LATTICE QUOTIENTS AND QUOTIENTOPES IN GENERALIZATIONS OF THE WEAK ORDER

V. PILAUD
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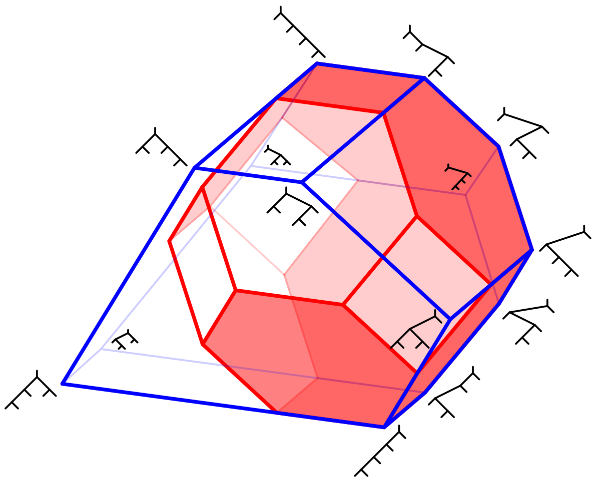
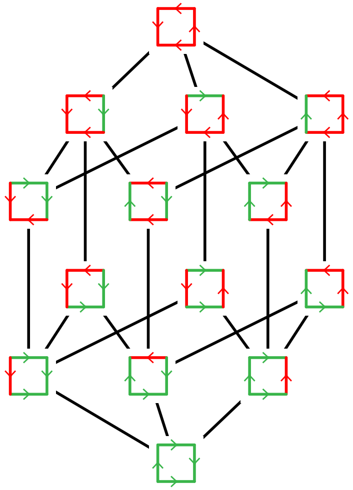
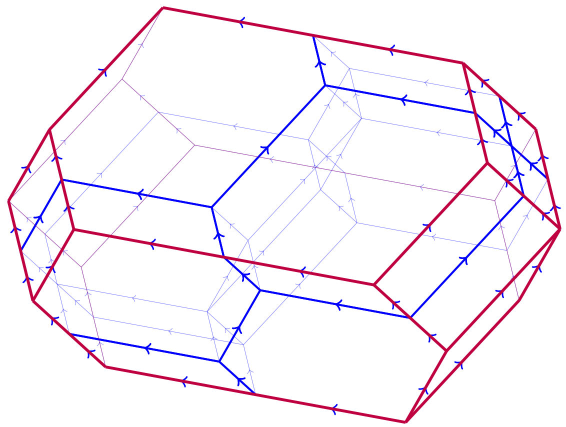
(Sorbonne Univ.)



