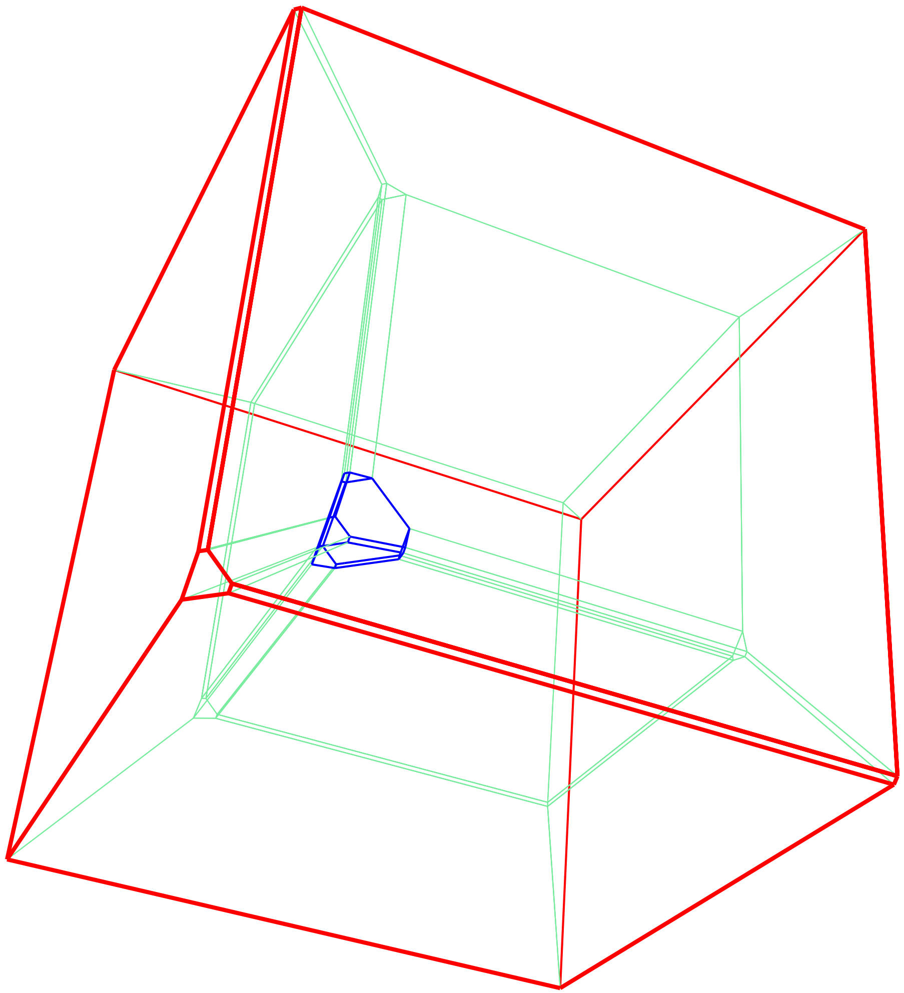
I. COMBINATORICS OF POLYTOPES



V. PILAUD (Universitat de Barcelona)

Osnabrück, Monday February 24th, 2025

slides available at: https://www.ub.edu/comb/vincentpilaud/documents/presentations/Osnabruck/1.pdf

COMBINATORIAL GEOMETRIES & GEOMETRIC COMBINATORICS 2025

When? October-November 2025

Where? Centre de Recerca Matematica, Barcelona, Spain

What? Intensive Research Program with

- Oct. 1–3: Recollections on polyhedral geometry and (oriented) matroids
- Oct. 6–17: research school (L. Anderson – C. Benedetti – R. Sanyal – G. Whittle)
- Oct. 20 – Nov. 21: research projects + seminars + visitors
- Nov. 24–28: conference

Why? good math + good food

How?

- full program registration on https://forms.gle/QGfi5XGR1592SMs2A by Feb. 14
- limited support for doctoral/postdoc students

Updates?

- https://www.ub.edu/comb/CGGC25/
- https://forms.gle/JGa79F4h9Xymd6sX8 for general announcements and registration deadlines

BEYOND PERMUTAHEDRA AND ASSOCIAHEDRA

When? December 1–5, 2025

Where? Centre International de Rencontres Mathématiques, Luminy, France

What?

2 mini courses.

- Nathan Reading
- Martha Yip

5 invited talks

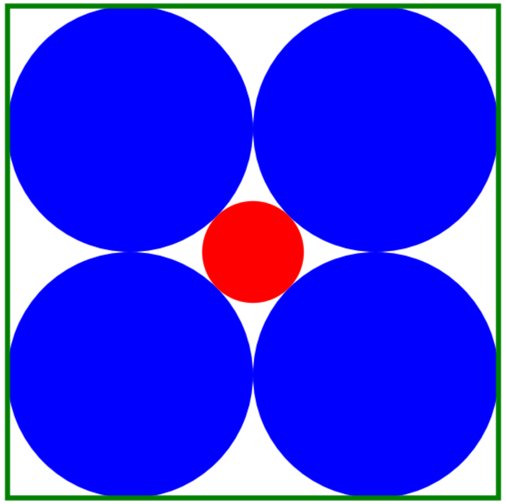
- Bérénice Delcroix-Oger
- Eléonore Faber
- Torsten Mütze
- Frédéric Patras
- Christian Stump

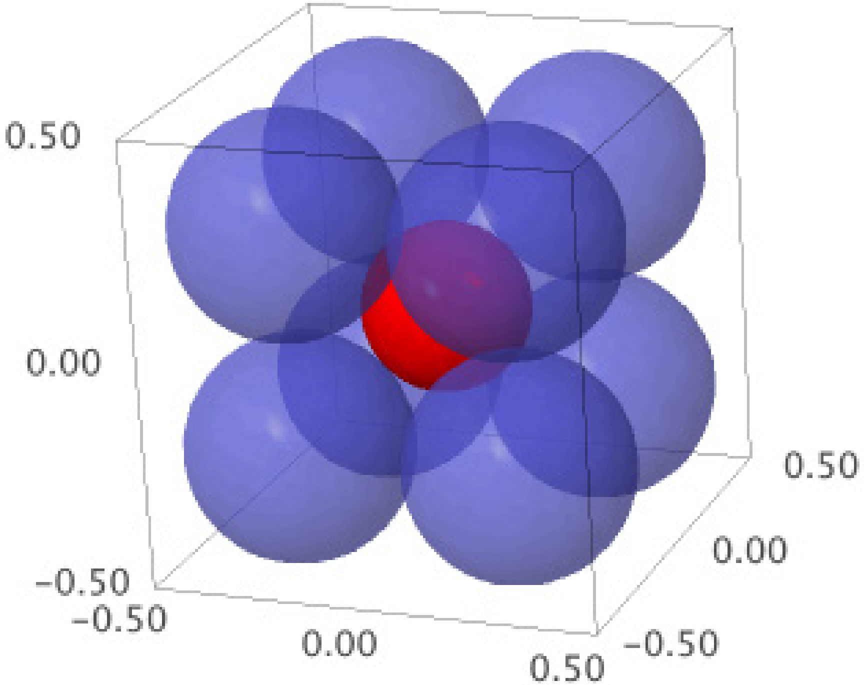
Why? good math + good food + good views

Updates?

- https://conferences.cirm-math.fr/3288.html
- announcement soon

BOULES DE PETANQUE & COCHONET

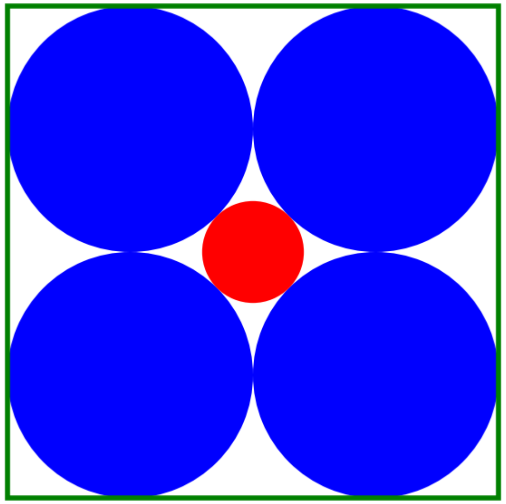


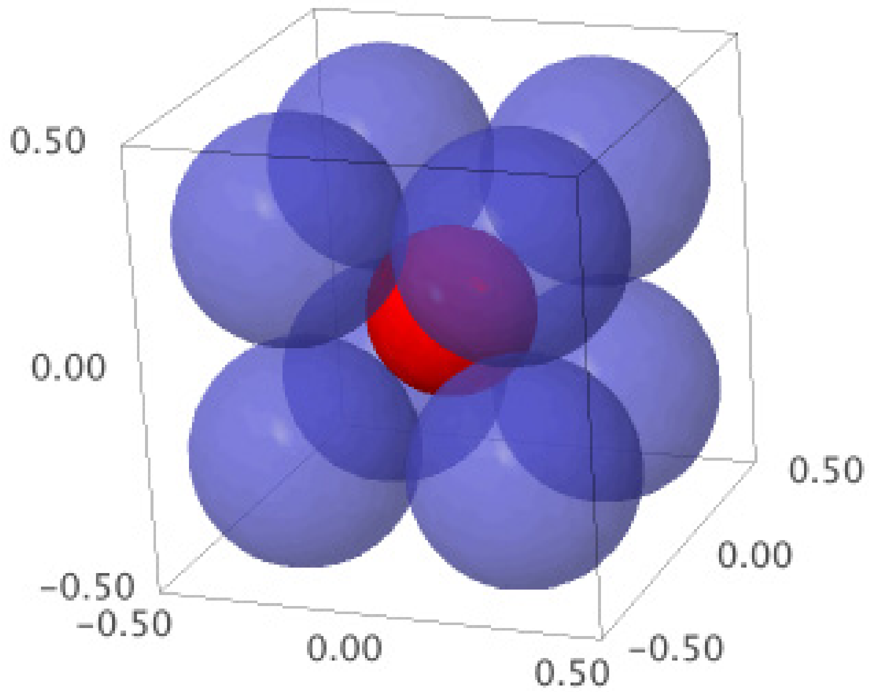


DEF. Pétanque = ... long story ... played with balls (blue) and a cochonet (red).

QU. What is the diameter of the cochonet ? and in dimension d? and in dimension 10?

COCHONET PARADOX

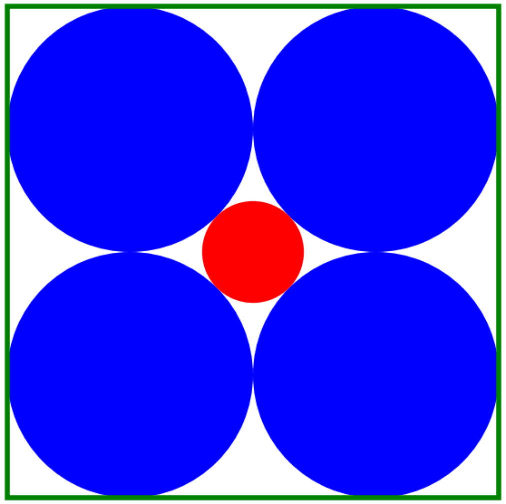


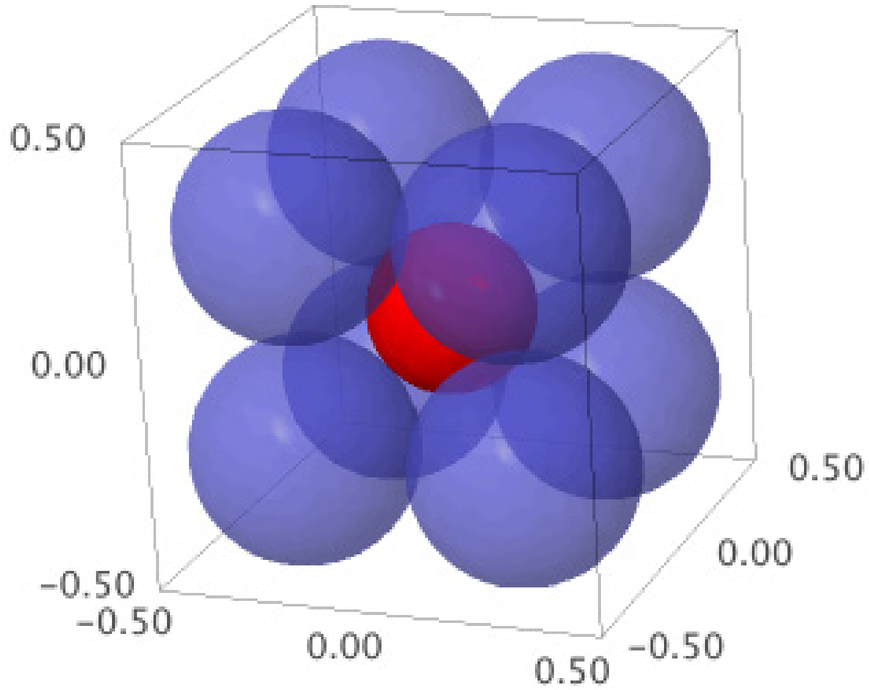


dimension d	1	2	3	...	9	10	11	...
diameter $= (\sqrt{d}-1)/2$	0	0.207	0.366	...	1	1.08	1.16	...
volume $= \dfrac{\left(\Gamma(1/2)\cdot(\sqrt{d}-1)/4\right)^d}{\Gamma(d/2+1)}$	0	0.0337	0.0257	...	0.00644	0.00543	0.00463	...

REM. In dimension ≥ 10, the cochonet is out of the box!!

COCHONET PARADOX





In high dimension, intuition is wrong, computations are correct.

SOME REFERENCES

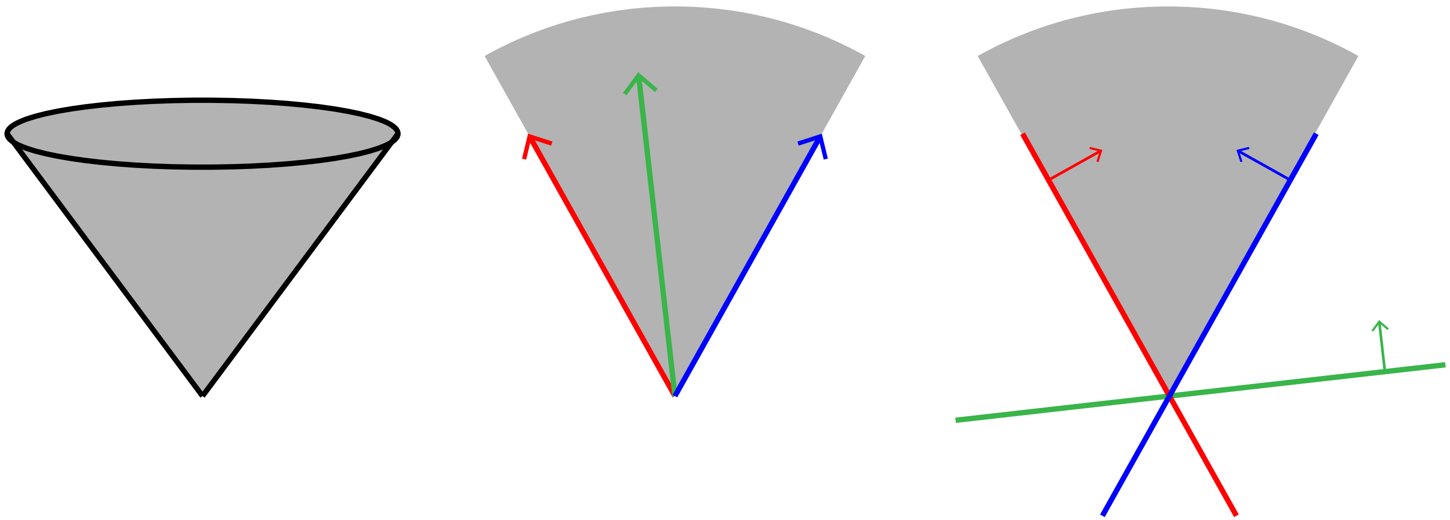
- Günter M. Ziegler. *Lectures on polytopes*.
  Vol. 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- Jiří Matoušek. *Lectures on discrete geometry*.
  Vol. 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

POLYHEDRAL CONES

CONES

DEF. $\mathbb{C} \subseteq \mathbb{R}^n$ convex cone $\iff$ $\mu \boldsymbol{u} + \nu \boldsymbol{v} \in \mathbb{C}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}$ and $\mu, \nu \in \mathbb{R}_{\geq 0}$.

DEF. dimension of $\mathbb{C}$ = dimension of its linear span.



DEF. $\mathcal{V}$-cone = convex cone generated by finitely many vectors

$$= \Big\{ \sum_{\boldsymbol{u} \in \boldsymbol{U}} \mu_{\boldsymbol{u}} \boldsymbol{u} \;\Big|\; \mu_{\boldsymbol{u}} \geq 0 \text{ for all } \boldsymbol{u} \in \boldsymbol{U} \Big\} \text{ for some finite } \boldsymbol{U}.$$

DEF. $\mathcal{H}$-cone = intersection of finitely many linear halfspaces

$$= \Big\{ \boldsymbol{u} \in \mathbb{R}^n \;\Big|\; \langle\, \boldsymbol{u} \mid \boldsymbol{v} \,\rangle \leq 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V} \Big\} \text{ for some finite } \boldsymbol{V}.$$

$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

remark: different proofs are possible.

Classical algorithmic proof = Fourier-Motzkin elimination procedure (projections on coordinate hyperplanes).

Here, induction + polarity...

THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

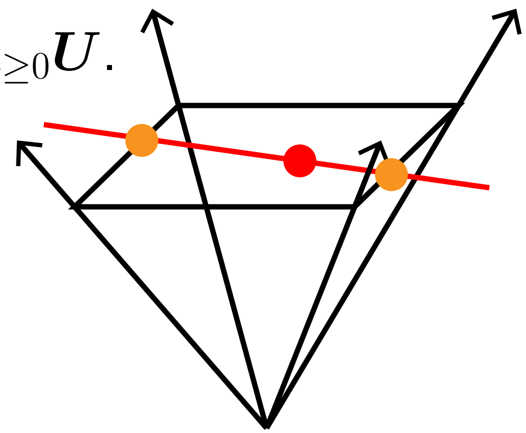
proof: $\mathcal{H}$-cone $\Longrightarrow$ $\mathcal{V}$-cone by induction on the dimension.

Consider an $\mathcal{H}$-cone $\mathbb{C} = \left\{ \boldsymbol{u} \in \mathbb{R}^n \;\middle|\; \langle\, \boldsymbol{u} \mid \boldsymbol{v} \,\rangle \le 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V} \right\}$.

It is clearly a $\mathcal{V}$-cone if $\dim(\mathbb{C}) = 0$ or if $\boldsymbol{V}$ does not contain two independent vectors.
Otherwise, there exist $\boldsymbol{v}, \boldsymbol{v}'$ in $\boldsymbol{V}$ and $\boldsymbol{w} \in \mathbb{R}^n$ st $\langle\, \boldsymbol{w} \mid \boldsymbol{v} \,\rangle \le 0$ and $\langle\, \boldsymbol{w} \mid \boldsymbol{v}' \,\rangle \ge 0$
(consider $\boldsymbol{w} = \langle\, \boldsymbol{v} \mid \boldsymbol{v}' \,\rangle \boldsymbol{v} + \langle\, \boldsymbol{v}' \mid \boldsymbol{v}' \,\rangle \boldsymbol{v} - \langle\, \boldsymbol{v} \mid \boldsymbol{v}' \,\rangle \boldsymbol{v}' - \langle\, \boldsymbol{v} \mid \boldsymbol{v} \,\rangle \boldsymbol{v}'$)

For $\boldsymbol{v} \in \boldsymbol{V}$, define $\mathbb{C}_{\boldsymbol{v}} = \mathbb{C} \cap \boldsymbol{v}^{\perp}$.
By induction, the $\mathcal{H}$-cone $\mathbb{C}_{\boldsymbol{v}}$ is the $\mathcal{V}$-cone generated by some finite set $\boldsymbol{U}_{\boldsymbol{v}}$.
We claim that the $\mathcal{H}$-cone $\mathbb{C}$ is the $\mathcal{V}$-cone generated by the finite set $\boldsymbol{U} = \bigcup_{\boldsymbol{v} \in \boldsymbol{V}} \boldsymbol{U}_{\boldsymbol{v}}$.

Let $\boldsymbol{u} \in \mathbb{C}$.
If $\boldsymbol{u}$ is on the boundary of $\mathbb{C}$, it belongs to some $\mathbb{C}_{\boldsymbol{v}} = \mathbb{R}_{\ge 0} \boldsymbol{U}_{\boldsymbol{v}} \subseteq \mathbb{R}_{\ge 0} \boldsymbol{U}$.
Otherwise, $(\boldsymbol{u} + \mathbb{R}\boldsymbol{w}) \cap \mathbb{C}$ is a segment $[\boldsymbol{u}^+, \boldsymbol{u}^-]$.
There is $\boldsymbol{v}^+, \boldsymbol{v}^- \in \boldsymbol{V}$ st $\boldsymbol{u}^+ \in \mathbb{C}_{\boldsymbol{v}^+}$ and $\boldsymbol{u}^- \in \mathbb{C}_{\boldsymbol{v}^-}$.
Thus $\boldsymbol{u} \in \mathbb{R}_{\ge 0}\{\boldsymbol{u}^+, \boldsymbol{u}^-\} \subseteq \mathbb{R}_{\ge 0}(\boldsymbol{U}_{\boldsymbol{v}^+} \cup \boldsymbol{U}_{\boldsymbol{v}^-}) \subseteq \mathbb{R}_{\ge 0}\boldsymbol{U}$.



$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

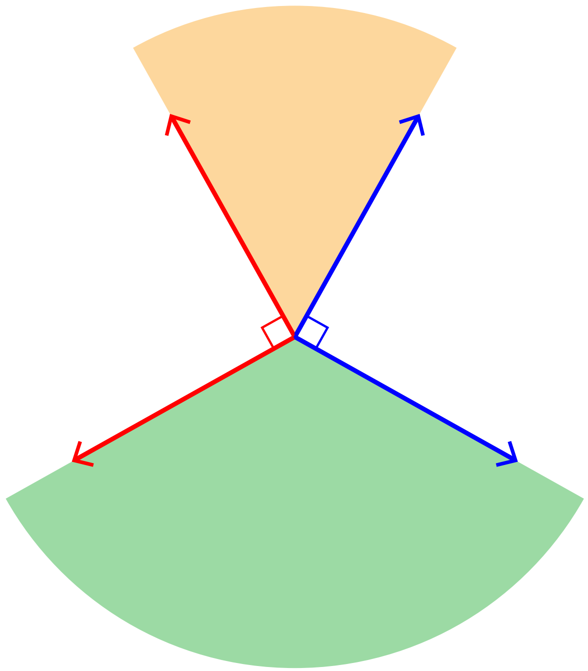
THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

proof: $\mathcal{V}$-cone $\Longrightarrow$ $\mathcal{H}$-cone by polarity.

DEF. linear polar $\mathbb{U}^\circ = \{\boldsymbol{v} \in \mathbb{R}^n \mid \langle\, \boldsymbol{u} \mid \boldsymbol{v}\,\rangle \leq 0$ for all $\boldsymbol{u} \in \mathbb{U}\}$.

PROP. $\mathbb{U}^\circ$ is a closed convex cone. If $\mathbb{U}$ is convex and closed, then $(\mathbb{U}^\circ)^\circ = \mathbb{U}$.

PROP. The polar of a $\mathcal{V}$-cone is an $\mathcal{H}$-cone.



$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

THM. (Minkowski-Weyl for cones) $\quad \mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

proof: $\mathcal{V}$-cone $\Longrightarrow$ $\mathcal{H}$-cone by polarity.

Consider an $\mathcal{V}$-cone $\mathbb{C}$.
Its polar $\mathbb{C}^\circ$ is an $\mathcal{H}$-cone, thus a $\mathcal{V}$-cone according to the first part of the proof.
Therefore, $\mathbb{C} = (\mathbb{C}^\circ)^\circ$ is an $\mathcal{H}$-cone.

DEF. linear polar $\mathbb{U}^\circ = \{\boldsymbol{v} \in \mathbb{R}^n \mid \langle\, \boldsymbol{u} \mid \boldsymbol{v}\,\rangle \le 0$ for all $\boldsymbol{u} \in \mathbb{U}\}$.

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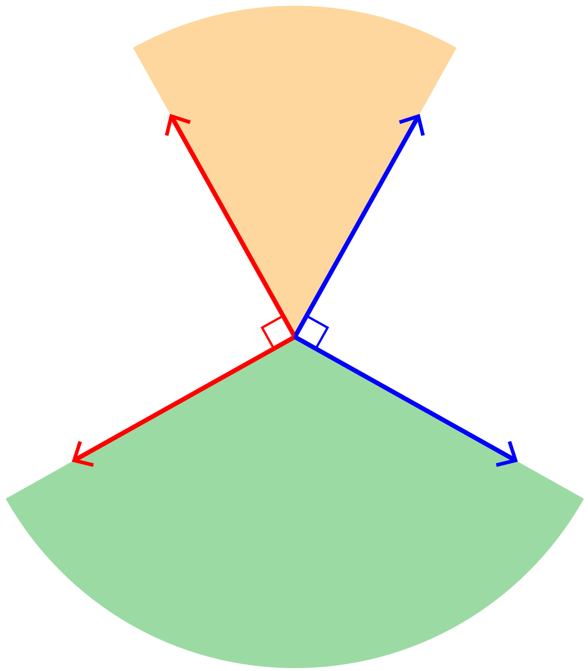
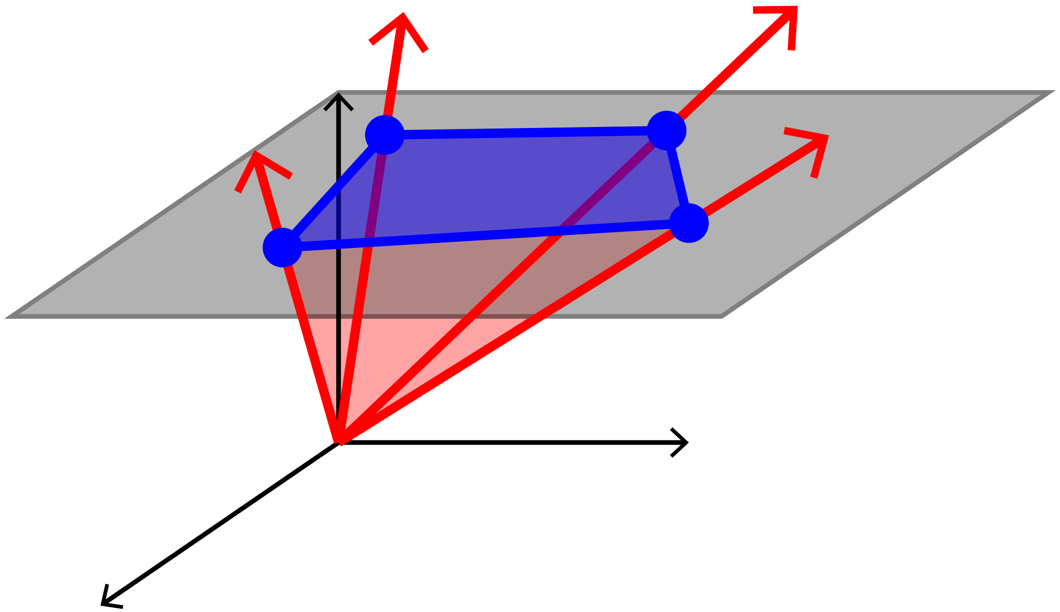
PROP. The polar of a $\mathcal{V}$-cone is an $\mathcal{H}$-cone.



