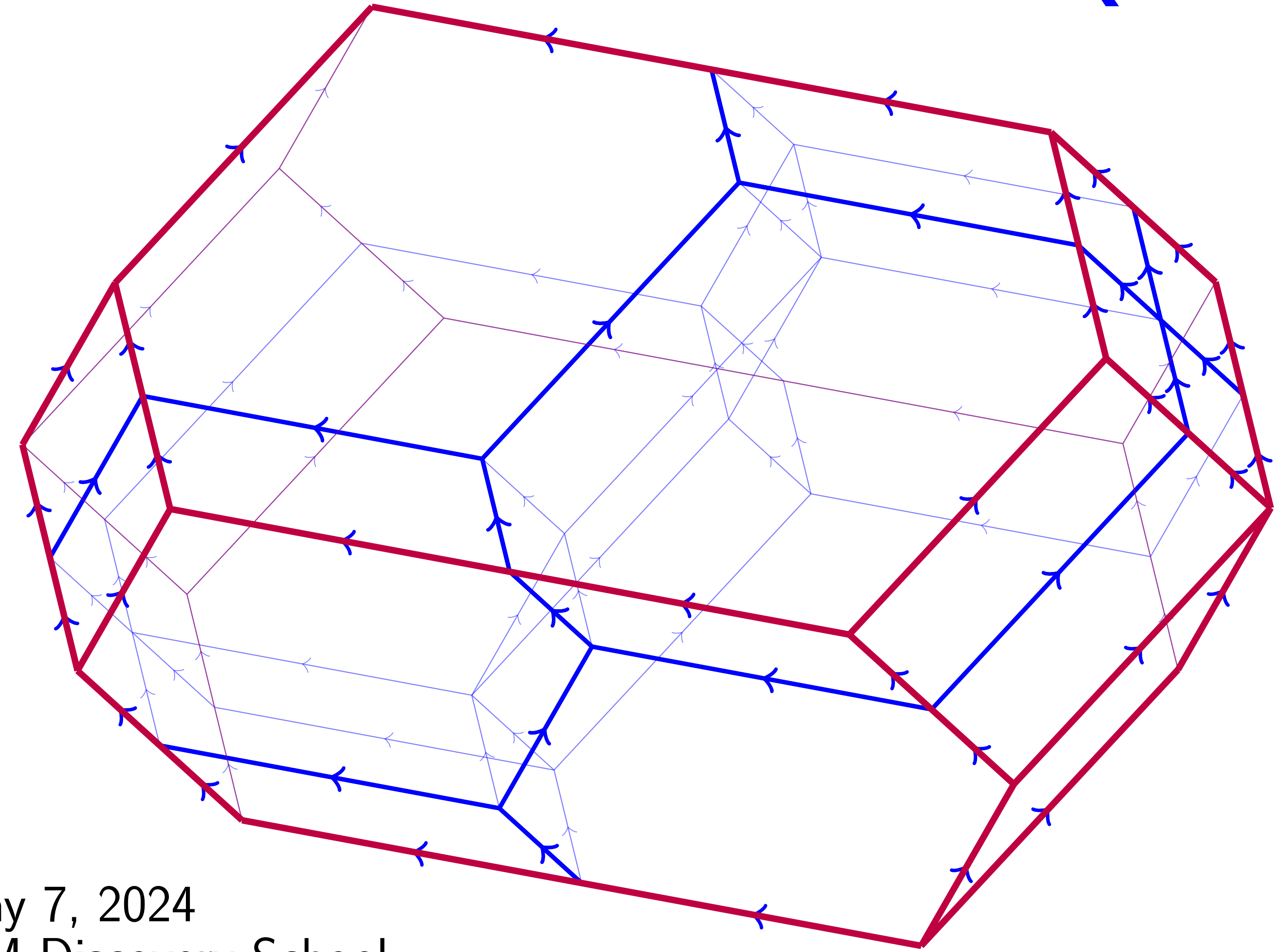
3. THE S-WEAK ORDER AND ITS QUOTIENTS



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ISM Discovery School

THE S-WEAK ORDER

Ceballos–Pons, *The s-weak order I & II* ('22^+)

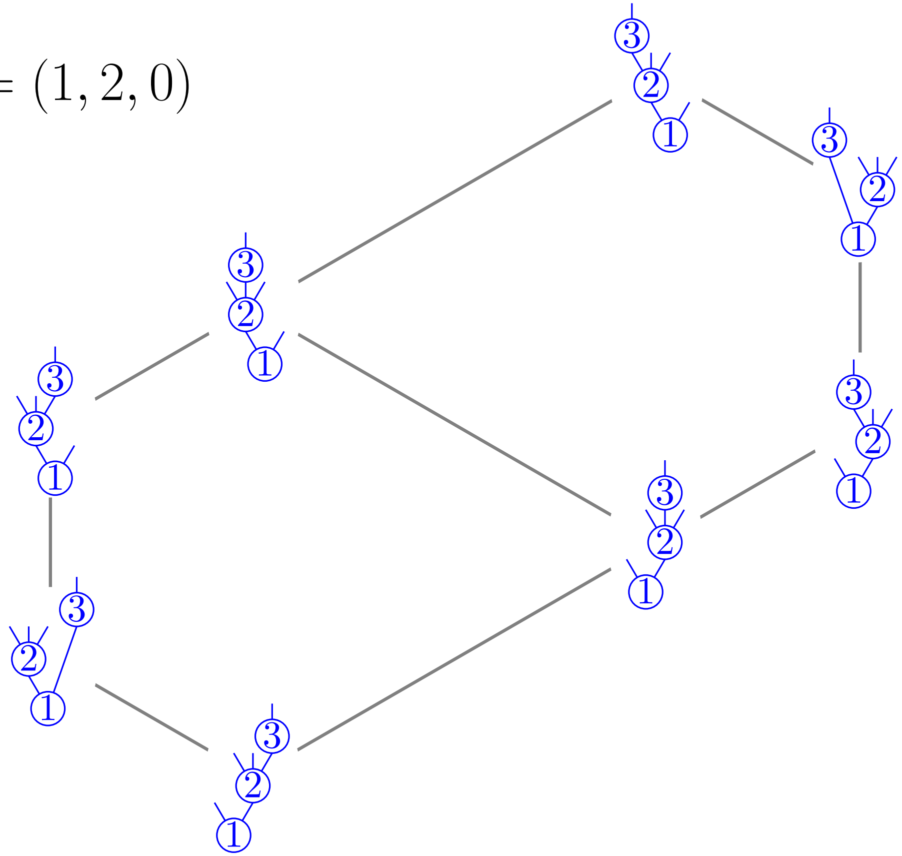
S-WEAK ORDER

$\boldsymbol{s} = (s_1, \dots, s_n)$ weak composition

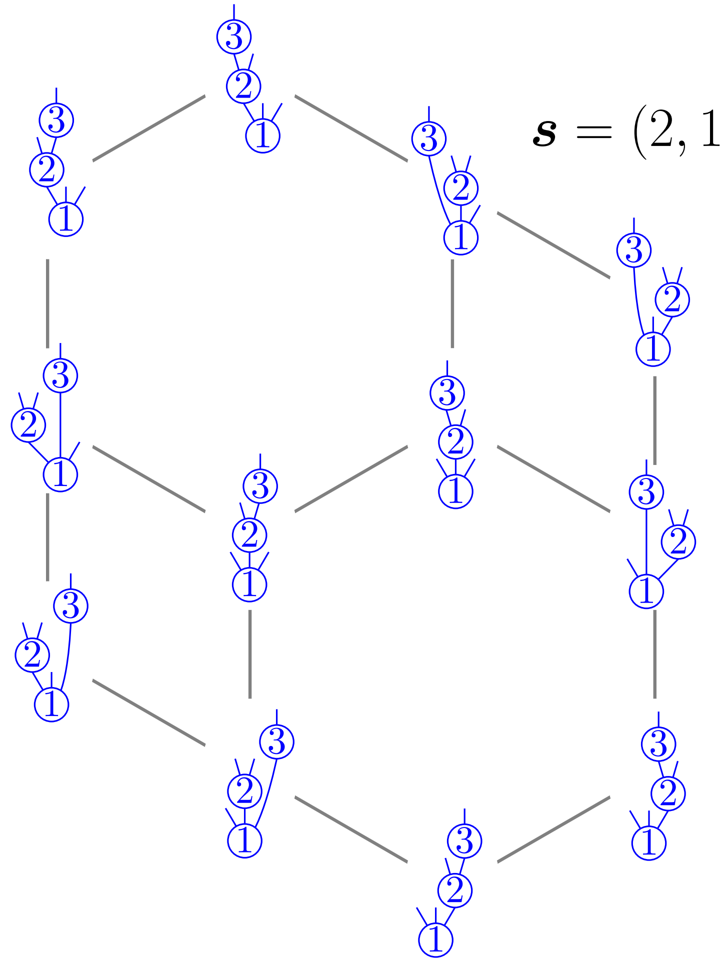
$\boldsymbol{s}$-tree = tree on $[n]$ where i has $s_i + 1$ children, which are either leaves of nodes $> i$

$\boldsymbol{s}$-weak order = $\boldsymbol{s}$-trees ordered by $\mathrm{T} \le \mathrm{T}'$ if $\mathrm{pos}(\mathrm{T}, i, j) \le \mathrm{pos}(\mathrm{T}', i, j)$ for all $1 \le i < j \le n$

$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$



THM. The $\boldsymbol{s}$-weak order is a polygonal, semidistributive, and congruence uniform lattice

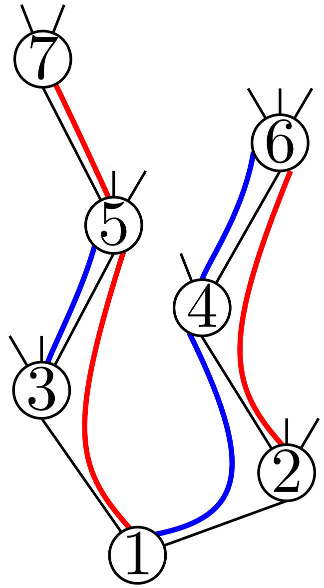
Ceballos–Pons, *The s-weak order I & II* ('22$^+$)

ASCENTS AND DESCENTS IN S-TREES

Fix $1 \leq i < j \leq n$

ascent (resp. descent) of an $\boldsymbol{s}$-tree $\mathrm{T} = (i, j)$ such that

- $1 \leq i < j \leq n$
- i is the greatest ancestor of j such that the increasing path from i to j in T takes the leftmost (resp. rightmost) outgoing edge at each node, except at node i
- either $s_j = 0$ or the leftmost (resp. rightmost) edge of j is a leaf



COVER RELATIONS IN S-WEAK ORDER

Fix $1 \leq i < j \leq n$

ascent (resp. descent) of an $\boldsymbol{s}$-tree $\mathrm{T} = (i, j)$ such that

- $1 \leq i < j \leq n$
- i is the greatest ancestor of j such that the increasing path from i to j in T takes the leftmost (resp. rightmost) outgoing edge at each node, except at node i
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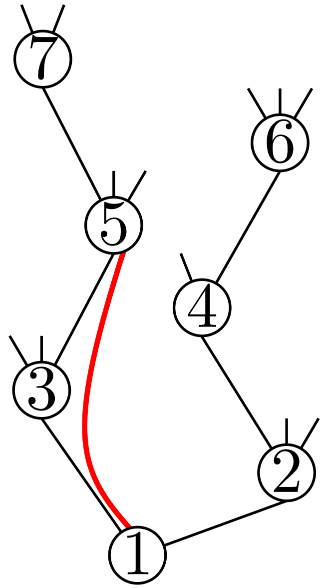
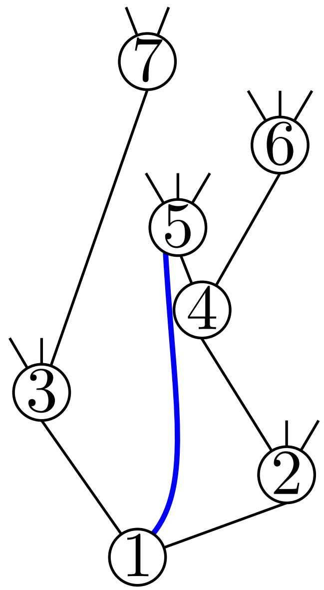




PROP. The $\boldsymbol{s}$-trees covered by (resp. covering) an $\boldsymbol{s}$-tree are obtained by flipping one of its descents (resp. ascents)

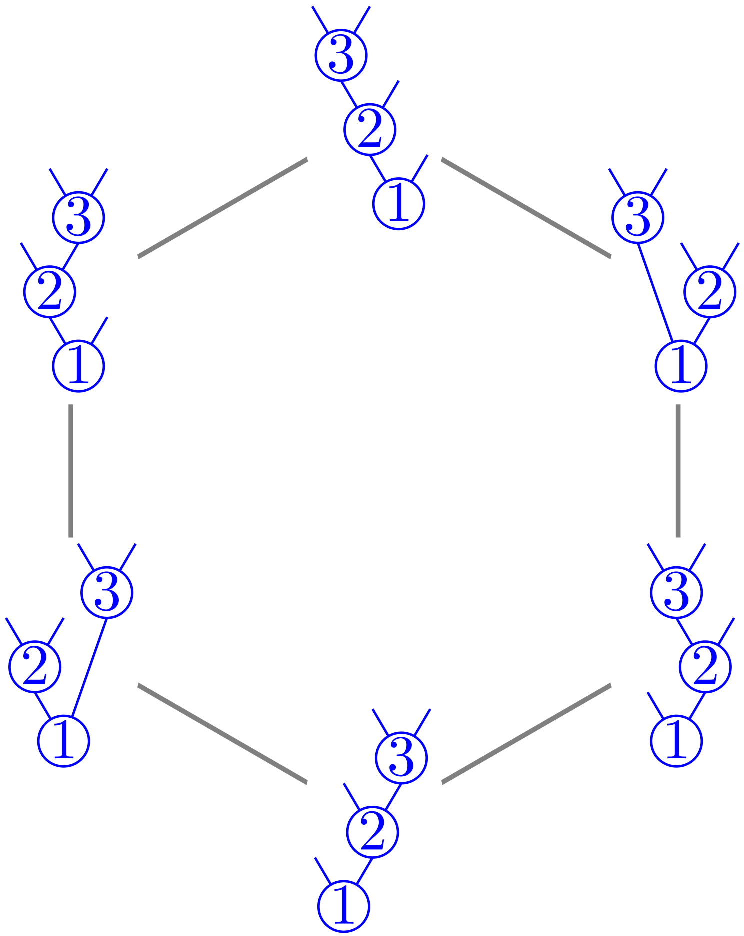
Ceballos–Pons, *The s-weak order I & II* ('22$^+$)

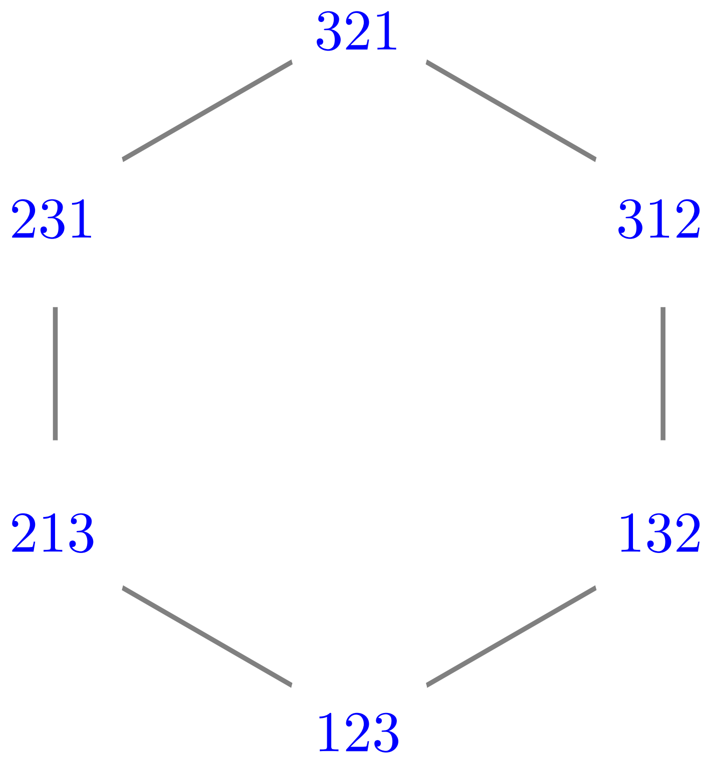
S-WEAK ORDER

When $\boldsymbol{s} = (1, 1, \ldots, 1)$,

$\boldsymbol{s}$-trees $\longleftrightarrow$ permutations

$\boldsymbol{s}$-weak order $\longleftrightarrow$ weak order





More generally, if $\boldsymbol{s}$ contains no 0,

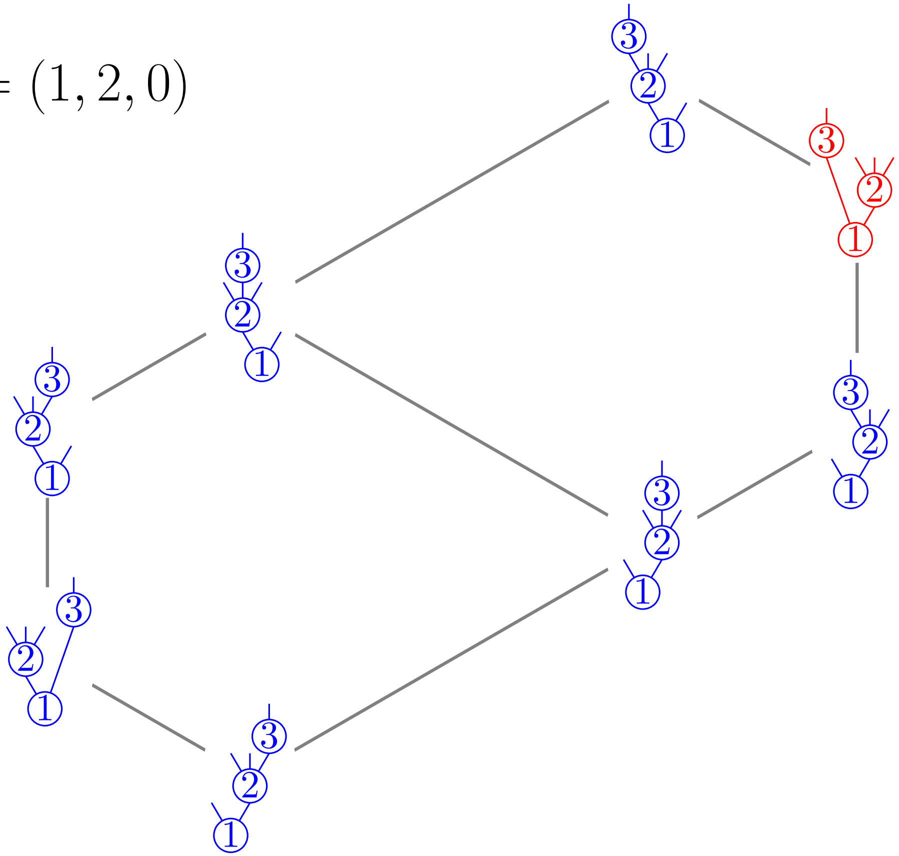
$\boldsymbol{s}$-trees $\longleftrightarrow$ Stirling $\boldsymbol{s}$-permutations