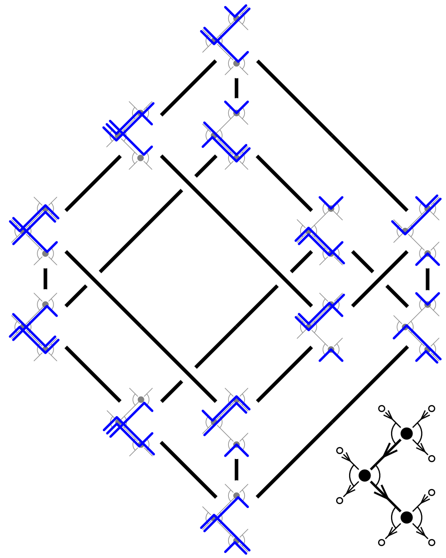
1. Quotientopes



2. Acyc. reorient. latt.



3. s-weak order



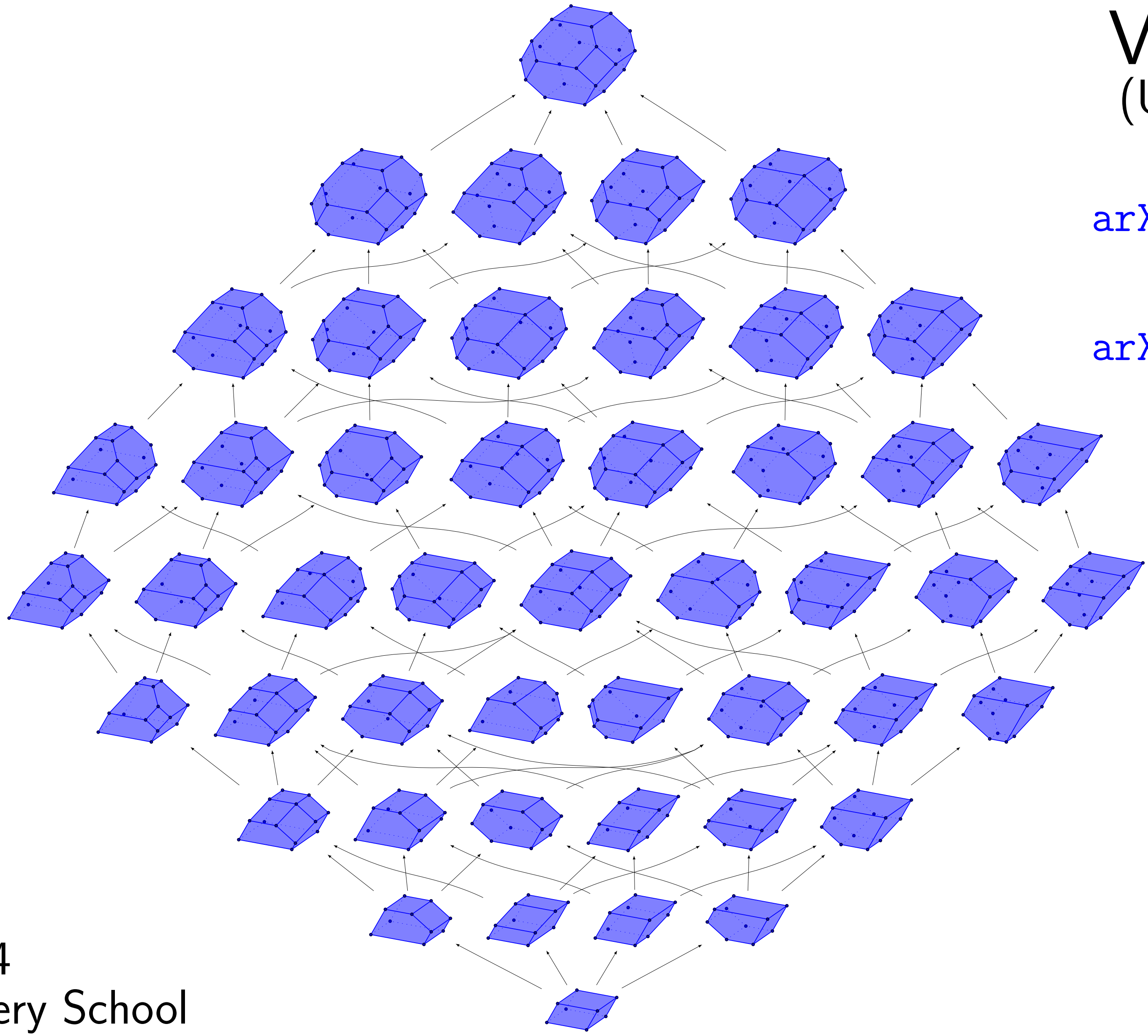
4. Gentle associahedra

May 6–8, 2024, ISM Discovery School, Montreal

"Lattice congruences on the weak order know a lot of combinatorics and geometry related to Coxeter groups."