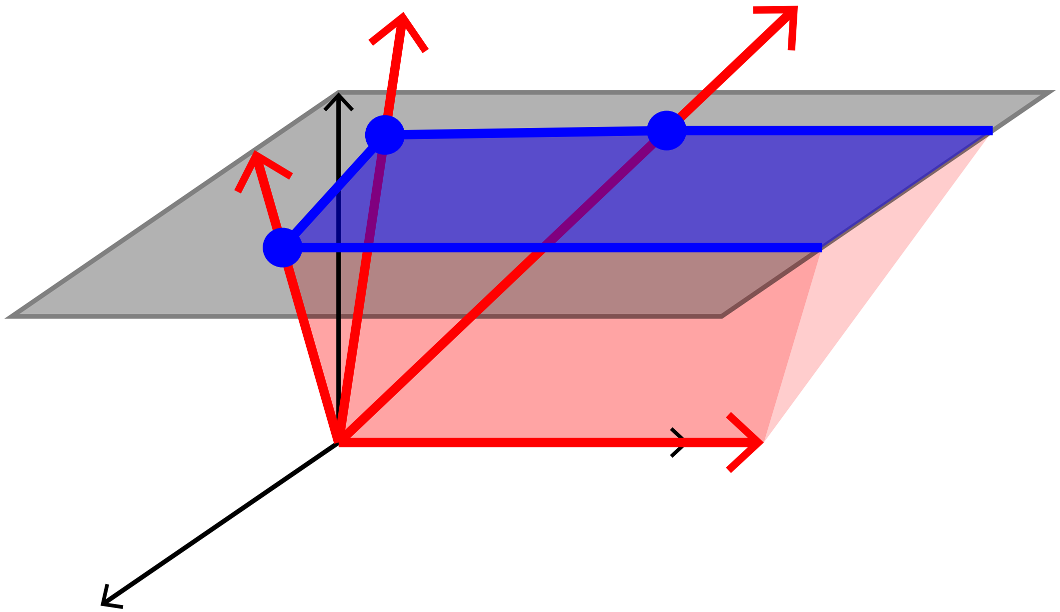
INTERSECTING A CONE BY A HYPERPLANE

DEF. polyhedral cone $=$ $\mathcal{V}$-cone $=$ $\mathcal{H}$-cone.

DEF. polyhedron $=$ intersection of a polyhedral cone by an affine hyperplane.



bounded
$=$ polytope



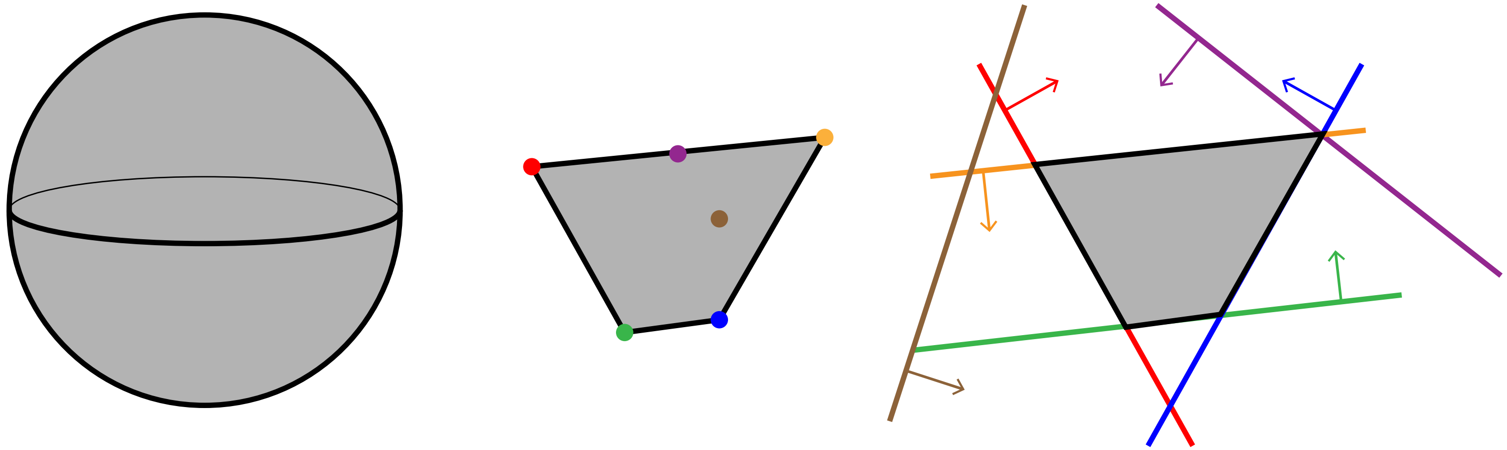
undounded
$=$ polytope $+$ recession cone

POLYTOPES

POLYTOPES

DEF. $\mathbb{P} \subseteq \mathbb{R}^n$ convex $\iff$ $\mu\boldsymbol{x}+\nu\boldsymbol{y} \in \mathbb{P}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{P}$ and $\mu, \nu \in \mathbb{R}_{\geq 0}$ with $\mu+\nu = 1$.

DEF. dimension of $\mathbb{P}$ = dimension of its affine span.



DEF. $\mathcal{V}$-polytope = convex hull of finite point set in $\mathbb{R}^n$

$$= \Big\{ \sum_{\boldsymbol{x}\in\boldsymbol{X}} \mu_{\boldsymbol{x}}\boldsymbol{x} \;\Big|\; \sum_{\boldsymbol{x}\in\boldsymbol{X}} \mu_{\boldsymbol{x}} = 1 \text{ and } \mu_{\boldsymbol{x}} \geq 0 \text{ for all } \boldsymbol{x} \in \boldsymbol{X} \Big\} \text{ for a finite } \boldsymbol{X}.$$

DEF. $\mathcal{H}$-polytope = bounded intersection of finitely many affine halfspaces of $\mathbb{R}^n$

$$= \big\{ \boldsymbol{x} \in \mathbb{R}^n \;\big|\; \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \leq c_{\boldsymbol{y}} \text{ for all } \boldsymbol{y} \in \boldsymbol{Y} \big\} \text{ for a finite } \boldsymbol{Y}.$$

$\mathcal{V}$-POLYTOPES VS $\mathcal{H}$-POLYTOPES

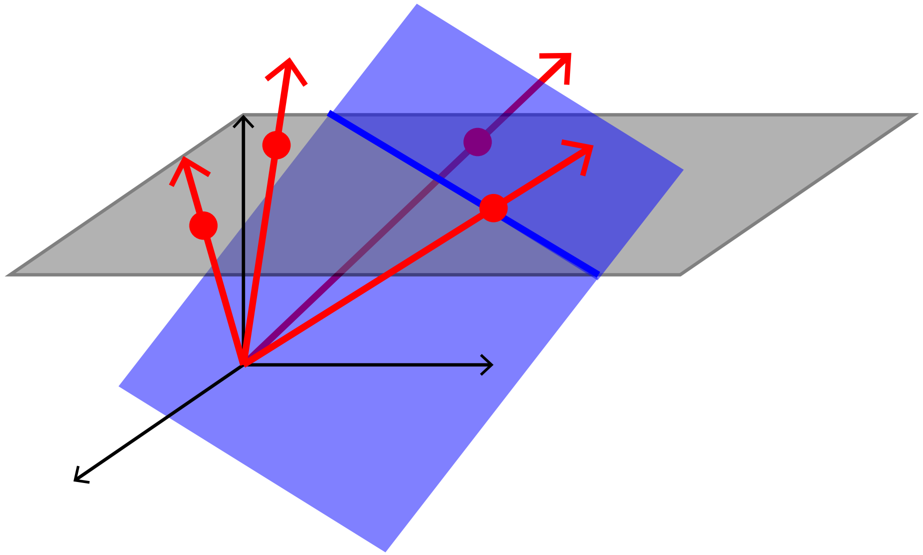
THM. (Minkowski-Weyl for polytopes) $\mathcal{V}$-polytope $\iff$ $\mathcal{H}$-polytope.

proof: embed the affine space $\mathbb{R}^n$ into the linear space $\mathbb{R}^{n+1}$.

$$\boldsymbol{x} \qquad\qquad \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le c_{\boldsymbol{y}}$$

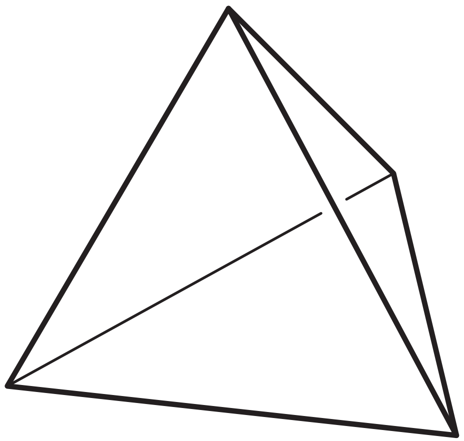
$$\updownarrow \qquad\qquad \updownarrow$$

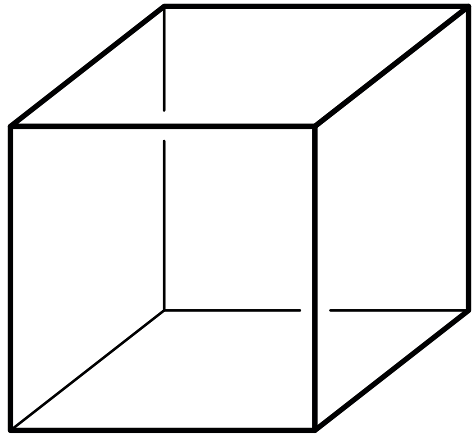
$$\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \qquad \left\langle \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \,\middle|\, \begin{bmatrix} \boldsymbol{y} \\ -c_{\boldsymbol{y}} \end{bmatrix} \right\rangle \le 0$$

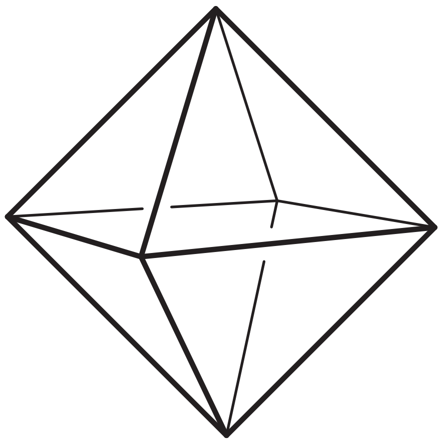


DEF. polytope $=$ $\mathcal{V}$-polytope $=$ $\mathcal{H}$-polytope.

CLASSICAL POLYTOPES







DEF. d-simplex = convex hull of $d+1$ affinely independent points.
standard d-simplex $\triangle_d = \mathrm{conv}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d+1}\}$
$= \Big\{ \boldsymbol{x} \in \mathbb{R}^{d+1} \;\Big|\; \sum_{i \in [d+1]} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \in [d+1] \Big\}$.

DEF. d-cube $\square_d = \mathrm{conv}(\{\pm 1\}^d) = \Big\{ \boldsymbol{x} \in \mathbb{R}^d \;\Big|\; -1 \leq x_i \leq 1 \text{ for all } i \in [d] \Big\}$.

DEF. d-cross-pol. $\lozenge_d = \mathrm{conv}\,\{\pm \boldsymbol{e}_i \mid i \in [d]\} = \Big\{ \boldsymbol{x} \in \mathbb{R}^d \;\Big|\; \sum_{i \in [d]} \varepsilon_i x_i \leq 1 \text{ for all } \varepsilon \in \{\pm 1\}^d \Big\}$.

AFFINE POLARITY

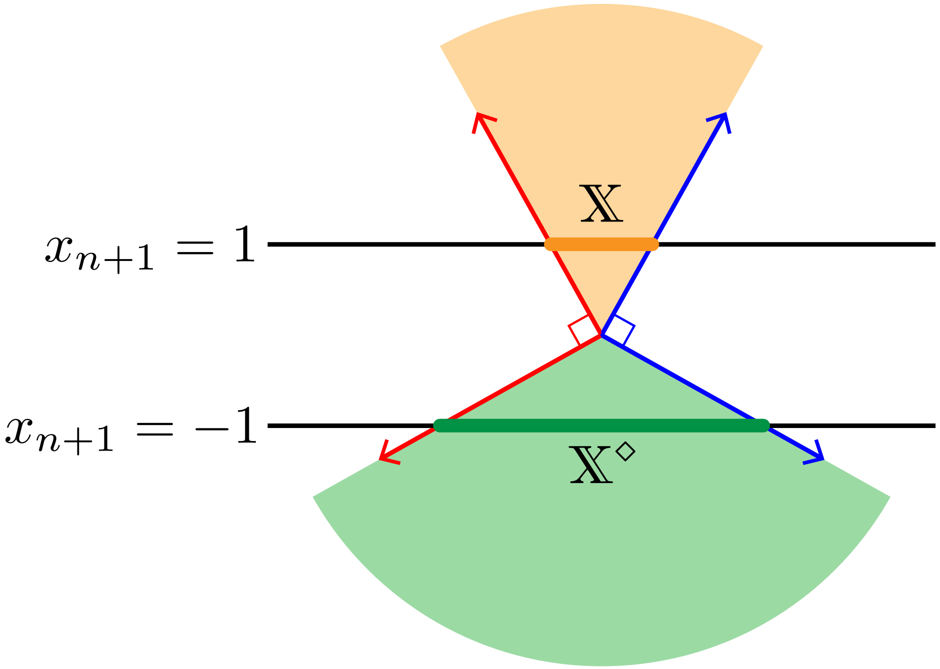
DEF. linear polar $\mathbb{U}^\circ = \left\{ \boldsymbol{v} \in \mathbb{R}^{n+1} \;\middle|\; \langle\, \boldsymbol{u} \mid \boldsymbol{v} \,\rangle \leq 0 \text{ for all } \boldsymbol{u} \in \mathbb{U} \right\}$.

DEF. affine polar $\mathbb{X}^\diamond = \{ \boldsymbol{y} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \leq 1 \text{ for all } \boldsymbol{x} \in \mathbb{X} \}$.

$$\langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \leq 1$$

$$\updownarrow$$

$$\left\langle \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \;\middle|\; \begin{bmatrix} \boldsymbol{y} \\ -1 \end{bmatrix} \right\rangle \leq 0$$

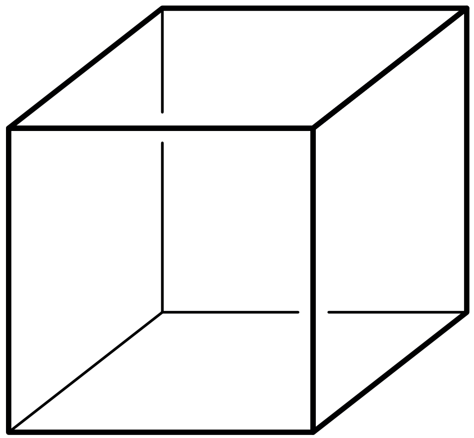


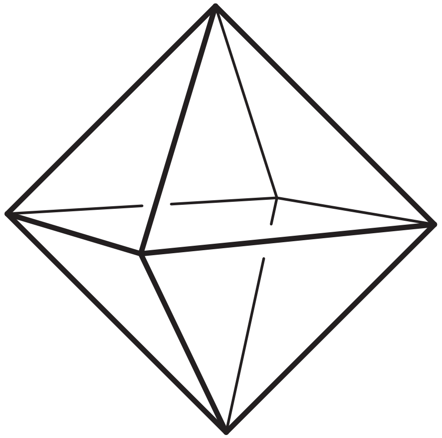
PROP. $\mathbb{X}^\diamond$ is closed and convex, and bounded iff $\mathbf{0} \in \operatorname{int}(\mathbb{X})$. If $\mathbb{X}$ is closed, convex and contains $\mathbf{0}$, then $(\mathbb{X}^\diamond)^\diamond = \mathbb{X}$.

POLAR POLYTOPE

DEF. affine polar $\mathbb{X}^\diamond = \{\boldsymbol{y} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1$ for all $\boldsymbol{x} \in \mathbb{X}\}$.

PROP. Assume $\mathbf{0} \in \operatorname{int}(\mathbb{P})$.
If $\mathbb{P} = \operatorname{conv}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1$ for all $\boldsymbol{y} \in \boldsymbol{Y}\}$,
then $\mathbb{P}^\diamond = \operatorname{conv}(\boldsymbol{Y}) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1$ for all $\boldsymbol{x} \in \boldsymbol{X}\}$.





EXM. d-cube $\square_d = \operatorname{conv}(\{\pm 1\}^d) = \left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; -1 \le x_i \le 1 \text{ for all } i \in [d]\right\}$.

d-cross-pol. $\Diamond_d = \operatorname{conv}\{\pm \boldsymbol{e}_i \mid i \in [d]\} = \left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; \sum_{i \in [d]} \varepsilon_i x_i \le 1 \text{ for all } \varepsilon \in \{\pm 1\}^d\right\}$.

EXM: MATCHING POLYTOPES

DEF. $G = (V, E)$ graph.
matching on G = subset of E with at most one edge incident to each vertex.
matching polytope $\mathbb{M}(G)$ = convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

QU. Consider the polytope $\mathbb{N}(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V.$$

- Show that $\mathbb{M}(G) \subseteq \mathbb{N}(G)$.
- Give an example where this inclusion is strict.
- Show that $\mathbb{M}(G) = \mathbb{N}(G)$ when G is bipartite.

EXM: MATCHING POLYTOPES

DEF. $G = (V, E)$ graph.
matching on G = subset of E with at most one edge incident to each vertex.
matching polytope $\mathbb{M}(G)$ = convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

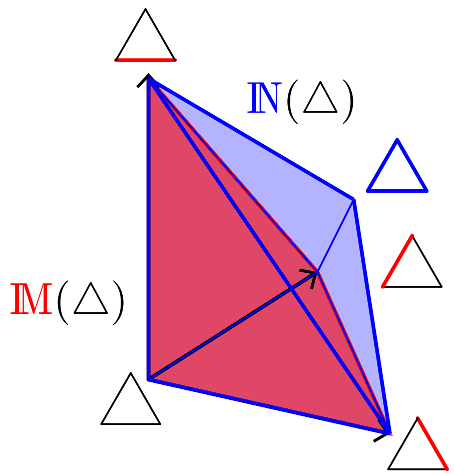
PROP. The matching polytope $\mathbb{M}(G)$ is contained in the polytope $\mathbb{N}(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V,$$

and $\mathbb{M}(G) = \mathbb{N}(G)$ when G is bipartite.

proof: $\mathbb{M}(G) \subseteq \mathbb{N}(G)$ as $(\chi_M)_e \geq 0$ and $\sum_{e \ni v} (\chi_M)_e \leq 1$ (at most one edge per vertex).
Strict inclusion in general:

$\mathbb{M}(\triangle) = \mathrm{conv}\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
$\mathbb{N}(\triangle) = \mathrm{conv}\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/2\}$



EXM: MATCHING POLYTOPES

DEF. $G = (V, E)$ graph.
matching on G = subset of E with at most one edge incident to each vertex.
matching polytope $\mathbb{M}(G)$ = convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

PROP. The matching polytope $\mathbb{M}(G)$ is contained in the polytope $\mathbb{N}(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V,$$

and $\mathbb{M}(G) = \mathbb{N}(G)$ when G is bipartite.

proof: $\mathbb{M}(G) \subseteq \mathbb{N}(G)$ as $(\chi_M)_e \geq 0$ and $\sum_{e \ni v} (\chi_M)_e \leq 1$ (at most one edge per vertex).
Assume now that G is bipartite, so that all its cycles are even.
For $\boldsymbol{x} \in \mathbb{N}(G)$, let $U(\boldsymbol{x}) = \{e \in E \mid 0 < \boldsymbol{x}_e < 1\}$.
If $U(\boldsymbol{x}) \neq \varnothing$, it contains a cycle $C = e_1, \dots, e_{2p}$, which is even since G is bipartite.
Let $\lambda = \min \{\boldsymbol{x}_e \mid e \in C\} \cup \{1 - \boldsymbol{x}_e \mid e \in C\}$.
Then $\boldsymbol{x}$ is in the middle of $\boldsymbol{x} + \lambda\chi_C$ and $\boldsymbol{x} - \lambda\chi_C$, which both belong to $\mathbb{N}(G)$.
Therefore, all vertices of $\mathbb{N}(G)$ belong to $\{0, 1\}^E$, and thus $\mathbb{M}(G) = \mathbb{N}(G)$.

OPERATIONS ON POLYTOPES

CARTESIAN PRODUCT

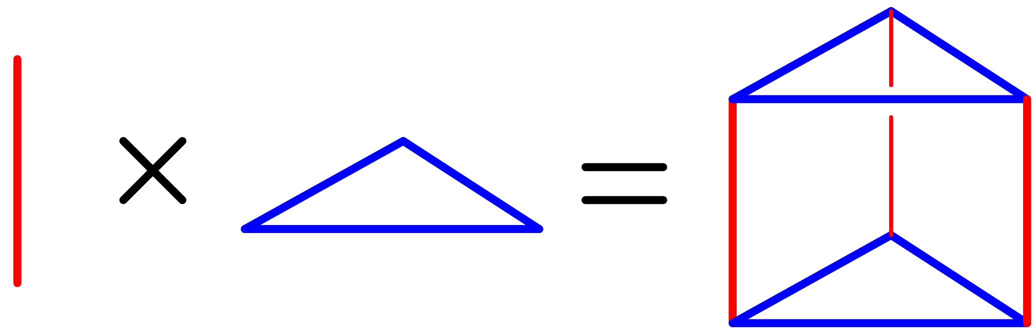
DEF. $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{X}' \subseteq \mathbb{R}^{n'}$.
Cartesian product $\mathbb{X} \times \mathbb{X}' = \{(\boldsymbol{x}, \boldsymbol{x'}) \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^{n+n'}$.