S-TAMARI LATTICE

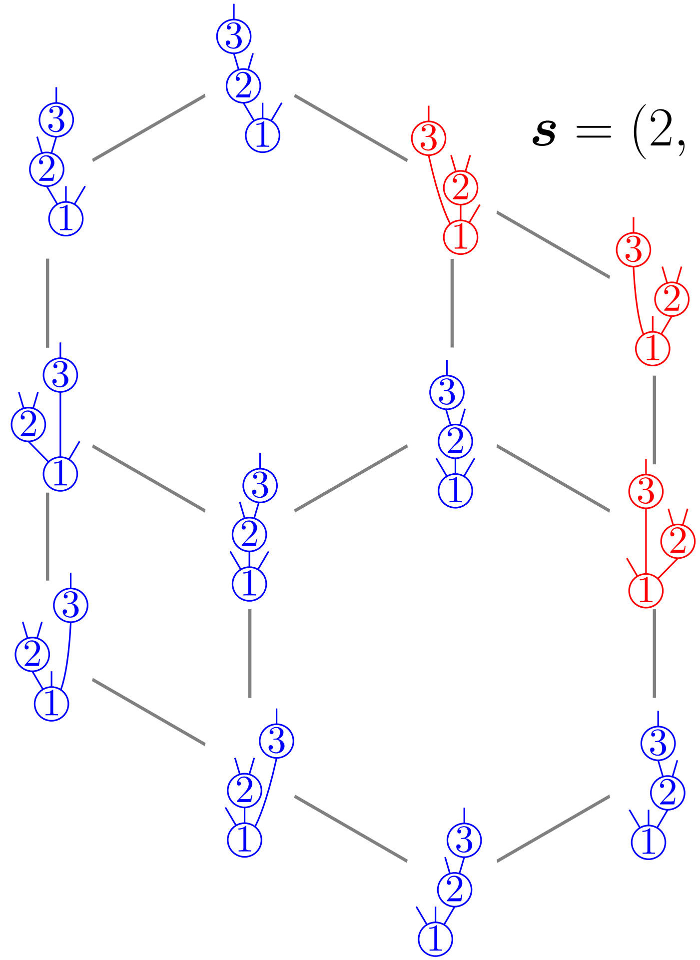
$\boldsymbol{s}$-Tamari tree $=$ $\boldsymbol{s}$-tree T such that $\mathrm{pos}(\mathrm{T}, a, b) \geq \mathrm{pos}(\mathrm{T}, a, c)$ for any $1 \leq a < b < c \leq n$

$\boldsymbol{s}$-Tamari lattice $=$ sublattice of the $\boldsymbol{s}$-weak order induced by $\boldsymbol{s}$-Tamari trees

$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$



THM. The $\boldsymbol{s}$-Tamari trees induce a sublattice of the $\boldsymbol{s}$-weak order

Ceballos–Pons, *The s-weak order I & II* ('22^{+})

S-TAMARI LATTICE

<u>$\boldsymbol{s}$-Tamari tree</u> $=$ $\boldsymbol{s}$-tree T such that $\mathrm{pos}(\mathrm{T}, a, b) \geq \mathrm{pos}(\mathrm{T}, a, c)$ for any $1 \leq a < b < c \leq n$

<u>$\boldsymbol{s}$-Tamari lattice</u> $=$ sublattice of the $\boldsymbol{s}$-weak order induced by $\boldsymbol{s}$-Tamari trees

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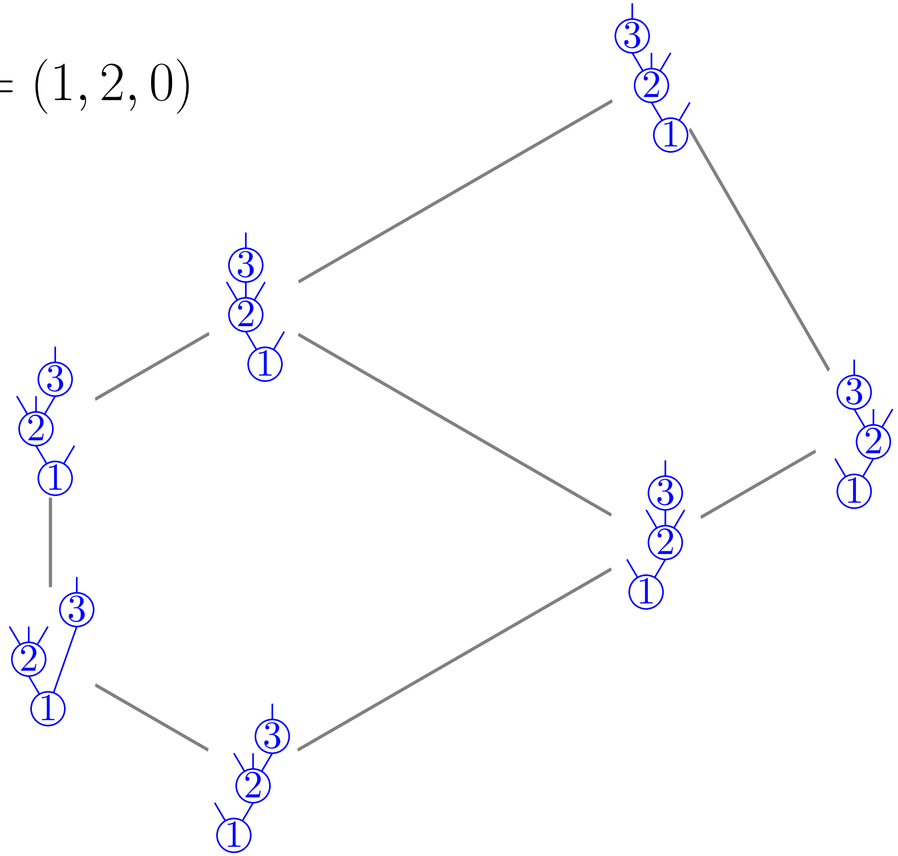
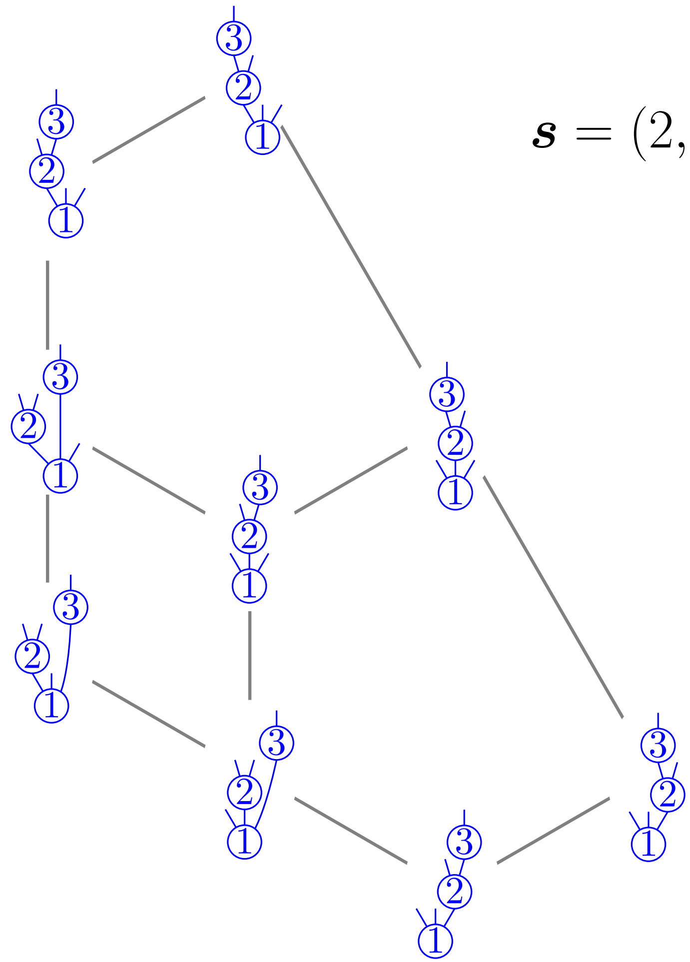




$\boldsymbol{s} = (2, 1, 0)$

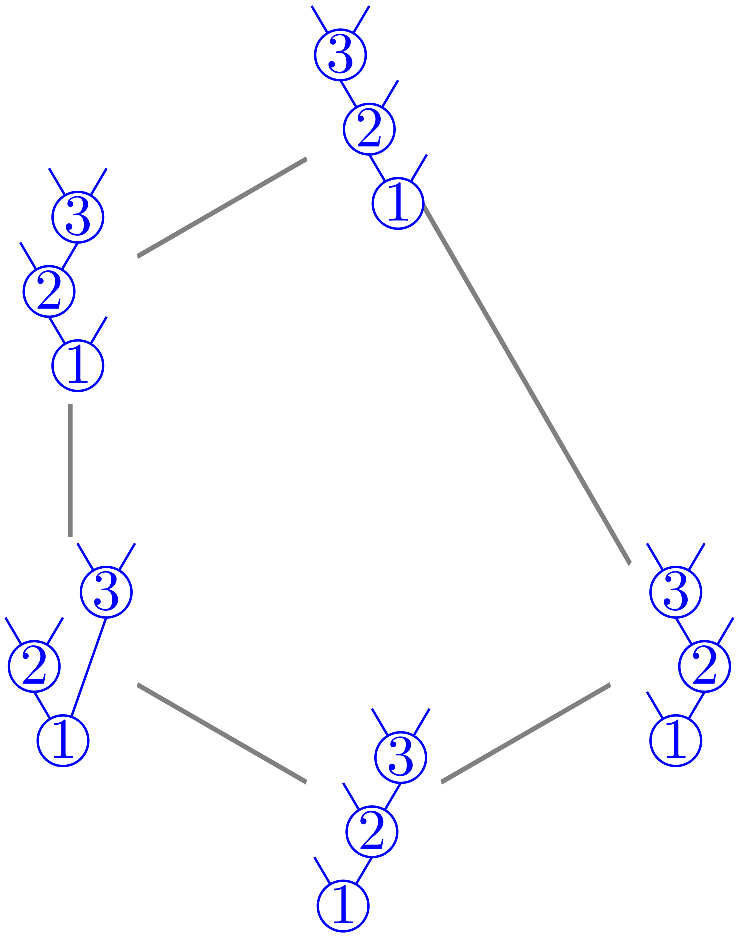
THM. The $\boldsymbol{s}$-Tamari trees induce a sublattice of the $\boldsymbol{s}$-weak order

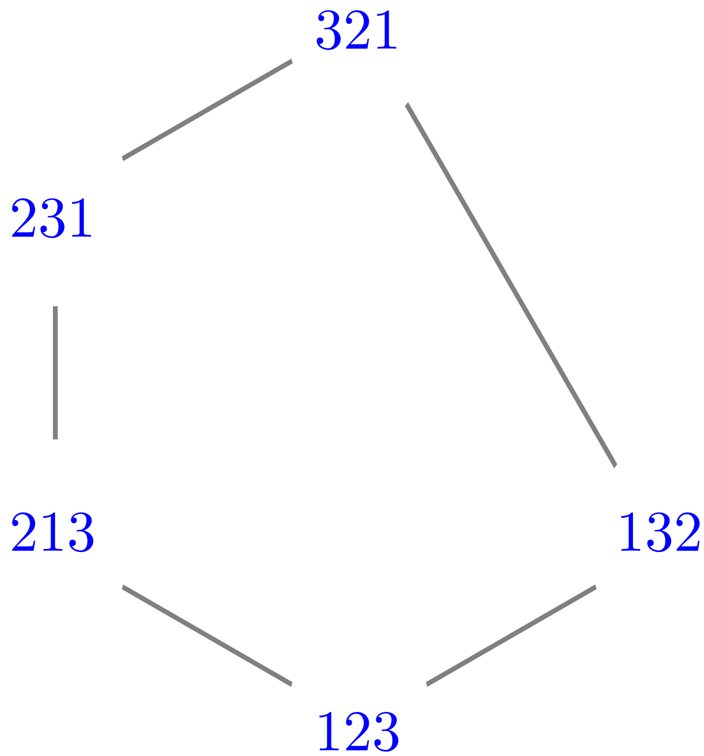
Ceballos–Pons, *The s-weak order I & II* ('22^{+})

S-TAMARI LATTICE

When $\boldsymbol{s} = (1, 1, \ldots, 1)$,

Tamari $\boldsymbol{s}$-trees	$\longleftrightarrow$	binary trees
$\boldsymbol{s}$-Tamari lattice	$\longleftrightarrow$	Tamari lattice





More generally, if $\boldsymbol{s}$ contains no 0,

$\boldsymbol{s}$-Tamari trees	$\longleftrightarrow$	Stirling $\boldsymbol{s}$-permutations avoiding 312

INSERTION IN BUSHES

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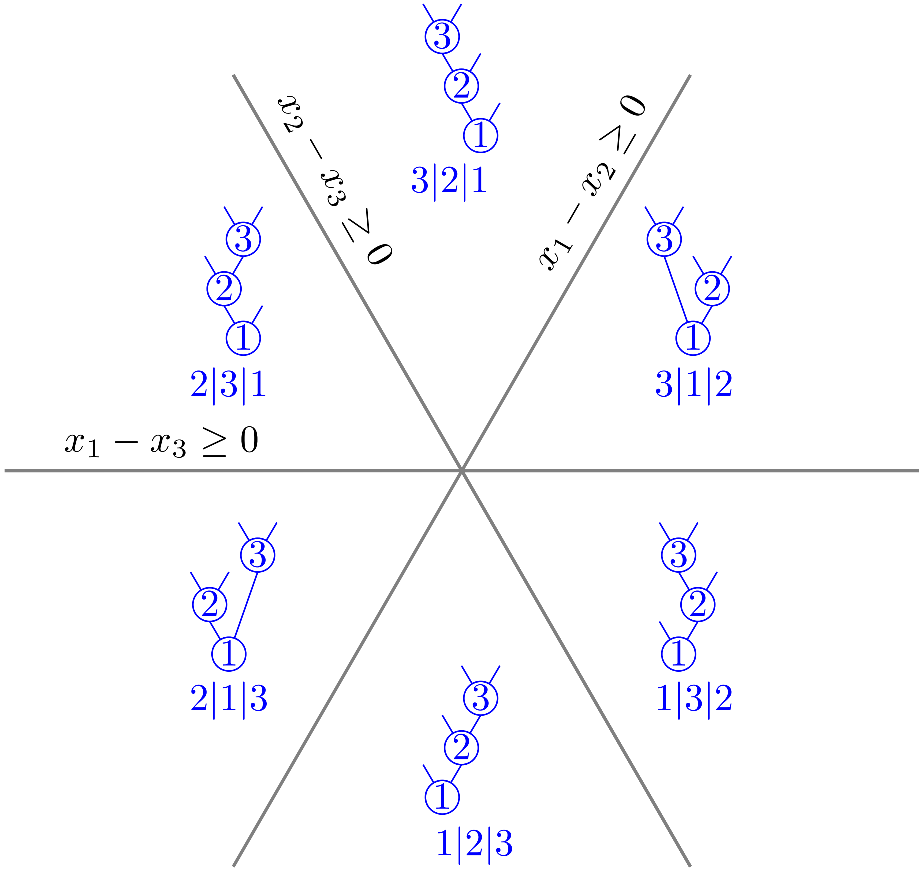
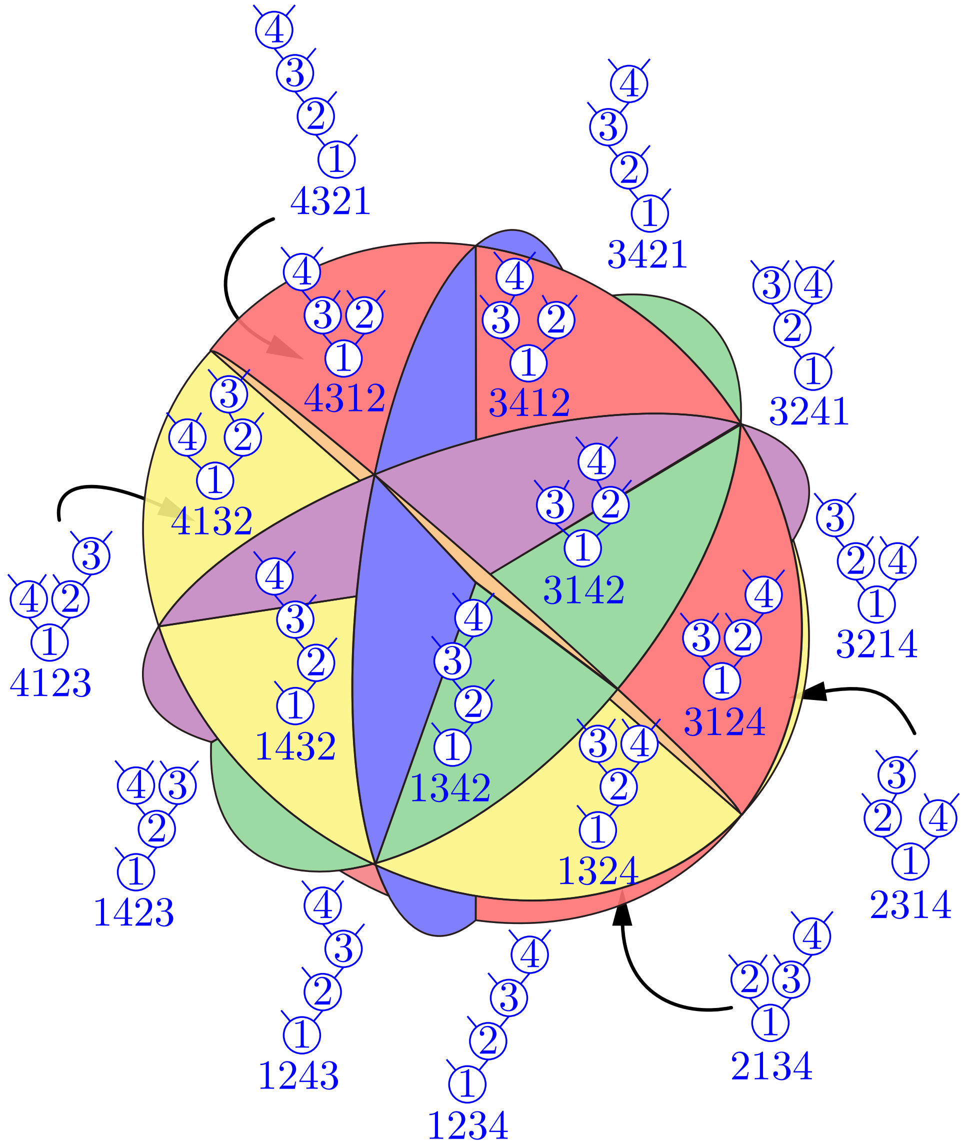
INSERTION IN INCREASING BINARY TREES

increasing binary tree = binary tree where each node is smaller than its children

increasing binary tree insertion of generic $\boldsymbol{x} \in \mathbb{R}^n$ =

- root $= 1$
- left subtree $= \{i \in [2, n] \mid x_i < x_1\}$
- right subtree $= \{j \in [2, n] \mid x_1 < x_j\}$



