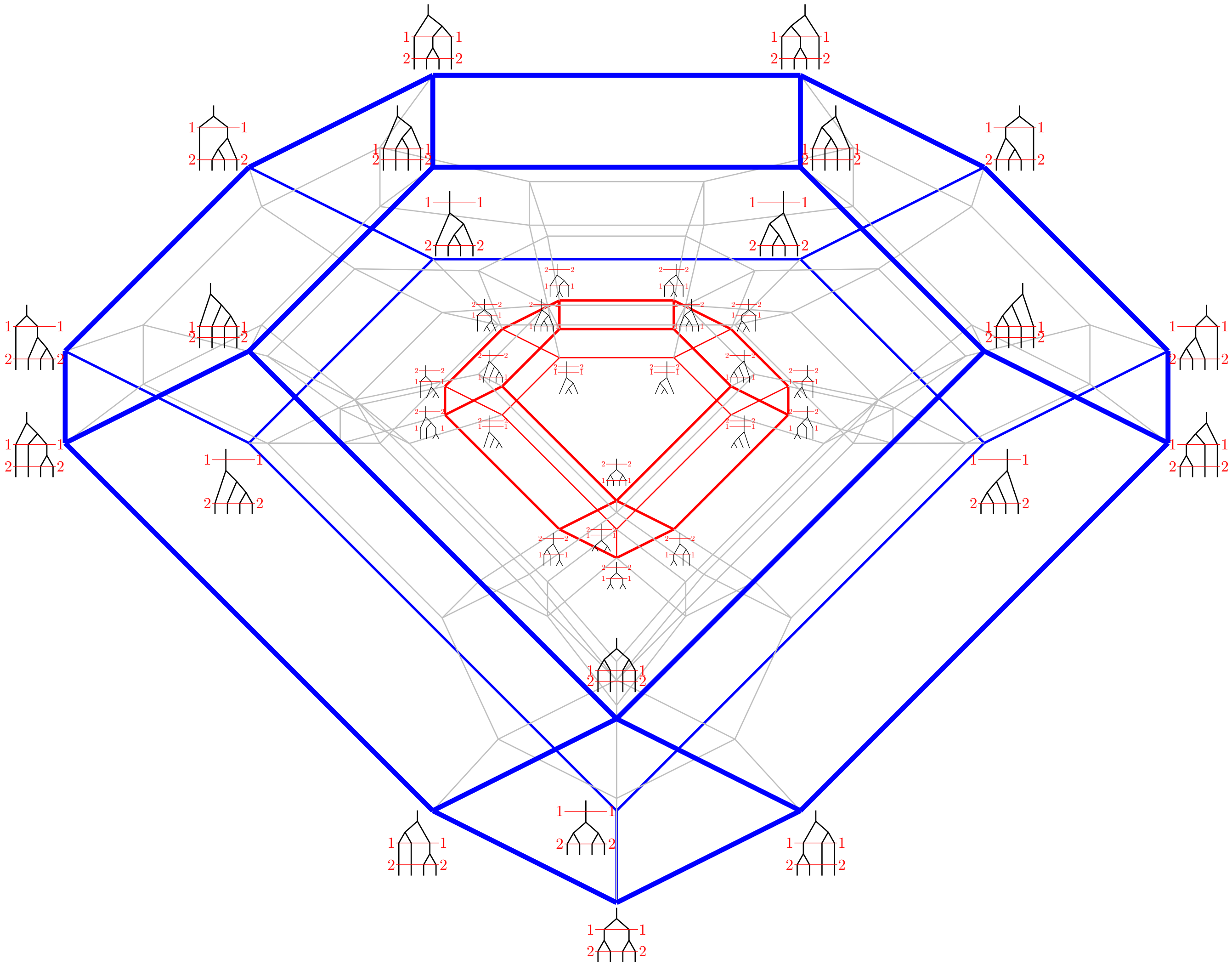
II. DEFORMED PERMUTAHEDRA



V. PILAUD (Universitat de Barcelona)

Osnabrück, Monday February 24th, 2025

slides available at: https://www.ub.edu/comb/vincentpilaud/documents/presentations/Osnabruck/2.pdf

"The biggest lesson I learned from Richard Stanley's work is,
combinatorial objects want to be partially ordered! [...]"

"A related lesson that Stanley has taught me is,
combinatorial objects want to belong to polytopes! [...]"

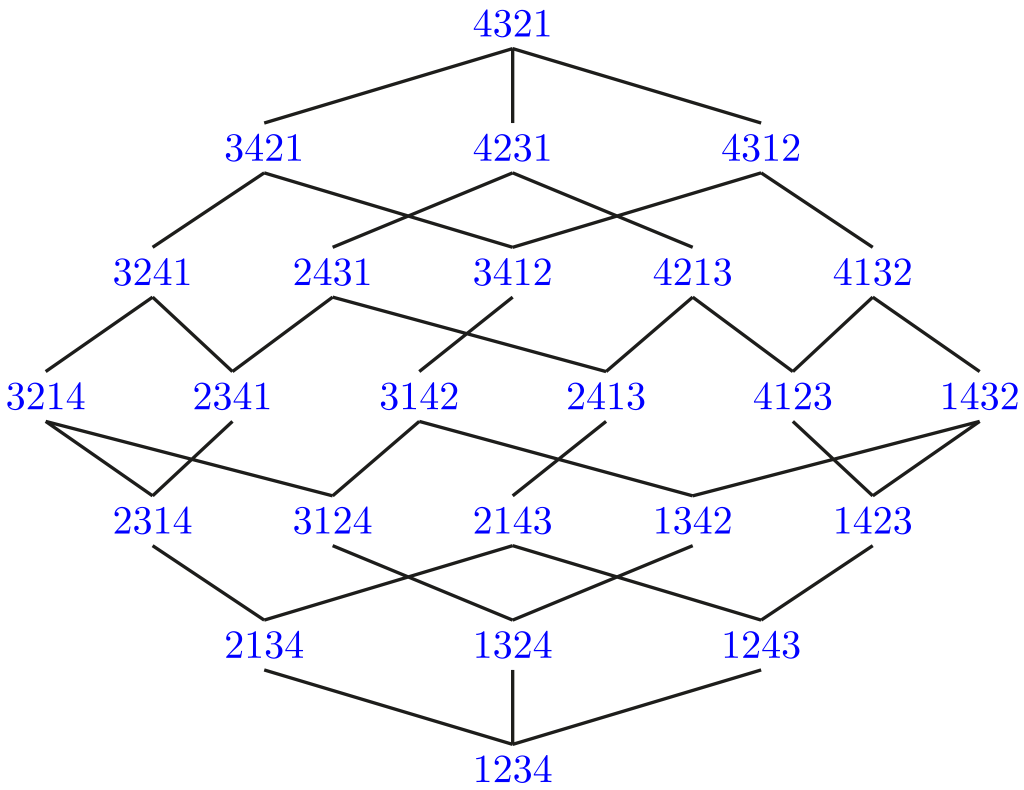
Propp, *Lessons I Learned from Richard Stanley* ('15)

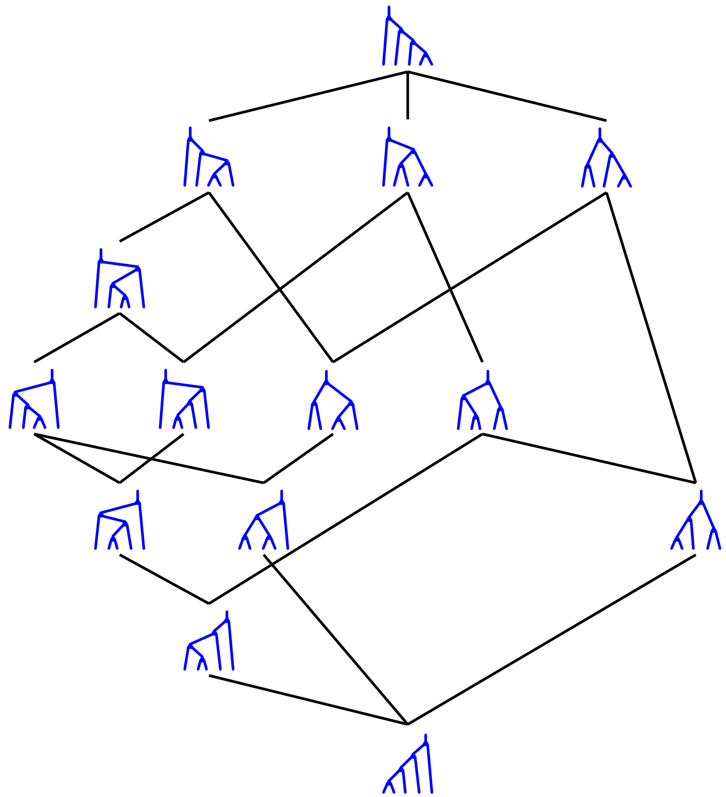
PERMUTAHEDRON & ASSOCIAHEDRON

Ceballos–Santos–Ziegler, *Many non-equivalent realizations of the associahedron* ('15)
P.–Santos–Ziegler, *Celebrating Loday's associahedron* ('23)

LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice $=$ partially ordered set L where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$





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ordered by paths of simple transpositions

Tamari lattice $=$ binary trees on $[n]$
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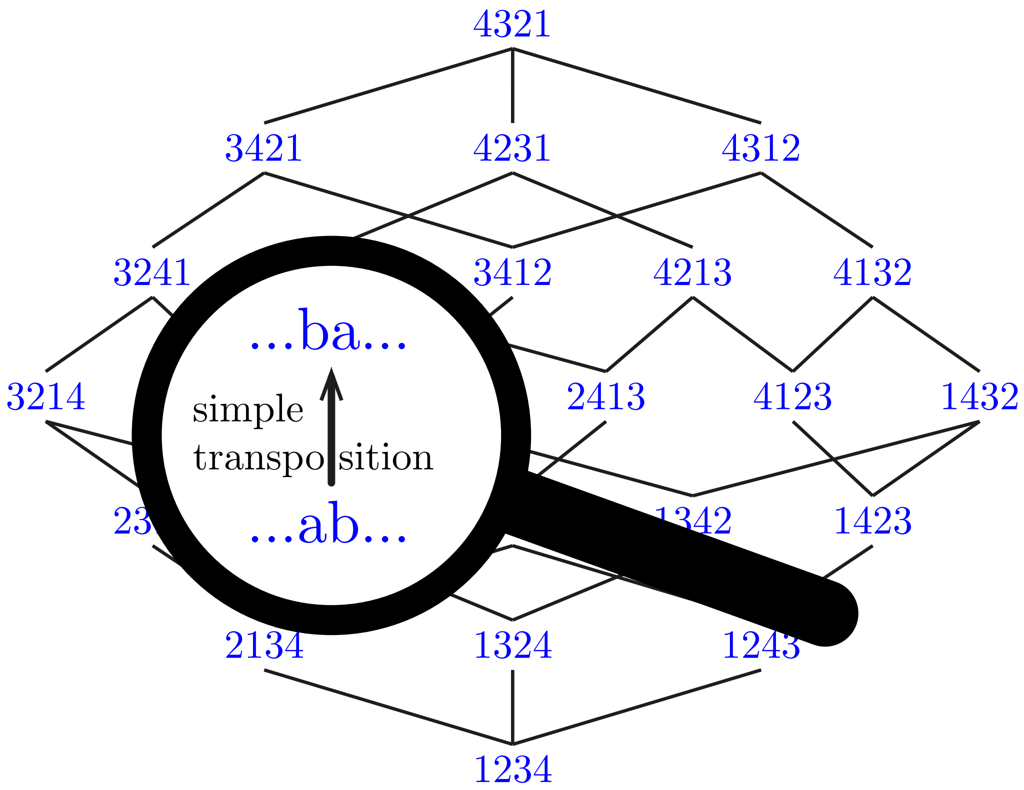

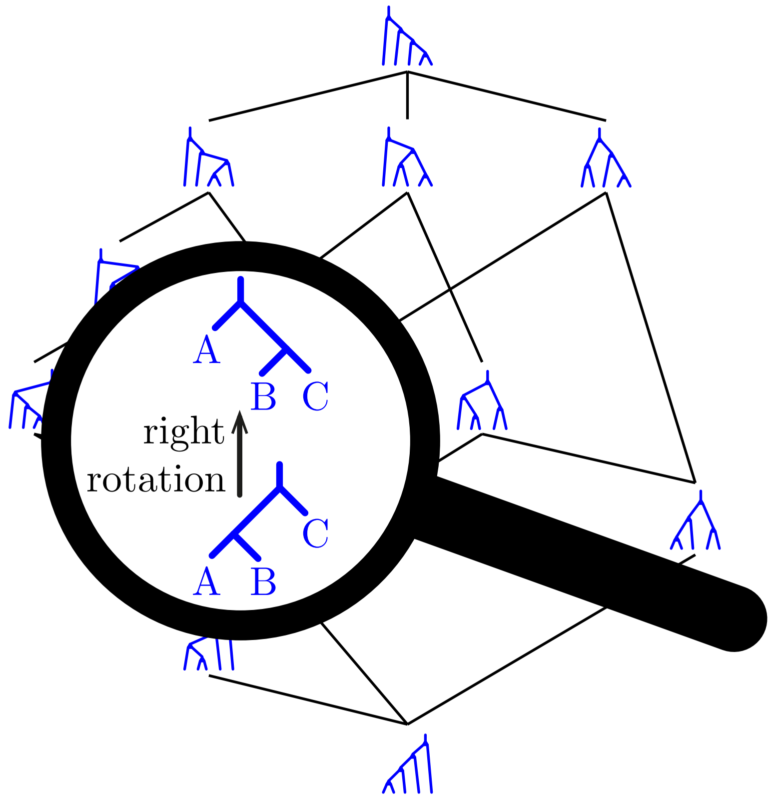

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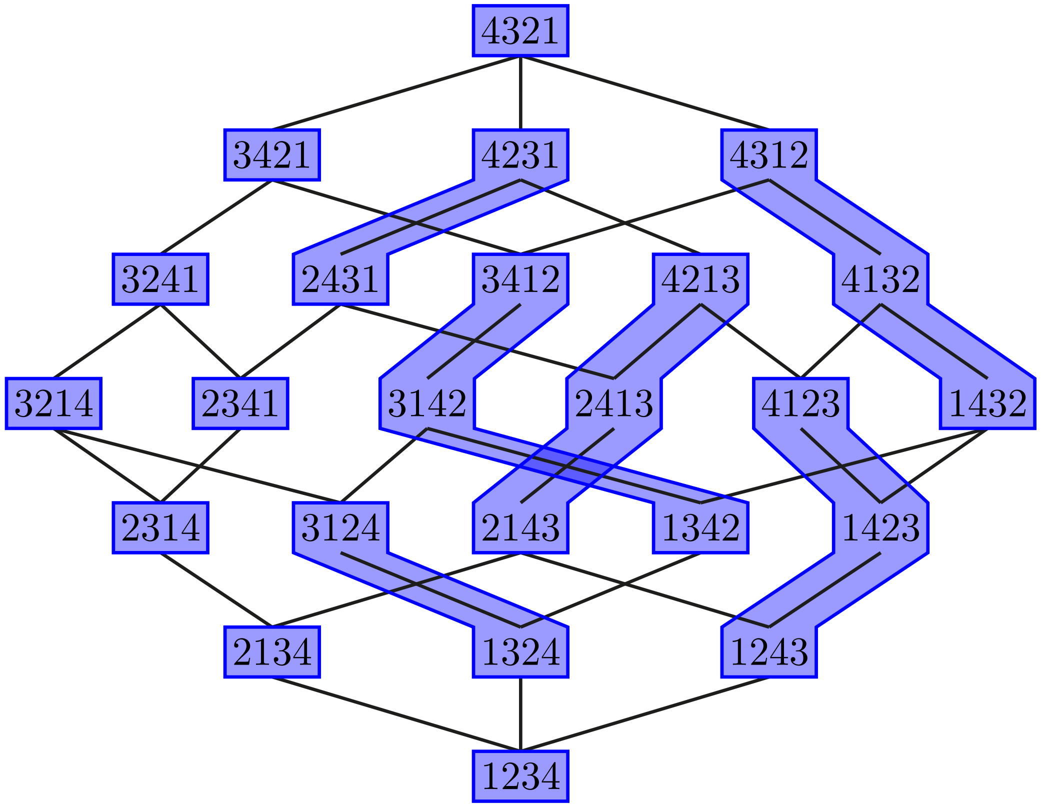

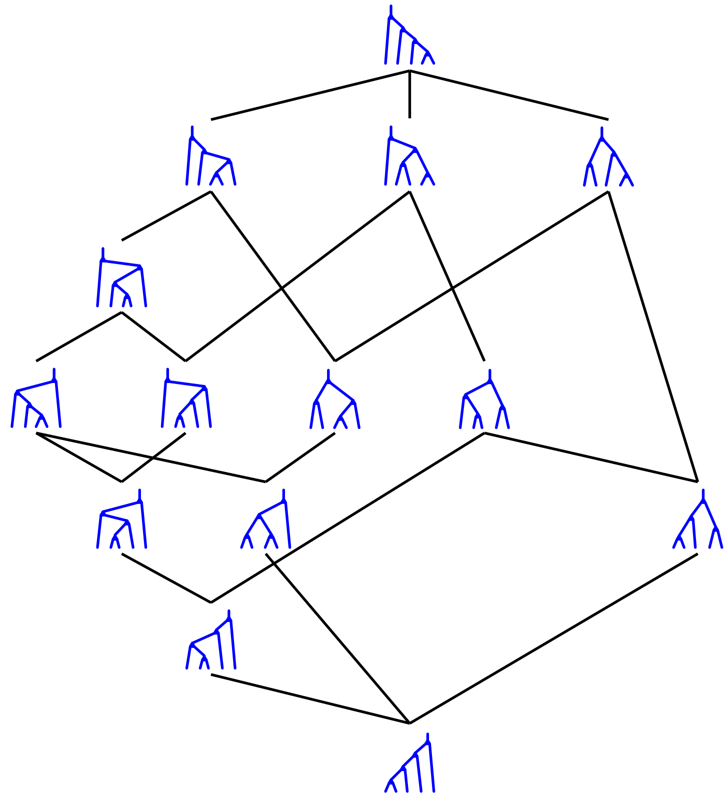

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$=$ equivalence classes are fibers of BST insertion
$=$ rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

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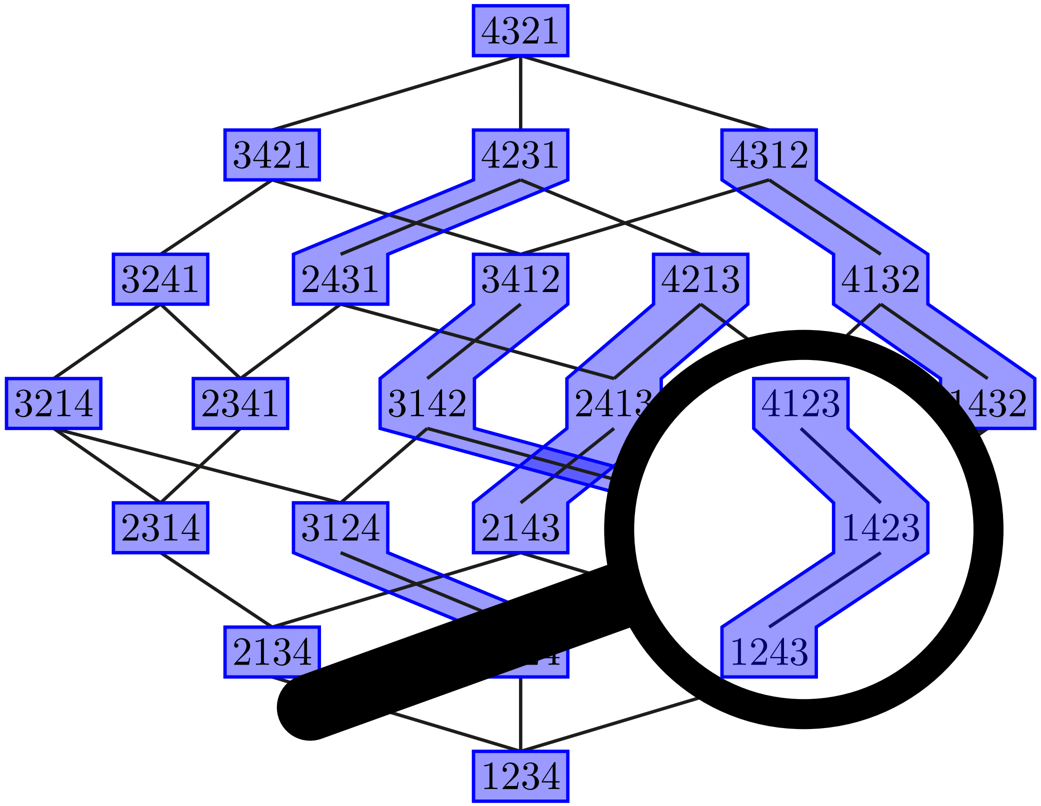

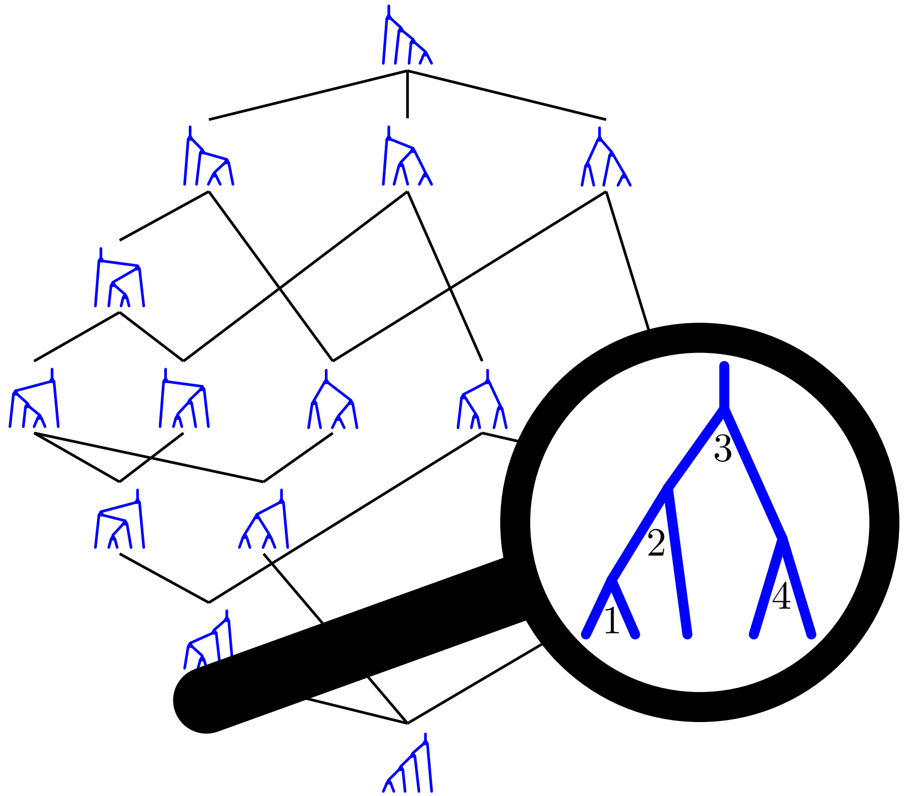

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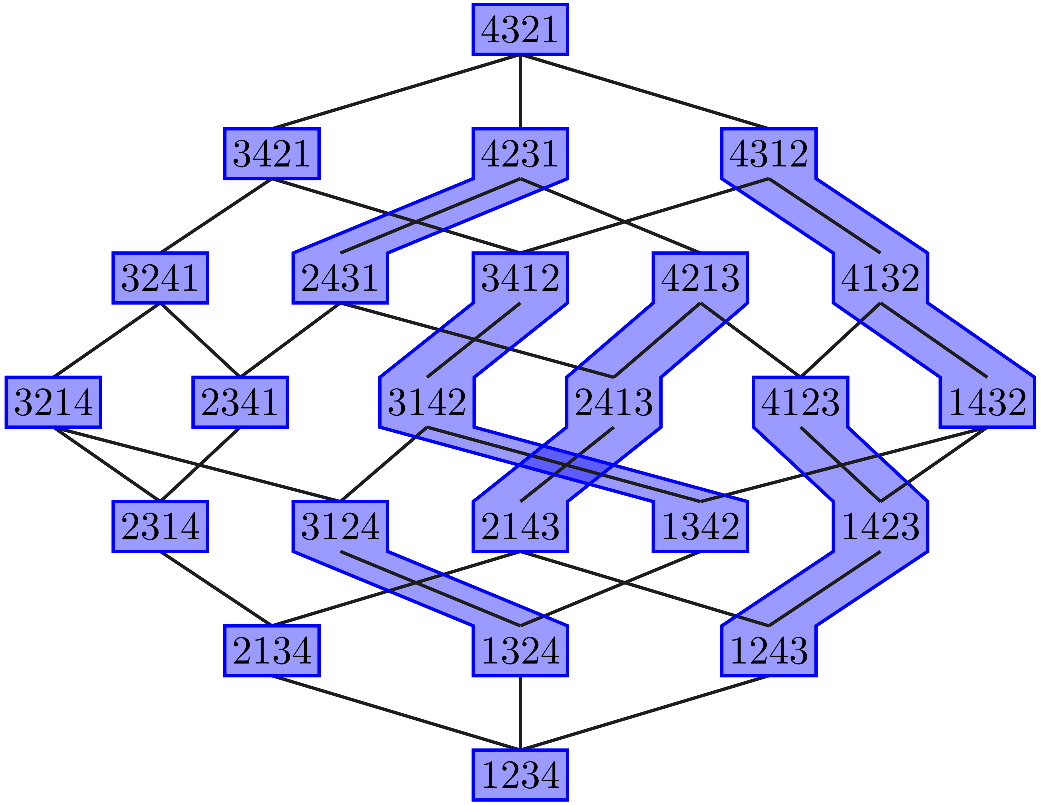

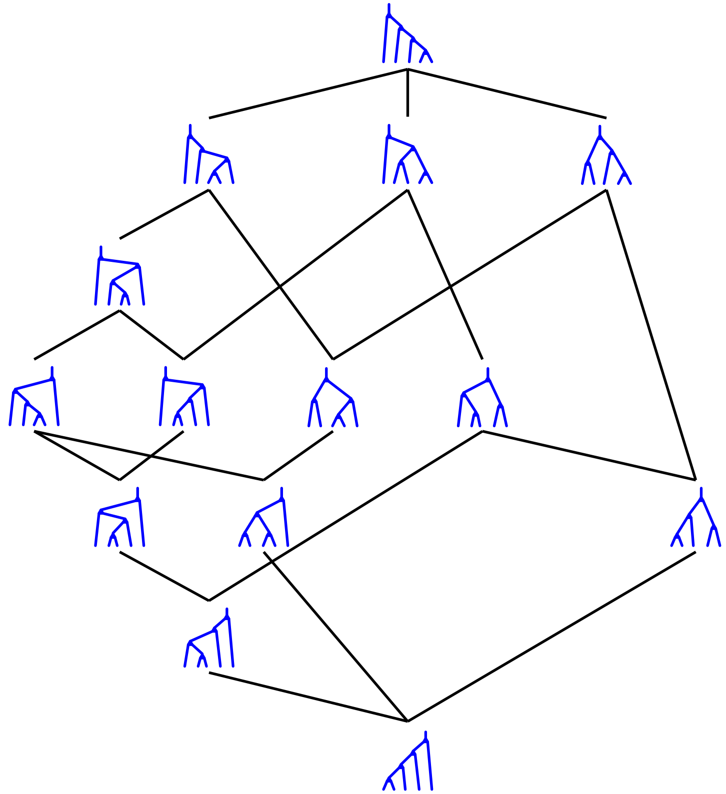

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lattice congruence $=$ equivalence relation $\equiv$ which respects meets and joins

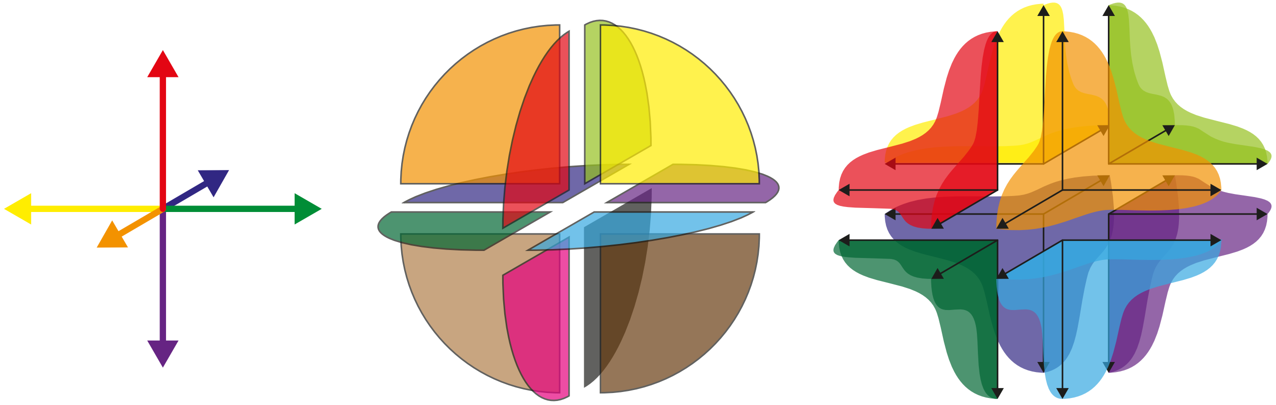
$x \equiv x'$ and $y \equiv y' \Longrightarrow x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$

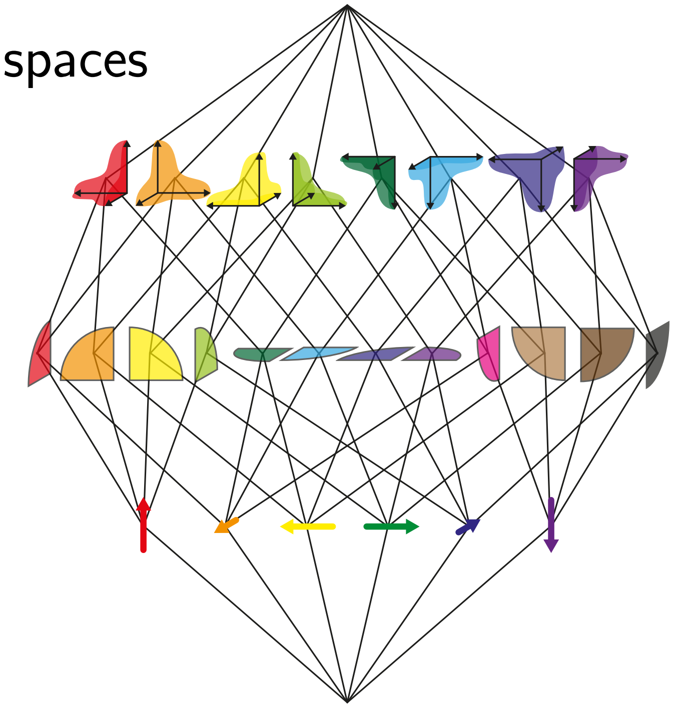
quotient lattice $=$ lattice on classes with $X \leq Y \iff \exists\, x \in X,\ y \in Y, x \leq y$

FANS: BRAID FAN AND SYLVESTER FAN

polyhedral cone = positive span of a finite set of vectors
= intersection of a finite set of linear half-spaces

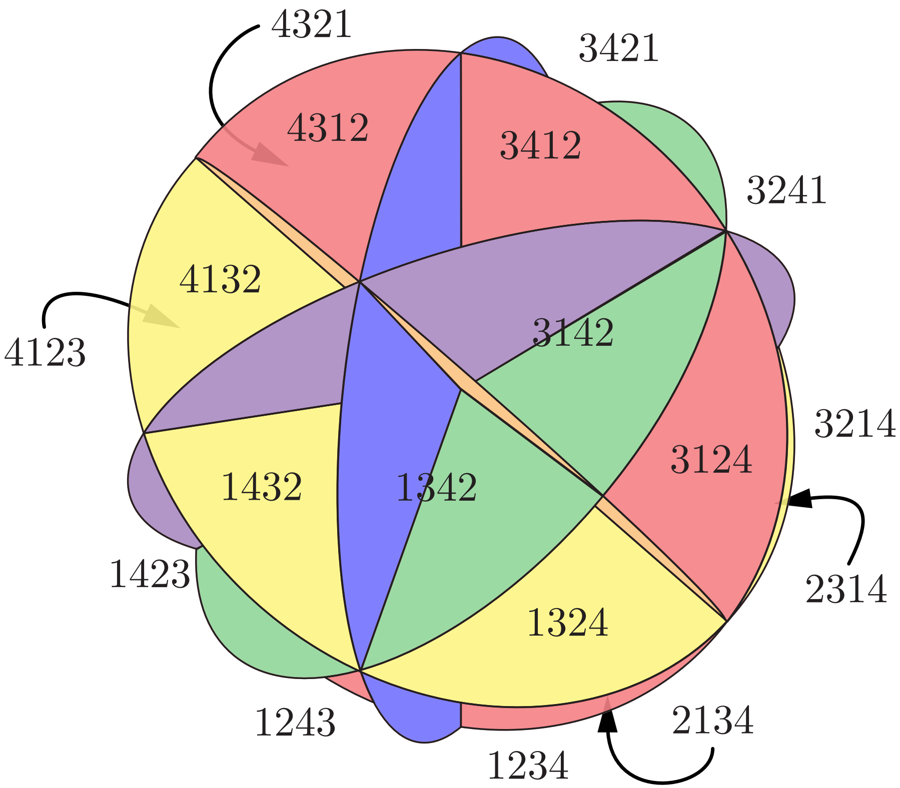
fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face

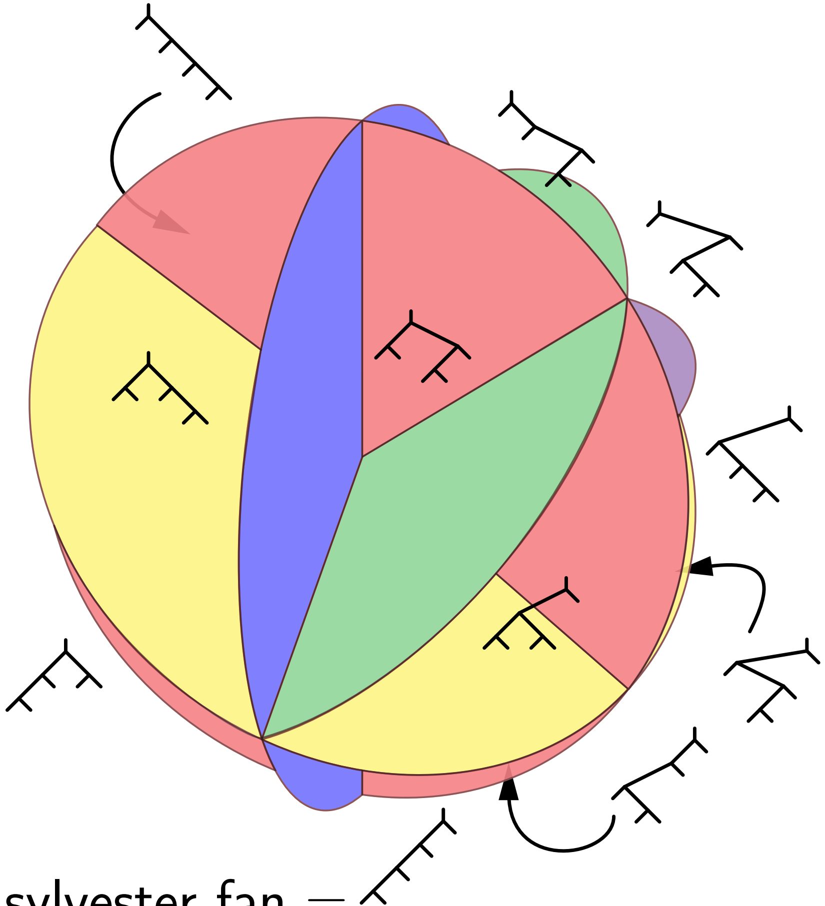




FANS: BRAID FAN AND SYLVESTER FAN

fan = collection of polyhedral cones closed by faces and intersecting along faces





braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$

sylvester fan =

$$\mathbb{C}(T) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T \}$$

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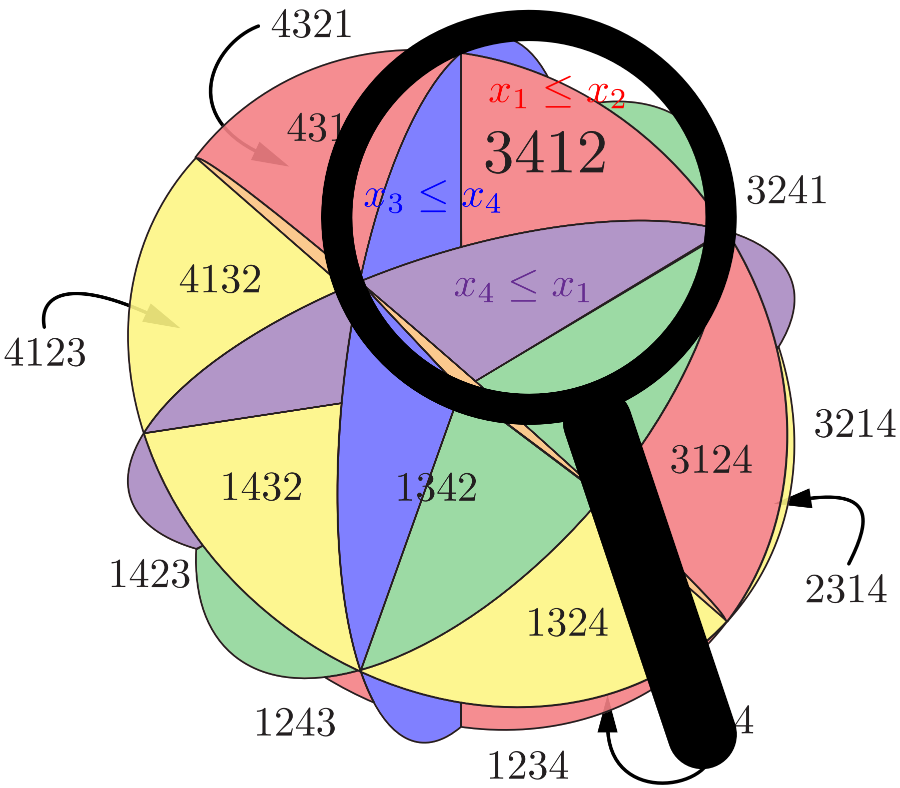

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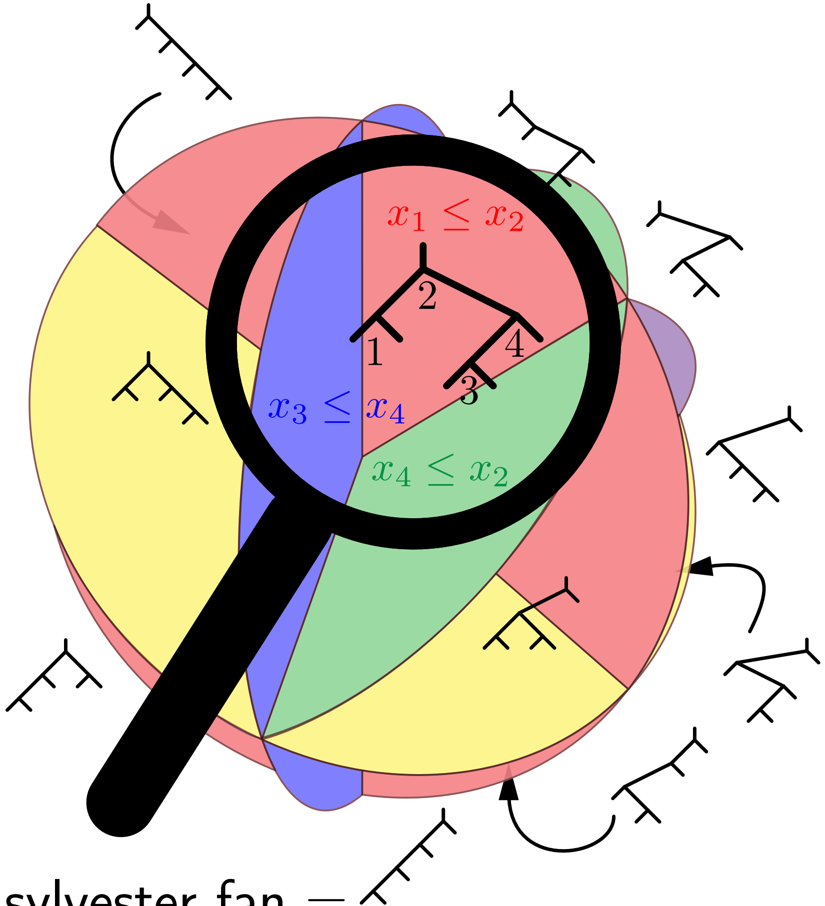

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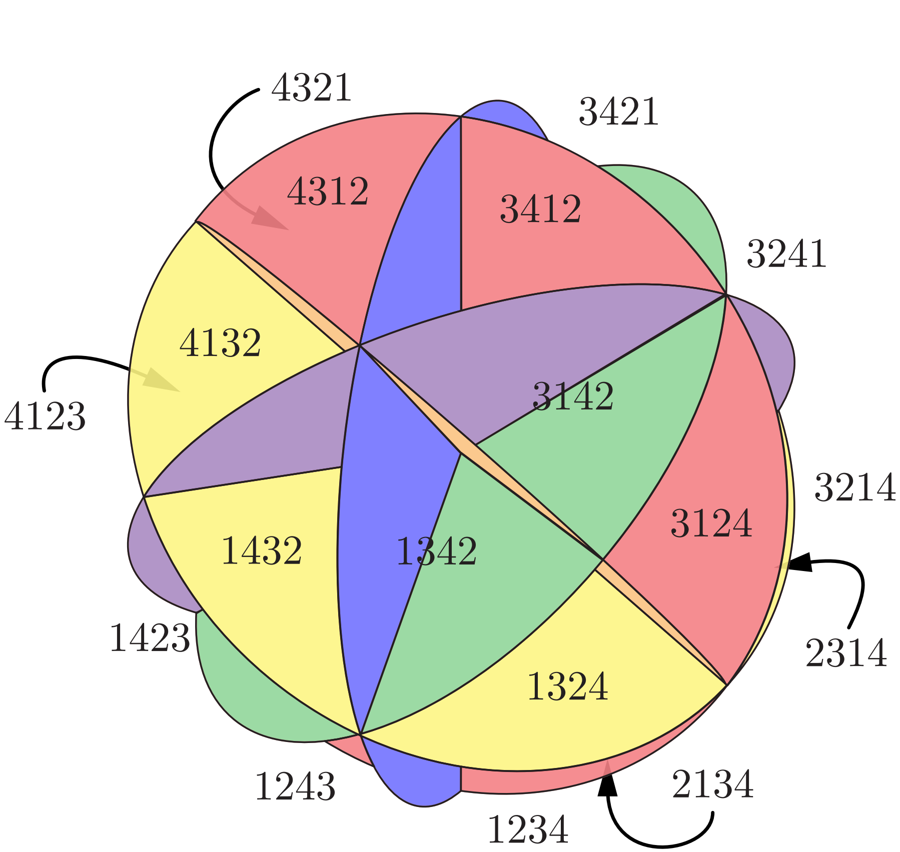

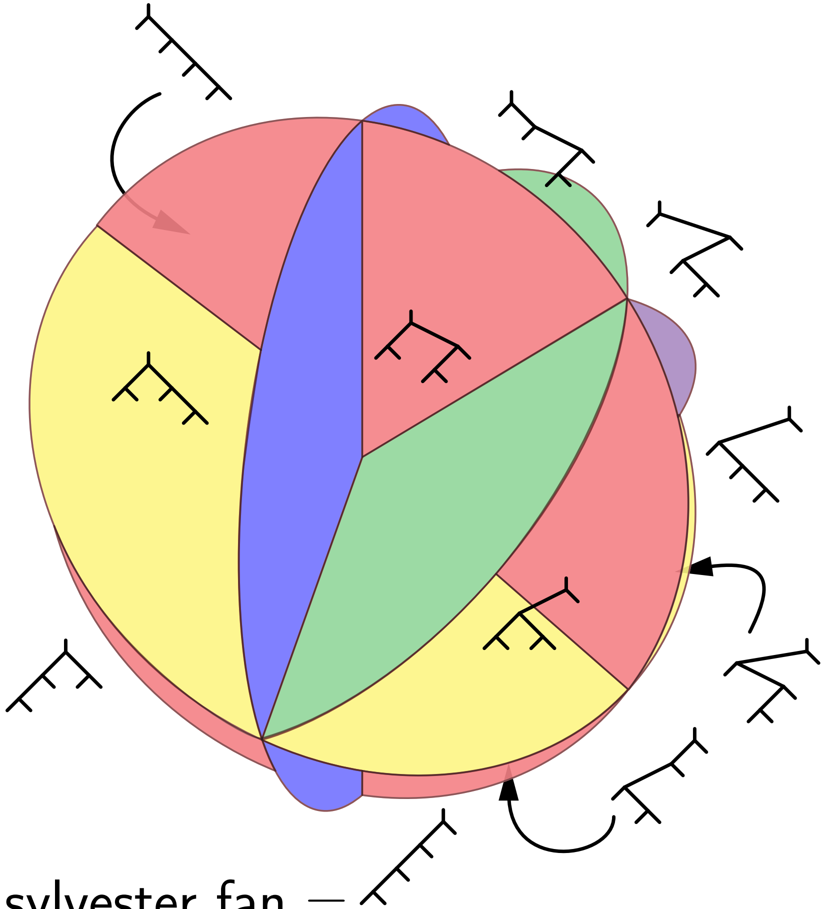

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quotient fan $=$ $\mathbb{C}(T)$ is obtained by glueing $\mathbb{C}(\sigma)$ for all linear extensions σ of T

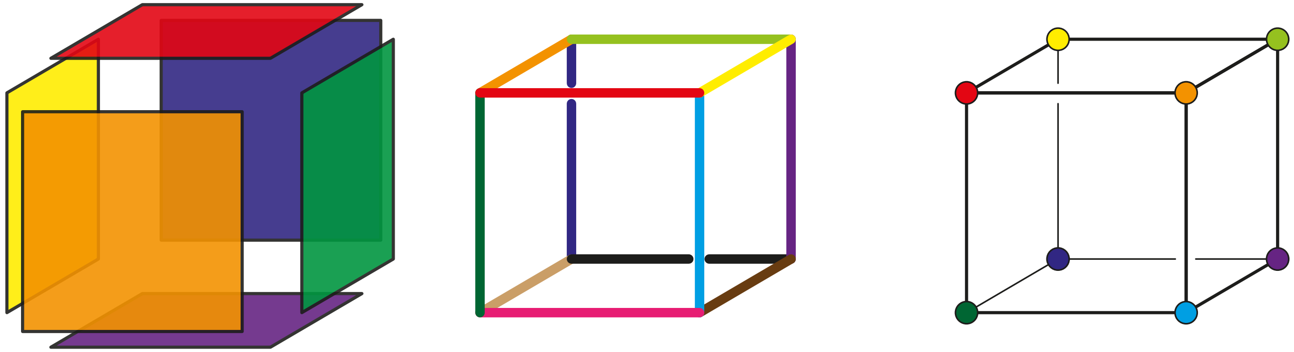
POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

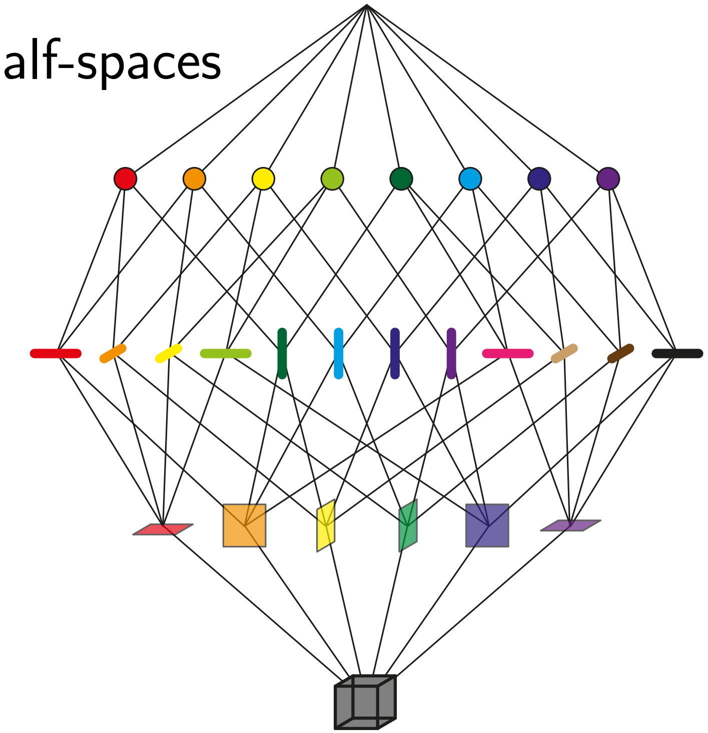
polytope = convex hull of a finite set of points

= bounded intersection of a finite set of affine half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations





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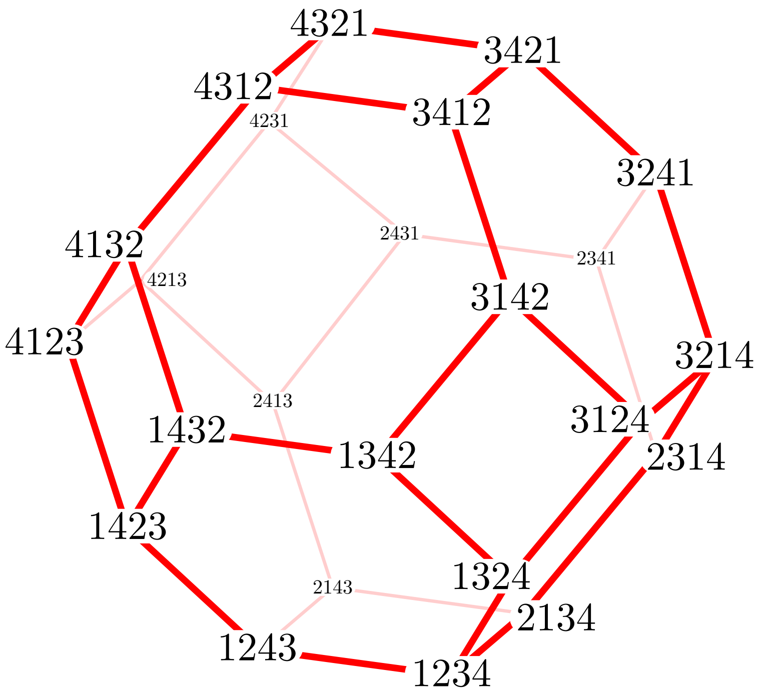

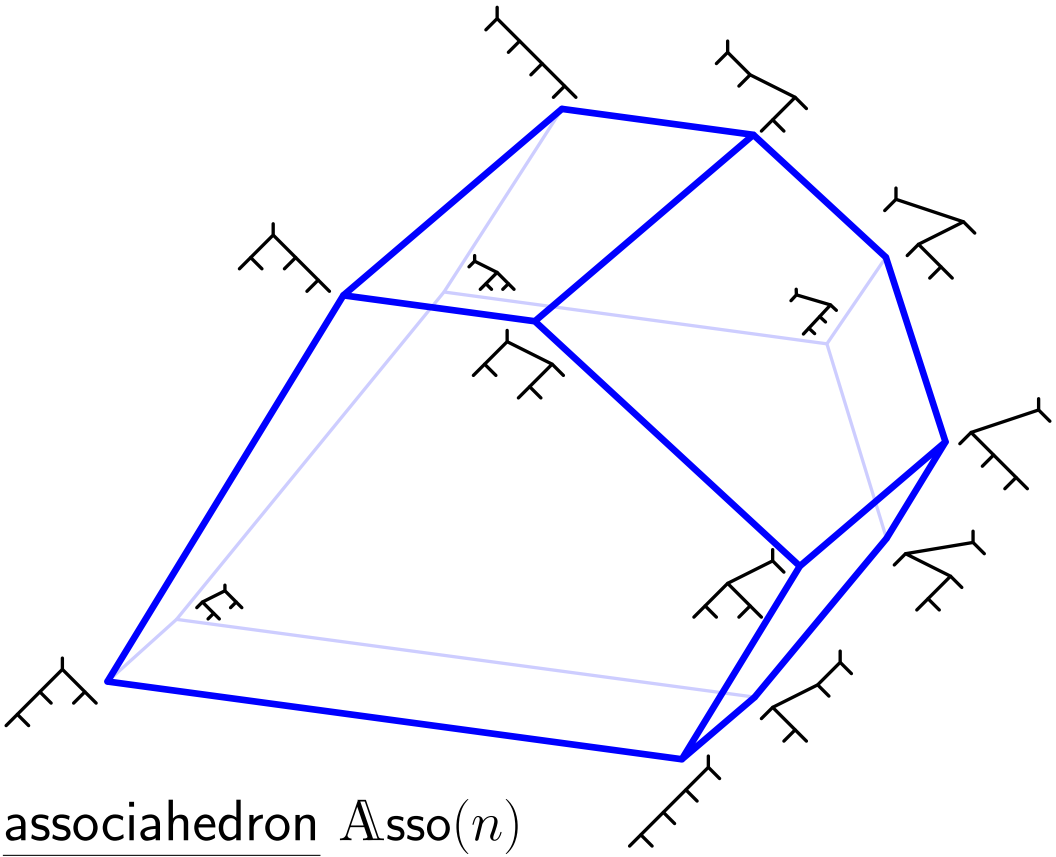

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$= \mathrm{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$

$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \geq \binom{|J|+1}{2} \right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$= \mathrm{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree} \right\}$

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Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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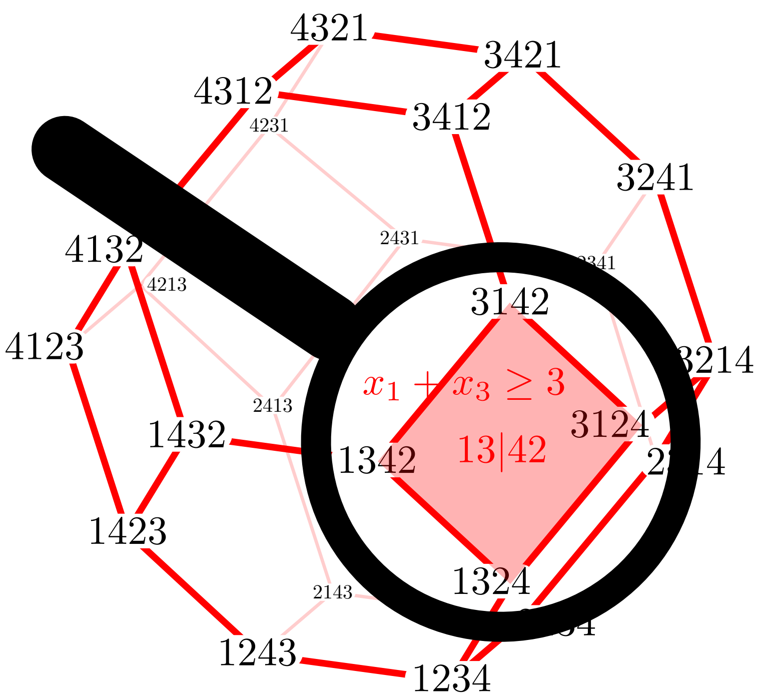

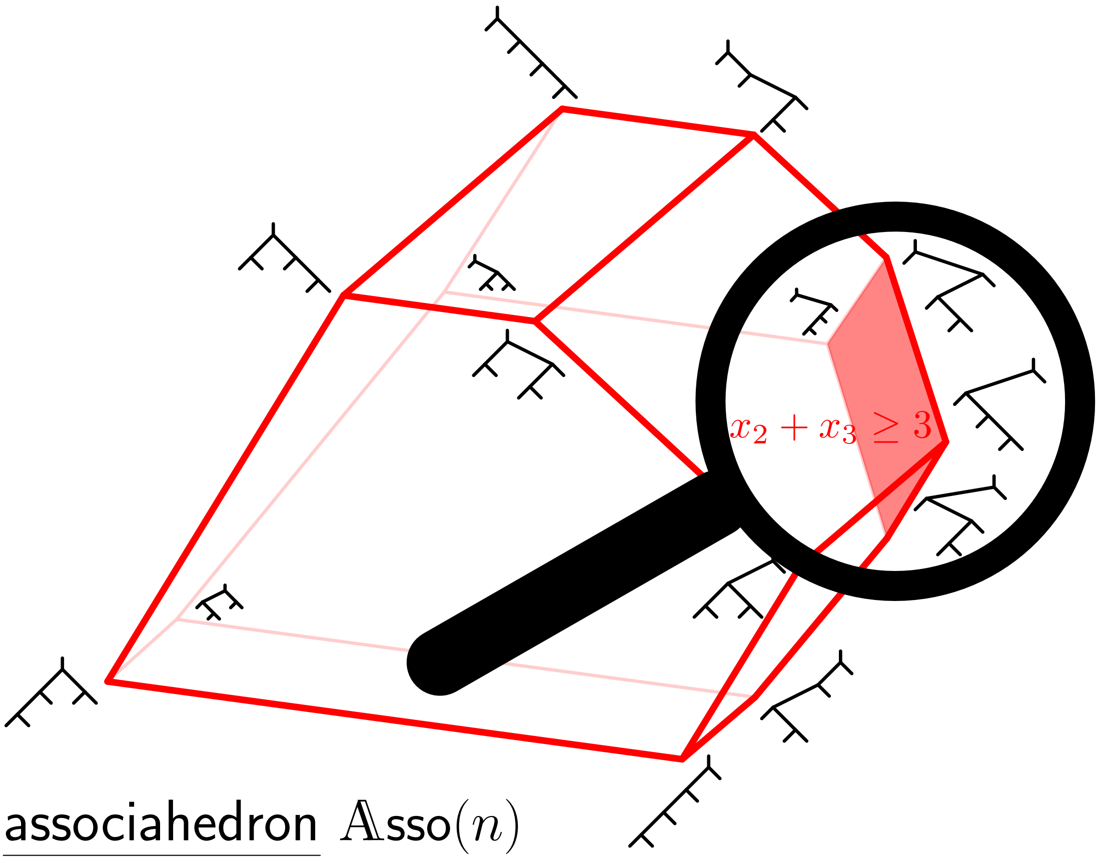

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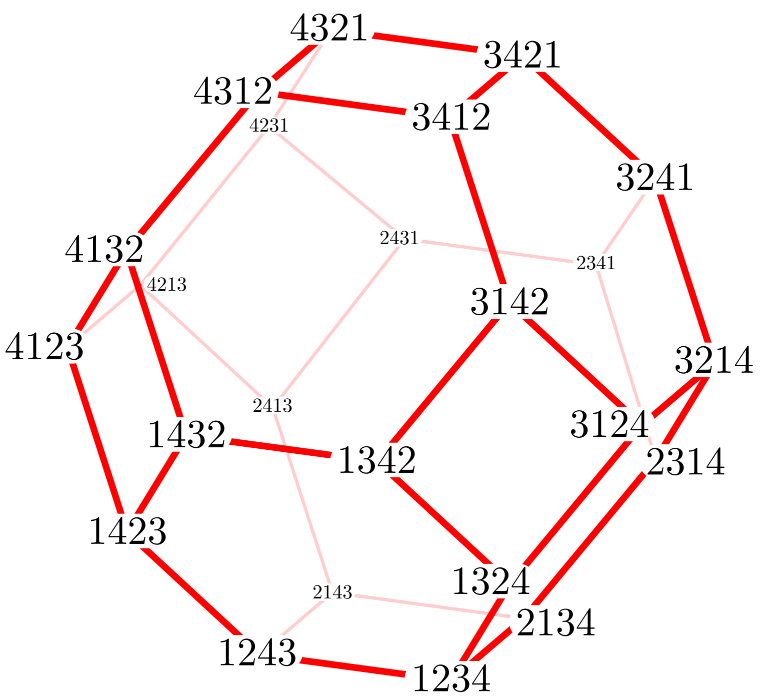