PROP. The Cartesian product $\mathbb{P} \times \mathbb{P}'$ of two polytopes $\mathbb{P}$ and $\mathbb{P}'$ is a polytope. Moreover

$$\begin{aligned}\mathbb{P} \times \mathbb{P}' &= \operatorname{conv}(\boldsymbol{X} \times \boldsymbol{X'}) \\ &= \left\{ (\boldsymbol{x}, \boldsymbol{x'}) \in \mathbb{R}^{n+n'} \;\middle|\; \begin{array}{l} \langle\, (\boldsymbol{x}, \boldsymbol{x'}) \mid (\boldsymbol{y}, \boldsymbol{0}) \,\rangle \le c_{\boldsymbol{y}} \text{ for all } \boldsymbol{y} \in \boldsymbol{Y} \\ \langle\, (\boldsymbol{x}, \boldsymbol{x'}) \mid (\boldsymbol{0}, \boldsymbol{y'}) \,\rangle \le c_{\boldsymbol{y'}} \text{ for all } \boldsymbol{y'} \in \boldsymbol{Y'} \end{array} \right\}\end{aligned}$$

where $\mathbb{P} = \operatorname{conv}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le c_{\boldsymbol{y}} \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}\}$.
and $\mathbb{P}' = \operatorname{conv}(\boldsymbol{X'}) = \left\{\boldsymbol{x'} \in \mathbb{R}^{n'} \;\middle|\; \langle\, \boldsymbol{x'} \mid \boldsymbol{y'} \,\rangle \le c_{\boldsymbol{y'}} \text{ for all } \boldsymbol{y'} \in \boldsymbol{Y'}\right\}$.

exm:
cube: $\square_d = [-1, 1]^d$
prism: $\mathbb{P}\mathrm{rism}(\mathbb{P}) = [-1, 1] \times \mathbb{P}$



DIRECT SUM

DEF. $\mathbb{P} \subset \mathbb{R}^n$ and $\mathbb{P}' \subset \mathbb{R}^{n'}$ two polytopes with $\mathbf{0} \in \operatorname{int} \mathbb{P}$ and $\mathbf{0} \in \operatorname{int} \mathbb{P}'$.
direct sum $\mathbb{P} \oplus \mathbb{P}' = \operatorname{conv}\big(\{(\boldsymbol{x}, \mathbf{0}) \mid \boldsymbol{x} \in \mathbb{P}\} \cup \{(\mathbf{0}, \boldsymbol{x}') \mid \boldsymbol{x}' \in \mathbb{P}'\} \big) \subset \mathbb{R}^{n+n'}$

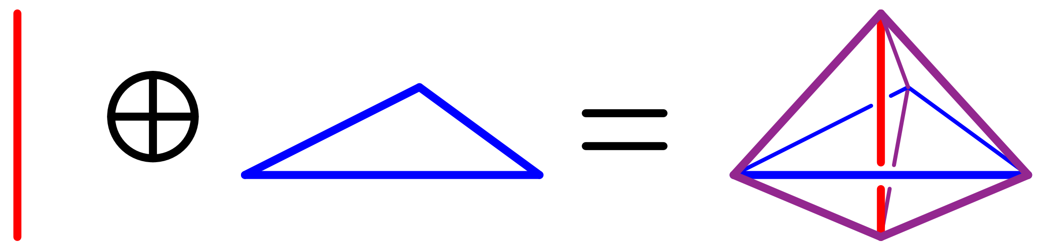
PROP.
$$\mathbb{P} \oplus \mathbb{P}' = \operatorname{conv}\big(\{(\boldsymbol{x}, \mathbf{0}) \mid \boldsymbol{x} \in \boldsymbol{X}\} \cup \{(\mathbf{0}, \boldsymbol{x}') \mid \boldsymbol{x}' \in \boldsymbol{X}'\} \big)$$
$$= \Big\{ (\boldsymbol{x}, \boldsymbol{x}') \in \mathbb{R}^{n+n'} \;\Big|\; \langle\, (\boldsymbol{x}, \boldsymbol{x}') \mid (\boldsymbol{y}, \boldsymbol{y}') \,\rangle \le 1 \text{ for all } \boldsymbol{y} \in \boldsymbol{Y} \text{ and } \boldsymbol{y}' \in \boldsymbol{Y}' \Big\}$$

where $\mathbb{P} = \operatorname{conv}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1 \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}\}$.
and $\mathbb{P}' = \operatorname{conv}(\boldsymbol{X}') = \Big\{\boldsymbol{x}' \in \mathbb{R}^{n'} \;\Big|\; \langle\, \boldsymbol{x}' \mid \boldsymbol{y}' \,\rangle \le 1 \text{ for all } \boldsymbol{y}' \in \boldsymbol{Y}'\Big\}$.

exm:

cross-poly.: $\Diamond_d = [-1,1] \oplus \cdots \oplus [-1,1]$

bipyramid: $\mathbb{B}\mathrm{ipyr}(\mathbb{P}) = [-1,1] \oplus \mathbb{P}$



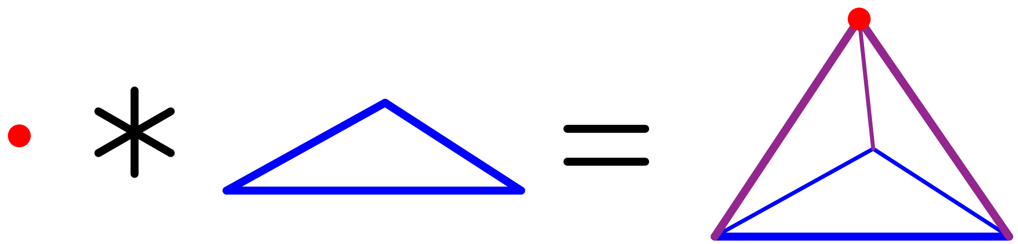
PROP. $(\mathbb{P} \oplus \mathbb{P}')^\diamond = \mathbb{P}^\diamond \times \mathbb{P}'^\diamond$.

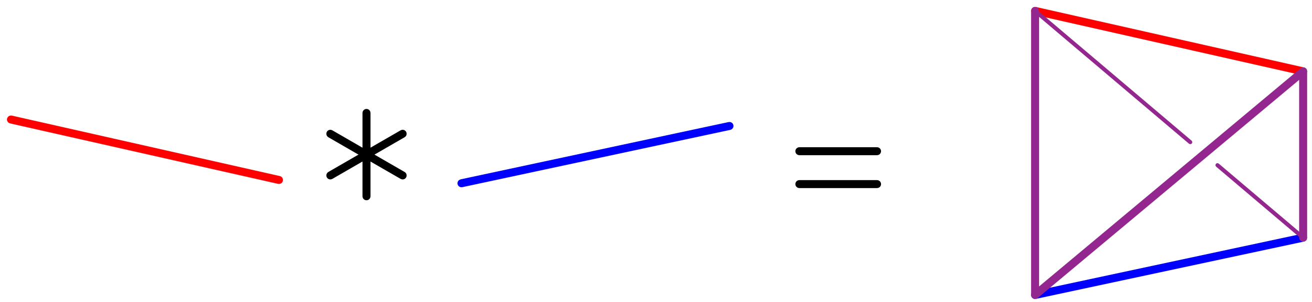
JOIN

DEF. $\mathbb{P} \subset \mathbb{R}^n$ and $\mathbb{P}' \subset \mathbb{R}^{n'}$ two polytopes.
join $\mathbb{P} * \mathbb{P}'$ = convex hull of $\mathbb{P}$ and $\mathbb{P}'$ in independent affine subspaces

$$= \mathrm{conv}\left(\{(\boldsymbol{x}, \boldsymbol{0}, 1) \mid \boldsymbol{x} \in \mathbb{P}\} \cup \{(\boldsymbol{0}, \boldsymbol{x'}, -1) \mid \boldsymbol{x'} \in \mathbb{P}'\} \right) \subset \mathbb{R}^{n+n'+1}$$

exm:
simplex: $\triangle_d = \triangle_i * \triangle_{d-i}$
pyramid: $\mathrm{Pyr}(\mathbb{P}) =$ point $* \mathbb{P}$
k-fold pyramid: $\mathrm{Pyr}_k(\mathbb{P}) = \triangle_{k-1} * \mathbb{P}$

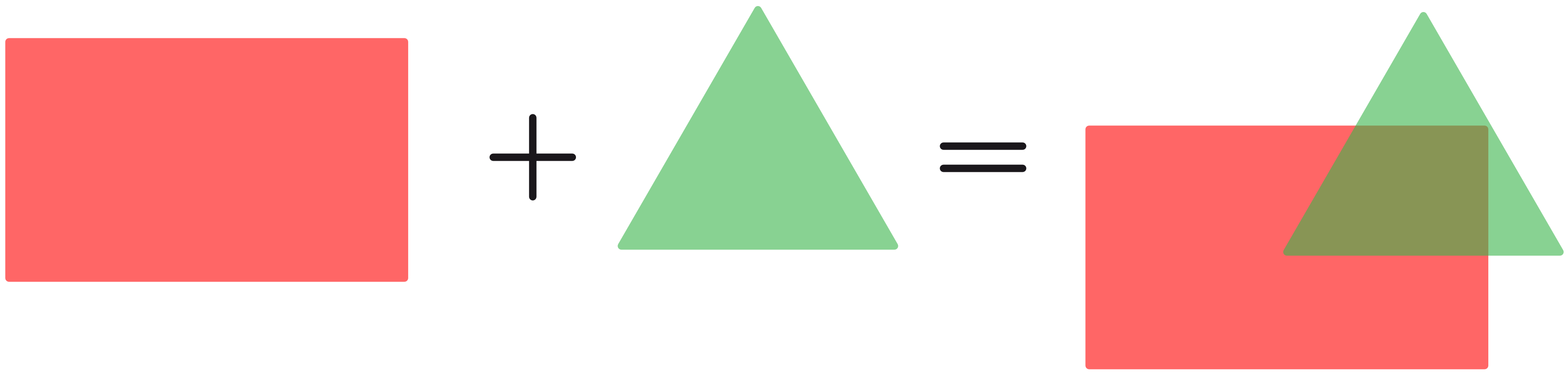




MINKOWSKI SUM

DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x'} \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

PROP. The Minkowski sum $\mathbb{P} + \mathbb{P}'$ of two polytopes $\mathbb{P}$ and $\mathbb{P}'$ is a polytope.



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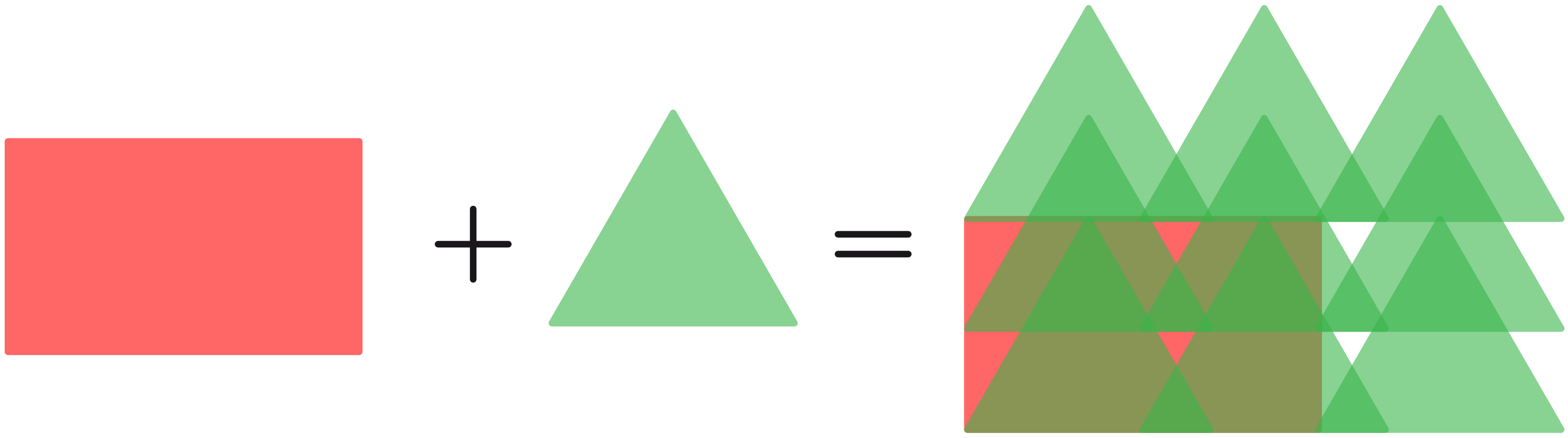

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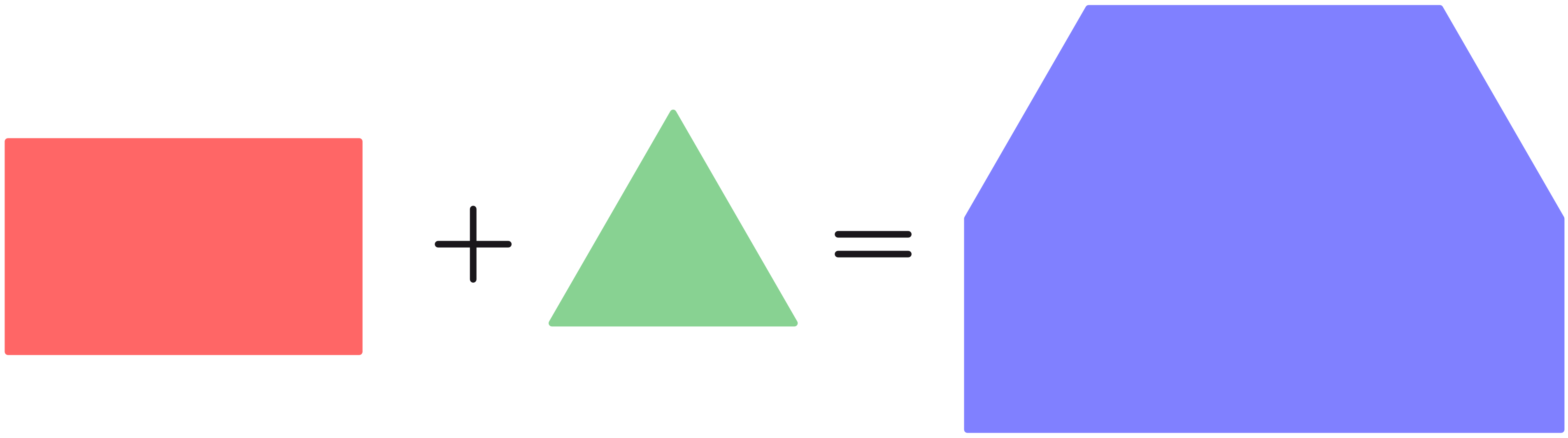


MINKOWSKI SUM

DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x'} \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

PROP. The Minkowski sum $\mathbb{P} + \mathbb{P}'$ is the image of the Cartesian product $\mathbb{P} \times \mathbb{P}'$ under the affine projection $(\boldsymbol{x}, \boldsymbol{x'}) \longmapsto \boldsymbol{x} + \boldsymbol{x'}$.

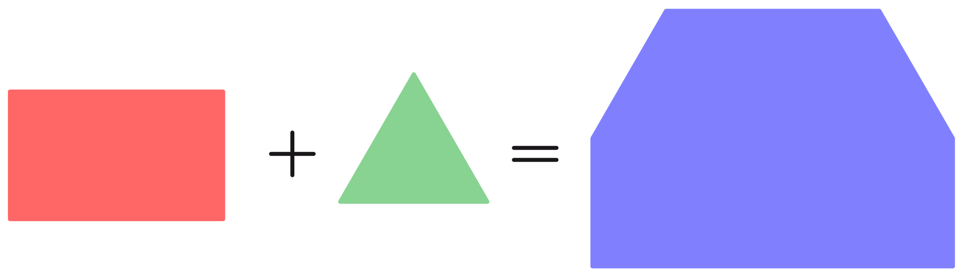
MINKOWSKI SUM

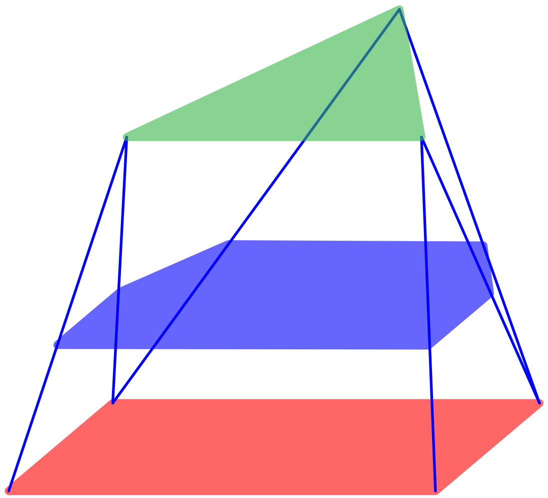
DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x'} \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

PROP. For any $-1 \le \lambda \le 1$, the section of the Cayley polytope

$$\mathbb{C}\mathrm{ay}(\mathbb{P}, \mathbb{P}') = \mathrm{conv}\left(\{(\boldsymbol{x}, -1) \mid \boldsymbol{x} \in \mathbb{P}\} \cup \{(\boldsymbol{x'}, 1) \mid \boldsymbol{x'} \in \mathbb{P}'\}\right) \subset \mathbb{R}^{n+1}$$

by the hyperplane $\left\{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid x_{n+1} = \lambda\right\}$ is the Minkowski sum $\frac{1-\lambda}{2} \cdot \mathbb{P} + \frac{1+\lambda}{2} \cdot \mathbb{P}'$.

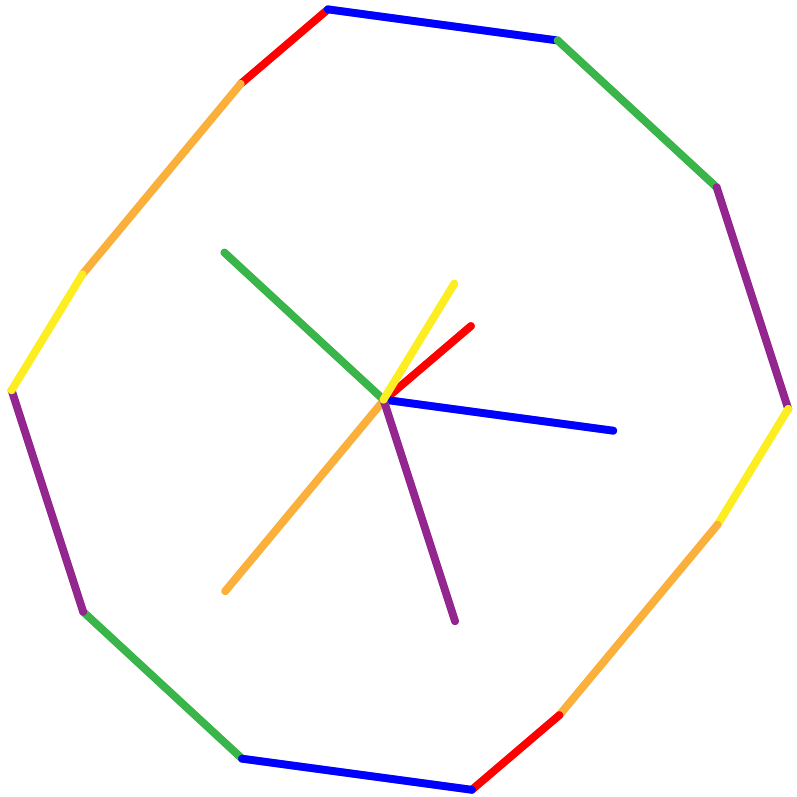


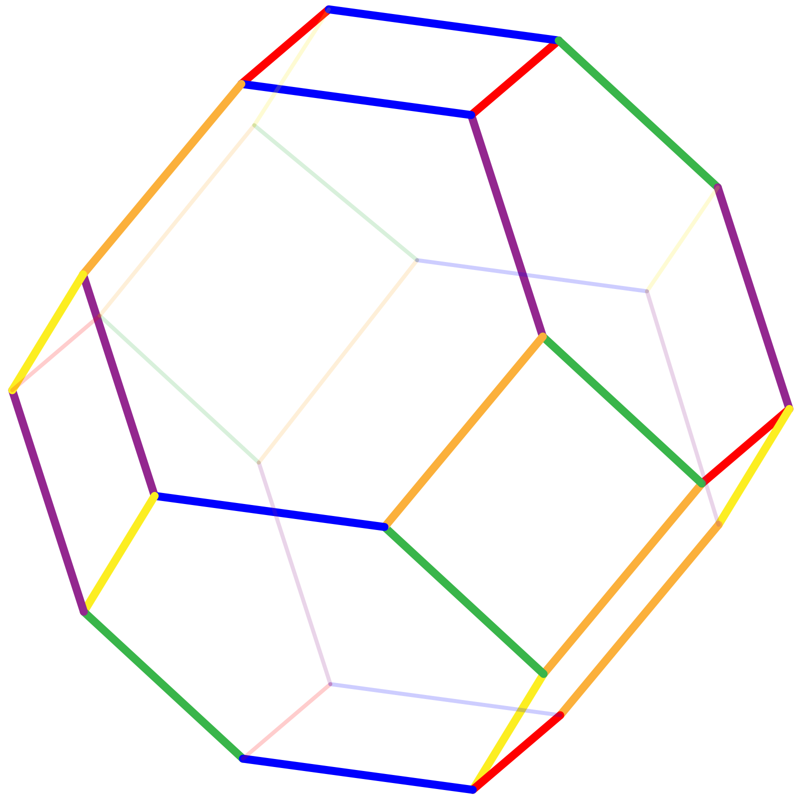


ZONOTOPE

DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x}' \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x}' \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

DEF. zonotope = Minkowki sum of segments
= projection of a cube $\square_d$





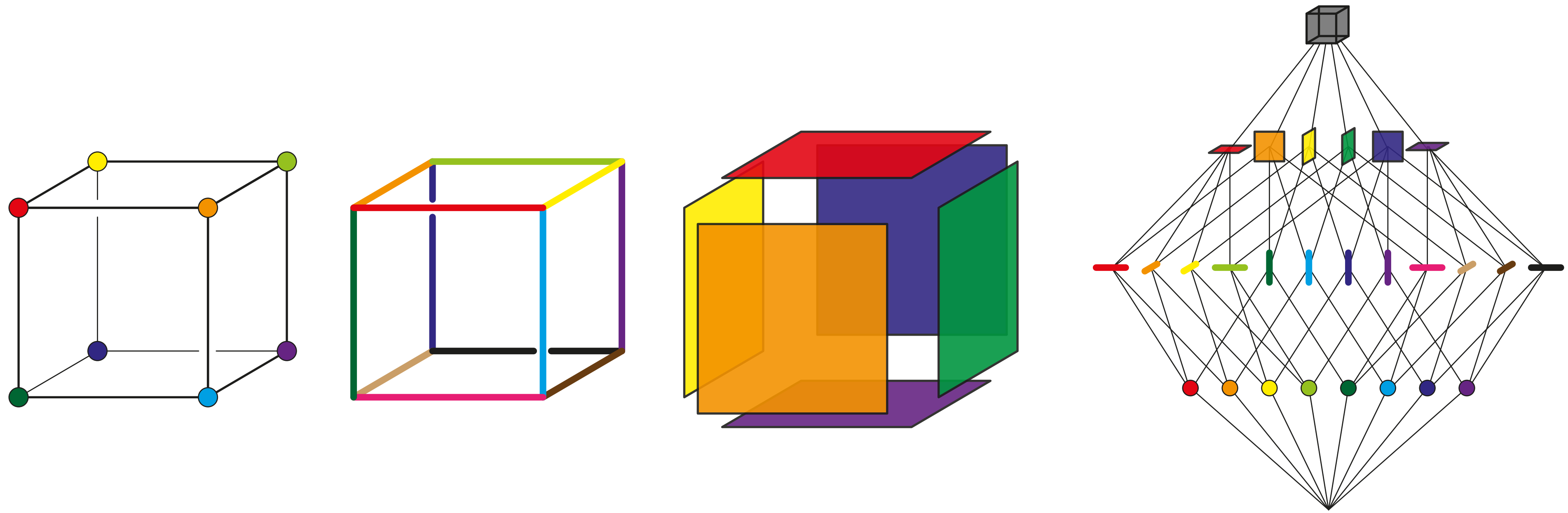
FACES

FACES

DEF. face of a polytope $\mathbb{P}$ =

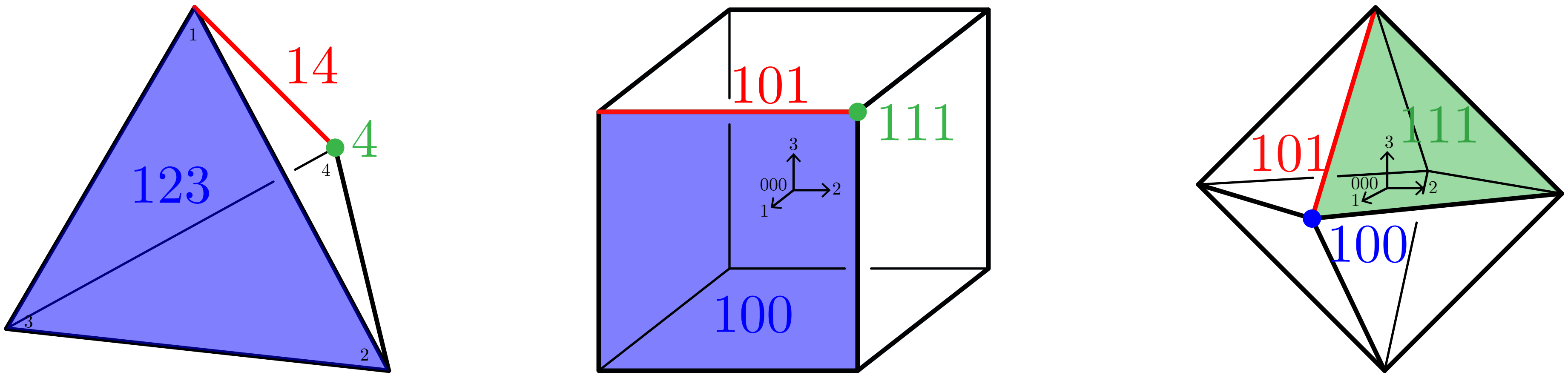
- either the polytope $\mathbb{P}$ itself,
- or the intersection of $\mathbb{P}$ with a supporting hyperplane of $\mathbb{P}$,
- or the empty set.

NOT. $\mathcal{F}(\mathbb{P}) = \{\text{faces of } \mathbb{P}\}$ and $\mathcal{F}_k(\mathbb{P}) = \{k\text{-dimensional faces of } \mathbb{P}\}$.



vertices $= \mathcal{F}_0(\mathbb{P})$ edges $= \mathcal{F}_1(\mathbb{P})$ ridges $= \mathcal{F}_{d-2}(\mathbb{P})$ facets $= \mathcal{F}_{d-1}(\mathbb{P})$

EXM: FACES OF CLASSICAL POLYTOPES



PROP. The faces of the d-simplex $\triangle_d$, the d-cube $\square_d$ and the d-cross-polytope $\lozenge_d$ are:

- d-simplex $\triangle_d$:

  subset I of $[d+1]$ $\longleftrightarrow$ face $\triangle_I = \operatorname{conv}\{\boldsymbol{e}_i \mid i \in I\}$.

- d-cube $\square_d$: the empty face $\varnothing$ and

  word w in $\{-1,0,1\}^d$ $\longleftrightarrow$ face $\square_w = \{\boldsymbol{x} \in \square_d \mid w_i(x_i - w_i) = 0$ for all $i \in [d]\}$.

- d-cross-polytope $\lozenge_d$: the d-cross-polytope $\lozenge_d$ itself and

  word w in $\{-1,0,1\}^d$ $\longleftrightarrow$ face $\triangle_w = \operatorname{conv}\{w_i\boldsymbol{e}_i \mid i \in [d]$ st $w_i \neq 0\}$.

FACE PROPERTIES

PROP. For a polytope $\mathbb{P}$,

- $\mathbb{P} = \mathrm{conv}(\mathcal{F}_0(\mathbb{P}))$ (a polytope is the convex hull of its vertices),
- $\mathbb{P} = \mathrm{conv}(\boldsymbol{X}) \Longrightarrow \mathcal{F}_0(\mathbb{P}) \subseteq \boldsymbol{X}$ (all vertices of a polytope are extreme).

PROP. For a face $\mathbb{F}$ of a polytope $\mathbb{P}$,

- $\mathbb{F}$ is a polytope,
- $\mathcal{F}_0(\mathbb{F}) = \mathcal{F}_0(\mathbb{P}) \cap \mathbb{F}$,
- $\mathcal{F}(\mathbb{F}) = \{\mathbb{G} \in \mathcal{F}(\mathbb{P}) \mid \mathbb{G} \subseteq \mathbb{F}\} \subseteq \mathcal{F}(\mathbb{P})$.

PROP. $\mathcal{F}(\mathbb{P})$ is stable by intersection: $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{P}) \Longrightarrow \mathbb{F} \cap \mathbb{G} \in \mathcal{F}(\mathbb{P})$.

proof ideas: separation theorems, finding a suitable supporting hyperplane, ...