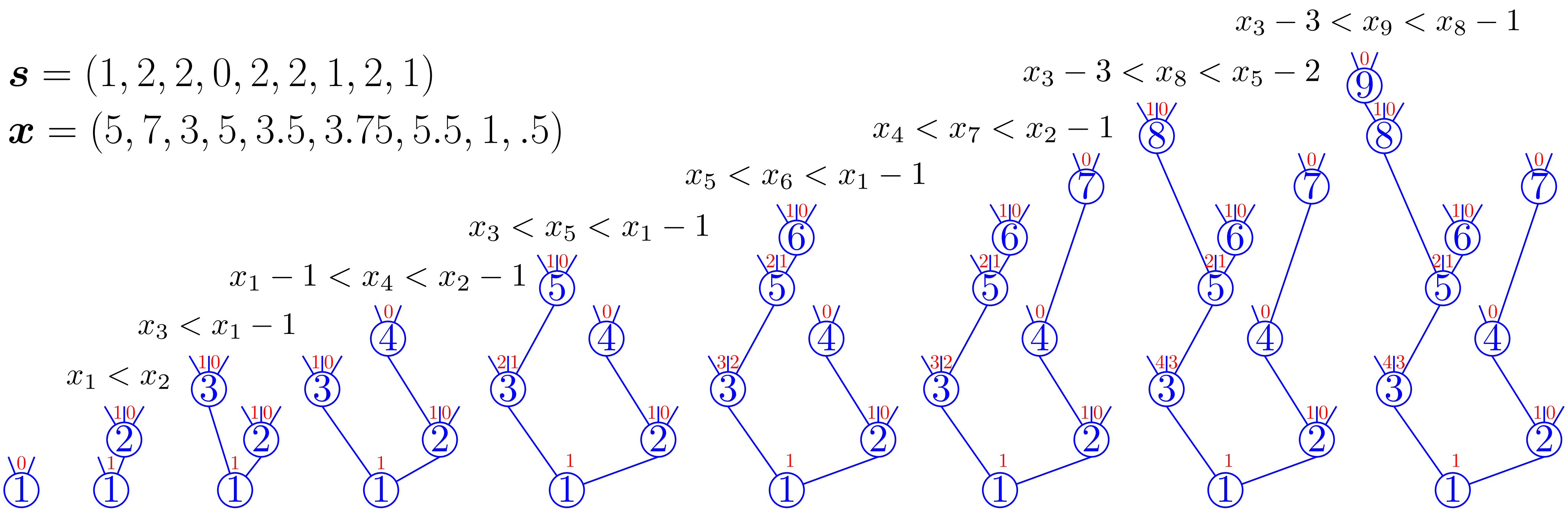


INSERTION IN S-TREES

For each generic $\boldsymbol{x} \in \mathbb{R}^n$, we construct an $\boldsymbol{s}$-tree $\mathrm{T}(\boldsymbol{s}, \boldsymbol{x})$ inductively as follows:

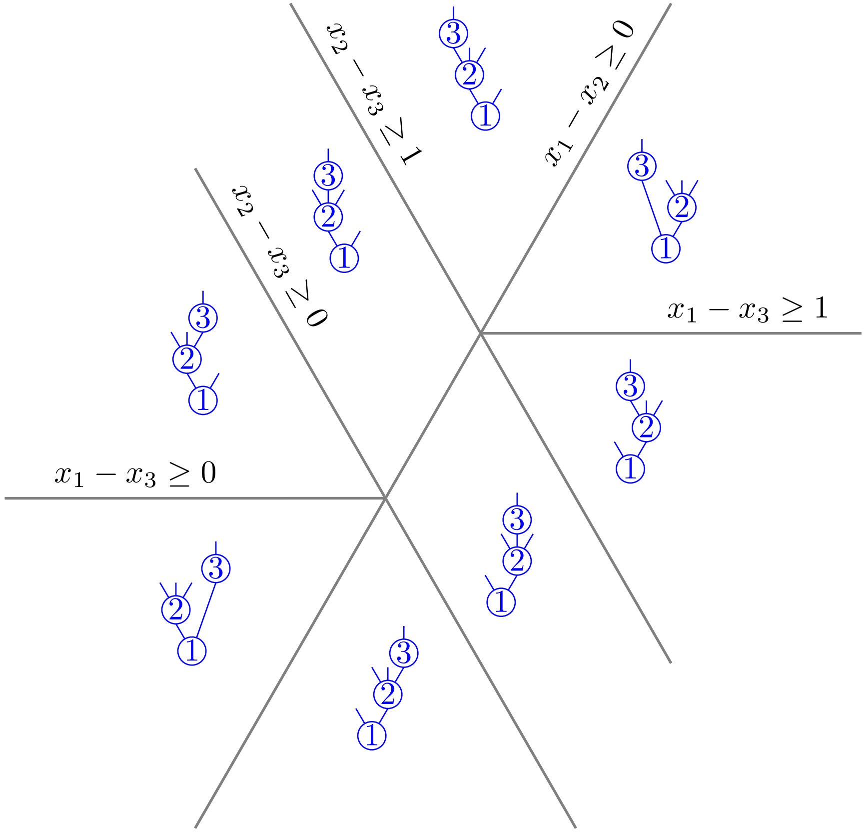
- start with a single leaf
- at step j,
  - attach a new node j to the leaf between two labels (u, ρ) and (v, σ) such that $x_u - \rho < x_j < x_v - \sigma$
  - attach $s_j + 1$ leaves to the node j, with gaps labeled by $(j, s_j - 1), \dots, (j, 1), (j, 0)$
  - add $\max(0, s_j - 1)$ to the second entry of all gap labels on the left of j

$$\boldsymbol{s} = (1, 2, 2, 0, 2, 2, 1, 2, 1)$$
$$\boldsymbol{x} = (5, 7, 3, 5, 3.5, 3.75, 5.5, 1, .5)$$

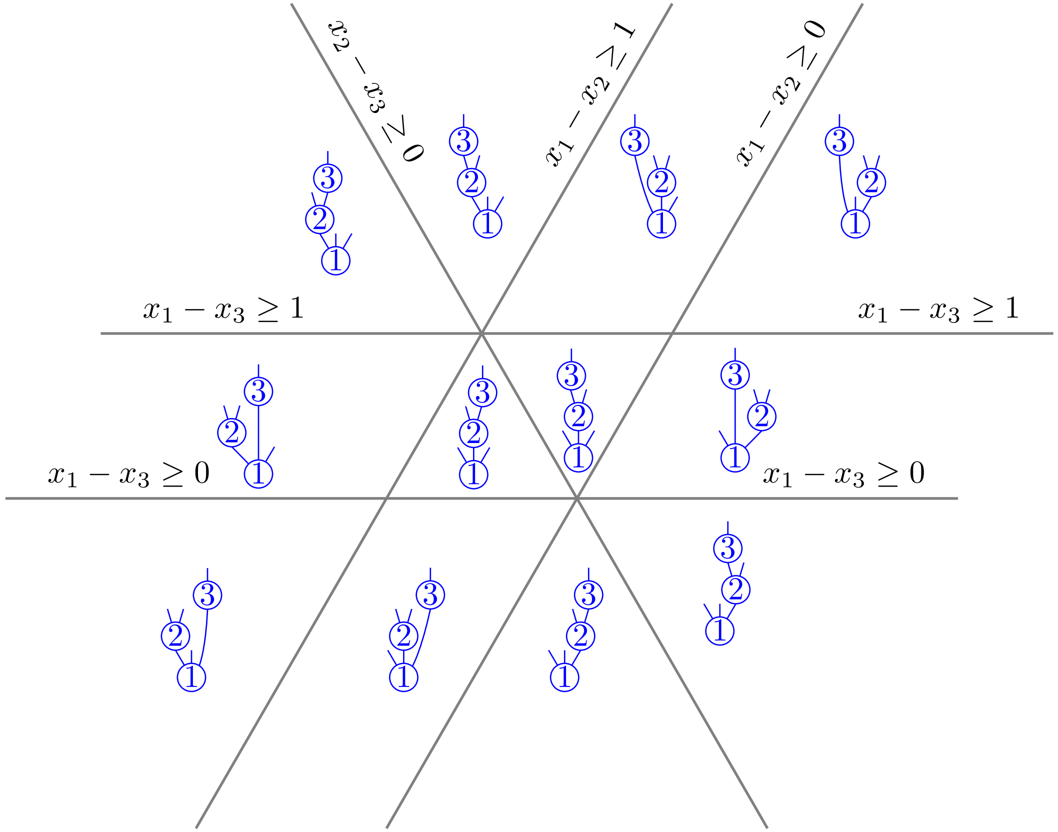


S-FOAM

$\boldsymbol{s}$-foam = polytopal complex formed by the fibers of the insertion in s-trees



$\boldsymbol{s} = (1, 2, 0)$



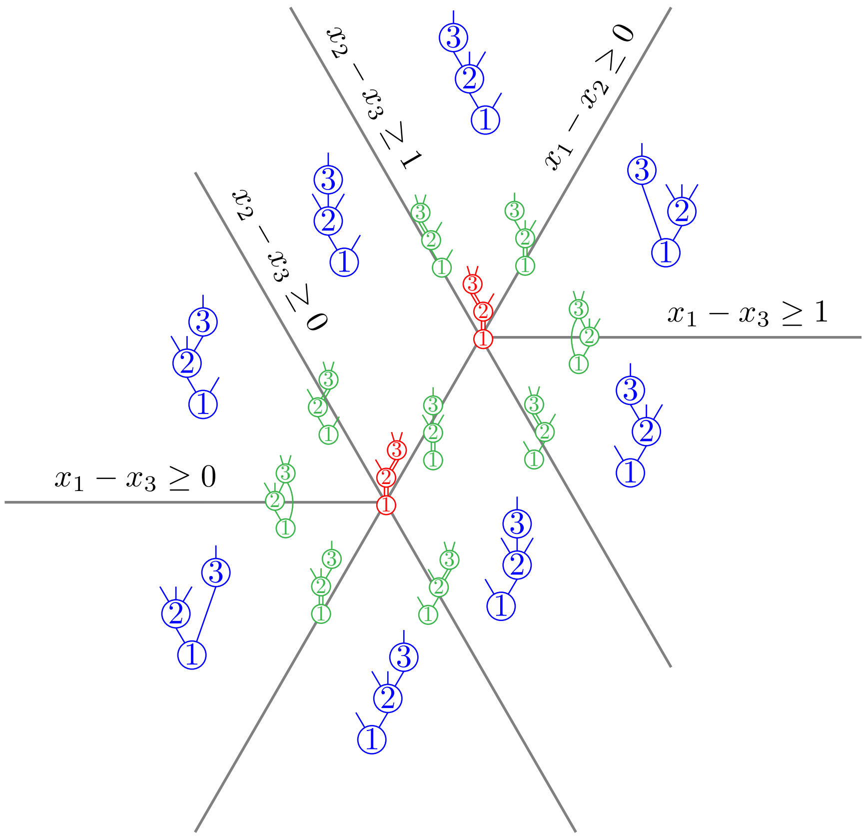
$\boldsymbol{s} = (2, 1, 0)$

PROP. Hasse diagram of the $\boldsymbol{s}$-weak order $\simeq$ oriented dual graph of the $\boldsymbol{s}$-foam

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24$^+$)

S-FOAM

$\boldsymbol{s}$-foam = complete polytopal complex formed by the fibers of the insertion in s-bushes



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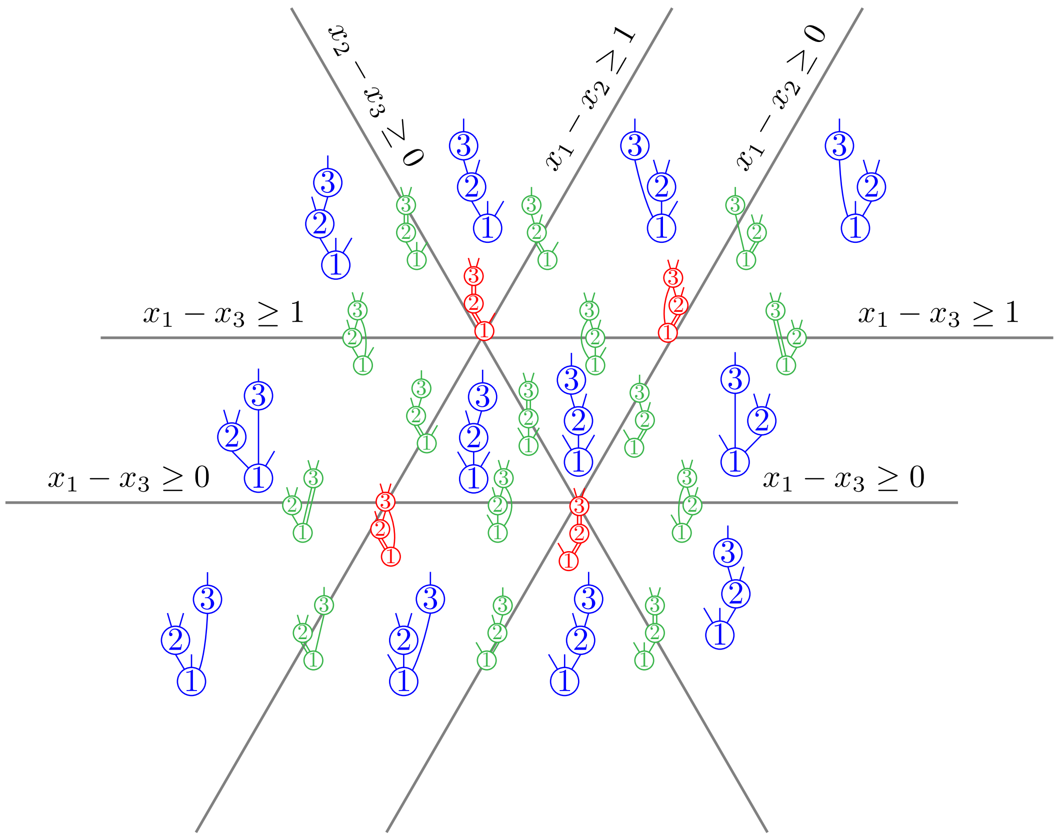

$\boldsymbol{s} = (2, 1, 0)$

PROP. Hasse diagram of the $\boldsymbol{s}$-weak order $\simeq$ oriented dual graph of the $\boldsymbol{s}$-foam

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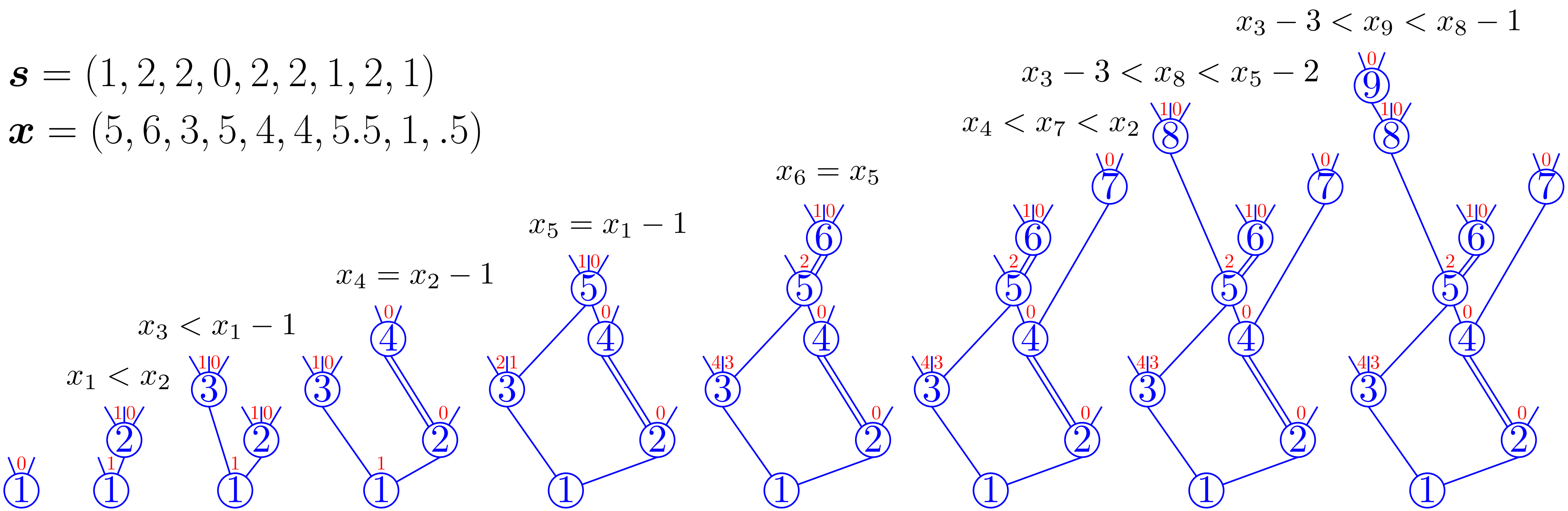
INSERTION IN S-BUSHES