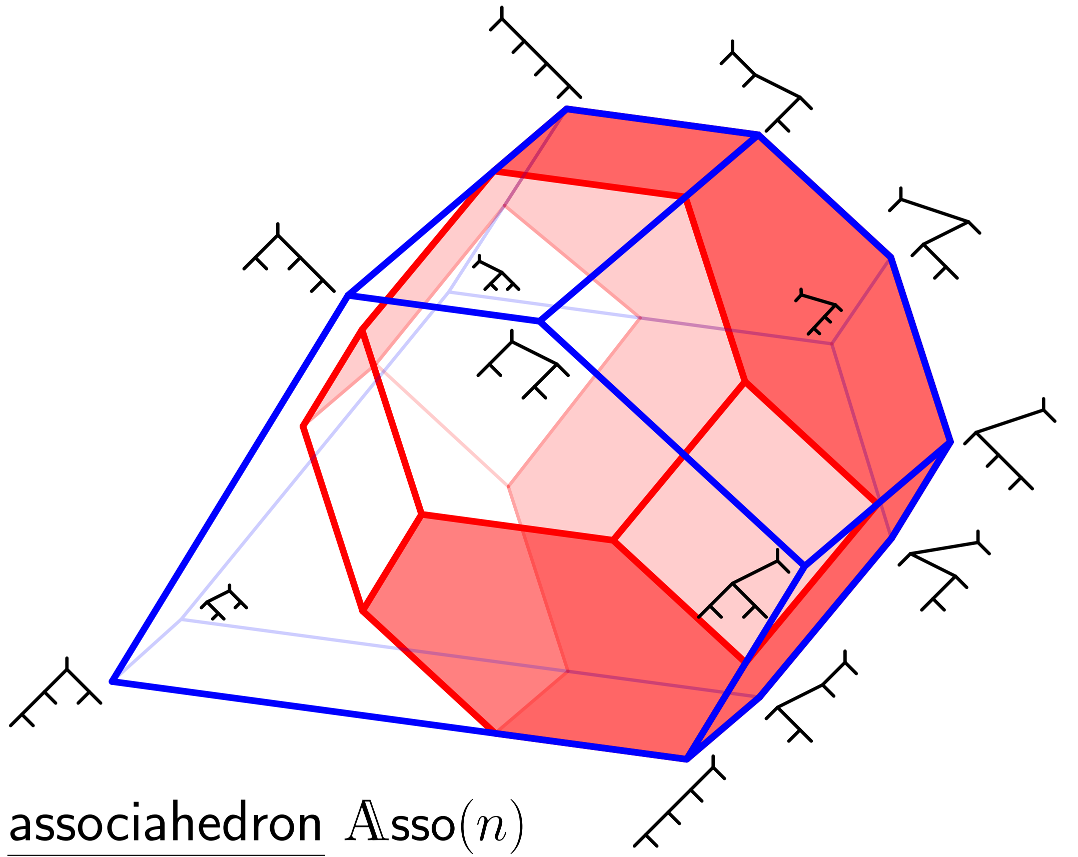

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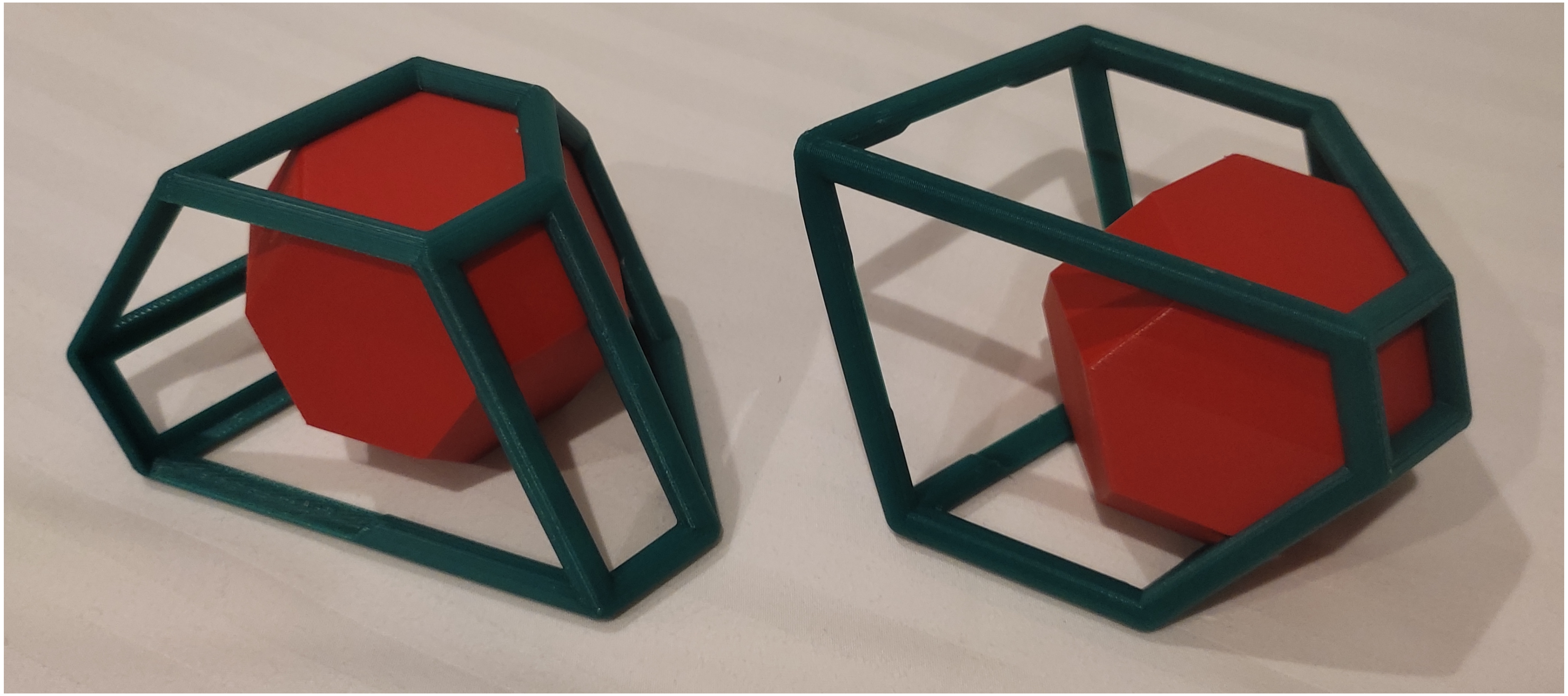

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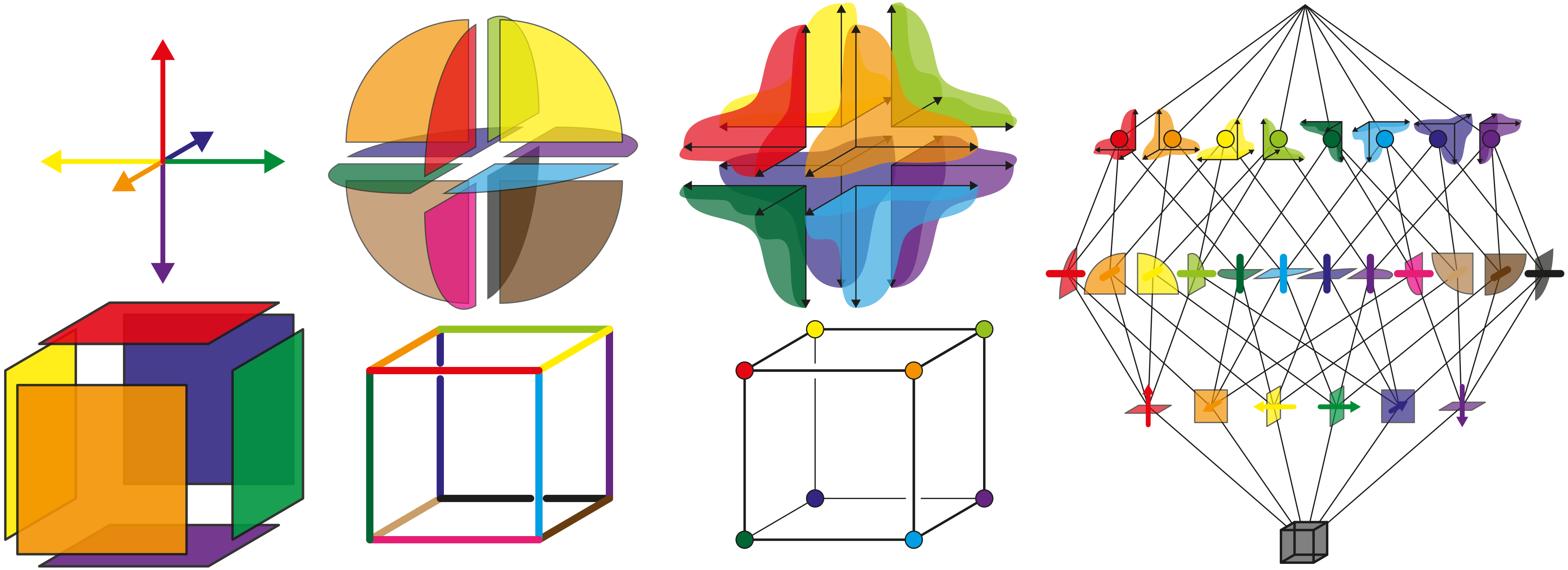
LATTICES – FANS – POLYTOPES



face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = $\{$ normal cone of $\mathbb{F}$ $|$ $\mathbb{F}$ face of $\mathbb{P}$ $\}$

LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{P}\mathrm{erm}(n)$

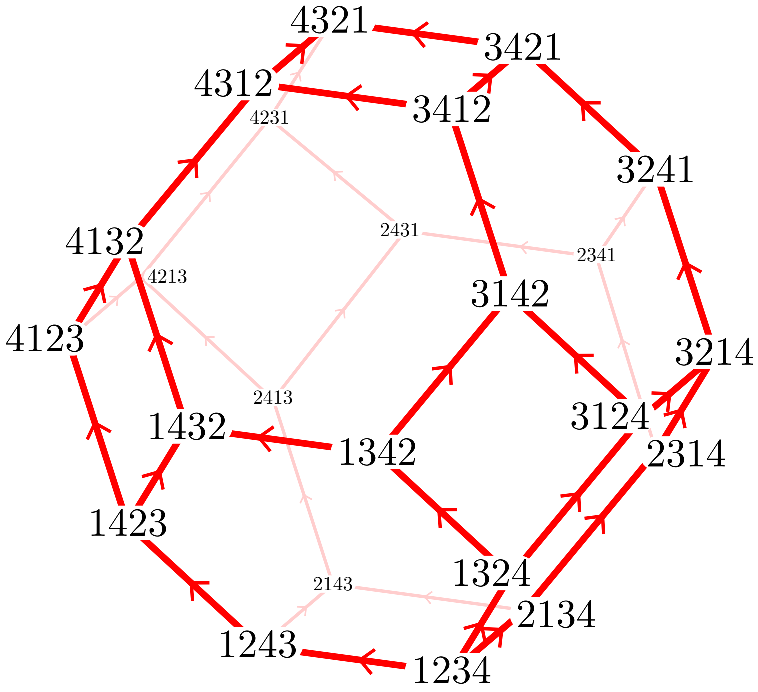
$\Longrightarrow$ braid fan

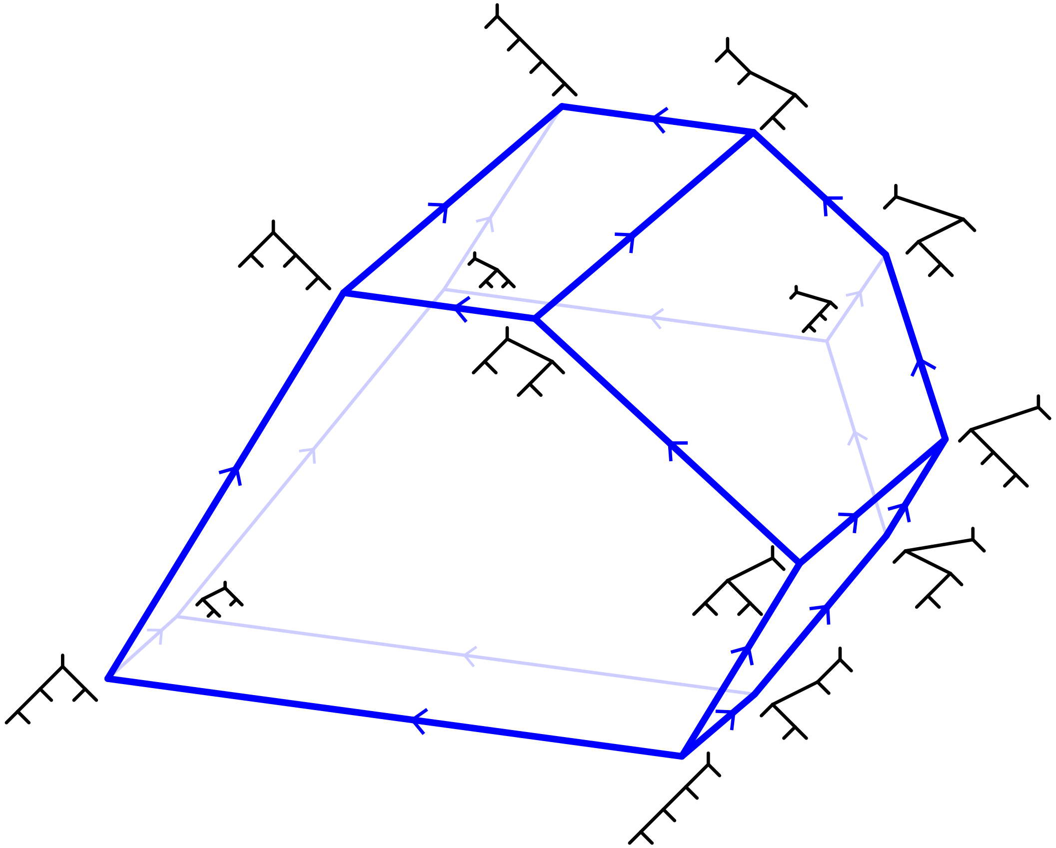
$\Longrightarrow$ weak order on permutations

associahedron $\mathbb{A}\mathrm{sso}(n)$

$\Longrightarrow$ Sylvester fan

$\Longrightarrow$ Tamari lattice on binary trees





DEFORMED PERMUTAHEDRA

Postnikov, *Permutohedra, associahedra, and beyond* ('09)
Postnikov–Reiner–Williams, *Faces of generalized permutohedra* ('08)

DEFORMED PERMUTAHEDRA

deformation of a polytope $\mathbb{P}$ = polytope $\mathbb{Q}$ such that

- $\mathbb{Q}$ is obtained from $\mathbb{P}$ by moving its vertices such that edge directions are preserved
- $\mathbb{Q}$ is obtained from $\mathbb{P}$ by translating its inequalities without passing through a vertex
- the normal fan of $\mathbb{P}$ refines the normal fan of $\mathbb{Q}$
- $\mathbb{Q}$ is a weak Minkowski summand of $\mathbb{P}$, i.e. there is $\mathbb{R}$ and $\lambda > 0$ such that $\lambda\mathbb{P} = \mathbb{Q} + \mathbb{R}$

POLYWOOD

deformed permutahedron = polymatroid = generalized permutahedron

Edmonds ('70) Postnikov ('09)

REMOVAHEDRA VS. DEFORMED PERMUTAHEDRA

deformation of $\mathbb{P}$ = obtained by translating inequalities in the facet description of $\mathbb{P}$

removahedron of $\mathbb{P}$ = obtained by removing inequalities in the facet description of $\mathbb{P}$

outsidahedra
removahedra
permutrees

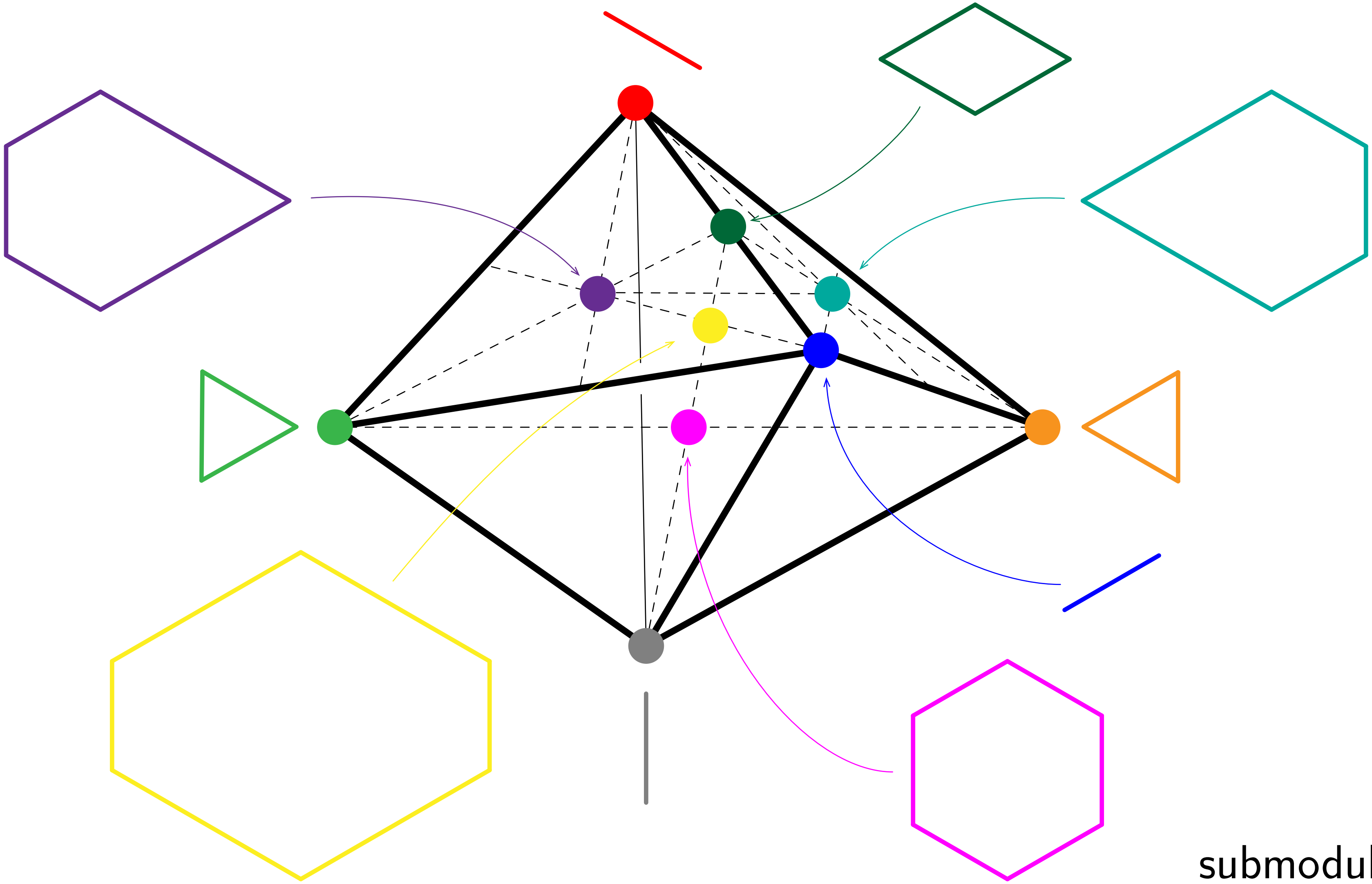
insidahedra
deformed permutahedra
quotientopes

POLYWOOD

DEFORMATION CONE

deformation of a polytope $\mathbb{P}$ = polytope $\mathbb{Q}$ such that $\lambda\mathbb{P} = \mathbb{Q} + \mathbb{R}$ for some $\mathbb{R}$ and $\lambda > 0$

deformation cone of $\mathbb{P}$ = all deformations of $\mathbb{P}$ (under dilations and Minkowski sums)



submodular cone

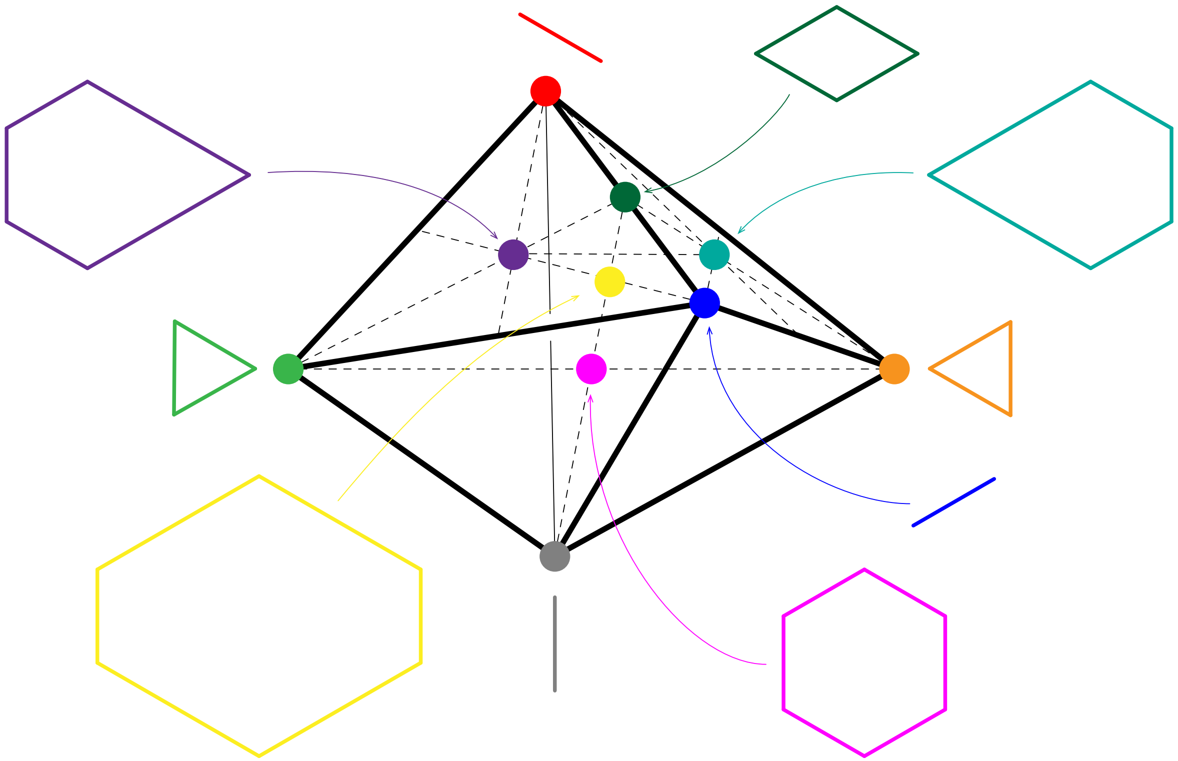
= deformation cone of $\mathbb{P}\mathrm{erm}(3)$

OPEN PROBLEM: RAYS OF THE DEFORMATION CONE

THM. The deformation cone of the permutahedron $\mathbb{P}\mathrm{erm}(n)$ is (isomorphic to) the set of submodular functions $h : 2^{[n]} \to \mathbb{R}_{\geq 0}$ satisfying $h(\varnothing) = h([n]) = 0$ and the submodular inequalities $h(I) + h(J) \geq h(I \cap J) + h(I \cup J)$ for all $I, J \subseteq [n]$.

THM. The facets correspond to submodular inequalities where $|I \smallsetminus J| = |J \smallsetminus I| = 1$.

PROB. Describe (or count) the rays of the submodular cone. Edmonds ('70)

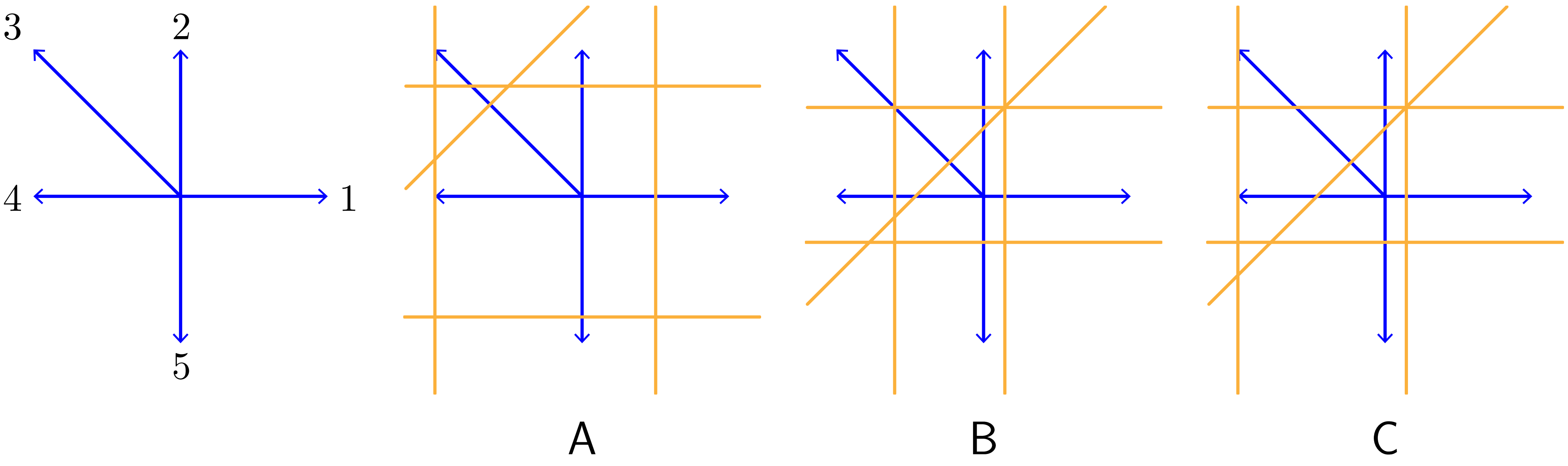


CHOOSING RIGHT-HAND-SIDES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

$\boldsymbol{G} = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$



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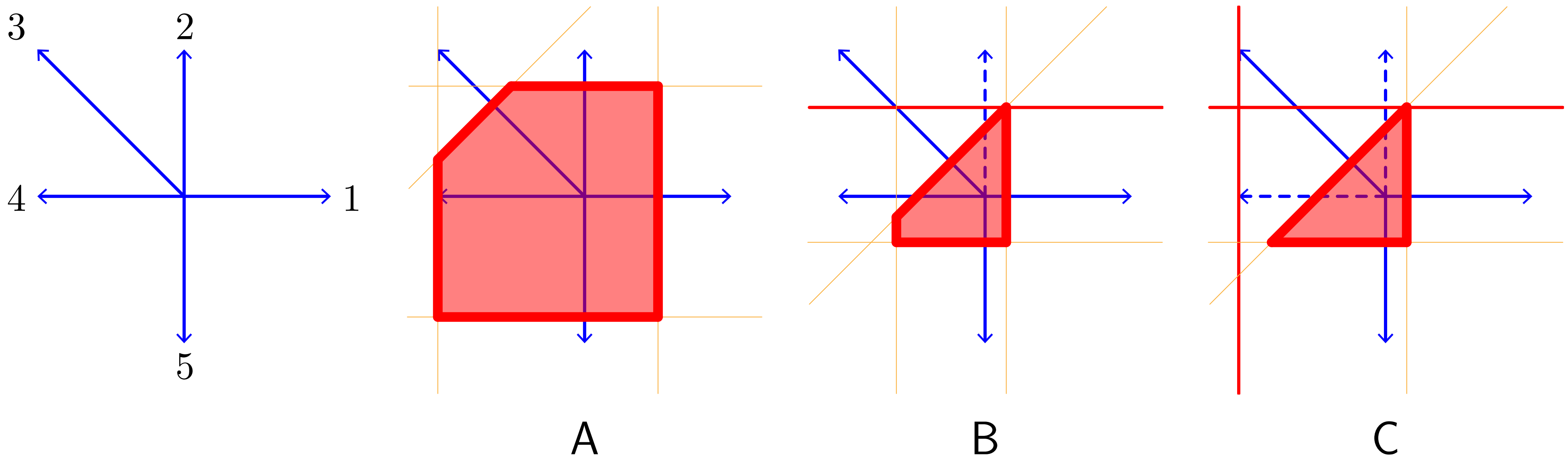

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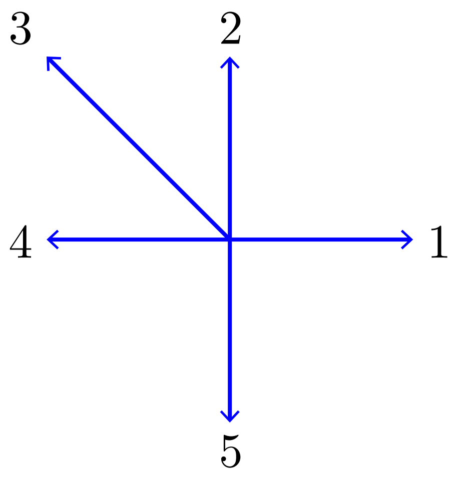

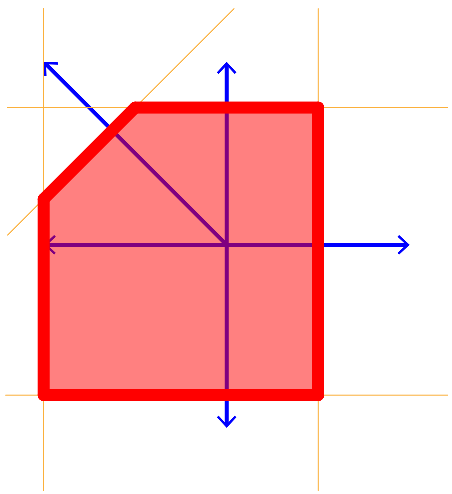

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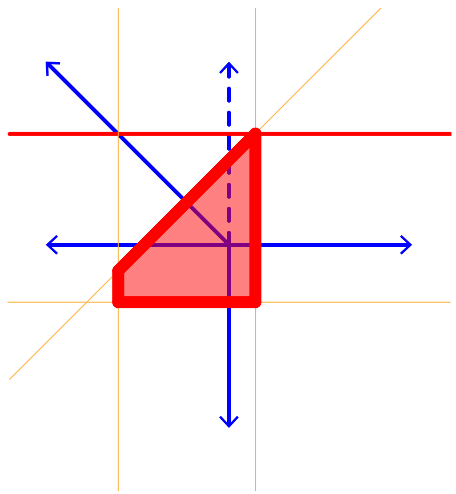
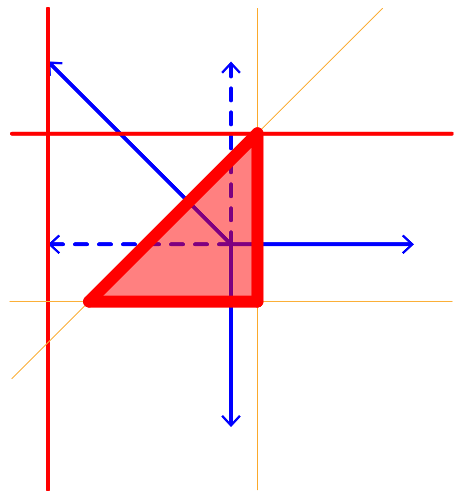


B



C

When is $\mathcal{F}$ the normal fan of $\mathbb{P}_{\boldsymbol{h}}$?

WALL-CROSSING INEQUALITIES

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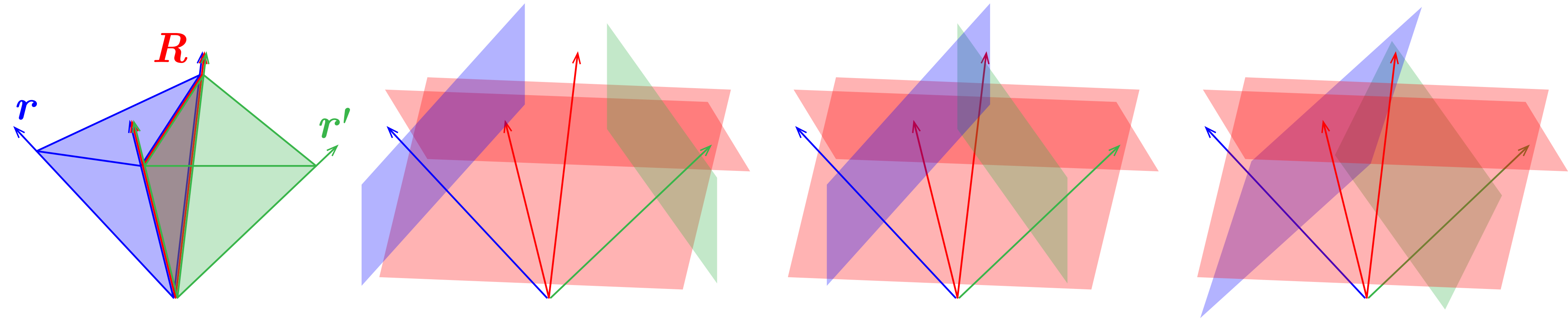

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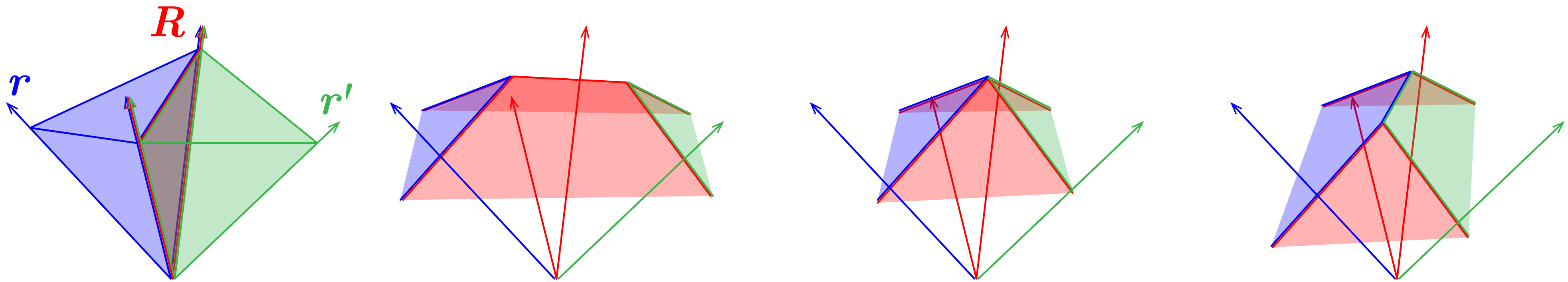
for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$



WALL-CROSSING INEQUALITIES

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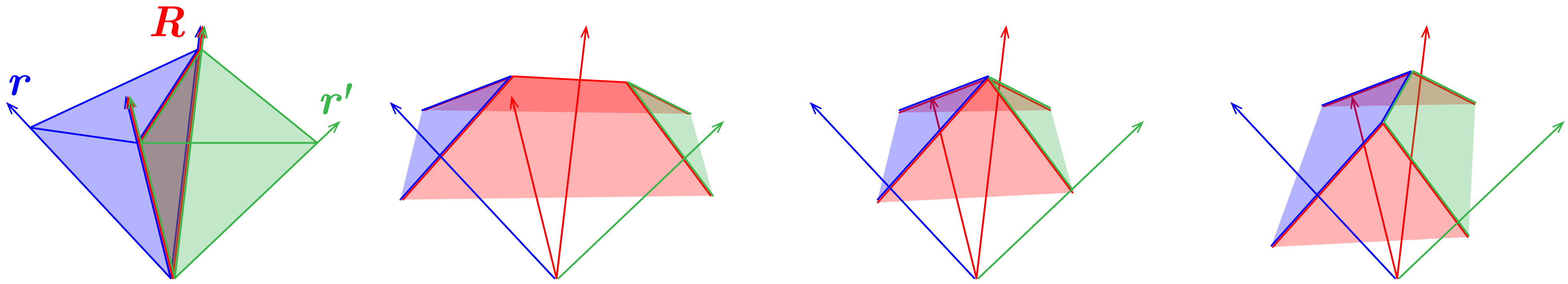

wall-crossing inequality for a wall $\boldsymbol{R}$ = $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, h_{\boldsymbol{s}} > 0$ where

- $\boldsymbol{r}, \boldsymbol{r'}$ = rays such that $\boldsymbol{R} \cup \{\boldsymbol{r}\}$ and $\boldsymbol{R} \cup \{\boldsymbol{r'}\}$ are chambers of $\mathcal{F}$
- $\alpha_{\boldsymbol{R},\boldsymbol{s}}$ = coeff. of unique linear dependence $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, \boldsymbol{s} = 0$ with $\alpha_{\boldsymbol{R},\boldsymbol{r}} + \alpha_{\boldsymbol{R},\boldsymbol{r'}} = 2$

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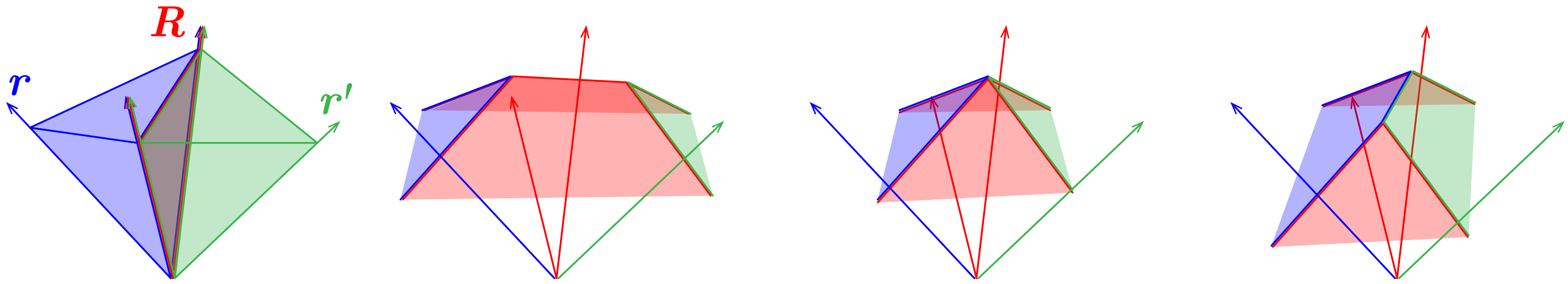

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$\mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}$ $\iff$ $\boldsymbol{h}$ satisfies all wall-crossing inequalities of $\mathcal{F}$

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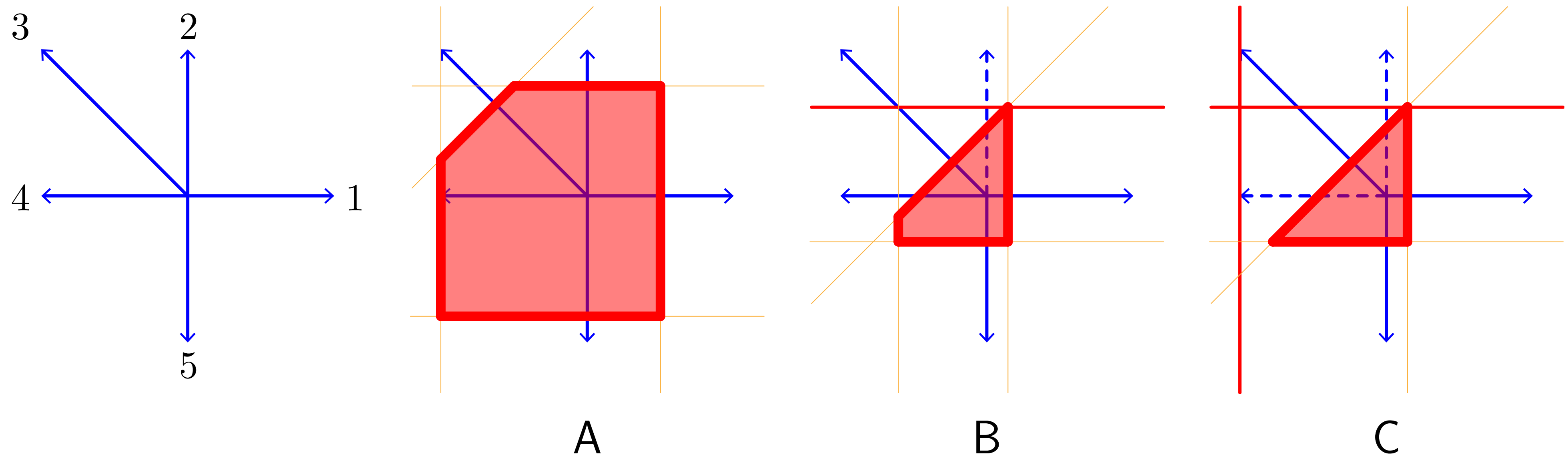
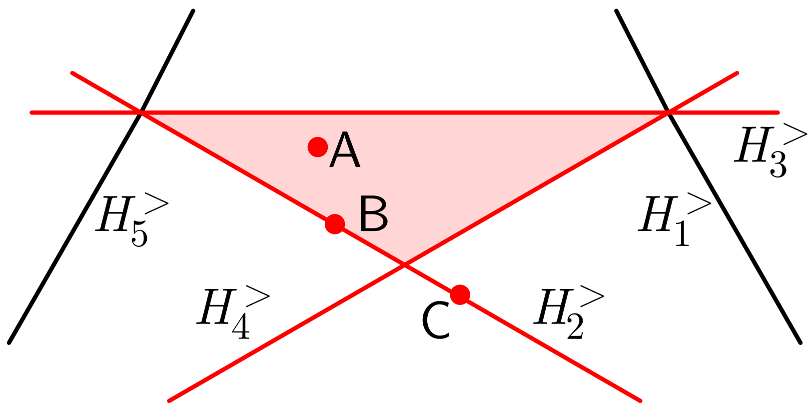


wall-crossing inequalities:

$$\begin{aligned}
\text{wall } 1 : &\quad h_2 + h_5 > 0 \\
\text{wall } 2 : &\quad h_1 + h_3 > h_2 \\
\text{wall } 3 : &\quad h_2 + h_4 > h_3 \\
\text{wall } 4 : &\quad h_3 + h_5 > h_4 \\
\text{wall } 5 : &\quad h_1 + h_4 > 0
\end{aligned}$$



TYPE CONE

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<u>type cone</u> $\mathbb{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ McMullen ('73)

$= \big\{\boldsymbol{h} \in \mathbb{R}^N \;\big|\; \mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}\big\}$

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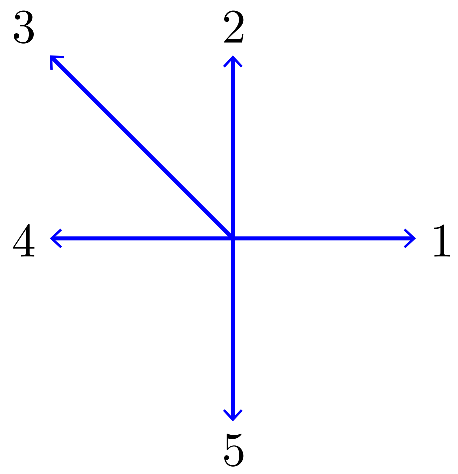

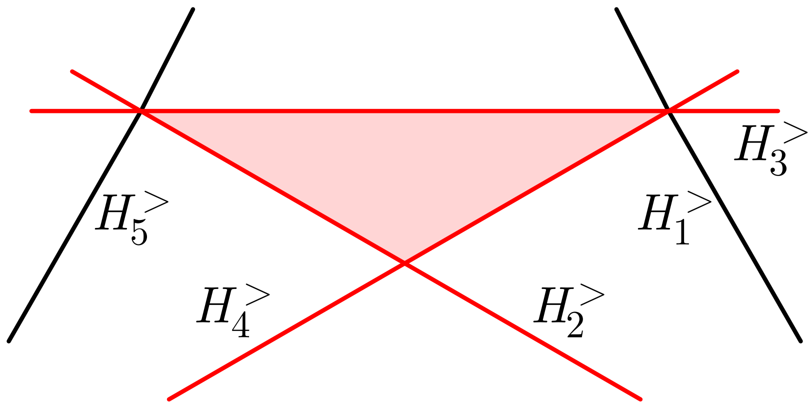

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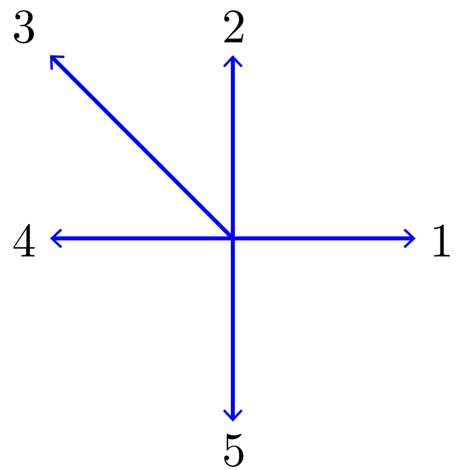
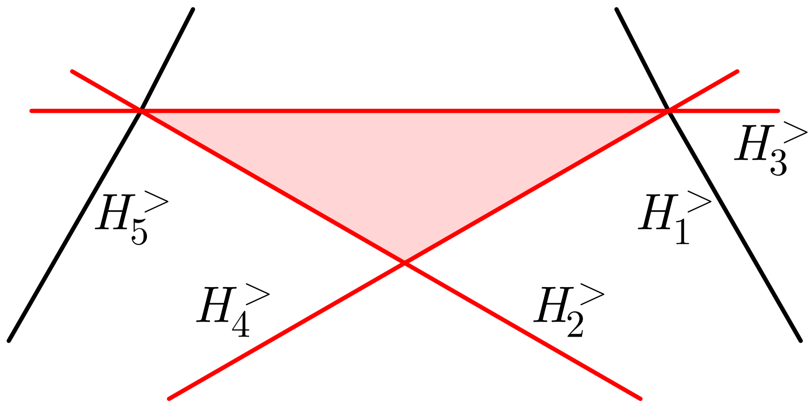
type cone $\mathbb{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ — McMullen ('73)
$= \{\boldsymbol{h} \in \mathbb{R}^N \mid \mathcal{F} \text{ is the normal fan of } \mathbb{P}_{\boldsymbol{h}}\}$
$= \{\boldsymbol{h} \in \mathbb{R}^N \mid \boldsymbol{h} \text{ satisfies all wall-crossing inequalities of } \mathcal{F}\}$





some properties of $\mathbb{TC}(\mathcal{F})$:

- $\mathbb{TC}(\mathcal{F})$ is an open cone (dilations preserve normal fans)
- $\mathbb{TC}(\mathcal{F})$ has lineality space $\boldsymbol{G}\,\mathbb{R}^n$ (translations preserve normal fans)
- dimension of $\mathbb{TC}(\mathcal{F})/\boldsymbol{G}\,\mathbb{R}^n = N - n$

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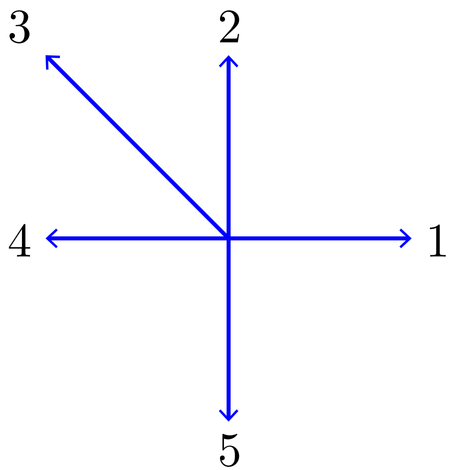
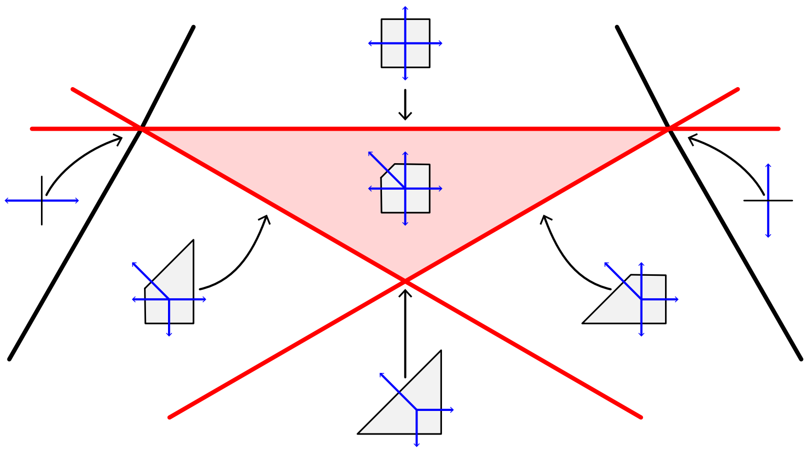
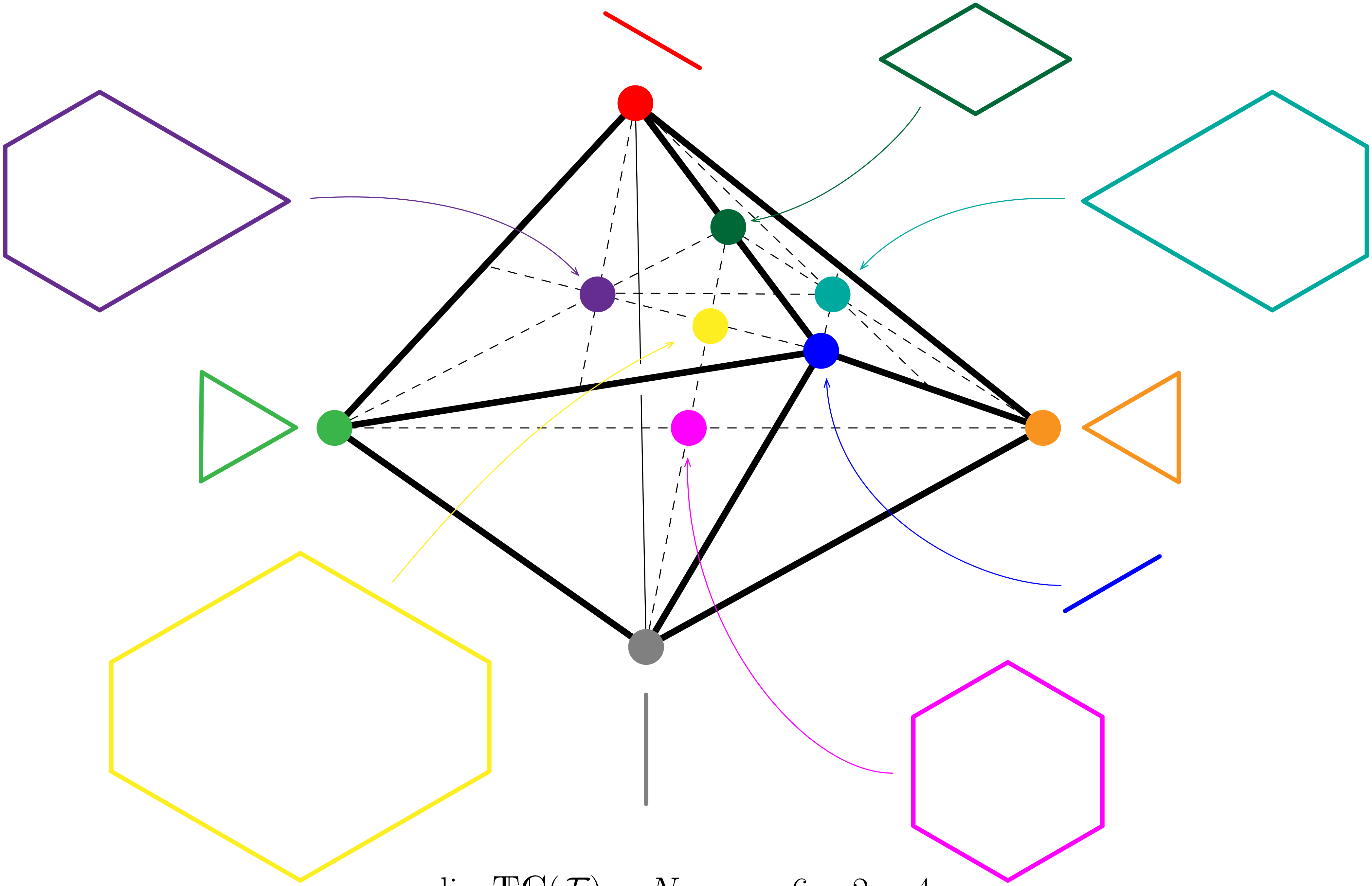




some properties of $\mathbb{TC}(\mathcal{F})$:

- closure of $\mathbb{TC}(\mathcal{F})$ = polytopes whose normal fan coarsens $\mathcal{F}$ = deformation cone
- Minkowski sums $\longleftrightarrow$ positive linear combinations

SUBMODULAR FUNCTIONS

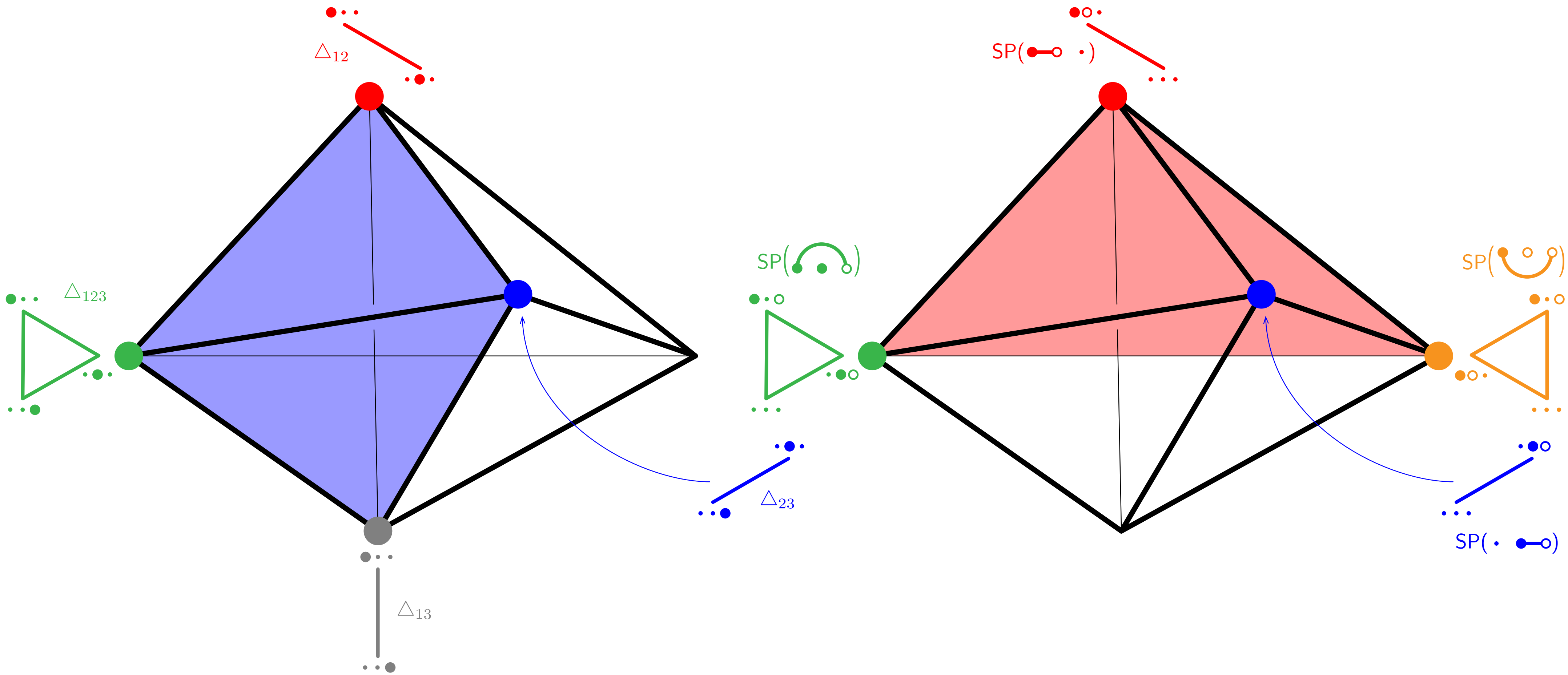


$$\dim \mathbb{TC}(\mathcal{F}) = N - n = 6 - 2 = 4$$

LINEAR BASES OF THE SUBMODULAR CONE

PROP. Linear bases of the submodular cone:

- $\{\triangle_I \mid I \subseteq [n]\}$ = faces of the standard simplex $\triangle_n$
- $\{\mathbb{SP}(i, j, A, B) \mid 1 \le i < j \le n,\ A \sqcup B = [n]\}$ = shard polytopes



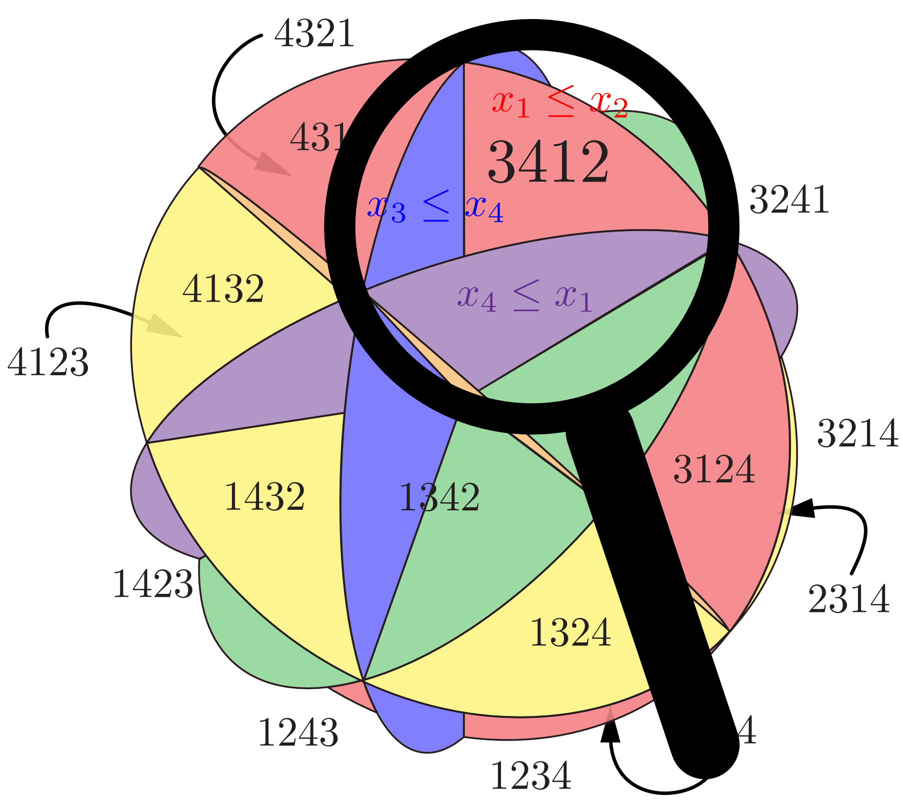
$\mathbb{P}$-POSETS

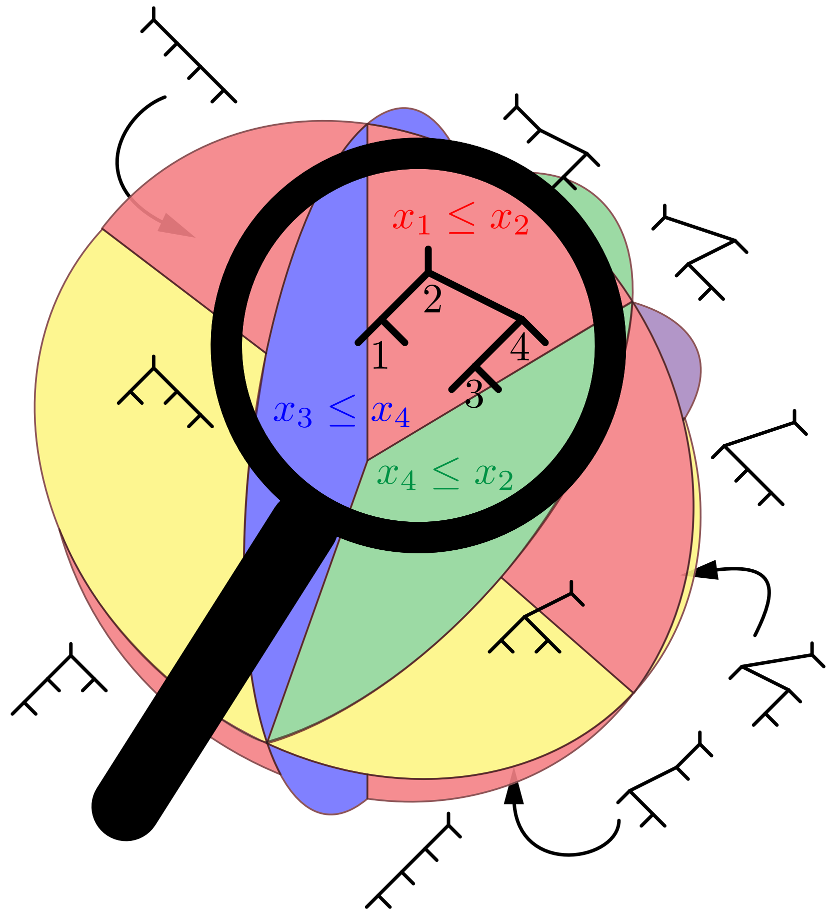
REM. poset $\preccurlyeq$ on $[n]$ $\longleftrightarrow$ full-dim. cone $\mathbb{C}_{\preccurlyeq} = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ for all } i \preccurlyeq j\}$

preposet $\preccurlyeq$ on $[n]$ $\longleftrightarrow$ cone $\mathbb{C}_{\preccurlyeq} = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ for all } i \preccurlyeq j\}$

DEF. $\mathbb{P}$ = deformed permutahedron

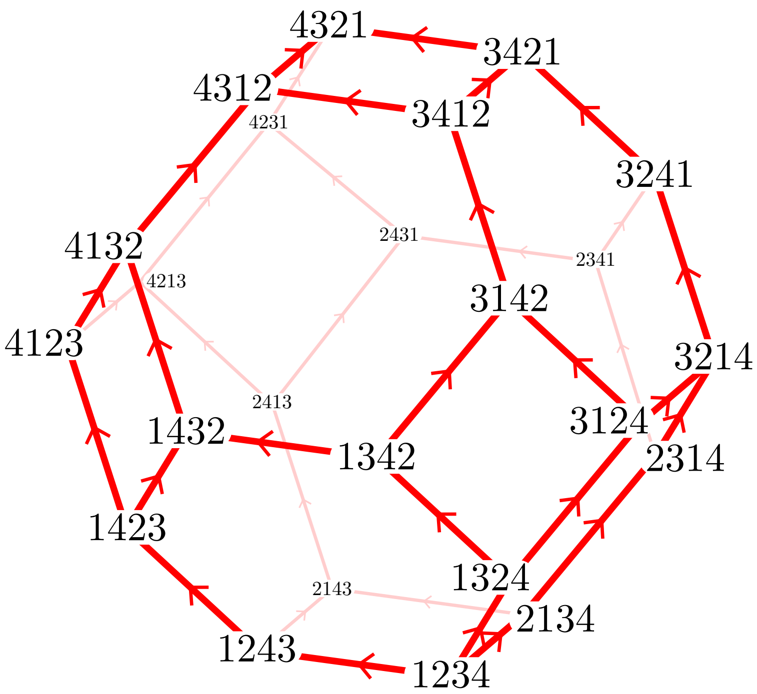
$\mathbb{P}$-posets = posets corresponding to maximal cones in the normal fan of $\mathbb{P}$

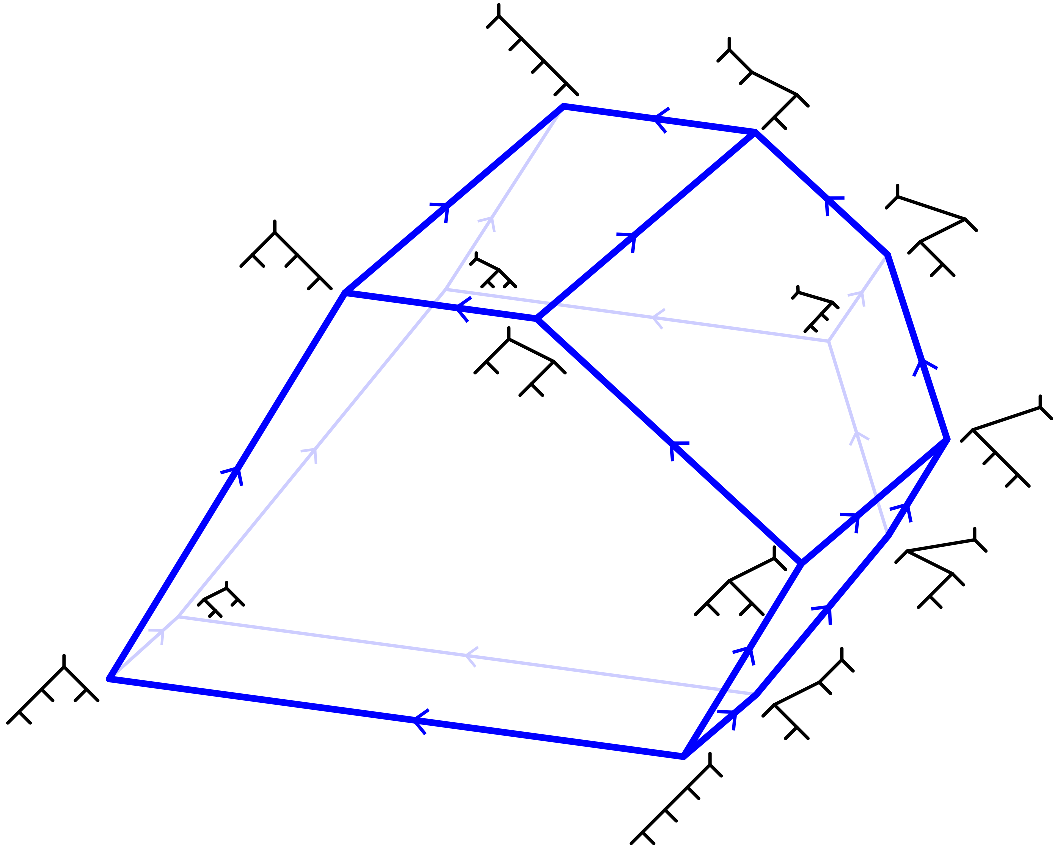




NATURAL ORIENTATION

DEF. $\mathbb{P}$ = deformed permutahedron
$\mathcal{P}_\mathbb{P}$ = transitive closure of the graph of $\mathbb{P}$ oriented in direction $(n, \dots, 2, 1) - (1, 2, \dots, n)$
$\mathcal{P}_\mathbb{P}$ is always a poset on $\mathbb{P}$-posets





PROB. When is $\mathcal{P}_\mathbb{P}$ a lattice? a distributive lattice? a semidistributive lattice?

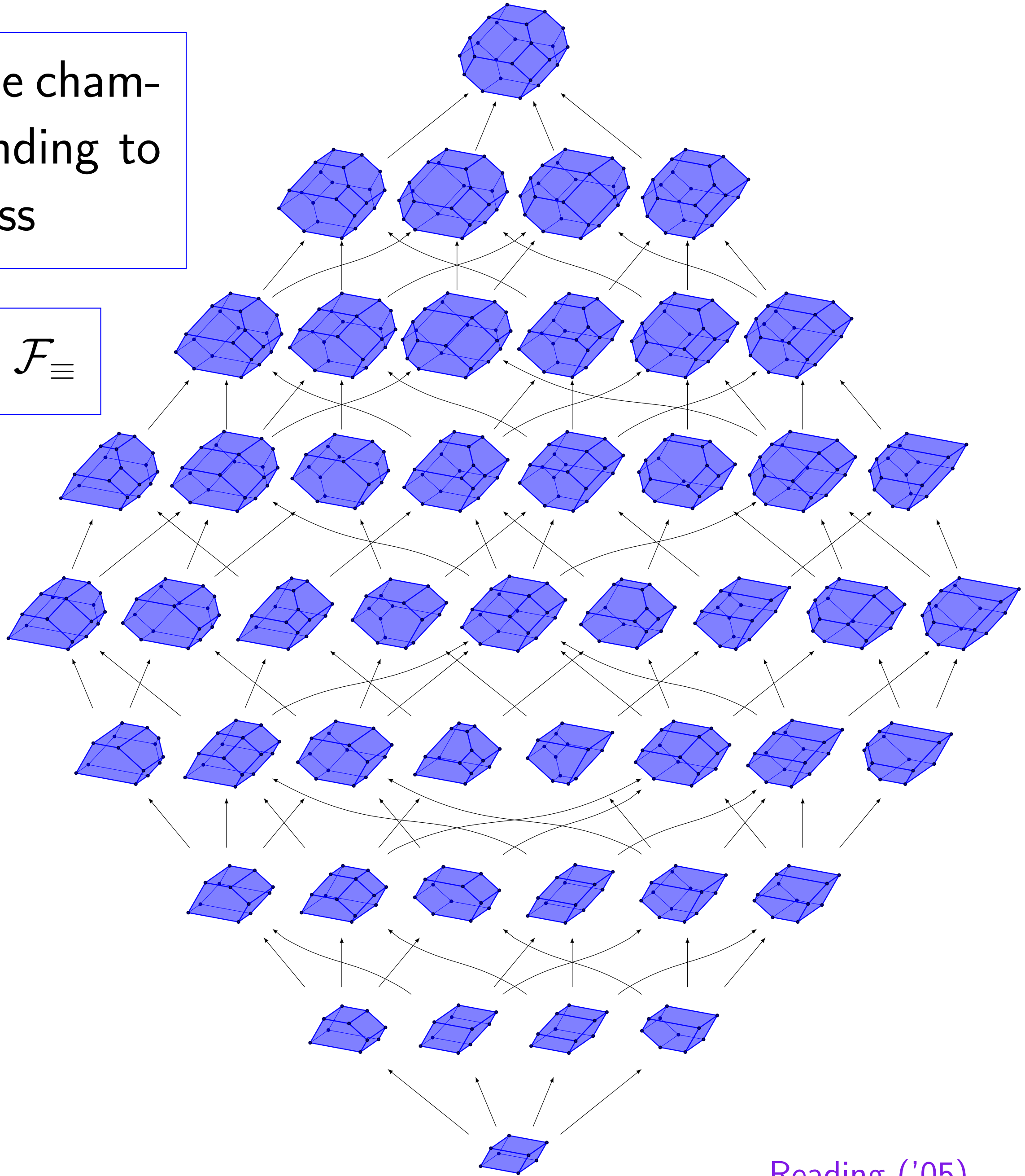
EXAMPLE 1: QUOTIENTOPES

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
P.–Santos, *Quotientopes* ('19)
Padrol–P.–Ritter, *Shard polytopes* ('22)

QUOTIENT FANS AND QUOTIENTOPES

quotient fan $\mathcal{F}_{\equiv}$ = obtained by glueing the chambers of the braid arrangement corresponding to chambers in the same $\equiv$-congruence class

quotientope = polytope with normal fan $\mathcal{F}_{\equiv}$



Reading ('05)
P.–Santos ('19)

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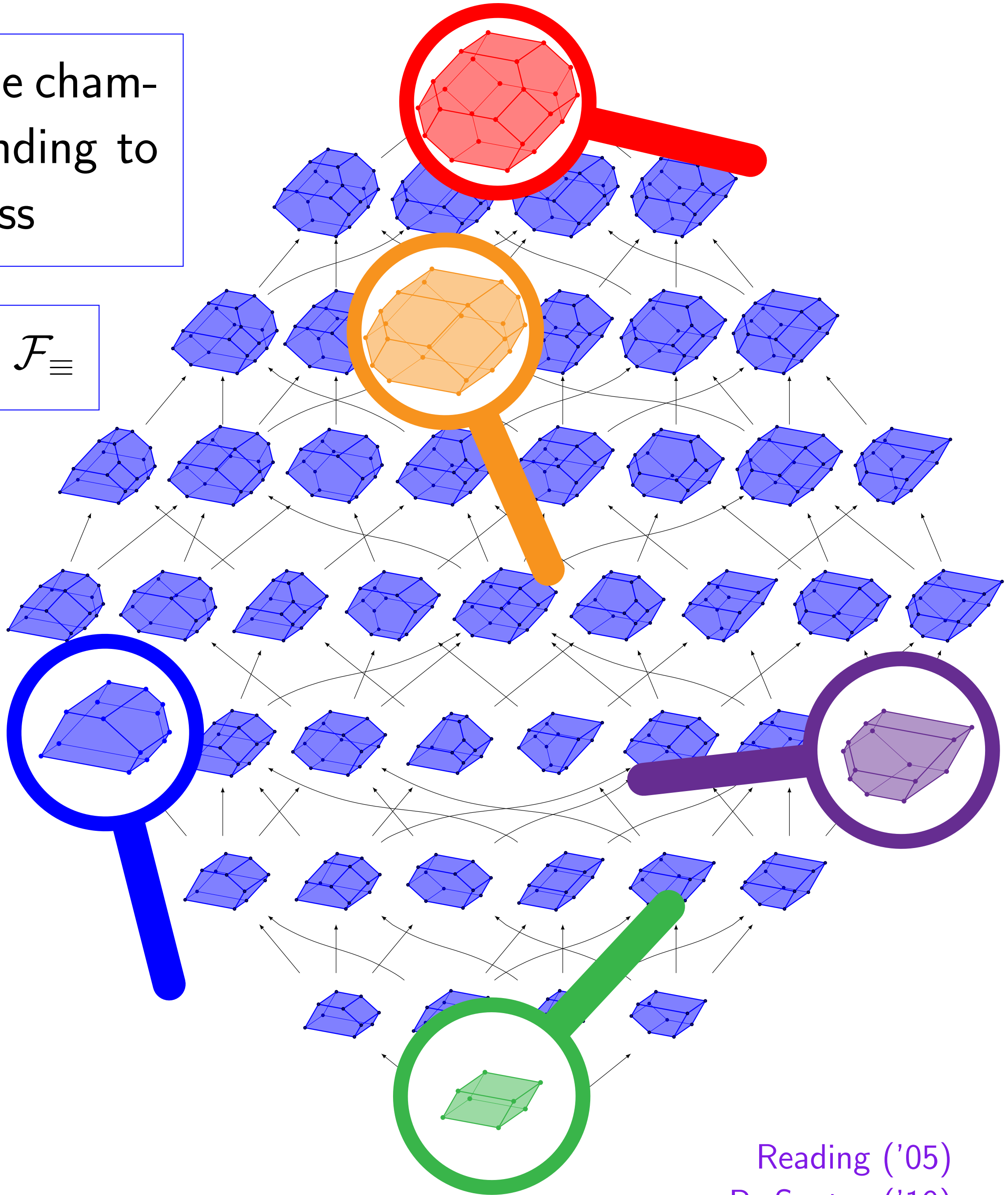

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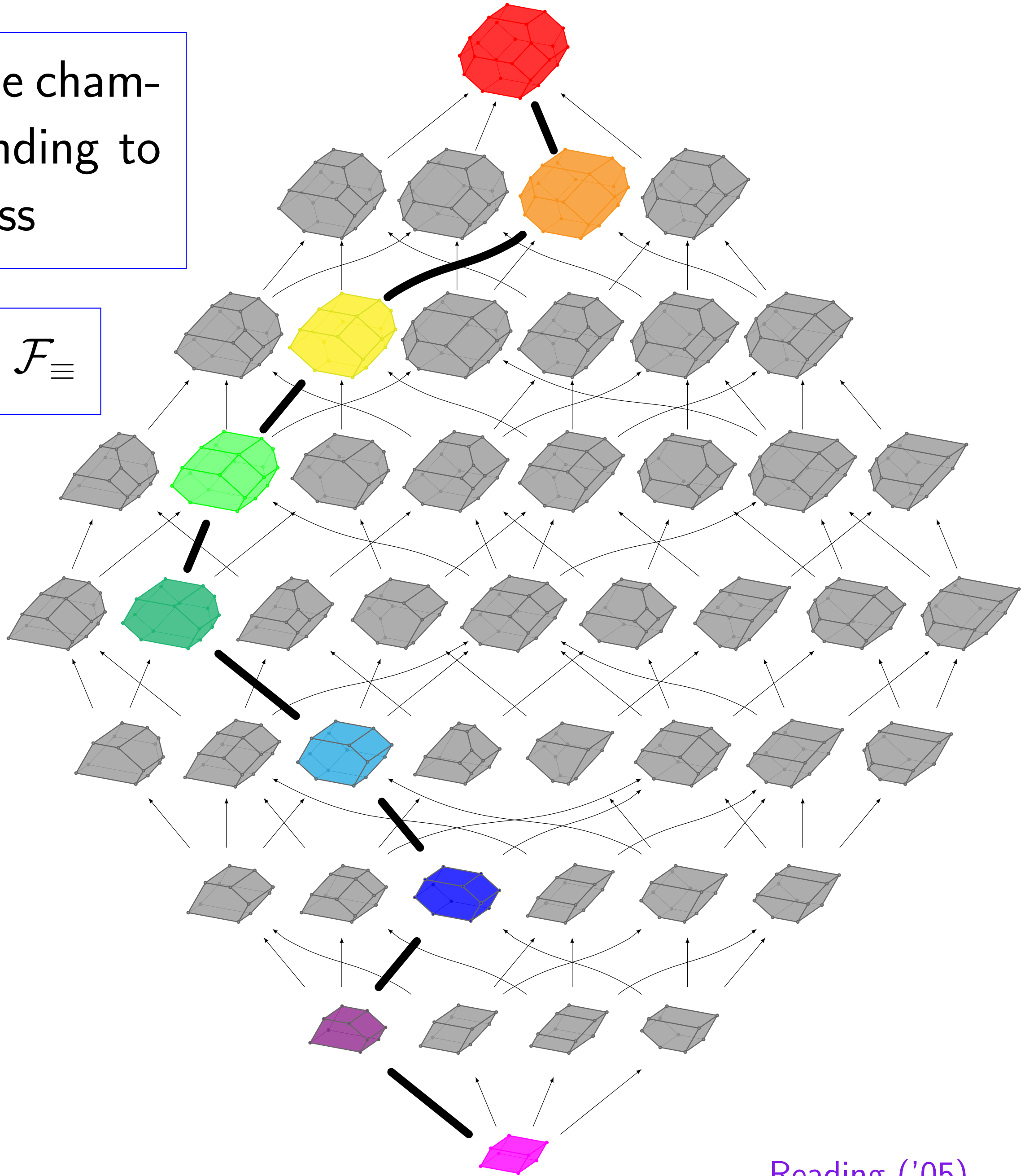


POLYWOOD

Reading ('05)
P.–Santos ('19)

EXAMPLE 2: GRAPHICAL ZONOTOPES

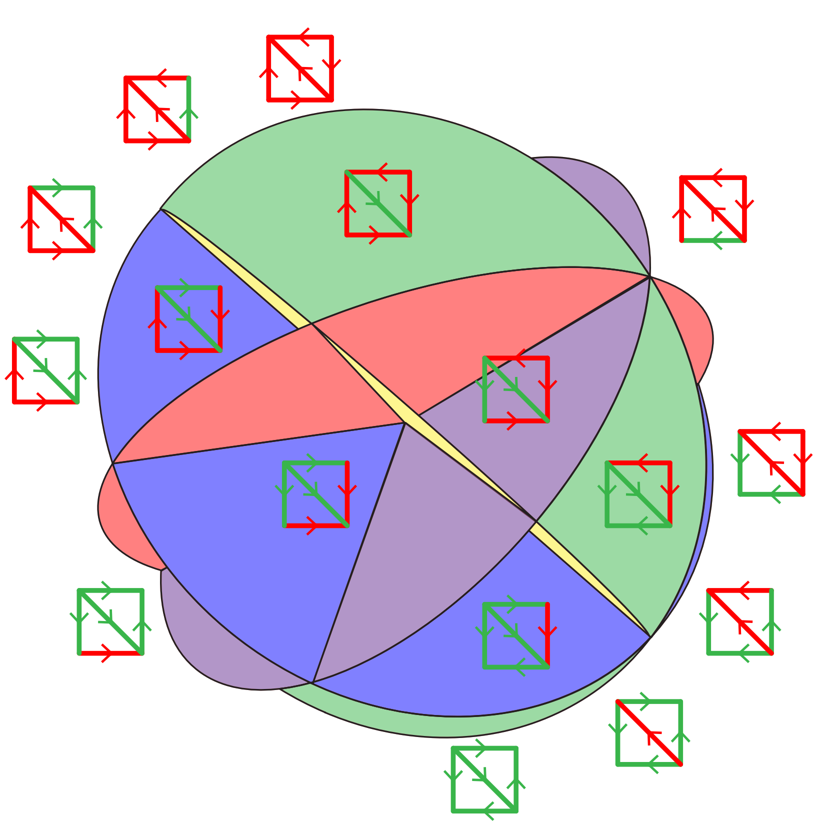
P., *Acyclic reorientation lattices and their lattice quotients* ('25)

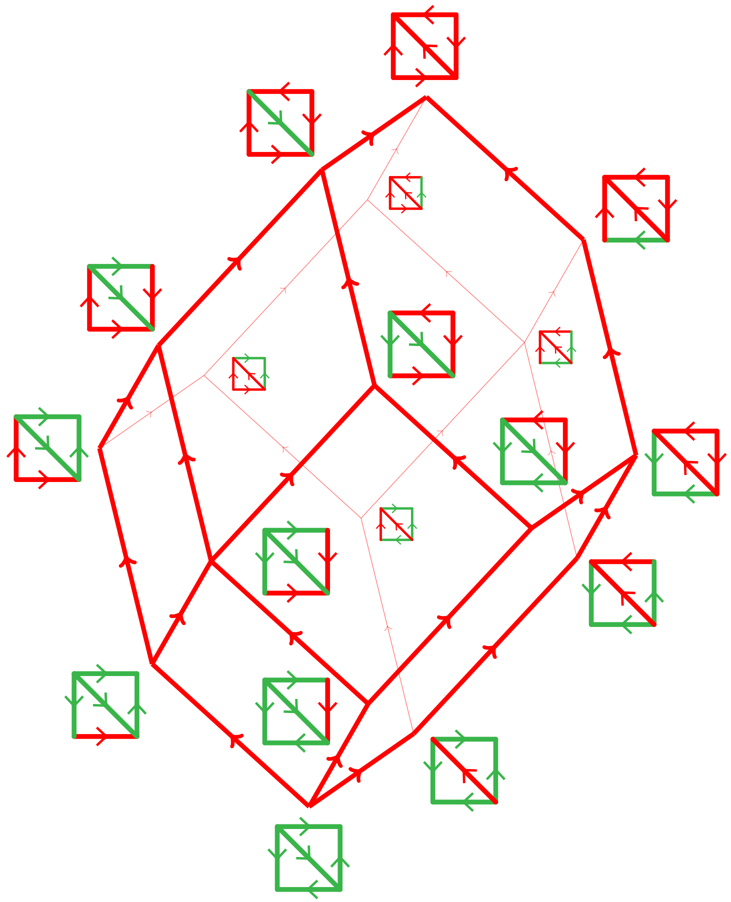
GRAPHICAL ZONOTOPES

D directed acyclic graph

graphical arrangement $\mathcal{H}_D$ = arrangement of hyperplanes $x_u = x_v$ for all arcs $(u, v) \in D$

graphical zonotope $\mathbb{Z}\text{ono}(D)$ = Minkowski sum of $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all arcs $(u, v) \in D$



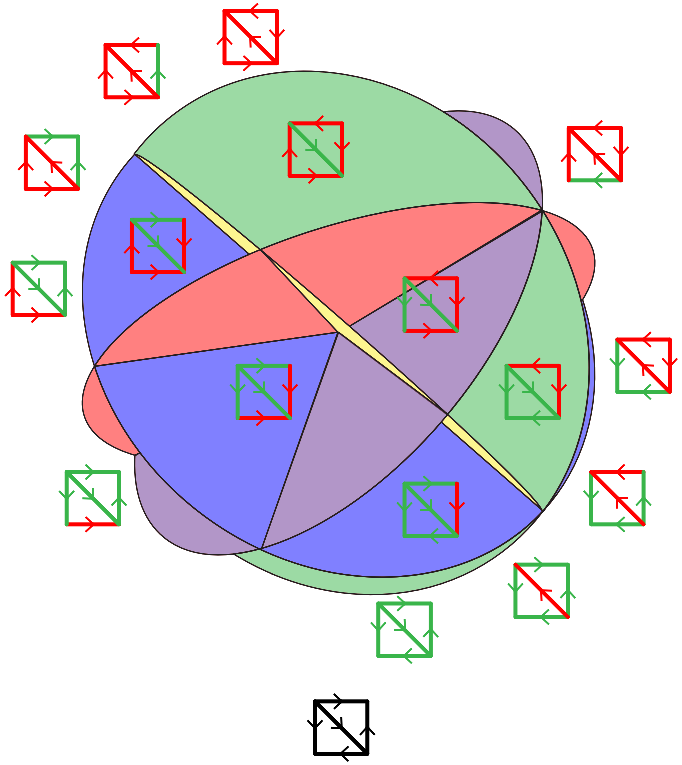


hyperplanes of $\mathcal{H}_D$	$\longleftrightarrow$	summands of $\mathbb{Z}\text{ono}(D)$	$\longleftrightarrow$	arcs of D
regions of $\mathcal{H}_D$	$\longleftrightarrow$	vertices of $\mathbb{Z}\text{ono}(D)$	$\longleftrightarrow$	acyclic reorientations of D
poset of regions of $\mathcal{H}_D$	$\longleftrightarrow$	oriented graph of $\mathbb{Z}\text{ono}(D)$	$\longleftrightarrow$	acyclic reorientation poset of D

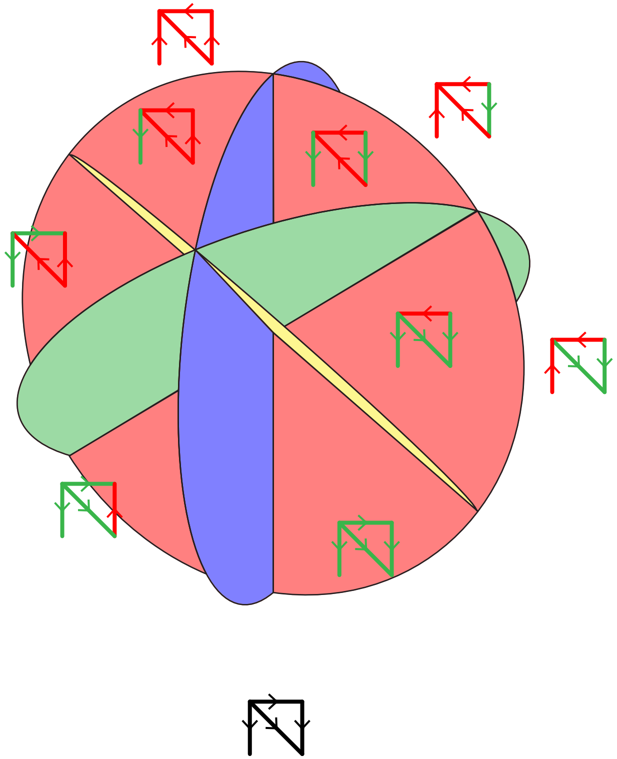
SIMPLE GRAPHICAL ZONOTOPES

PROP. TFAE:

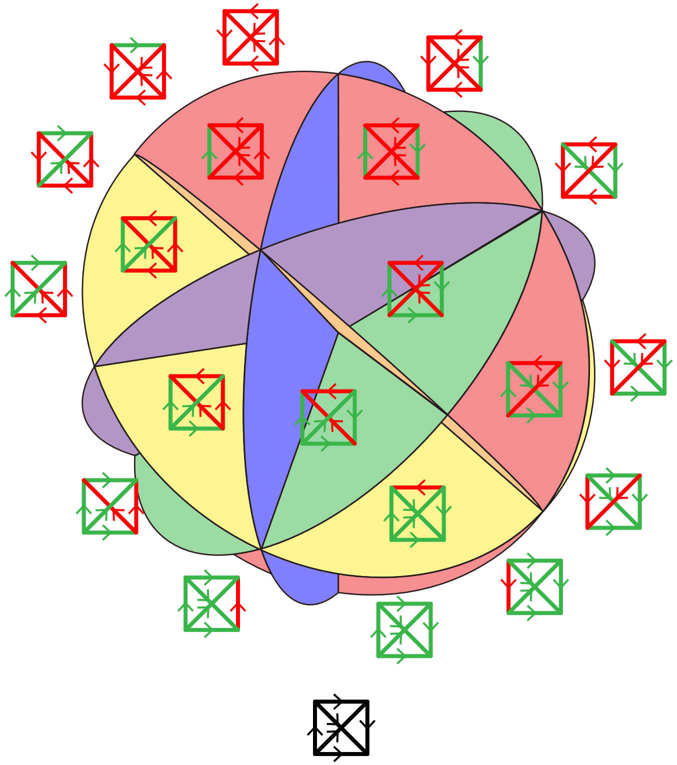
- the graphical arrangement $\mathcal{H}_D$ is simplicial
- the graphical zonotope $\mathbb{Z}\mathrm{ono}(D)$ is simple
- the transitive reduction of any acyclic reorientation of D is a forest
- D is chordful (a.k.a. block graph) $=$ any cycle of D induces a clique



not simplicial



simplicial



simplicial

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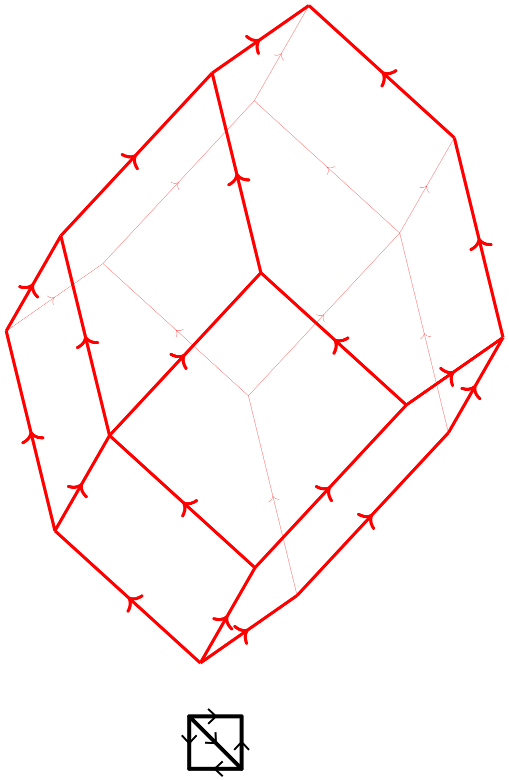
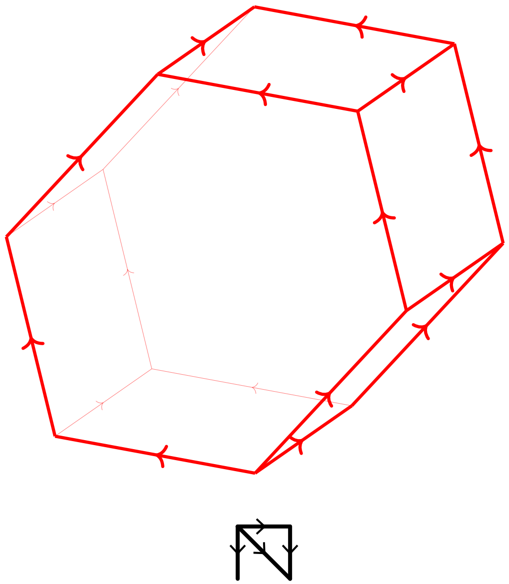
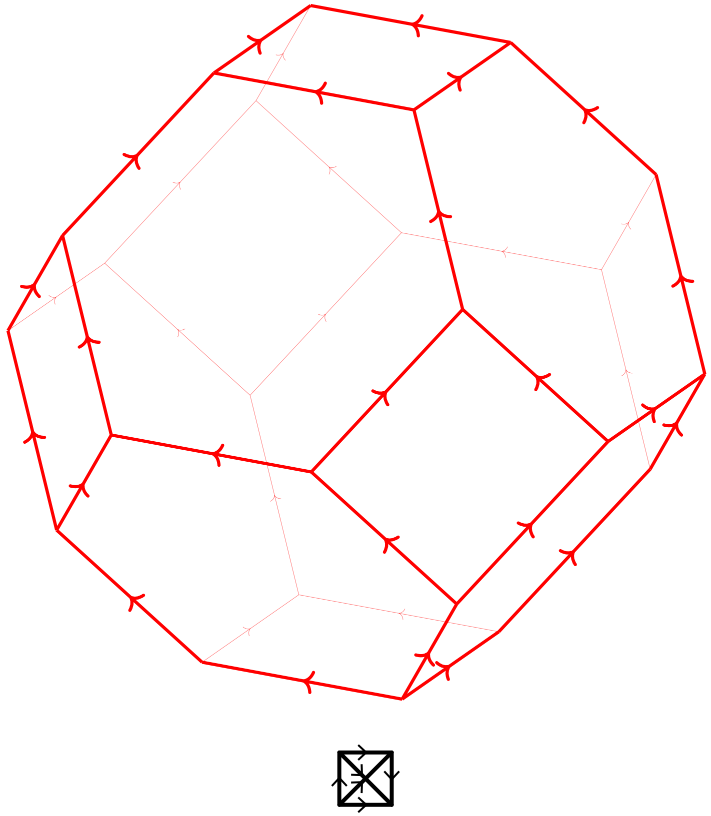


not simple



simple

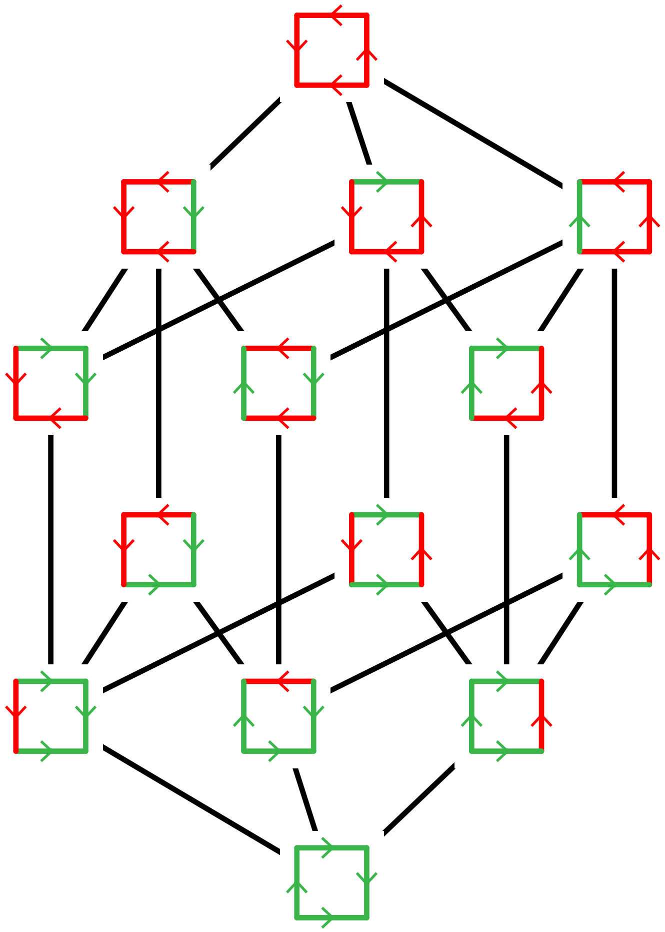


simple

ACYCLIC REORIENTATION POSETS

D directed acyclic graph

$\mathcal{AR}_D =$ all acyclic reorientations of D, ordered by inclusion of their sets of reversed arcs



minimal element D

maximal element $\bar{D}$

self-dual under reversing all arcs

cover relations = flipping a single arc

flippable arcs of E = <u>transitive reduction</u> of E

$= E \smallsetminus \{(u, v) \in E \mid \exists$ directed path $u \rightsquigarrow v$ in $E\}$

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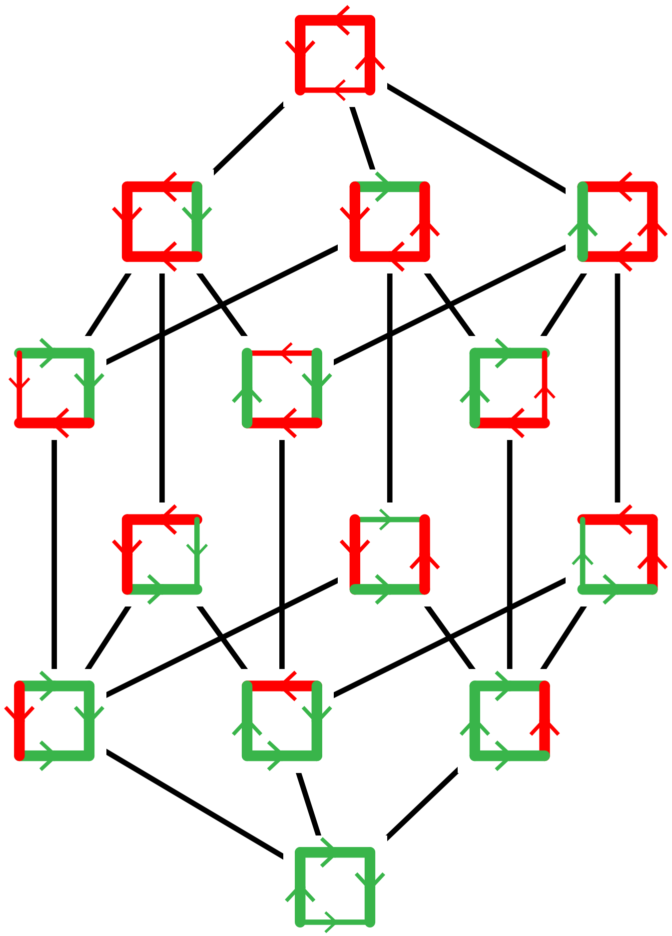

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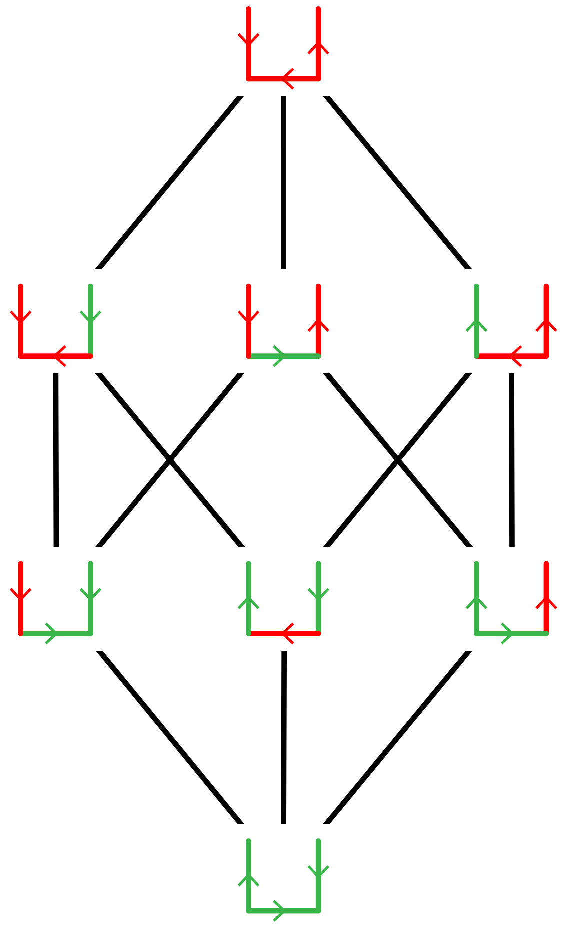
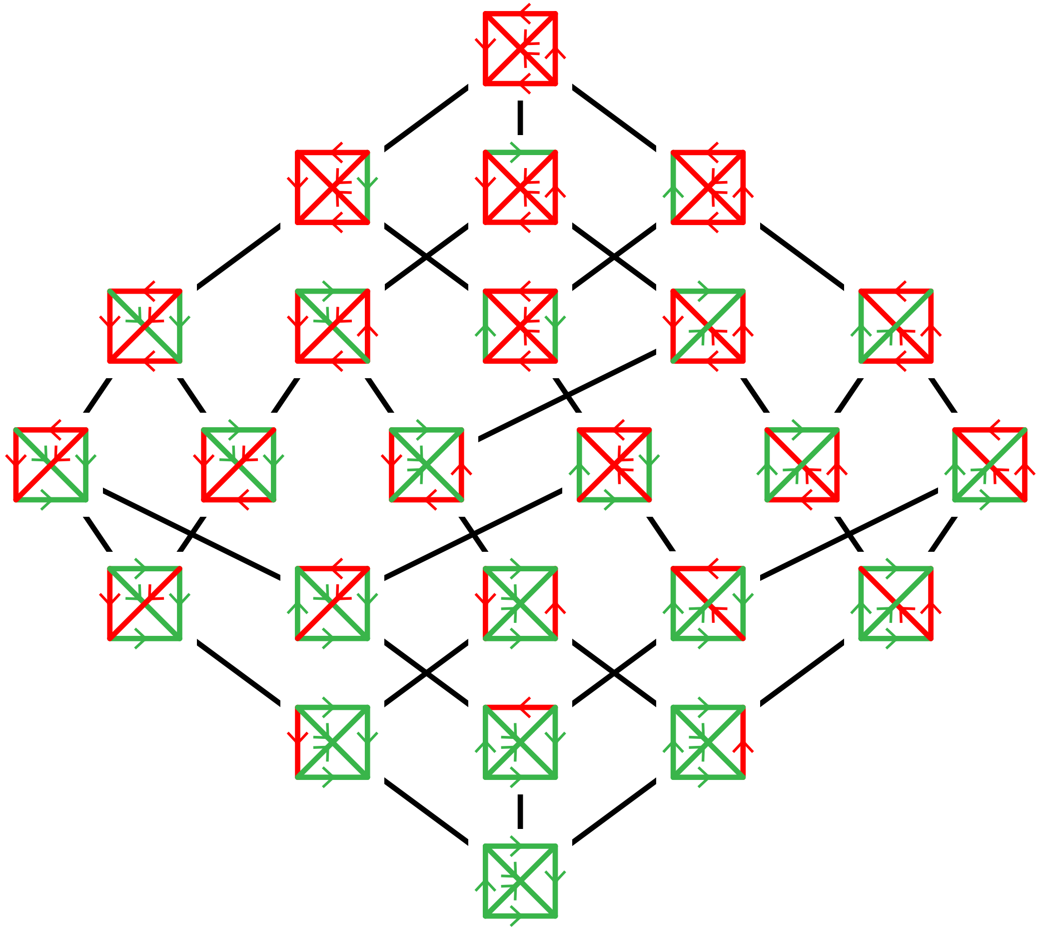
D forest



boolean lattice

D tournament

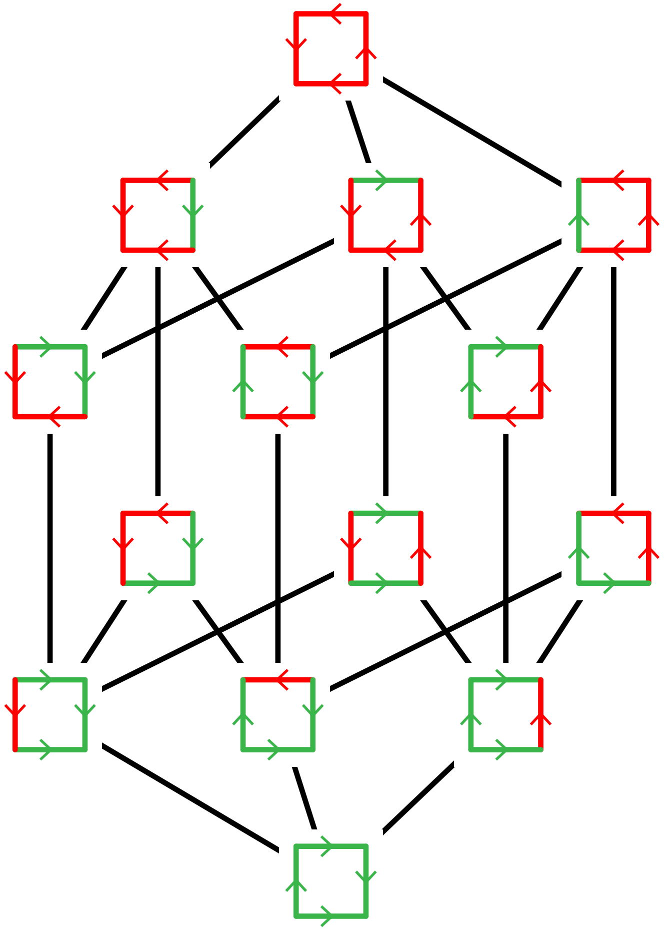


weak order

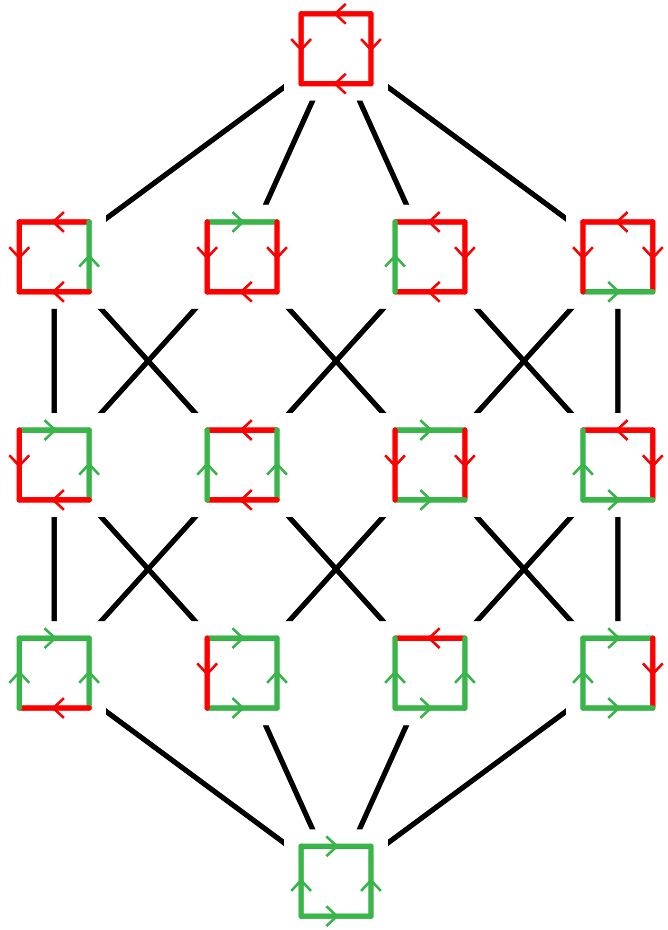
ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate P. ('25)



lattice



not lattice

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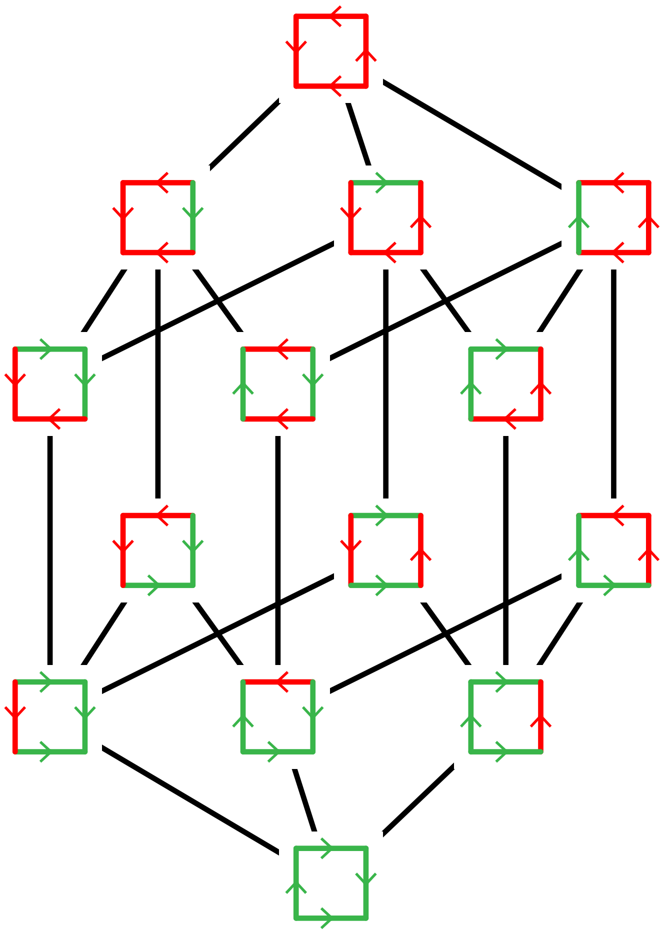
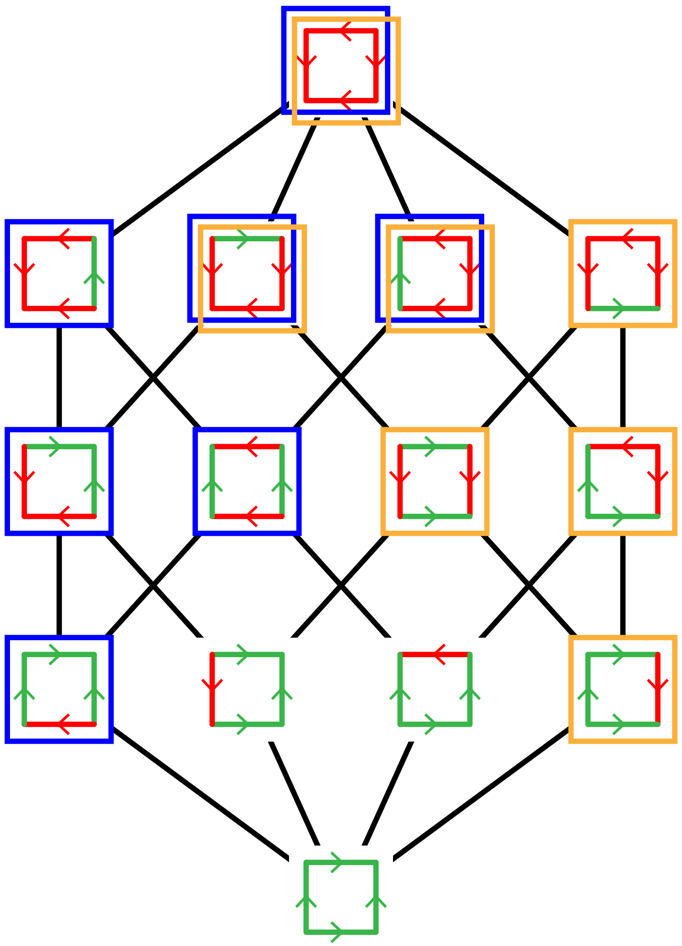


lattice



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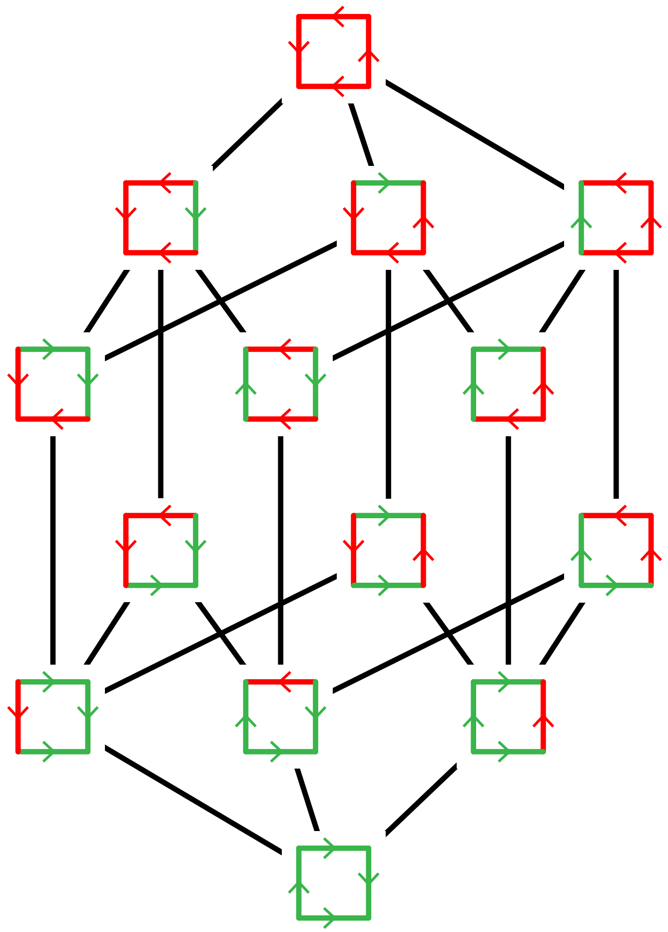

X subset of arcs of D is

- closed if all arcs of D in the transitive closure of X also belong to X
- coclosed if its complement is closed
- biclosed if it is closed and coclosed

PROP. If D vertebrate, P. ('25)
X biclosed $\iff$ the reorientation of X is acyclic

ACYCLIC REORIENTATION LATTICES

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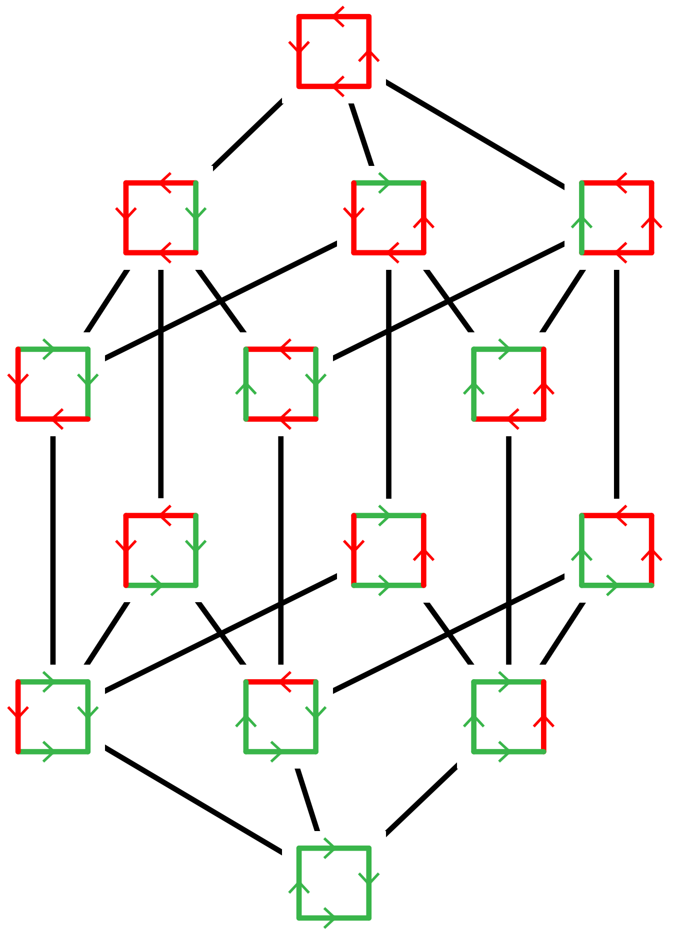
THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate — P. ('25)