LATTICE

DEF. <u>lattice</u> = partially ordered set $(\mathcal{L}, \leq)$ where any subset $\mathcal{X} \subseteq \mathcal{L}$ admits

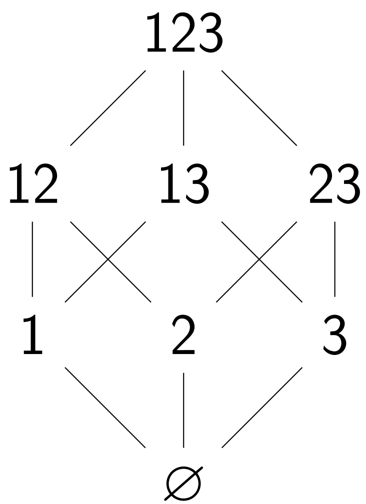
- a <u>meet</u> $\bigwedge \mathcal{X}$ = greatest lower bound

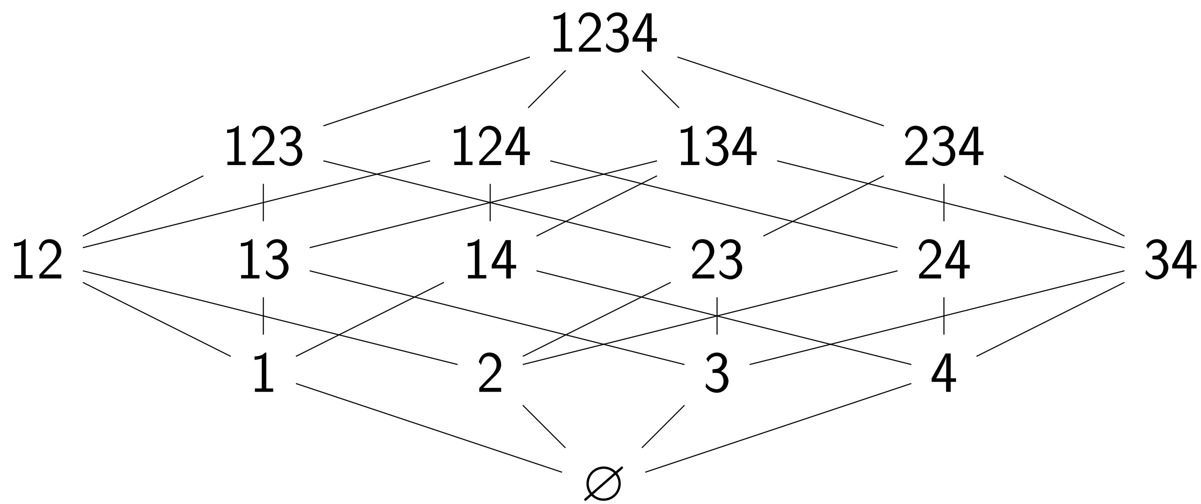
  $$\bigwedge \mathcal{X} \leq X \text{ for all } X \in \mathcal{X} \quad \text{and} \quad Y \leq X \text{ for all } X \in \mathcal{X} \text{ implies } Y \leq \bigwedge \mathcal{X}.$$

- a <u>join</u> $\bigvee \mathcal{X}$ = least upper bound

  $$X \leq \bigwedge \mathcal{X} \text{ for all } X \in \mathcal{X} \quad \text{and} \quad X \leq Y \text{ for all } X \in \mathcal{X} \text{ implies } \bigwedge \mathcal{X} \leq Y.$$

EXM. <u>boolean lattice</u> $\mathcal{B}(Y)$ = subsets of Y ordered by inclusion



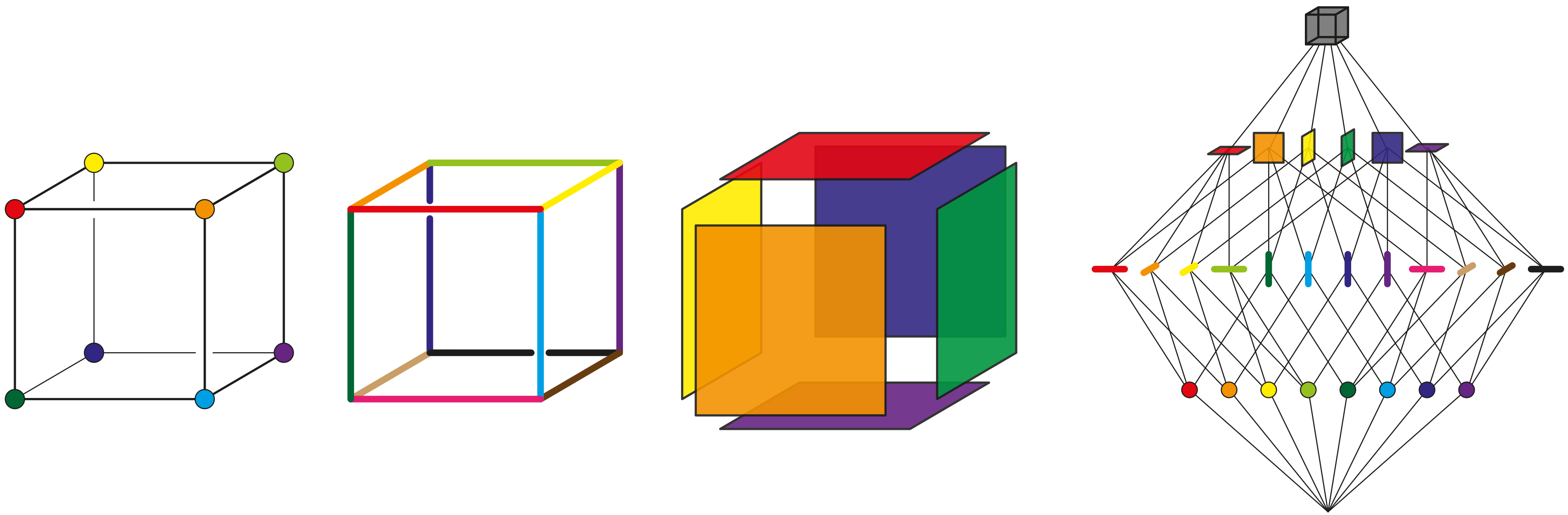


$$\bigwedge \mathcal{X} = \bigcap_{X \in \mathcal{X}} X \qquad \text{and} \qquad \bigvee \mathcal{X} = \bigcup_{X \in \mathcal{X}} X.$$

FACE LATTICE

PROP. The inclusion poset $\mathcal{F}(\mathbb{P})$ of faces of $\mathbb{P}$

- is a graded lattice (with rank function $\operatorname{rank}(\mathbb{F}) = \dim(\mathbb{F}) + 1$),
- is atomic (every face is the join of its vertices) and coatomic (every face is the meet of the facets containing it),
- every interval of $\mathcal{F}(\mathbb{P})$ is the face lattice of a polytope,
- has the diamond property (every interval of rank 2 has 4 elements).

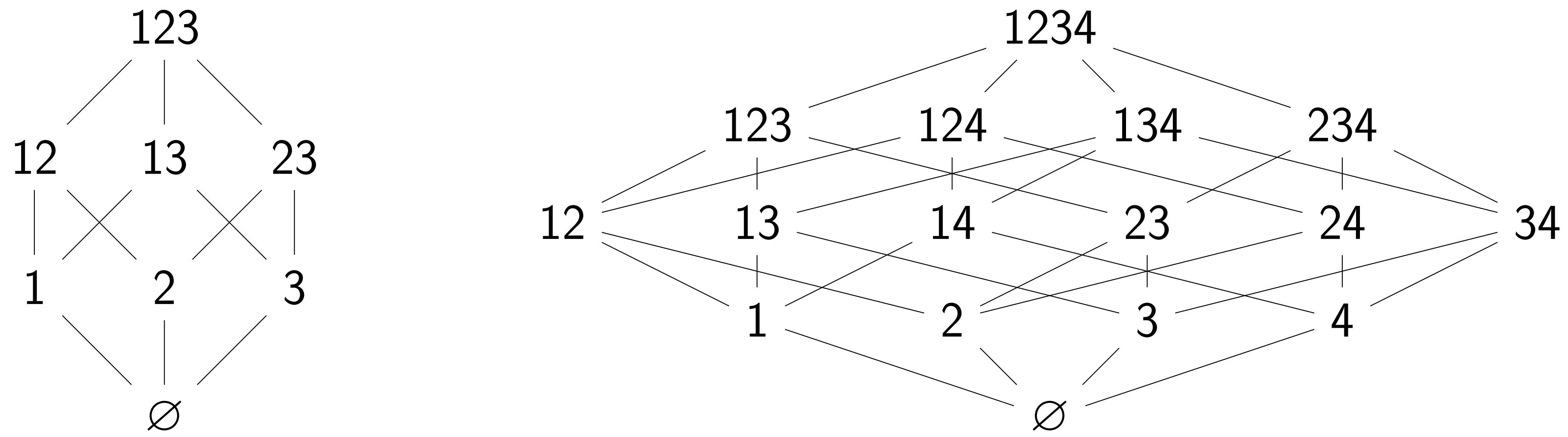


EXM: FACE LATTICES OF SIMPLICES

remark:

- any subset $I \subseteq [d+1]$ corresponds to a face $\triangle_I = \mathrm{conv}\,\{\boldsymbol{e}_i \mid i \in I\}$ of $\triangle_d$,
- $I \subseteq J \iff \triangle_I \subseteq \triangle_J$.

The face lattice of $\triangle_d$ is thus the boolean lattice on subsets of $[d+1]$:

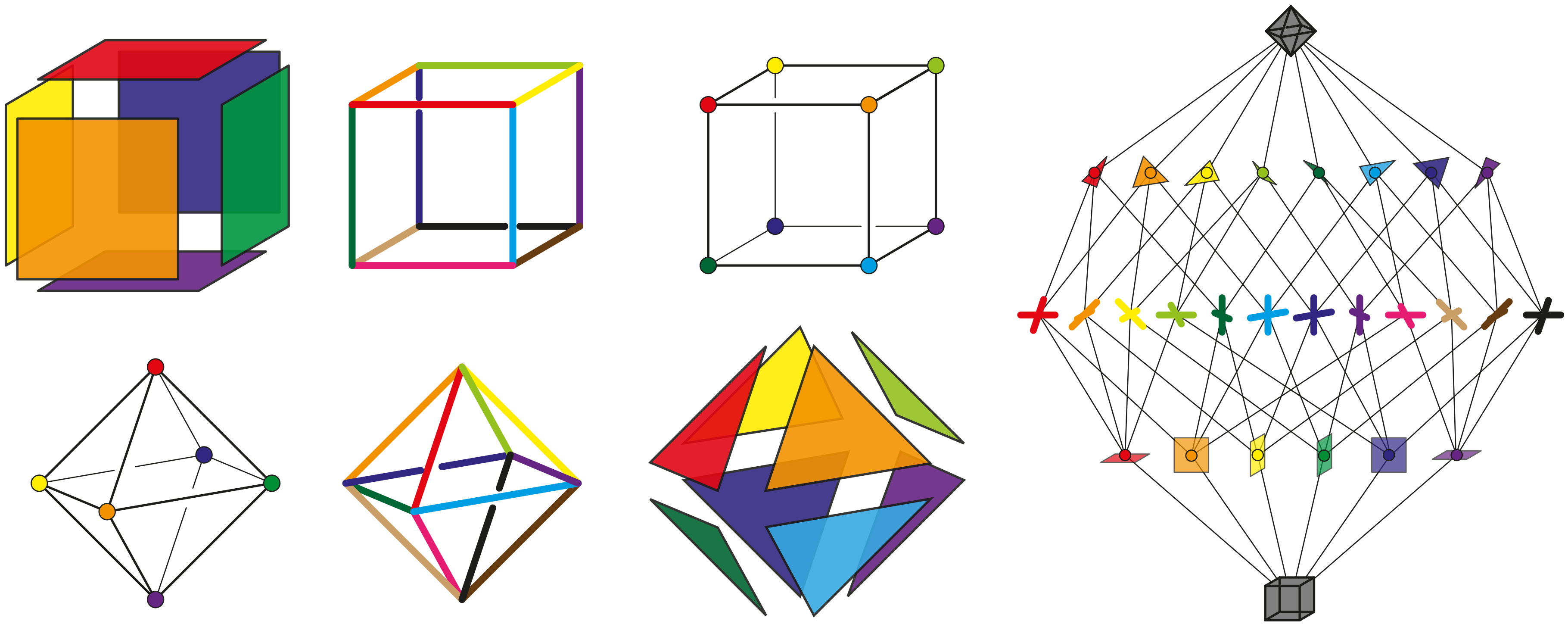


POLARITY AND FACES

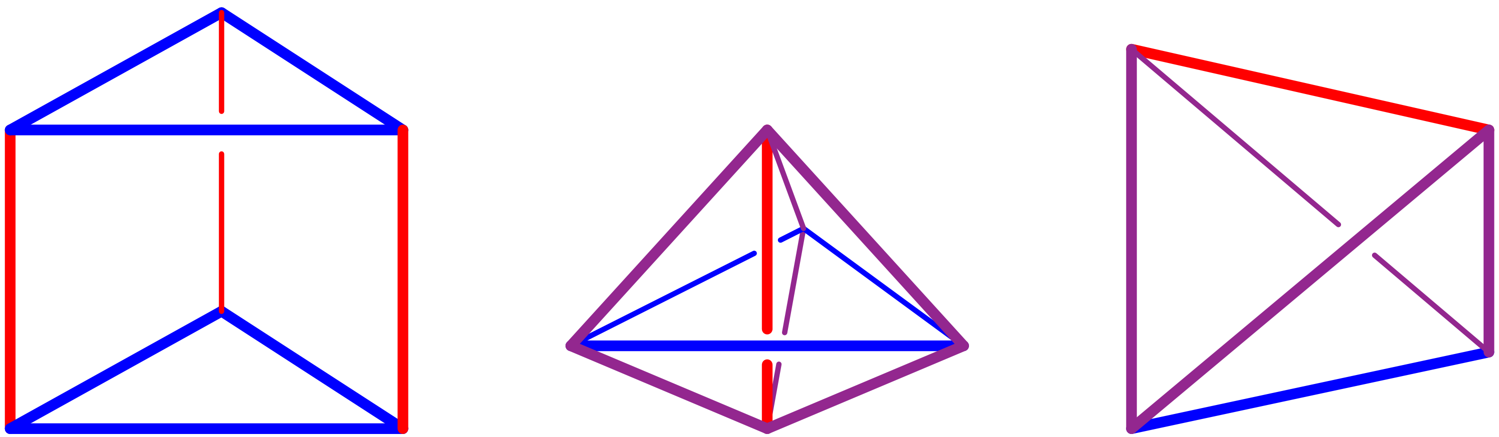
Assume $\mathbf{0} \in \mathrm{int}(\mathbb{P})$.

DEF. A face $\mathbb{F}$ of $\mathbb{P}$ defines a polar face $\mathbb{F}^\diamond = \{\boldsymbol{y} \in \mathbb{P}^\diamond \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y}\,\rangle = 1 \text{ for all } \boldsymbol{x} \in \mathbb{F}\}$.

PROP. The map $\mathbb{F} \longmapsto \mathbb{F}^\diamond$ is a lattice anti-isomorphism $\mathcal{F}(\mathbb{P}) \longrightarrow \mathcal{F}(\mathbb{P}^\diamond)$.



OPERATIONS AND FACES

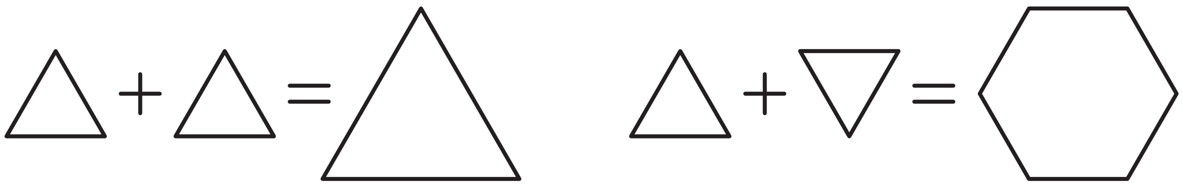


$\mathbb{P} \times \mathbb{P}'$ $\qquad$ $\mathbb{P} \oplus \mathbb{P}'$ $\qquad$ $\mathbb{P} * \mathbb{P}'$

PROP. Define $\mathcal{F}_\star(\mathbb{P}) = \mathcal{F}(\mathbb{P}) \smallsetminus \{\varnothing\}$ and $\mathcal{F}^\star(\mathbb{P}) = \mathcal{F}(\mathbb{P}) \smallsetminus \{\mathbb{P}\}$. Then

$$\mathcal{F}_\star(\mathbb{P} \times \mathbb{P}') = \{\mathbb{F} \times \mathbb{F}' \mid \mathbb{F} \in \mathcal{F}_\star(\mathbb{P}) \text{ and } \mathbb{F}' \in \mathcal{F}_\star(\mathbb{P}')\}$$
$$\mathcal{F}^\star(\mathbb{P} \oplus \mathbb{P}') = \{\mathbb{F} * \mathbb{F}' \mid \mathbb{F} \in \mathcal{F}^\star(\mathbb{P}) \text{ and } \mathbb{F}' \in \mathcal{F}^\star(\mathbb{P}')\}$$
$$\mathcal{F}(\mathbb{P} * \mathbb{P}') = \{\mathbb{F} * \mathbb{F}' \mid \mathbb{F} \in \mathcal{F}(\mathbb{P}) \text{ and } \mathbb{F}' \in \mathcal{F}(\mathbb{P}')\}$$

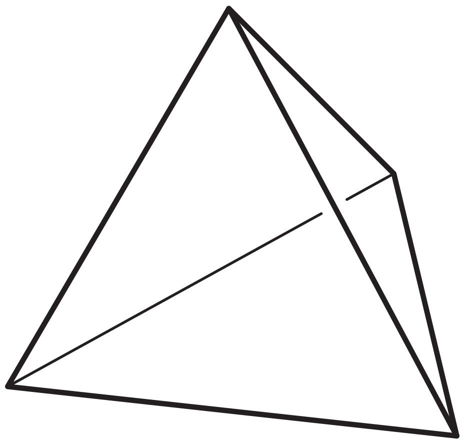
remark: the combinatorial structure of $\mathbb{P} + \mathbb{P}'$ depends on the geometry of $\mathbb{P}$ and $\mathbb{P}'$.



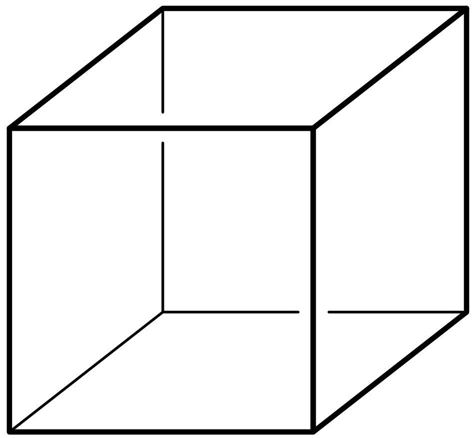
SIMPLE OR SIMPLICIAL POLYTOPES

DEF. A d-polytope $\mathbb{P}$ is

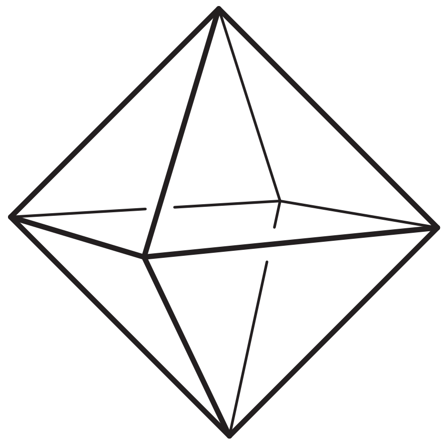
- simplicial if its vertices are in general position,
- simple if its facets are in general position.



simple and simplicial



simple but not simplicial



not simple but simplicial

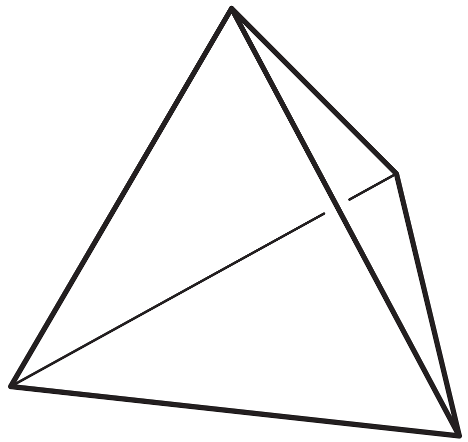
PROP. $\mathbb{P}$ is simple $\iff$ $\mathbb{P}^\diamond$ is simplicial.

QU. Show that a simple and simplicial polytope is a polygon or a simplex.

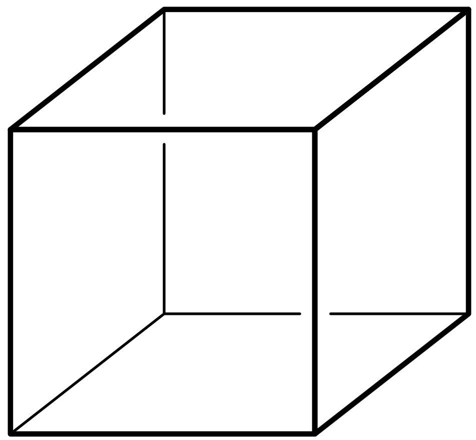
SIMPLE OR SIMPLICIAL POLYTOPES

DEF. A d-polytope $\mathbb{P}$ is

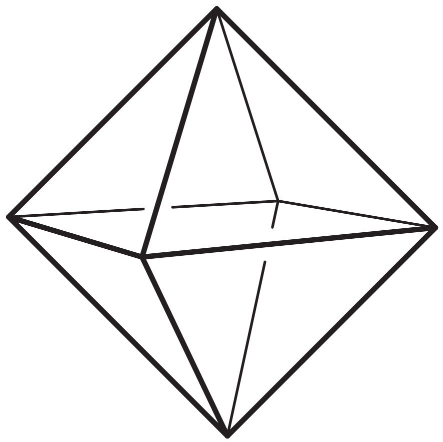
- simplicial if each facet contains d vertices (ie. is a simplex),
- simple if each vertex is contained in d edges (or equiv. in d facets).



simple and simplicial



simple but not simplicial

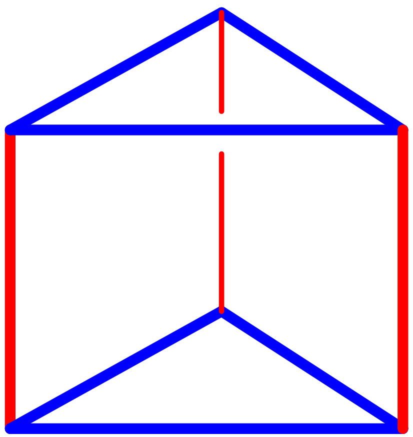


not simple but simplicial

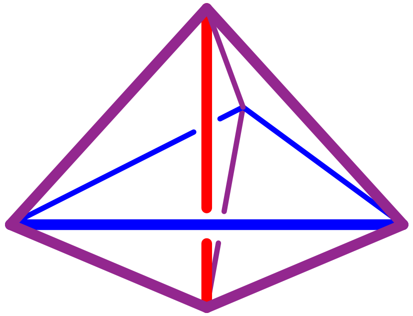
PROP. $\mathbb{P}$ is simple $\iff$ $\mathbb{P}^\diamond$ is simplicial.

QU. Show that a simple and simplicial polytope is a polygon or a simplex.

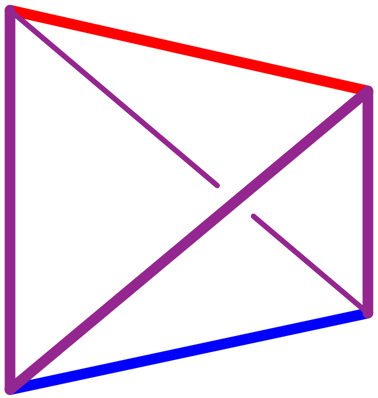
SIMPLE OR SIMPLICIAL POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$



$\mathbb{P} * \mathbb{P}'$

PROP.