For each $\boldsymbol{x} \in \mathbb{R}^n$, we construct an $\boldsymbol{s}$-bush $\mathrm{B}(\boldsymbol{s}, \boldsymbol{x})$ inductively as follows:

- start with a single leaf
- at step j,
  - attach a new node j to
    - either the leaf between two labels (u, ρ) and (v, σ) such that $x_u - \rho < x_j < x_v - \sigma$
    - or the two leaves around a gap label (w, τ) such that $x_j = x_w - \tau$
  - attach $s_j + 1$ leaves to the node j, with gaps labeled by $(j, s_j - 1), \dots, (j, 1), (j, 0)$ (except, if $s_j = 0$ and j has indegree 2, then we attach 2 leaves with gap label $(j, 0)$)
  - add $\max(0, s_j - 1)$ to the second entry of all gap labels on the left of j

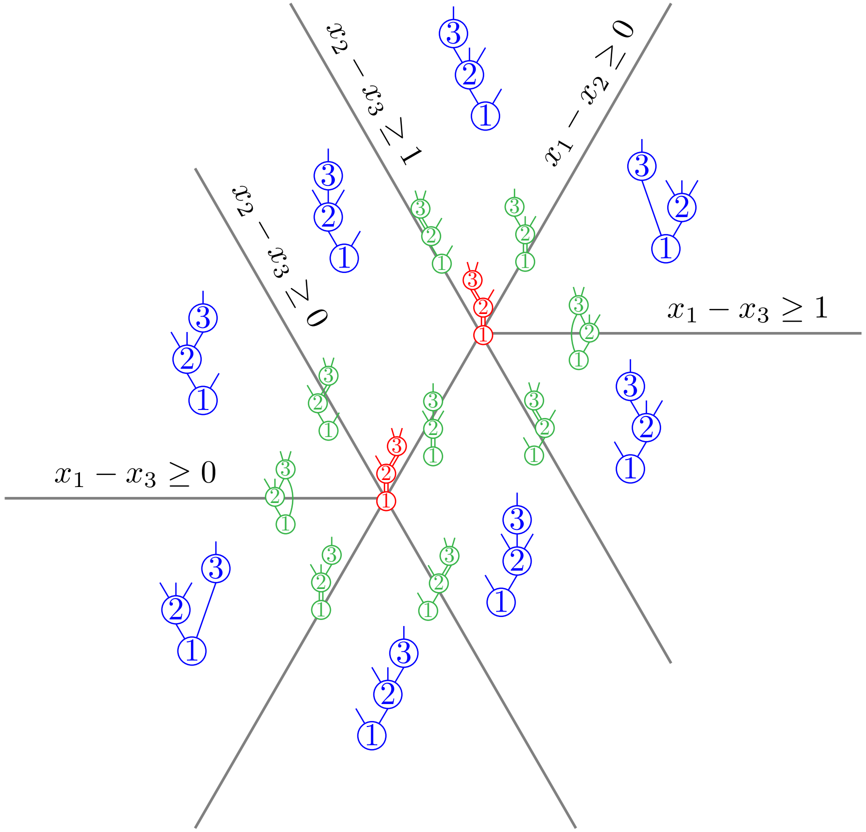
$\boldsymbol{s} = (1, 2, 2, 0, 2, 2, 1, 2, 1)$

$\boldsymbol{x} = (5, 6, 3, 5, 4, 4, 5.5, 1, .5)$

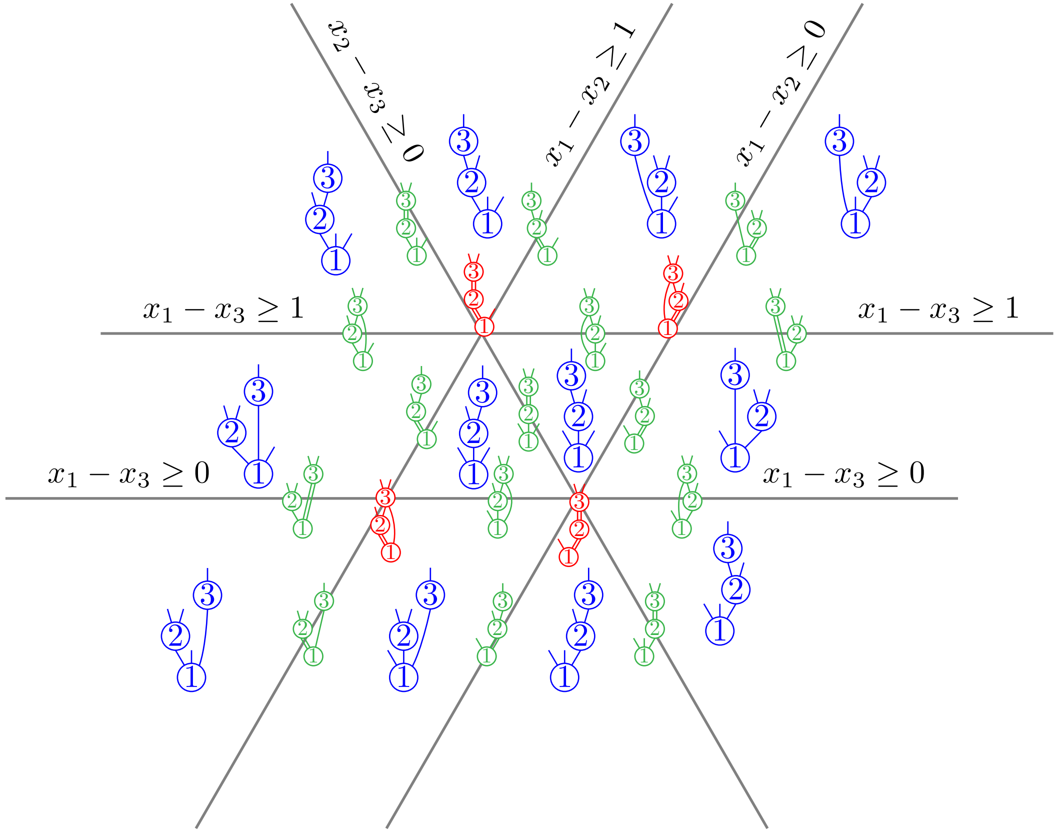


S-FOAM

$\boldsymbol{s}$-foam = complete polytopal complex formed by the fibers of the insertion in s-bushes



$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$

PROP. Hasse diagram of the $\boldsymbol{s}$-weak order $\simeq$ oriented dual graph of the $\boldsymbol{s}$-foam

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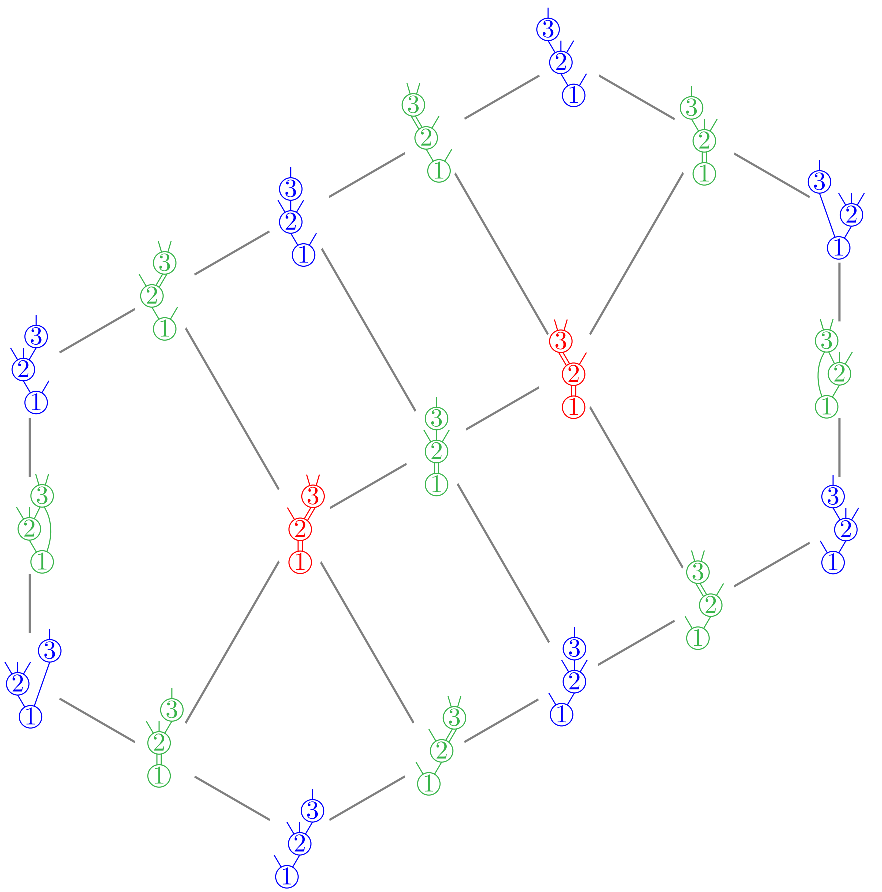
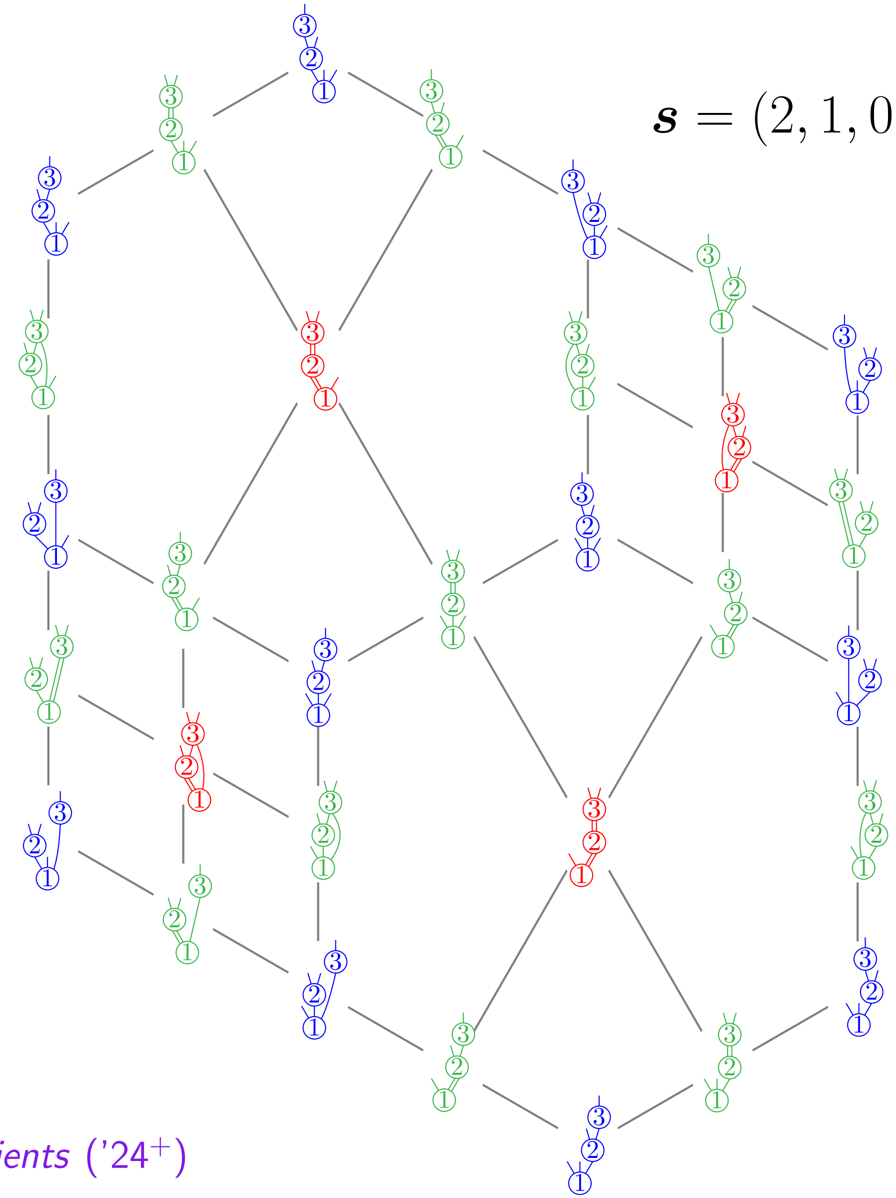
FACIAL S-WEAK ORDER

facial $\boldsymbol{s}$-weak order $=$ $\boldsymbol{s}$-bushes ordered by $\mathrm{B} \le \mathrm{B}'$ if for $1 \le i < j \le n$,
$\mathrm{lpos}(\mathrm{B}, i, j) \ge \mathrm{lpos}(\mathrm{B}', i, j)$ and $\mathrm{rpos}(\mathrm{B}, i, j) \le \mathrm{rpos}(\mathrm{B}', i, j)$

$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$



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CANONICAL REPRESENTATIONS OF S-TREES

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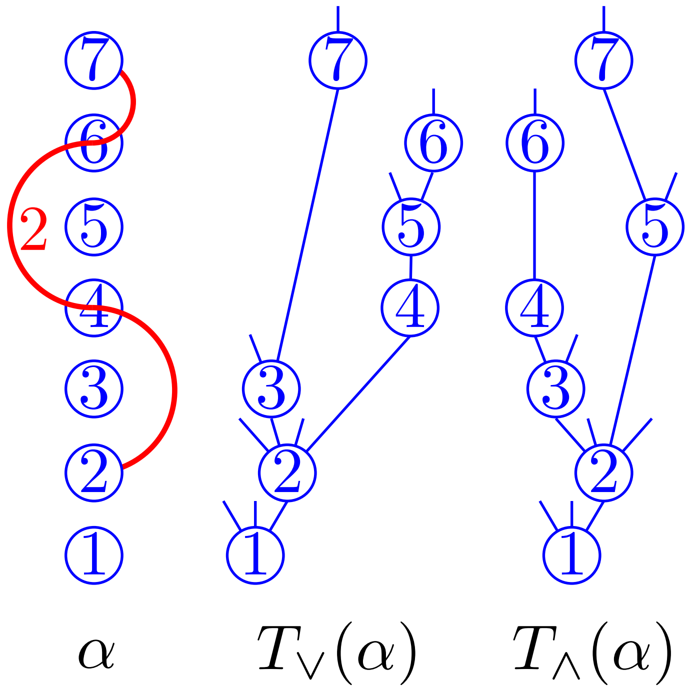
S-ARCS

$\underline{\boldsymbol{s}\text{-arc}}$ = quintuple (i, j, A, B, r) with

- $1 \le i < j \le n$
- A and B form a partition of $\{k \in]i, j[\mid s_k \neq 0\}$
- $r \in [s_i]$

$$\# \ \boldsymbol{s}\text{-arcs} = \sum_{1 \le i < j \le n} s_i \, 2^{\#\{k \in]i,j[\mid s_k \neq 0\}}$$



PROP. join irreducible $\boldsymbol{s}$-trees $\longleftarrow$ $\boldsymbol{s}$-arcs $\longrightarrow$ meet irreducible $\boldsymbol{s}$-trees

$$\mathrm{T}_\vee(\alpha) \longleftarrow\!\!\shortmid \ \alpha \ \longmapsto \mathrm{T}_\wedge(\alpha)$$

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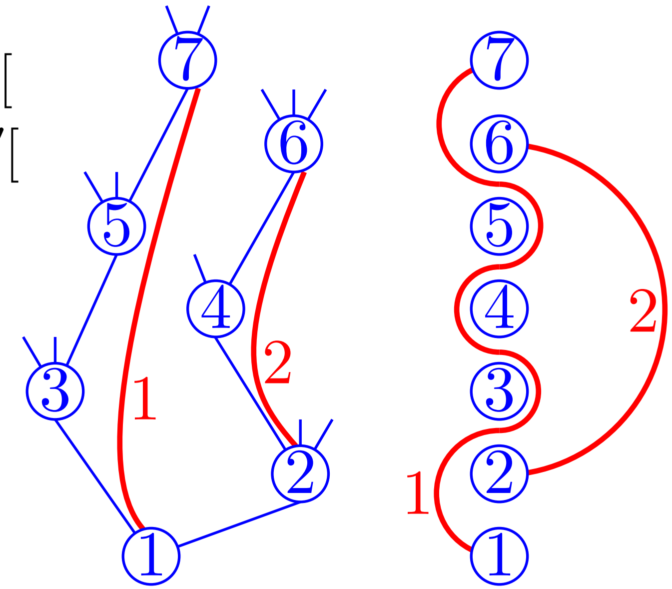
NON-CROSSING S-ARC DIAGRAMS

two s-arcs (i, j, A, B, r) and (i', j', A', r') with $j \le j'$ are non-crossing if $j < j'$ and either

- $j \le i'$
- $i < i' < j$ and $i' \in A$ and $j \notin A'$ and $A' \cap]i, j[\subseteq A \cap]i', j'[$
- $i < i' < j$ and $i' \in B$ and $j \notin B'$ and $A' \cap]i, j[\supseteq A \cap]i', j'[$
- $i = i'$ and $r < r'$ and $j \notin A'$ and $A' \cap]i, j[\subseteq A \cap]i', j'[$
- $i = i'$ and $r = r'$ and $s_j = 0$ and $A' \cap]i, j[= A \cap]i', j'[$
- $i = i'$ and $r > r'$ and $j \notin B'$ and $A' \cap]i, j[\supseteq A \cap]i', j'[$
- $i' < i$ and $i \in A'$ and $j \notin B'$ and $A' \cap]i, j[\supseteq A \cap]i', j'[$
- $i' < i$ and $i \in B'$ and $j \notin A'$ and $A' \cap]i, j[\subseteq A \cap]i', j'[$



PROP. bijection s-trees $\longrightarrow$ non-crossing s-arc diagrams

$$\mathrm{T} \longmapsto \delta_{\vee}(\mathrm{T}) = \{(i, j, A, B, r) \mid (i, j) \text{ descent of } \mathrm{T}\}$$

encoding canonical join representations

$$\mathrm{T} = \bigvee_{\alpha \in \delta_{\vee}(\mathrm{T})} \mathrm{T}_{\vee}(\alpha)$$

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NON-CROSSING S-ARC DIAGRAMS

PROP. bijection $\mathrm{T} \mapsto \delta_{\vee}(\mathrm{T})$ encoding canonical join representations $\mathrm{T} = \bigvee_{\alpha \in \delta_{\vee}(\mathrm{T})} \mathrm{T}_{\vee}(\alpha)$