Reading, *Finite Coxeter groups and the weak order* ('16)

1. QUOTIENTOPES



V. PILAUD
(Univ. Barcelona)

arXiv:1711.05353
F. SANTOS

arXiv:2007.01008
A. PADROL
J. RITTER

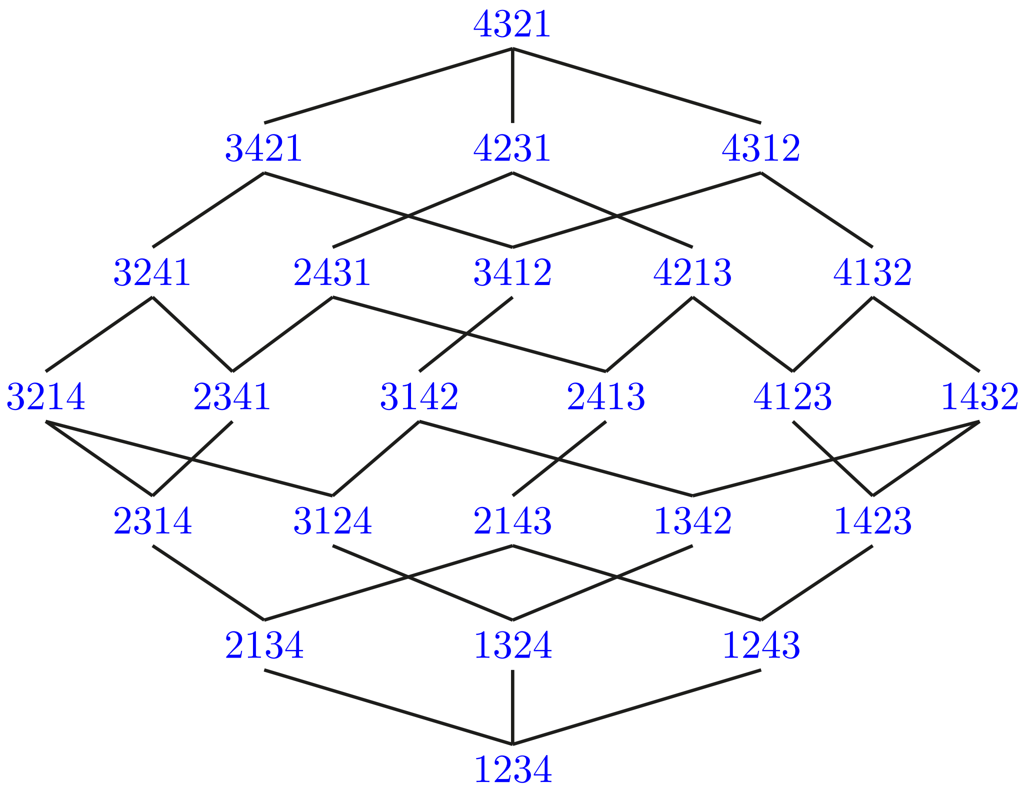
May 6, 2024
ISM Discovery School

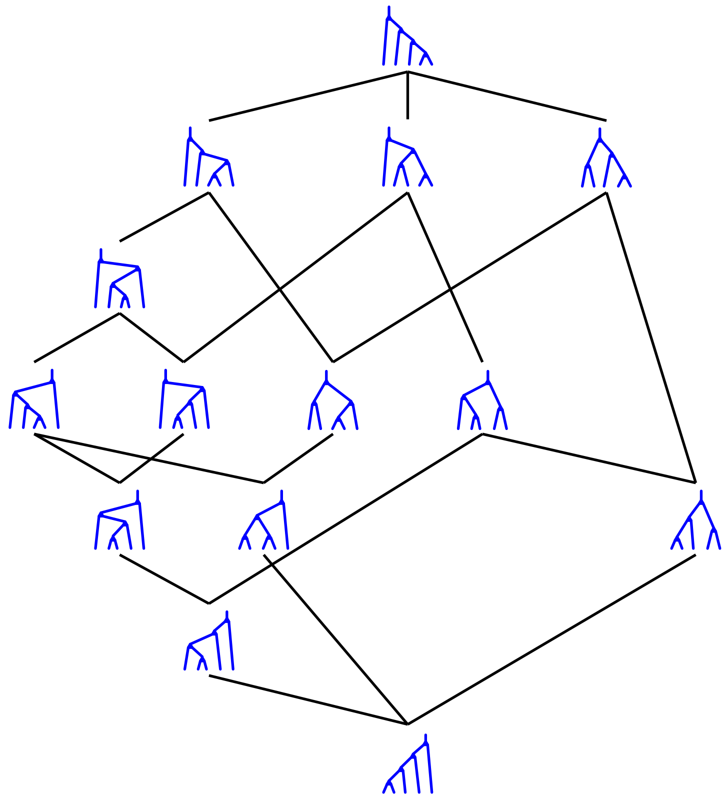
PERMUTAHEDRA & ASSOCIAHEDRA

P.–Santos–Ziegler, *Celebrating Loday's associahedron* ('23)

LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice = partially ordered set L where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$





weak order = permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

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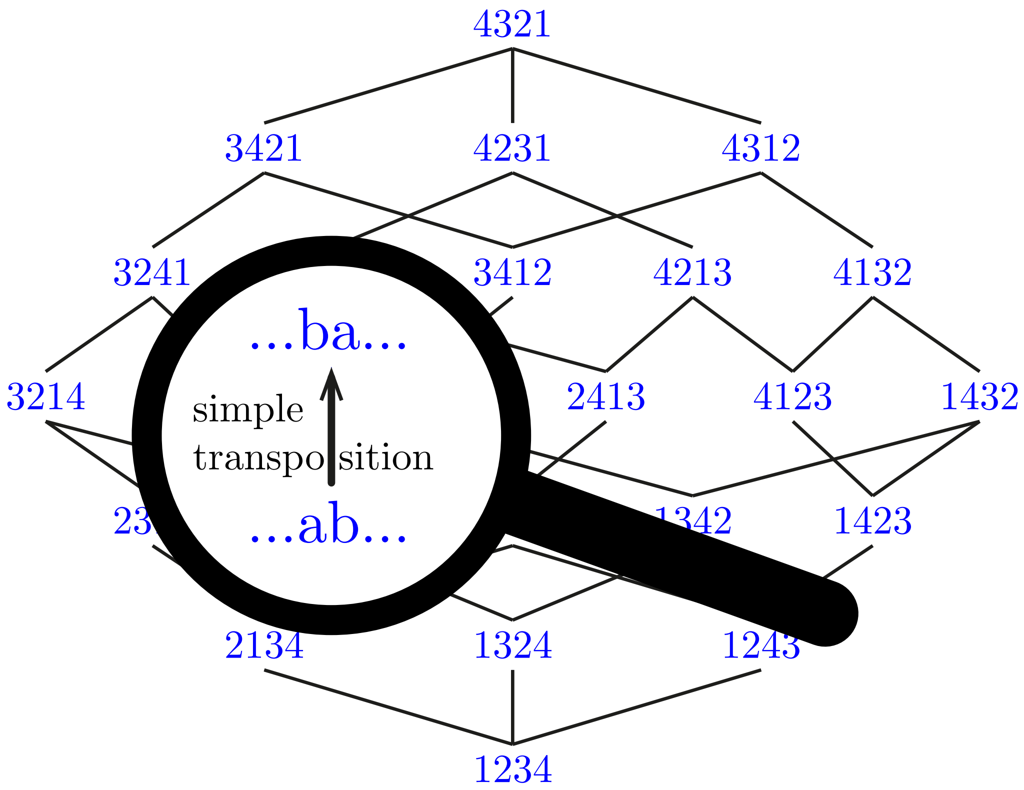