$\mathbb{P}$ and $\mathbb{P}'$ simple $\iff$ $\mathbb{P} \times \mathbb{P}'$ simple

$\mathbb{P}$ and $\mathbb{P}'$ simplicial $\iff$ $\mathbb{P} \oplus \mathbb{P}'$ simplicial

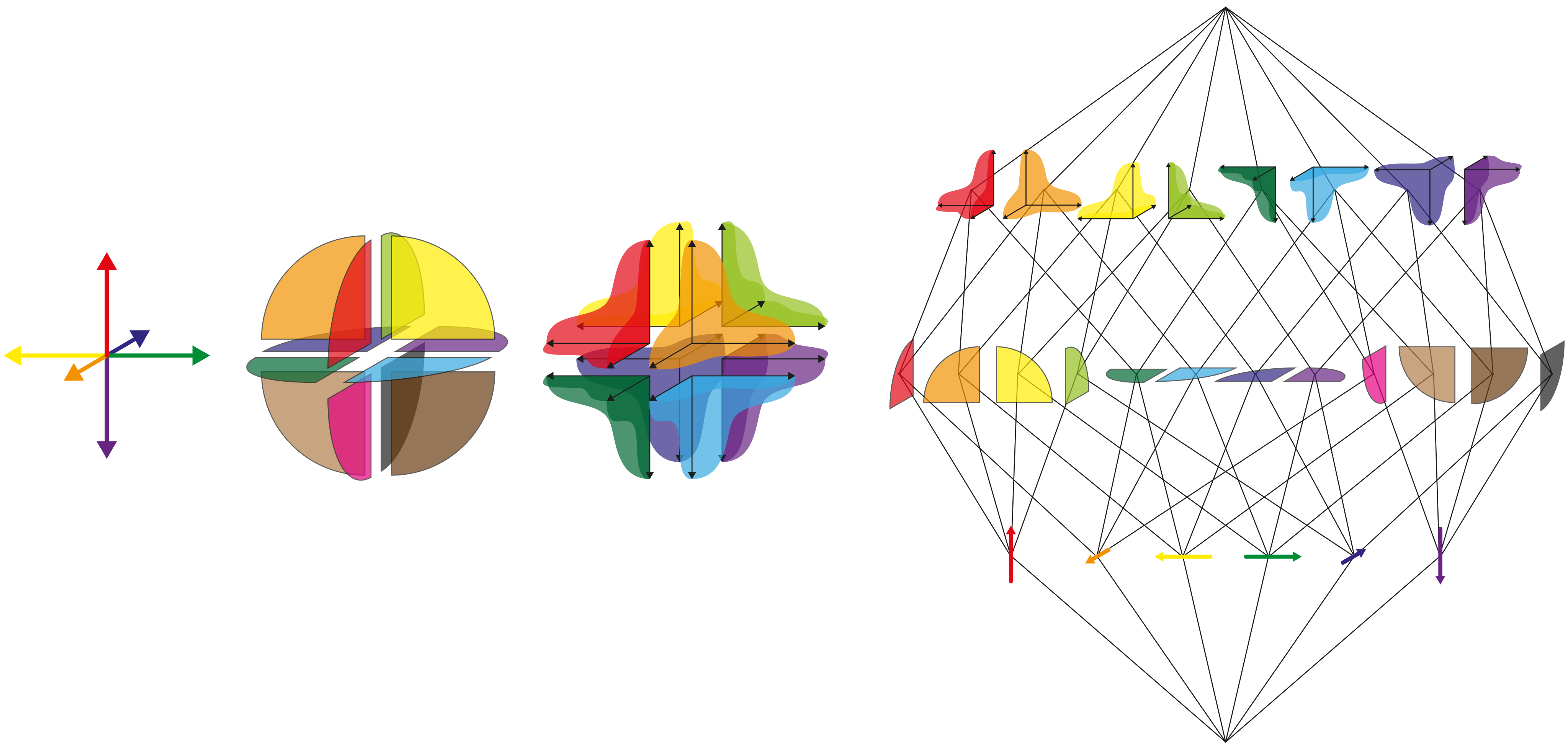
$\mathbb{P}$ and $\mathbb{P}'$ simplices $\iff$ $\mathbb{P} * \mathbb{P}'$ simple (or simplicial)

FANS

FAN

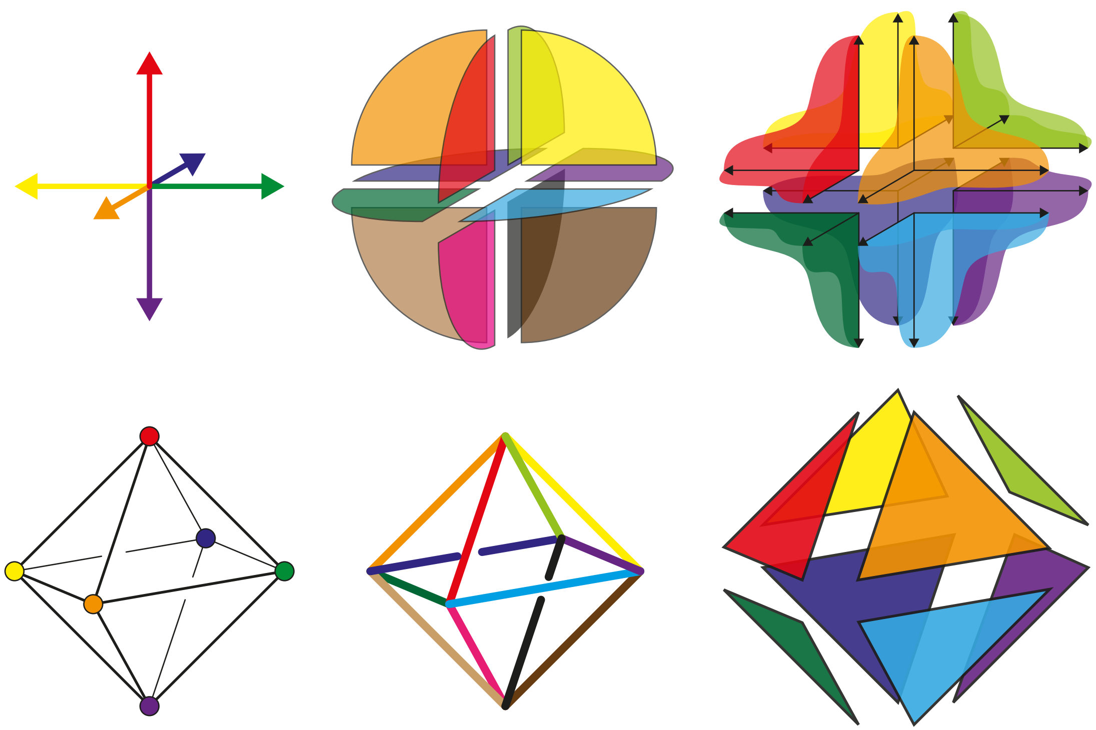
DEF. fan $\mathcal{F}$ = collection of polyhedral cones st

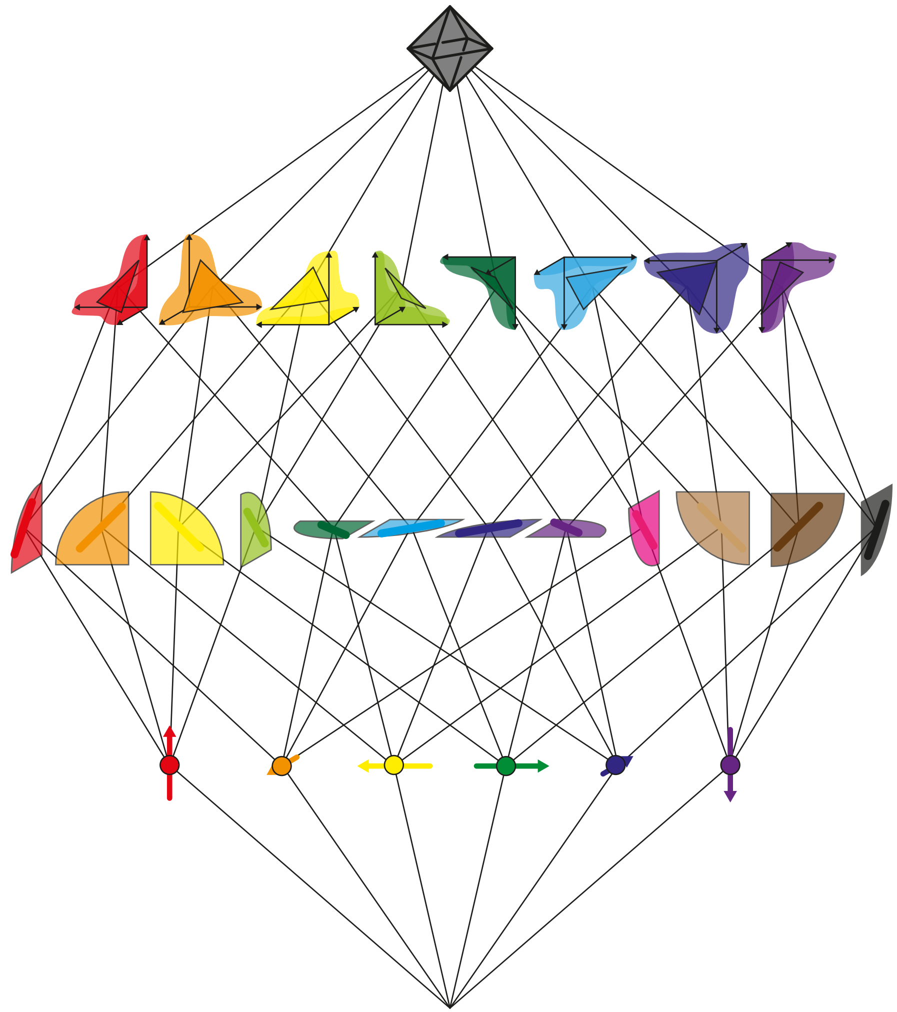
- closed by faces: if $\mathbb{C} \in \mathcal{F}$ and $\mathbb{C}'$ is a face of $\mathbb{C}$, then $\mathbb{C}' \in \mathcal{F}$,
- intersecting properly: if $\mathbb{C}, \mathbb{C}' \in \mathcal{F}$, the intersection $\mathbb{C} \cap \mathbb{C}'$ is a face of $\mathbb{C}$ and $\mathbb{C}'$.



FACE FAN

DEF. $\mathbb{P}$ polytope with $\mathbf{0} \in \operatorname{int}(\mathbb{P})$. $\mathbb{F}$ face of $\mathbb{P}$.
face cone of $\mathbb{F}$ = cone $\mathbb{R}_{\geq 0}\mathbb{F}$ generated by $\mathbb{F}$.
face fan of $\mathbb{P}$ = collection of face cones of all faces of $\mathbb{P}$.



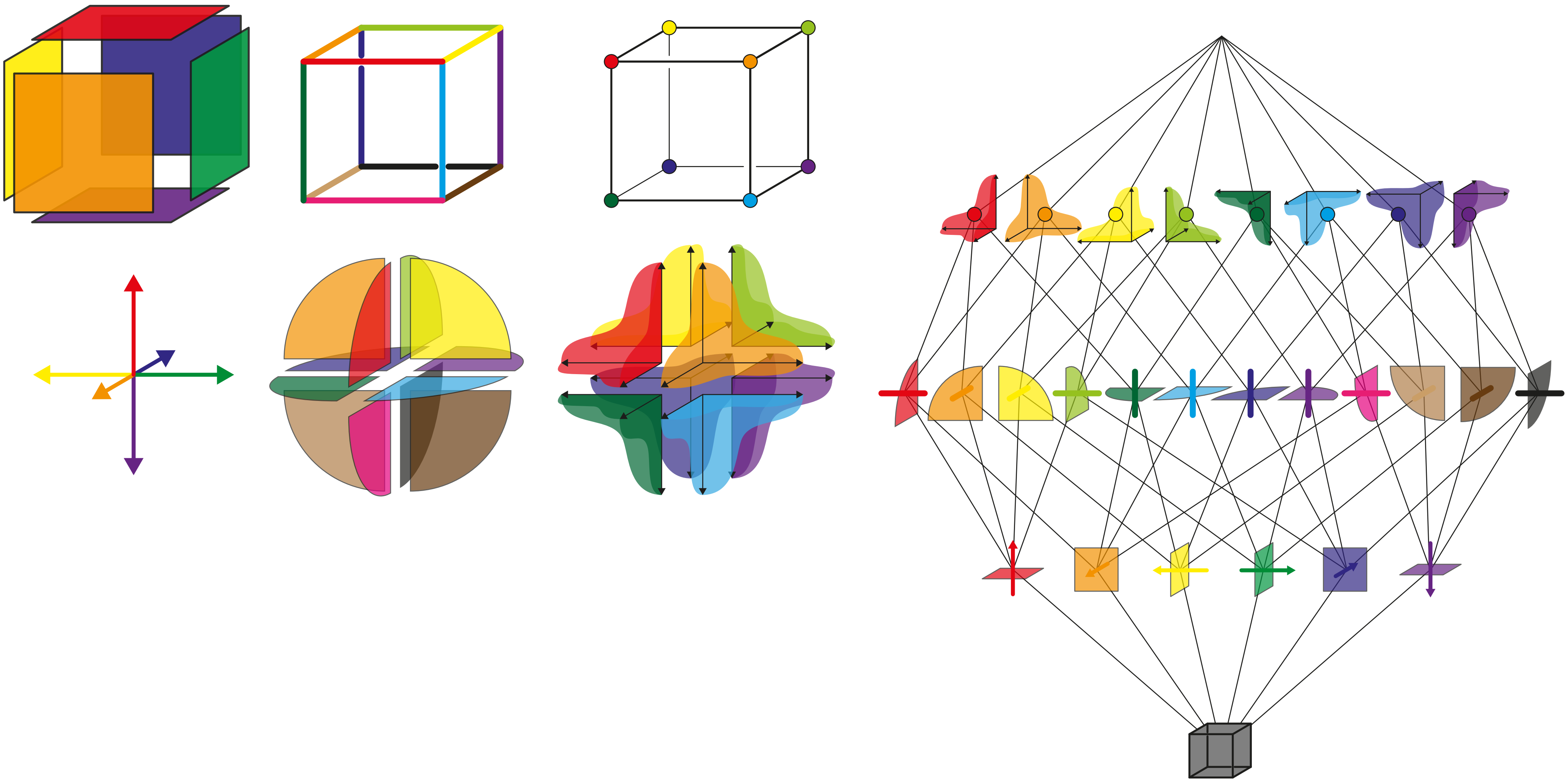


NORMAL FAN

DEF. $\mathbb{P}$ polytope. $\mathbb{F}$ face of $\mathbb{P}$.

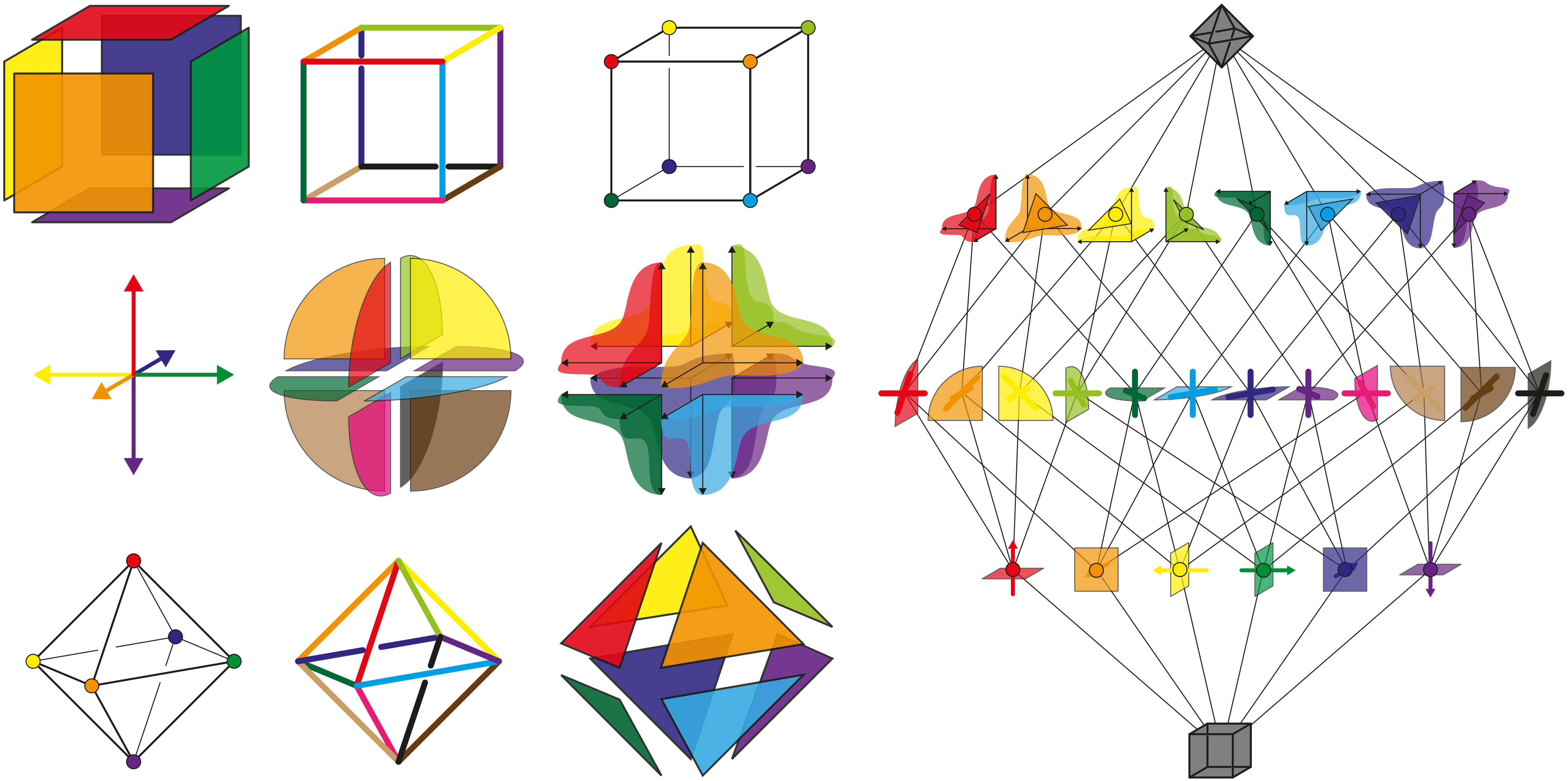
normal cone of $\mathbb{F}$ $=$ cone generated by outer normal vectors to facets of $\mathbb{P}$ containing $\mathbb{F}$.

normal fan of $\mathbb{P}$ $=$ collection of normal cones of all faces of $\mathbb{P}$.



FACE FAN VS NORMAL FAN

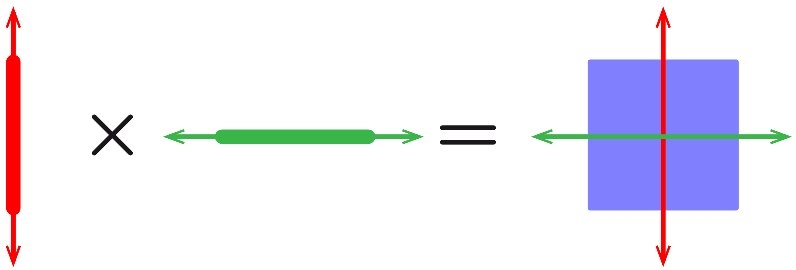
PROP. If $\mathbf{0} \in \operatorname{int}(\mathbb{P})$, then the face fan of $\mathbb{P}$ coincides with the normal fan of $\mathbb{P}^\diamond$.



NORMAL FANS AND POLYTOPE OPERATIONS

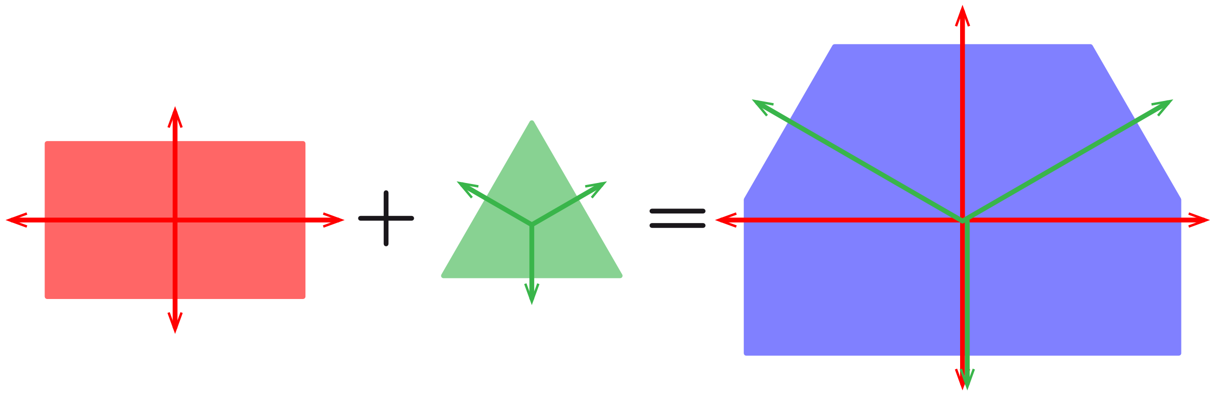
DEF. <u>direct sum</u> $\mathcal{F} \oplus \mathcal{F}' = \{\mathbb{C} \times \mathbb{C}' \mid \mathbb{C} \in \mathcal{F} \text{ and } \mathbb{C}' \in \mathcal{F}'\}$

PROP. normal fan of $\mathbb{P} \times \mathbb{P}'$ = direct sum of normal fans of $\mathbb{P}$ and $\mathbb{P}'$.



DEF. <u>common refinement</u> $\mathcal{F} \wedge \mathcal{F}' = \{\mathbb{C} \cap \mathbb{C}' \mid \mathbb{C} \in \mathcal{F} \text{ and } \mathbb{C}' \in \mathcal{F}'\}$

PROP. normal fan of $\mathbb{P} + \mathbb{P}'$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{P}'$.

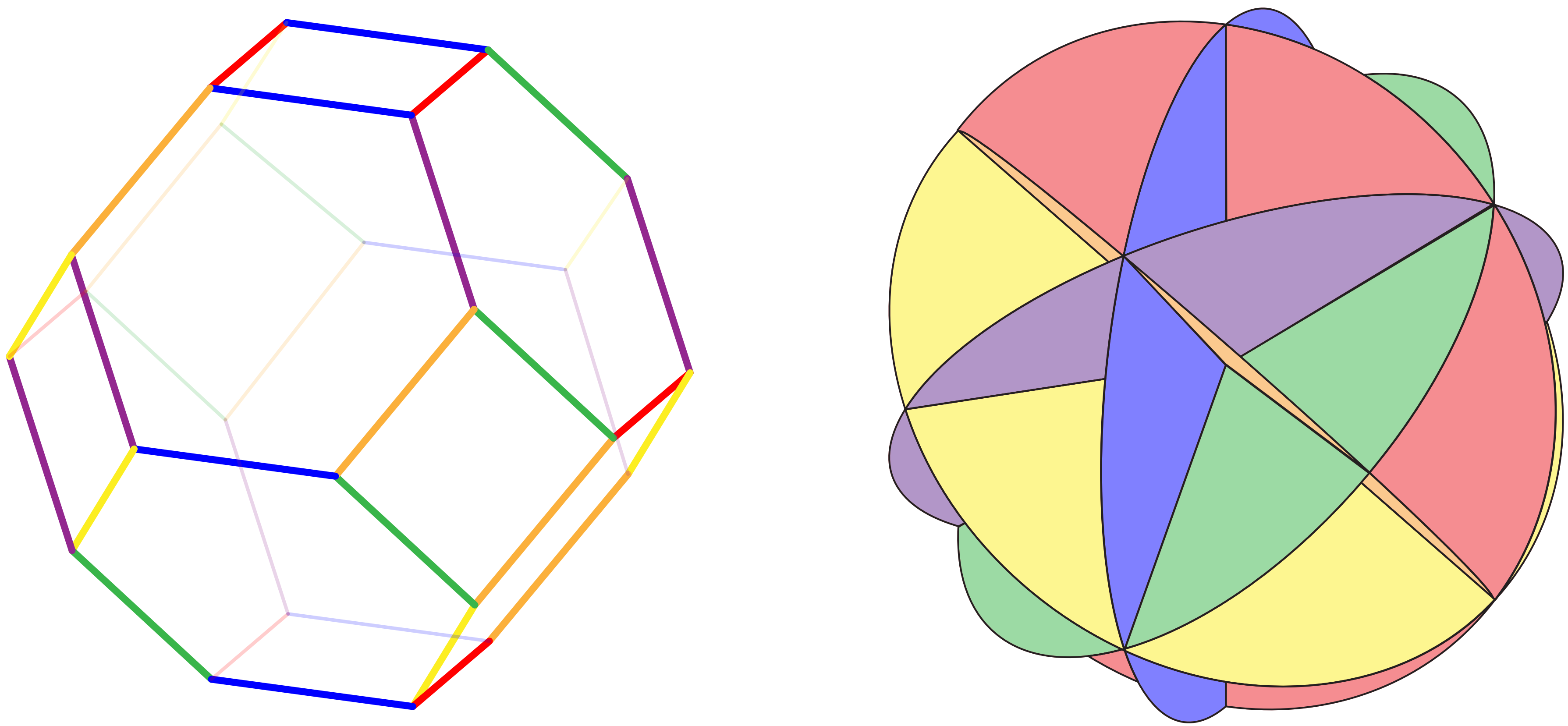


NORMAL FANS OF ZONOTOPES

DEF. common refinement $\mathcal{F} \wedge \mathcal{F}' = \{\mathbb{C} \cap \mathbb{C}' \mid \mathbb{C} \in \mathcal{F} \text{ and } \mathbb{C}' \in \mathcal{F}'\}$

PROP. normal fan of $\mathbb{P} + \mathbb{P}'$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{P}'$.

PROP. normal fans of zonotopes $\iff$ fans defined by hyperplane arrangements.