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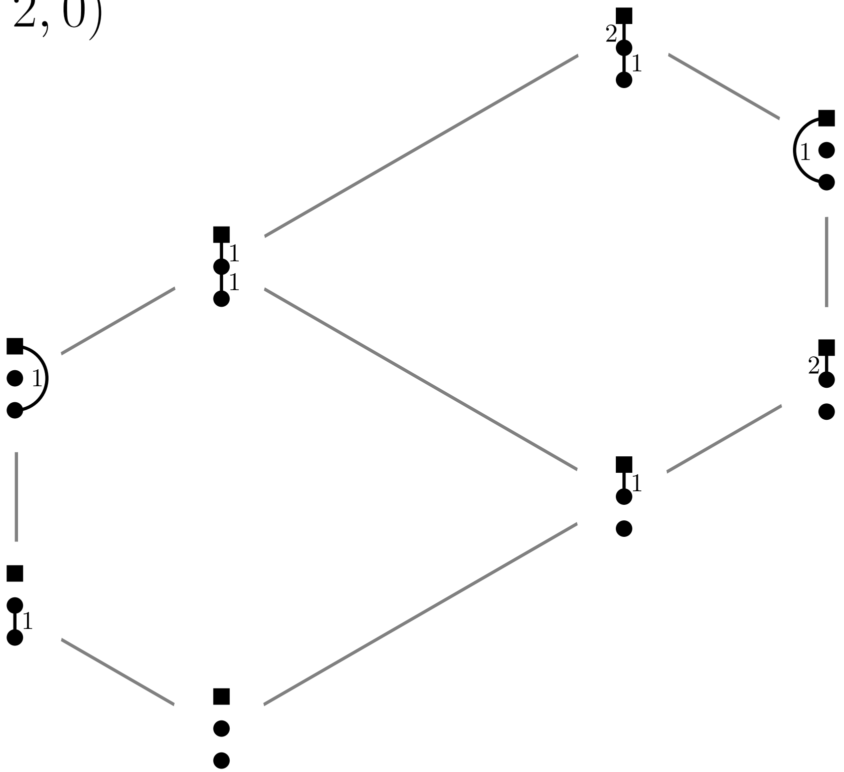
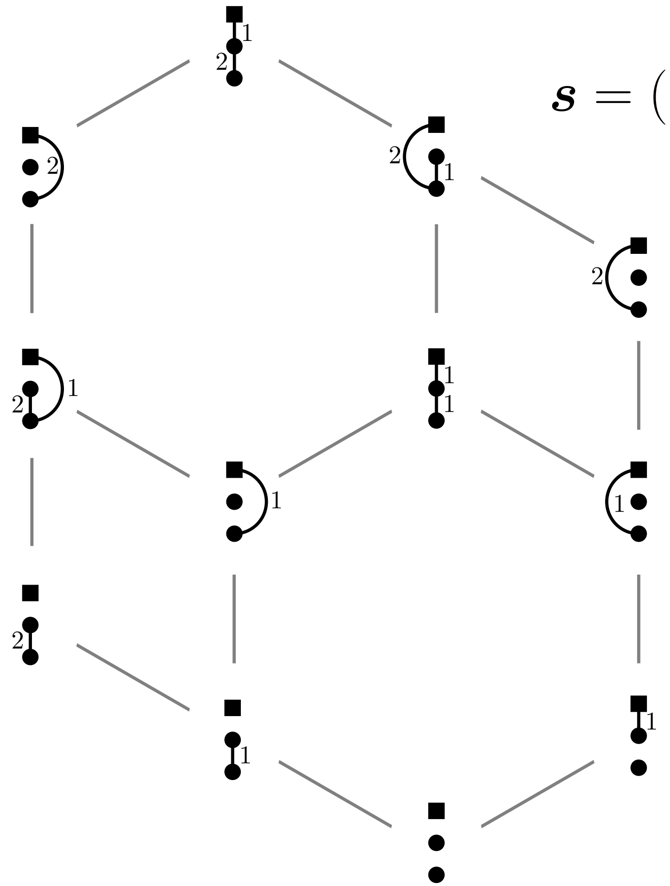
$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$



CANONICAL COMPLEX OF THE S-WEAK ORDER

PROP.

canonical join complex of the s-weak order $\longleftrightarrow$ non-crossing s-arc diagram complex
canonical complex of the s-weak order $\longleftrightarrow$ semi-crossing s-arc bidiagram complex

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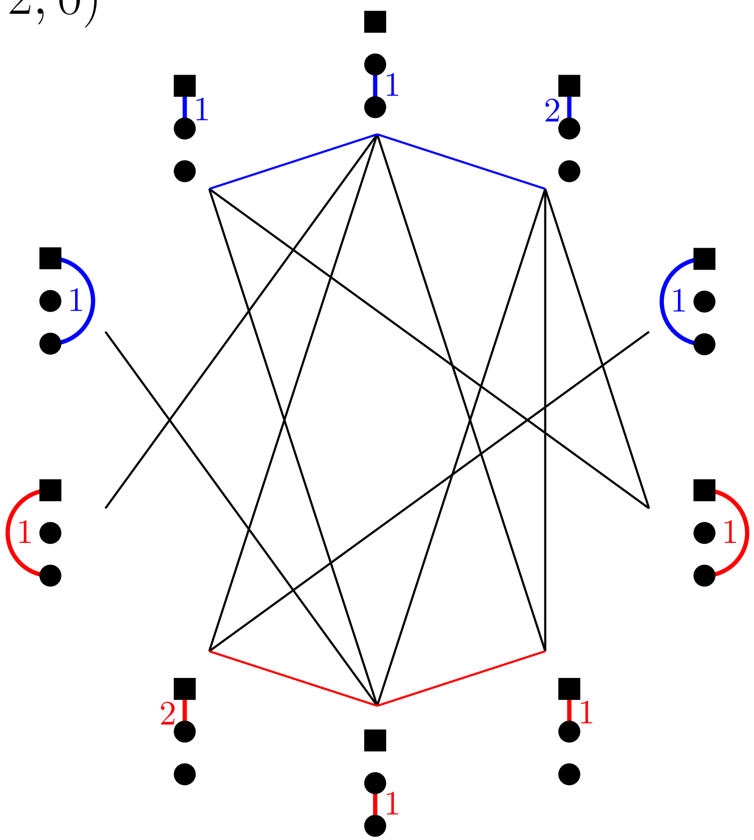
$s = (1, 2, 0)$



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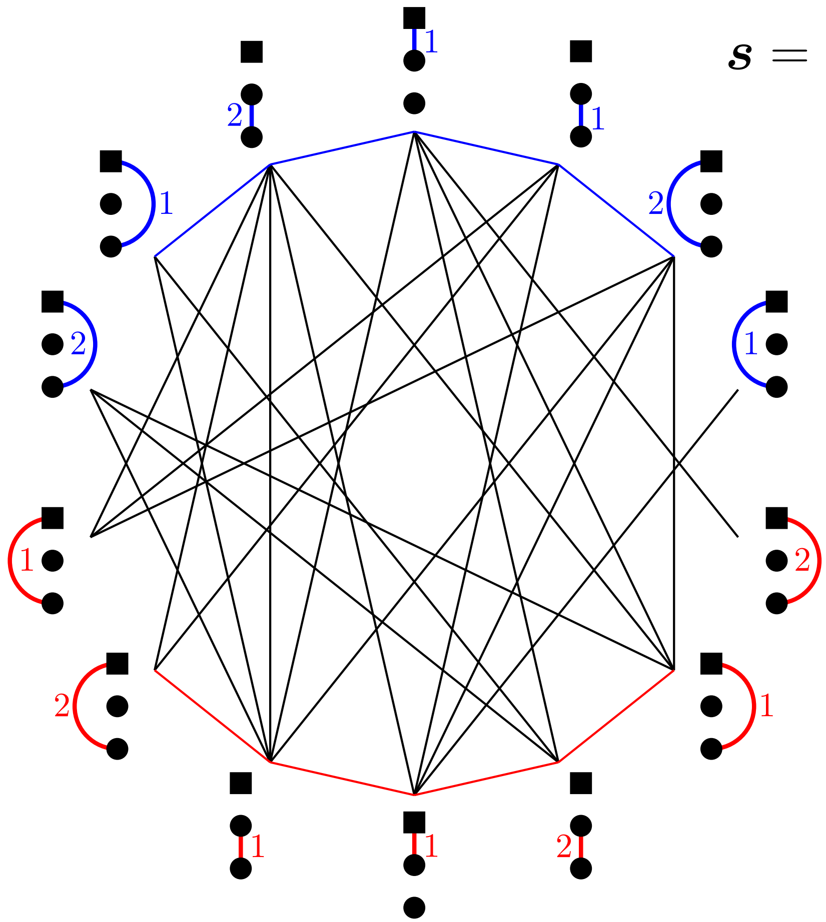

QUOTIENTS OF THE S-WEAK ORDER

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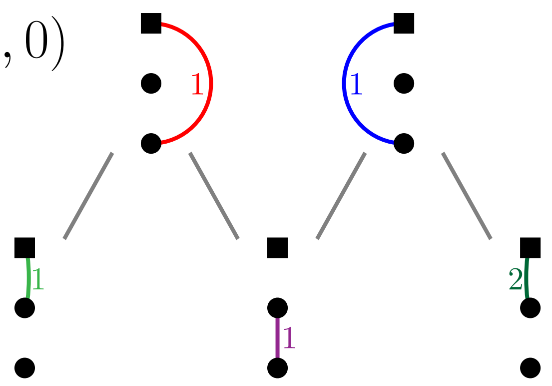
SUBARC ORDER

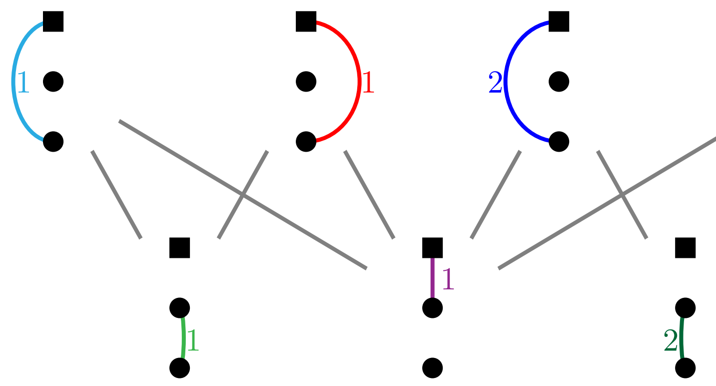
(i, j, A, B, r) subarc of (i', j', A', B', r') if all the following conditions hold

- $i' \leq i < j \leq j'$
- $A \subseteq A'$ and $B \subseteq B'$
- if $s_j = 0$ then $j = j'$
- if $i' = i$ then $r = r'$
- if $i' < i$ then either $i \in A'$ and $r = 1$, or $i \in B'$ and $r = s_i$

subarc order $= \boldsymbol{s}$-arcs ordered by the subarc relation

$\boldsymbol{s} = (1, 2, 0)$





$\boldsymbol{s} = (2, 1, 0)$

PROP. forcing order on join irreducible $\boldsymbol{s}$-tree $\longleftrightarrow$ subarc order on $\boldsymbol{s}$-arcs

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CONGRUENCE LATTICE OF THE S-WEAK ORDER

CORO. congruence of the s-weak order $\longleftrightarrow$ down set of the subarc order on s-arcs

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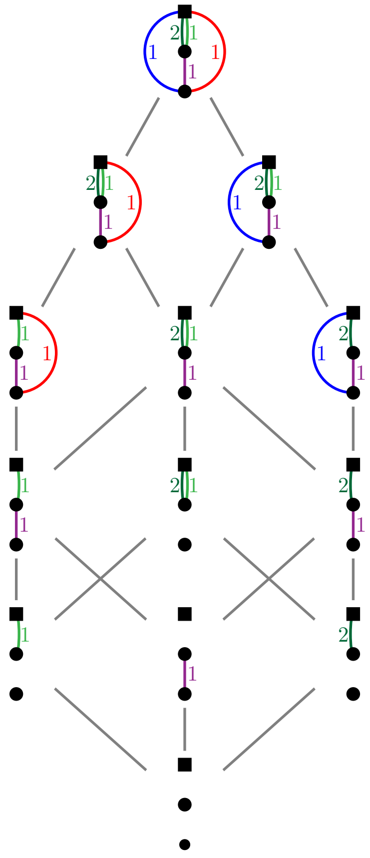
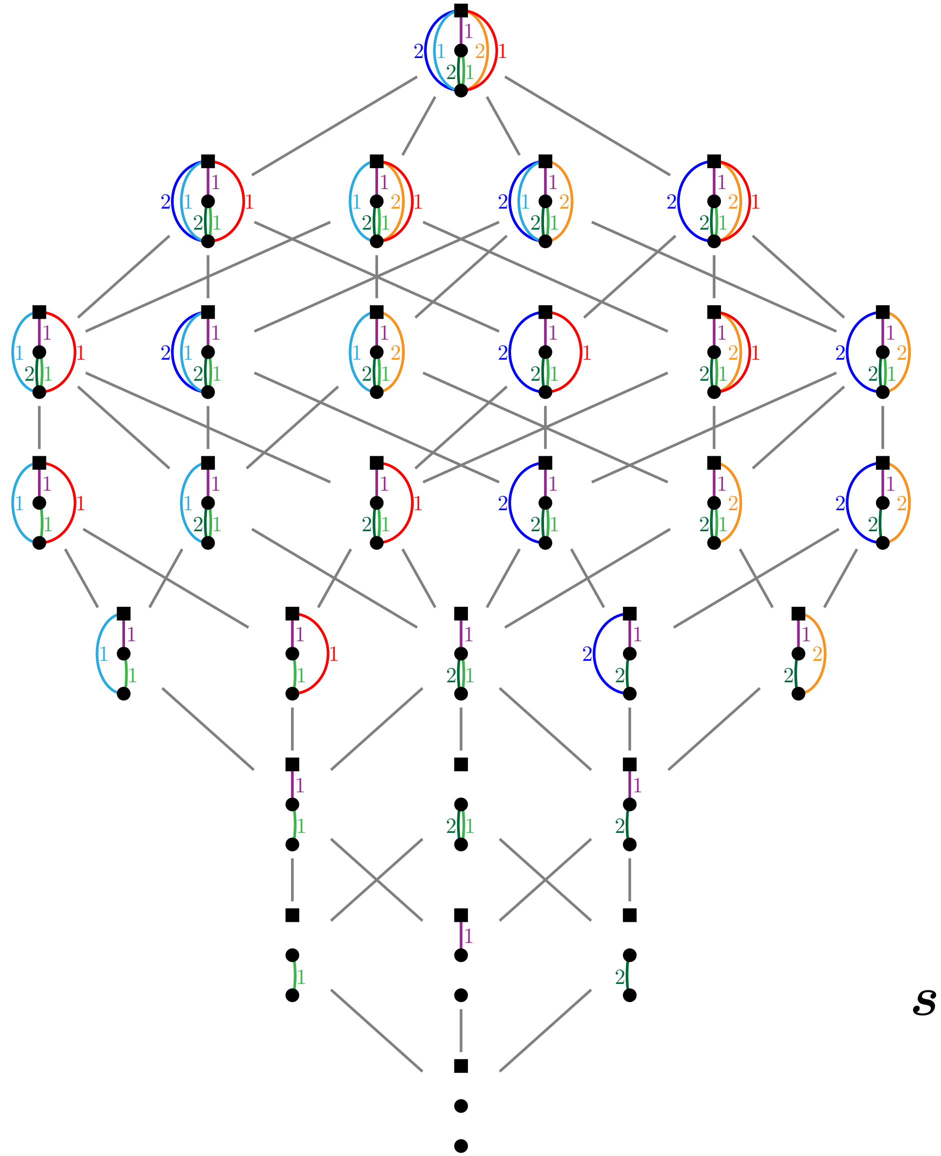


$s = (1, 2, 0)$



$s = (2, 1, 0)$

SOME RELEVANT CONGRUENCES

$\boldsymbol{s}$-sylvester congruence = down set of arcs is the set of right arcs $(i, j, \{k \in \,]i, j[\; | \; s_k \neq 0\}$
quotient = $\boldsymbol{s}$-Tamari lattice...

... but different from that of Ceballos–Pons when $\boldsymbol{s}$ contains some 0

α-Cambrian congruence = down set of arcs is the set of subarcs of α

CONJ. The following only depend on the endpoints of α:

- the cardinality of the α-Cambrian lattice
- the f-vector of the canonical join complex of the α-Cambrian lattice
- the undirected cover graph of the α-Cambrian lattice
- the face poset of the α-Cambrian foam, or dually, of the α-Cambrian quotientoplex

$\boldsymbol{\delta} : \{k \in [n] \mid s_k \neq 0\} \to \{⦶, ☮, Ⓨ, \otimes\}$ decoration
$\boldsymbol{\delta}$-permutree congruence = down set of arcs which do not pass on the right of a point j with $\boldsymbol{\delta}(j) \in \{Ⓨ, \otimes\}$ nor on the left of the points j with $\boldsymbol{\delta}(j) \in \{☮, \otimes\}$.