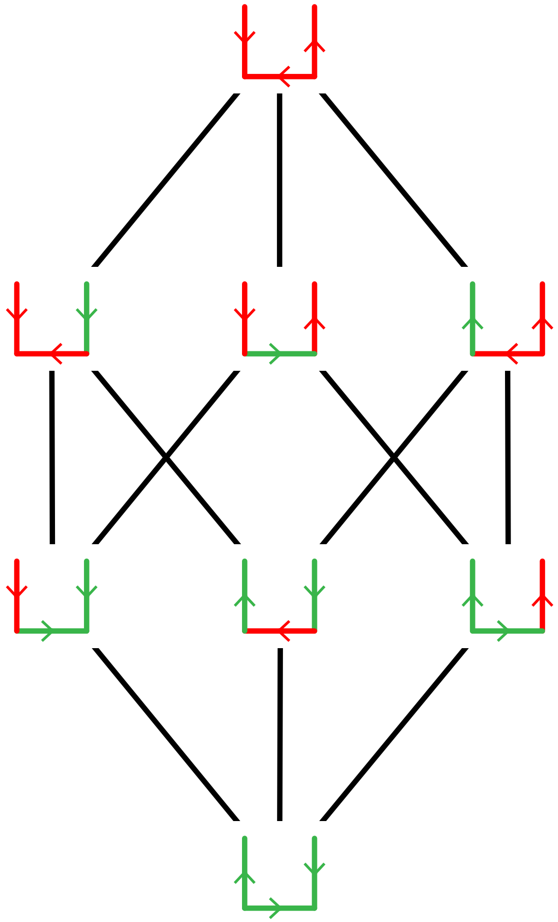
PROP. If D vertebrate, — P. ('25)

$$\mathrm{bwd}(E_1 \vee \ldots \vee E_k) =$$
transitive closure of $\mathrm{bwd}(E_1) \cup \cdots \cup \mathrm{bwd}(E_k)$

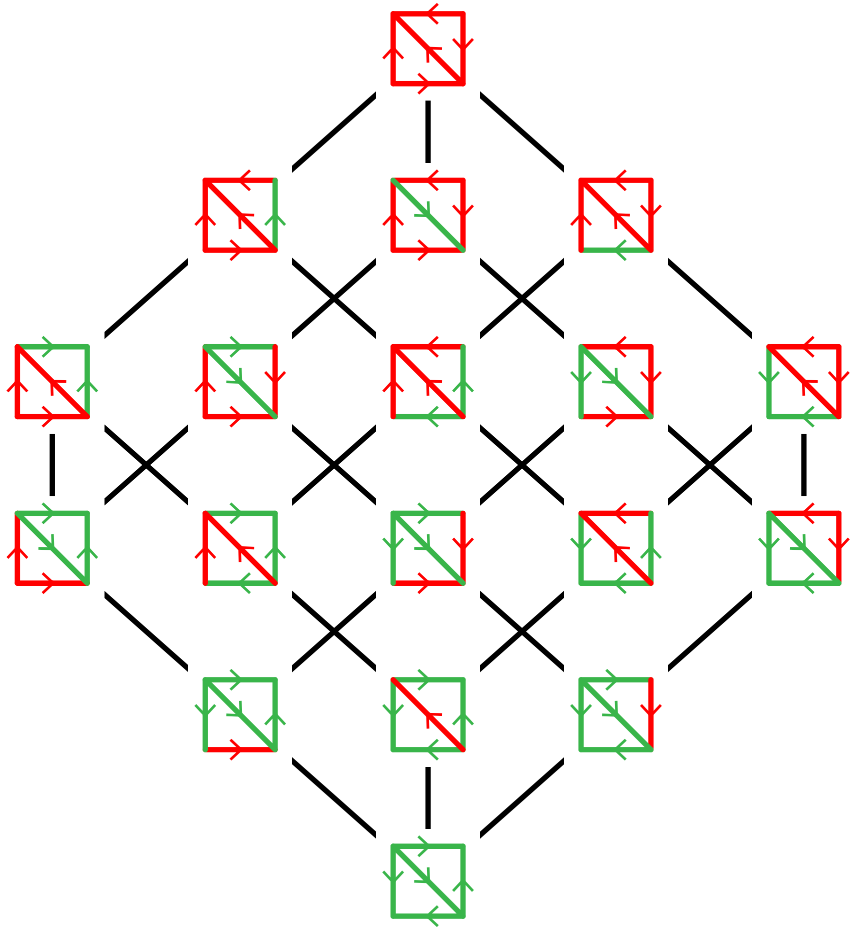
$$\mathrm{fwd}(E_1 \wedge \ldots \wedge E_k) =$$
transitive closure of $\mathrm{fwd}(E_1) \cup \cdots \cup \mathrm{fwd}(E_k)$

DISTRIBUTIVE ACYCLIC REORIENTATION POSETS

THM. $\mathcal{AR}_D$ distributive lattice $\Longleftrightarrow$ D forest $\Longleftrightarrow$ $\mathcal{AR}_D$ boolean lattice P. ('25)



distributive



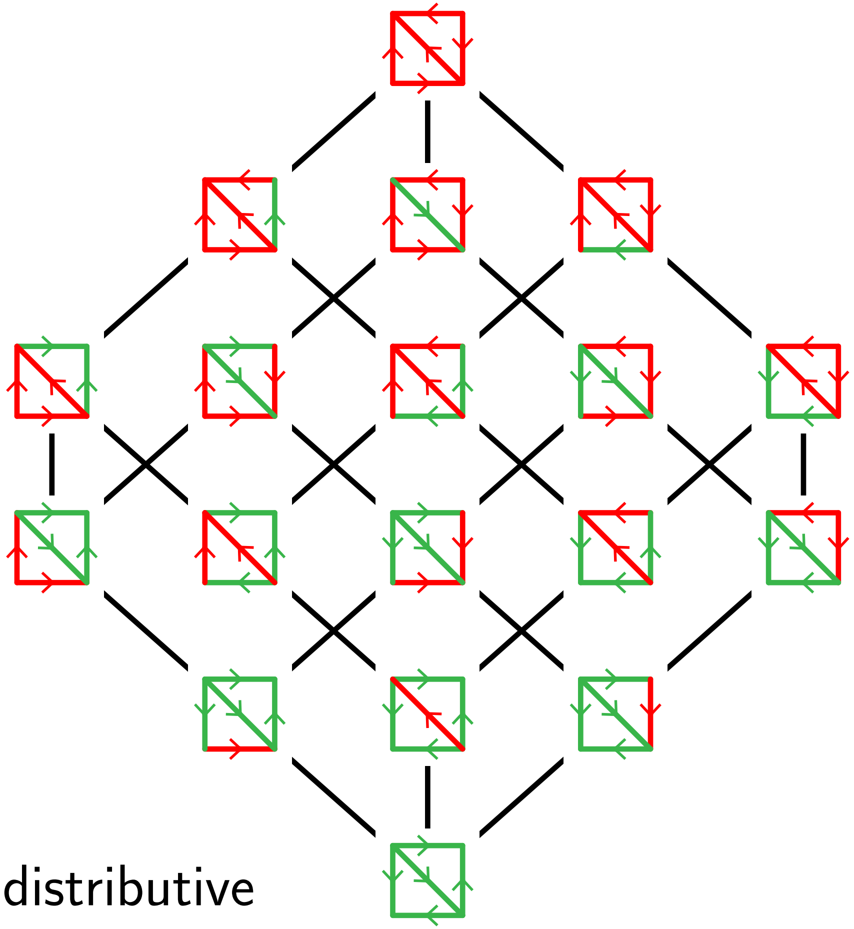
not distributive

SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

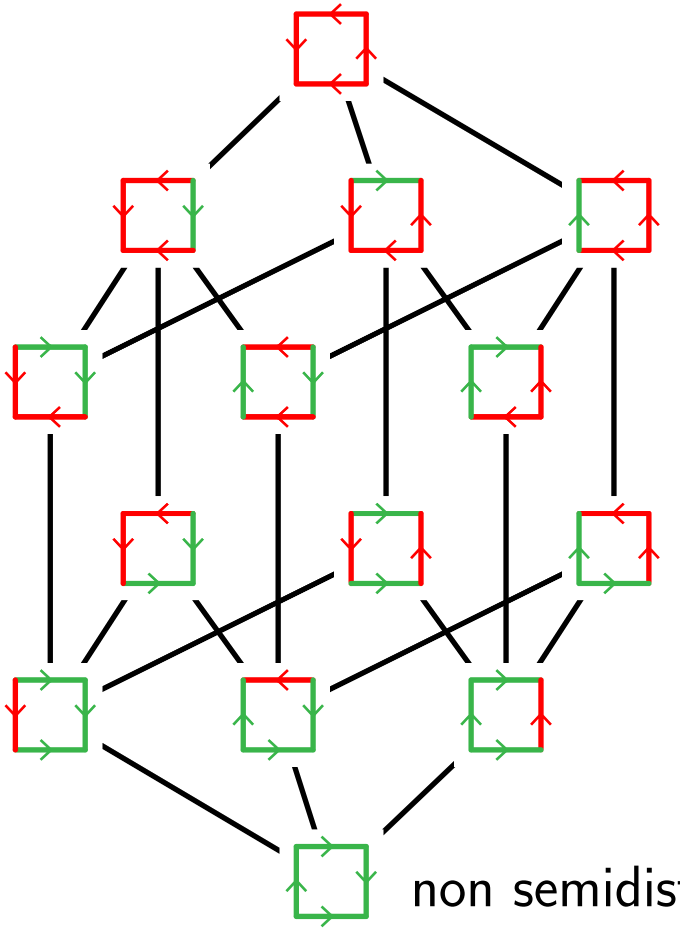
D skeletal =

- D vertebrate = transitive reduction of any induced subgraph of D is a forest
- D filled = any directed path joining the endpoints of an arc in D induces a tournament

THM. $\mathcal{AR}_D$ semidistributive lattice $\iff$ D is skeletal P. ('25)



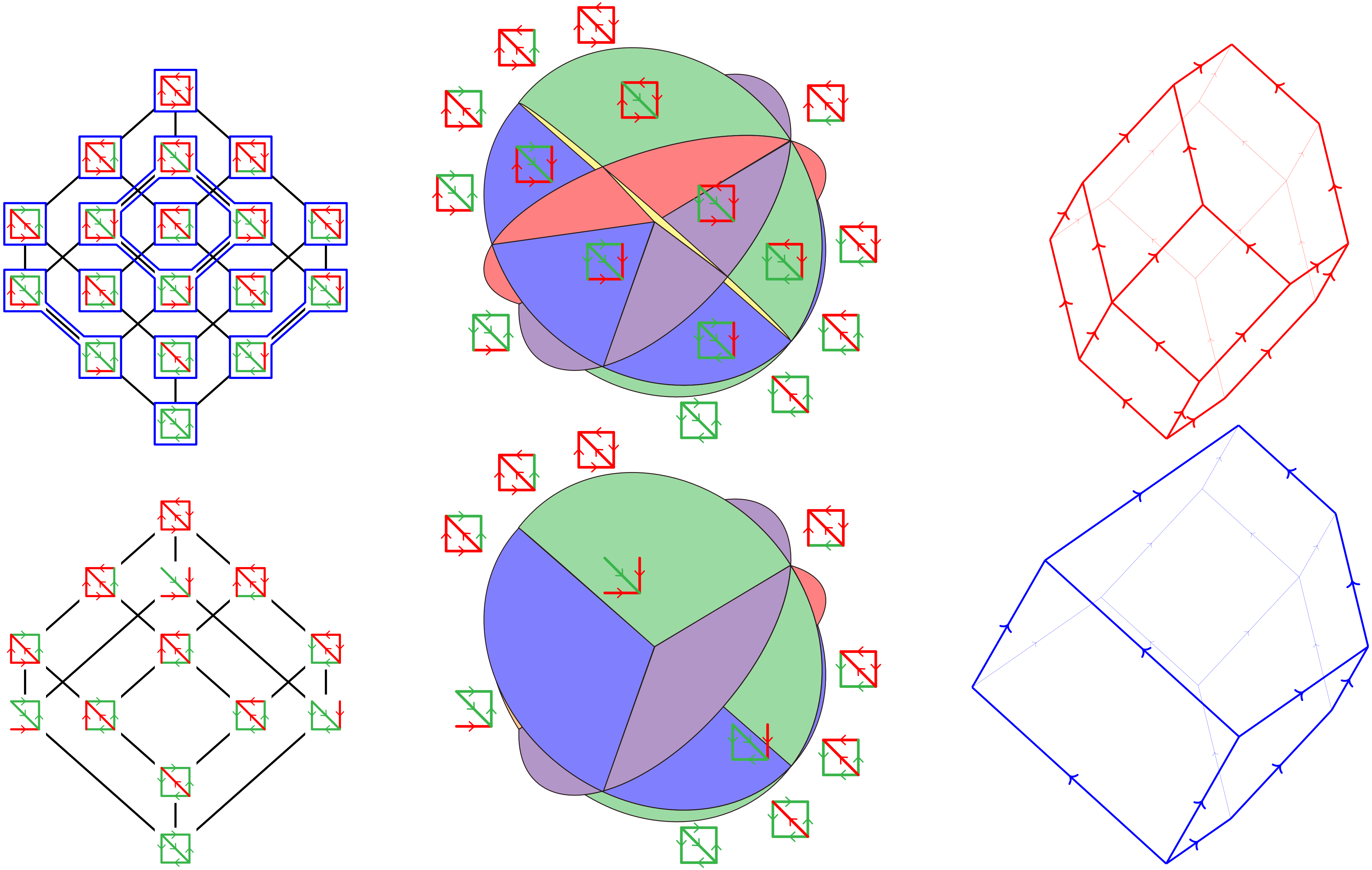
semidistributive



non semidistributive

QUOTIENTS OF SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

D skeletal $\implies$ $\mathcal{AR}_D$ semidistributive lattice $\implies$ quotient fans & quotientopes



EXAMPLE 3: NESTOHEDRA

Feichtner–Sturmfels, *Matroid polytopes, nested sets and Bergman fans* ('05)
Postnikov, *Permutohedra, associahedra, and beyond* ('09)

NESTOHEDRA

DEF. building set on $[n]$ = collection $\mathcal{B}$ of non-empty subsets of $[n]$ such that

- $\mathcal{B}$ contains all singletons $\{s\}$ for $s \in [n]$
- if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \varnothing$, then $B \cup B' \in \mathcal{B}$

$\kappa(\mathcal{B})$ = connected components of $\mathcal{B}$ = inclusion maximal elements of $\mathcal{B}$

DEF. nested set on $\mathcal{B}$ = subset $\mathcal{N}$ of $\mathcal{B} \smallsetminus \kappa(\mathcal{B})$ such that

- for any $B, B' \in \mathcal{N}$, either $B \subseteq B'$ or $B' \subseteq B$ or $B \cap B' = \varnothing$
- for any $k \geq 2$ pairwise disjoint $B_1, \dots, B_k \in \mathcal{N}$, the union $B_1 \cup \dots \cup B_k$ is not in $\mathcal{B}$

nested complex of $\mathcal{B}$ = simplicial complex of nested sets on $\mathcal{B}$

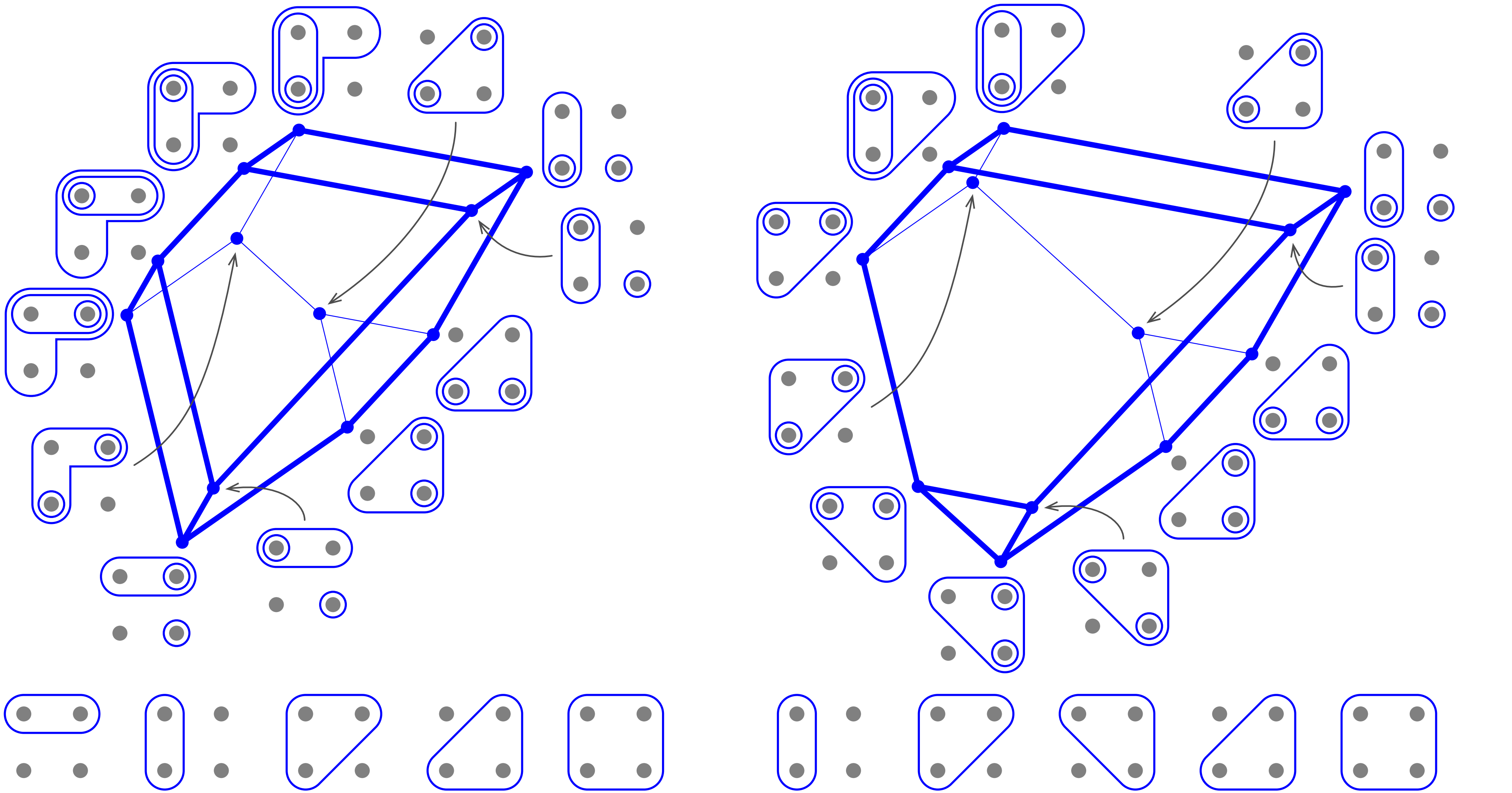
THM. The nested complex of $\mathcal{B}$ is isomorphic to the boundary complex of the polar of the nestohedron $\displaystyle\sum_{B \in \mathcal{B}} \lambda_B \triangle_B$ where

- $\triangle_B := \operatorname{conv}\{\boldsymbol{e}_b \mid b \in B\}$ face of the standard simplex $\triangle_{[n]} = \operatorname{conv}\{\boldsymbol{e}_s \mid s \in [n]\}$
- λ_B arbitrary strictly positive coefficients

Feichtner–Koslov ('04), Feichtner–Sturmfels ('05), Postnikov ('09), Zelevinski ('06)

NESTOHEDRA

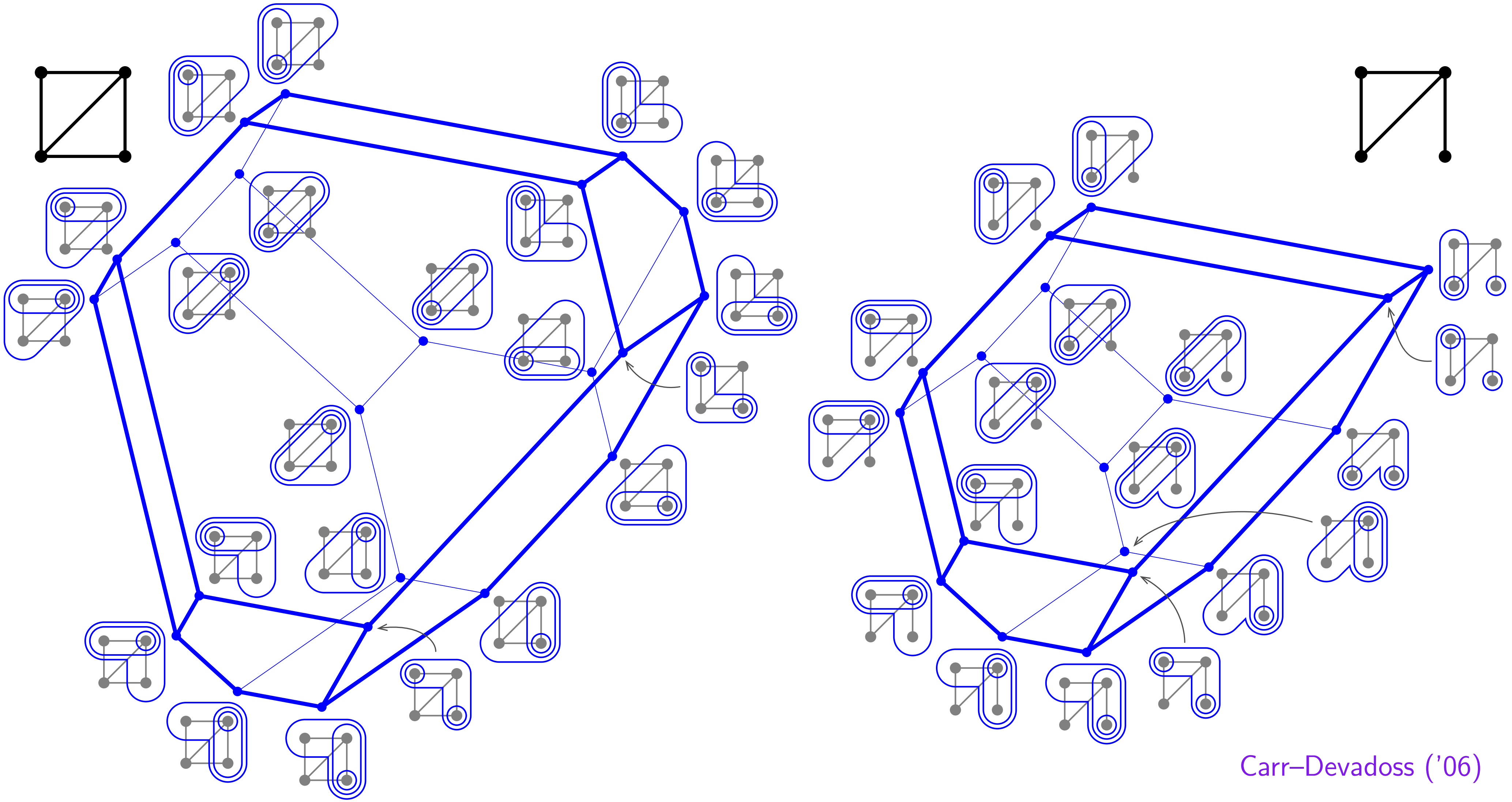
THM. The nested complex of $\mathcal{B}$ is isomorphic to the boundary complex of the polar of the nestohedron $\sum_{B \in \mathcal{B}} \lambda_B \triangle_B$



Feichtner–Koslov ('04), Feichtner–Sturmfels ('05), Postnikov ('09), Zelevinski ('06)

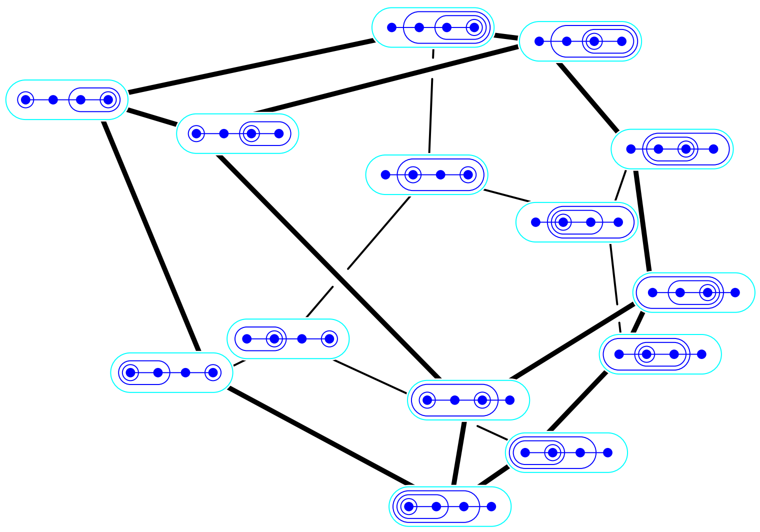
EXM. graphical building set of G = collection of all tubes of G

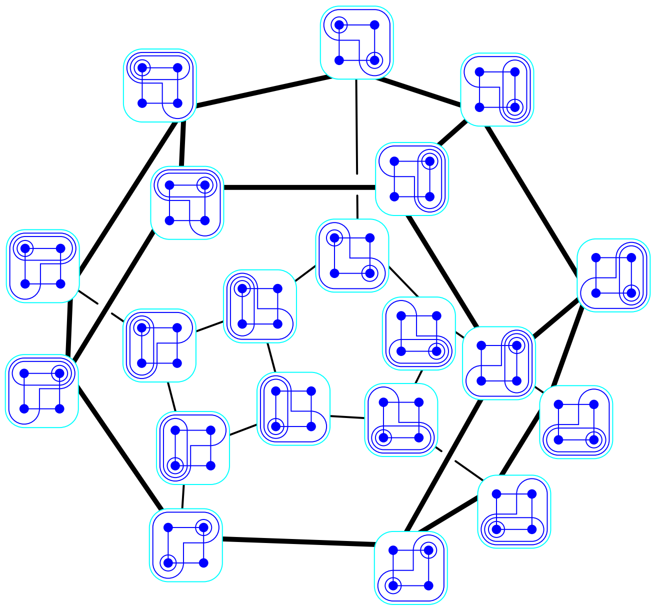
graphical nested set of G = simplicial complex of tubings on G

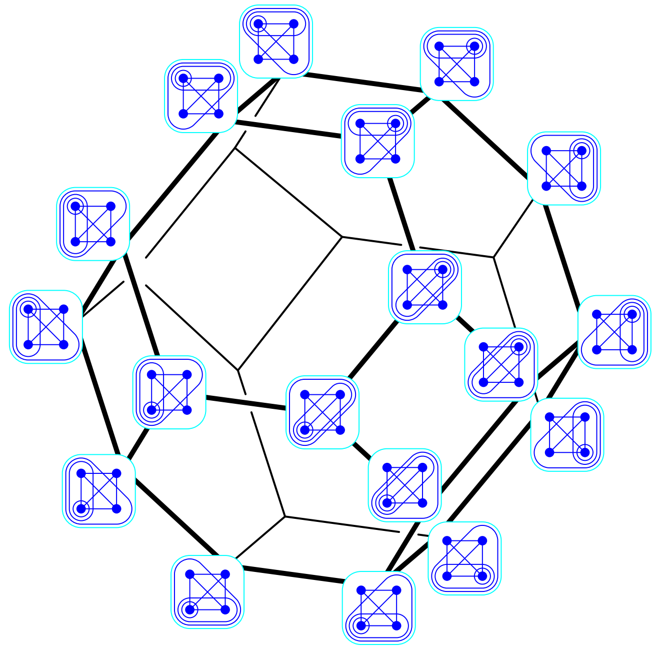


Carr–Devadoss ('06)

SPECIAL GRAPH ASSOCIAHEDRA



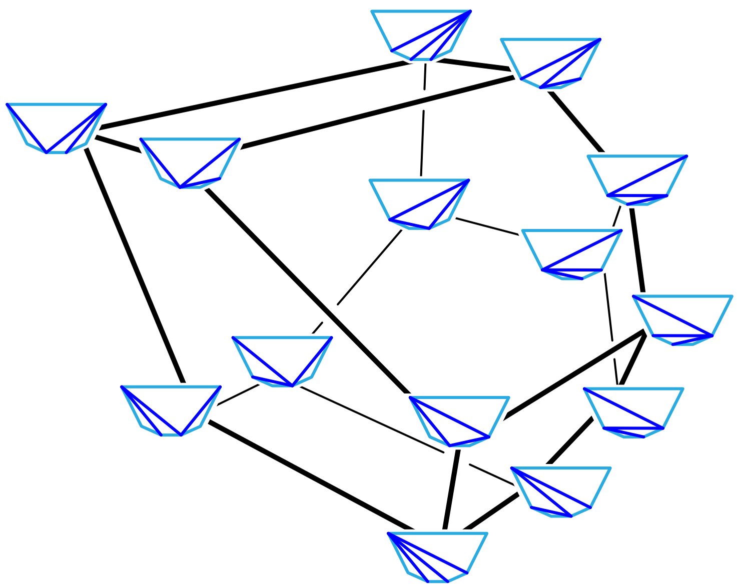


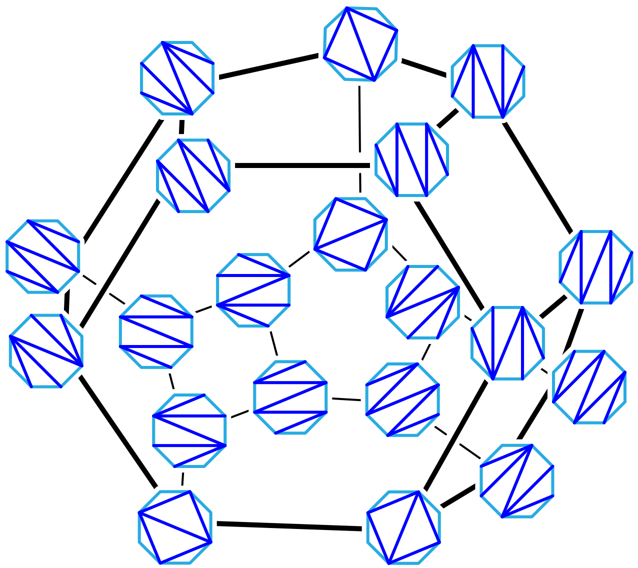


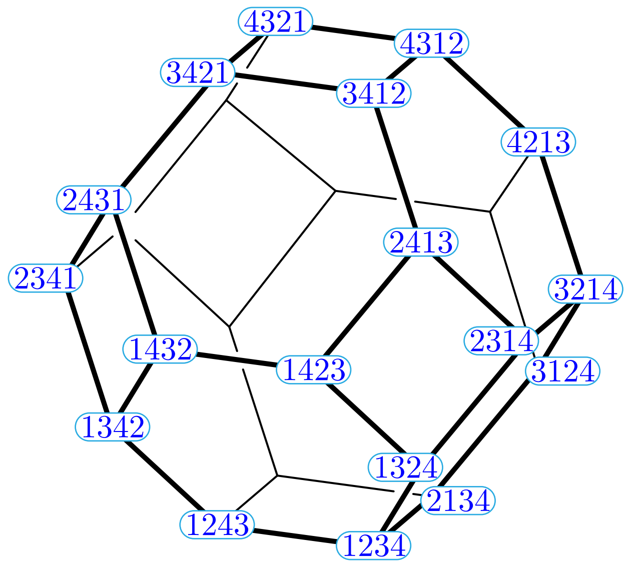
path associahedron
= associahedron

cycle associahedron
= cyclohedron

complete graph associahedron
= permutahedron







EXAMPLE 4: HYPERGRAPHIC POLYTOPES

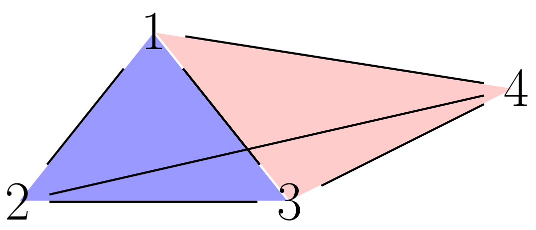
Benedetti–Bergeron–Machacek, *Hypergraphic polytopes* ('19)
Bergeron–P., *Interval hypergraphic polytopes* ('24$^+$)

HYPERGRAPHIC POLYTOPES

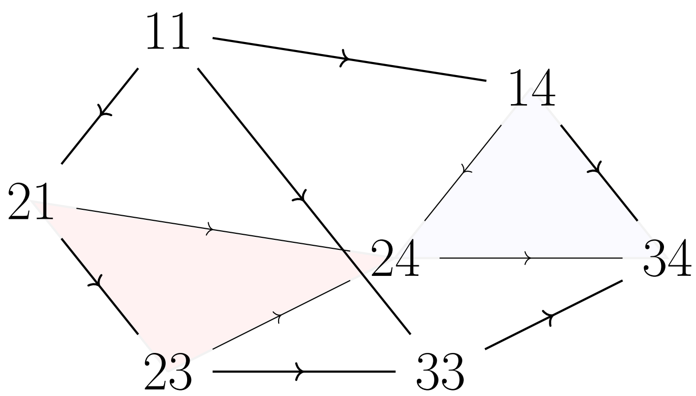
$\mathbb{H}$ = hypergraph on $[n]$

hypergraphic polytope $\triangle(\mathbb{H}) = \sum_{H \in \mathbb{H}} \triangle_H$ where $\triangle_H = \{\boldsymbol{e}_h \mid h \in H\}$

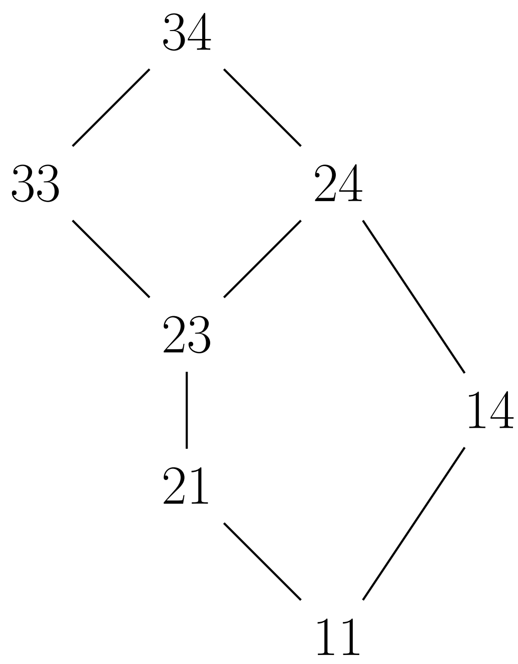
hypergraphic poset $P_{\mathbb{H}}$ = transitive closure of the graph of $\triangle(\mathbb{H})$
oriented in direction $\boldsymbol{\omega} = (n, \dots, 1) - (1, \dots, n)$



$\triangle_{123}$ and $\triangle_{134}$



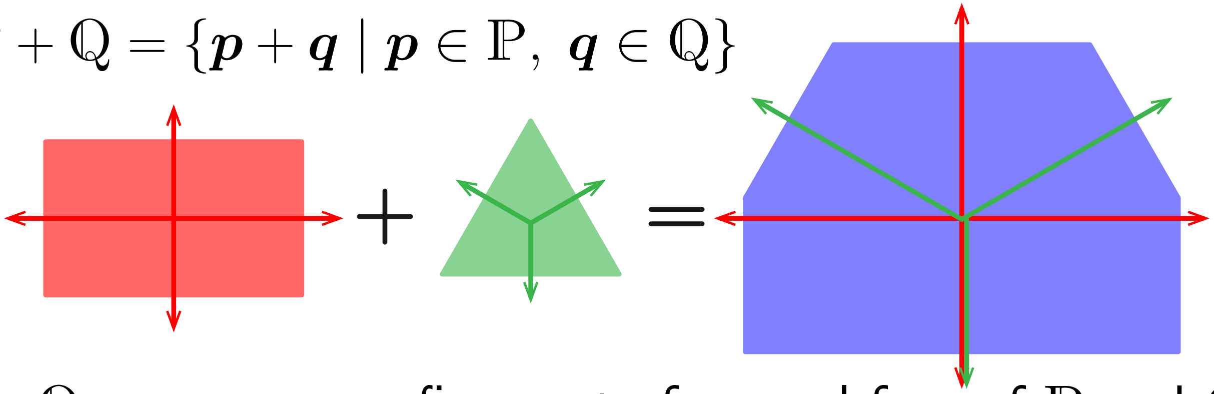
$\triangle(\mathbb{H}) = \triangle_{123} + \triangle_{134}$



$P_{\mathbb{H}}$

ACYCLIC ORIENTATIONS OF $\mathbb{H}$

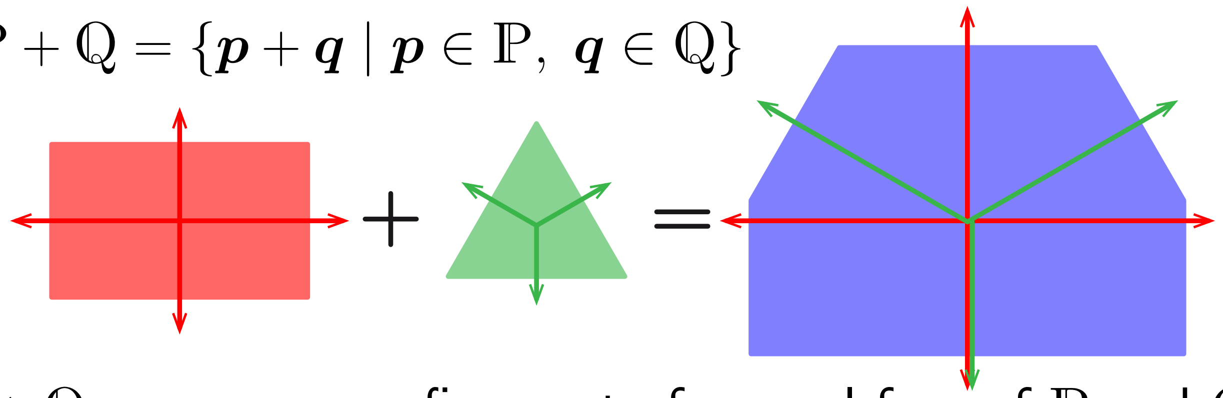
Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$



Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$

ACYCLIC ORIENTATIONS OF $\mathbb{H}$

Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$



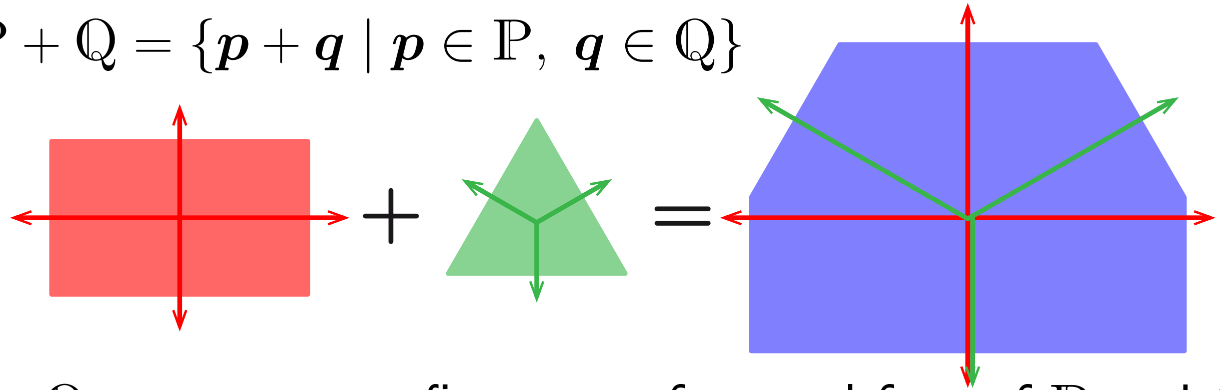
Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$

vertices of $\triangle(\mathbb{H})$ = acyclic orientations of $\mathbb{H}$ = maps O from $\mathbb{H}$ to $[n]$ such that

- $O(H) \in H$ for all $H \in \mathbb{H}$
- there is no $H_1, \ldots, H_k$ with $k \geq 2$ such that $O(H_{i+1}) \in H_i \smallsetminus \{O(H_i)\}$ for $i \in [k-1]$ and $O(H_1) \in H_k \smallsetminus \{O(H_k)\}$

ACYCLIC ORIENTATIONS OF $\mathbb{H}$

Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$



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edges of $\triangle(\mathbb{H})$ = orientation flips = pairs of acyclic orientations $O \neq O'$ of $\mathbb{H}$ such that there exist $1 \leq i < j \leq n$ such that for all $H \in \mathbb{H}$,

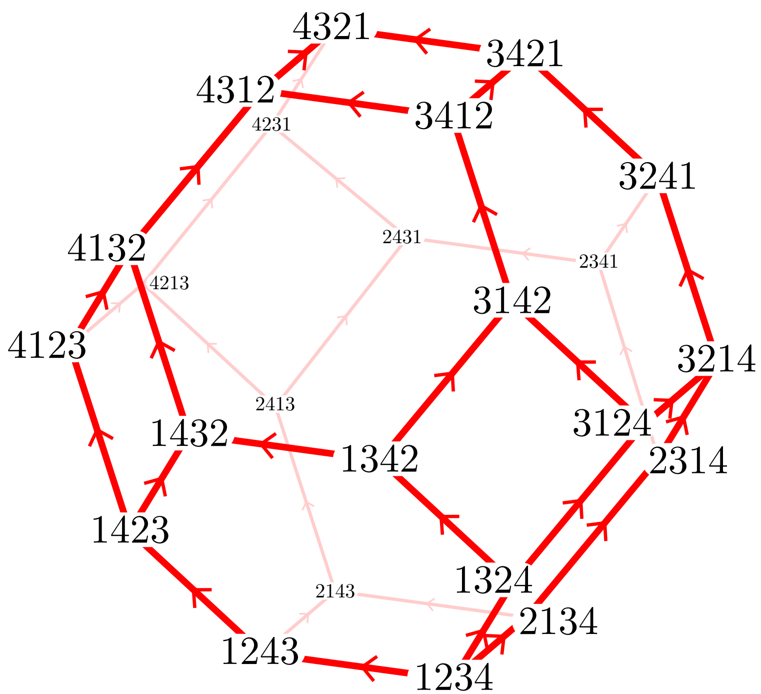
- if $O(H) \neq O'(H)$, then $O(H) = i$ and $O'(H) = j$
- if $\{i, j\} \subseteq H$, then $O(H) = i \iff O'(H) = j$

PERMUTAHEDRON & ASSOCIAHEDRON

$\mathbb{H}$ = all 2-element subsets of $[n]$

$\triangle(\mathbb{H})$ = permutahedron $\mathbb{P}\mathrm{erm}(n)$

$P_{\mathbb{H}}$ = weak order on permutations



$$= \mathrm{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

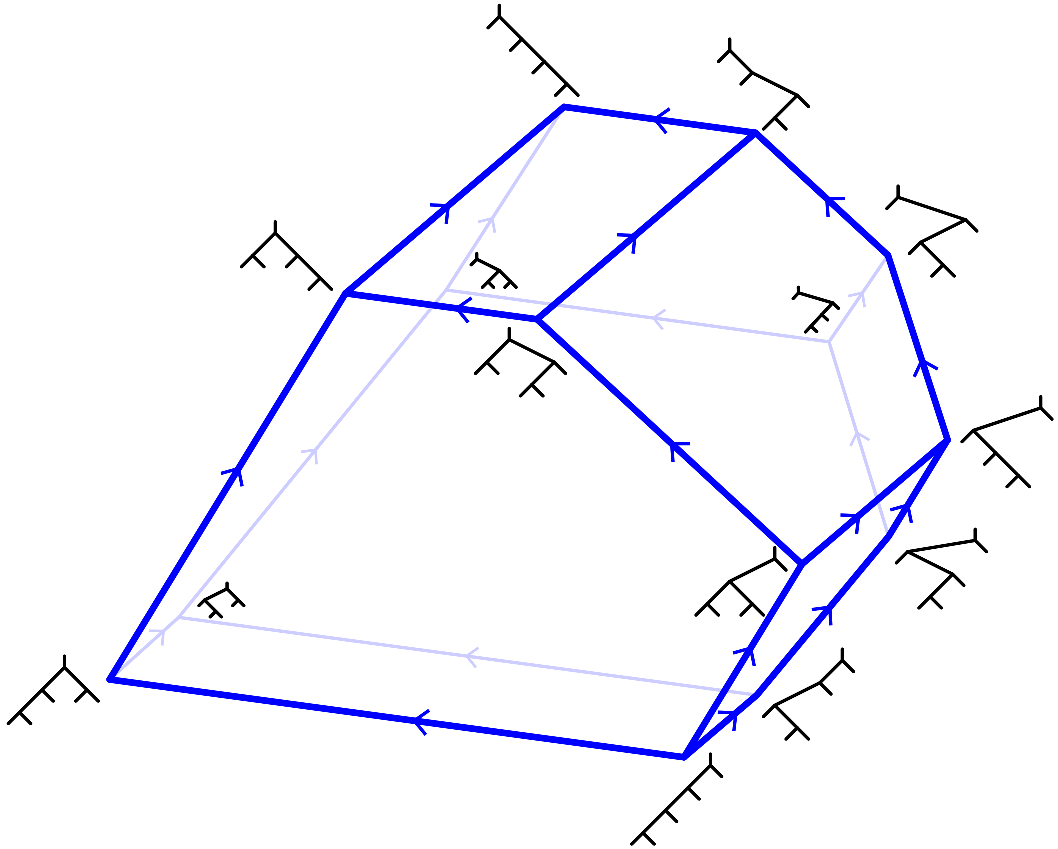
$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \geq \binom{|J|+1}{2} \right\}$

$\mathbb{H}$ = all intervals $[i,j]$ of $[n]$

$\triangle(\mathbb{H})$ = associahedron $\mathbb{A}\mathrm{sso}(n)$

$P_{\mathbb{H}}$ = Tamari lattice on binary trees



$$= \mathrm{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree} \right\}$$

$$= \mathbb{H} \cap \bigcap_{1\leq i<j\leq n} \mathbb{H}_{[i,j]}$$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

GRAPHICAL ZONOTOPE AND GRAPHICAL ASSOCIAHEDRON

$\mathbb{H}$ = edges of G

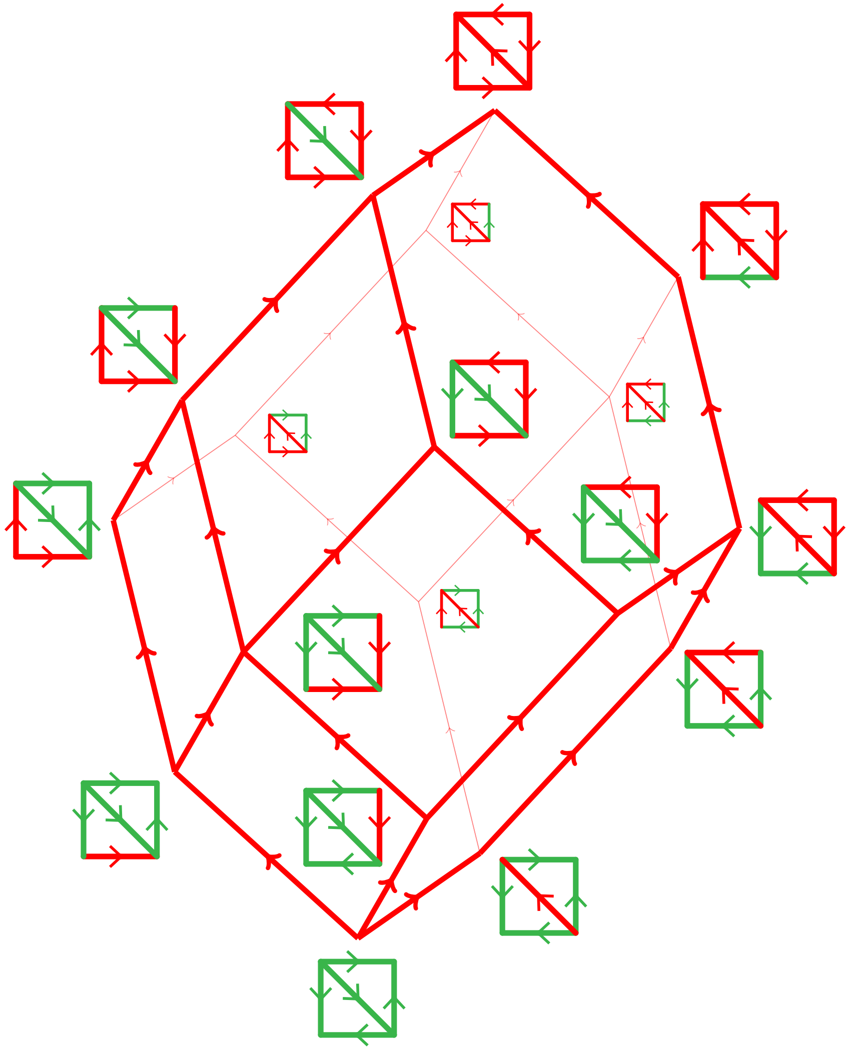
$\triangle(\mathbb{H})$ = graphical zonotope $\mathbb{Z}\text{ono}(G)$

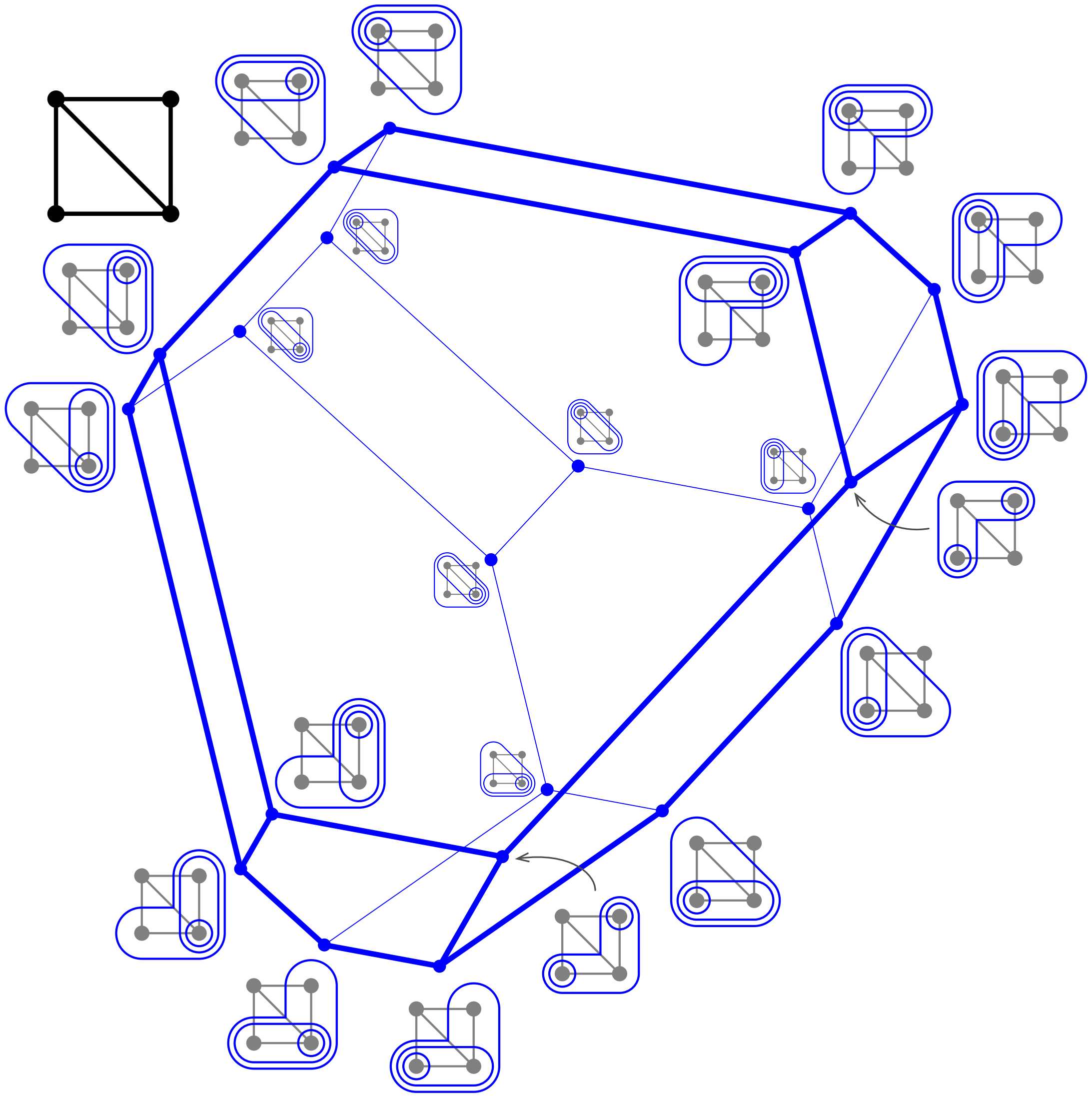
$P_{\mathbb{H}}$ = acyclic reorientation poset

$\mathbb{H}$ = tubes of G

$\triangle(\mathbb{H})$ = graph associahedron $\mathbb{A}\text{sso}(G)$

$P_{\mathbb{H}}$ = graph associahedron poset



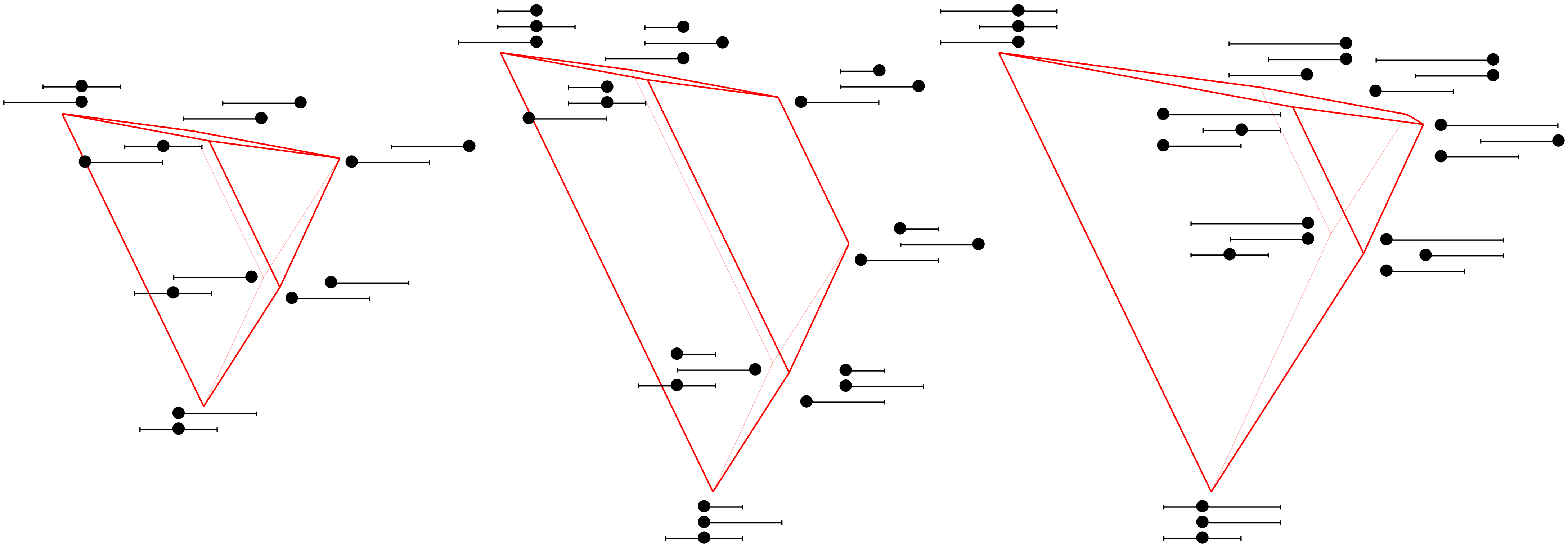


SIMPLE HYPERGRAPHIC POLYTOPES

OBS. TFAE:

- the hypergraphic fan of $\mathbb{H}$ is simplicial
- the hypergraphic polytope $\triangle(\mathbb{H})$ is simple
- the transitive reduction of any acyclic orientation of $\mathbb{H}$ is a forest