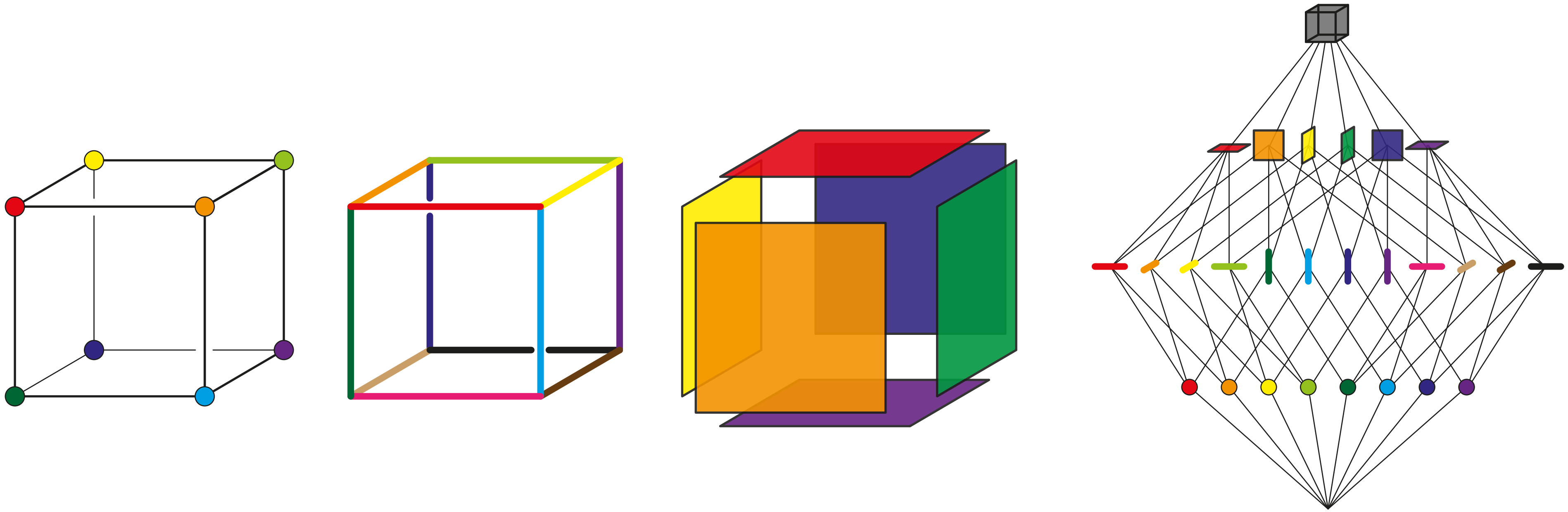


F-VECTOR & EULER RELATION

F-VECTOR & F-POLYNOMIAL

DEF. For a d-polytope $\mathbb{P}$,

- $f_i(\mathbb{P})$ = number of i-faces of $\mathbb{P}$,
- f-vector $f(\mathbb{P}) = \big(f_0(\mathbb{P}), \dots, f_d(\mathbb{P})\big)$,
- f-polynomial $f(\mathbb{P}, x) = \sum_{i=0}^{d} f_i(\mathbb{P})\, x^i$.



$$f(\square_3) = 8 + 12\,x + 6\,x^2 + x^3$$

In fact, it is convenient to define

$$F(\mathbb{P}, x) = \sum_{i=-1}^{d} f_i(\mathbb{P})\, x^{i+1}$$

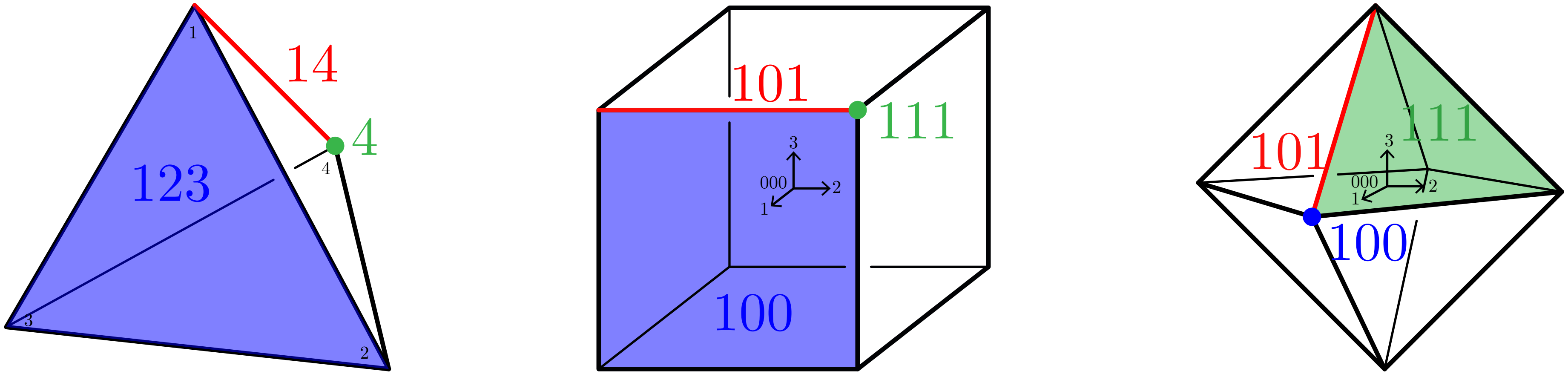
and to consider

$$f(\mathbb{P}, x) = \sum_{i=0}^{d} f_i(\mathbb{P})\, x^{i} = \frac{F(\mathbb{P}, x) - 1}{x}$$

and

$$\bar{f}(\mathbb{P}, x) = \sum_{i=-1}^{d-1} f_i(\mathbb{P})\, x^{i+1} = F(\mathbb{P}, x) - x^{d+1}$$

EXM: F-VECTOR OF CLASSICAL POLYTOPES



PROP. The f-vectors and F-polynomials of the d-simplex $\triangle_d$, the d-cube $\square_d$ and the d-cross-polytope $\lozenge_d$ are given by

$$f_i(\triangle_d) = \binom{d+1}{i+1} \qquad f_i(\square_d) = \binom{d}{i} 2^{d-i} \qquad f_i(\lozenge_d) = \binom{d}{i+1} 2^{i+1}$$

$$F(\triangle_d, x) = (x+1)^{d+1} \qquad F(\square_d, x) = 1 + x(x+2)^d \qquad F(\lozenge_d, x) = x^{d+1} + (2x+1)^d$$

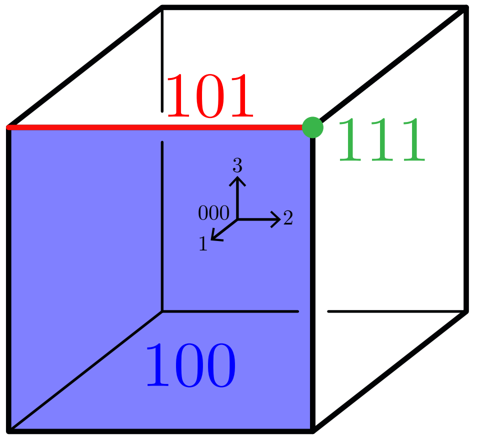
REM. In other words,

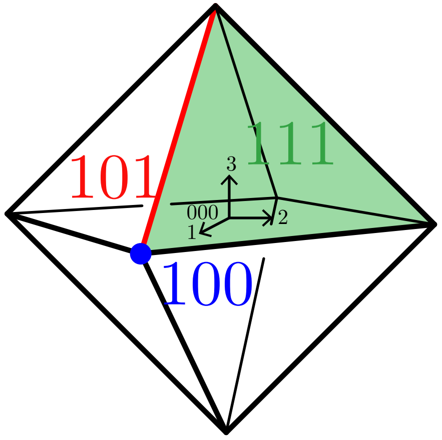
$$F(\triangle_d, x) = (x+1)^{d+1} \qquad f(\square_d, x) = (x+2)^d \qquad \bar{f}(\lozenge_d, x) = (2x+1)^d$$

EXM: F-VECTOR & POLARITY

PROP. $F(\mathbb{P}, x) = x^{d+1} F(\mathbb{P}^\diamond, 1/x)$

proof: $\mathbb{F} \longmapsto \mathbb{F}^\diamond$ anti-isomorphism, thus $f_i(\mathbb{P}) = f_{d-i-1}(\mathbb{P}^\diamond)$, thus $F_i(\mathbb{P}) = F_{d+1-i}(\mathbb{P}^\diamond)$.

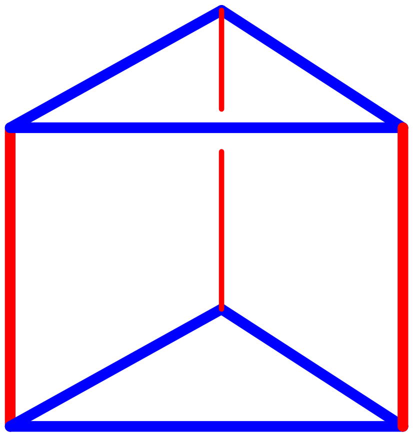




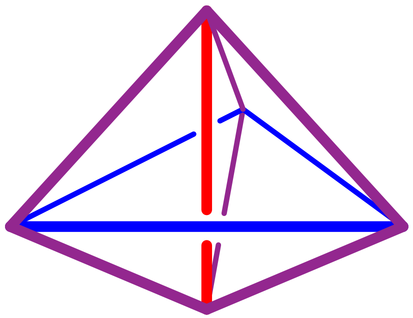
remark: sanity check on classical polytopes

$$F(\square_d, x) = 1 + x(x+2)^d \qquad F(\Diamond_d, x) = x^{d+1} + (2x+1)^d \qquad F(\triangle_d, x) = (x+1)^{d+1}$$

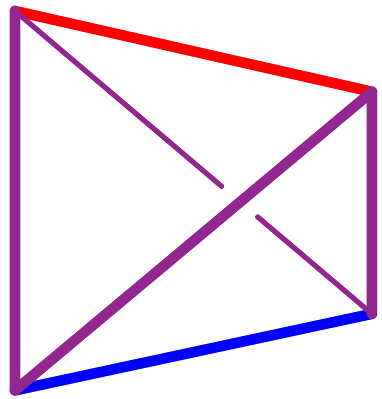
EXM: F-VECTORS & POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$



$\mathbb{P} * \mathbb{P}'$

PROP. The f-vectors and f-polynomials of the Cartesian product $\mathbb{P} \times \mathbb{P}'$, the direct sum $\mathbb{P} \oplus \mathbb{P}'$ and the join $\mathbb{P} * \mathbb{P}'$ are given by

$$f(\mathbb{P} \times \mathbb{P}', x) = f(\mathbb{P}, x) \cdot f(\mathbb{P}', x)$$
$$\bar{f}(\mathbb{P} \oplus \mathbb{P}', x) = \bar{f}(\mathbb{P}, x) \cdot \bar{f}(\mathbb{P}', x)$$
$$F(\mathbb{P} * \mathbb{P}', x) = F(\mathbb{P}, x) \cdot F(\mathbb{P}', x)$$

remark: sanity check on classical polytopes

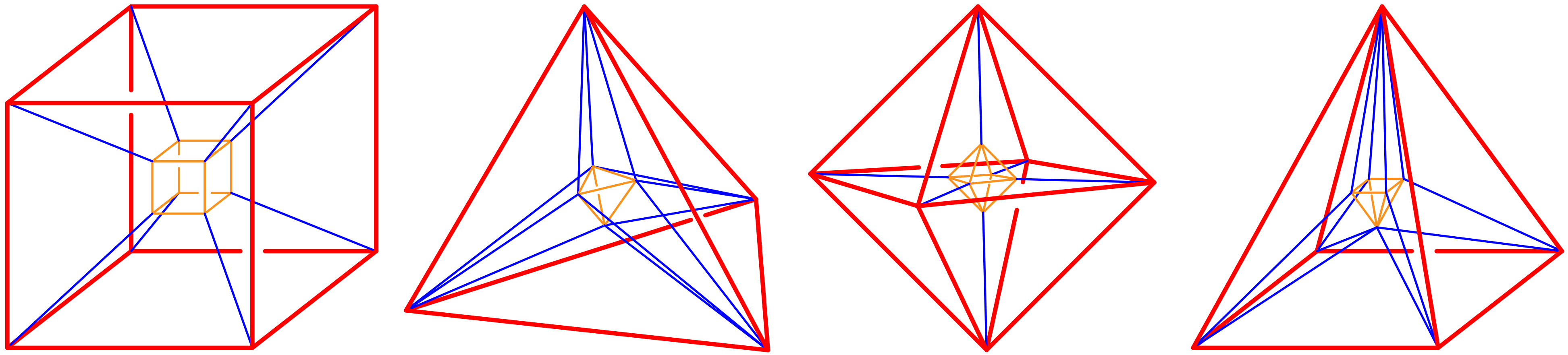
$$f(\square_d, x) = (x+2)^d \qquad \bar{f}(\Diamond_d, x) = (2x+1)^d \qquad F(\triangle_d, x) = (x+1)^{d+1}$$

HANNER POLYTOPES

DEF. Hanner polytope $=$ either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

EXM. The small dimensional Hanner polytopes are:

- $d = 1$: interval I,
- $d = 2$: square $I \oplus I \sim I \times I$,
- $d = 3$: cube $I^{\times 3} := I \times I \times I$ and cross-polytope $I^{\oplus 3} := I \oplus I \oplus I$,
- $d = 4$: cube $I^{\times 4}$, cross-polytope $I^{\oplus 4}$, prism over an octahedron $I^{\oplus 3} \times I$ and bipyramid over a cube $I^{\times 3} \oplus I$.



(Schlegel diagrams...)

HANNER POLYTOPES

DEF. Hanner polytope = either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

EXM. The small dimensional Hanner polytopes are:

- $d = 1$: interval I,
- $d = 2$: square $I \oplus I \sim I \times I$,
- $d = 3$: cube $I^{\times 3} := I \times I \times I$ and cross-polytope $I^{\oplus 3} := I \oplus I \oplus I$,
- $d = 4$: cube $I^{\times 4}$, cross-polytope $I^{\oplus 4}$, prism over an octahedron $I^{\oplus 3} \times I$ and bipyramid over a cube $I^{\times 3} \oplus I$.

REM. The Hanner polytope $P := (I \times I \times I) \oplus (I \times I \times I)$ cannot be

- a bipyramid: it has 16 vertices each of degree 11,
- a prism: it has 36 facets each of degree 8.

3^D CONJECTURE

DEF. Hanner polytope = either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

PROP. For any d-dimensional Hanner polytope $\mathbb{H}$,

$$\sum_{i=0}^{d} f_i(\mathbb{H}) = 3^d.$$

proof: $\sum_{i=0}^{d} f_i(\mathbb{H}) = f(\mathbb{H}, 1) = \bar{f}(\mathbb{H}, 1)$ together with

$$f(\mathbb{P} \times \mathbb{P}', x) = f(\mathbb{P}, x) \cdot f(\mathbb{P}', x) \qquad \text{and} \qquad \bar{f}(\mathbb{P} \oplus \mathbb{P}', x) = \bar{f}(\mathbb{P}, x) \cdot \bar{f}(\mathbb{P}', x).$$

CONJ. (Kalai's 3^d conjecture) If $\mathbb{P}$ is centrally symmetric (meaning $\mathbb{P} = -\mathbb{P}$), then

$$\sum_{i=0}^{d} f_i(\mathbb{P}) \geq 3^d,$$

with equality if and only if $\mathbb{P}$ is a Hanner polytope.

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

PROP. For any polytope $\mathbb{P}$ and hyperplane $\mathbb{H}$,

$$\chi(\mathbb{P}) = \chi(\mathbb{P}^+) + \chi(\mathbb{P}^-) - \chi(\mathbb{P}^\circ).$$

where $\mathbb{P}^+ = \mathbb{P} \cap \mathbb{H}^+$, $\mathbb{P}^- = \mathbb{P} \cap \mathbb{H}^-$ and $\mathbb{P}^\circ = \mathbb{P} \cap \mathbb{H}$.

PROP. For any polytopes $\mathbb{P}, \mathbb{Q} \subset \mathbb{R}^n$ st $\mathbb{P} \cup \mathbb{Q}$ is a polytope,

$$\chi(\mathbb{P} \cup \mathbb{Q}) + \chi(\mathbb{P} \cap \mathbb{Q}) = \chi(\mathbb{P}) + \chi(\mathbb{Q}).$$

remark: These conditions define weak valuations and strong valuations.
For polytopes, any weak valuation is a strong valuation.
Exm: indicator function, volume, number of integer points, etc.

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1$.

proof: Induction on the dimension.

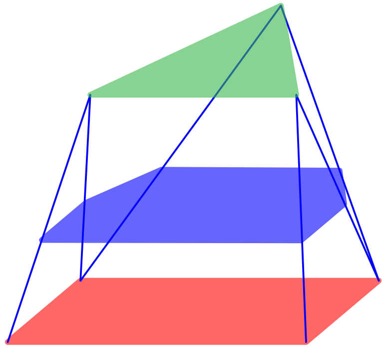
EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d}(-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1.$

proof: Induction on the dimension.

1. Observe first that it holds for Cayley polytopes (in particular for pyramids):



$$\chi(\mathbb{C}\mathrm{ay}(\mathbb{P}, \mathbb{R})) = \chi(\mathbb{P}) + \chi(\mathbb{R}) + (-1) \cdot \chi(\mathbb{Q})$$
$$= 1 + 1 - 1 = 1$$

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

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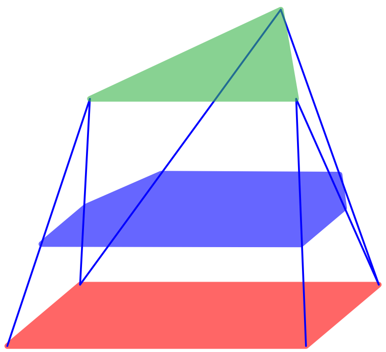
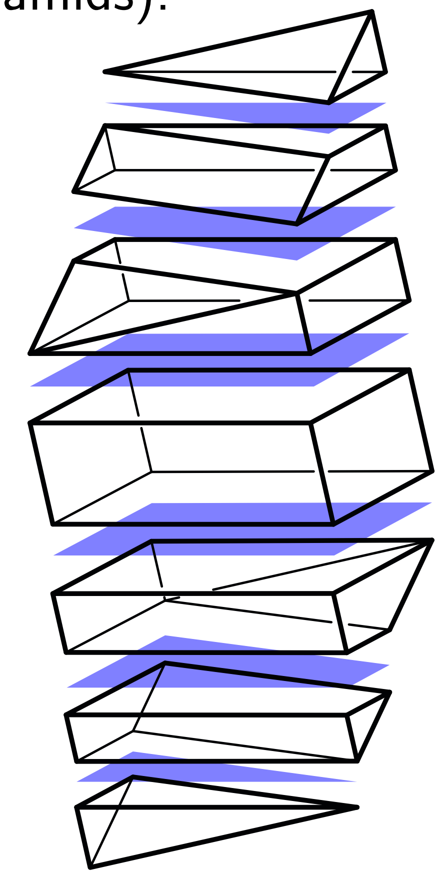
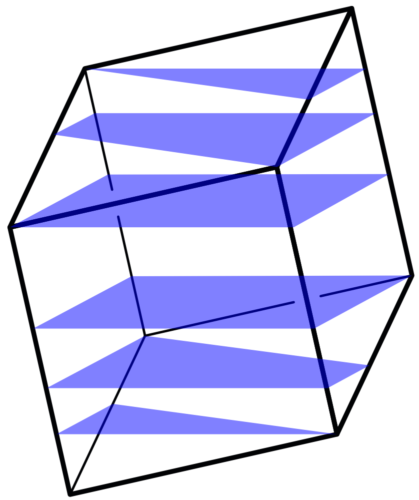


$$\begin{aligned}\chi(\mathbb{C}\mathrm{ay}(\mathbb{P}, \mathbb{R})) &= \chi(\mathbb{P}) + \chi(\mathbb{R}) + (-1) \cdot \chi(\mathbb{Q}) \\ &= 1 + 1 - 1 = 1\end{aligned}$$



2. Choose a Morse function ϕ,
slice the polytope $\mathbb{P}$ into Cayley polytopes,
and apply the valuation property:

$$\begin{aligned}\chi(\mathbb{P}) &= \chi(\mathbb{P}_0) - \chi(\mathbb{S}_1) + \cdots - \chi(\mathbb{S}_k) + \chi(\mathbb{P}_k) \\ &= 1 - 1 + \cdots - 1 + 1 = 1\end{aligned}$$



EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1$.

PROP. Let $\mathbb{P}_{i,d} = \mathbb{P}\mathrm{yr}^{d-i}(\square_i)$ for $i \in [d]$. The f-vectors $f(\mathbb{P}_{i,d})$ are affinely independent.

proof: induction on the dimension d.

Affine dependance among f-vectors $\longleftrightarrow$ affine dependance among F-polynomials.

$\mathbb{P}_{i,d} = \square_i * \triangle_{d-i} \implies F(\mathbb{P}_{i,d}, x) = F(\square_i, x) \cdot F(\triangle_{d-i}, x) = (1 + x(x+2)^i) \cdot (x+1)^{d-i+1}$.

Assume $0 = \sum_{i=0}^{d} \lambda_i F(\mathbb{P}_{i,d}, x)$. Two cases:

- if $\lambda_d = 0$, then $0 = \sum_{i=0}^{d-1} \lambda_i F(\mathbb{P}_{i,d}, x) = (x+1) \cdot \sum_{i=0}^{d-1} \lambda_i F(\mathbb{P}_{i,d-1}, x)$ and induction.
- if $\lambda_d \neq 0$, then $$F(\mathbb{P}_{d,d}, x) = -\sum_{i=0}^{d-1} \lambda_i/\lambda_d \, F(\mathbb{P}_{i,d}, x)$$ $$(1 + x(x+2)^d) \cdot (x+1) = -(x+1)^2 \cdot \sum_{i=0}^{d-1} \lambda_i/\lambda_d \, (1 + x(x+2)^i) \cdot (x+1)^{d-i-1}$$

  a contradiction since -1 is a simple root on the left and a double root on the right.

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1.$

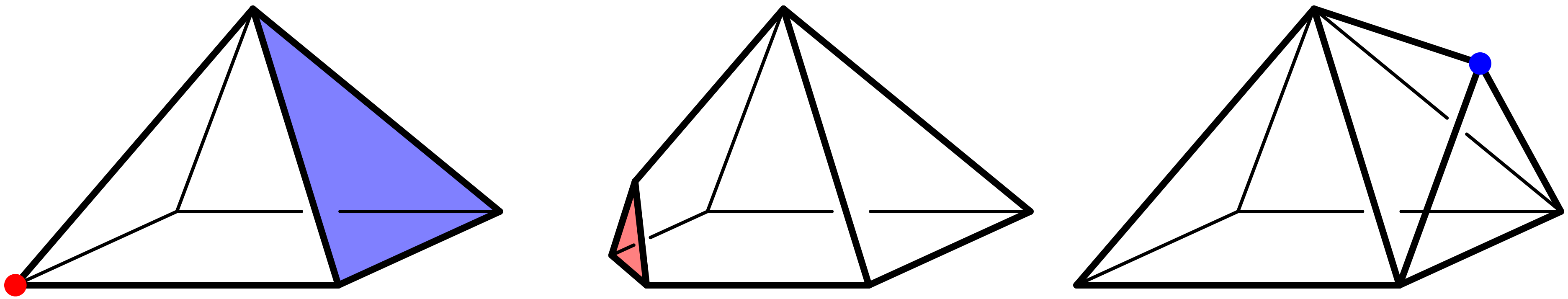
PROP. Let $\mathbb{P}_{i,d} = \mathbb{P}\mathrm{yr}^{d-i}(\square_i)$ for $i \in [d]$. The f-vectors $f(\mathbb{P}_{i,d})$ are affinely independent.

CORO. The Euler relation is the only relation among f-vectors of general polytopes.

F-VECTORS OF 3-POLYTOPES

QU. Describe the effect on the f-vector of the following (polar) operations:

- simple vertex truncation: cut a vertex whose vertex figure is a simplex,
- simplicial facet stacking: stack a vertex on a facet which is a simplex.



QU. What is the f-vector of a pyramid over a p-gon?

QU. Prove that the f-vectors of 3-polytopes are the integer vectors $(f_0, f_1, f_2, 1)$ st

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \le 2f_2 - 4 \qquad \text{and} \qquad f_2 \le 2f_0 - 4.$$

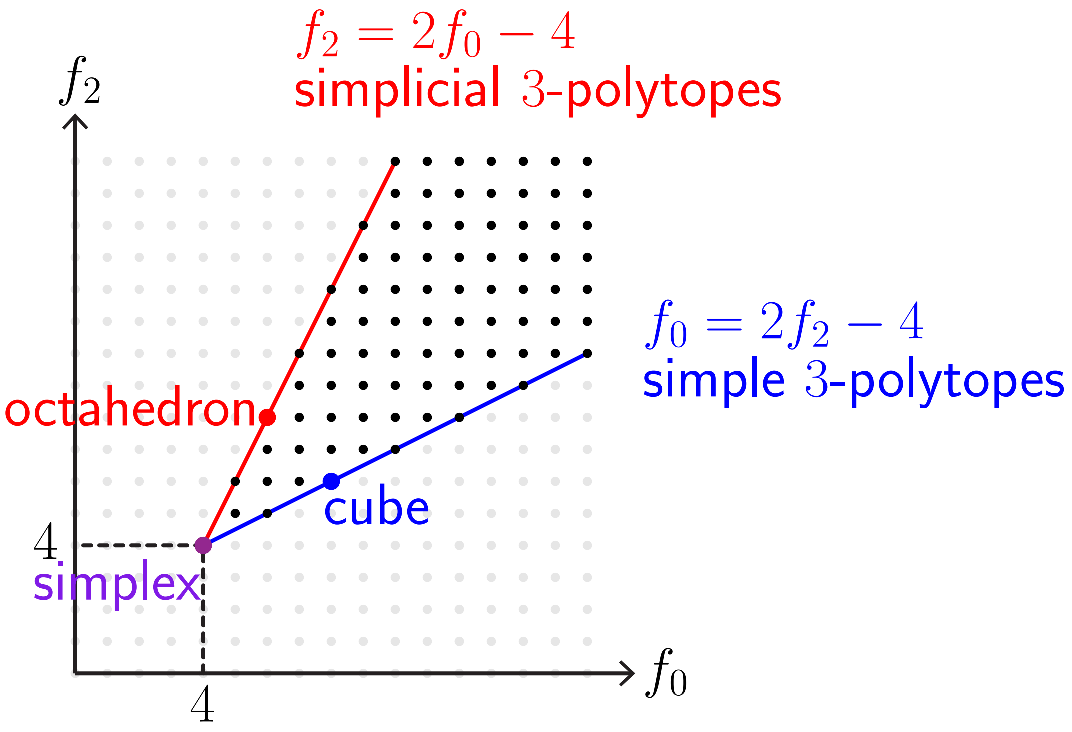
F-VECTORS OF 3-POLYTOPES

THM. The f-vectors of 3-polytopes are the integer vectors $(f_0, f_1, f_2, 1)$ st

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \le 2f_2 - 4 \qquad \text{and} \qquad f_2 \le 2f_0 - 4.$$

proof: For one direction, combine the inequalities

- $f_0 - f_1 + f_2 = 2$ (Euler relation),
- $2f_1 \ge 3f_0$ (every vertex is contained in at least 3 edges, every edge contains 2 vertices),
- $2f_1 \ge 3f_2$ (every face contains at least 3 edges, every edge is contained in 2 faces).