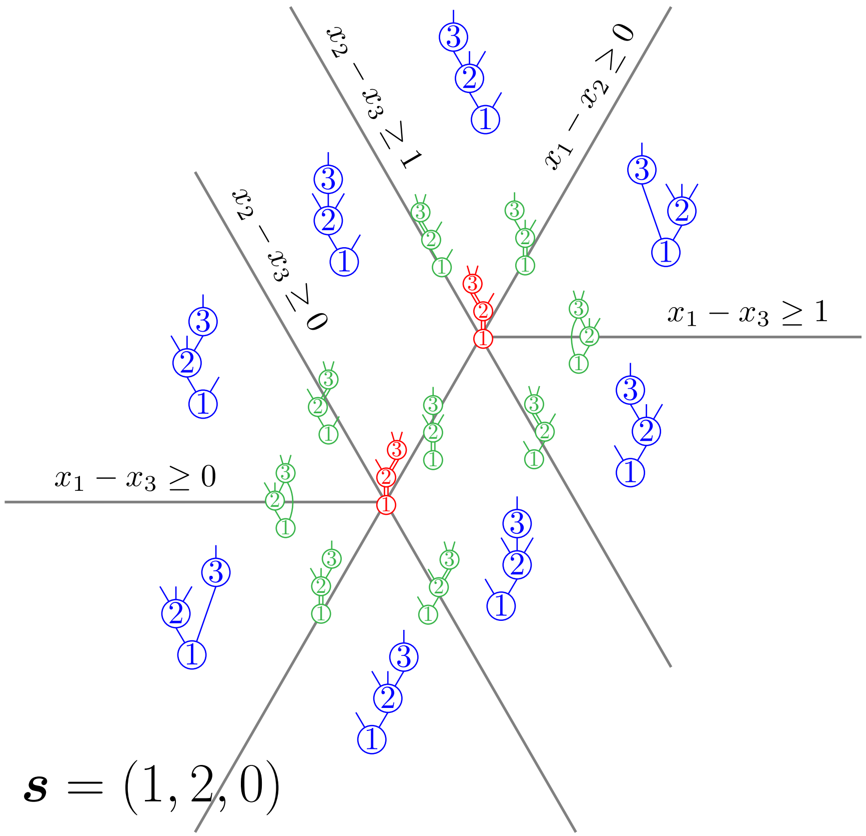
GEOMETRIC REALIZATIONS

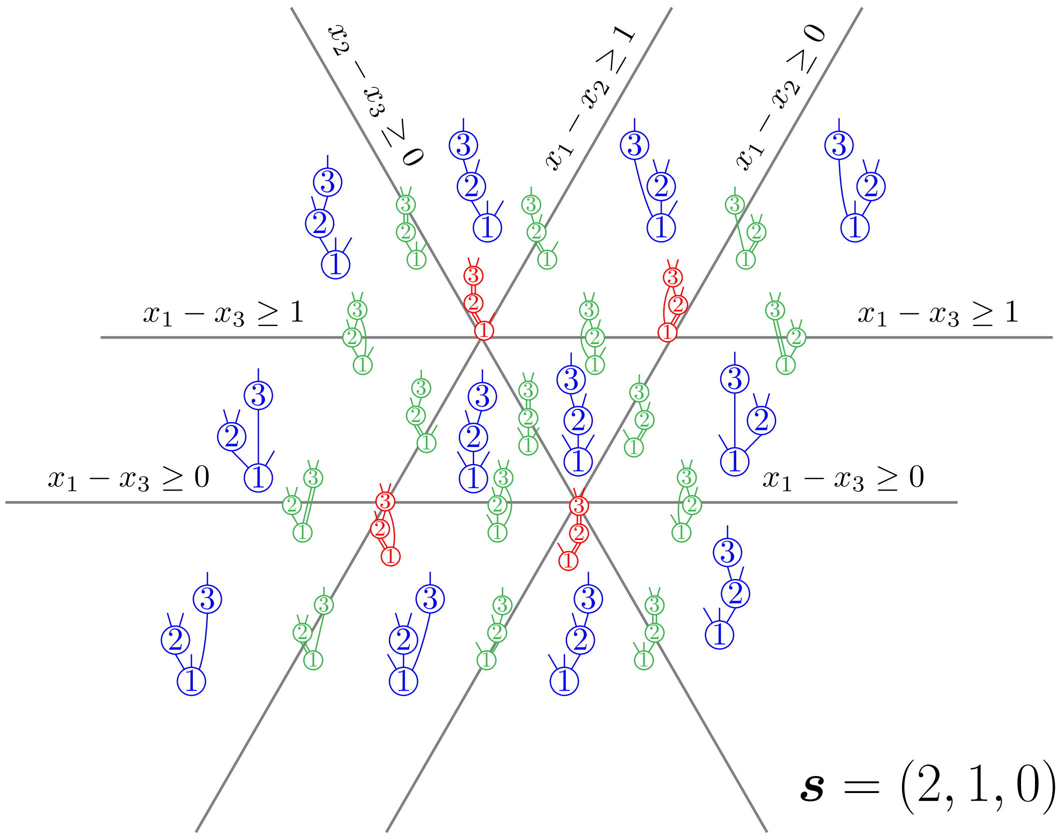
Ceballos–Pons, *The s-weak order I & II* ('22$^+$)
González-D'León–Morales–Philippe–Tamayo-Jiménez–Yip, *Realizing the s-permutahedron via flow polytopes* ('23$^+$)
Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24$^+$)

S-FAOM

PROP. Hasse diagram of the s-weak order $\simeq$ oriented dual graph of the s-foam

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24^{+})



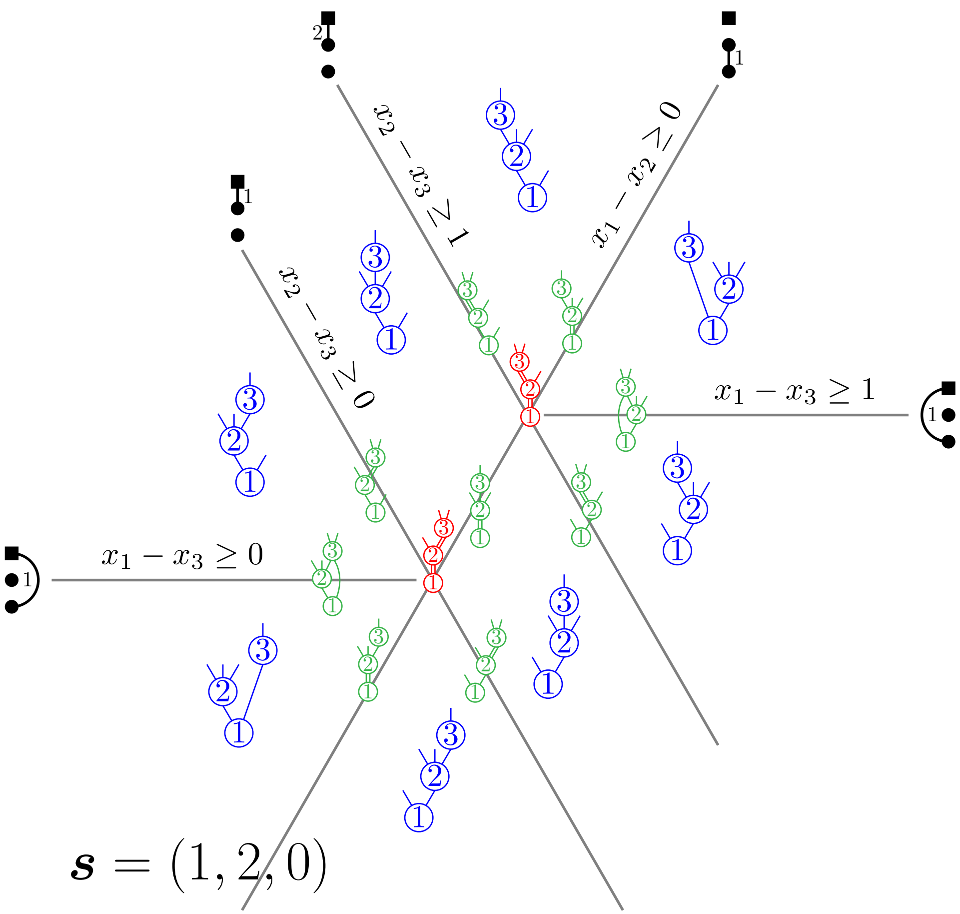


QUOTIENT FAOMS

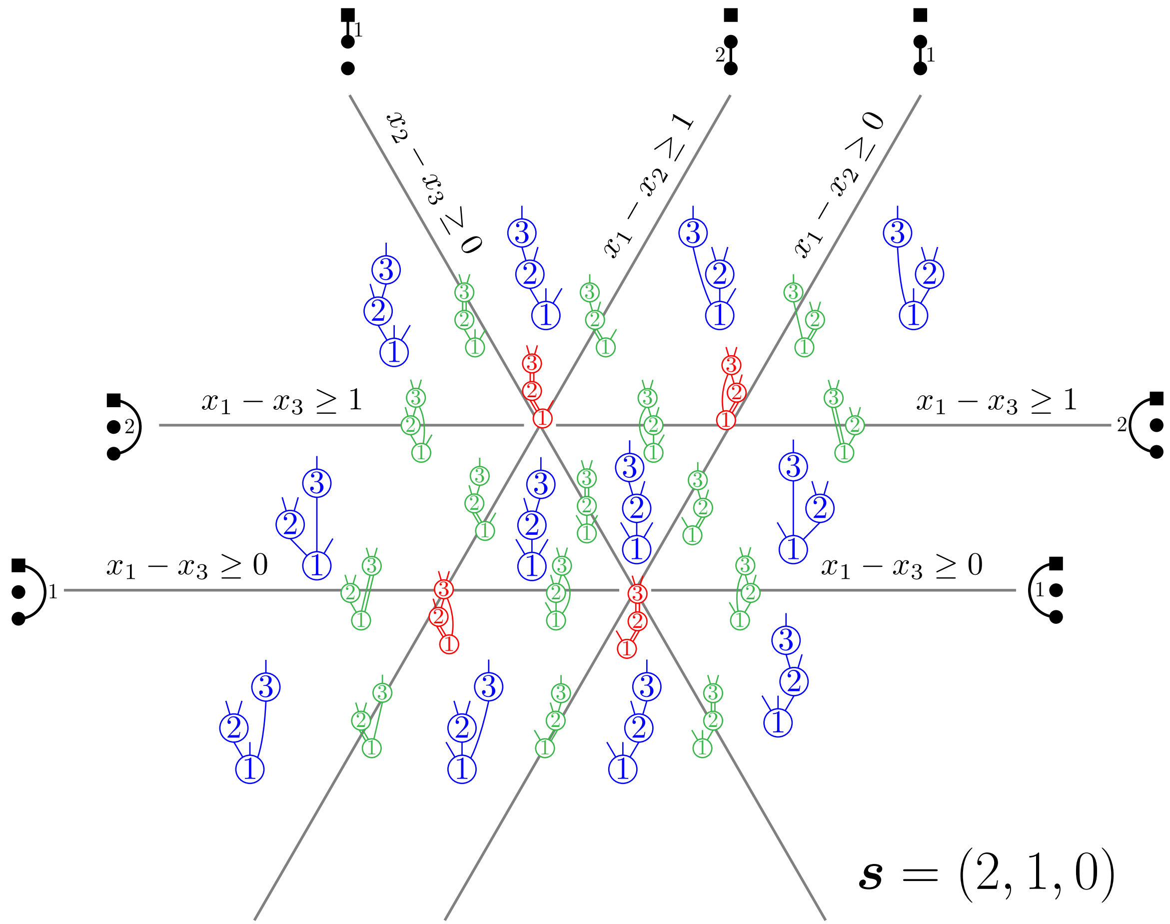
$\boldsymbol{s}$-arc $\alpha = (i, j, A, B, r)$

<u>α-shard</u> Σ_α = polyhedron of $\mathbb{R}^n$ defined by

- the equality $x_i - x_j = r - 1 + \sum_{k \in B} \max(0, s_k - 1)$,
- the inequalities $x_i - x_a \geq r - 1 + \sum_{k \in B \cap]i,a[} \max(0, s_k - 1)$ for all $a \in A$, and
- the inequalities $x_i - x_b \leq r - 1 + \sum_{k \in B \cap]i,b[} \max(0, s_k - 1)$ for all $b \in B$.



$\boldsymbol{s} = (1, 2, 0)$



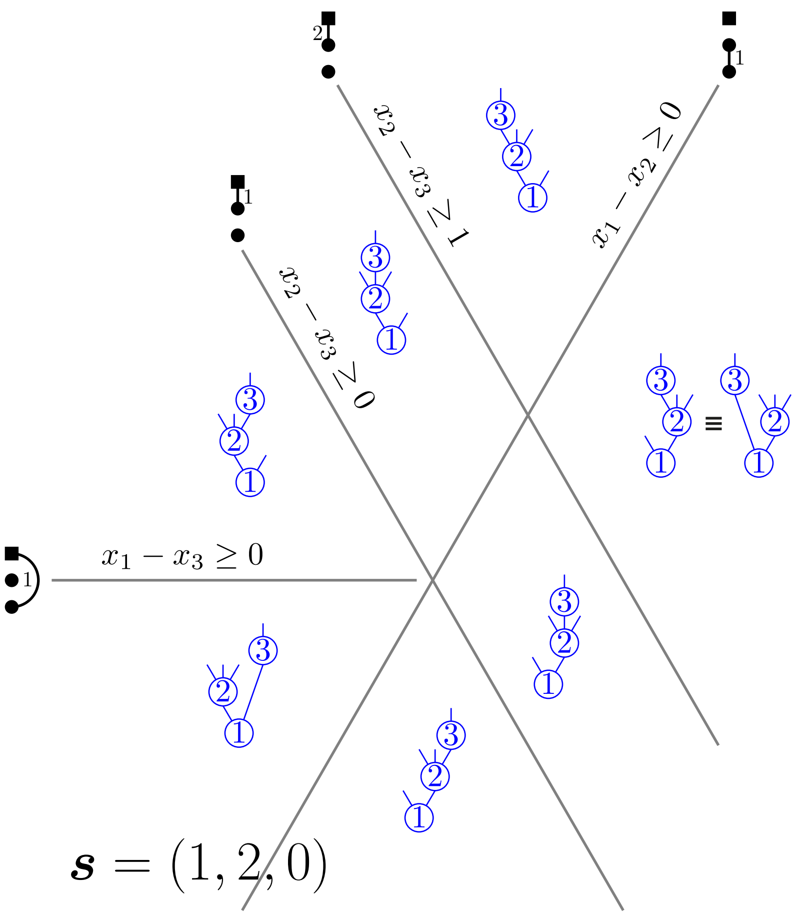
$\boldsymbol{s} = (2, 1, 0)$

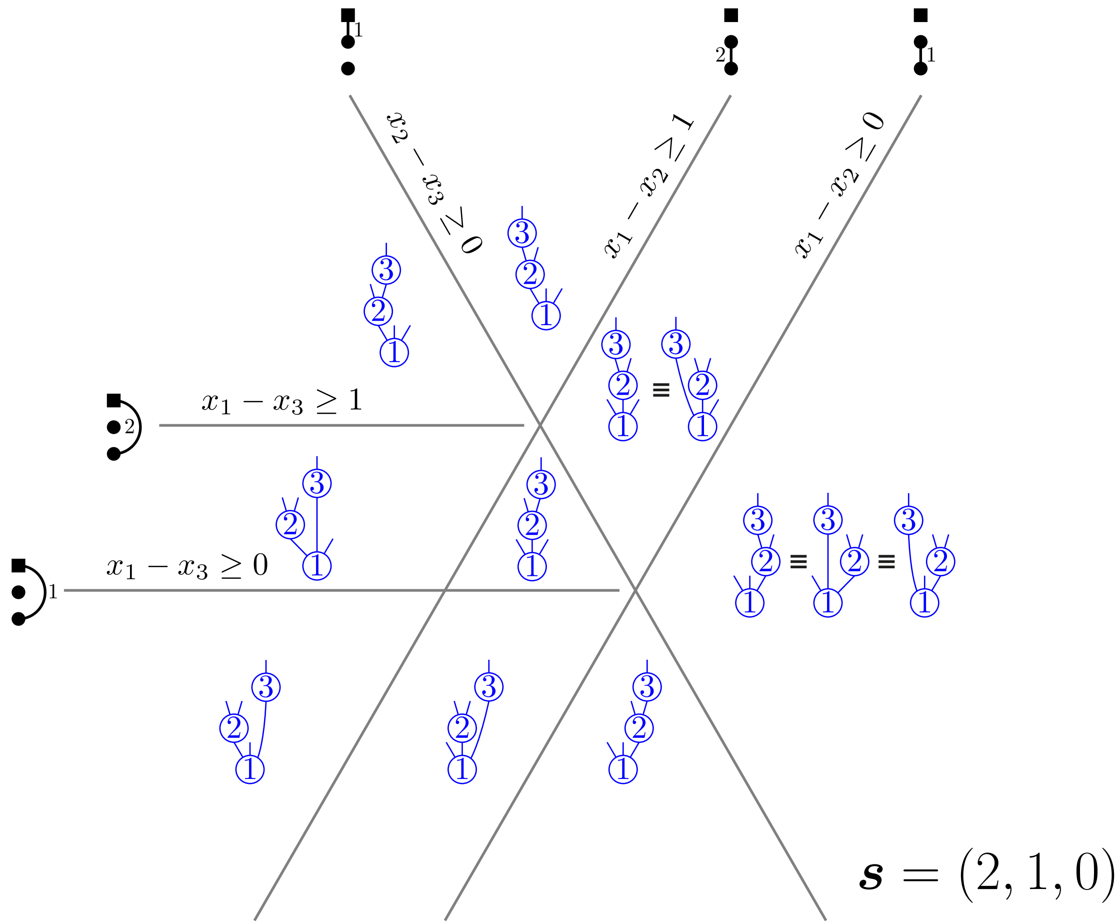
QUOTIENT FAOMS

THM. Hasse diagram of the quotient of the s-weak order by a congruence $\equiv$
$\longleftrightarrow$ oriented dual graph of the polyhedral complex obtained equivalently by

- glueing the maximal cells of the s-foam corresponding to congruent s-trees for $\equiv$
- keeping only the α-shards for α uncontracted by $\equiv$

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24+)