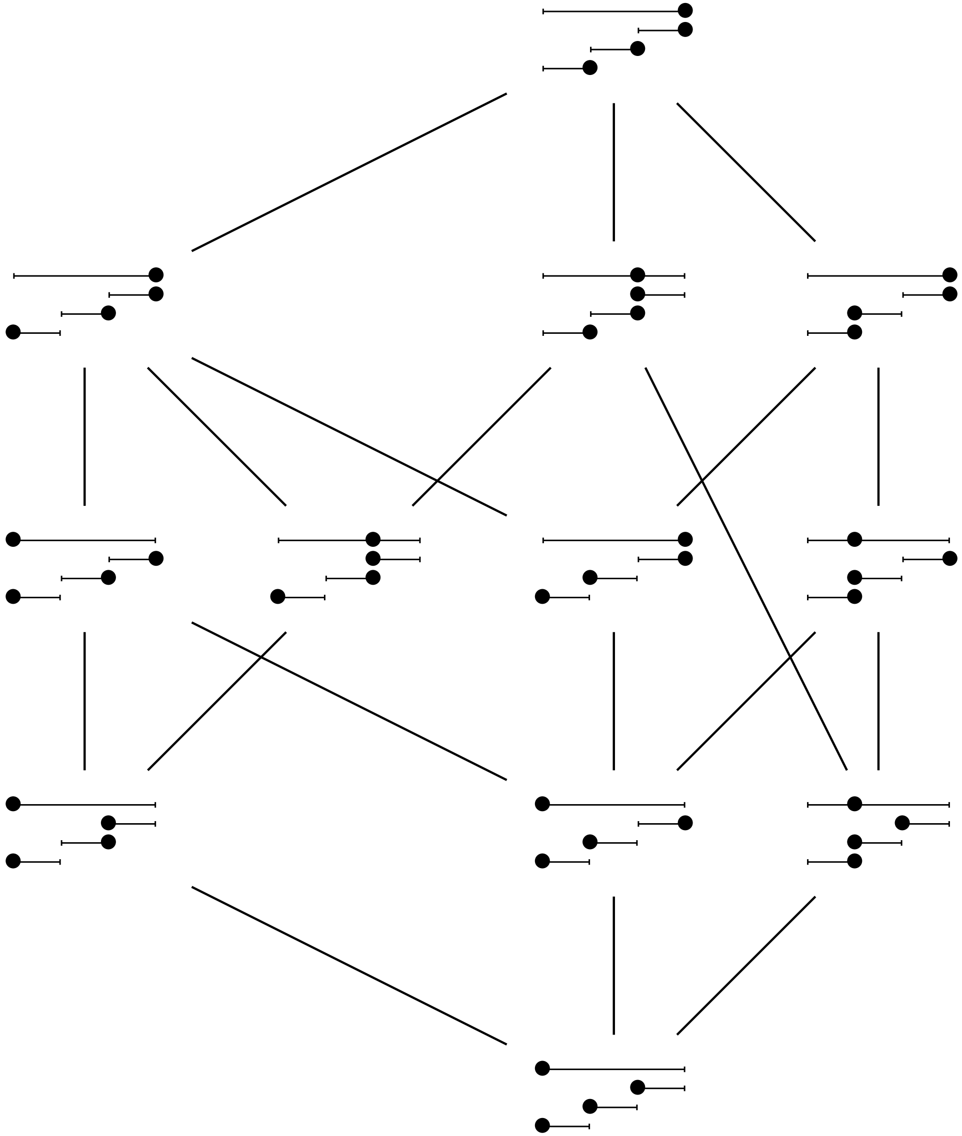
PROB. Characterize hypergraphs $\mathbb{H}$ whose hypergraphic polytope $\triangle(\mathbb{H})$ is simple



not simple　　simple　　simple

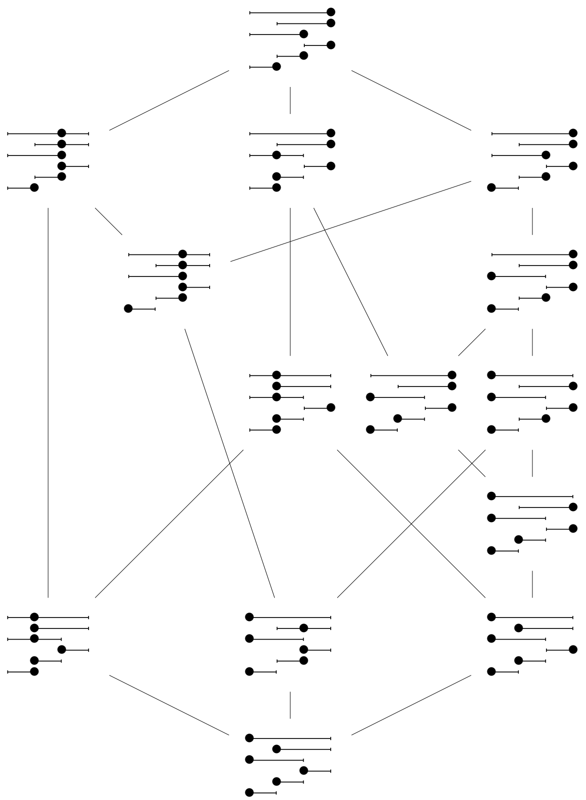
INTERVAL HYPERGRAPHIC POSETS

interval hypergraph $\mathbb{I}$ = subset of the set of intervals $[i, j]$ of $[n]$



SOME INTERVAL HYPERGRAPHIC POLYTOPES

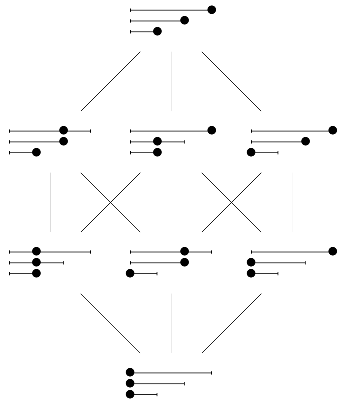
all intervals



associahedron

Tamari ('51), Loday ('04)

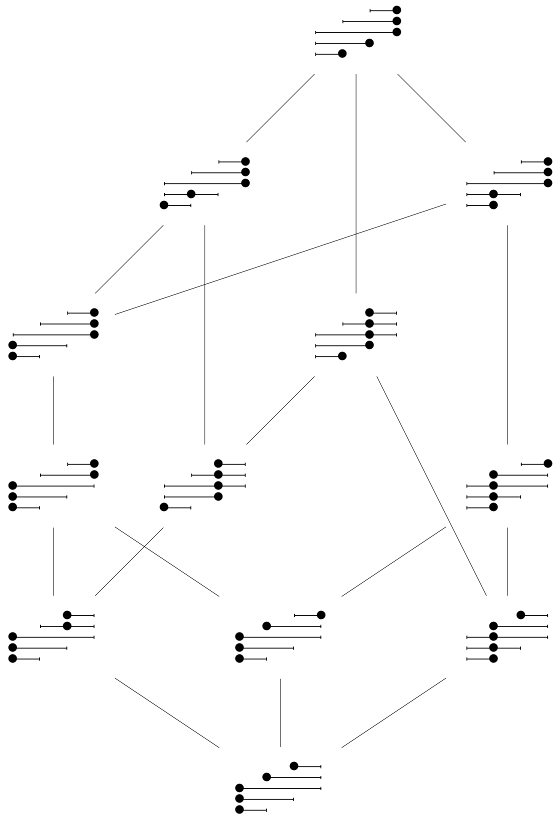
initial intervals



cube

Pitman–Stanley ('02)

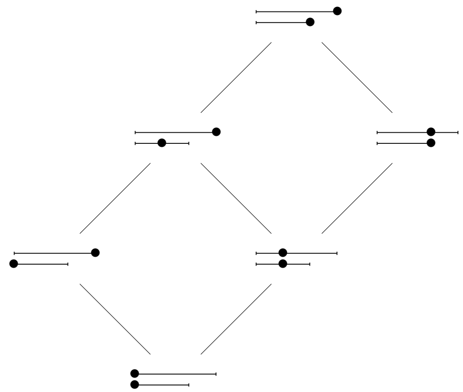
initial and final intervals



freehedron

Saneblidze ('09)

nested or disjoint

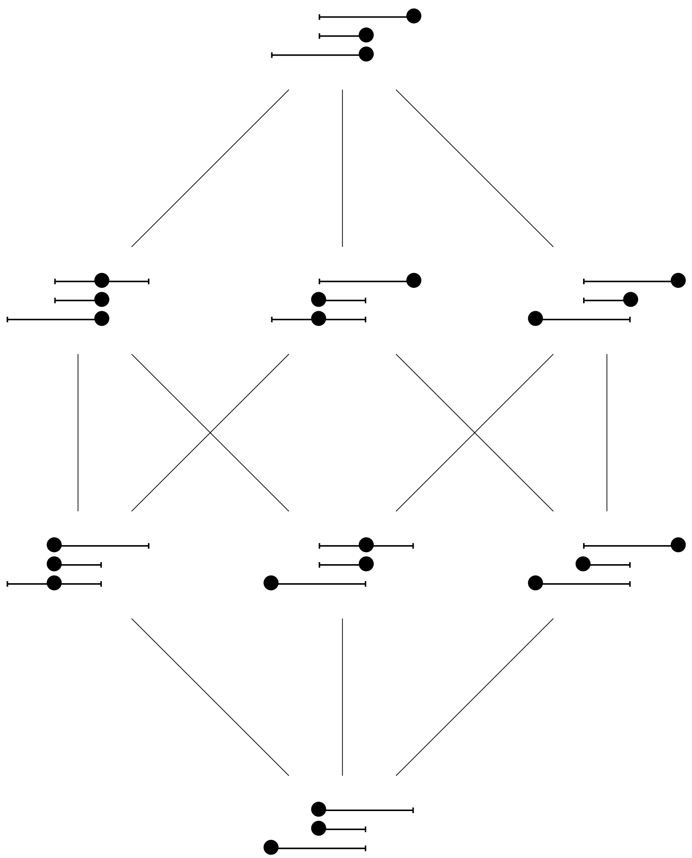


fertilotope

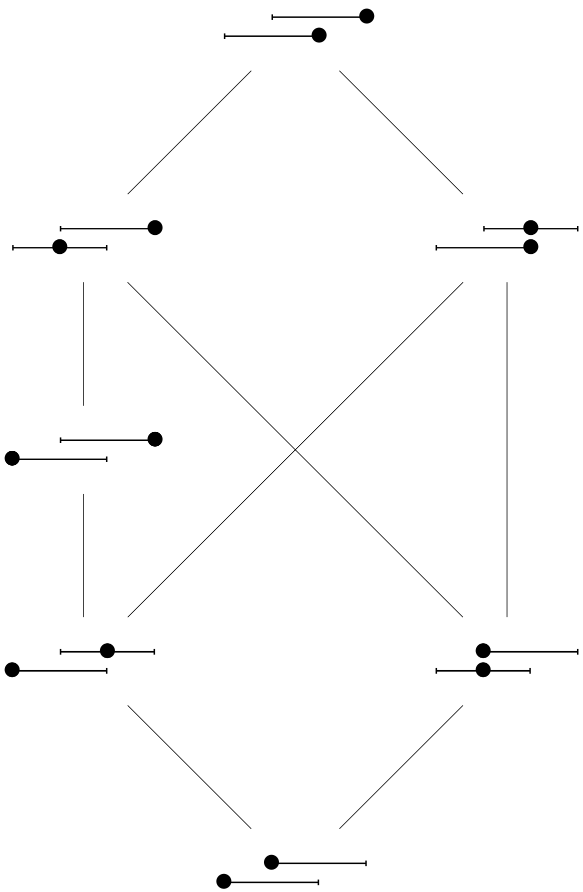
Defant ('23)

INTERVAL HYPERGRAPHIC LATTICES

THM. $P_{\mathbb{I}}$ lattice $\iff$ $\mathbb{I}$ is closed under intersection Bergeron–P. ('24$^+$)



lattice

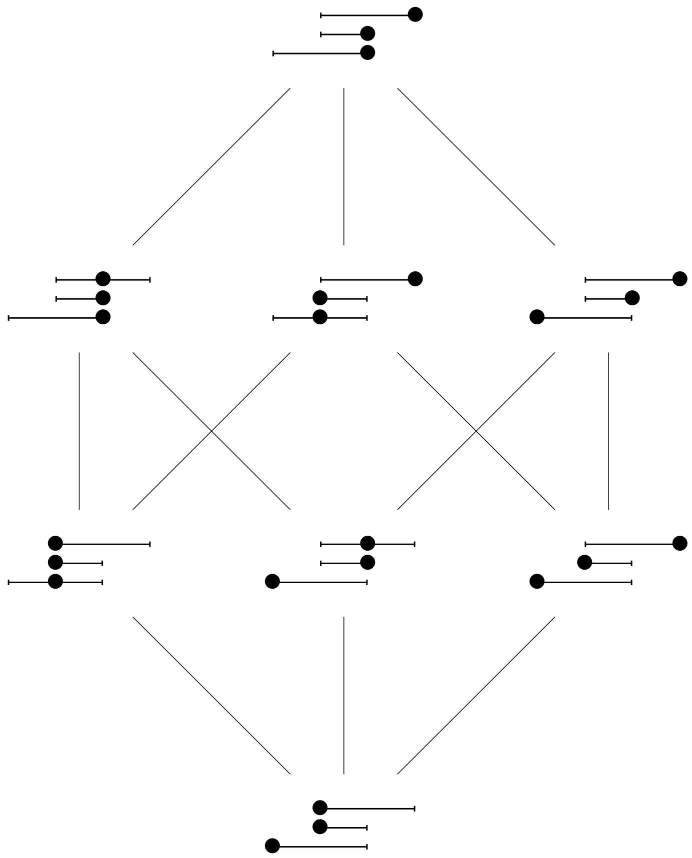


not lattice

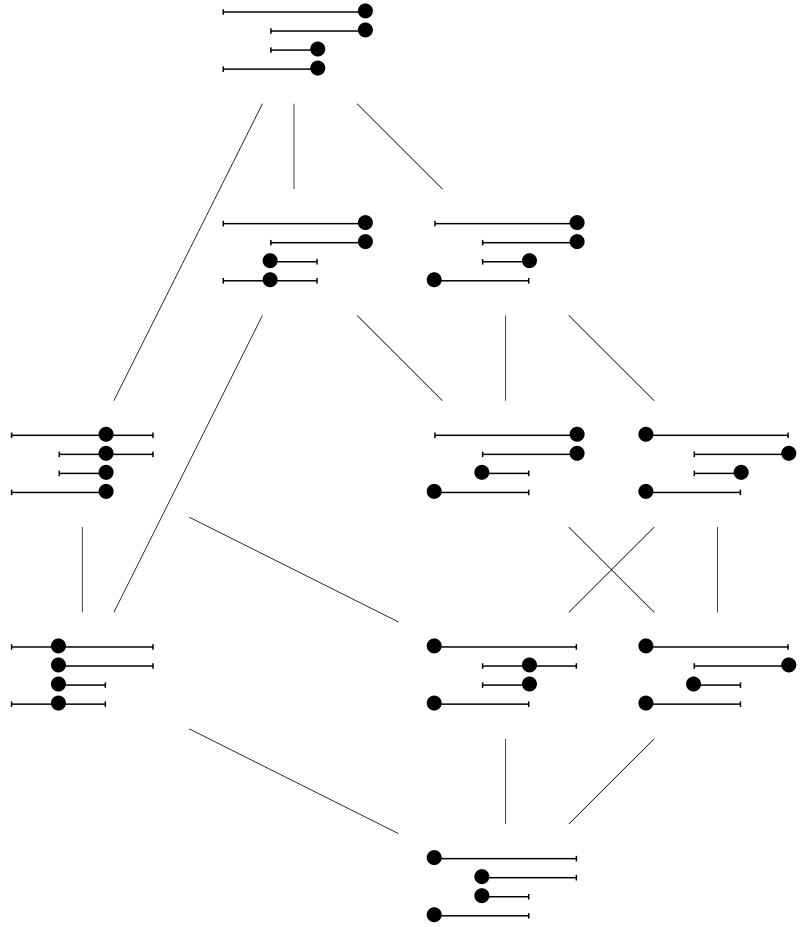
DISTRIBUTIVE INTERVAL HYPERGRAPHIC LATTICES

THM. $P_{\mathbb{I}}$ distributive lattice $\iff$ for all $I, J \in \mathbb{I}$ such that $I \not\subseteq J$, $I \not\supseteq J$ and $I \cap J \neq \varnothing$, the intersection $I \cap J$ is in $\mathbb{I}$ and is initial or final in any $K \in \mathbb{I}$ with $I \cap J \subseteq K$

Bergeron–P. ('24^{+})



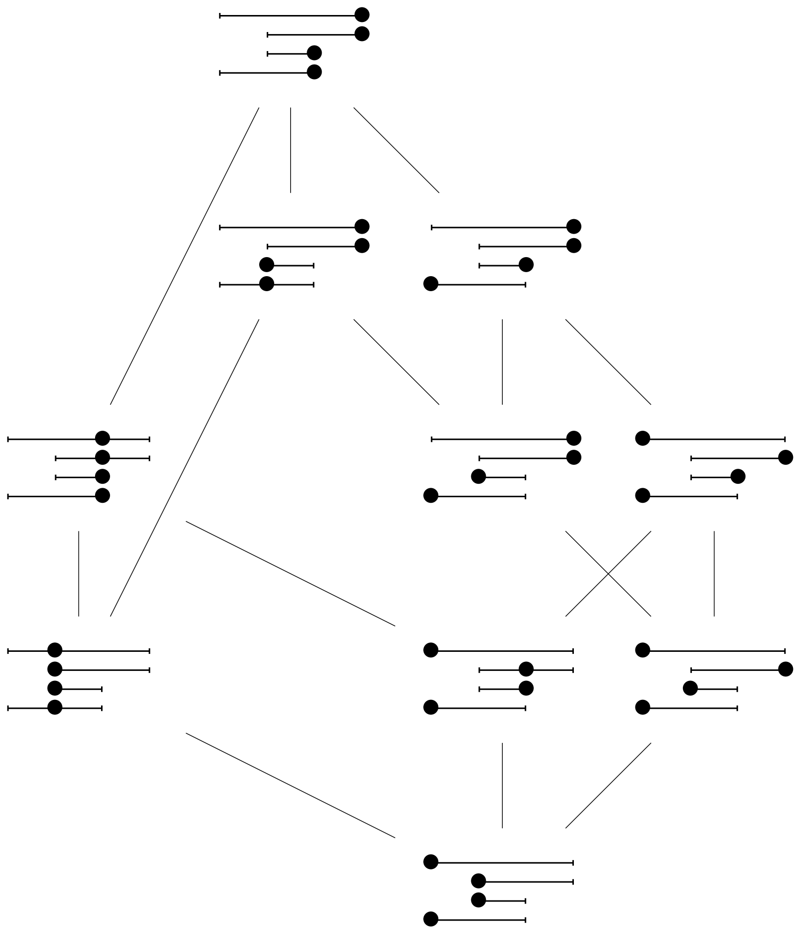
distributive



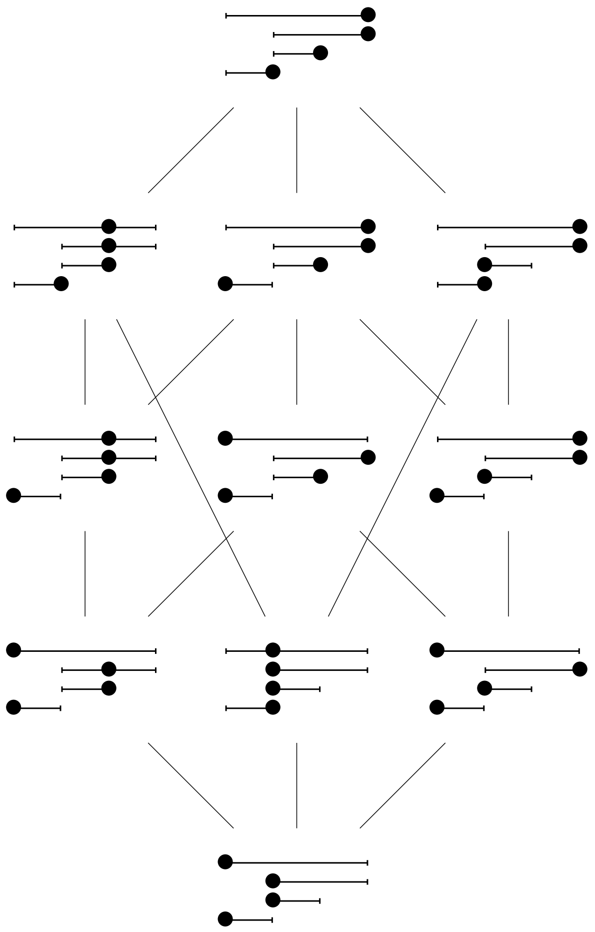
not distributive

SEMIDISTRIBUTIVE INTERVAL HYPERGRAPHIC LATTICES

THM. $P_{\mathbb{I}}$ join semidistributive lattice $\iff$ $\mathbb{I}$ is closed under intersection and for all $[r, r'], [s, s'], [t, t'], [u, u'] \in \mathbb{I}$ with $r < s \leq r' < s'$, $r < t \leq s' < t'$, $u < \min(s, t)$, $s' < u'$, there is $[v, v'] \in \mathbb{I}$ such that $v < s$ and $s' < v' < t'$ Bergeron–P. ('24$^+$)



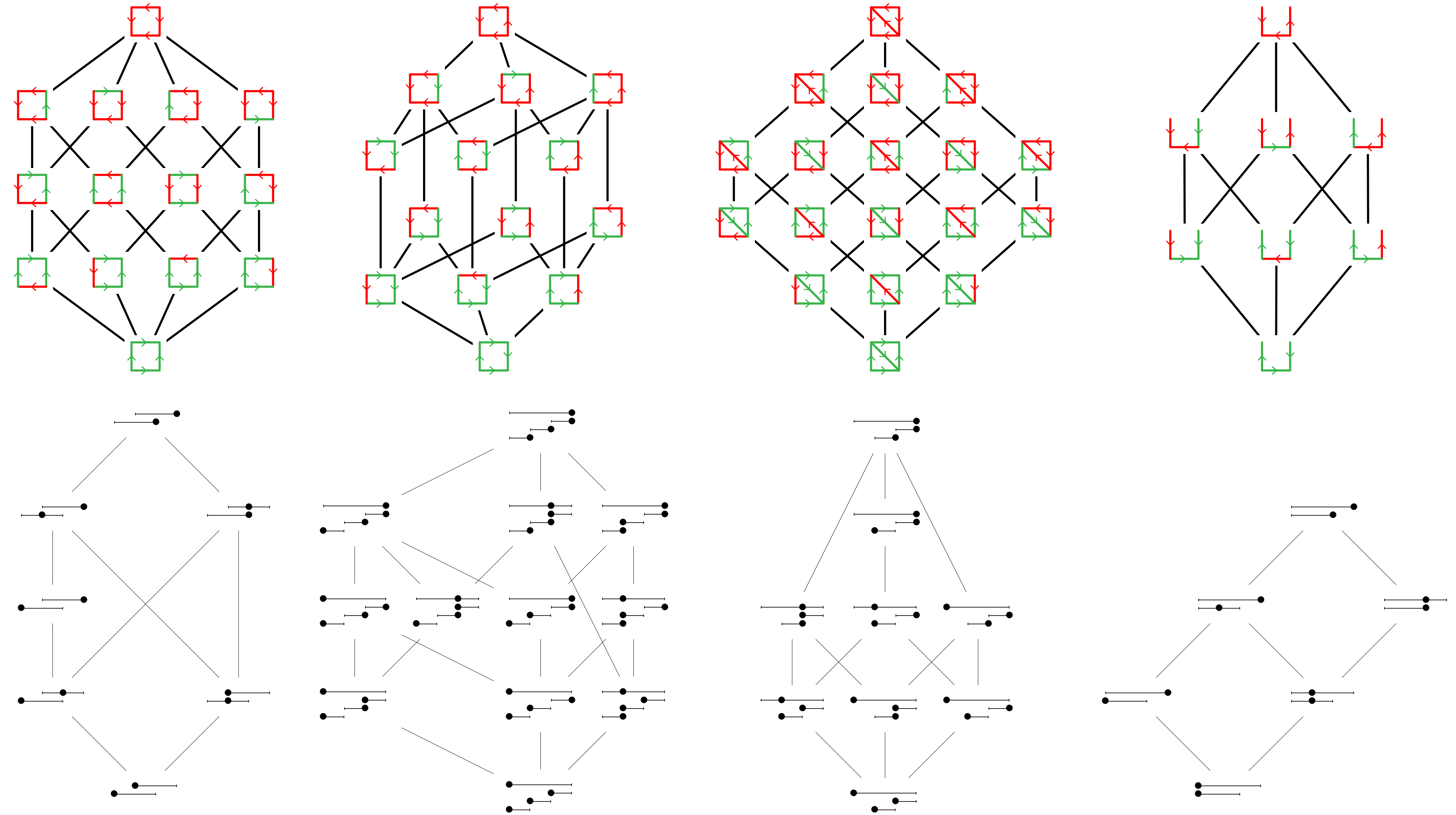
semidistributive



not join semidistributive

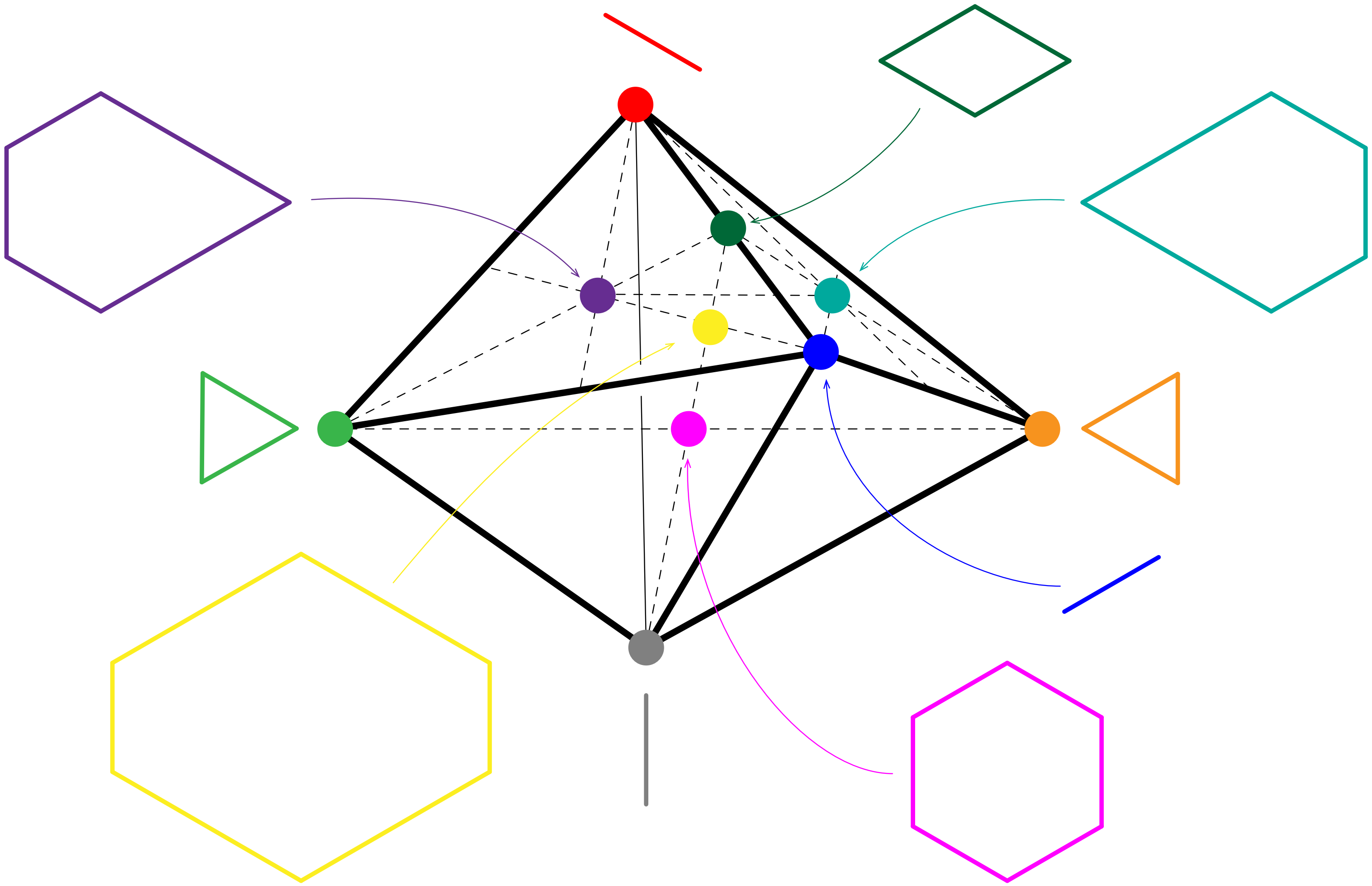
SUMMARY

not a lattice | lattice | semidistributive lattice | distributive lattice



PROB. Provide a similar classification for arbitrary hypergraphs $\mathbb{H}$

SUMMARY OF LAST EPISODE

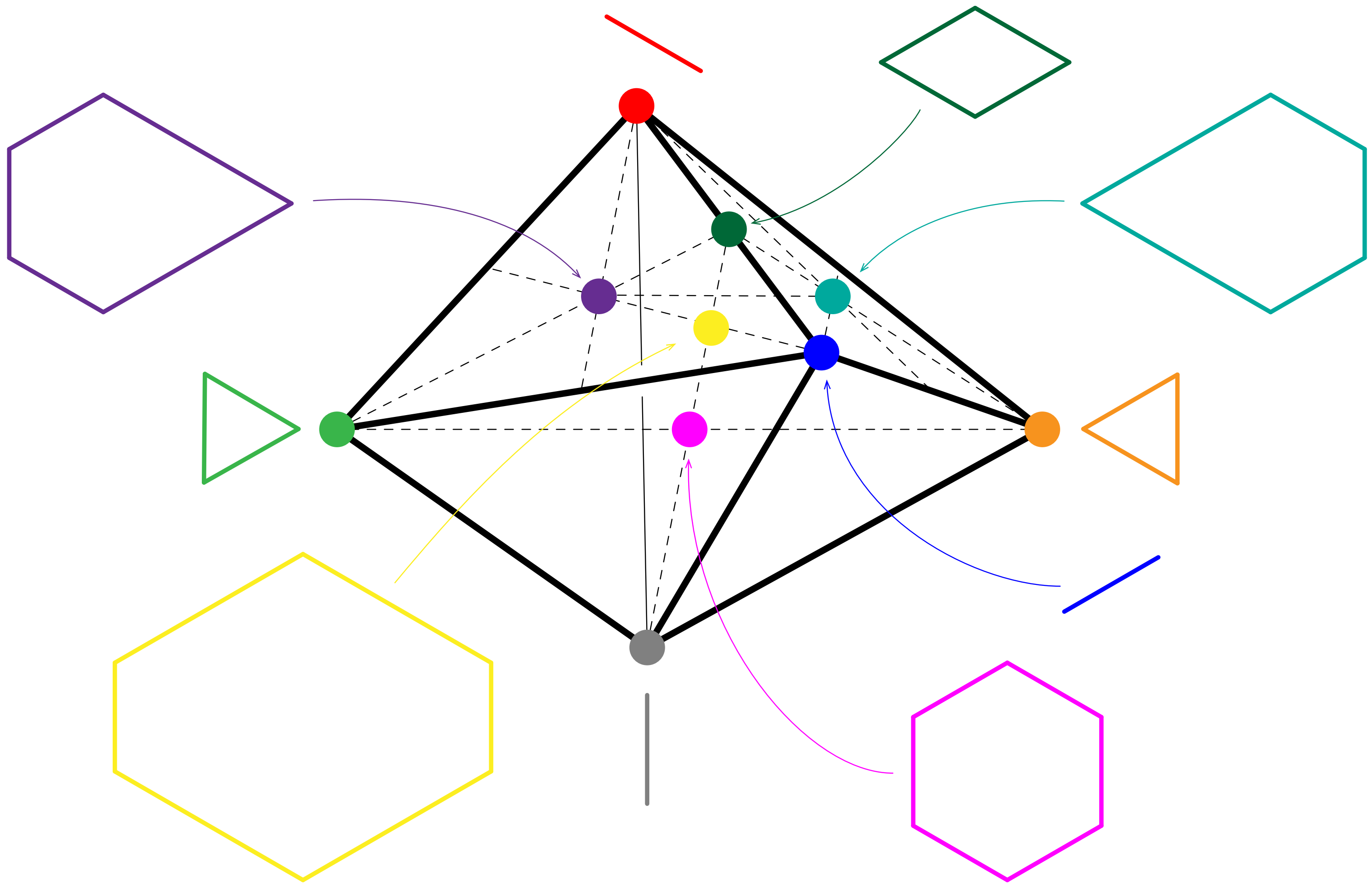


SUMMARY

permutahedron $= \mathrm{conv}\left\{[\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n\right\}$

deformed permutahedron = polytope whose normal fan coarsens the braid fan

deformation cone = cone of deformations (under Minkowski sum and dilation)
polyhedral cone parametrized by heights of inequalities (or length of edges)



SUMMARY

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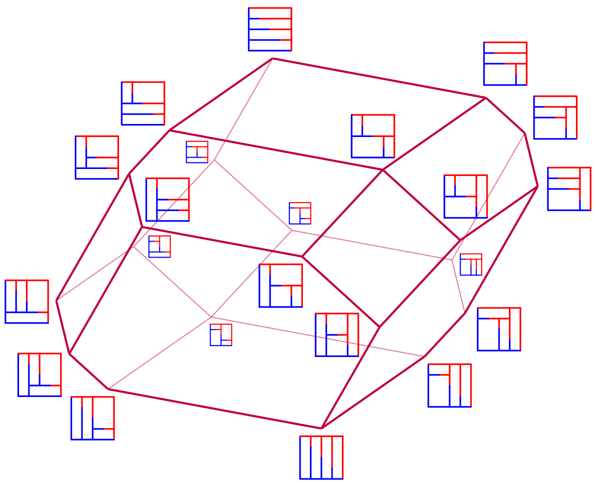
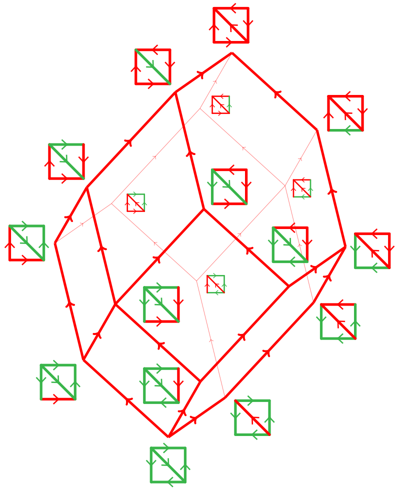
deformation cone = cone of deformations (under Minkowski sum and dilation)
polyhedral cone parametrized by heights of inequalities (or length of edges)

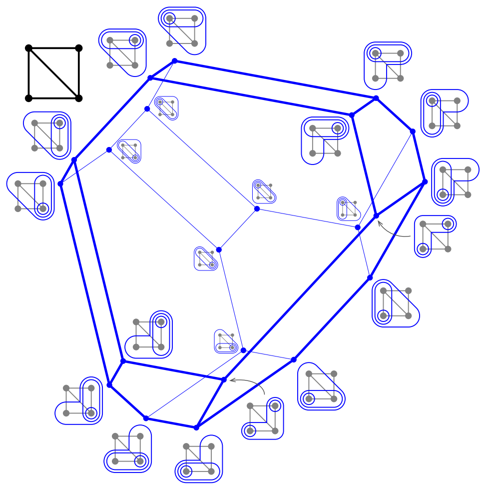
4 families of examples



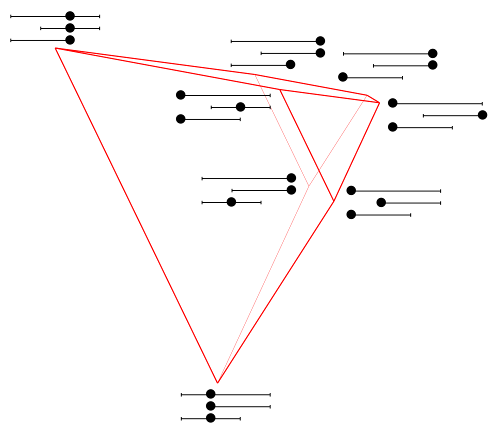
quotientopes



graphical zonotopes



nestohedra
graph associahedra



hypergraphic polytopes

SHUFFLE OF DEFORMED PERMUTAHEDRA

Chapoton–P., *Shuffles of deformed permutahedra, multiplihedra, constrainahedra, and biassociahedra* ('24)

SHUFFLE OF DEFORMED PERMUTAHEDRA

shuffle of $\mathbb{P} \subseteq \mathbb{R}^m$ and $\mathbb{Q} \subseteq \mathbb{R}^n$ =

$$\mathbb{P} \star \mathbb{Q} = \mathbb{P} \times \mathbb{Q} + \sum_{i \in [m], j \in [n]} [\boldsymbol{e}_i, \boldsymbol{e}_{m+j}] \subseteq \mathbb{R}^{m+n}$$

SHUFFLE OF DEFORMED PERMUTAHEDRA

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$$\mathbb{P} \star \mathbb{Q} = \mathbb{P} \times \mathbb{Q} + \sum_{i \in [m], j \in [n]} [\boldsymbol{e}_i, \boldsymbol{e}_{m+j}] \subseteq \mathbb{R}^{m+n}$$

EXM. $\bullet \star \bullet \star \cdots \star \bullet = \mathbb{P}\mathrm{erm}(n)$

EXM. Chapoton–P. ('24)

1. graphical zonotopes $\mathbb{Z}\mathrm{ono}(G) \star \mathbb{Z}\mathrm{ono}(H) = \mathbb{Z}\mathrm{ono}(G \circledast H)$
2. hypergraphic polytopes $\triangle(\mathbb{G}) \star \triangle(\mathbb{H}) = \triangle(\mathbb{G} \circledast \mathbb{H})$

SHUFFLE OF DEFORMED PERMUTAHEDRA

shuffle of $\mathbb{P} \subseteq \mathbb{R}^m$ and $\mathbb{Q} \subseteq \mathbb{R}^n$ =

$$\mathbb{P} \star \mathbb{Q} = \mathbb{P} \times \mathbb{Q} + \sum_{i \in [m], j \in [n]} [\boldsymbol{e}_i, \boldsymbol{e}_{m+j}] \subseteq \mathbb{R}^{m+n}$$

EXM. $\bullet \star \bullet \star \cdots \star \bullet = \mathbb{P}\mathrm{erm}(n)$

EXM. Chapoton–P. ('24)

1. graphical zonotopes $\mathbb{Z}\mathrm{ono}(G) \star \mathbb{Z}\mathrm{ono}(H) = \mathbb{Z}\mathrm{ono}(G \circledast H)$
2. hypergraphic polytopes $\triangle(\mathbb{G}) \star \triangle(\mathbb{H}) = \triangle(\mathbb{G} \circledast \mathbb{H})$

PROP. Chapoton–P. ('24)

1. $\mathbb{P} \subseteq \mathbb{R}^m$ and $\mathbb{Q} \subseteq \mathbb{R}^n$ defo. perm. $\implies$ $\mathbb{P} \star \mathbb{Q} \subseteq \mathbb{R}^{m+n}$ defo. perm.
2. $\star$ is associative (but not commutative, except if considered up to coordinate swap)
3. $\star$ does not preserve simplicity of polytopes
4. $\star$ does not preserve lattice property
5. $\mathbb{P}$ has lattice property $\implies$ $\mathbb{P} \star \mathbb{P}\mathrm{erm}(n)$ has lattice property for any $n \geq 1$

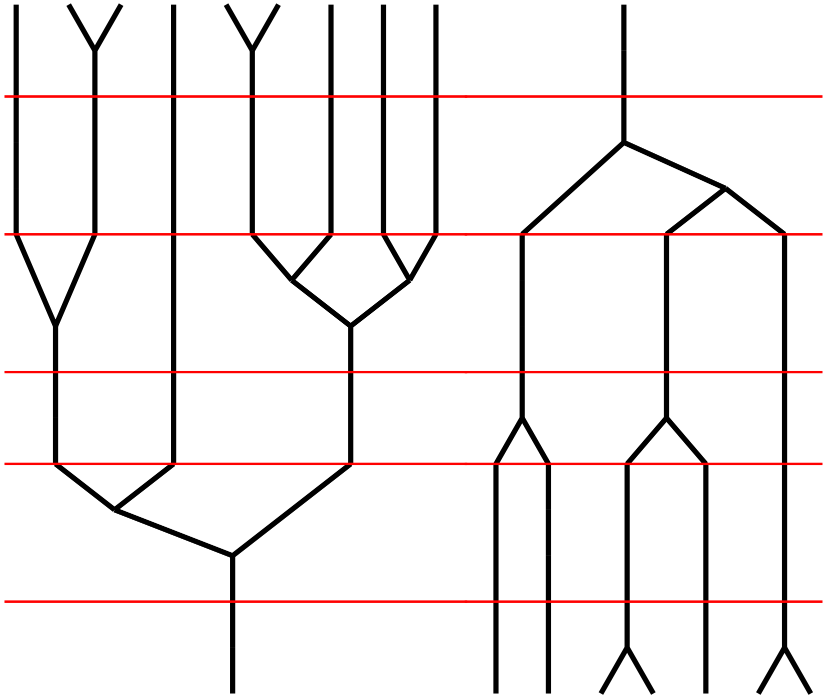
SHUFFLE OF DEFORMED PERMUTAHEDRA

$(\mathbb{P}, \mathbb{Q})$-biposet $\preccurlyeq_{\boldsymbol{p},\boldsymbol{q},\mu}$ = poset on $[m+n]$ defined from

- a $\mathbb{P}$-poset $\preccurlyeq_{\boldsymbol{p}}$ and a $\mathbb{Q}$-poset $\preccurlyeq_{\boldsymbol{q}}$,
- an ordered partition μ of $[m+n]$ with parts alternatively contained in $[m]$ and $[n]^{+m}$,

by $i \preccurlyeq_{\boldsymbol{p},\boldsymbol{q},\mu} j$ iff

- either there is an oriented path from i to j in $\preccurlyeq_{\boldsymbol{p}}$ or in ${\preccurlyeq_{\boldsymbol{q}}}^{+m}$,
- or i is in a block lower than j.

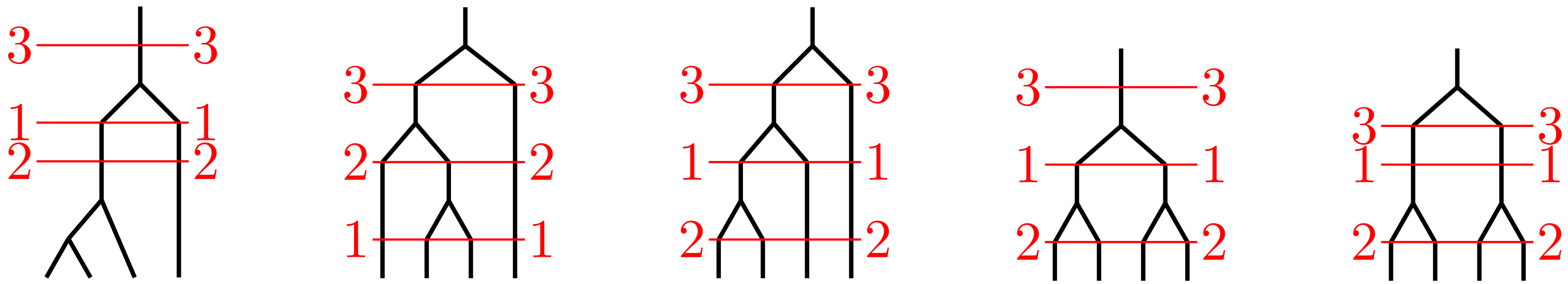


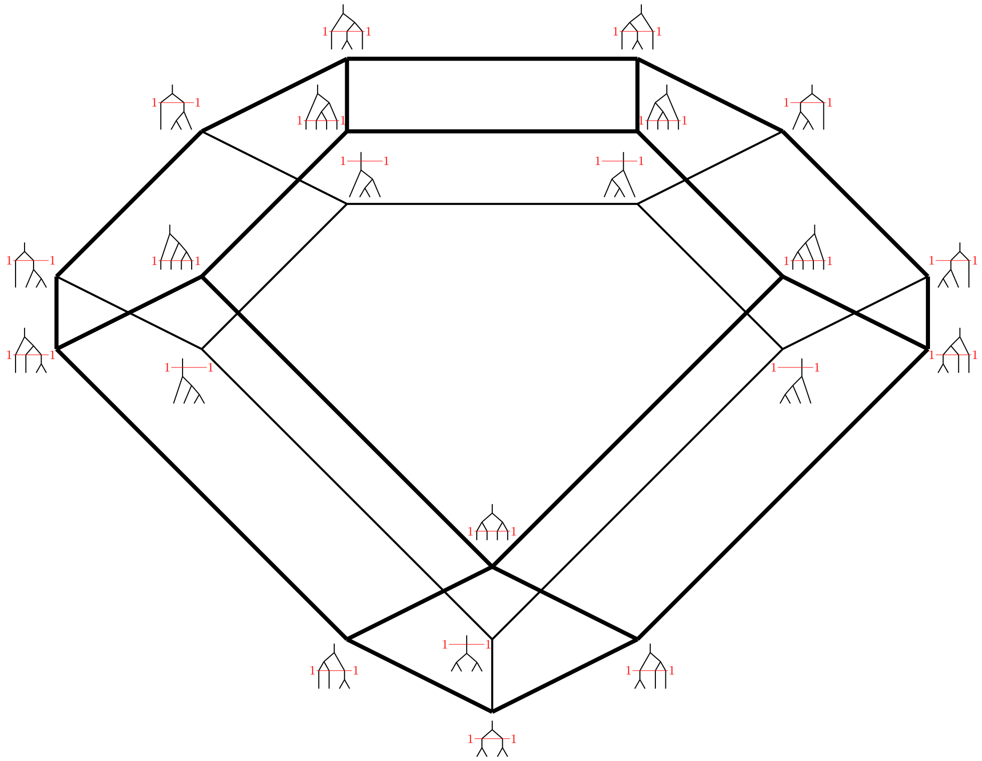
PROP. $(\mathbb{P} \star \mathbb{Q})$-posets = $(\mathbb{P}, \mathbb{Q})$-biposets   Chapoton–P. ('24)

MULTIPLIHEDRON

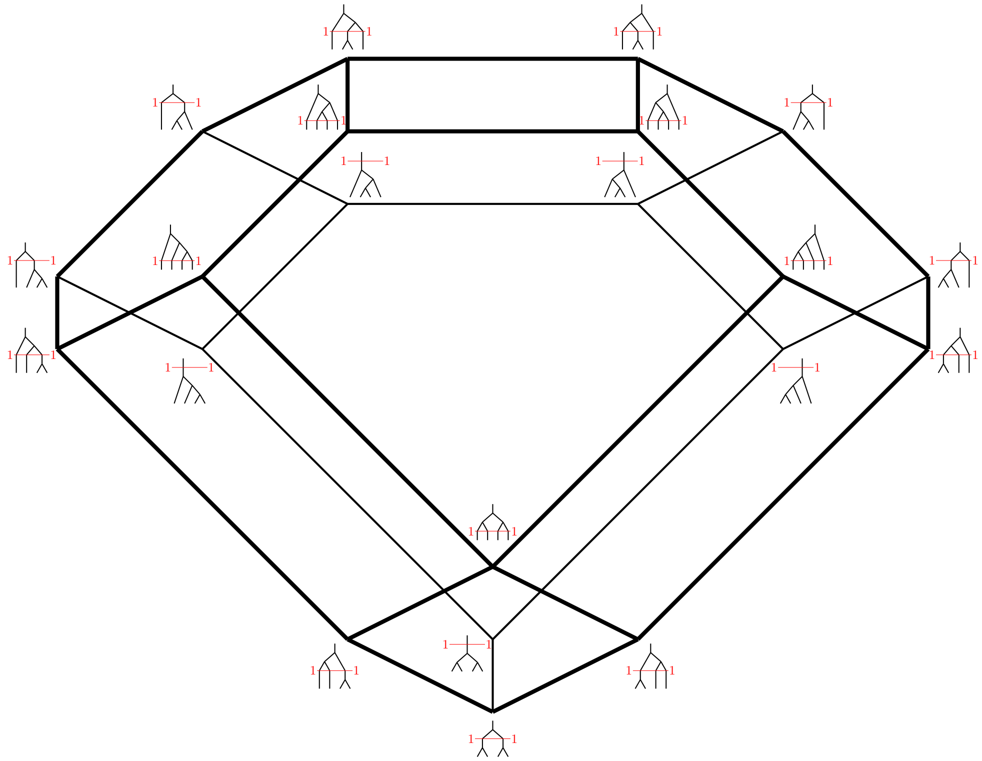
(m,n)-multiplihedron $\mathbb{M}\mathsf{ul}(m,n) = \mathbb{P}\mathsf{erm}(m) \star \mathbb{A}\mathsf{sso}(n)$ (not simple, lattice property)

vertices of $\mathbb{M}\mathsf{ul}(m,n)$ = m-painted n-trees



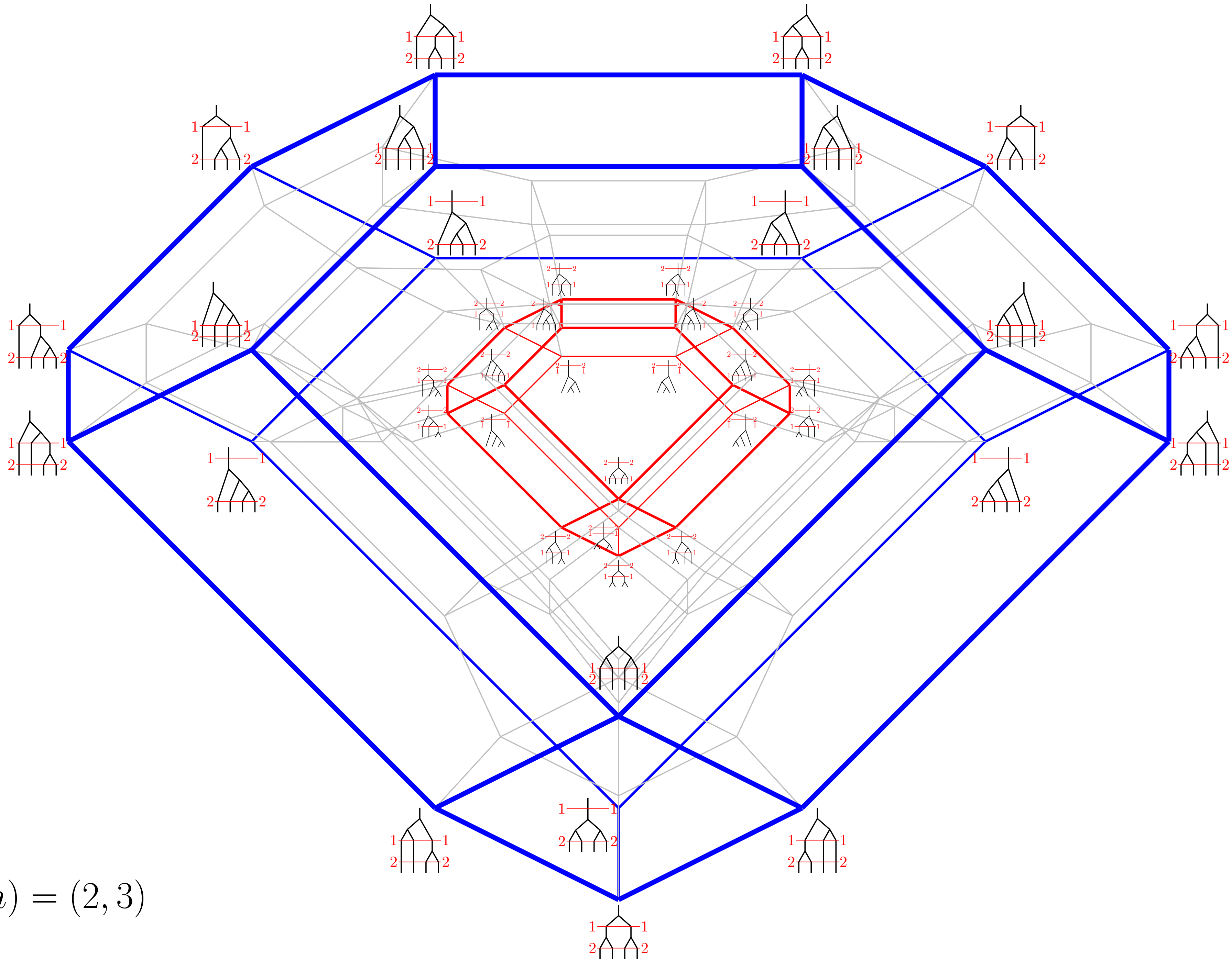


$(m,n) = (1,3)$



$(m,n) = (2,2)$

MULTIPLIHEDRON

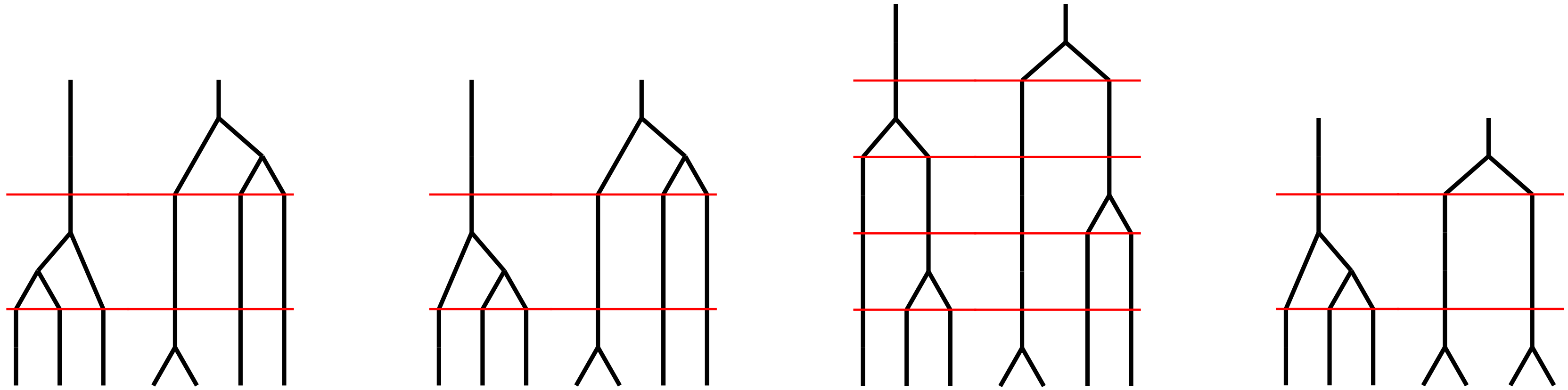


$(m, n) = (2, 3)$

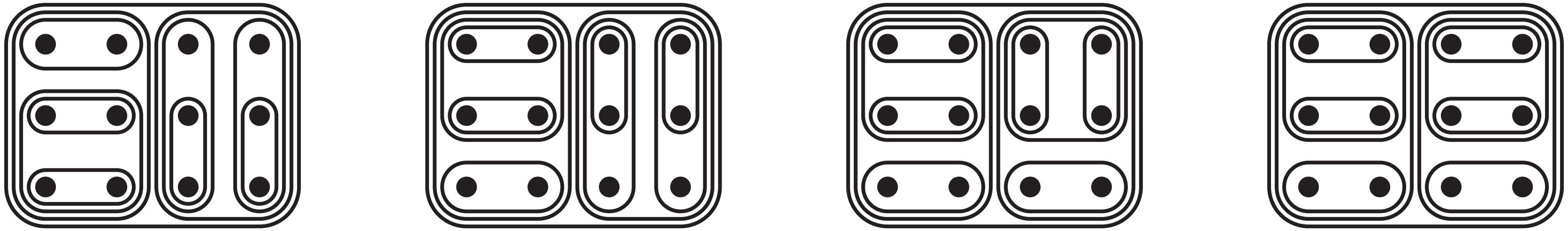
CONSTRAINAHEDRON

(m,n)-constrainahedron $\mathbb{C}\text{onstr}(m,n) = \mathbb{A}\text{sso}(m) \star \mathbb{A}\text{sso}(n)$

vertices of $\mathbb{C}\text{onstr}(m,n) = (m,n)$-cotrees



in bijection with good rectangular bracketings... (connection with colliding particles)

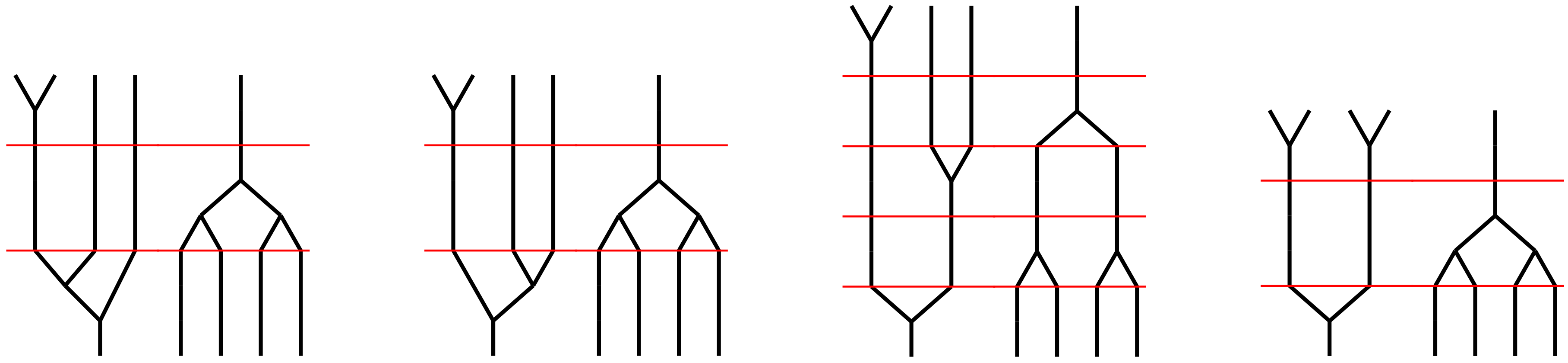


(not simple, not lattice property)

BIASSOCIAHEDRON

(m, n)-biassociahedron $\mathbb{B}\mathsf{ias}(m, n) = \mathbb{A}\overline{\mathsf{sso}}(m) \star \mathbb{A}\mathsf{sso}(n)$

vertices of $\mathbb{B}\mathsf{ias}(m, n) = (m, n)$-bitrees



(connections to bialgebras up-to-homotopy)

(not simple, not lattice property)

REM. $\#$ vertices $\mathbb{C}\mathsf{onstr}(m, n) = \#$ vertices $\mathbb{B}\mathsf{ias}(m, n)$ but

$$f(\mathbb{C}\mathsf{onstr}(3, 3)) = (1, 606, 1550, 1384, 498, 60, 1)$$
$$f(\mathbb{B}\mathsf{ias}(3, 3)) = (1, 606, 1549, 1382, 497, 60, 1)$$

