonical join complex of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is a generalization of non-crossing partitions

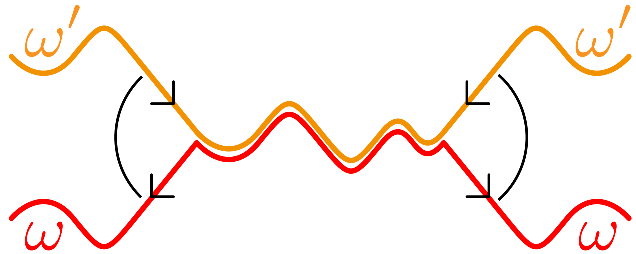




SUMMARY

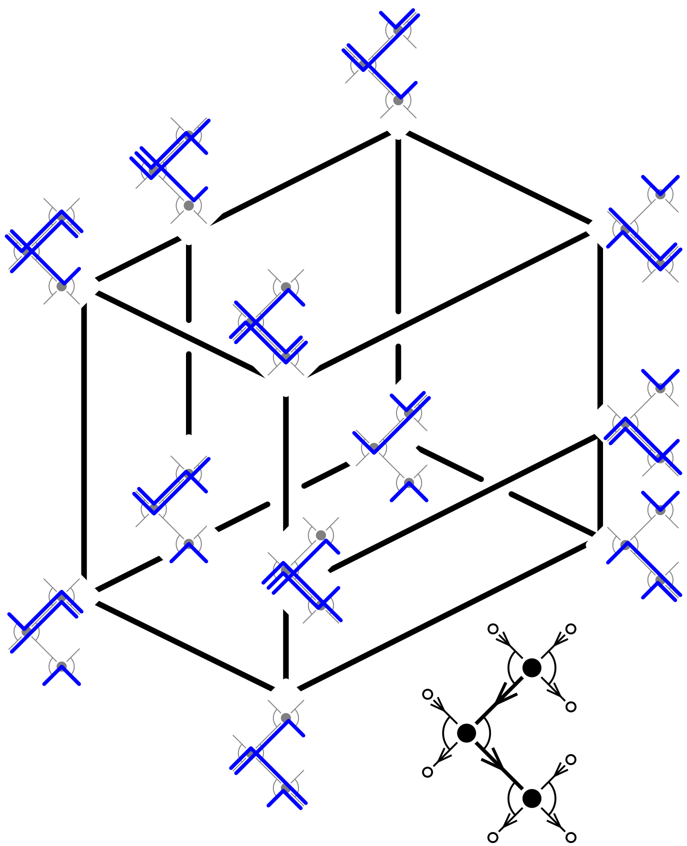
non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q}) =$

- vertices $=$ walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces $=$ collections of pairwise non-kissing walks in $\bar{Q}^{❀}$