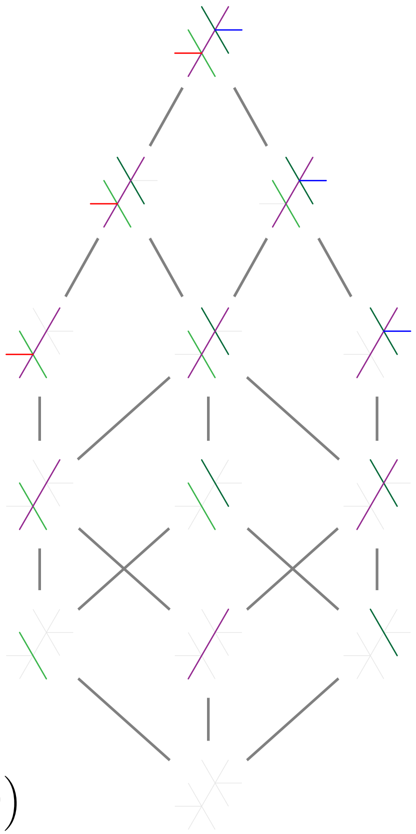




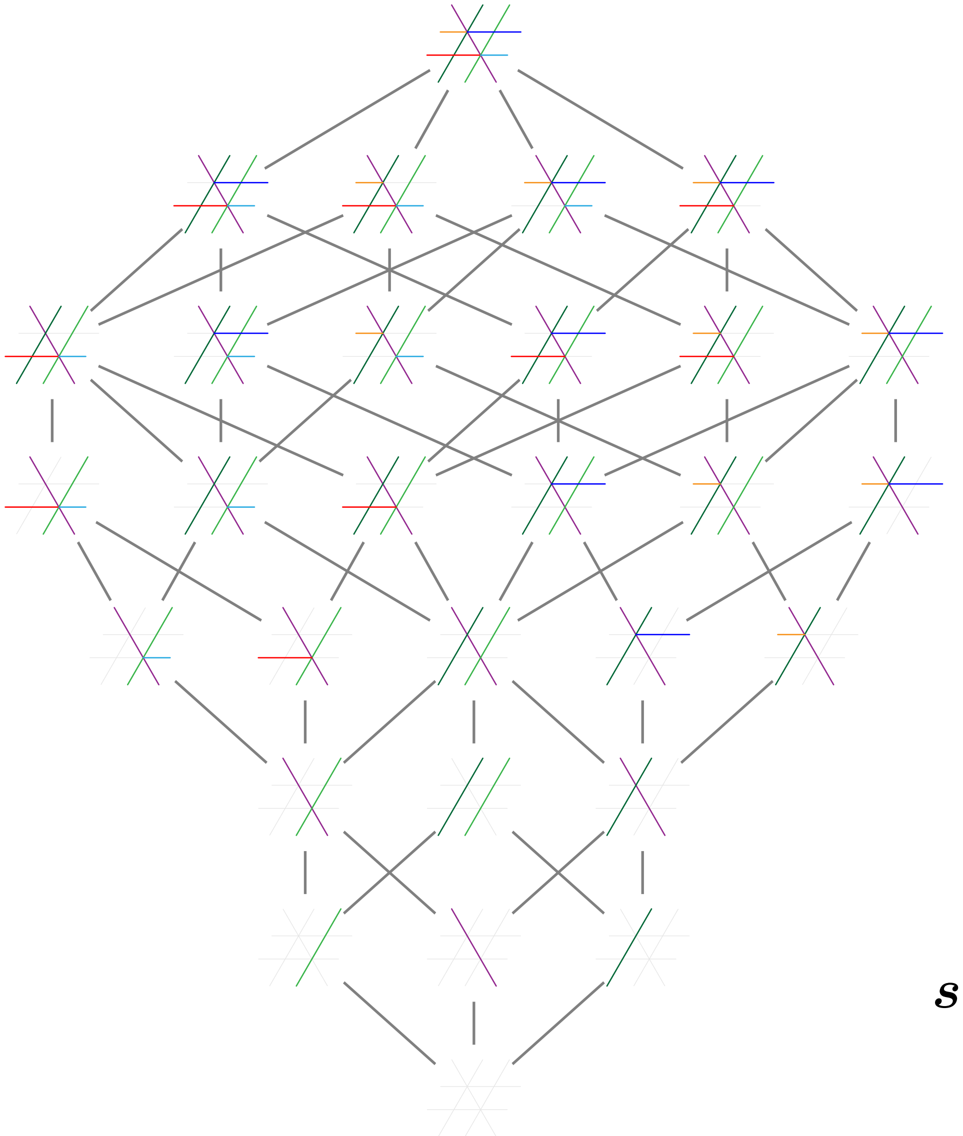
QUOTIENT FAOMS

THM. Hasse diagram of the quotient $\simeq$ oriented dual graph of the quotient foam

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24$^+$)



$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$

SHARDOPLEXES

$\boldsymbol{s}$-arc $\alpha = (i, j, A, B, r) \longrightarrow$ arc $\tilde{\alpha} = (i, j, A, B)$

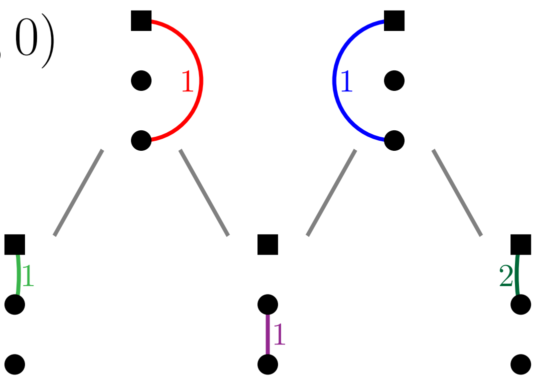
$\boldsymbol{q}$ vertex of the $\boldsymbol{s}$-foam

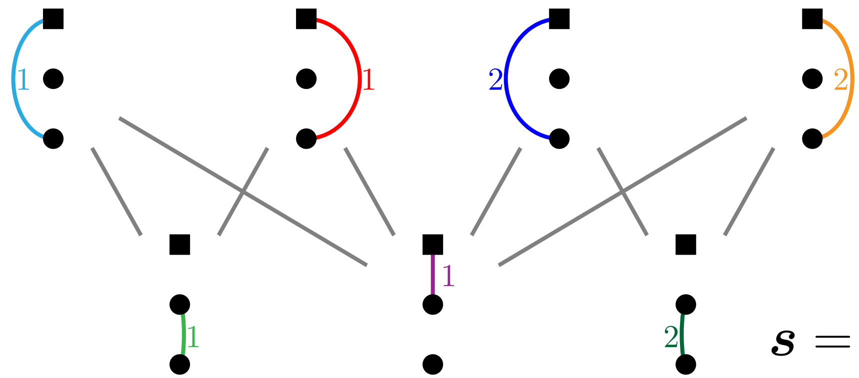
local shard polytope $\mathbb{S}_{\alpha}^{\boldsymbol{q}}$ = face of the shard polytope $\mathbb{SP}_{\tilde{\alpha}}$ maximizing the scalar product with the vector

$$\sum_{\ell \in]i,j]} \Big(\boldsymbol{q}_{\ell} - \boldsymbol{q}_{i} - r + 1 - \sum_{k \in B \cap]i,\ell[} \max(0, s_k - 1) \Big) \boldsymbol{e}_{\ell}.$$

shardoplex = polytopal complex formed by the local shard polytopes $\mathbb{S}_{\alpha}^{\boldsymbol{q}}$ for all vertices $\boldsymbol{q}$ of the quotient foam

$\boldsymbol{s} = (1, 2, 0)$





$\boldsymbol{s} = (2, 1, 0)$

SHARDOPLEXES

$\boldsymbol{s}$-arc $\alpha = (i, j, A, B, r) \longrightarrow$ arc $\tilde{\alpha} = (i, j, A, B)$

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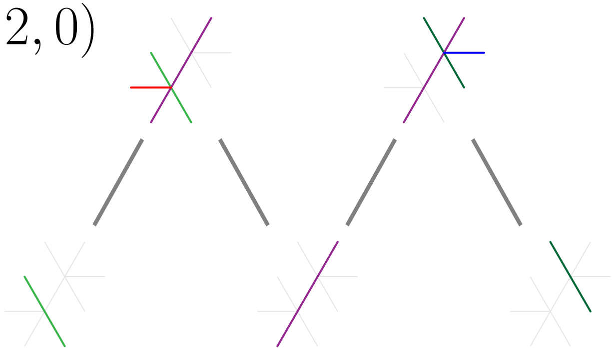

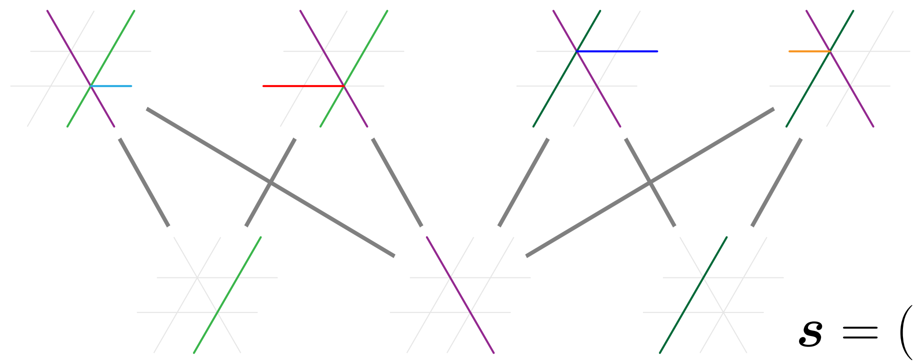

$\boldsymbol{s} = (2, 1, 0)$

SHARDOPLEXES

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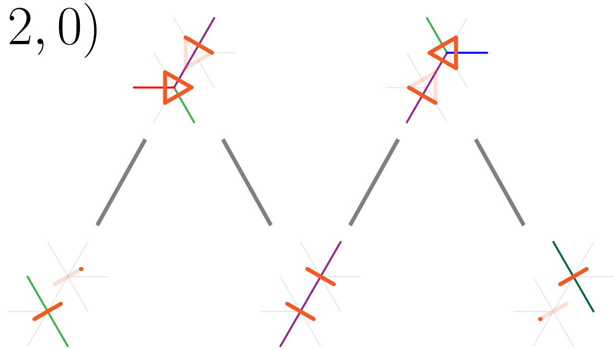
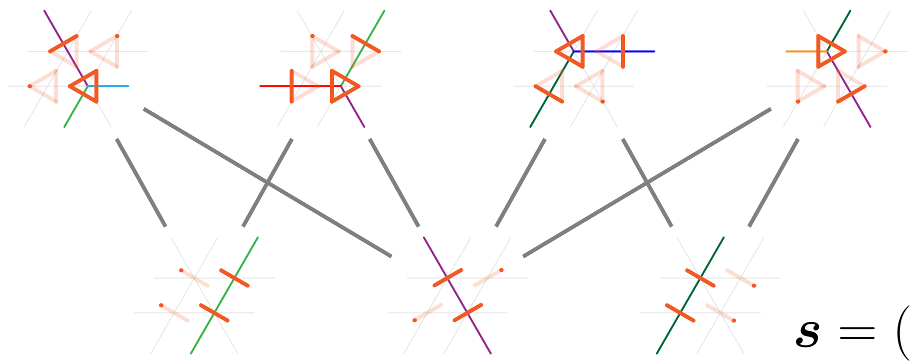
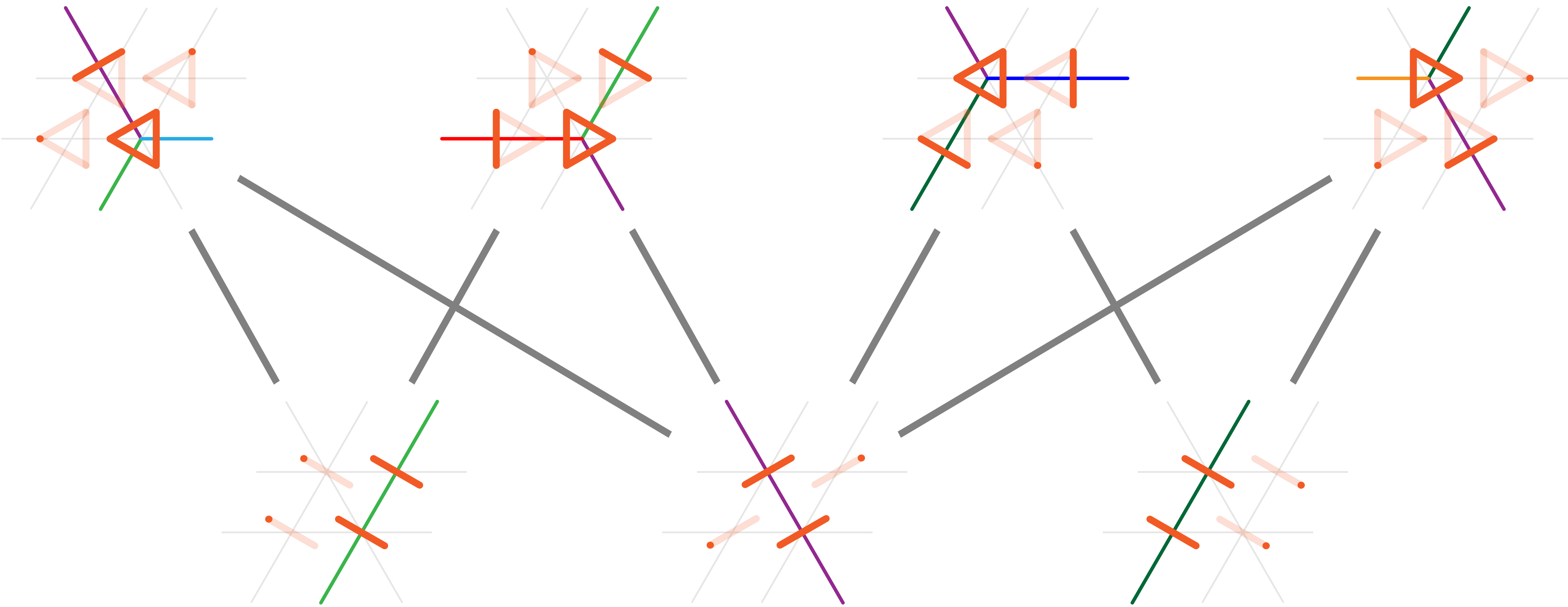
$\boldsymbol{s} = (1, 2, 0)$





$\boldsymbol{s} = (2, 1, 0)$

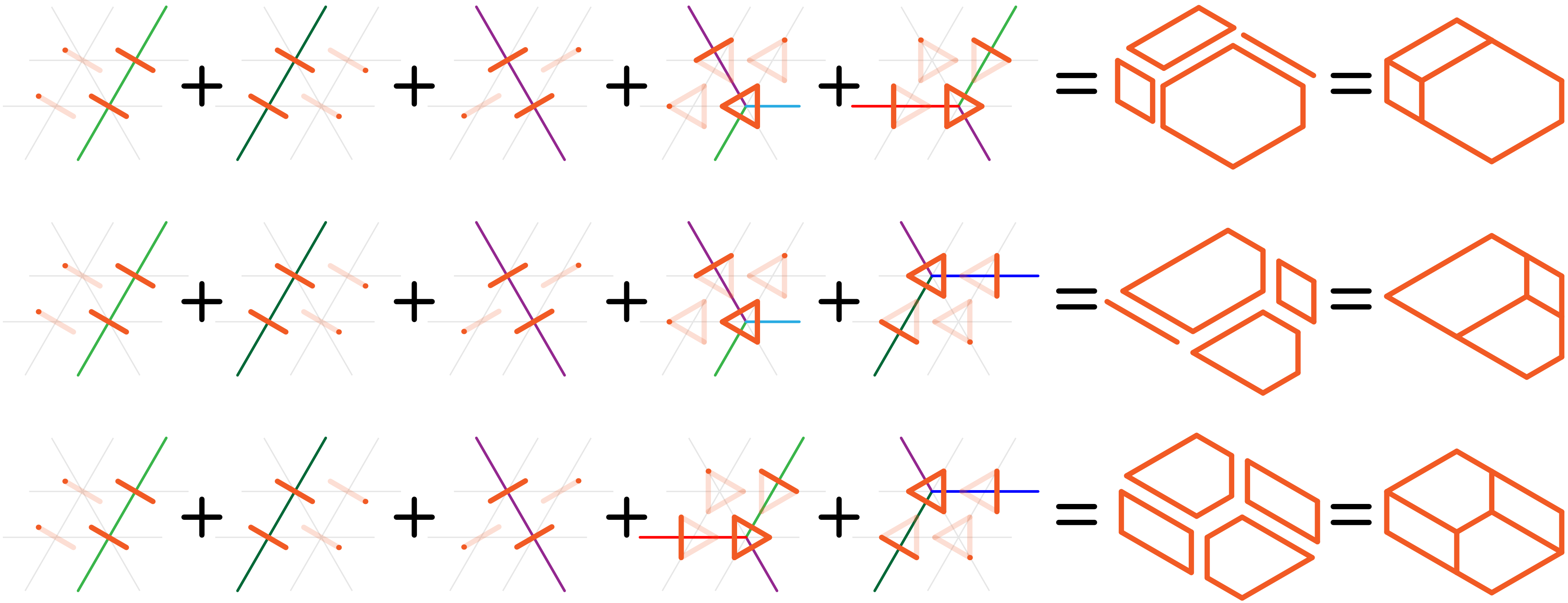
SHARDOPLEXES



QUOTIENTOPLEXES

$\equiv$ congruence of the s-weak order

quotientoplex $\mathbb{Q}_{\equiv}$ = polytopal complex obtained as the Minkowski sum of the shardoplexes $\mathbb{S}_\alpha$ of the s-arcs α uncontracted by $\equiv$



THM. • Hasse diagram of the quotient $\simeq$ oriented skeleton of the quotientoplex $\mathbb{Q}_{\equiv}$
• quotientoplex $\mathbb{Q}_{\equiv}$ is a polytopal subdivision of the quotientope $\mathbb{Q}_{\tilde{\equiv}}$

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24$^+$)