APPLICATION 1: PIVOT POLYTOPES OF PRODUCTS OF SIMPLICES

Black–De Loera–Lütjeharms–Sanyal, *The polyhedral geometry of pivot rules and monotone paths* ('23)
Black–Lütjeharms–Sanyal, *From linear programming to colliding particles* ('24$^+$)
P.–Poullot, *Pivot polytopes of products of simplices and shuffles of associahedra* ('24$^+$)

PIVOT POLYTOPE

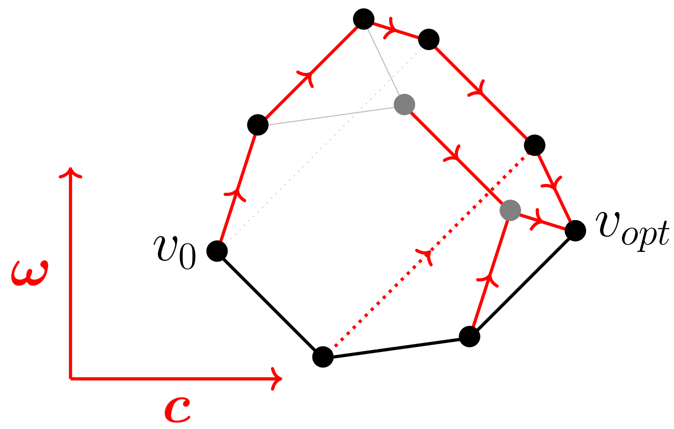
linear optimization = maximize linear function $\boldsymbol{c}$ over a polyhedron $\mathbb{P}$

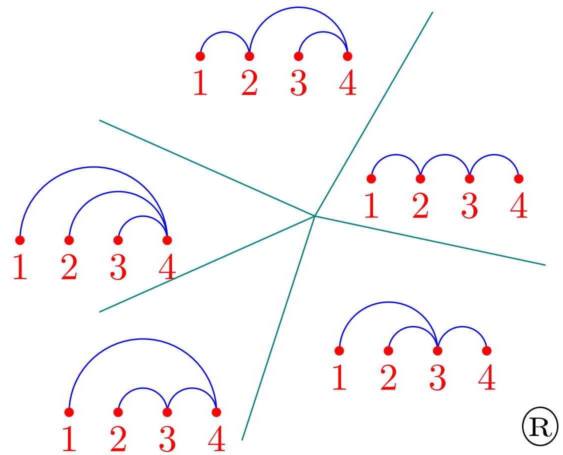
simplex algorithm = start from any vertex
choose an improving neighbor according to a pivot rule
stop at maximal vertex

max-slope pivot rule wrt generic weight $\boldsymbol{\omega}$ = chooses the improving neighbor maximizing the slope on the plane defined by $\boldsymbol{c}$ and $\boldsymbol{\omega}$

PROP. Black–De Loera–Lütjeharms–Sanyal ('23)

- max-slope pivot rule is memoryless $\Longrightarrow$ behavior encoded by an arborescence A_ω
- the fibers of $\boldsymbol{\omega} \mapsto A_\omega$ define the pivot fan





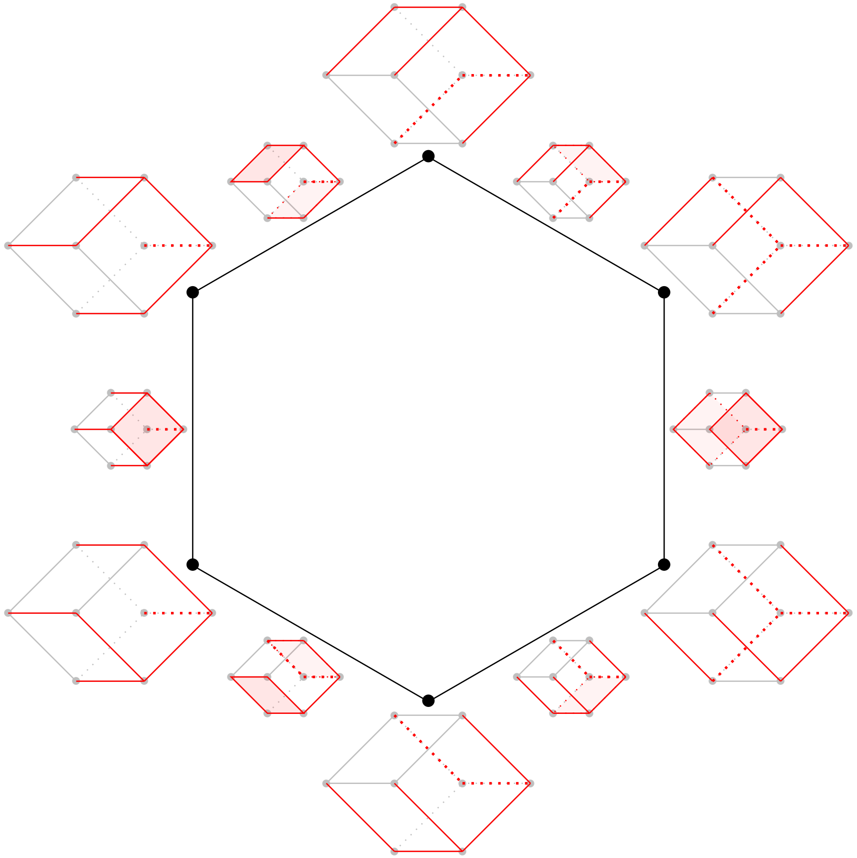
Ⓡ Germain Poullot

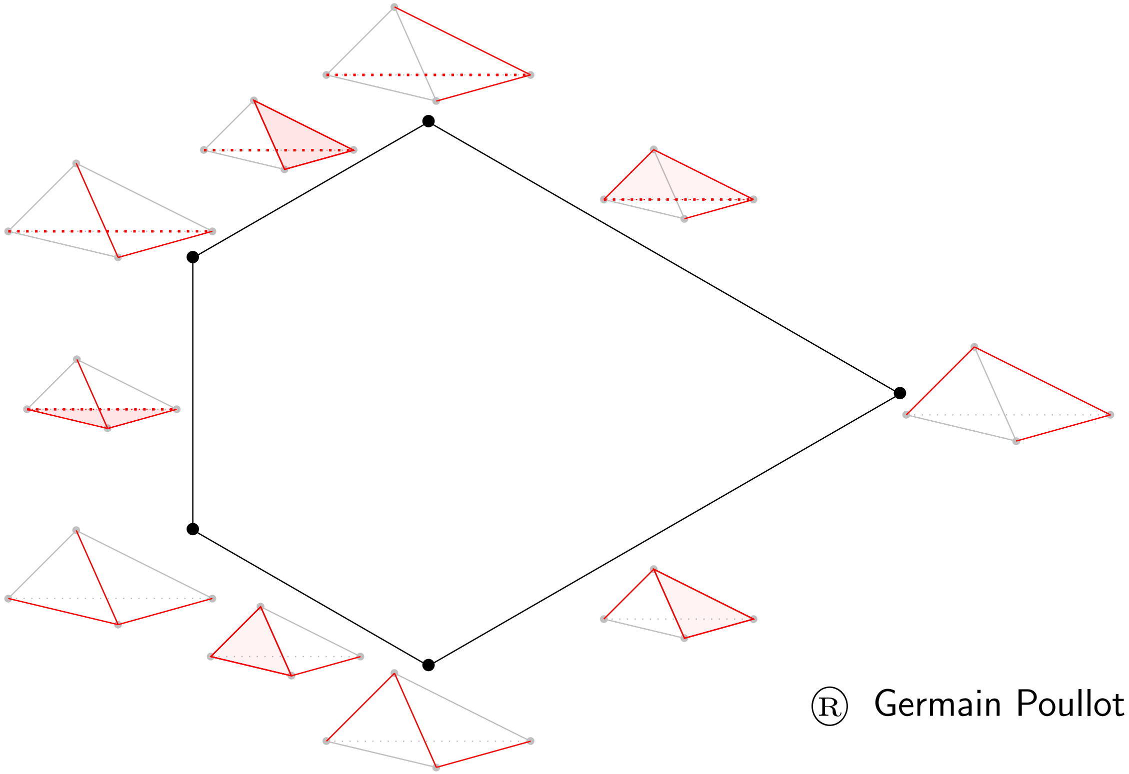
PIVOT POLYTOPE

DEF. $\mathbb{P}$ = polytope in $\mathbb{R}^d$ Black–De Loera–Lütjeharms–Sanyal ('23)

pivot polytope $\Pi(\mathbb{P})$ = polytope whose normal fan is the pivot fan
= polytope encoding arborescences of max-slope pivot rule

EXM. $\Pi(\square_m) = \mathbb{P}\mathrm{erm}(m)$ and $\Pi(\square_n) \simeq \mathbb{A}\mathrm{sso}(n)$ Black–De Loera–Lütjeharms–Sanyal ('23)





Ⓡ Germain Poullot

PIVOT POLYTOPE OF PRODUCT OF SIMPLICES

THM. $\triangle_1 \subset \mathbb{R}^{n_1}, \dots, \triangle_r \subset \mathbb{R}^{n_r}$ full-dimensional simplices — P.–Poullot ('24$^+$)

$\Longrightarrow$ $\Pi(\triangle_1 \times \dots \times \triangle_r)$ is combinatorially equivalent to $\mathbb{A}\mathsf{sso}(n_1) \star \dots \star \mathbb{A}\mathsf{sso}(n_r)$

CORO. Black–De Loera–Lütjeharms–Sanyal ('23) Black–Lütjeharms–Sanyal ('24$^+$) P.–Poullot ('24$^+$)

1. $\Pi(\square_m) \simeq \mathbb{P}\mathsf{erm}(m)$
2. $\Pi(\square_n) \simeq \mathbb{A}\mathsf{sso}(n)$
3. $\Pi(\square_m \times \triangle_n) \simeq (m, n)$-multiplihedron
4. $\Pi(\triangle_m \times \triangle_n) \simeq (m, n)$-constrainahedron

APPLICATION 2: HOCHSCHILD POLYTOPES

P.–Polyakova, *Hochschild polytopes* ('25)

LATTICES: PAINTED TREES & LIGHTED SHADES

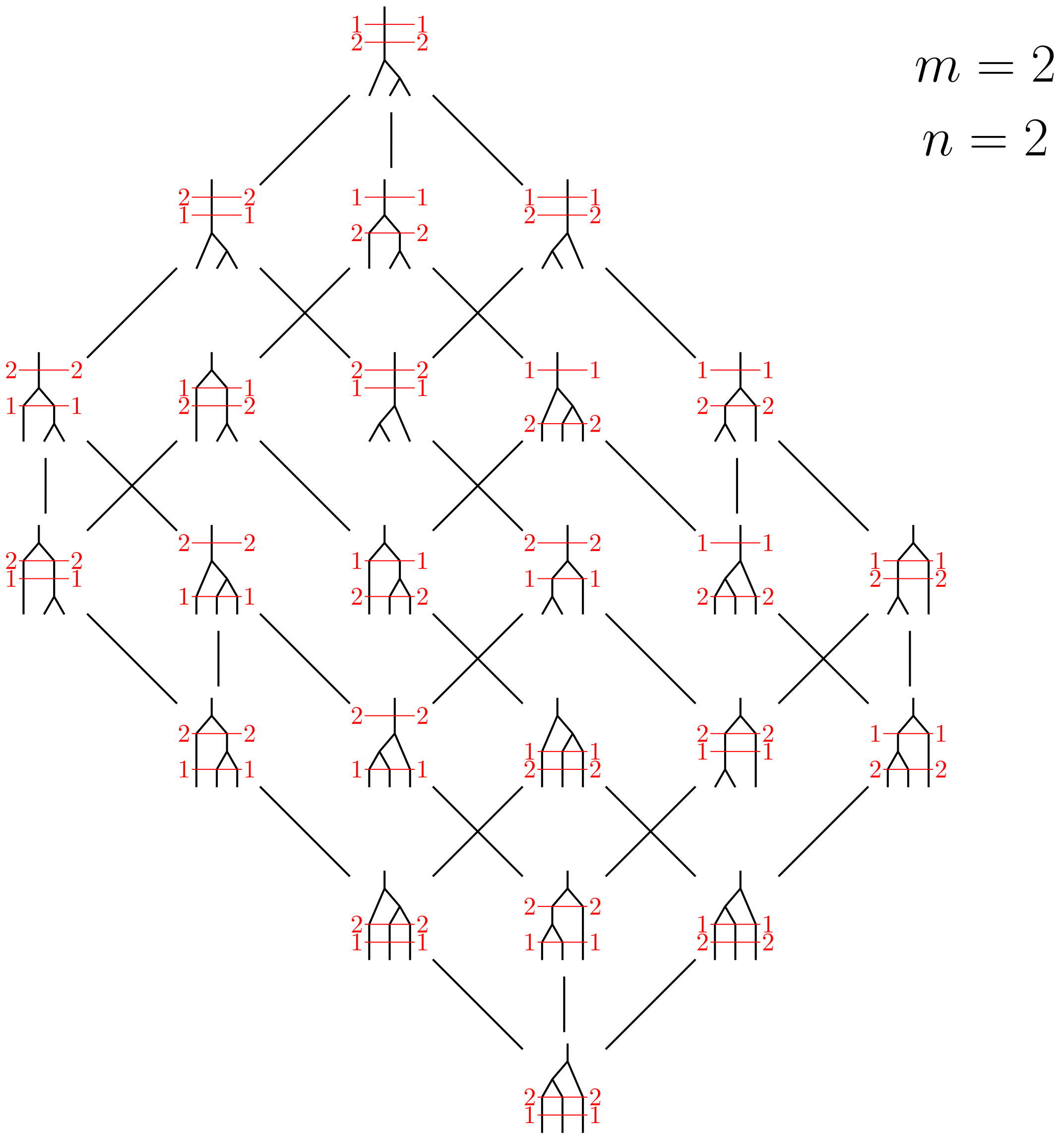
painted tree

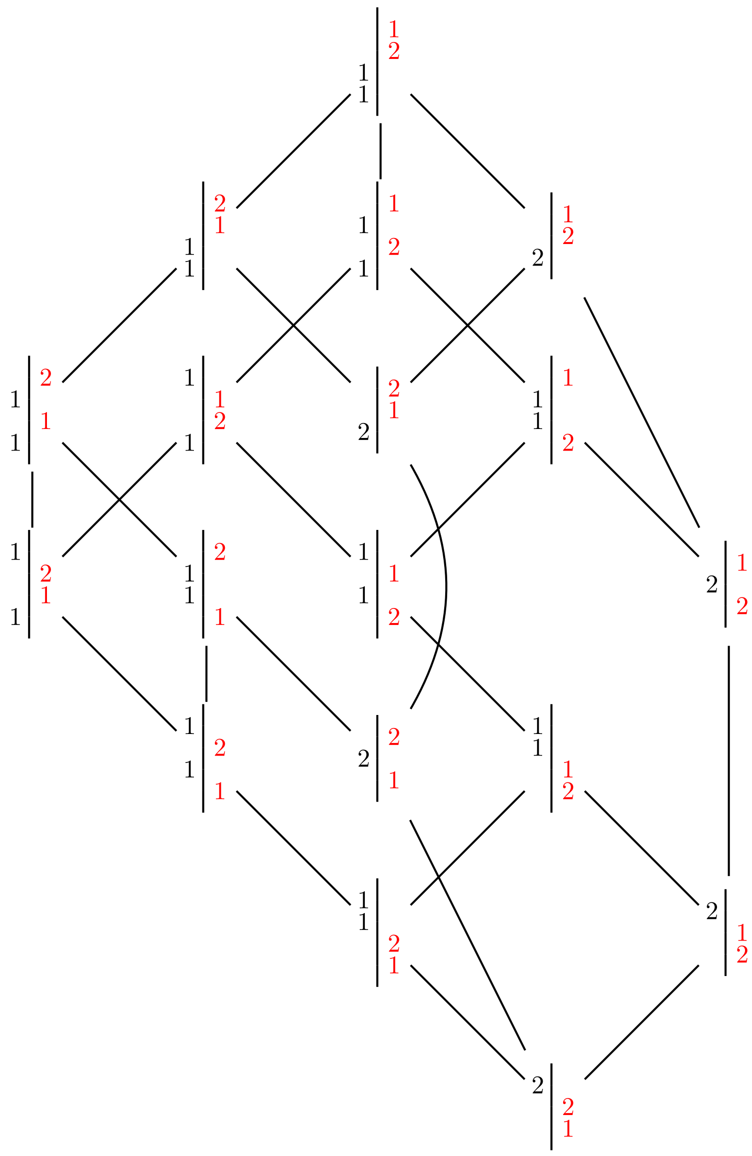
® Konstantin Dimopoulos

lighted shade

® Narae Kim

LATTICES: PAINTED TREES & LIGHTED SHADES





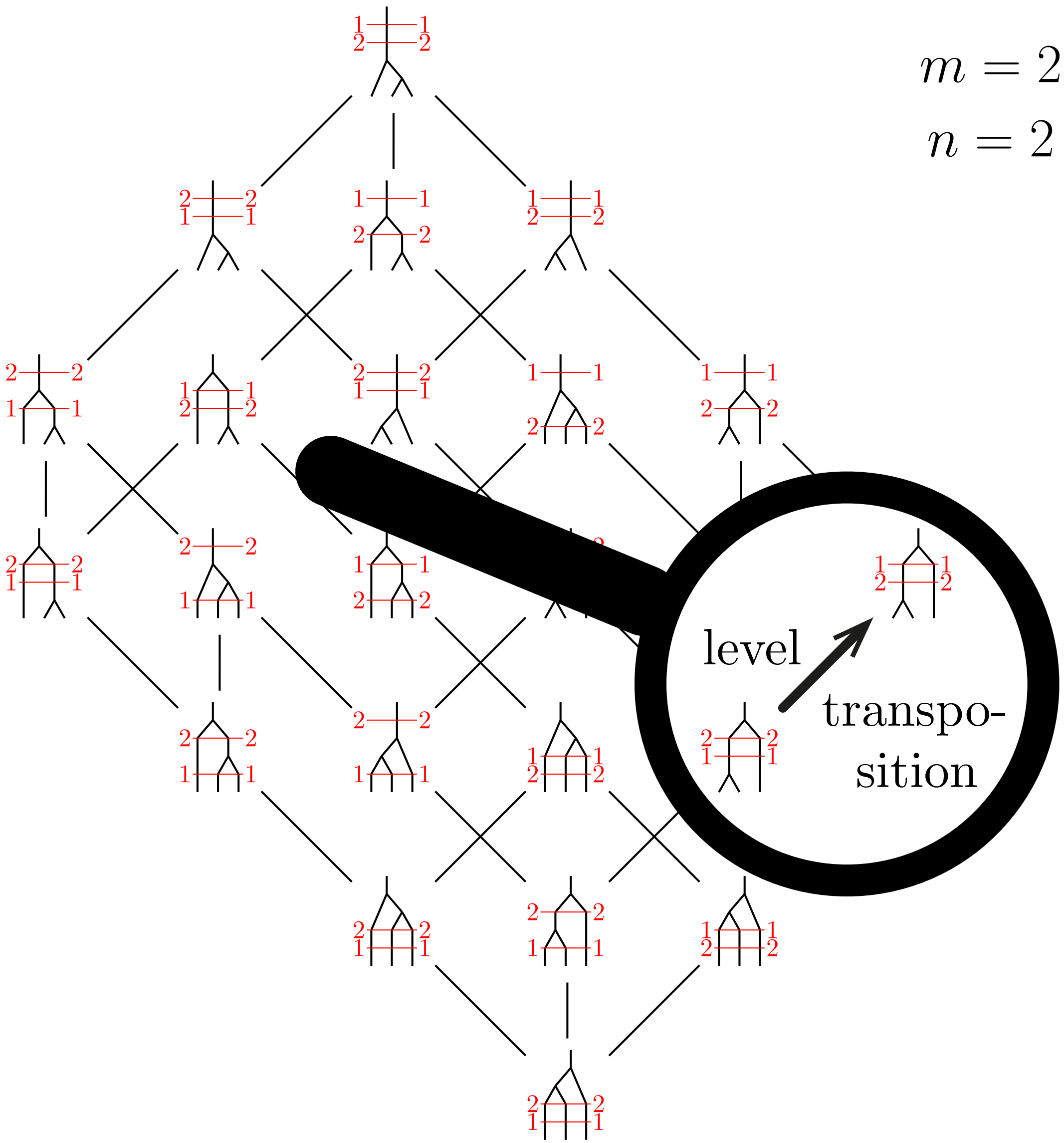
m-painted n-tree $=$ binary tree with n nodes and m levels

Chapoton–P. ('24)

m-lighted n-shade $=$ partition of $[n]$ with m levels

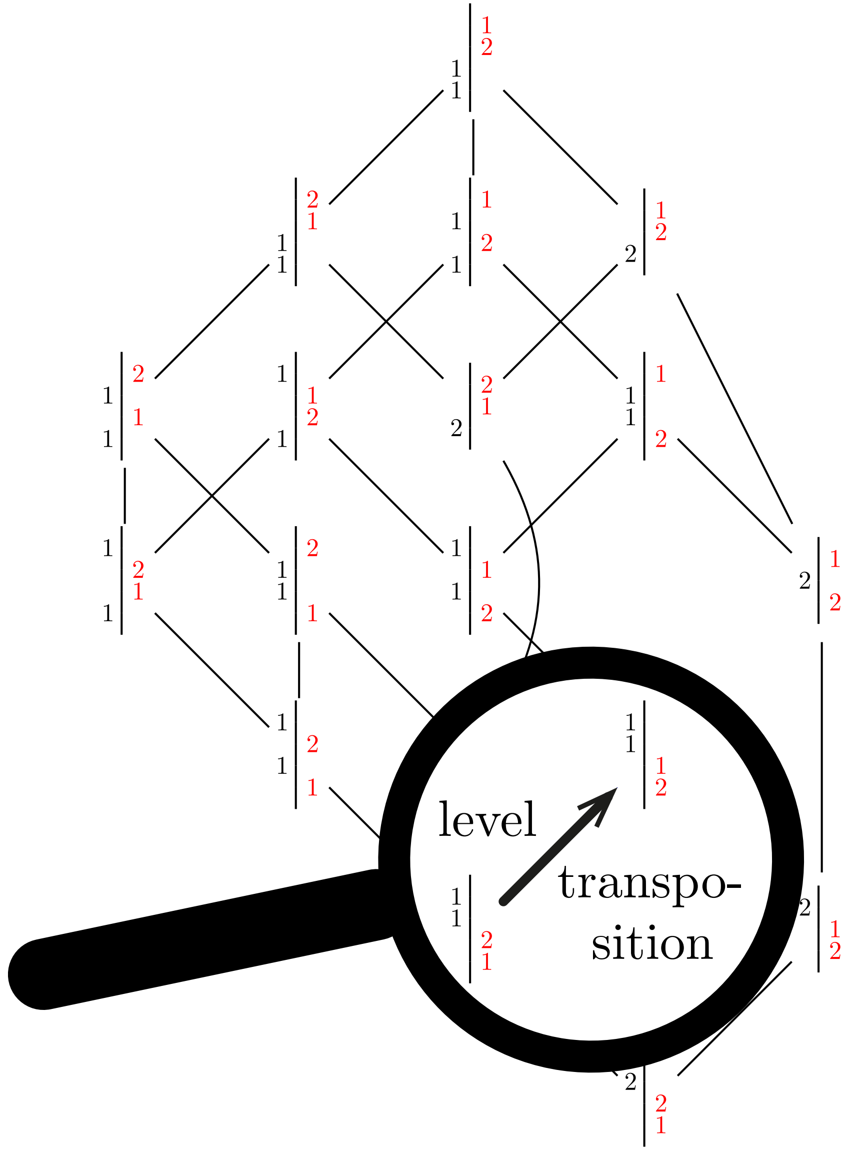
P.–Polyakova ('25)

LATTICES: PAINTED TREES & LIGHTED SHADES



$m = 2$

$n = 2$



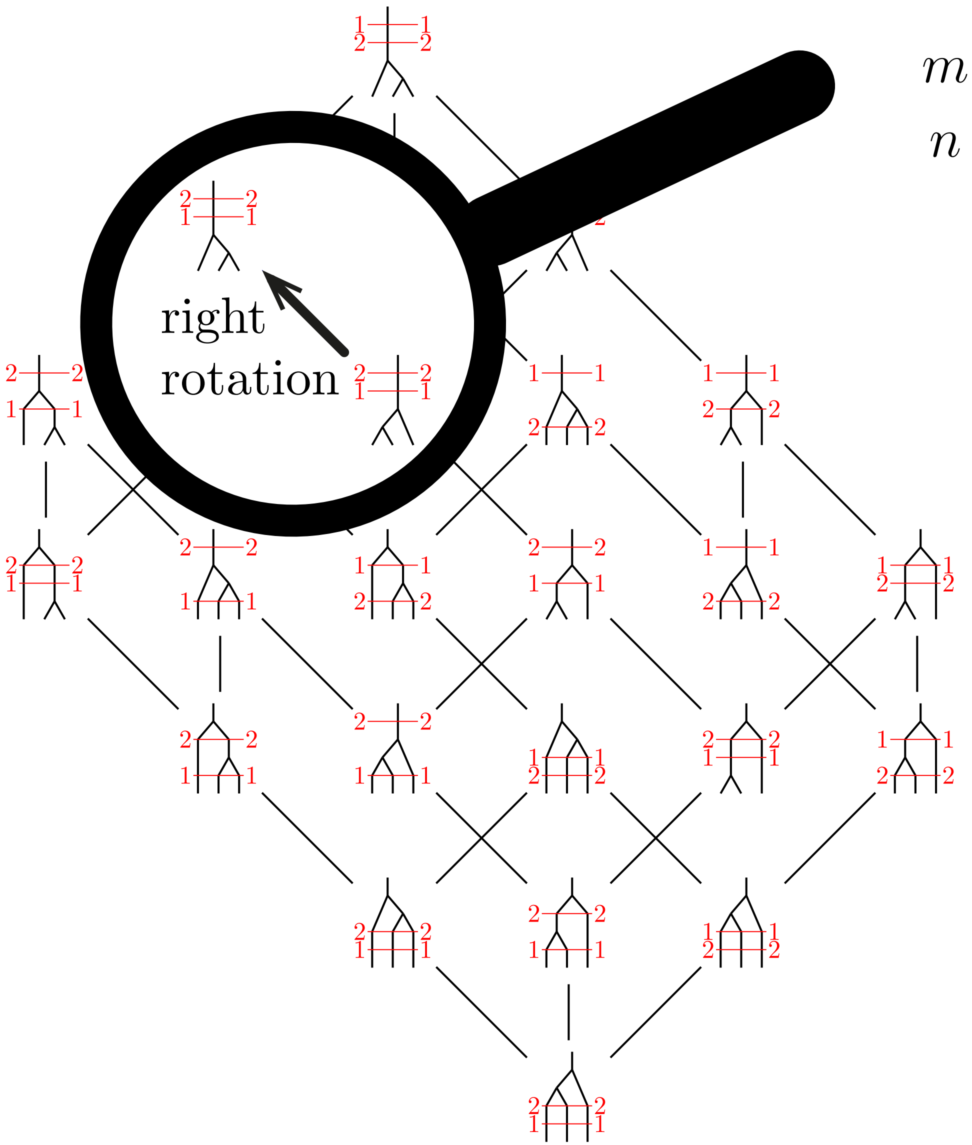
m-painted n-tree $=$ binary tree with n nodes and m levels

Chapoton–P. ('24)

m-lighted n-shade $=$ partition of $[n]$ with m levels

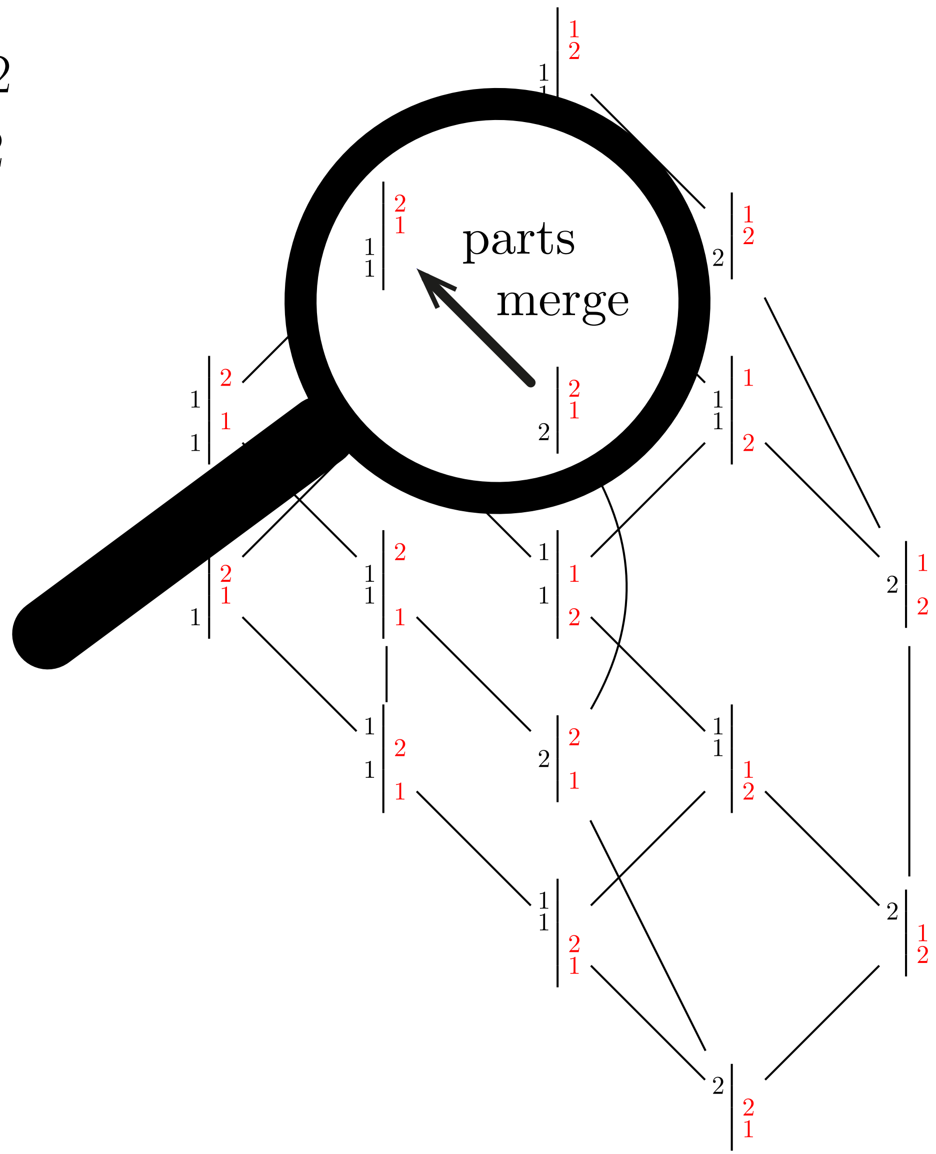
P.–Polyakova ('25)

LATTICES: PAINTED TREES & LIGHTED SHADES



$m = 2$

$n = 2$



m-painted n-tree = binary tree with n nodes and m levels

Chapoton–P. ('24)

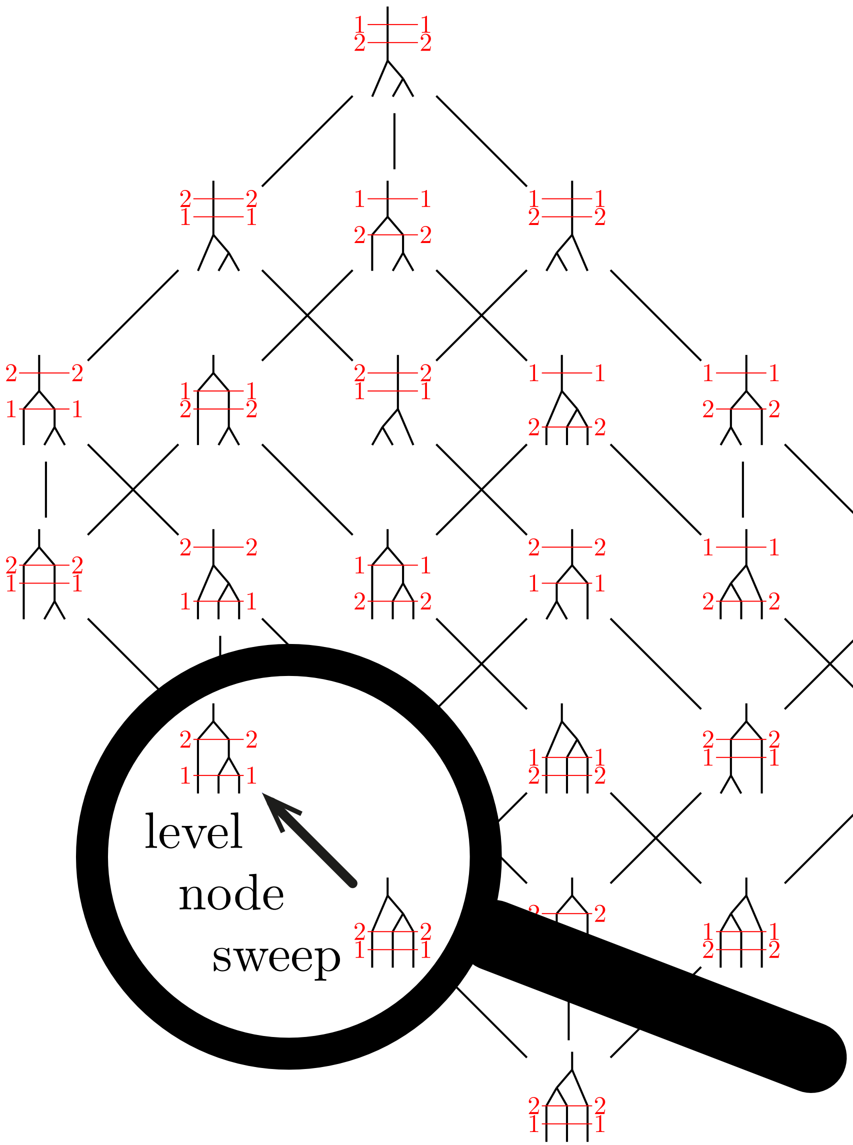
m-lighted n-shade = partition of $[n]$ with m levels

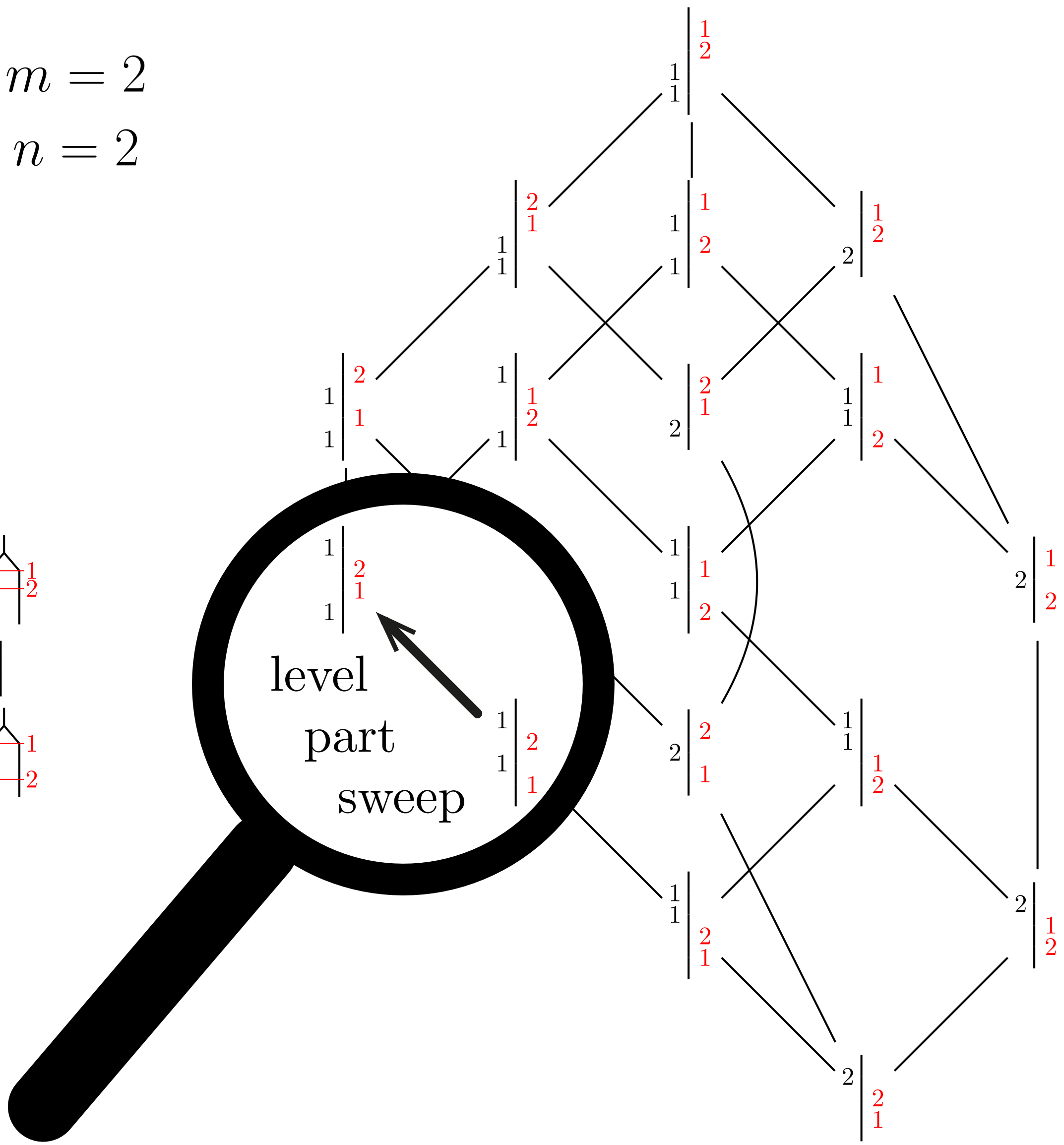
P.–Polyakova ('25)

LATTICES: PAINTED TREES & LIGHTED SHADES

$m = 2$

$n = 2$





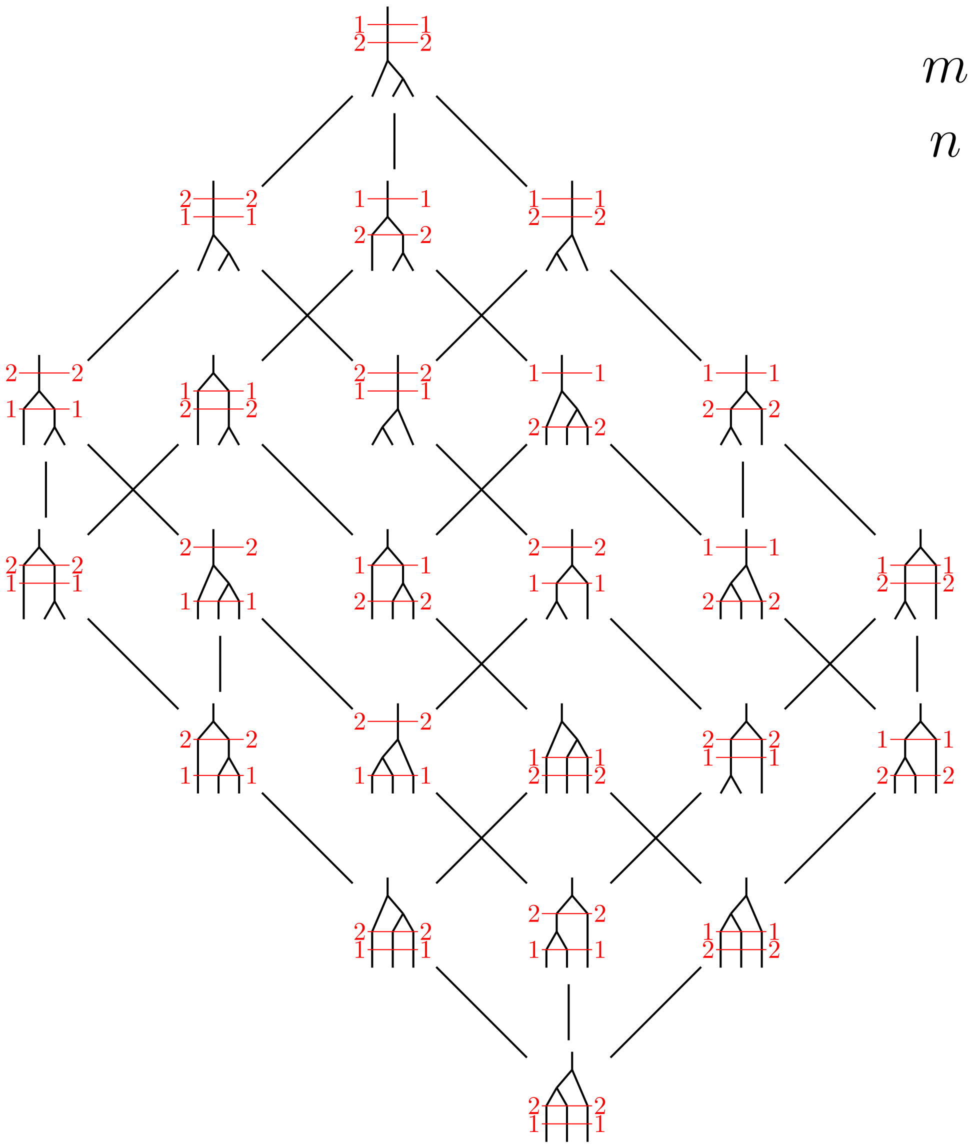
m-painted n-tree = binary tree with n nodes and m levels

Chapoton–P. ('24)

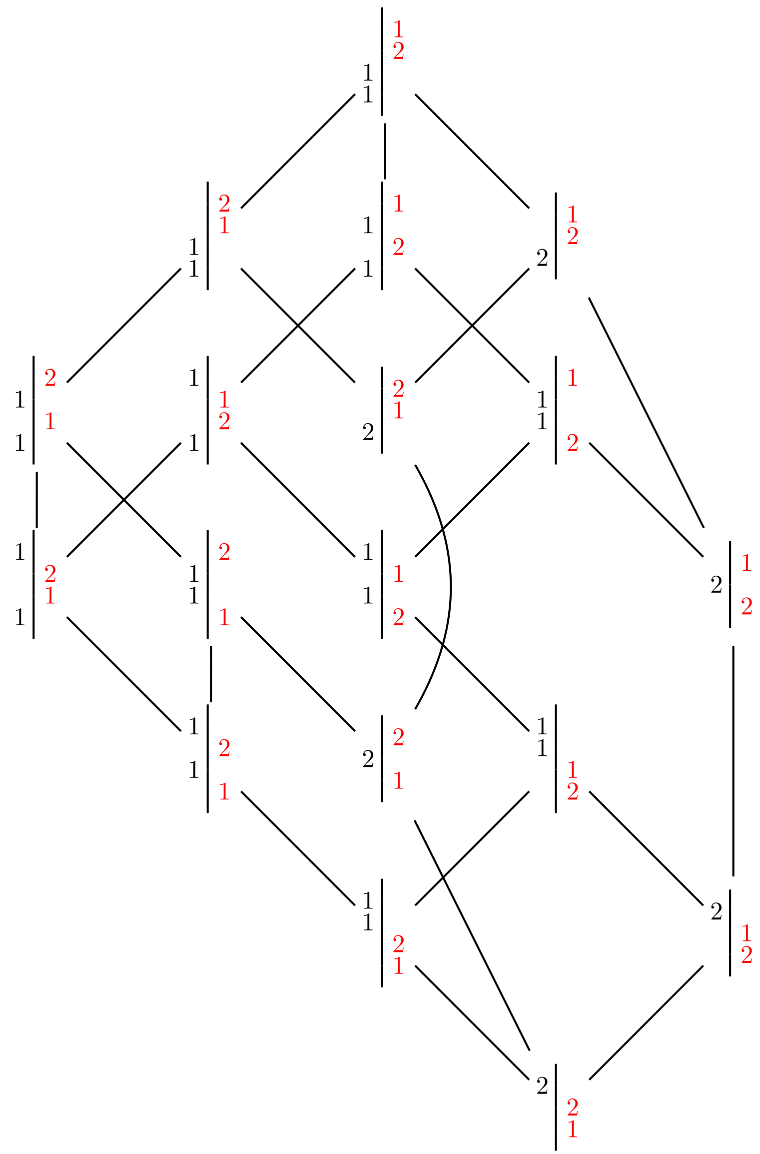
m-lighted n-shade = partition of $[n]$ with m levels

P.–Polyakova ('25)

LATTICES: PAINTED TREES & LIGHTED SHADES



$$m = 2$$
$$n = 2$$



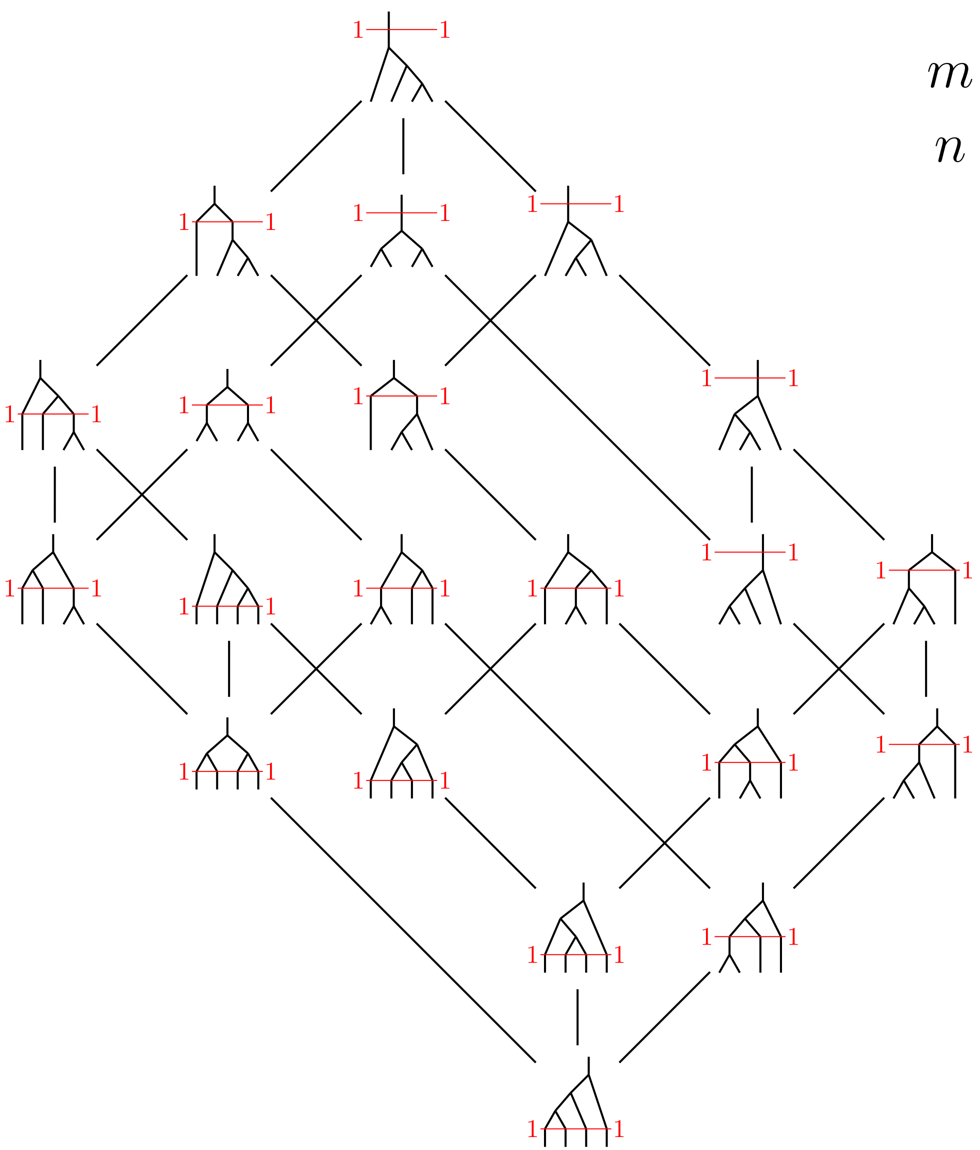
shadow map = arity sequence on the right branch

meet semilattice morphism, but not lattice morphism

LATTICES: PAINTED TREES & LIGHTED SHADES

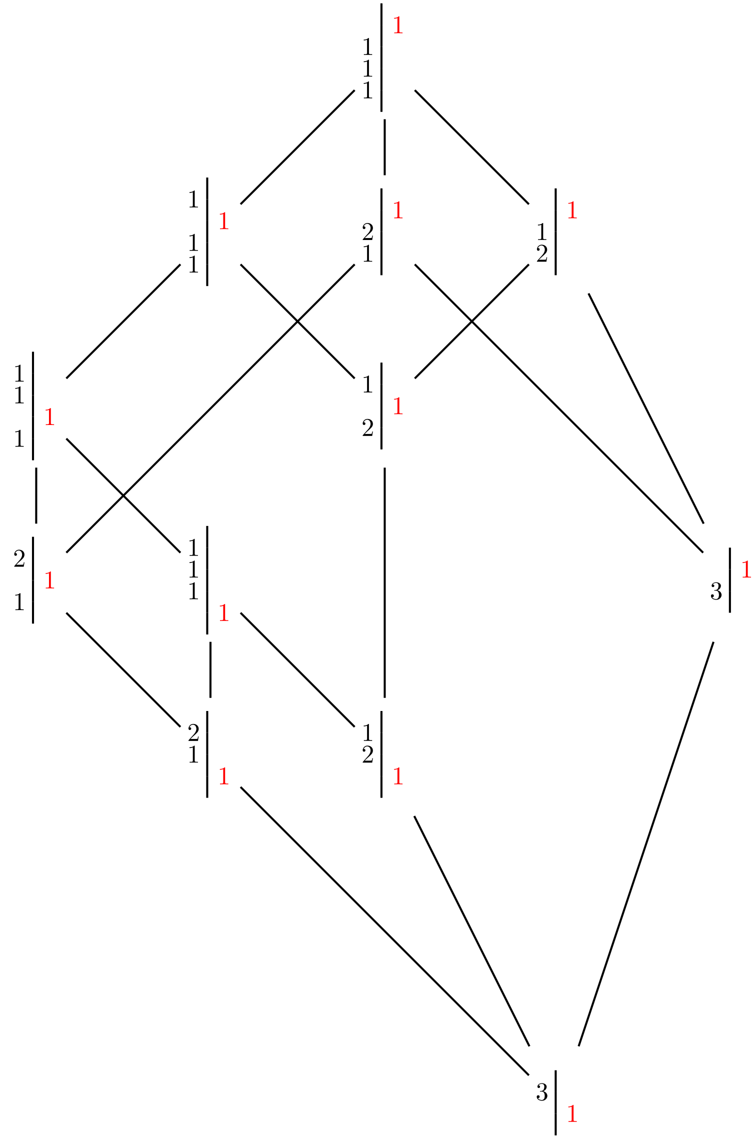
$m = 1$

$n = 3$



multiplihedron lattice

Stasheff ('63) — Forcey ('08) — Ardila–Doker ('13)



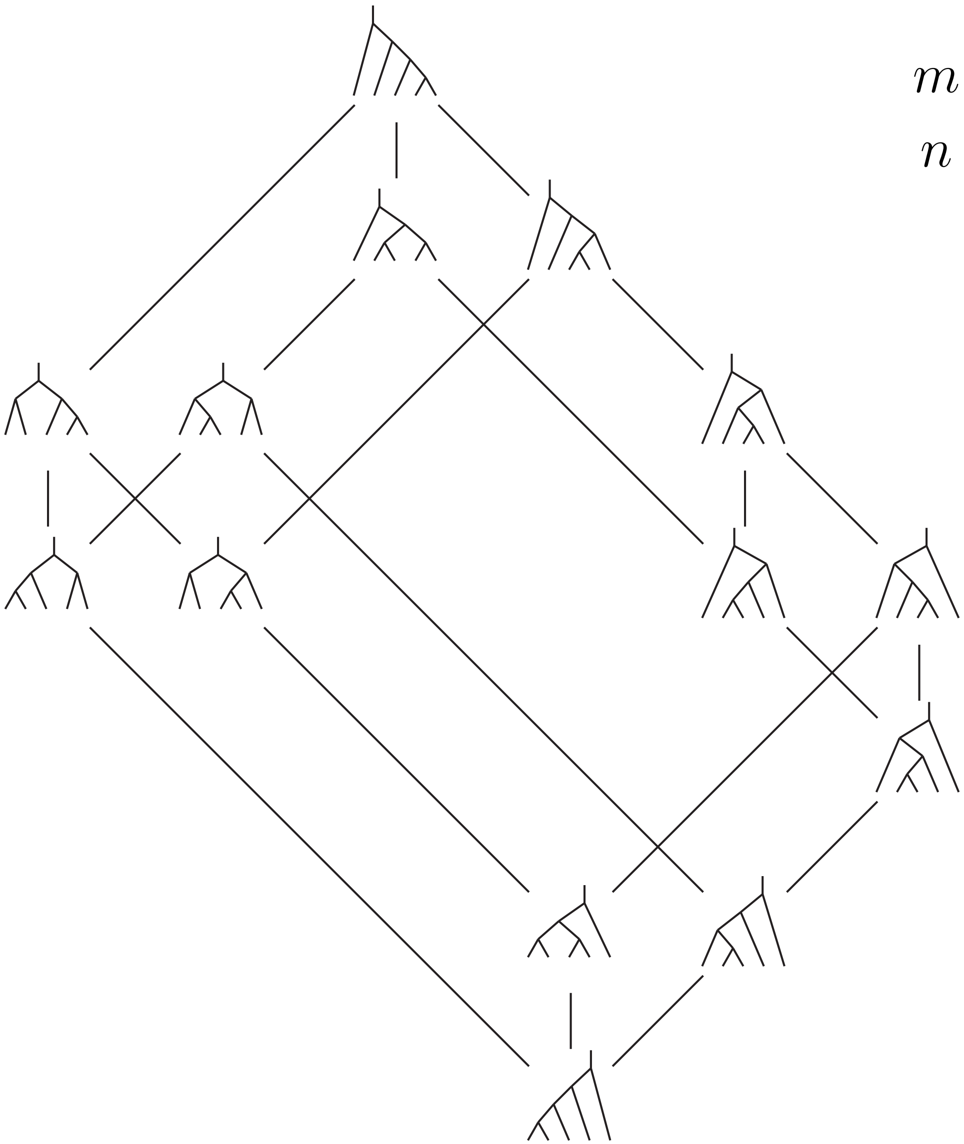
Hochschild lattice

Abad–Crainic–Dherin ('11) — Poliakova ('20$^+$)
Chapoton ('20) — Combe ('21) — Mühle ('22)

LATTICES: PAINTED TREES & LIGHTED SHADES

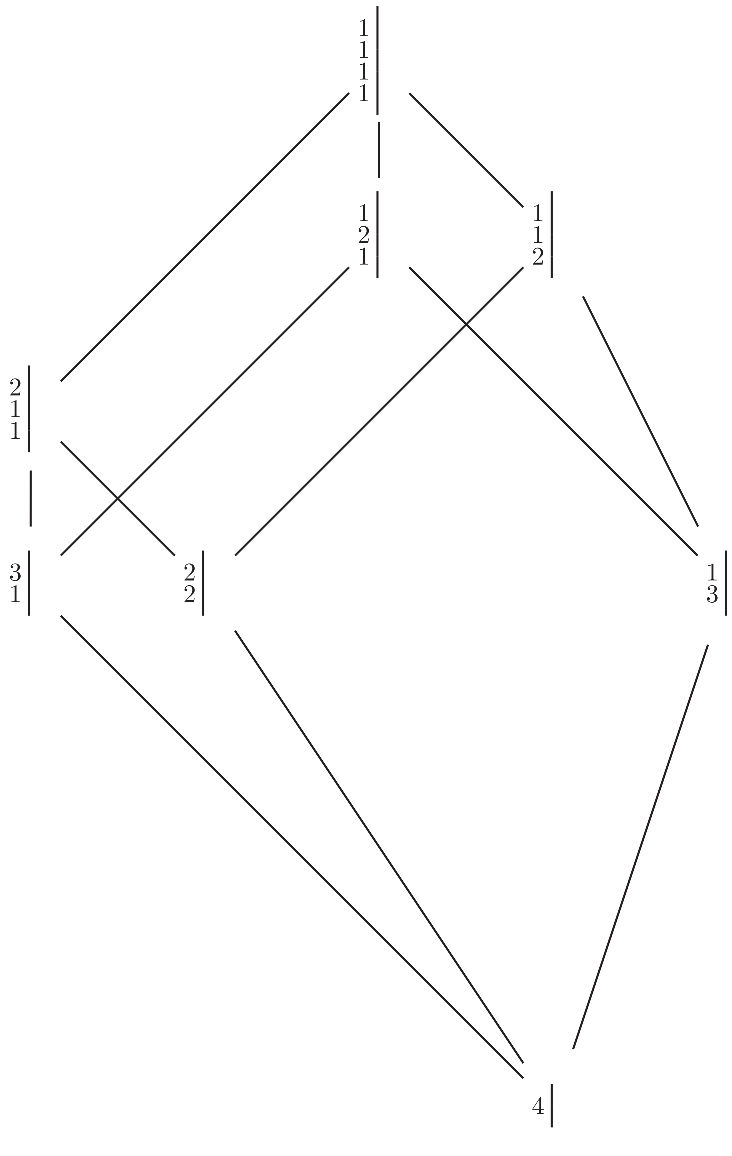
$m = 0$

$n = 4$



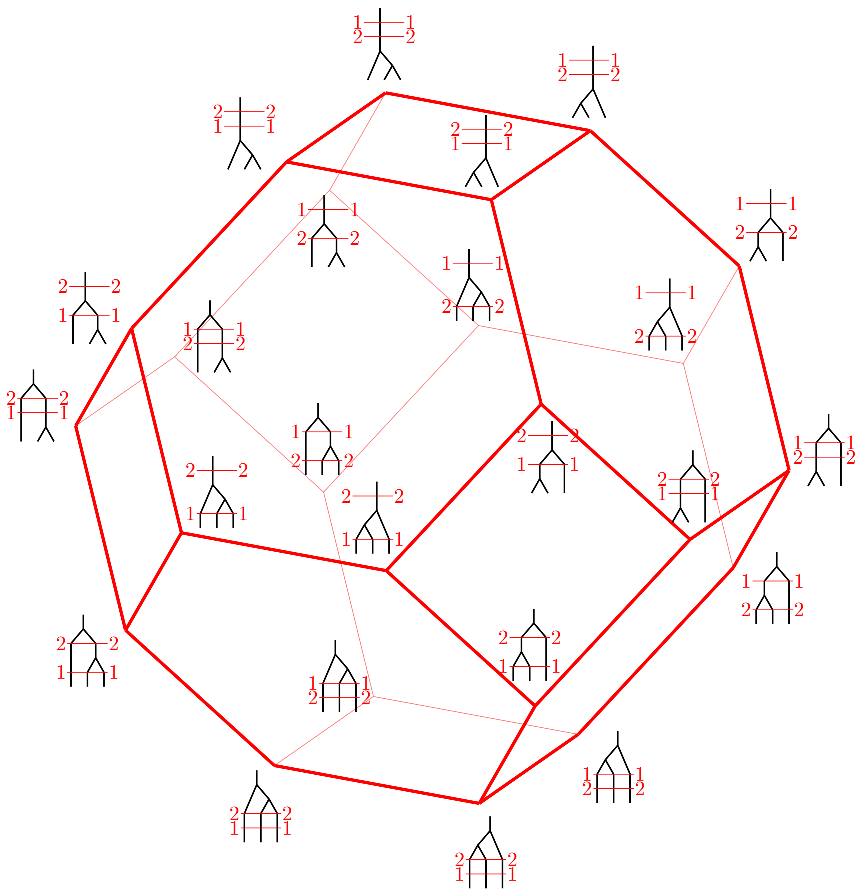
Tamari lattice

Tamari ('51)