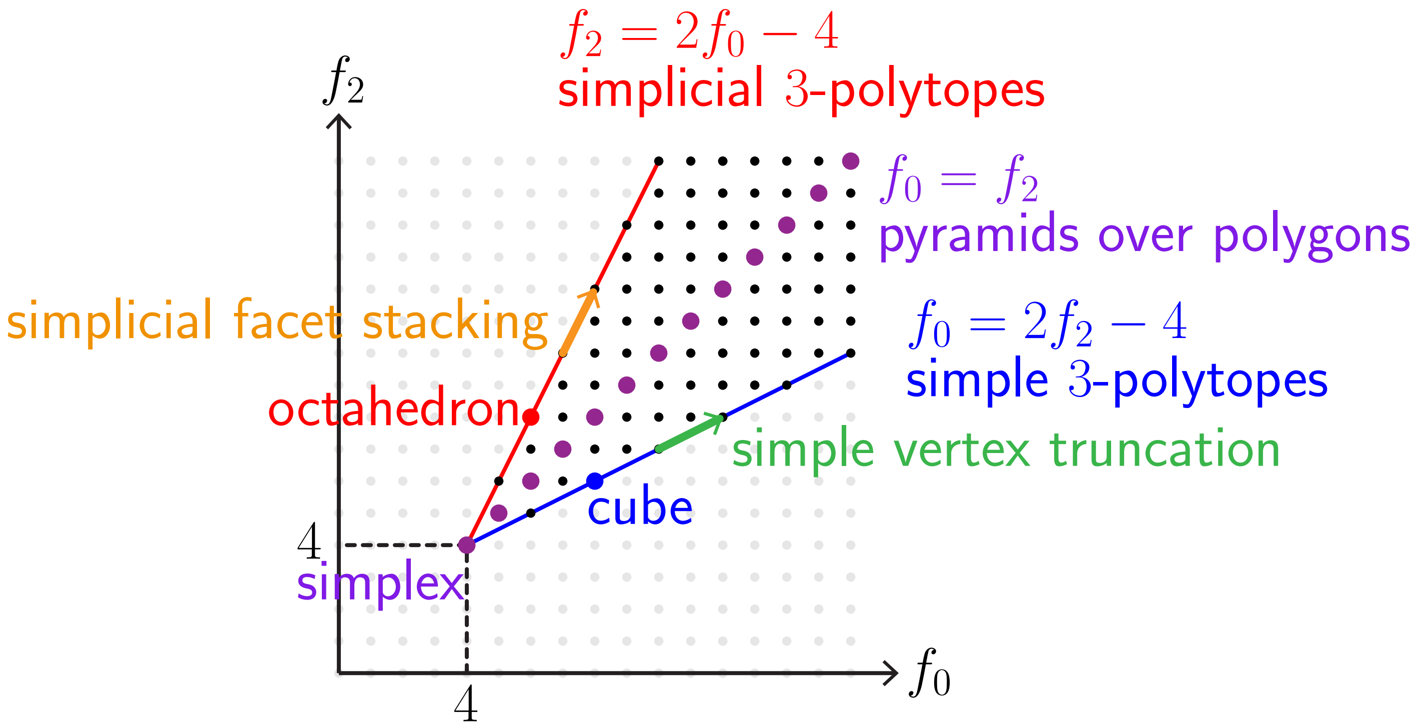
F-VECTORS OF 3-POLYTOPES

THM. The f-vectors of 3-polytopes are the integer vectors $(f_0, f_1, f_2, 1)$ st

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \le 2f_2 - 4 \qquad \text{and} \qquad f_2 \le 2f_0 - 4.$$

proof: For the other direction, observe that

- the f-vector of a pyramid over a p-gon is $(p+1, 2p, p+1, 1)$,
- a simple vertex truncation adds $(2, 3, 1, 0)$ to the f-vector,
- a simplicial facet stacking adds $(1, 3, 2, 0)$ to the f-vector.



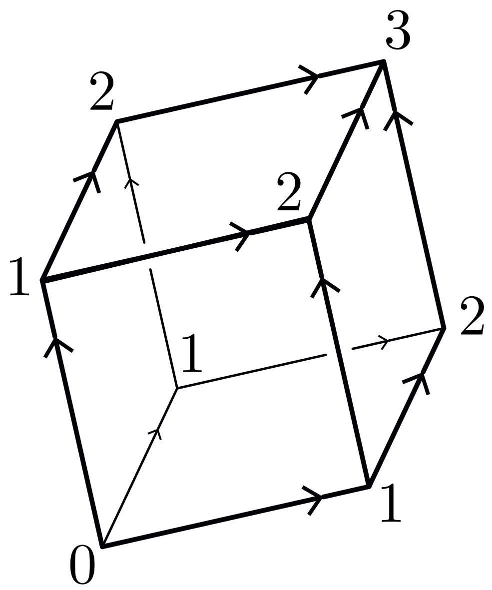
H-VECTOR & DEHN-SOMMERVILLE RELATIONS

H-VECTOR & H-POLYNOMIAL

DEF. A d-polytope is simple if each vertex is contained in d facets, or equiv. d edges.

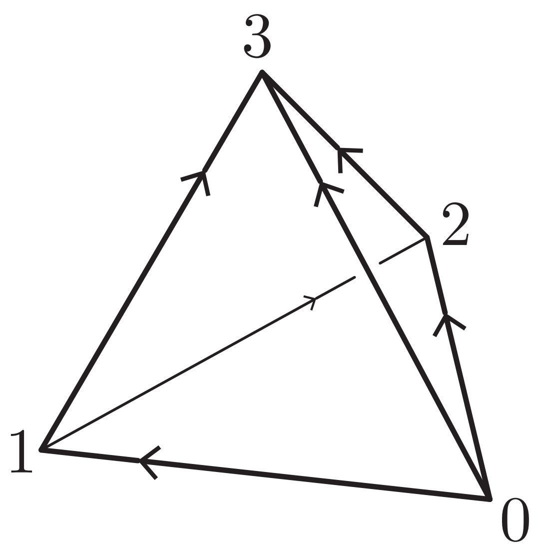
DEF. $\mathbb{P}$ = simple d-polytope,
ϕ = Morse function ($\phi(u) \neq \phi(v)$ for any edge (u, v) of $\mathbb{P}$)
Orient the edges of $\mathbb{P}$ according to ϕ and define

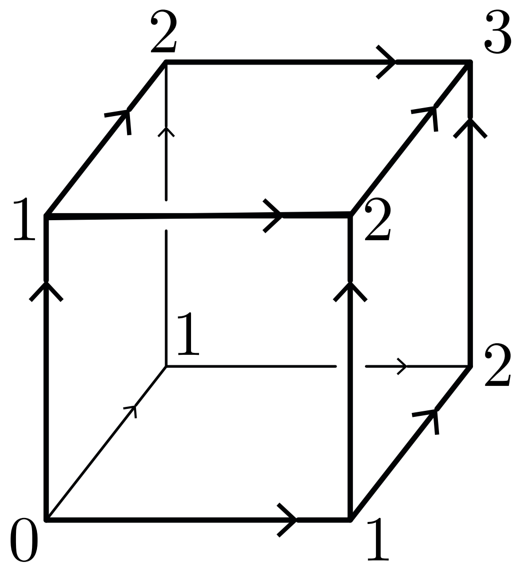
- $h_j(\mathbb{P})$ = number of vertices of $\mathbb{P}$ with indegree j,
- h-vector $h(\mathbb{P}) = \big(h_0(\mathbb{P}), \dots, h_d(\mathbb{P})\big)$,
- h-polynomial $h(\mathbb{P}, x) = \sum_{j=0}^{d} h_j(\mathbb{P})\, x^j$.



$$h(\square_3) = 1 + 3\,x + 3\,x^2 + x^3$$

EXM: F-VECTOR OF CLASSICAL POLYTOPES





PROP. The h-vectors and h-polynomials of the d-simplex $\triangle_d$ and the d-cube $\square_d$ are given by

$$h_j(\triangle_d) = 1 \qquad\qquad h_j(\square_d) = \binom{d}{j}$$

$$h(\triangle_d, x) = \sum_{j=0}^{d} x^j = \frac{x^{d+1}-1}{x-1} \qquad\qquad h(\square_d, x) = \sum_{j=0}^{d} \binom{d}{j} x^j = (x+1)^d$$

F-VECTOR VS H-VECTOR

THM. The f-vector and h-vector of any simple d-polytope $\mathbb{P}$ are related by

$$f_i(\mathbb{P}) = \sum_{j=0}^{d} \binom{j}{i} h_j(\mathbb{P}) \qquad \text{and} \qquad h_j(\mathbb{P}) = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i(\mathbb{P})$$

and the f-polynomial and h-polynomial are related by

$$f(\mathbb{P}, x) = h(\mathbb{P}, x+1) \qquad \text{and} \qquad h(\mathbb{P}, x) = f(\mathbb{P}, x-1).$$

remark: sanity check on classical polytopes

$$f(\triangle_d, x) = \frac{(x+1)^{d+1} - 1}{x} = h(\triangle_d, x+1) \quad \text{and} \quad f(\square_d, x) = (x+2)^d = h(\square_d, x+1)$$

F-VECTOR VS H-VECTOR

THM. The f-vector and h-vector of any simple d-polytope $\mathbb{P}$ are related by

$$f_i(\mathbb{P}) = \sum_{j=0}^{d} \binom{j}{i} h_j(\mathbb{P}) \qquad \text{and} \qquad h_j(\mathbb{P}) = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i(\mathbb{P})$$

and the f-polynomial and h-polynomial are related by

$$f(\mathbb{P}, x) = h(\mathbb{P}, x+1) \qquad \text{and} \qquad h(\mathbb{P}, x) = f(\mathbb{P}, x-1).$$

proof: double counting the set $\mathcal{S}(i, \phi)$ of pairs $(\boldsymbol{v}, \mathbb{F})$ where $\mathbb{F}$ is an i-face of $\mathbb{P}$ and $\boldsymbol{v}$ is the ϕ-maximal vertex of $\mathbb{F}$:

$$f_i(\mathbb{P}) = \sum_{\mathbb{F} \in \mathcal{F}_i(\mathbb{P})} 1 = |\mathcal{S}(i, \phi)| = \sum_{\boldsymbol{v} \in \mathcal{F}_0(\mathbb{P})} \binom{\operatorname{indeg}(\boldsymbol{v})}{i} = \sum_{j=0}^{d} \binom{j}{i} h_j(\mathbb{P}).$$

This implies all other relations since

LEM. $f_i = \sum_{j=0}^{d} \binom{j}{i} h_j \iff f(x) = h(x+1) \iff h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i.$

F-VECTOR VS H-VECTOR

LEM. $f_i = \sum_{j=0}^{d} \binom{j}{i} h_j \iff f(x) = h(x+1) \iff h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i.$

proof:

$$f_i = \sum_{j=0}^{d} \binom{j}{i} h_j \qquad\qquad h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{aligned} h(x+1) &= \sum_{j=0}^{d} h_j (x+1)^j \\ &= \sum_{j=0}^{d} h_j \sum_{i=0}^{j} \binom{j}{i} x^i \\ &= \sum_{i=0}^{d} \left(\sum_{j=0}^{d} \binom{j}{i} h_j \right) x^i \\ &= \sum_{i=0}^{d} f_i x^i = f(x). \end{aligned}$$

$$\begin{aligned} f(x-1) &= \sum_{i=0}^{d} f_i (x-1)^i \\ &= \sum_{i=0}^{d} f_i \sum_{j=0}^{d} \binom{i}{j} (-1)^{i+j} x^j \\ &= \sum_{j=0}^{d} \left(\sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i \right) x^j \\ &= \sum_{j=0}^{d} h_j x^j = h(x) \end{aligned}$$

DEHN-SOMMERVILLE RELATIONS

THM. (Dehn-Sommerville relations)
The h-vector of a simple d-polytope $\mathbb{P}$ is symmetric:

$$h_j(\mathbb{P}) = h_{d-j}(\mathbb{P}) \qquad \text{for all } 0 \leq j \leq d.$$

In terms of f-vectors,

$$\sum_{i=j}^{d} (-1)^{i+j} \binom{i}{j} f_i(\mathbb{P}) = \sum_{i=d-j}^{d} (-1)^{d+i-j} \binom{i}{d-j} f_i(\mathbb{P}) \qquad \text{for all } 0 \leq j \leq d.$$

proof: consider the Morse functions ϕ and $-\phi$ ...
A degree with ϕ-indegree j has $(-\phi)$-indegree $d-j$.

remark: for $j = 0$, $h_0(\mathbb{P}) = h_d(\mathbb{P})$ is the Euler relation.

DEHN-SOMMERVILLE RELATIONS

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PROP. The f-vectors $f(\mathbb{C}\mathrm{yc}^{\diamond}_{d,d+i})$ for $i \in [\lfloor d/2 \rfloor + 1]$ are affinely independent.

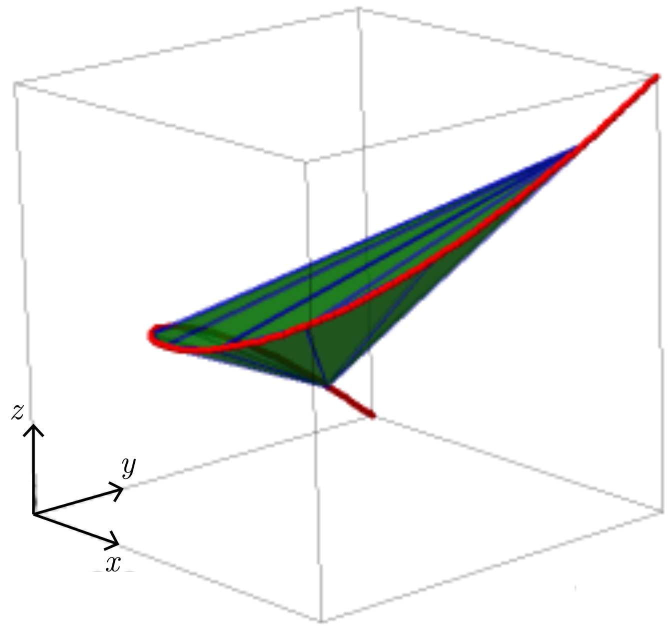
CORO. The Dehn-Sommerville relations are the only relations among f-vectors of simple polytopes.

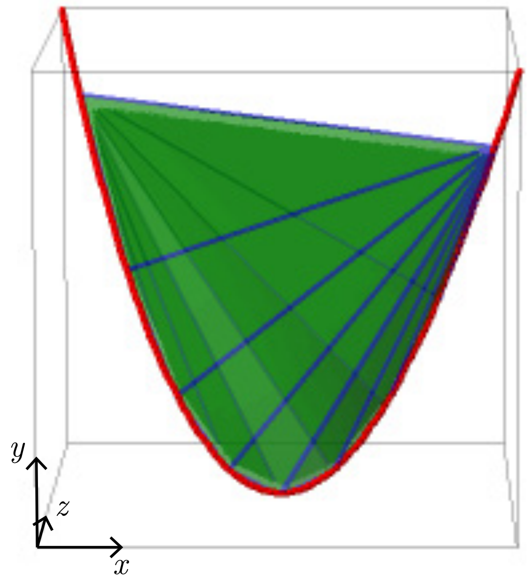
MANY FACES: CYCLIC POLYTOPES

MOMENT CURVE & CYCLIC POLYTOPES

DEF. moment curve = curve parametrized by $\mu_d : t \mapsto (t, t^2, \dots, t^d) \in \mathbb{R}^d$.
cyclic polytope $\mathbb{C}\mathrm{yc}_d(n) = \mathrm{conv}\{\mu_d(t_i) \mid i \in [n]\}$ for arbitrary reals $t_1 < \dots < t_n$.

exm: two views of $\mathbb{C}\mathrm{yc}_3(9)$





remark: we will see later that the combinatorics of $\mathbb{C}\mathrm{yc}_d(n)$ is independent of $t_1 < \dots < t_n$.

CYCLIC POLYTOPES ARE NEIGHBORLY

DEF. moment curve = curve parametrized by $\mu_d : t \mapsto (t, t^2, \dots, t^d) \in \mathbb{R}^d$.
cyclic polytope $\mathbb{C}\mathrm{yc}_d(n) = \mathrm{conv}\,\{\mu_d(t_i) \mid i \in [n]\}$ for arbitrary reals $t_1 < \dots < t_n$.

THM. The cyclic polytope $\mathbb{C}\mathrm{yc}_d(n)$ is

- simplicial: all facets are simplices,
- neighborly: all j-subsets of vertices define a $(j-1)$-face of $\mathbb{C}\mathrm{yc}_d(n)$ for $j \le \lfloor d/2 \rfloor$.

proof: use polynomials!

- If $\mu_d(s_1), \dots, \mu_d(s_{d+1})$ belong to an affine hyperplane $\sum_{i \in [d]} \alpha_i \, x_i = -\alpha_0$, then $s_1, \dots, s_{d+1}$ are all roots of the polynomial $\sum_{i=0}^{d} \alpha_i \, t^i$. A contradiction.
- For $j \le \lfloor d/2 \rfloor$ and $s_1, \dots, s_j \in \{t_1, \dots, t_n\}$, the polynomial $\sum_{i=0}^{d} \alpha_i \, t^i = \prod_{i \in [j]} (t - s_i)^2$ is non-negative and vanishes on $s_1, \dots, s_j$. Thus the hyperplane $\sum_{i \in [d]} \alpha_i \, x_i = -\alpha_0$ supports a face of $\mathbb{C}\mathrm{yc}_d(n)$ with vertices $\mu_d(s_1), \dots, \mu_d(s_j)$.

H-VECTORS OF POLAR CYCLIC POLYTOPES

CORO. The polar of the cyclic polytope $\mathbb{C}\mathrm{yc}_d(n)^\diamond$ is simple and its h-vector is given by

$$h_j = \binom{n-d+j-1}{j} \text{ for } j \le \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad h_j = \binom{n-j-1}{d-j} \text{ for } j > \left\lfloor \frac{d}{2} \right\rfloor.$$

proof: $\mathbb{C}\mathrm{yc}_d(n)$ is neighborly $\Longrightarrow f_i\big(\mathbb{C}\mathrm{yc}_d(n)\big) = \binom{n}{i}$ for $i \le \lfloor d/2 \rfloor$

$$\Longrightarrow f_i\big(\mathbb{C}\mathrm{yc}_d(n)^\diamond\big) = \binom{n}{d-i} \text{ for } i > \lfloor d/2 \rfloor.$$

Therefore

$$h_j\big(\mathbb{C}\mathrm{yc}_d(n)^\diamond\big) = \sum_{i=j}^{d} (-1)^{i+j} \binom{i}{j} \binom{n}{d-i} = \binom{n-j-1}{d-j}. \qquad \text{if } j > \left\lfloor \frac{d}{2} \right\rfloor \qquad (\star)$$

$$= h_{d-j}\big(\mathbb{C}\mathrm{yc}_d(n)^\diamond\big) = \binom{n-d+j-1}{j} \qquad \text{if } j \le \left\lfloor \frac{d}{2} \right\rfloor$$

For $(\star)$, check that

- it holds when $j = 0$ and $j = d$, and
- if it holds for (j, d) and $(j+1, d)$ then it holds for $(j+1, d+1)$.

UPPER BOUND THEOREM

THM. (Upper Bound Theorem, McMullen) For any d-polytope $\mathbb{P}$ with n vertices:

$$f_i(\mathbb{P}) \le f_i(\mathbb{C}\mathrm{yc}_d(n)).$$

remark:

- clear for $i \le \lfloor d/2 \rfloor$ since $f_i(\mathbb{C}\mathrm{yc}_d(n)) = \binom{n}{i+1}$,
- equivalent to polar version $f_i(\mathbb{P}) \le f_i(\mathbb{C}\mathrm{yc}_d(n)^\diamond)$ for any d-polytope $\mathbb{P}$ with n facets,
- enough to prove it for simplicial/simple polytopes,
- thus implied by h-vector version:

THM. (Upper Bound Theorem, McMullen) For any simple d-polytope $\mathbb{P}$ with n facets:

$$h_j(\mathbb{P}) \le \binom{n-d+j-1}{j} \text{ for } j \le \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad h_j(\mathbb{P}) \le \binom{n-j-1}{d-j} \text{ for } j > \left\lfloor \frac{d}{2} \right\rfloor.$$

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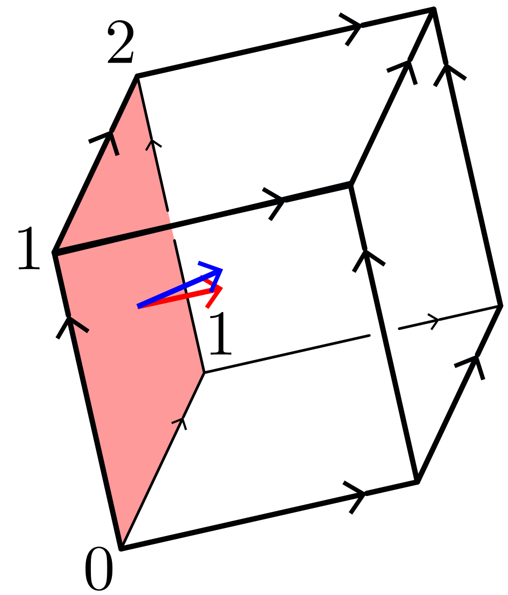
proof:

1. $h_i(\mathbb{F}) \leq h_i(\mathbb{P})$ for $\mathbb{F} \in \mathcal{F}_{d-1}(\mathbb{P})$
   ϕ obtained by perturbation of the inner normal of $\mathbb{F}$
   then $\mathrm{indeg}_{\mathbb{F}}(\boldsymbol{v}) = \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{F}$



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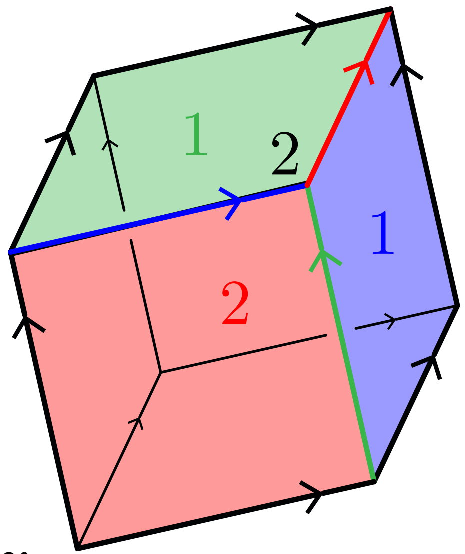
   ϕ obtained by perturbation of the inner normal of $\mathbb{F}$
   then $\mathrm{indeg}_{\mathbb{F}}(\boldsymbol{v}) = \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{F}$

2. $$\sum_{\mathbb{F} \in \mathcal{F}_{d-1}(\mathbb{P})} h_i(\mathbb{F}) = (d-i)\, h_i(\mathbb{P}) + (i+1)\, h_{i+1}(\mathbb{P})$$
   Let $\boldsymbol{v} \in \mathbb{F}$, and e the edge of $\mathbb{P}$ st $\boldsymbol{v} \in e \not\subset \mathbb{F}$

   then $\mathrm{indeg}_{\mathbb{F}}(\boldsymbol{v}) = i \iff \begin{cases} \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v}) = i \text{ and } e \text{ leaving } \boldsymbol{v}, \text{ or} \\ \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v}) = i+1 \text{ and } e \text{ entering } \boldsymbol{v}. \end{cases}$



UPPER BOUND THEOREM

THM. (Upper Bound Theorem, McMullen) For any simple d-polytope $\mathbb{P}$ with n facets:

$$h_j(\mathbb{P}) \le \binom{n-d+j-1}{j} \text{ for } j \le \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad h_j(\mathbb{P}) \le \binom{n-j-1}{d-j} \text{ for } j > \left\lfloor \frac{d}{2} \right\rfloor.$$

proof:

1. $h_i(\mathbb{F}) \le h_i(\mathbb{P})$ for $\mathbb{F} \in \mathcal{F}_{d-1}(\mathbb{P})$
   ϕ obtained by perturbation of the inner normal of $\mathbb{F}$
   then $\mathrm{indeg}_{\mathbb{F}}(\boldsymbol{v}) = \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{F}$

2. $$\sum_{\mathbb{F} \in \mathcal{F}_{d-1}(\mathbb{P})} h_i(\mathbb{F}) = (d-i)\, h_i(\mathbb{P}) + (i+1)\, h_{i+1}(\mathbb{P})$$
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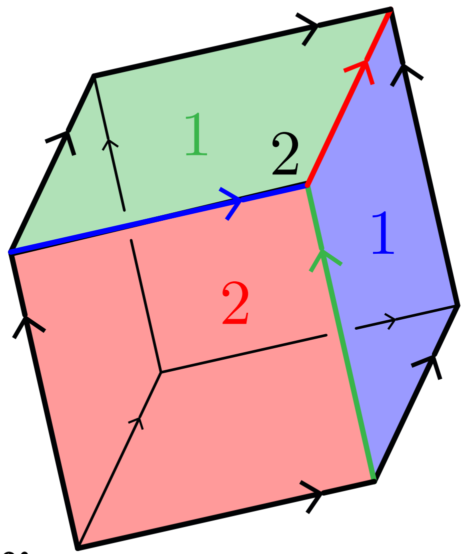


$1+2 \implies (d-i)\, h_i(\mathbb{P}) + (i+1)\, h_{i+1}(\mathbb{P}) \le n\, h_i(\mathbb{P}) \implies h_{i+1}(\mathbb{P}) \le \dfrac{n+d-i}{i+1} h_i(\mathbb{P}).$

and induction...

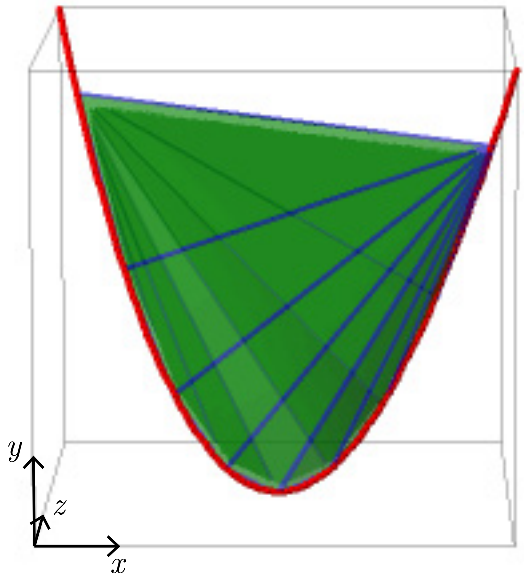
GALE'S EVENNESS CRITERION

DEF. For $I \subseteq [n] = \{1, \dots, n\}$, define

- block of I = intervals of I,
- even block of I = block of I of even size,
- internal block of I = block of I that does not contain 1 or n.