QUOTIENTOPLEXES

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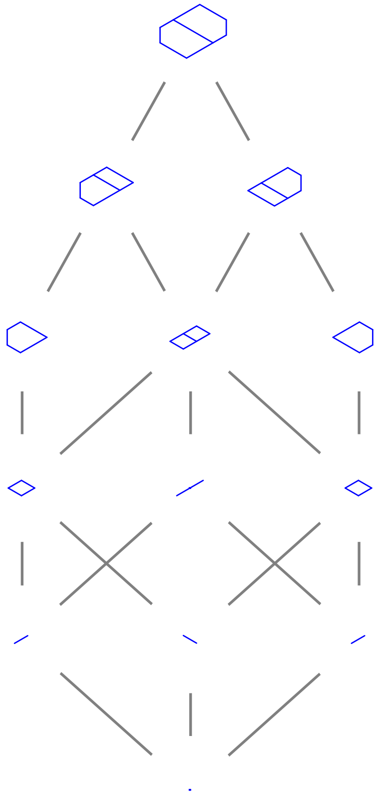
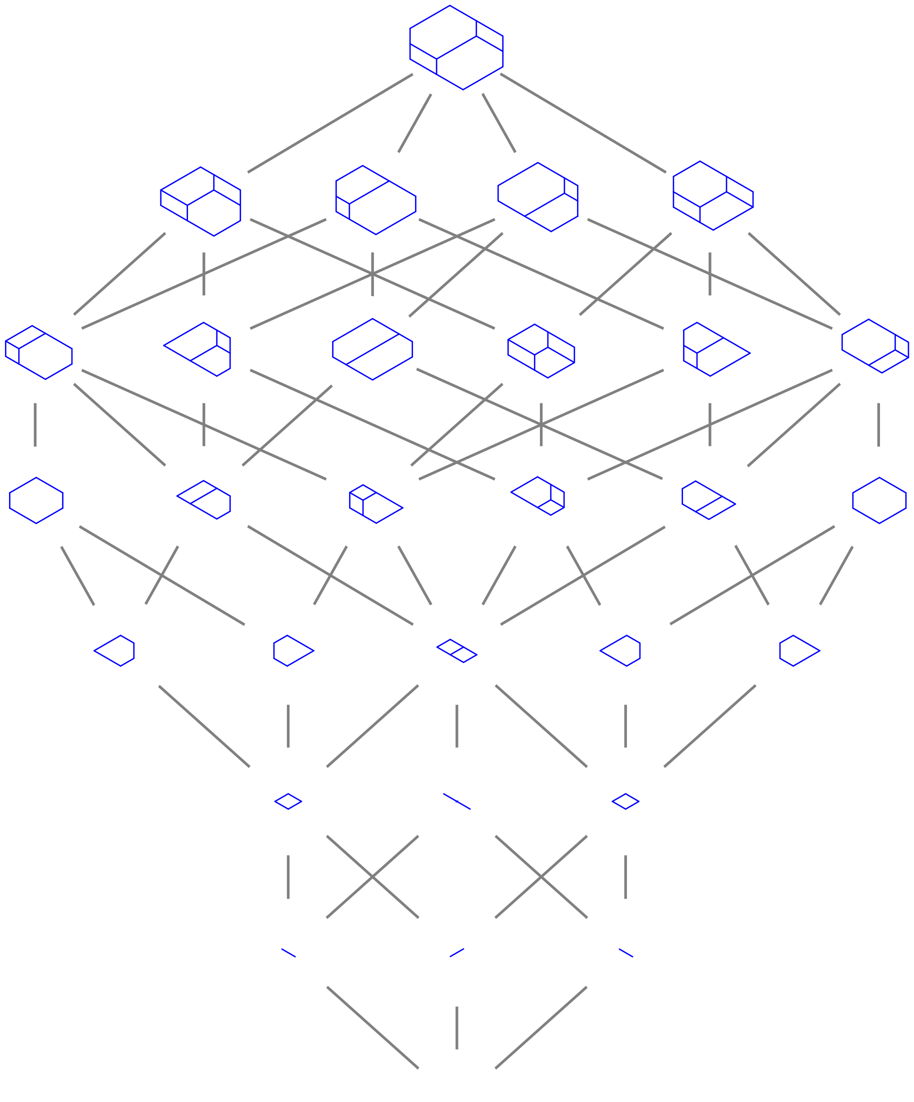
Philippe–P., *Geometric realizations of the* s*-weak order and its quotients* ('24$^+$)



$\boldsymbol{s} = (1, 2, 0)$



$\boldsymbol{s} = (2, 1, 0)$

$\boldsymbol{s} = (1, 2, 0)$

POLYWOOD

QUOTIENTOPLEXES

THM. • Hasse diagram of the quotient $\simeq$ oriented skeleton of the quotientoplex $\mathbb{Q}_{\equiv}$

• quotientoplex $\mathbb{Q}_{\equiv}$ is a polytopal subdivision of the quotientope $\mathbb{Q}_{\tilde{\equiv}}$

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24$^+$)

For the trivial congruence, this solves the conjecture of Ceballos–Pons

THM. Hasse diagram of the s-weak order $\simeq$ oriented skeleton of polytopal subdivision of a graphical zonotope combinatorially equivalent to $\sum_{1\le i<j\le n} s_i \operatorname{conv}\{\boldsymbol{e}_i, \boldsymbol{e}_k\}$

González-D'León–Morales–Philippe–Tamayo-Jiménez–Yip, *Realizing the s-permutahedron via flow polytopes* ('23$^+$)

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24$^+$)

QUOTIENTOPLEXES

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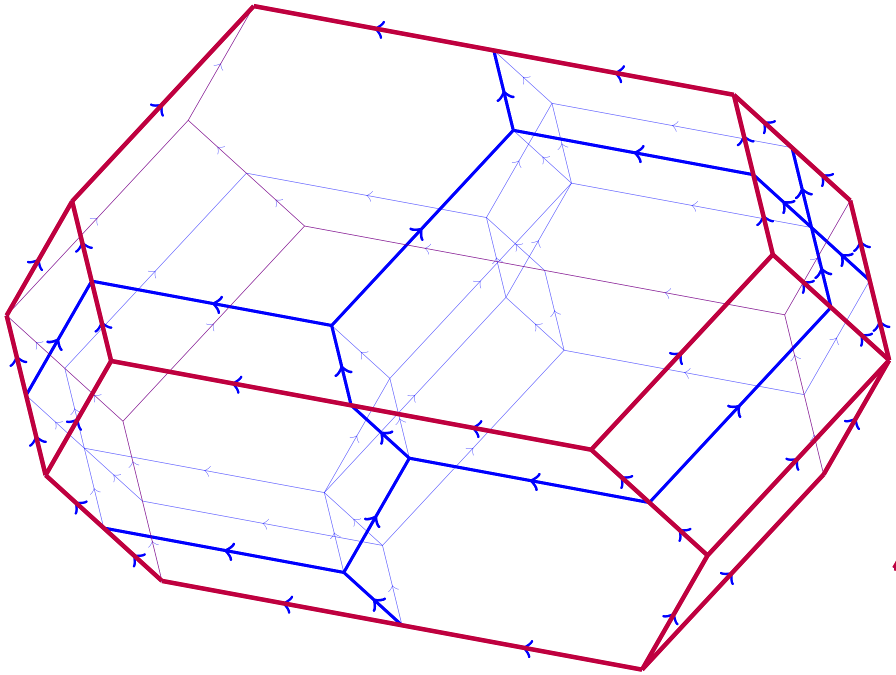
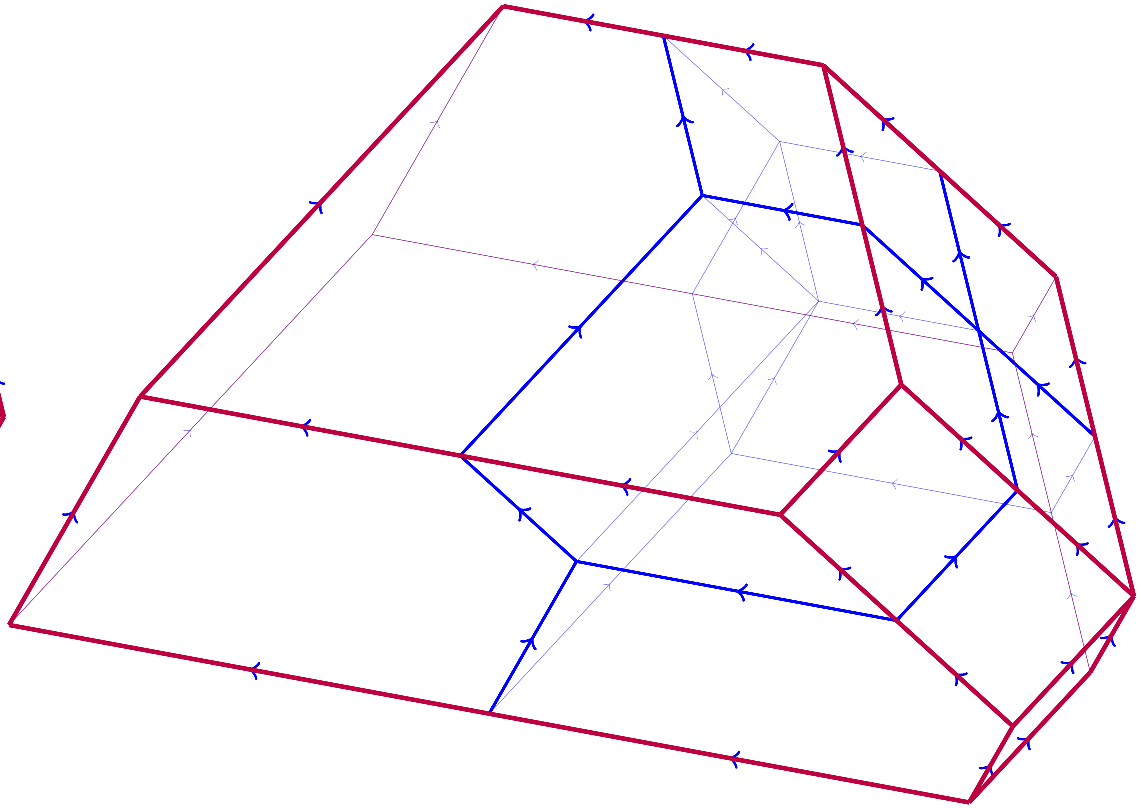
Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24^{+})

$\boldsymbol{s} = (2, 1, 0, 1)$



trivial congruence



sylvester congruence

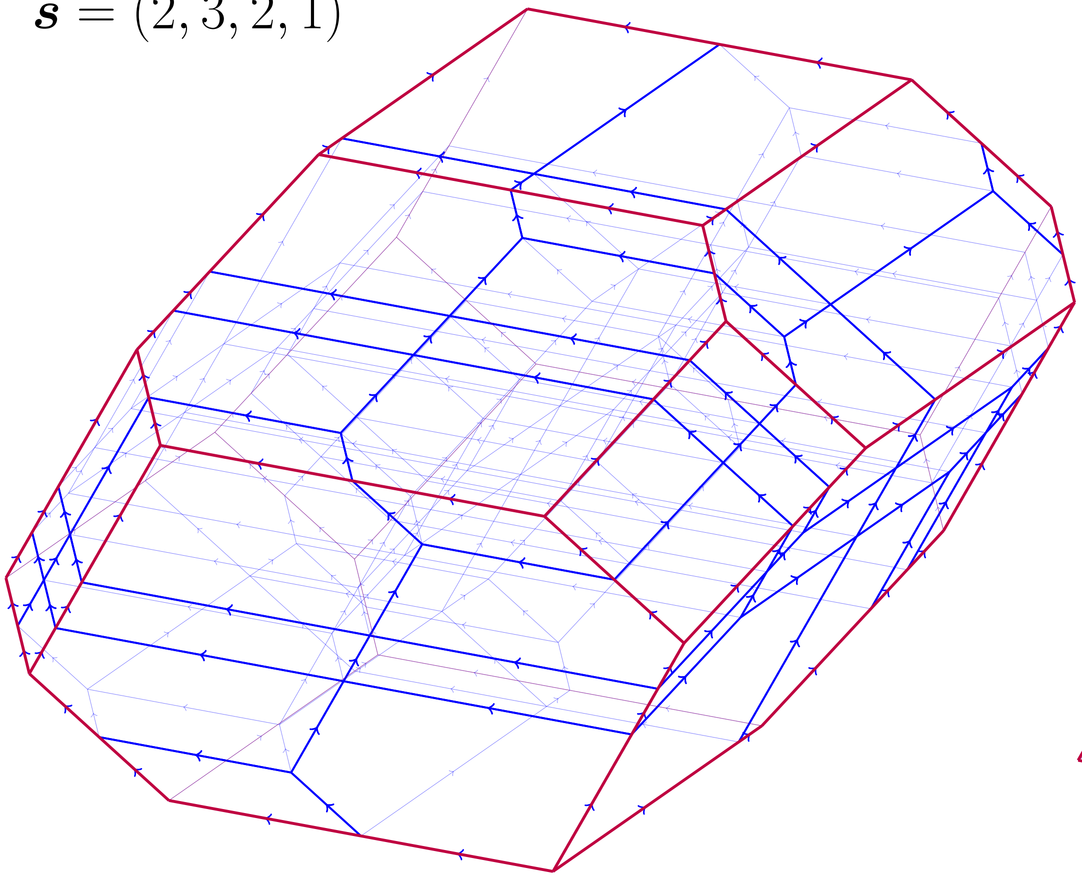
QUOTIENTOPLEXES

THM. • Hasse diagram of the quotient $\simeq$ oriented skeleton of the quotientoplex $\mathbb{Q}_{\equiv}$

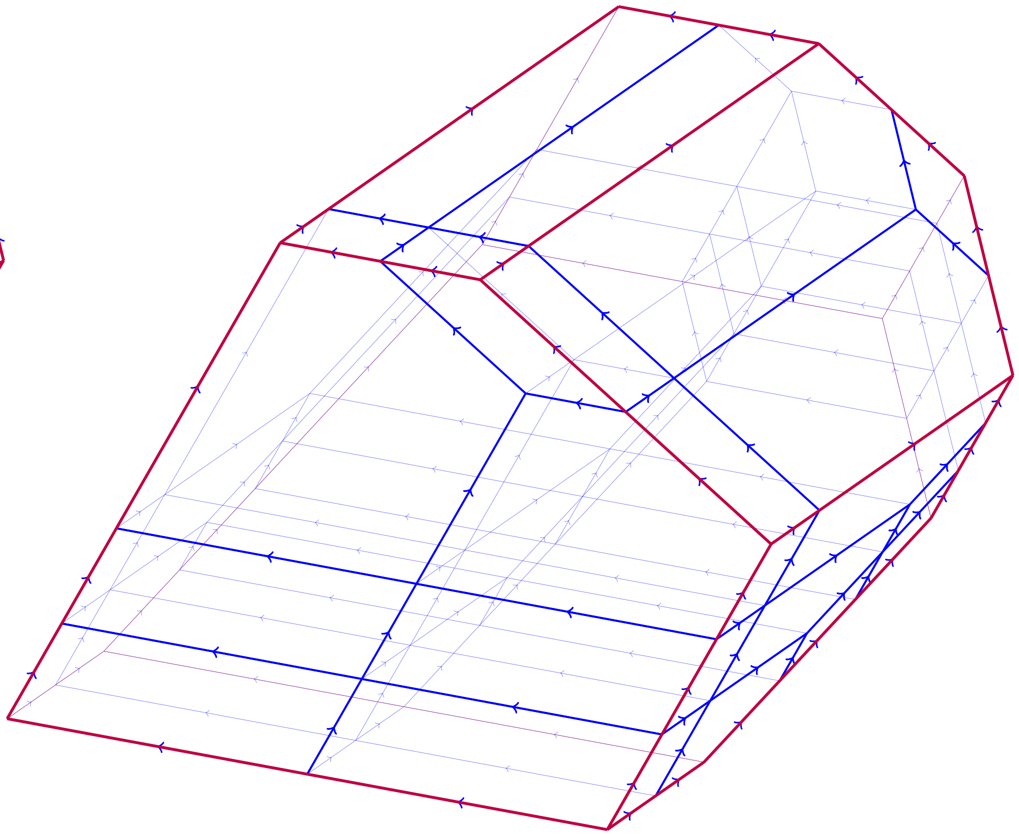
• quotientoplex $\mathbb{Q}_{\equiv}$ is a polytopal subdivision of the quotientope $\mathbb{Q}_{\tilde{\equiv}}$

Philippe–P., *Geometric realizations of the s-weak order and its quotients* ('24^{+})

$\boldsymbol{s} = (2, 3, 2, 1)$



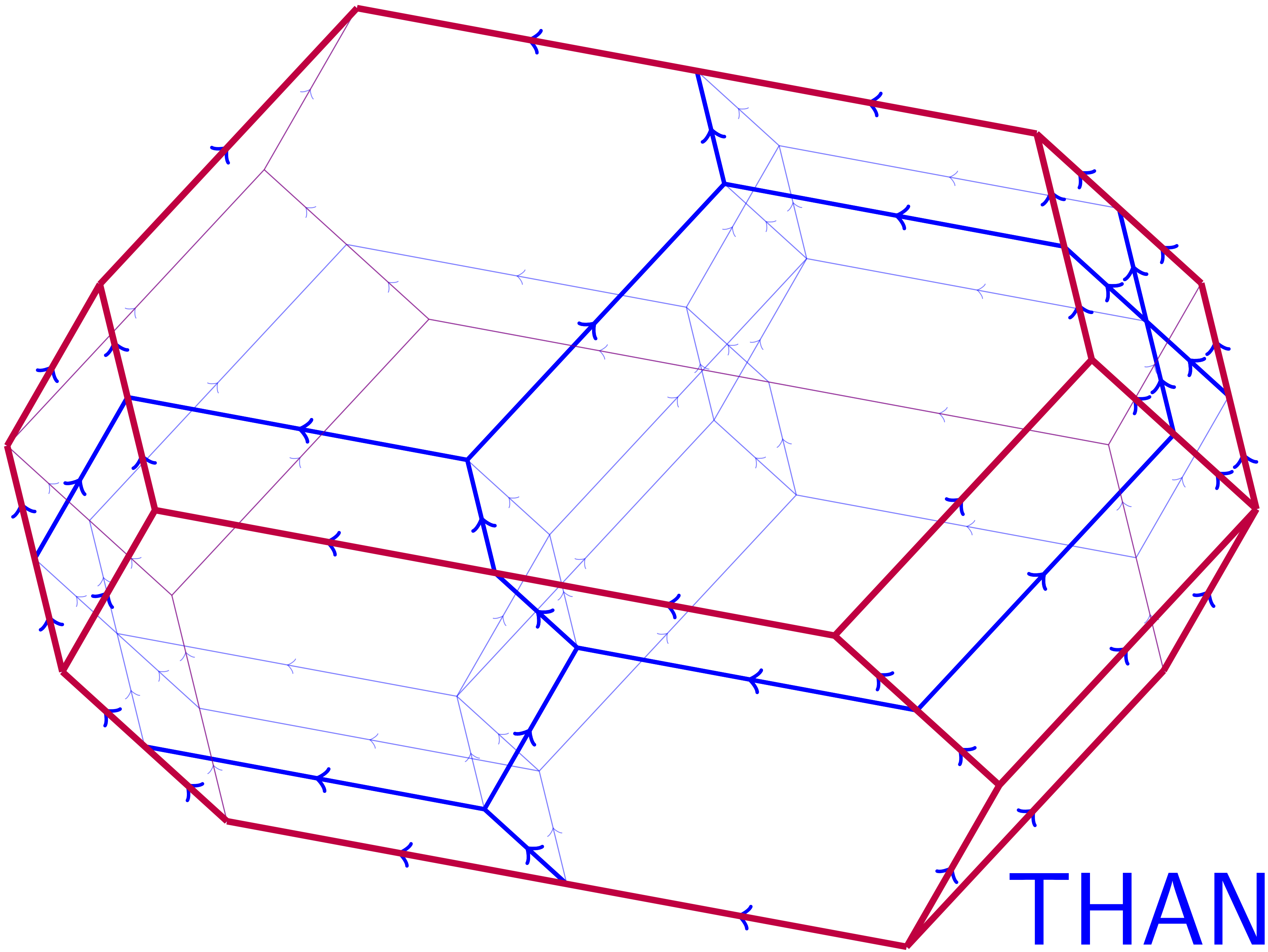
trivial congruence



sylvester congruence

$\boldsymbol{s} = (1, 2, 1, 1)$

POLYWOOD



THANKS