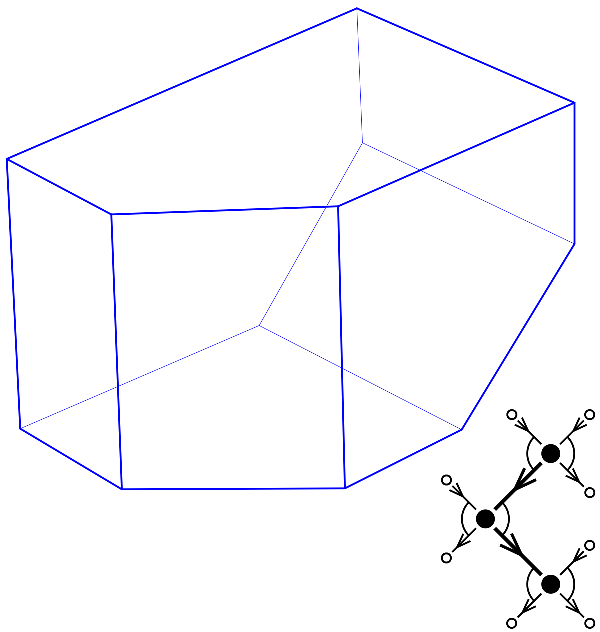


... generalizing the associahedron

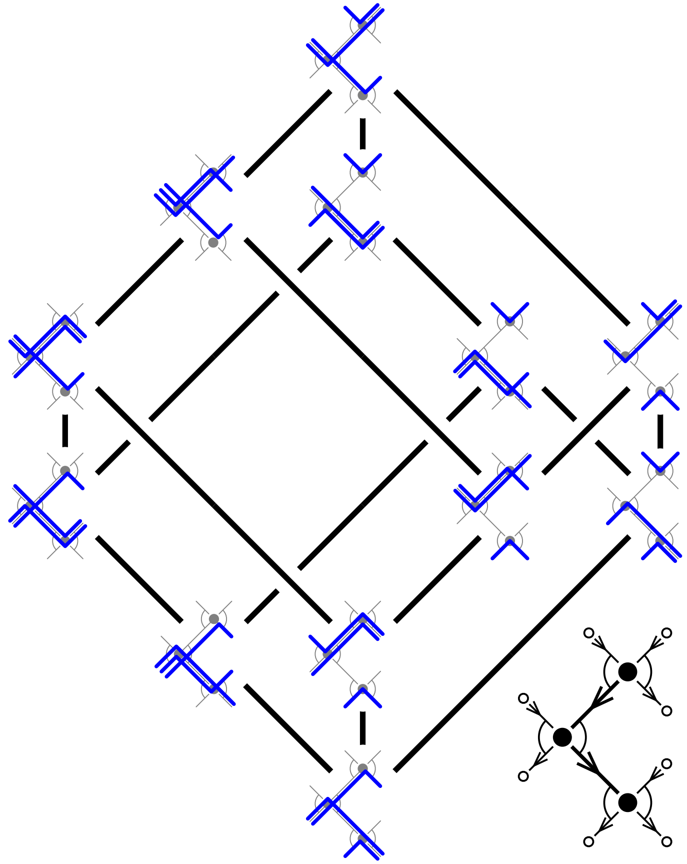
Flip graph

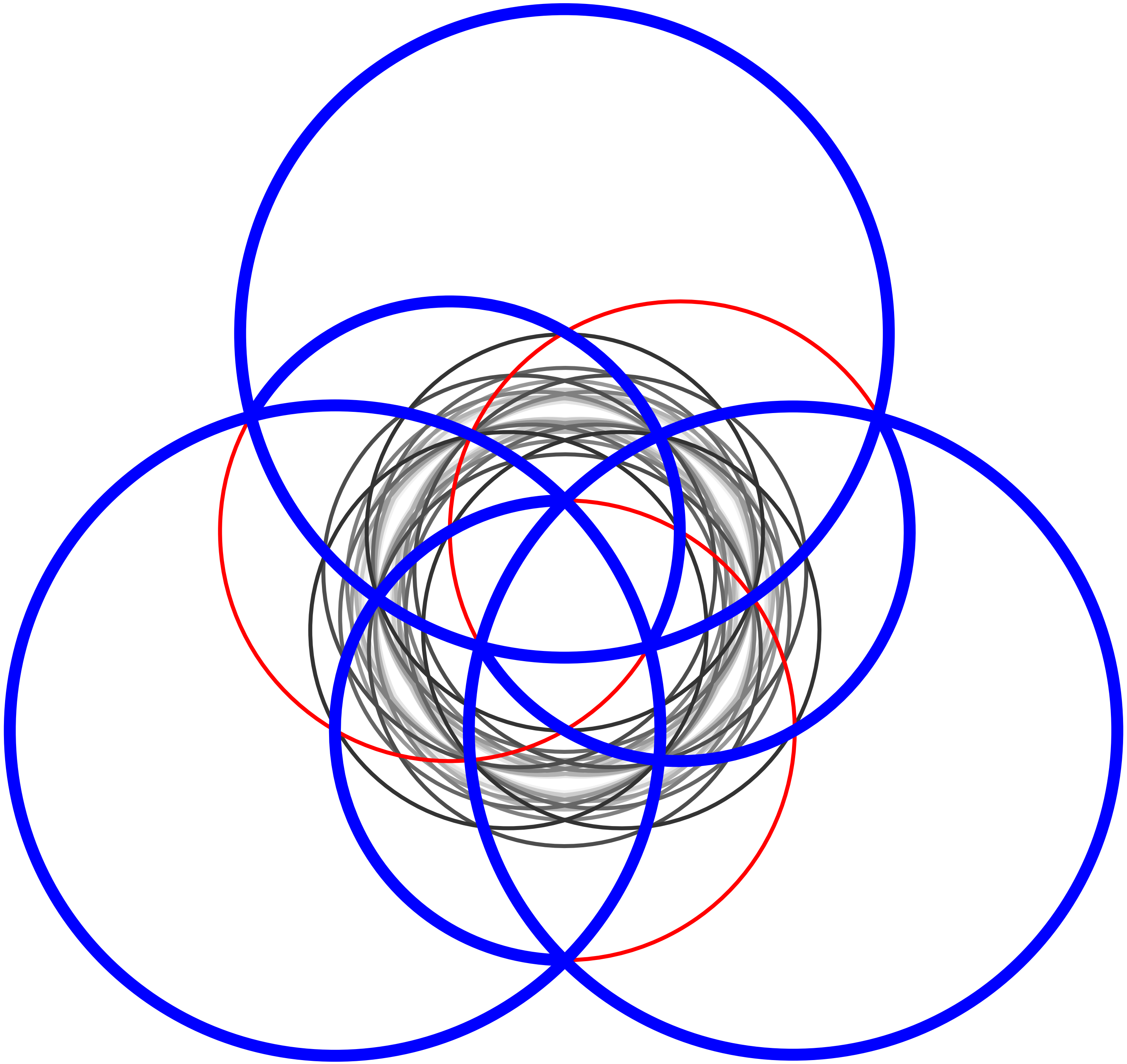


Associahedron



Tamari lattice





THANK YOU