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\operatorname{conv}\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n)$ $\iff$ all internal blocks of I are even.

exm: The facets $\mathbb{C}\mathrm{yc}_3(n)$ correspond to $\{i, i+1, n\}$ and $\{1, i+1, i+2\}$ for $i \in [n-2]$.



GALE'S EVENNESS CRITERION

DEF. For $I \subseteq [n] = \{1, \dots, n\}$, define

- block of I = maximal intervals of I,
- even block of I = block of I of even size,
- internal block of I = block of I that does not contain 1 or n.

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\operatorname{conv}\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n)$ $\iff$ all internal blocks of I are even.

proof: For any $I = \{i_1, \dots, i_d\} \subseteq [n]$ and $k \in [n]$, the position of $\mu_d(t_k)$ with respect to the hyperplane $\mathbb{H}$ containing $\mu_d(t_{i_1}), \dots, \mu_d(t_{i_d})$ is given by the sign of the Vandermonde determinant

$$\det \begin{bmatrix} 1 & \dots & 1 & 1 \\ t_{i_1} & \dots & t_{i_d} & t_k \\ \vdots & \ddots & \vdots & \vdots \\ t_{i_1}^d & \dots & t_{i_d}^d & t_k^d \end{bmatrix} = \prod_{1 \le p < q \le d} (t_{i_q} - t_{i_p}) \prod_{1 \le p \le d} (t_k - t_{i_p}).$$

which is 0 if $k \in I$ and -1 to the parity of the number of $p \in [d]$ such that $i_p > k$. Therefore, all points $\mu_d(t_k)$ lie on the same side of $\mathbb{H}$ iff all internal blocks of I are even.

GALE'S EVENNESS CRITERION

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\mathrm{conv}\,\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n) \iff$ all internal blocks of I are even.

CORO. $\mathbb{C}\mathrm{yc}_d(n)$ is neighborly and independent of the choice of $t_1 < \cdots < t_n$.

proof:

- neighborly since for any $<\leq \lfloor d/2 \rfloor$, any j-subset can be completed into a d-subset satisfying Gale's evenness criterion (complete all odd blocks and add the remaining elements at the end).
- independent of the choice of $t_1 < \cdots < t_n$ since Gale's evenness criterion tells the vertices-facets incidences, which determine the whole combinatorics.

GALE'S EVENNESS CRITERION

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\operatorname{conv}\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n)$ $\iff$ all internal blocks of I are even.

CORO. $\mathbb{C}\mathrm{yc}_d(n)$ is neighborly and independent of the choice of $t_1 < \dots < t_n$.

CORO. $f_{d-1}\big(\mathbb{C}\mathrm{yc}_d(n)\big) = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor}$.

proof: number of $2k$-subsets of $[n]$ where all blocks are even $= \binom{n-k}{k}$

$$\circ \bullet \bullet \circ \bullet \bullet \bullet \bullet \circ \circ \bullet \bullet \quad \longleftrightarrow \quad \circ \bullet \circ \bullet \bullet \circ \circ \bullet$$

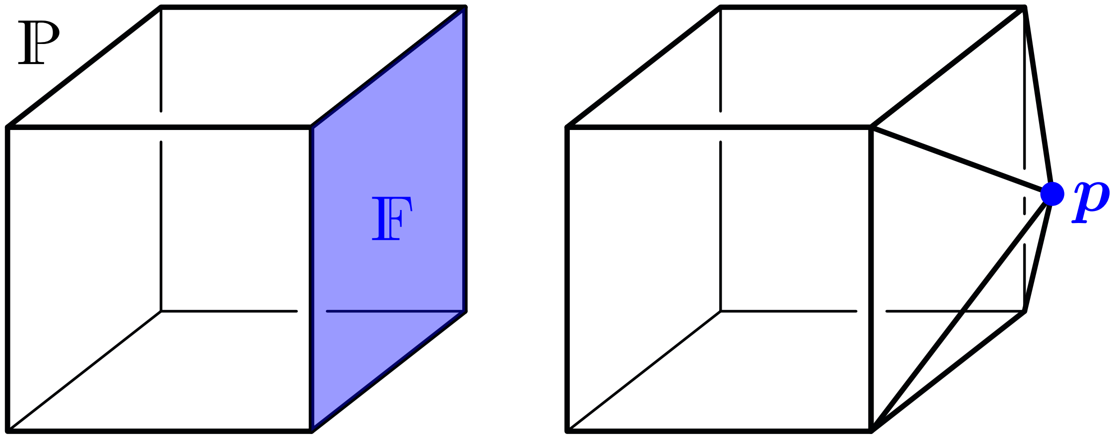
Then case analysis:

	1 in an odd block	otherwise
n in an odd block	d even $\binom{n-2-\frac{d-2}{2}}{\frac{d-2}{2}}$	d odd $\binom{n-1-\frac{d-1}{2}}{\frac{d-1}{2}}$
otherwise	d odd $\binom{n-1-\frac{d-1}{2}}{\frac{d-1}{2}}$	d even $\binom{n-\frac{d}{2}}{\frac{d}{2}}$

FEW FACES: STACKED POLYTOPES

STACKING OVER A FACET

DEF. stacking over a facet $\mathbb{F}$ of $\mathbb{P}$ = constructing $\mathrm{conv}(\mathbb{P} \cup \{\boldsymbol{p}\})$ where $\boldsymbol{p}$ is beyond $\mathbb{F}$ but beneath all other facets of $\mathbb{P}$.

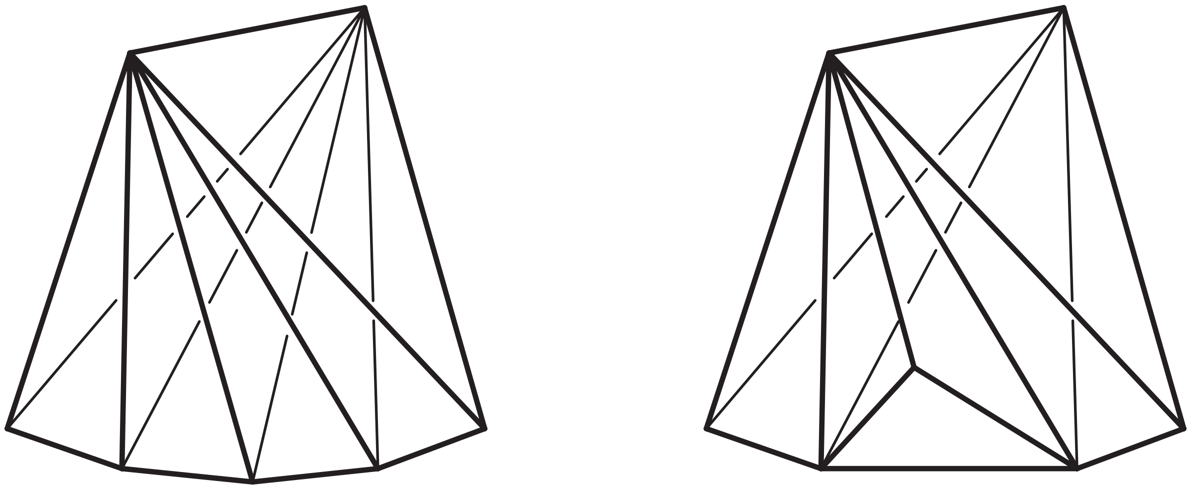


LEM. If $\mathbb{P}'$ is obtained from $\mathbb{P}$ by staking on $\mathbb{F}$, then

$$f_0(\mathbb{P}') = f_0(\mathbb{P}) + 1,$$
$$f_i(\mathbb{P}') = f_i(\mathbb{P}) + f_{i-1}(\mathbb{F}), \qquad \text{for } 0 \leq i \leq d-2,$$
$$f_{d-1}(\mathbb{P}') = f_{d-1}(\mathbb{P}) + f_{d-2}(\mathbb{F}) - 1.$$

F-VECTORS OF STACKED POLYTOPES

DEF. stacked polytope $=$ polytope arising from a d-simplex by stacking n times.



LEM. The f-vector of a stacked polytope on $d+n$ vertices is

$$f_0 = d+1+n,$$
$$f_i = \binom{d+1}{i+1} + n\binom{d}{i} \quad \text{for } 0 \le i \le d-2,$$
$$f_{d-1} = d+1+n(d-1).$$

LOWER BOUND THEOREM

THM. (Lower Bound Theorem, Barnette) For any simplicial d-polytope $\mathbb{P}$ with n vertices:

$$f_i(\mathbb{P}) \geq f_i(\mathbb{Q})$$

where $\mathbb{Q}$ is a stacked polytope on n vertices.
Moreover, equality holds $\iff$ $d = 3$ or $d \geq 4$ and $\mathbb{P}$ is stacked.

GRAPHS OF POLYTOPES

POLYTOPE SKELETA

DEF. $\mathbb{P}$ d-polytope, $k \leq d$.

graph of $\mathbb{P}$ $=$ graph with same vertices and edges as $\mathbb{P}$.

k-skeleton of $\mathbb{P}$ $=$ all $\leq k$-dimensional faces of $\mathbb{P}$.

POLYTOPAL GRAPHS

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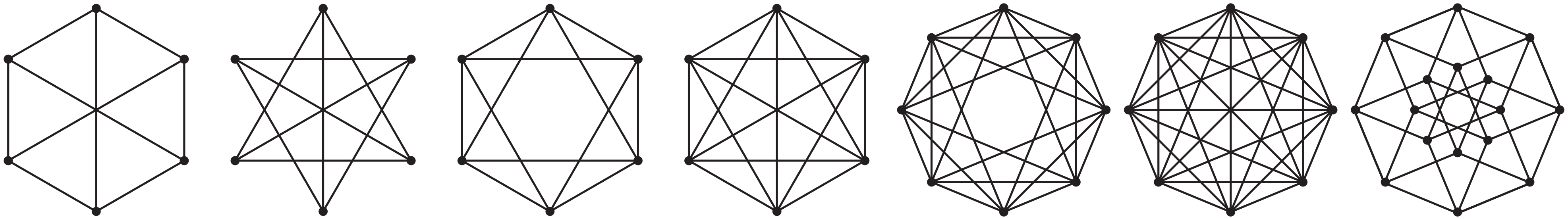
graph of $\mathbb{P}$ = graph with same vertices and edges as $\mathbb{P}$.
k-skeleton of $\mathbb{P}$ = all $\leq k$-dimensional faces of $\mathbb{P}$.

QU. Which of the following graphs are graphs of polytopes? In which dimension?



POLYTOPAL GRAPHS

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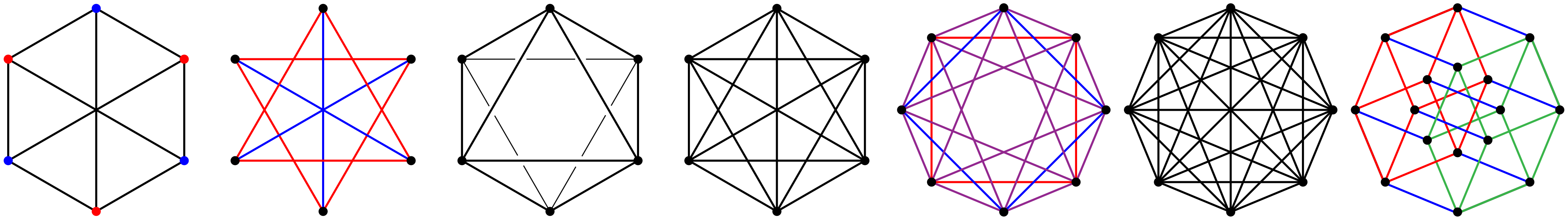
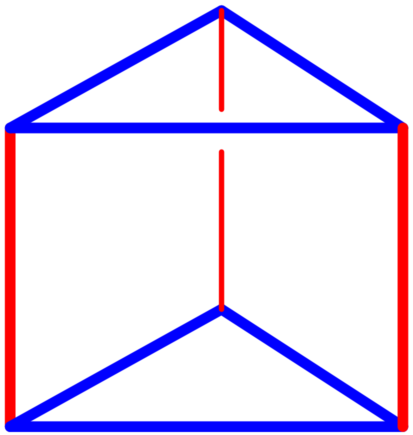
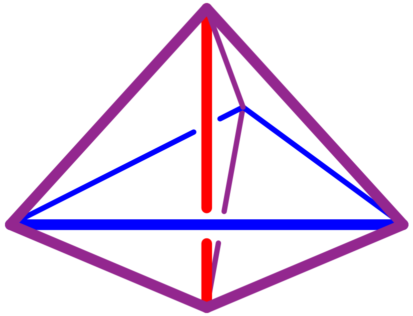


$\varnothing$	$\triangle_2 \times \triangle_1$	$\lozenge_3$	$\mathbb{C}\mathrm{yc}_4(6)$, $\triangle_5$	$\lozenge_4$, $\lozenge_2 * \lozenge_2$	$\mathbb{C}\mathrm{yc}_d(8)$	$\square_4$
$\varnothing$	3	3	$\{4, 5\}$	$\{4, 5\}$	$\{4, 5, 6, 7\}$	4

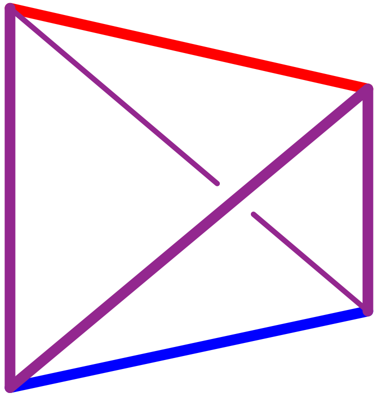
GRAPHS & POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$



$\mathbb{P} * \mathbb{P}'$

PROP. Define $E^\star(\mathbb{P}) = E(\mathbb{P}) \smallsetminus \{\mathbb{P}\}$ (if $\dim \mathbb{P} = 1$, then $E^\star(\mathbb{P}) = \varnothing$).

$$V(\mathbb{P} \times \mathbb{P}') = V(\mathbb{P}) \times V(\mathbb{P}') \qquad E(\mathbb{P} \times \mathbb{P}') = \big(V(\mathbb{P}) \times E(\mathbb{P}')\big) \cup \big(E(\mathbb{P}) \times V(\mathbb{P}')\big)$$

$$V(\mathbb{P} \oplus \mathbb{P}') = V(\mathbb{P}) \cup V(\mathbb{P}') \qquad E(\mathbb{P} \oplus \mathbb{P}') = E^\star(\mathbb{P}) \cup E^\star(\mathbb{P}') \cup \big(V(\mathbb{P}) \times V(\mathbb{P}')\big)$$

$$V(\mathbb{P} * \mathbb{P}') = V(\mathbb{P}) \cup V(\mathbb{P}') \qquad E(\mathbb{P} * \mathbb{P}') = E(\mathbb{P}) \cup E(\mathbb{P}') \cup \big(V(\mathbb{P}) \times V(\mathbb{P}')\big)$$

GRAPHS OF 3-POLYTOPES

THM. (Steinitz) 3-polytopal $\iff$ planar and 3-connected.

Different proofs are possible:

- See Ziegler, Lect. 4 for the proof based on ΔY operations.
- Lift Tutte's barycentric embedding.

THM. (Mnëv, Richter-Gebert) Polytopality of graphs is NP-hard.

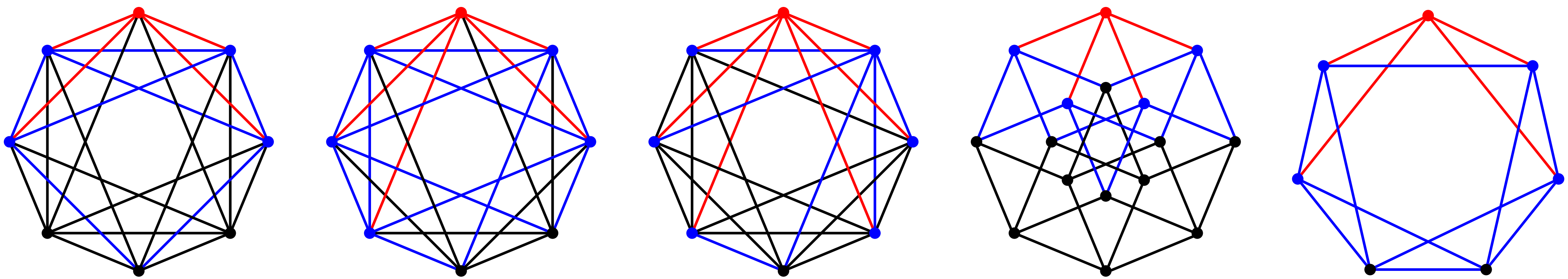
SOME NECESSARY CONDITIONS

THM. If G is the graph of a d-polytope, then

(1) Balinski's Theorem: G is d-connected.

(2) Principal Subdivision Property: Every vertex of G is the principal vertex of a principal subdivision of K_{d+1}.

(3) Separation Property: The maximal number of components into which G may be separated by removing $n > d$ vertices equals $f_{d-1}\big(\mathbb{C}\mathrm{yc}_d(n)\big)$.



DEDUCING THE FACES FROM THE GRAPH

THM. (Whitney) In a 3-polytope, graphs of faces $=$ non-separating induced cycles.

REM. In general, the graph does not determine the face lattice of the polytope (even for a fixed dimension).

THM. (Blind & Mani-Levitska, Kalai)
Two simple polytopes with isomorphic graphs have isomorphic face lattices.

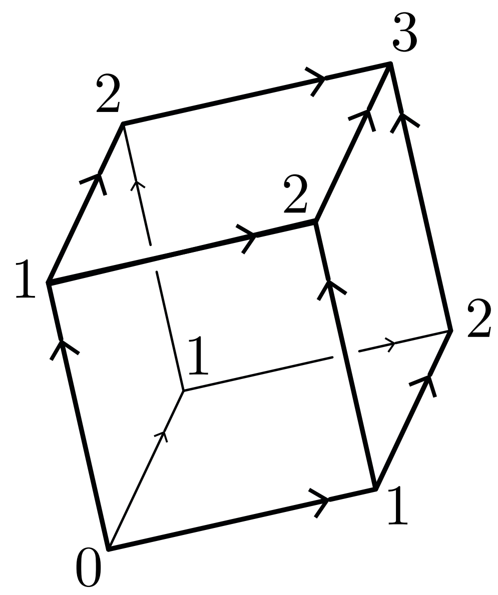
DEDUCING THE FACES FROM THE GRAPH

THM. (Blind & Mani-Levitska, Kalai)
Two simple polytopes with isomorphic graphs have isomorphic face lattices.

proof: G graph of a simple d-polytope $\mathbb{P}$. An orientation $\mathcal{O}$ of G is:

- acyclic $=$ no oriented cycle,
- good $=$ each face of $\mathbb{P}$ has a unique sink.

Intuitively, good acyclic orientations of $G \quad \longleftrightarrow \quad$ linear orientations of $\mathbb{P}$



DEDUCING THE FACES FROM THE GRAPH

THM. (Blind & Mani-Levitska, Kalai)
Two simple polytopes with isomorphic graphs have isomorphic face lattices.

proof: G graph of a simple d-polytope $\mathbb{P}$. An orientation $\mathcal{O}$ of G is:

- acyclic = no oriented cycle,
- good = each face of $\mathbb{P}$ has a unique sink.

1. Good acyclic orientations can be recognized from G:
   $h_j(\mathcal{O})$ = number indegree j vertices for $\mathcal{O}$.
   $F(\mathcal{O}) := h_0(\mathcal{O}) + 2\,h_1(\mathcal{O}) + \cdots + 2^d\,h_d(\mathcal{O})$.
   Since $\mathbb{P}$ is simple, each indegree j vertex is a sink in 2^j faces.
   Thus $F(\mathcal{O}) \geq$ number of faces of $\mathbb{P}$ with equality iff $\mathcal{O}$ is good.

DEDUCING THE FACES FROM THE GRAPH

THM. (Blind & Mani-Levitska, Kalai)
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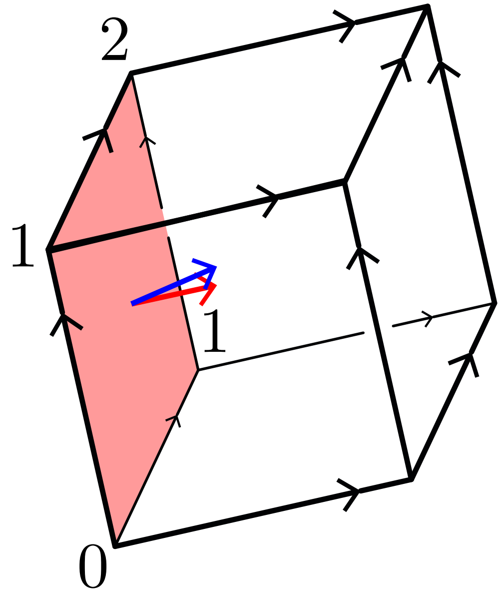


1. Good acyclic orientations can be recognized from G
2. Faces of $\mathbb{P}$ can be determined from good acyclic orientations:

   H regular induced subgraph of G, with vertices W.

   H is the graph of a face of $\mathbb{P}$

   $\iff$ W is initial wrt some good acyclic orientation.

$\Longrightarrow$ perturb a linear functional defining the face

$\Longleftarrow$ assume H k-regular subgraph of G induced by W initial for $\mathcal{O}$.

Let $\boldsymbol{v}$ be a sink of H, and $\mathbb{F}$ be the k-face containing the k edges of H incident to $\boldsymbol{v}$.

Since $\mathcal{O}$ is good, $\boldsymbol{v}$ is the unique sink of the graph of $\mathbb{F}$.

Since W is initial, all vertices of $\mathbb{F}$ are in W.

Since H and the graph of $\mathbb{F}$ are k-regular, they coincide.

DIAMETERS OF POLYTOPES & THE SIMPLEX METHOD

DEF. diameter of G = minimum δ such that any two vertices are connected by a path with at most δ edges.
$\Delta(d,n)$ = maximal diameter of a d-polytope with at most n facets.

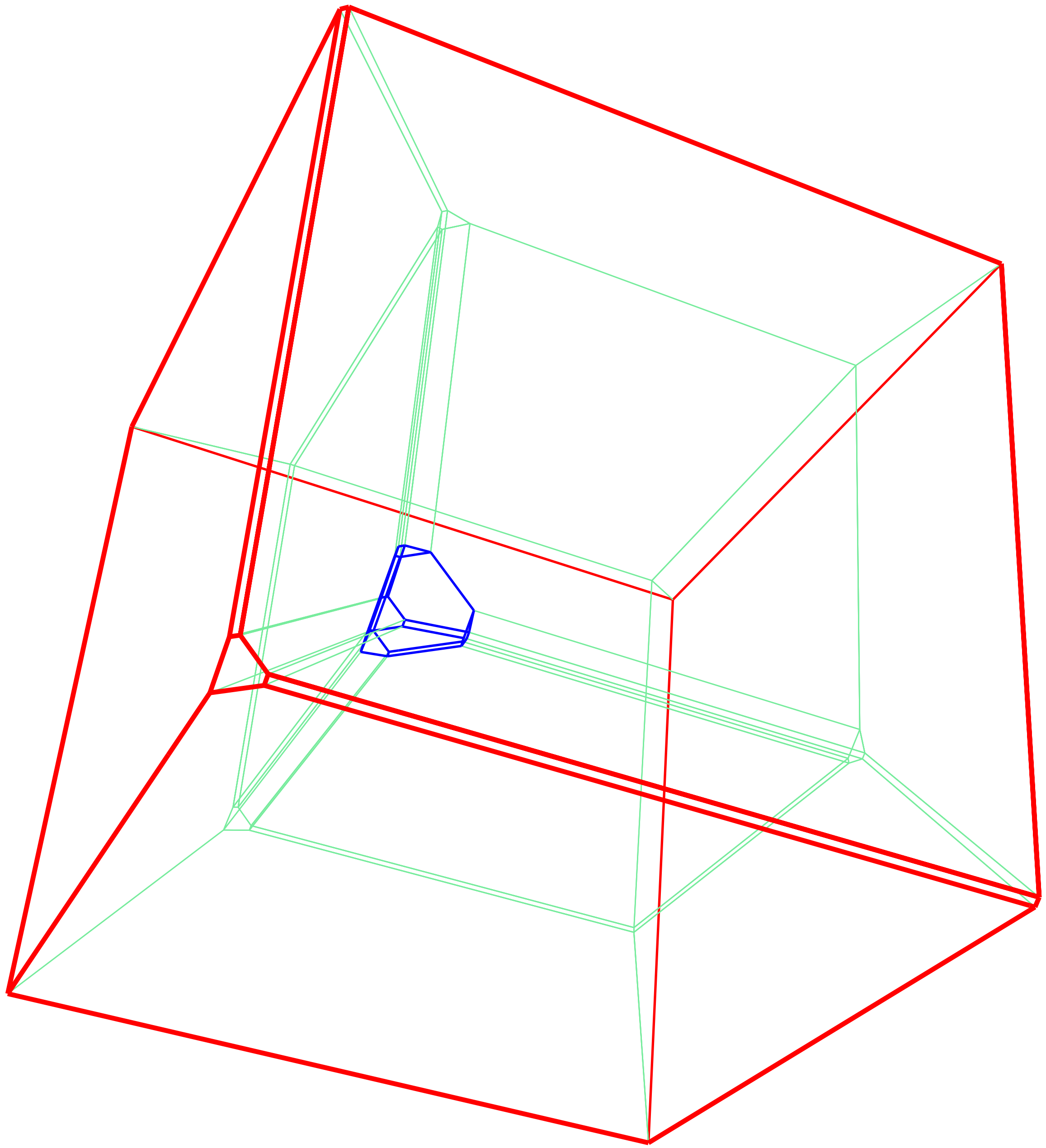
remark: diameters of polytopes are important in linear programming and its resolution via the classical simplex algorithm.

CONJ. (Hirsh, disproved by Santos) $\Delta(d,n) \le n-d$.

PROB. Is $\Delta(d,n)$ bounded polynomially in both n and d.

THM. (Kalai and Kleitman) $\Delta(d,n) \le n^{\log_2(d)+1}$.

THM. (Barnette, Larman) $\Delta(d,n) \le \dfrac{2^{d-2}}{3} n$.



THANK YOU