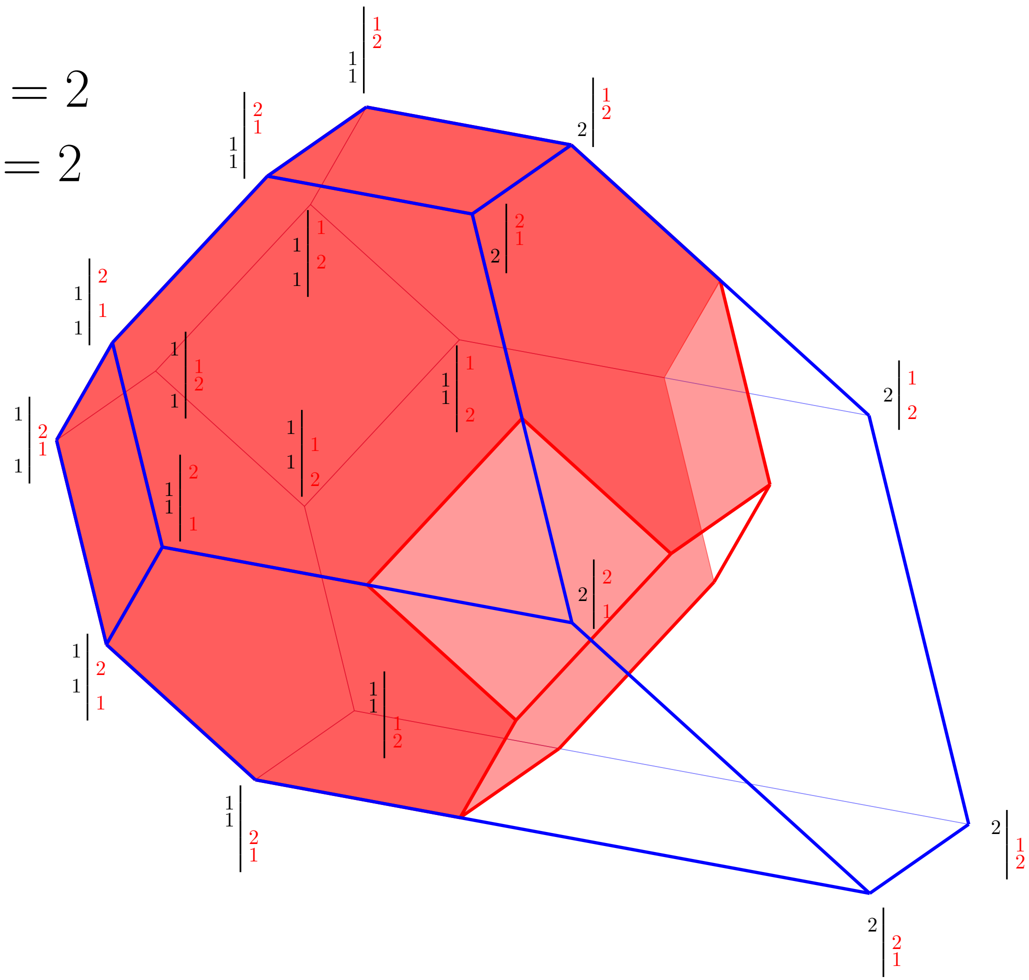
boolean lattice

POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 2$

$n = 2$



(m, n)-multiplihedron

$= $ shuffle of $\mathbb{P}\mathrm{erm}(m)$ and $\mathbb{A}\mathrm{sso}(n)$

$$= \mathbb{P}\mathrm{erm}(m) \times \mathbb{A}\mathrm{sso}(n) + \sum_{i \in [m], j \in [n]} [\boldsymbol{e}_i, \boldsymbol{e}_{m+j}]$$

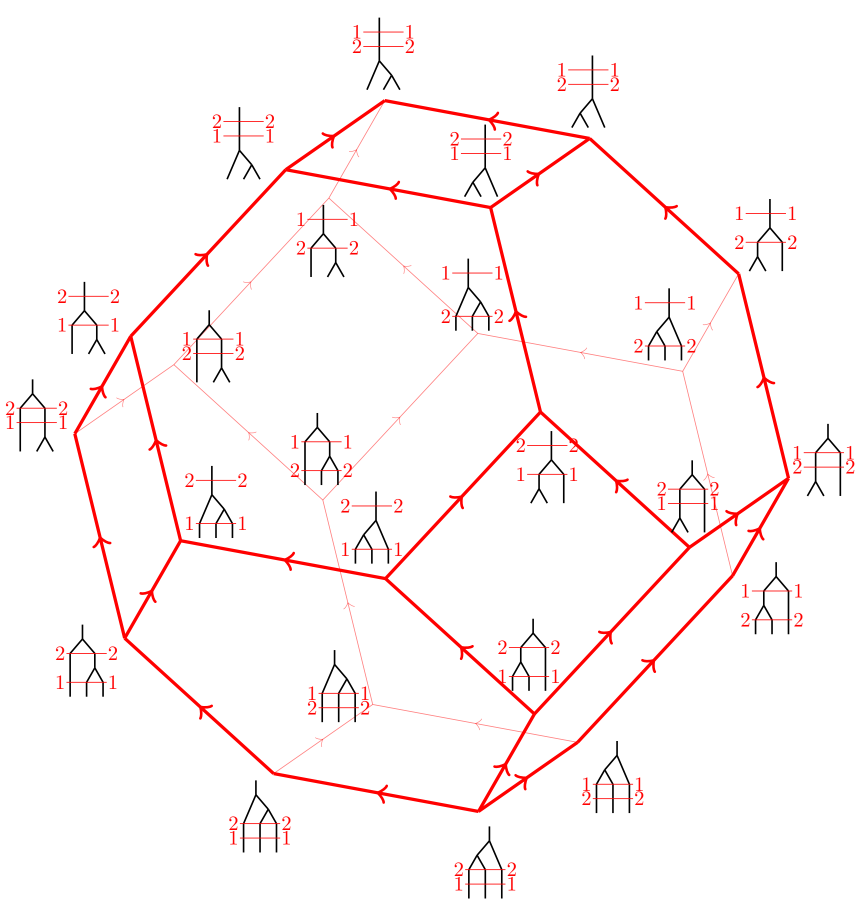
Chapoton–P. ('24)

(m, n)-Hochschild polytope

$= $ removahedron of $\mathbb{M}\mathrm{ul}(m, n)$

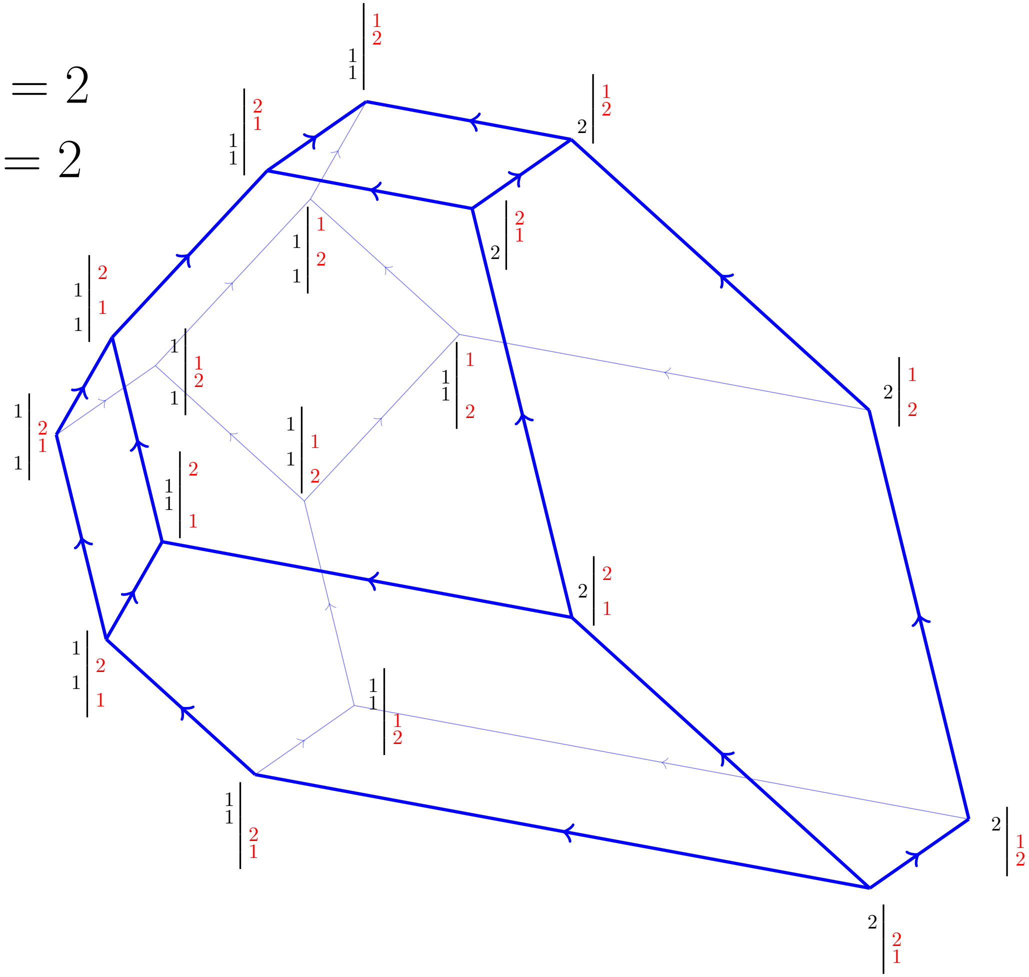
P.–Polyakova ('25)

POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 2$

$n = 2$



(m, n)-multiplihedron

$\implies (m, n)$-multiplihedron lattice

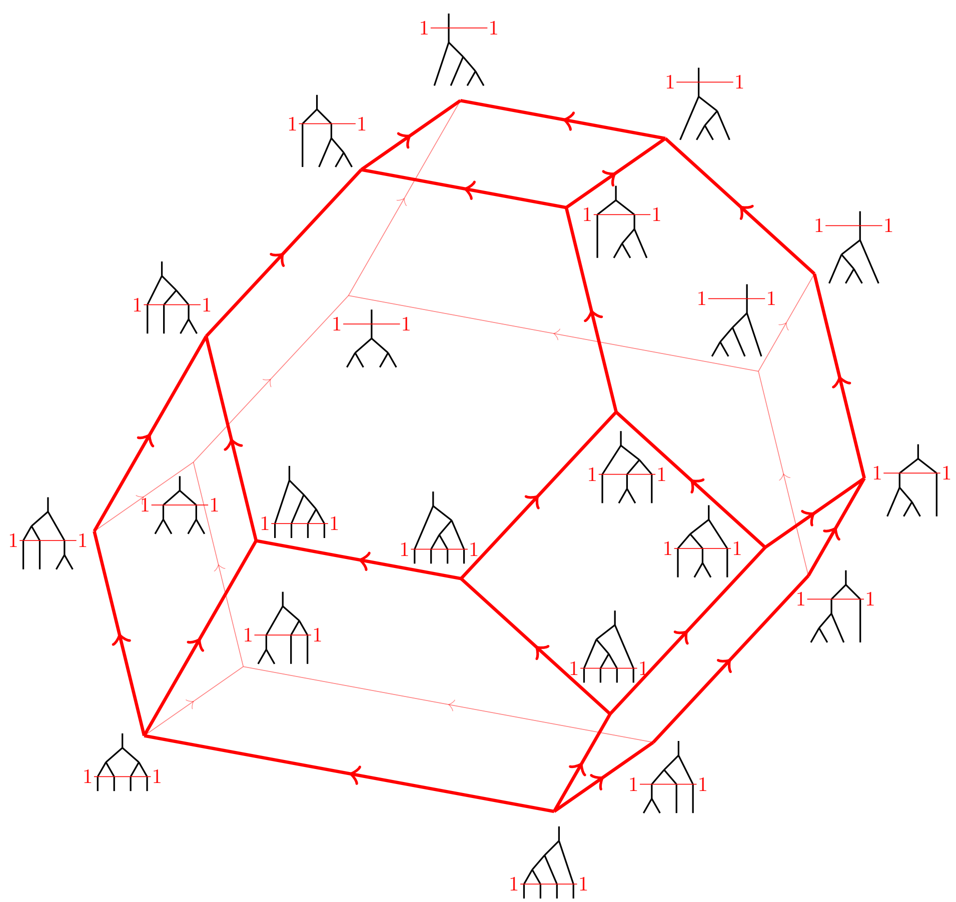
Chapoton–P. ('24)

(m, n)-Hochschild polytope

$\implies (m, n)$-Hochschild lattice

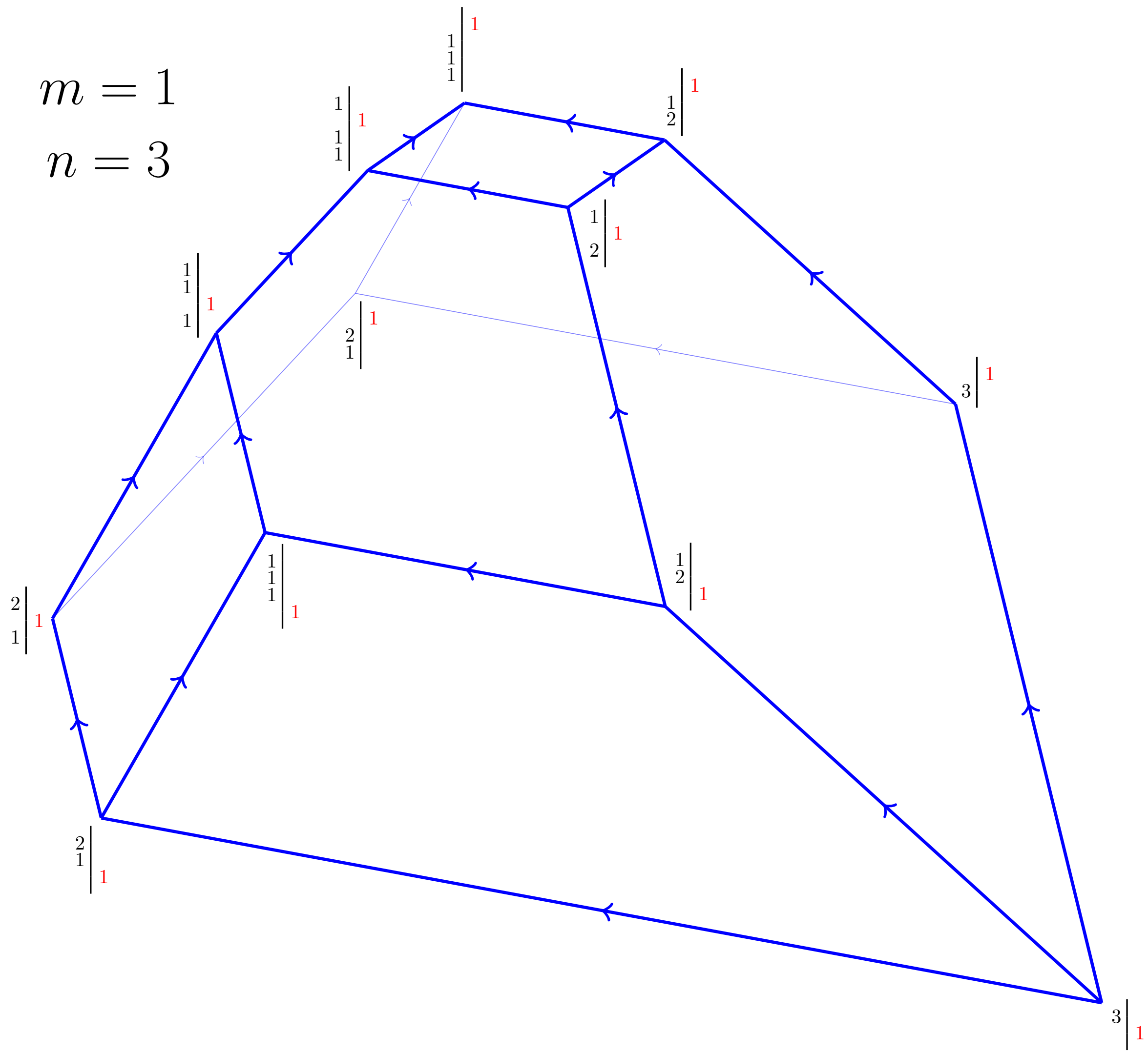
P.–Polyakova ('25)

POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 1$

$n = 3$



(m, n)-multiplihedron

$\Longrightarrow (m, n)$-multiplihedron lattice

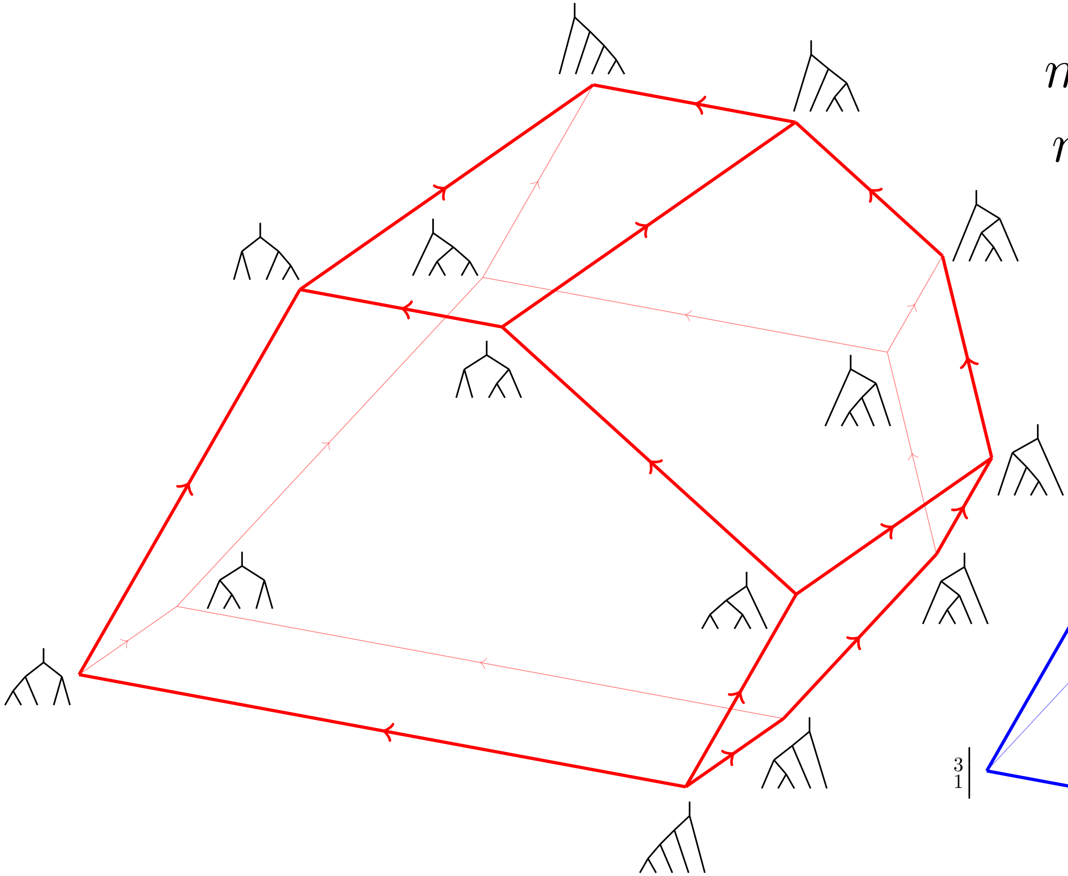
Chapoton–P. ('24)

(m, n)-Hochschild polytope

$\Longrightarrow (m, n)$-Hochschild lattice

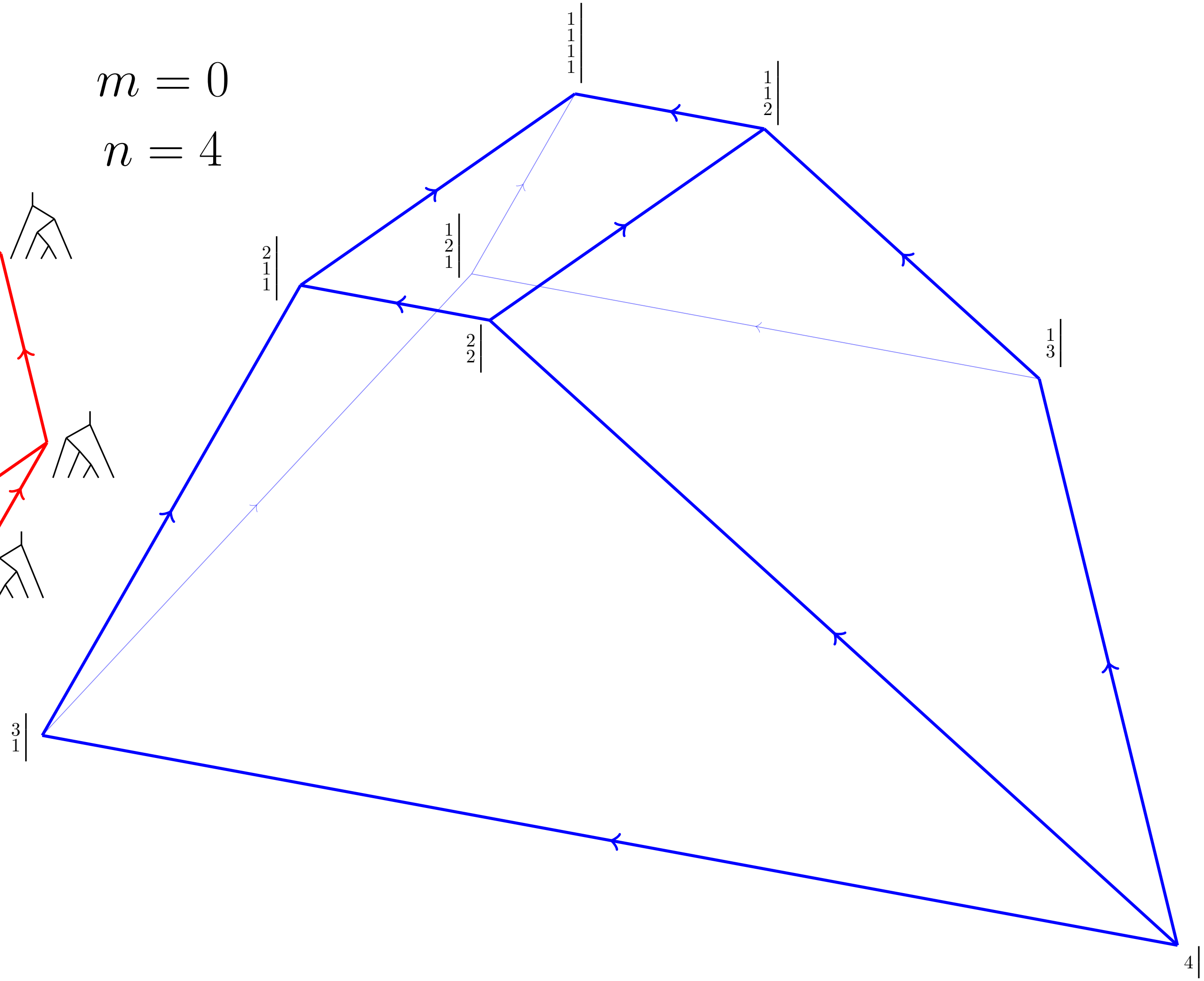
P.–Polyakova ('25)

POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 0$

$n = 4$



(m, n)-multiplihedron

$\Longrightarrow (m, n)$-multiplihedron lattice

Chapoton–P. ('24)

(m, n)-Hochschild polytope

$\Longrightarrow (m, n)$-Hochschild lattice

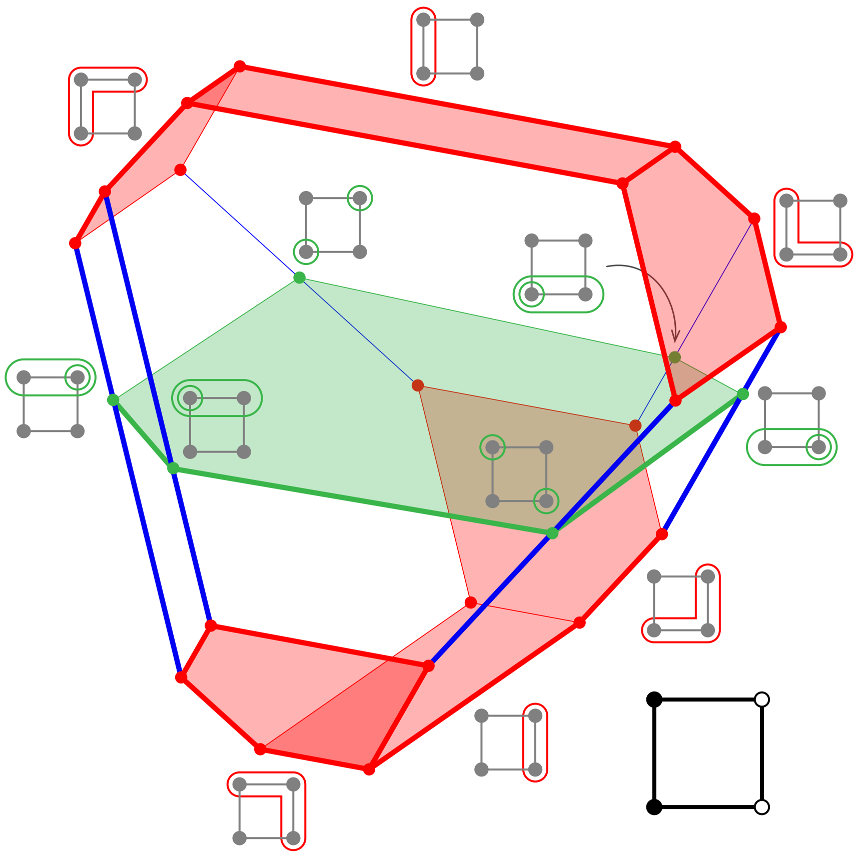
P.–Polyakova ('25)

PROJECTIONS AND SECTIONS OF DEFORMED PERMUTAHEDRA

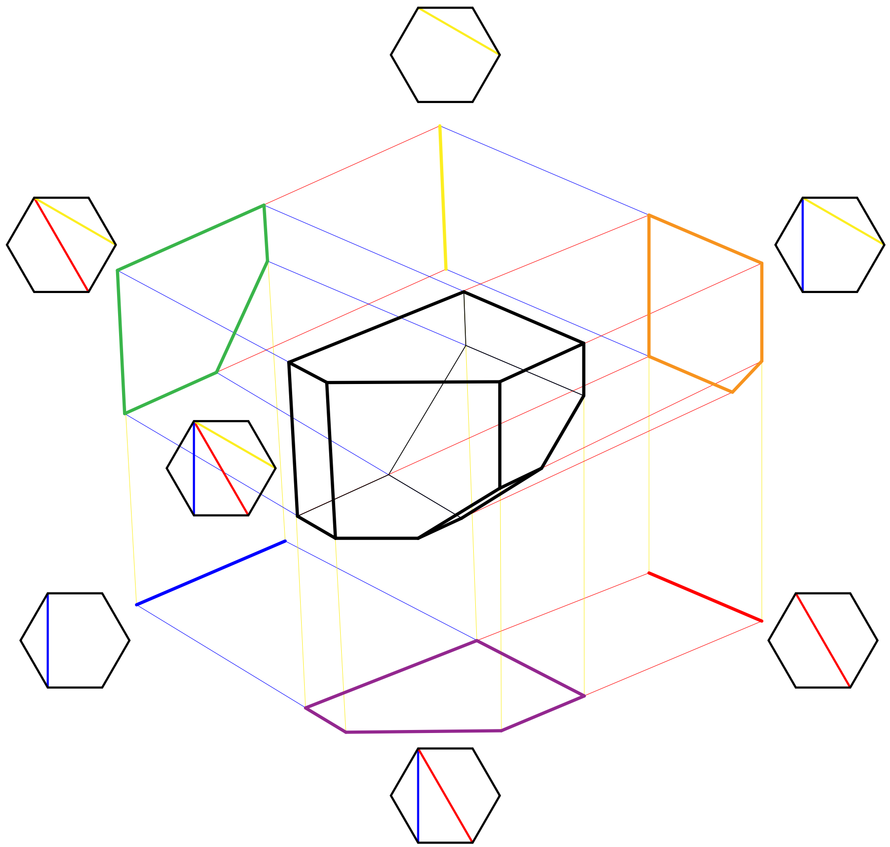
Sometimes, life is unfair...

not all nice combinatorial spheres can be realized as deformed permutahedra
(for instance if some 2-faces are not triangles, squares, pentagons, hexagons...)

then try projections or sections of deformed permutahedra



sections: poset associahedra

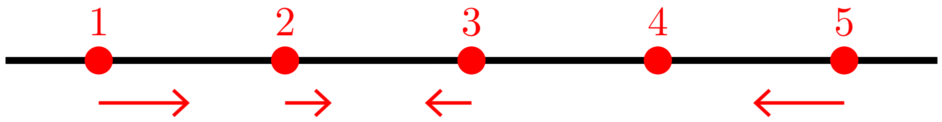


projections: accordiohedra

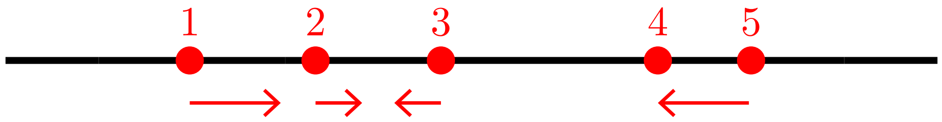
SECTIONS: POSET ASSOCIAHEDRA

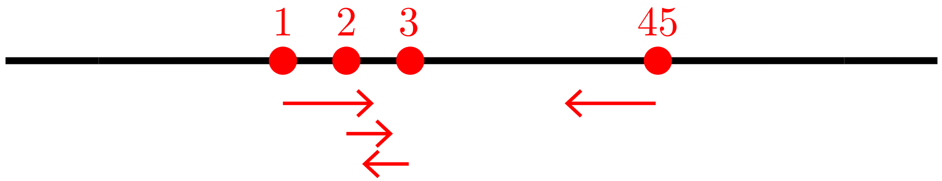
Galashin, *P-associahedra* ('21)
Sack, *A realization of poset associahedra* ('25)
Mantovani–Padrol–P., *Acyclonestohedra* ('23$^+$)

COLLAPSING LINE

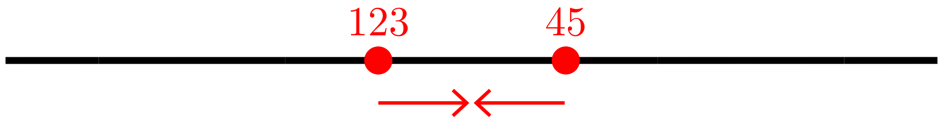


COLLAPSING LINE



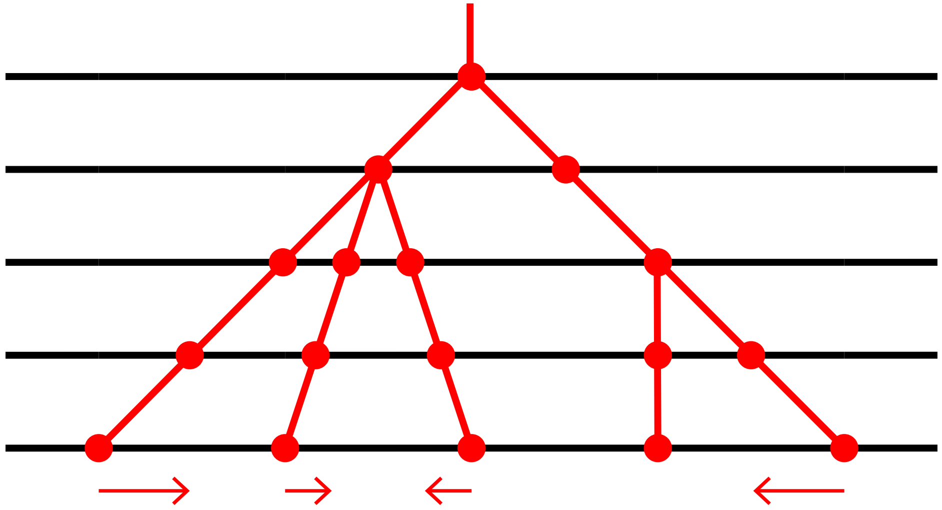

1
2
3
45

COLLAPSING LINE

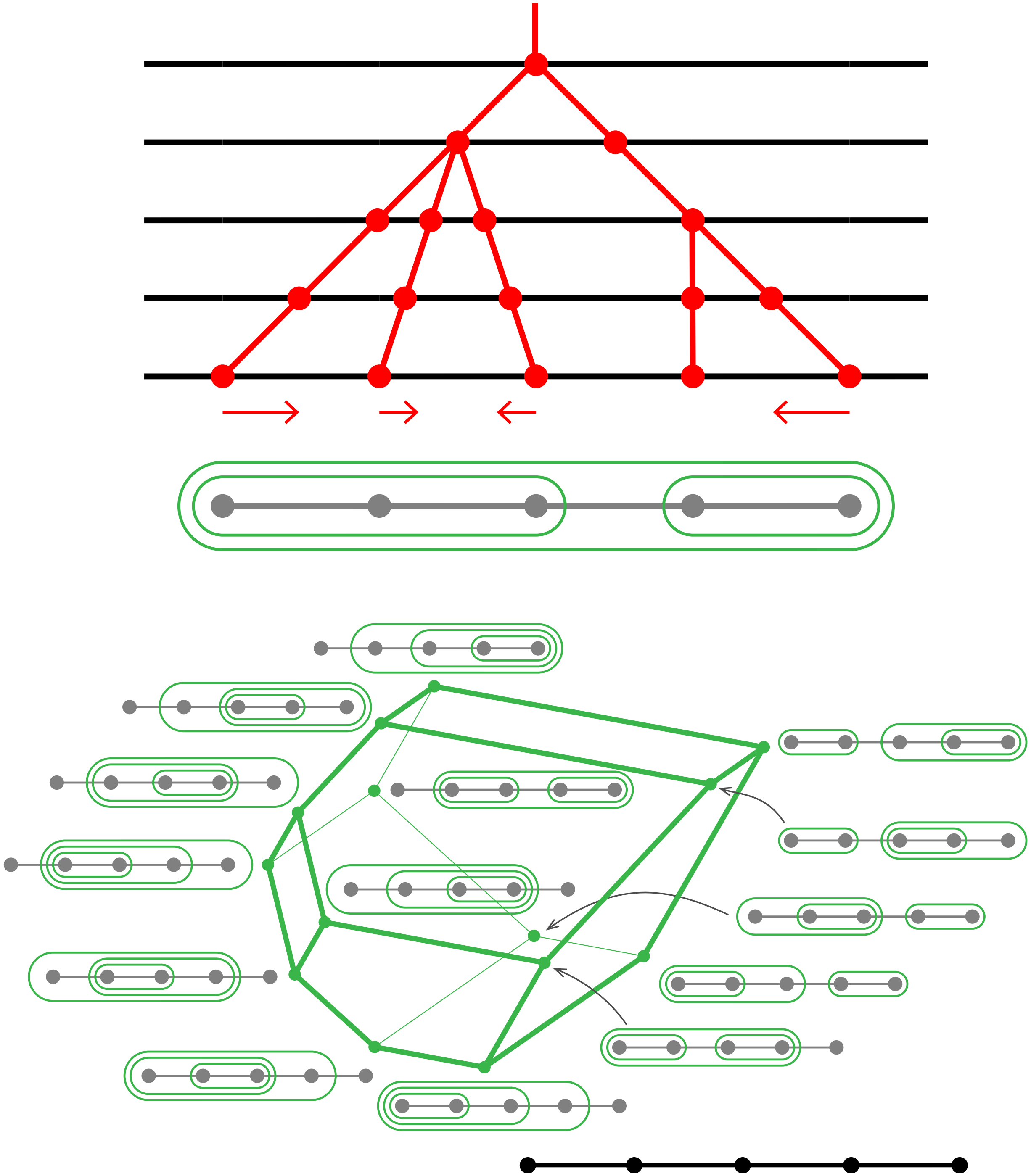


COLLAPSING LINE

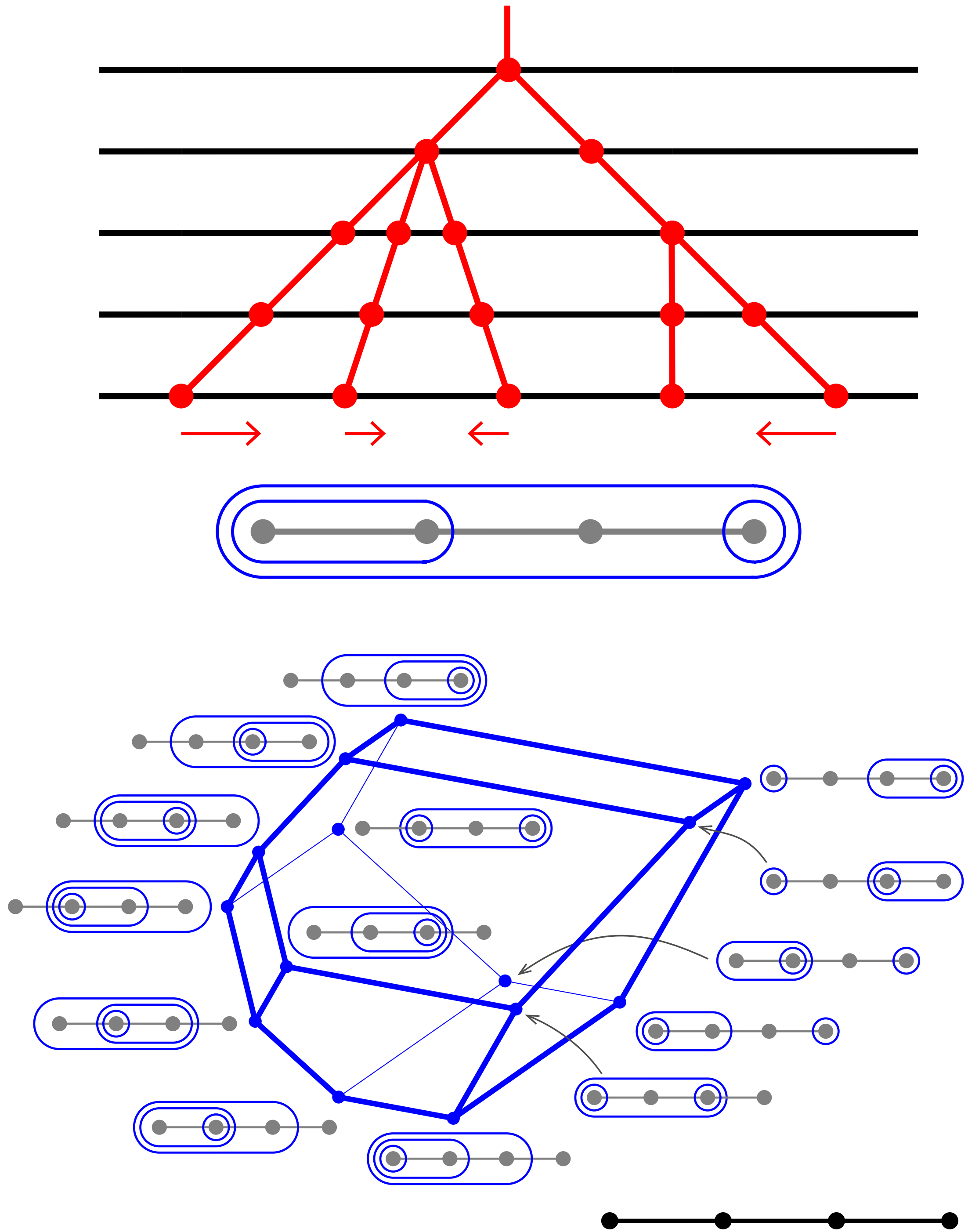
COLLAPSING LINE



PIPINGS



TUBINGS



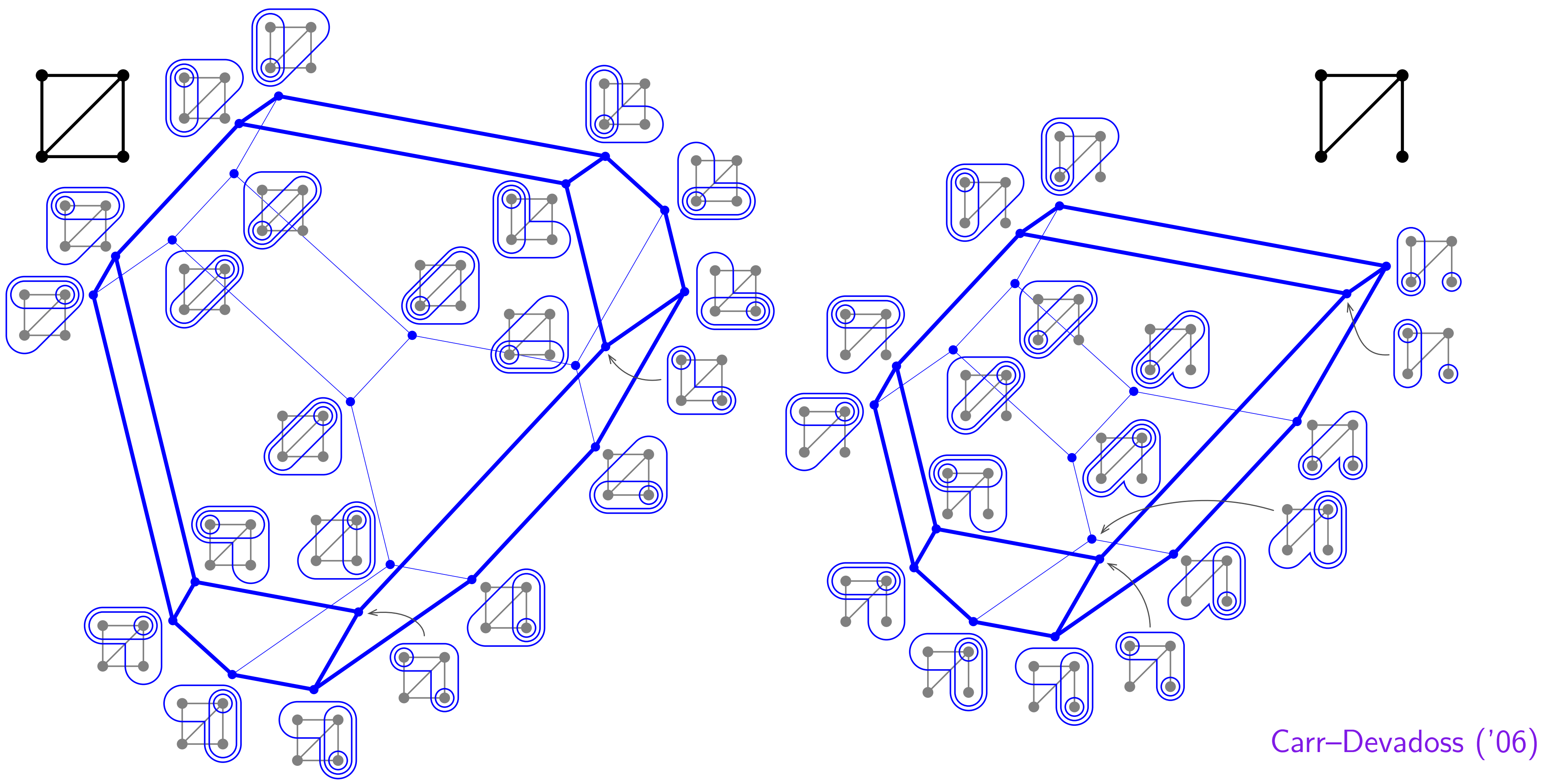
GRAPH ASSOCIAHEDRA

PROP. tubes of G = connected induced subgraphs of G

tubings of G = colletions of tubes pairwise nested or disjoint and non-adjacent

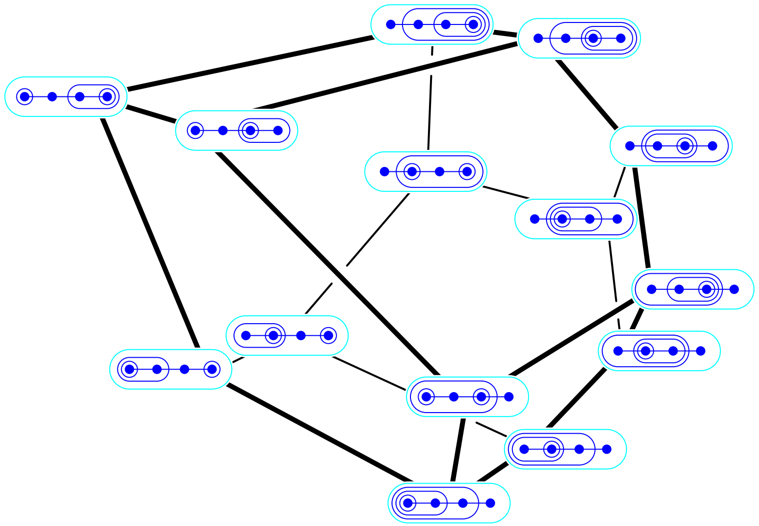
graph associahedron $\mathbb{A}\mathrm{sso}(G) = \sum_{t \text{ tube of } G} \triangle_{\boldsymbol{T}}$

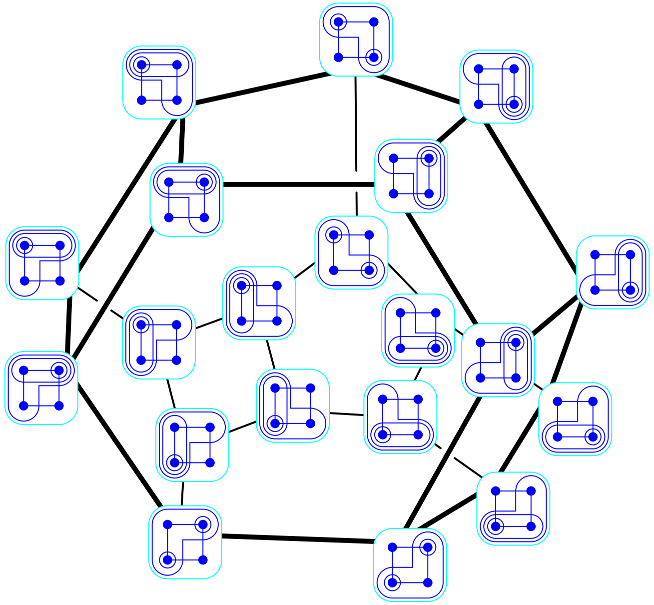
face lattice of $\mathbb{A}\mathrm{sso}(G) \longleftrightarrow$ inclusion lattice on tubings of G

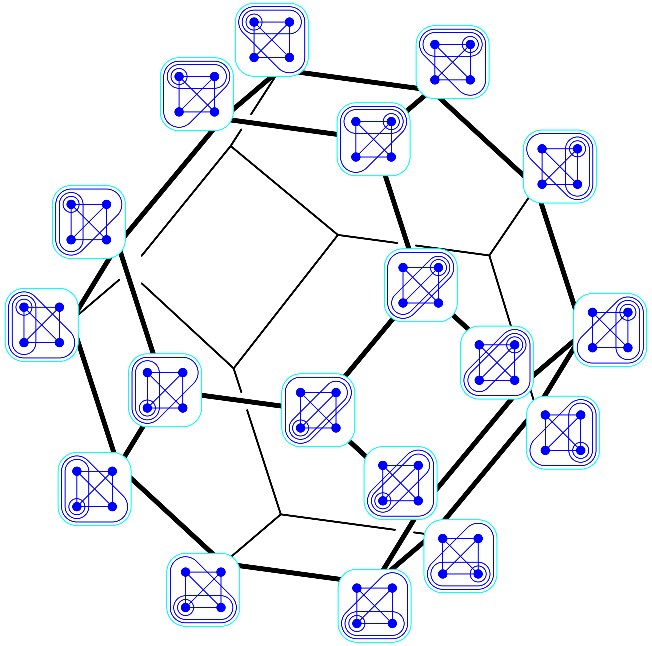


Carr–Devadoss ('06)

SPECIAL GRAPH ASSOCIAHEDRA



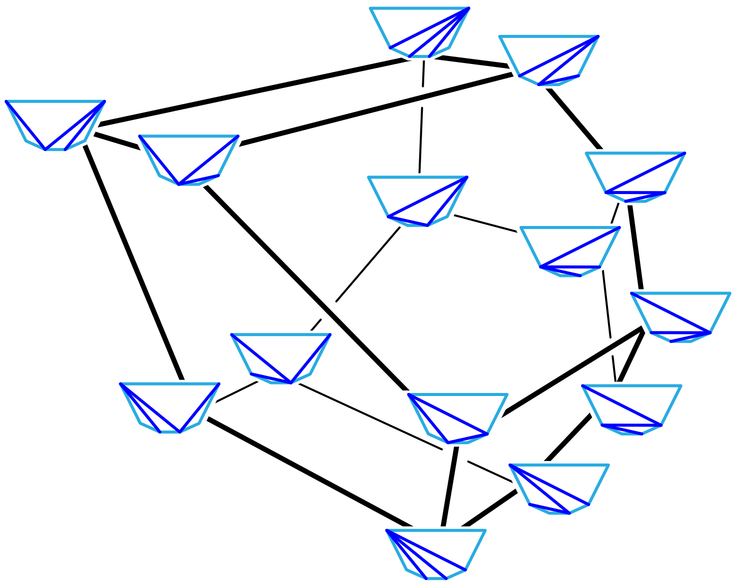


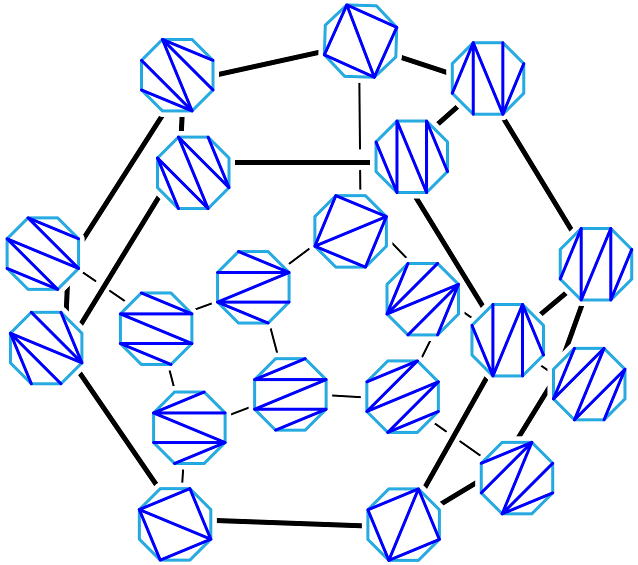


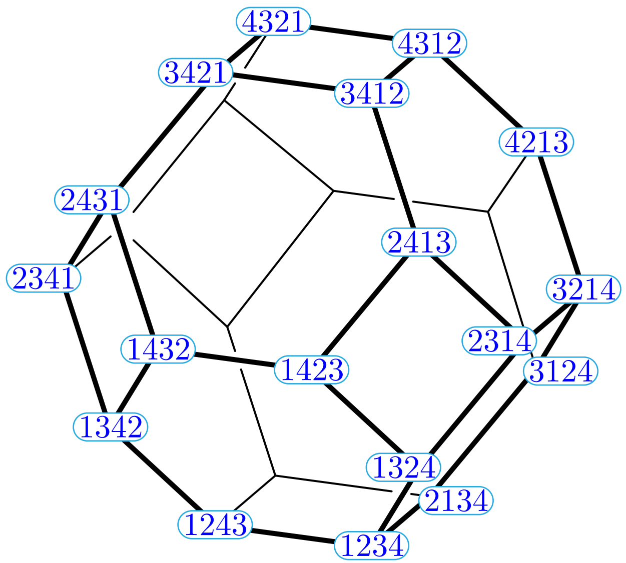
path associahedron
= associahedron

cycle associahedron
= cyclohedron

complete graph associahedron
= permutahedron







COLLAPSING POSET

P poset

$f : P \times [0,1] \to \mathbb{R}$ with $f(p,-)$ continuous, $f(-,t)$ order preserving, and $|f(P,1)| = 1$

Galashin ('21)

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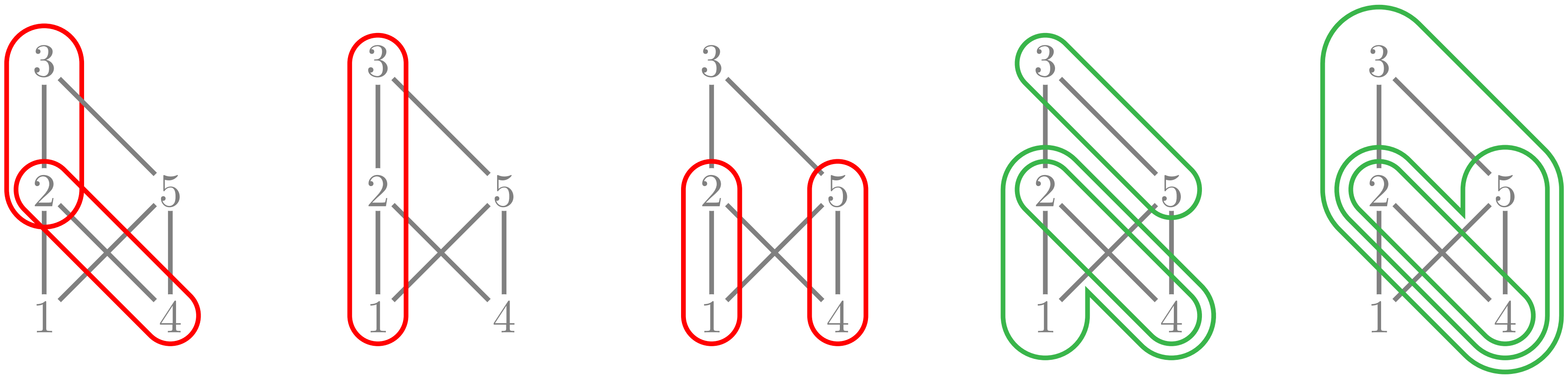
As before, remember collapsing events



PIPING COMPLEX

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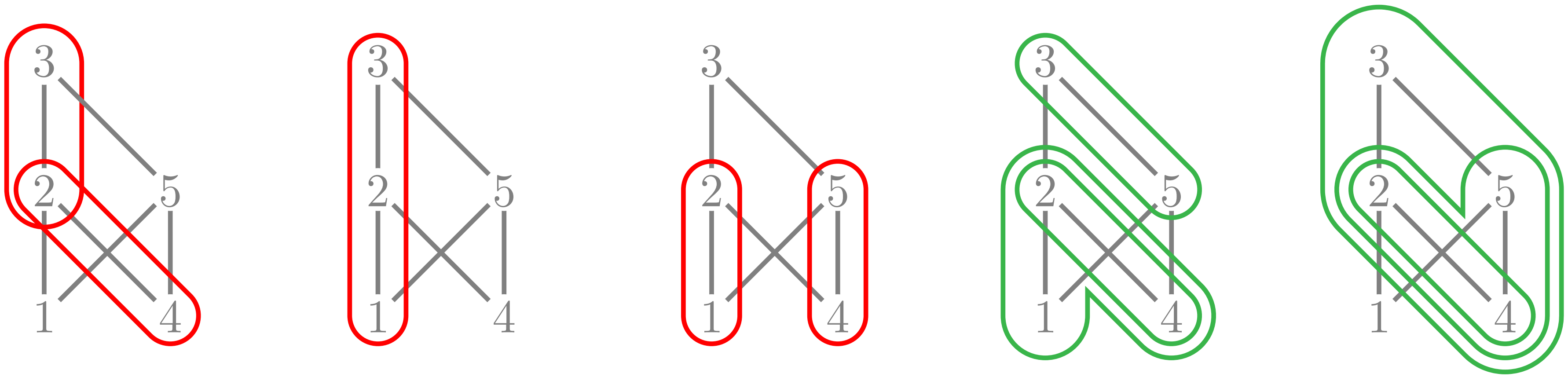

DEF. pipe of P = connected subset of P of size ≥ 2

piping of P = collection Y of pipes of P such that

- pipes are pairwise disjoints or nested
- $P_{/\bigcup X}$ acyclic for any $X \subseteq Y$

piping complex of P = simplicial complex of pipings of P

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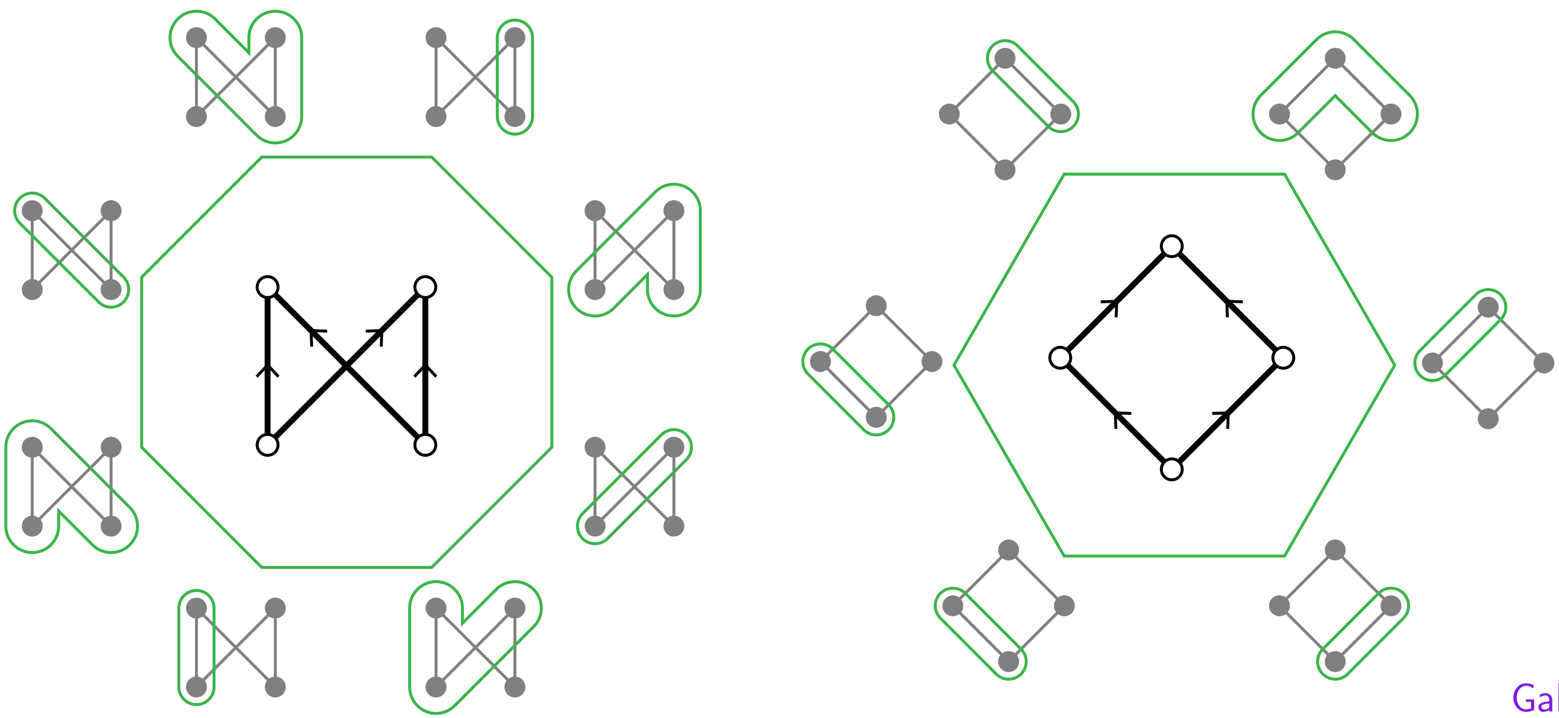

Galashin ('21)

POSET ASSOCIAHEDRON

DEF. pipe of P = connected subset of P of size ≥ 2

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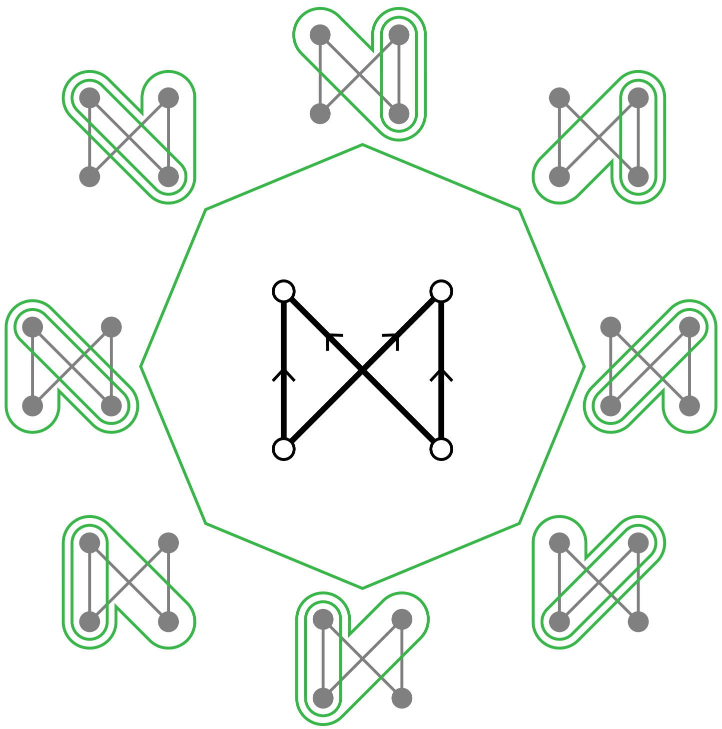
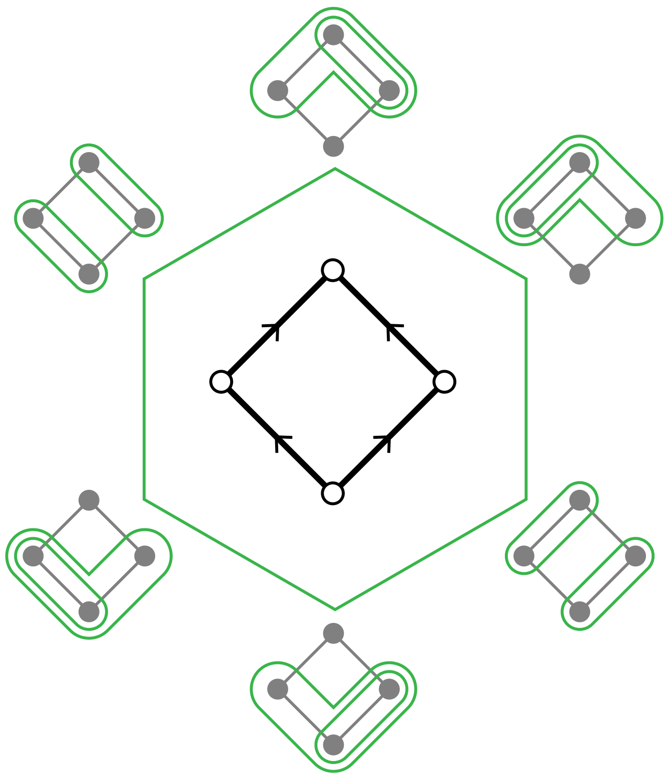
P-associahedron = simple polytope whose polar is the piping complex of P



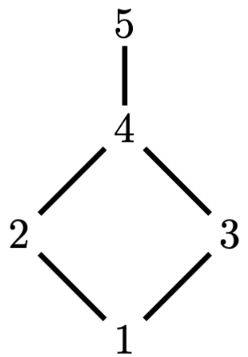


Galashin ('21)

POSET ASSOCIAHEDRON

THM. P-associahedra can be obtained by truncations of the order polytope of P

Galashin ('21)



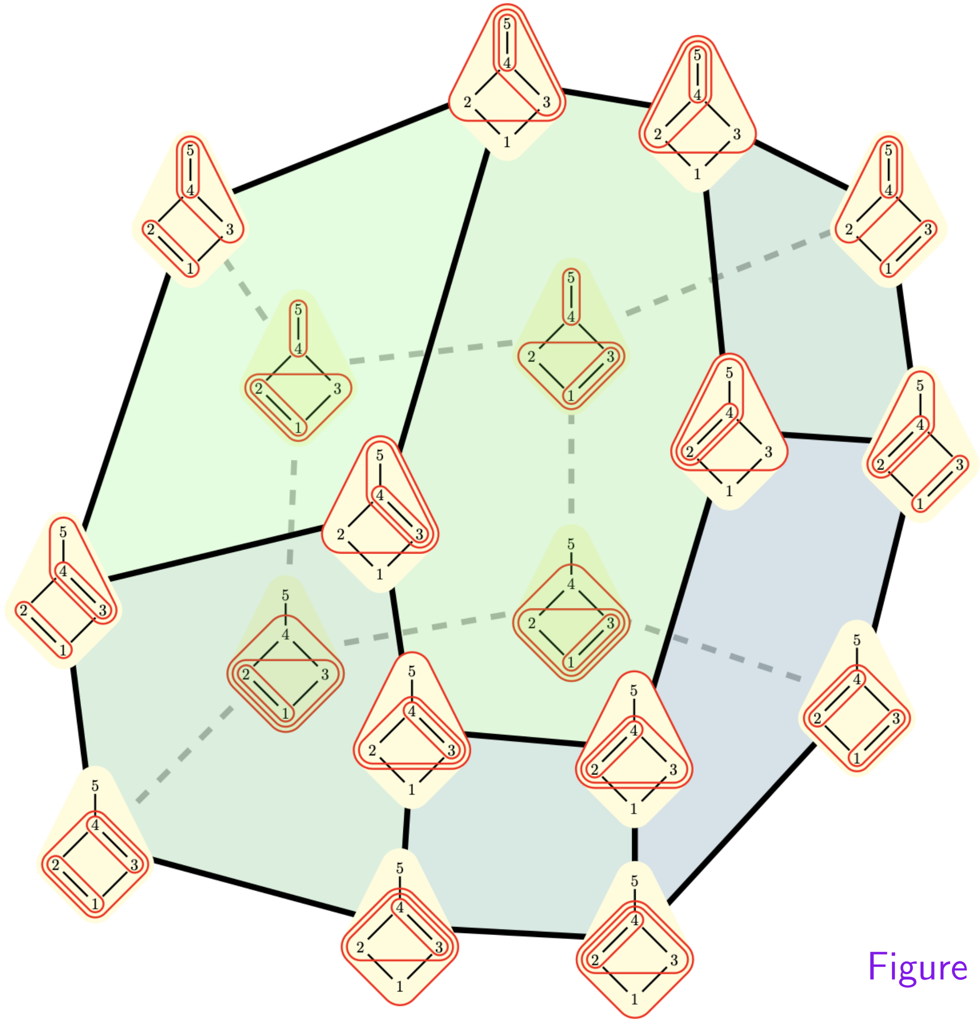


Figure from Galashin ('21)

QU. Find nice realizations

POSET ASSOCIAHEDRON

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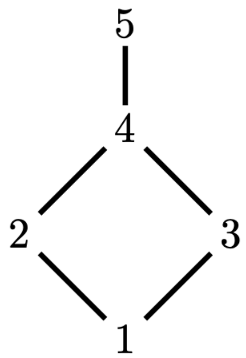
Galashin ('21)



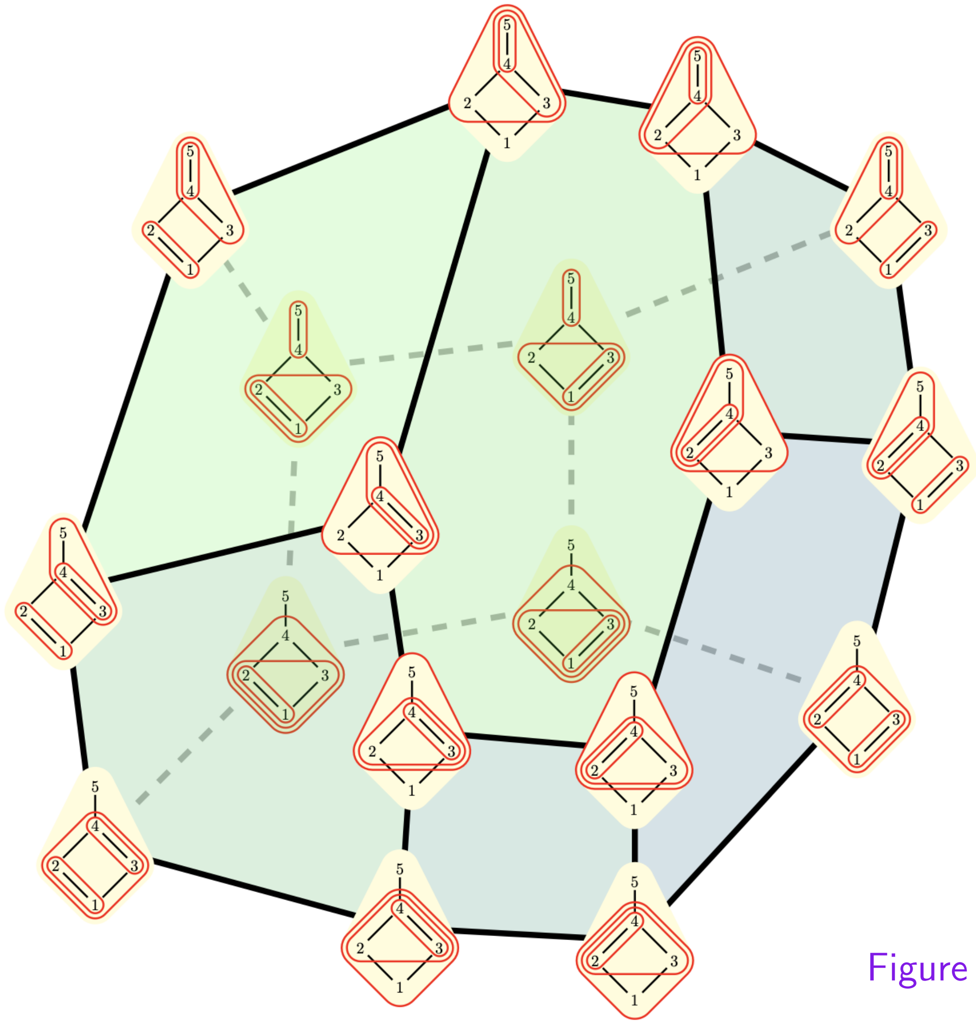
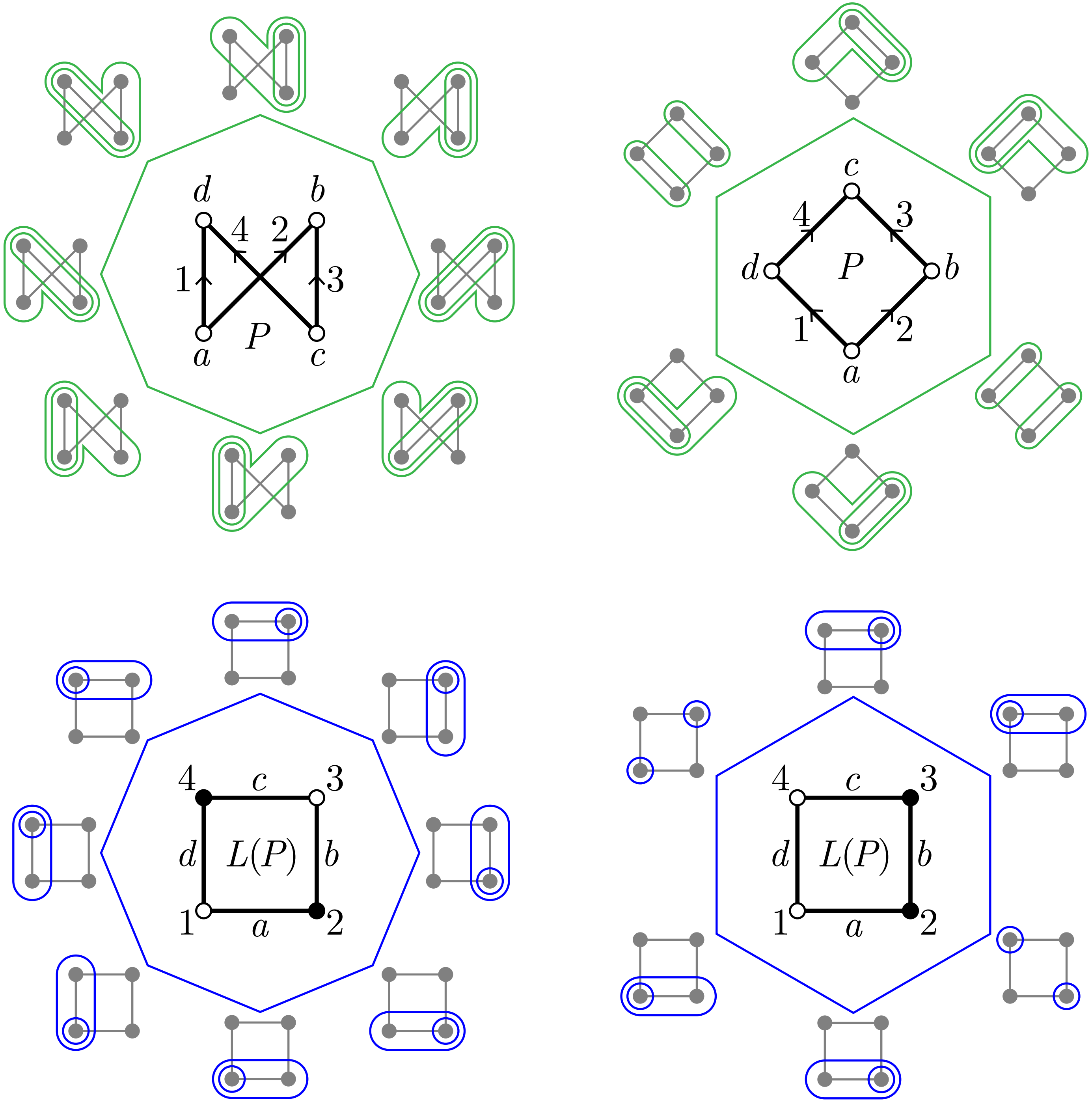


Figure from Galashin ('21)

QU. Find nice realizations

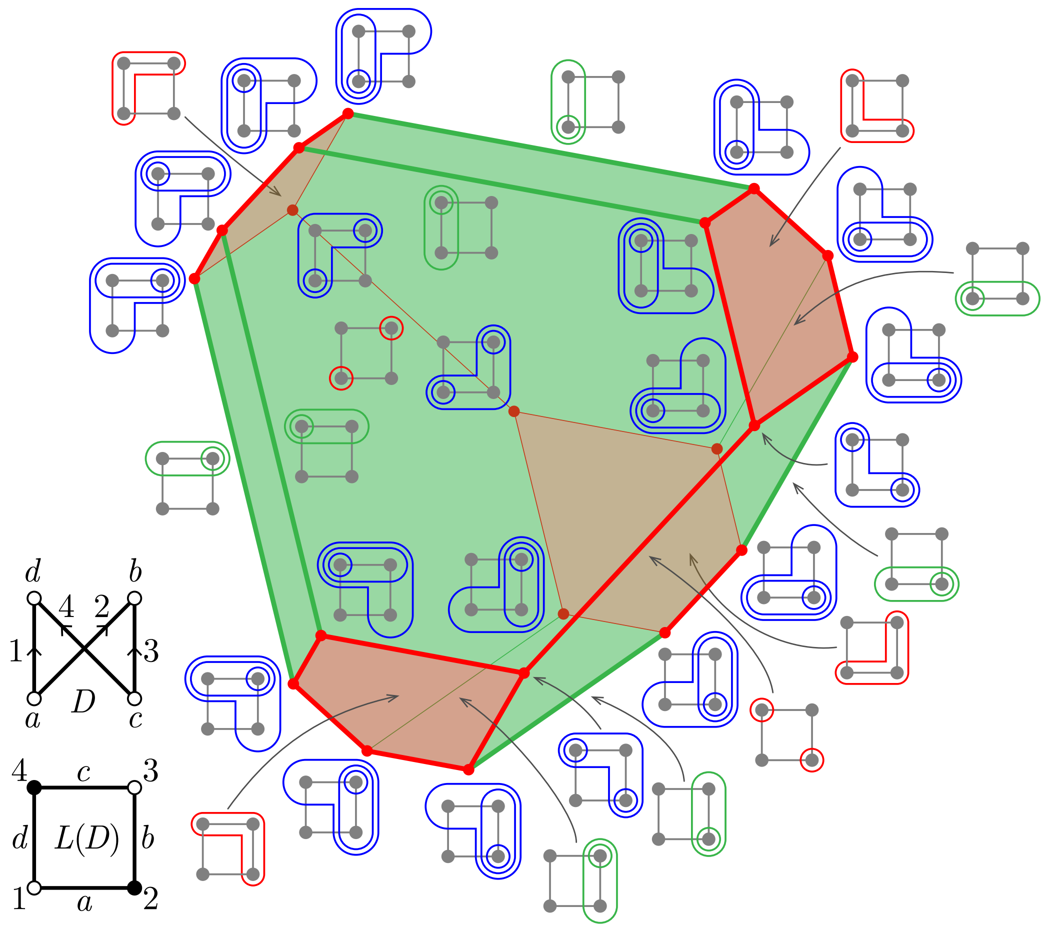
Sack ('23) Mantovani–Padrol–P. ('23$^+$)

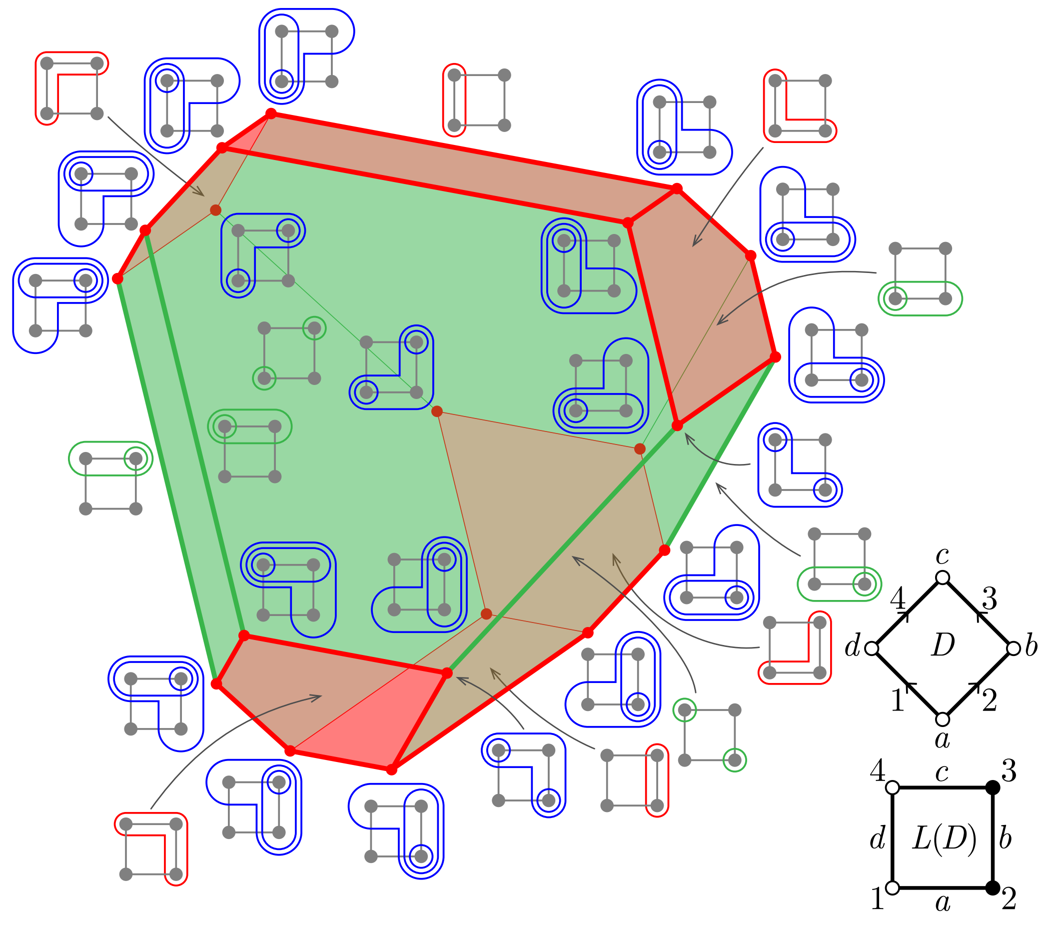
POSET ASSOCIAHEDRON



POSET ASSOCIAHEDRON

OBS. The acyclic part of the nested complex of $L(P)$ is the piping complex of P



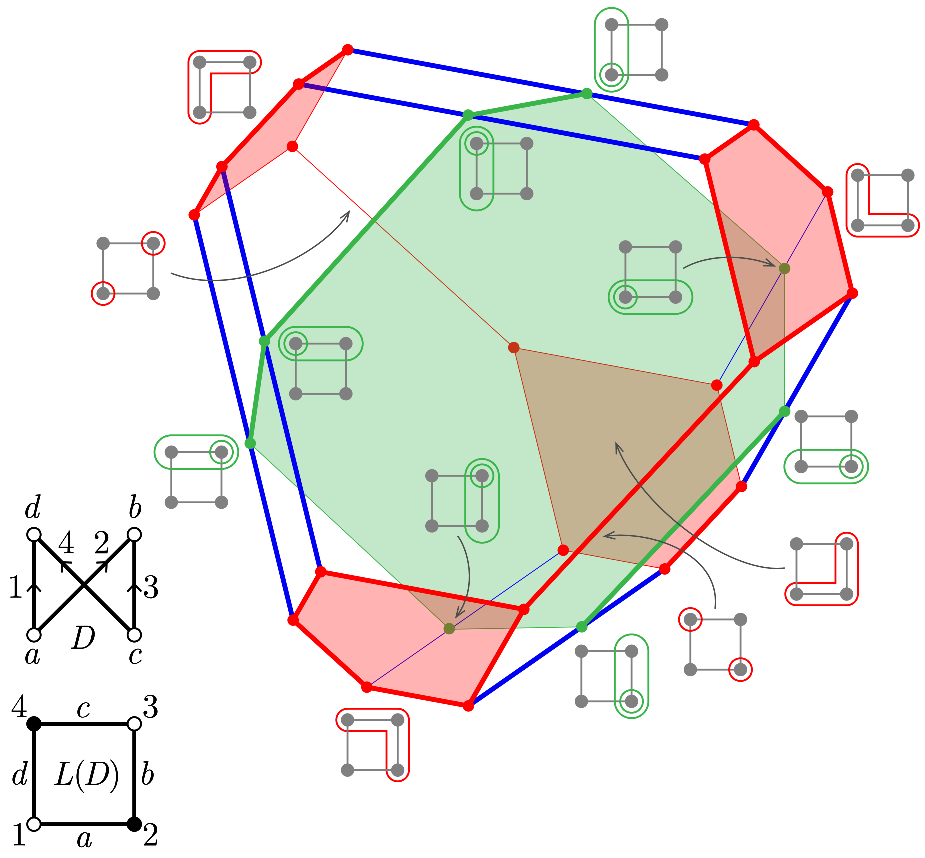


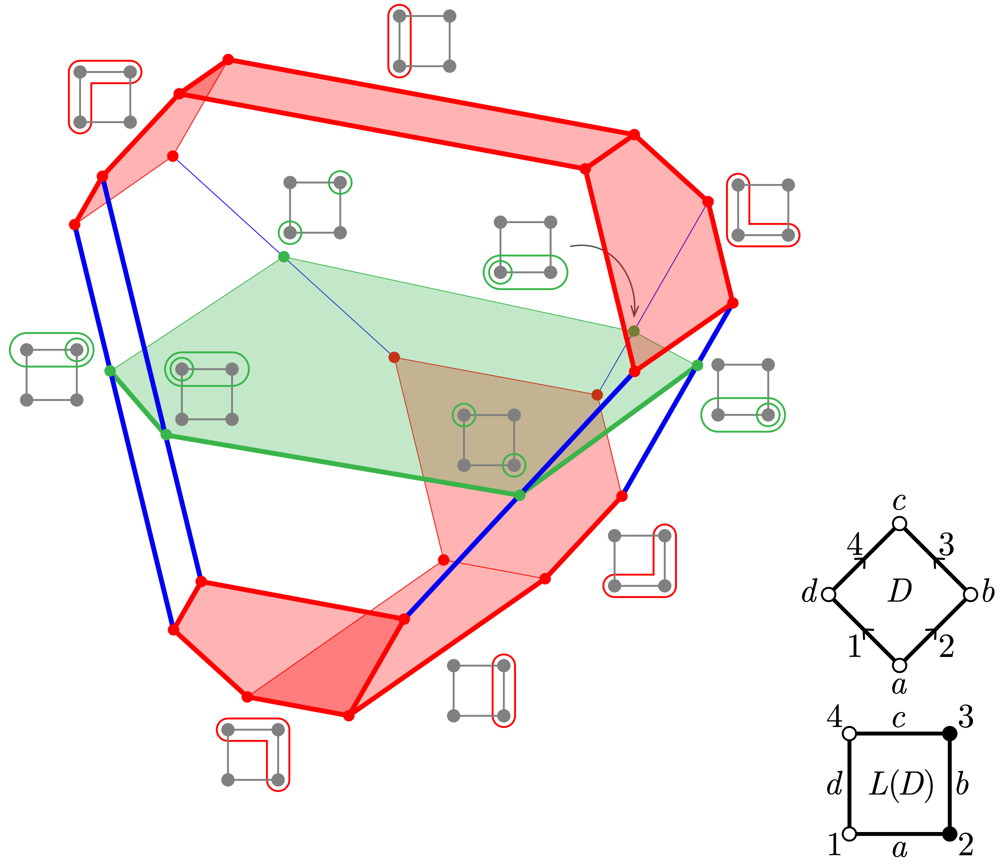
Mantovani–Padrol–P. ('23$^+$)

POSET ASSOCIAHEDRON

OBS. The acyclic part of the nested complex of $L(P)$ is the piping complex of P

THM. A section of an $L(P)$-associahedron is a P-associahedron





Mantovani–Padrol–P. ('23$^+$)

PROJECTIONS: ACCORDIOHEDRA

Garver-McConville, *Oriented flip graphs and noncrossing tree partitions* ('16$^+$)
Manneville–P., *Geometric realizations of the accordion complex* ('19)

$\color{red}{D_\circ}$-ACCORDION COMPLEX

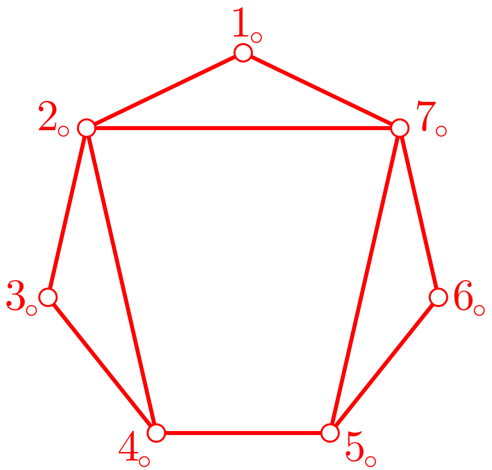
$2n$ points of the unit circle labeled counterclockwise by $1_\circ, 1_\bullet, 2_\circ, 2_\bullet, \dots, n_\circ, n_\bullet$

Fix a dissection $D_\circ$ of the red hollow polygon

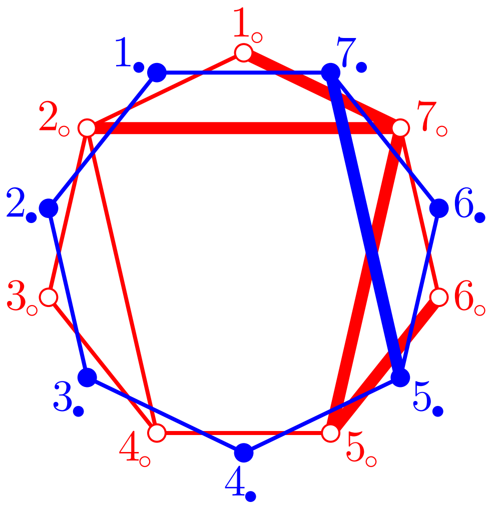
$D_\circ$-accordion diagonal = diagonal of the blue solid polygon that crosses an accordion of $D_\circ$

$D_\circ$-accordion dissection = set of non-crossing $D_\circ$-accordion diagonals

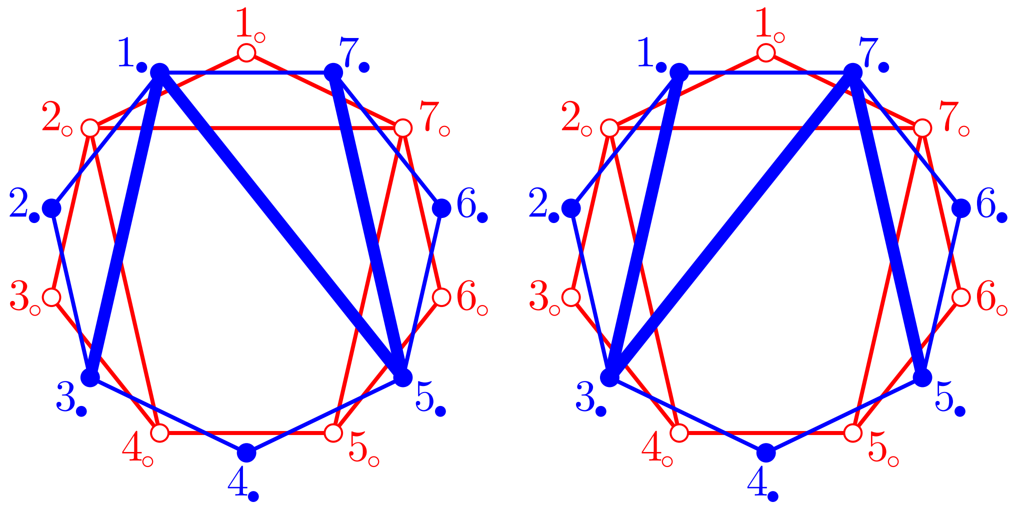
$D_\circ$-accordion complex = simplicial complex of $D_\circ$-accordion dissections



dissection $D_\circ$

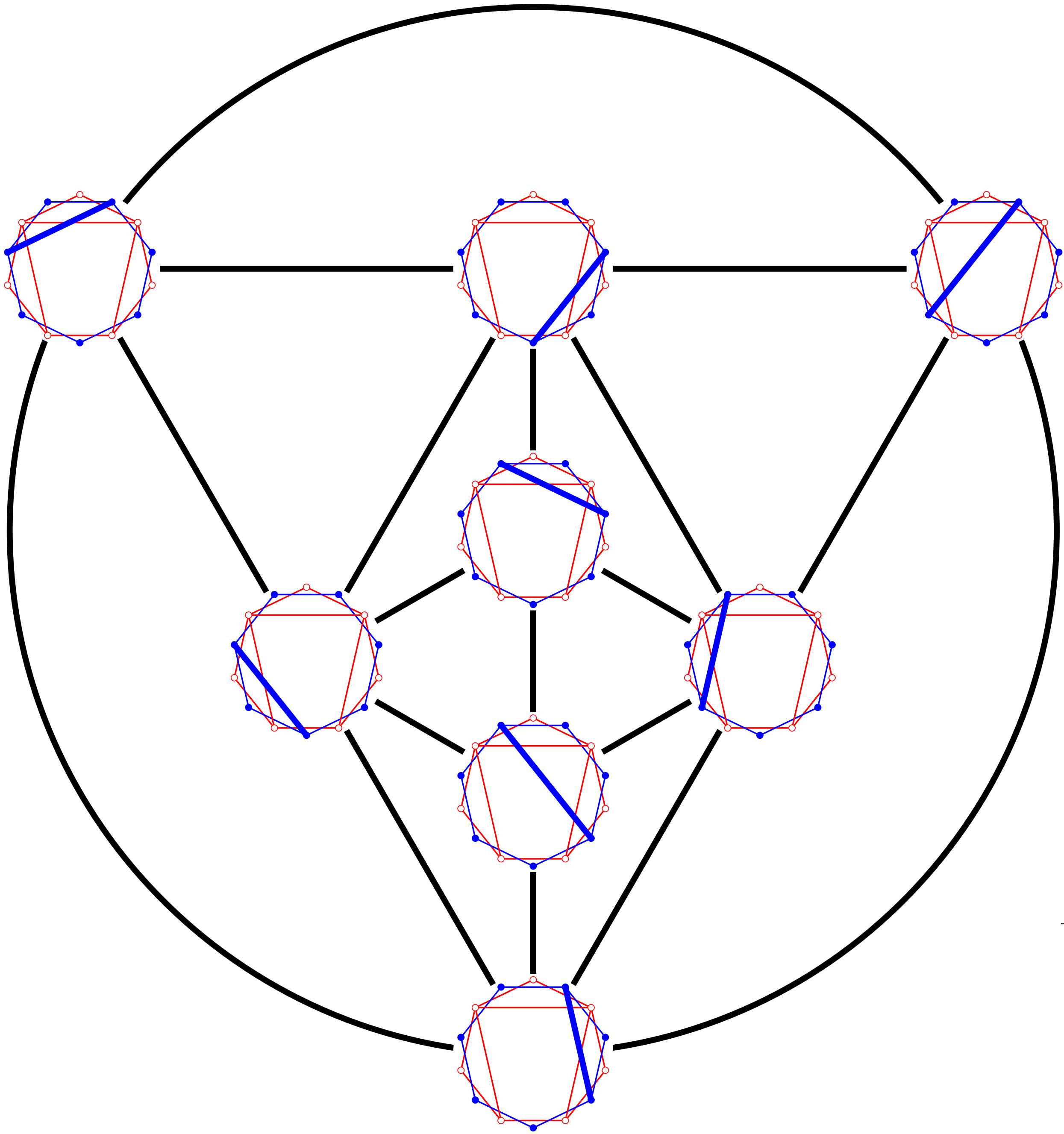


$D_\circ$-accordion diagonal



two maximal $D_\circ$-accordion dissections

$\mathrm{D}_\circ$-ACCORDION COMPLEX



$\mathrm{D}_\circ$-accordion complex = simplicial complex of $\mathrm{D}_\circ$-accordion dissections

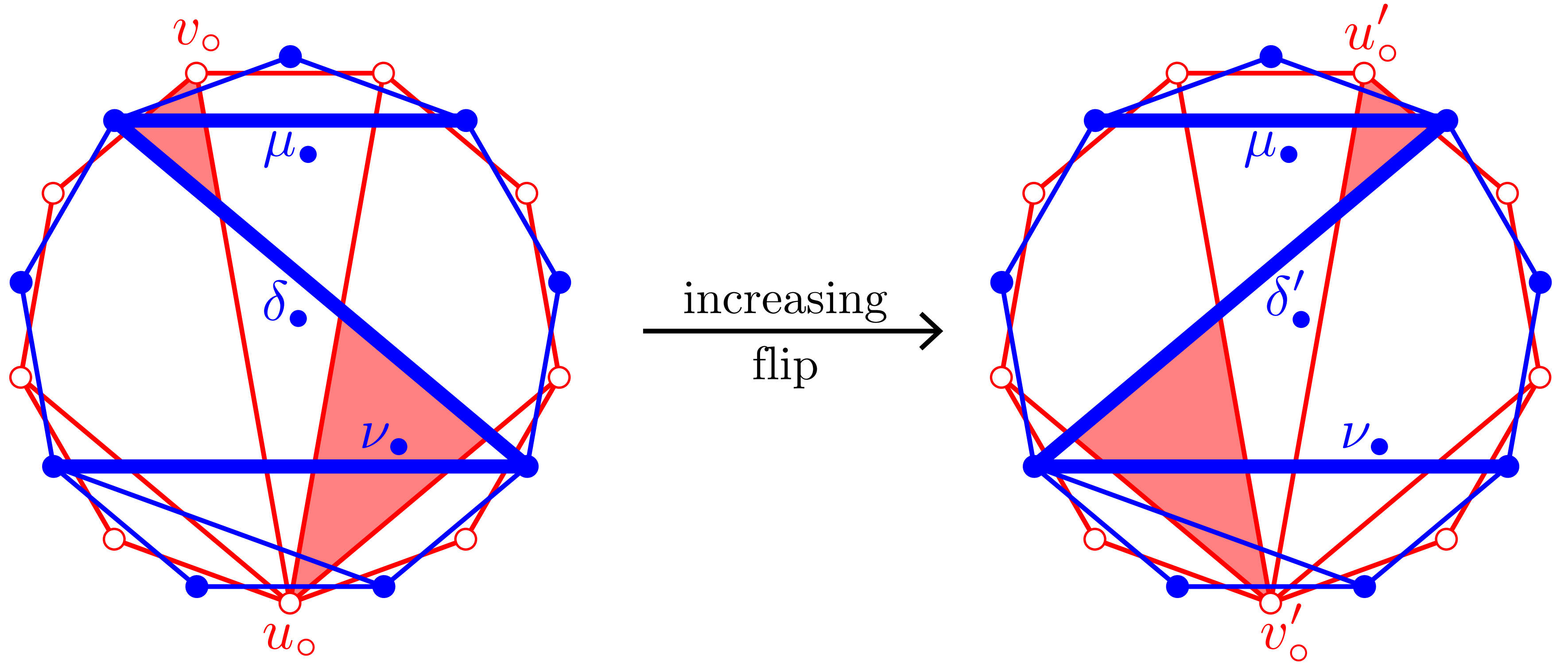
Exm: for a triangulation $\mathrm{T}_\circ$, the $\mathrm{T}_\circ$-accordion complex is a simplicial associahedron

FLIPS

PROP. The $\mathrm{D}_\circ$-accordion complex is a pseudomanifold:

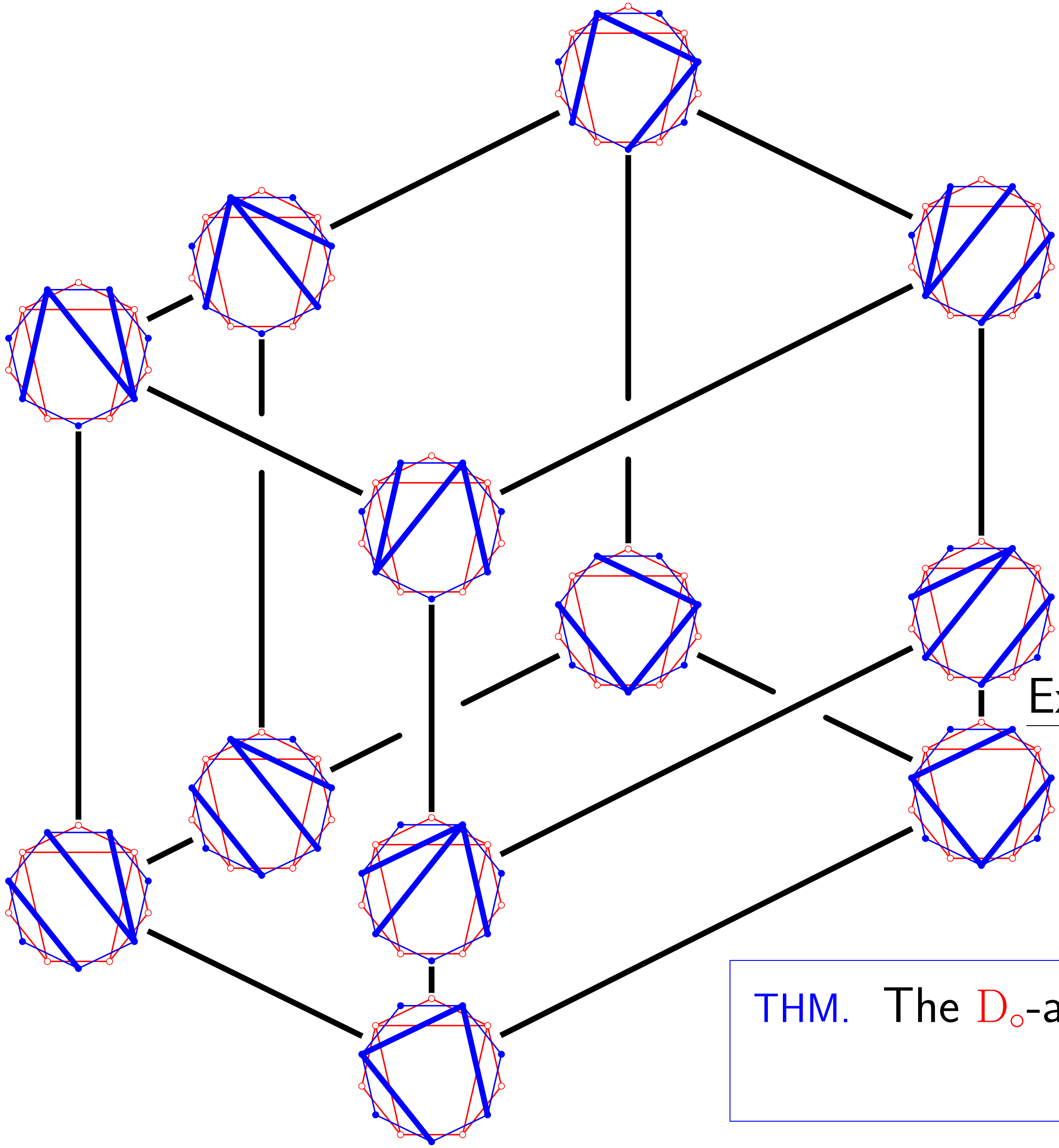
- pure: any maximal $\mathrm{D}_\circ$-accordion dissection has $|\mathrm{D}_\circ|$ diagonals
- thin: for any maximal $\mathrm{D}_\circ$-accordion dissection $\mathrm{D}_\bullet$ and any $\delta_\bullet \in \mathrm{D}_\bullet$, there is a unique $\delta'_\circ \neq \delta_\circ$ such that $\mathrm{D}_\circ \triangle \{\delta_\circ, \delta'_\circ\}$ is again a $\mathrm{D}_\circ$-accordion dissection

Garver-McConville ('16$^+$)



increasing flip = flip that changes a Σ to a Z

$\mathrm{D}_\circ$-ACCORDION LATTICE



increasing flip = flip that changes a Σ to a Z

$\mathrm{D}_\circ$-accordion poset = increasing flip poset on maximal $\mathrm{D}_\circ$-accordion dissections

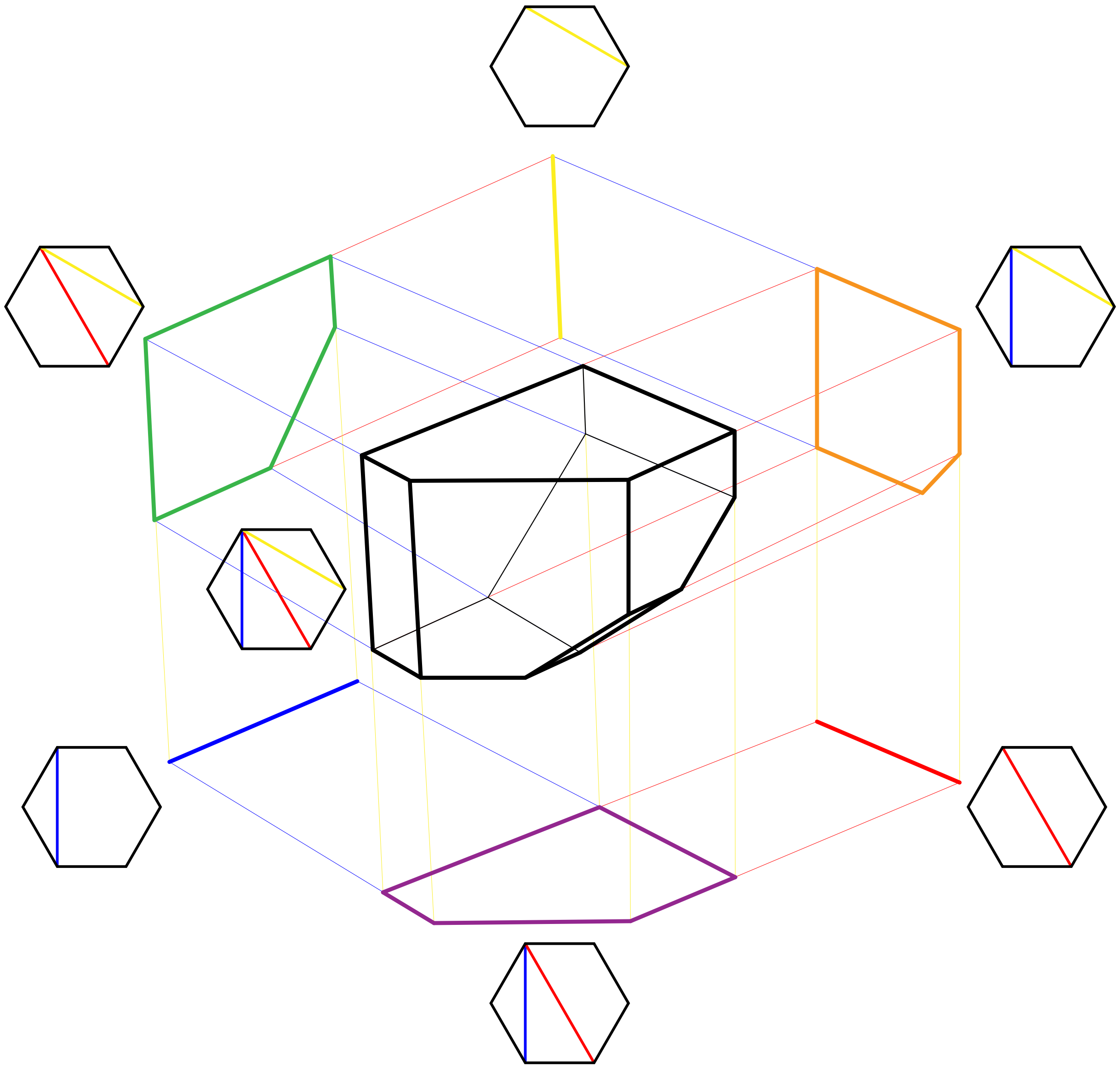
Exm: for a comb triangulation $\mathrm{T}_\circ$, the $\mathrm{T}_\circ$-accordion poset is the Tamari lattice

THM. The $\mathrm{D}_\circ$-accordion poset is a lattice

Garver-McConville ('16$^+$)

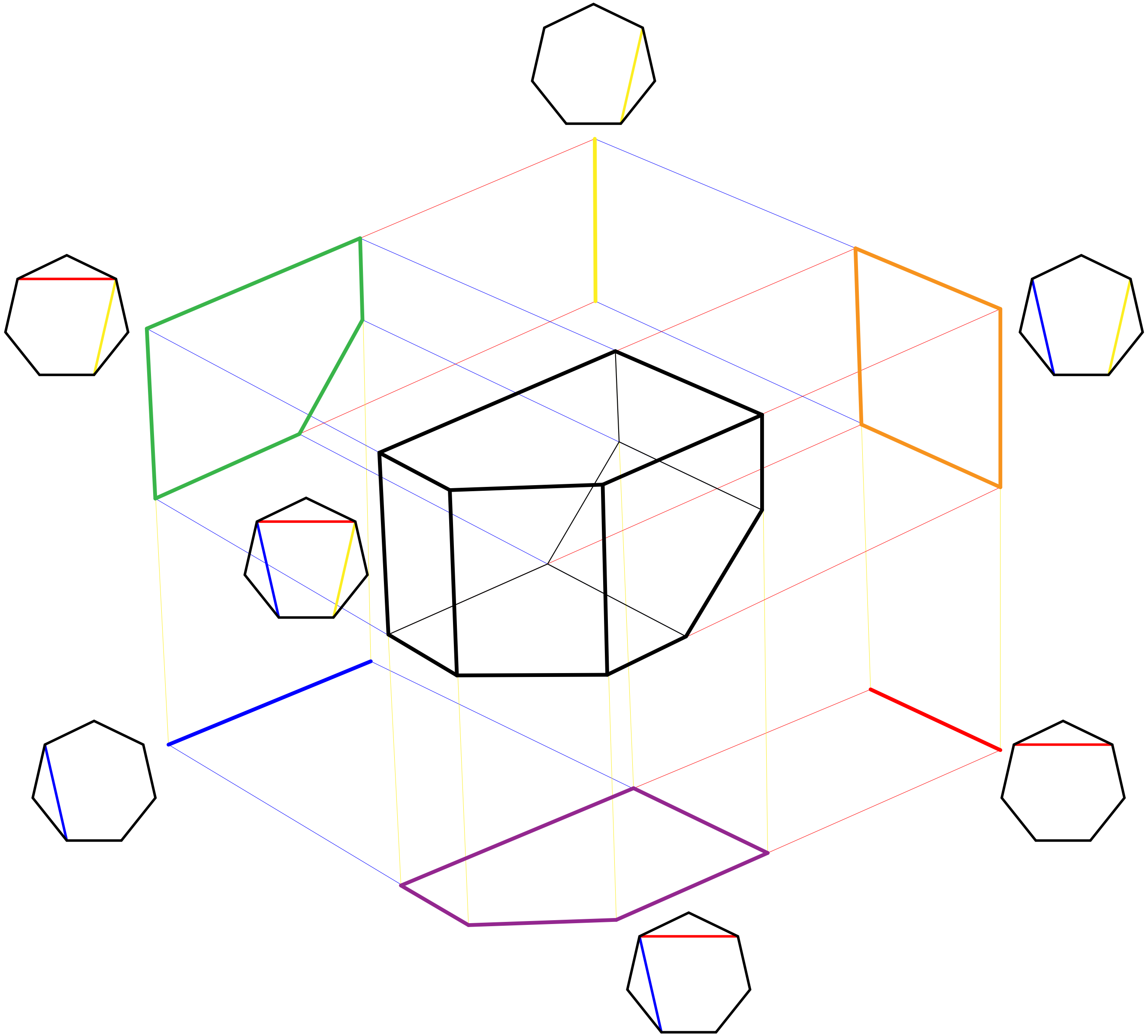
$\mathrm{D}_{\circ}$-ACCORDIOHEDRON

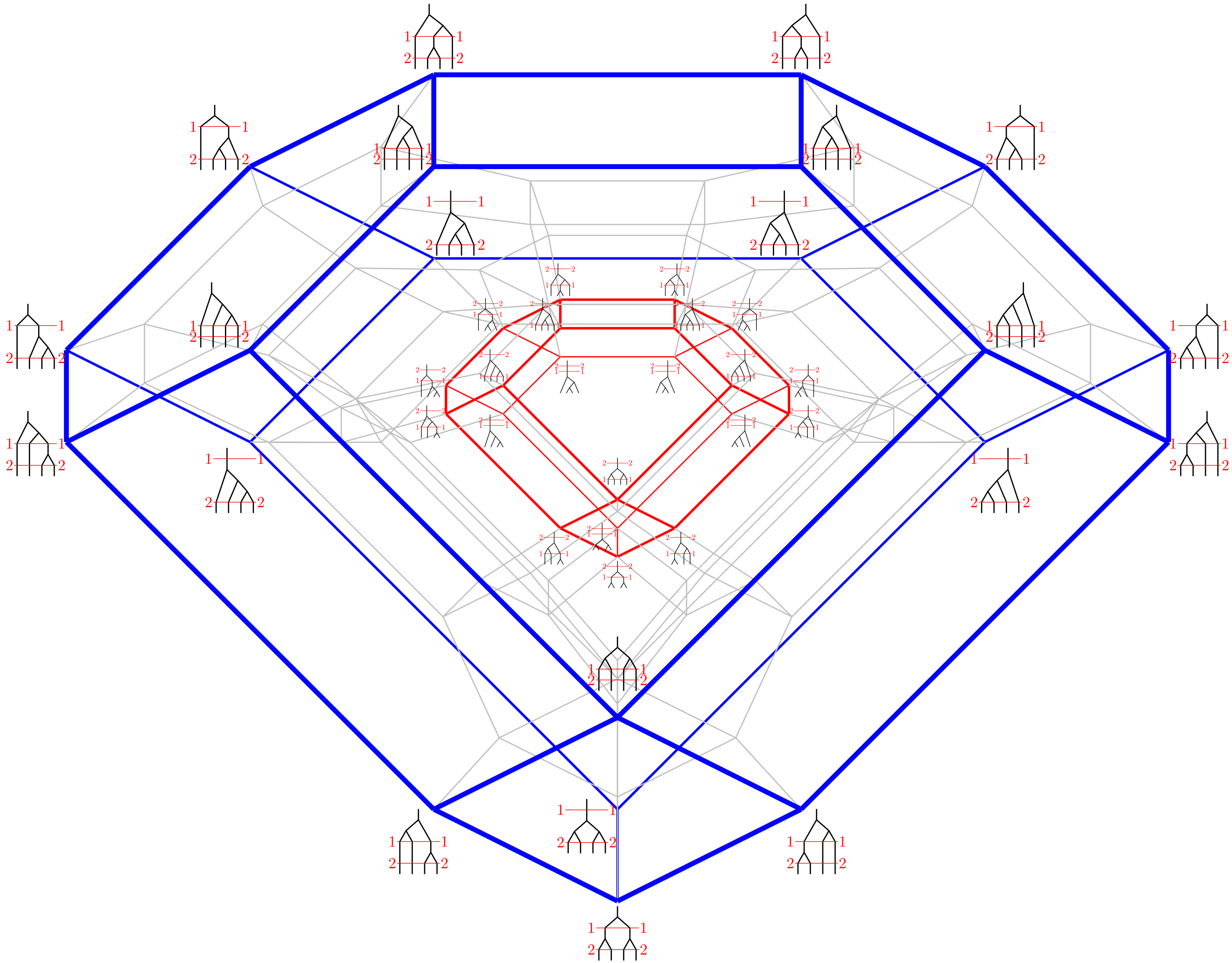
THM. The $\mathrm{D}_{\circ}$-accordion complex is the boundary complex of the polar of the accordiohedron, obtained as a projection of an associahedron. Manneville–P. ('19)



PROJECTIONS OF PROJECTIONS

PROP. If $\mathrm{D}_\circ \subseteq \mathrm{D}'_\circ$, then $\mathbb{A}\mathrm{sso}(\mathrm{D}'_\circ)$ is the projection of $\mathbb{A}\mathrm{sso}(\mathrm{D}_\circ)$ on $\langle \boldsymbol{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$.





THANK YOU