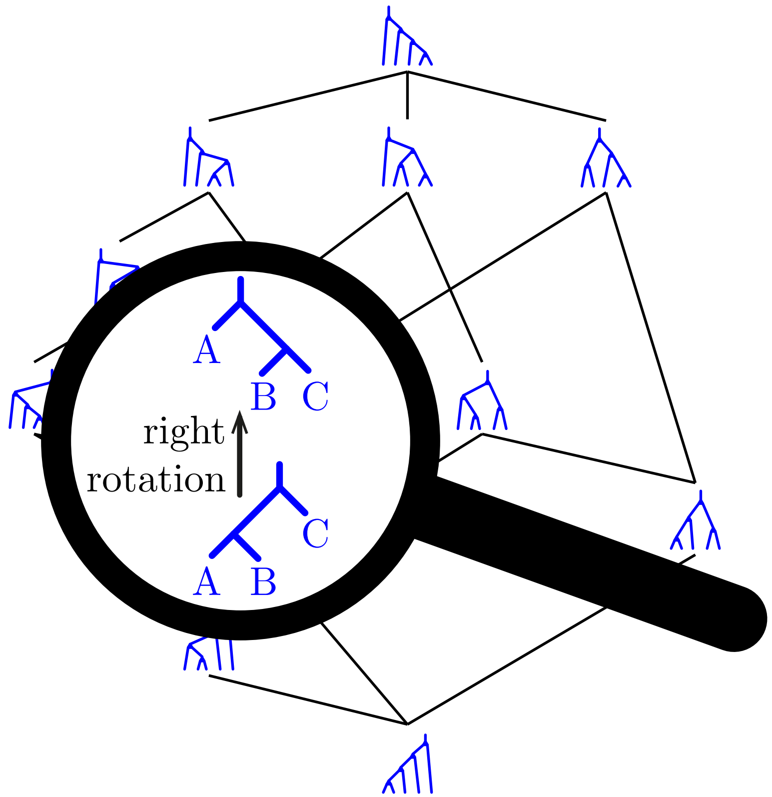

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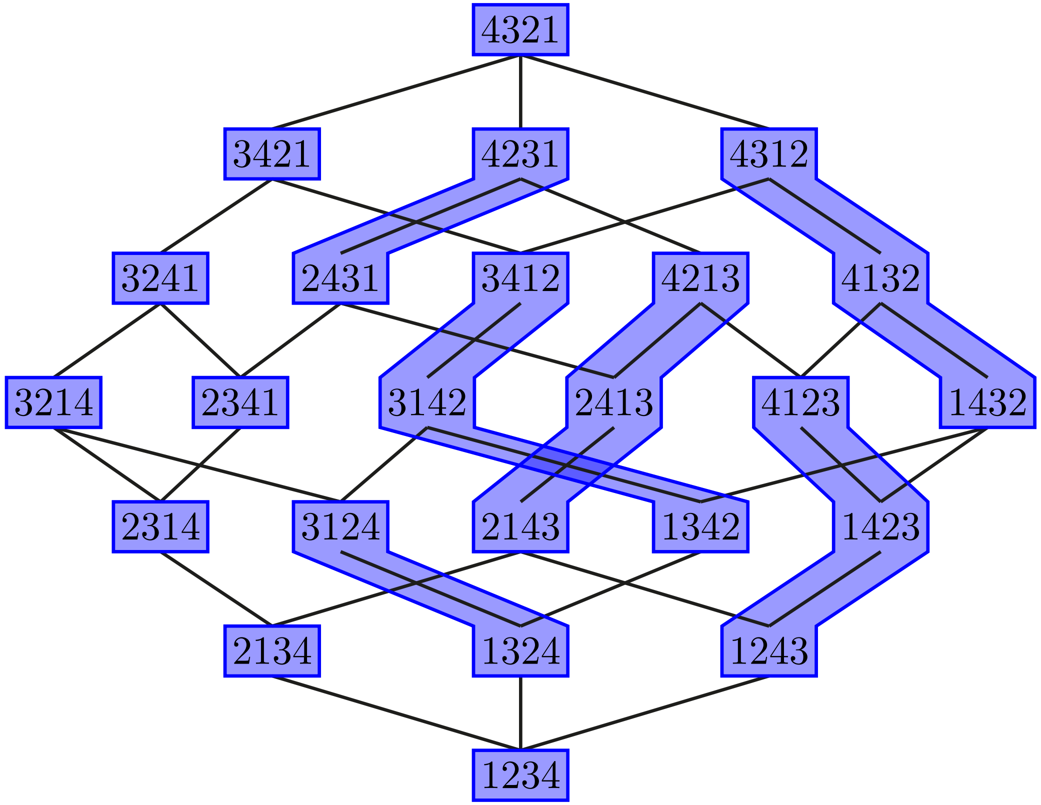

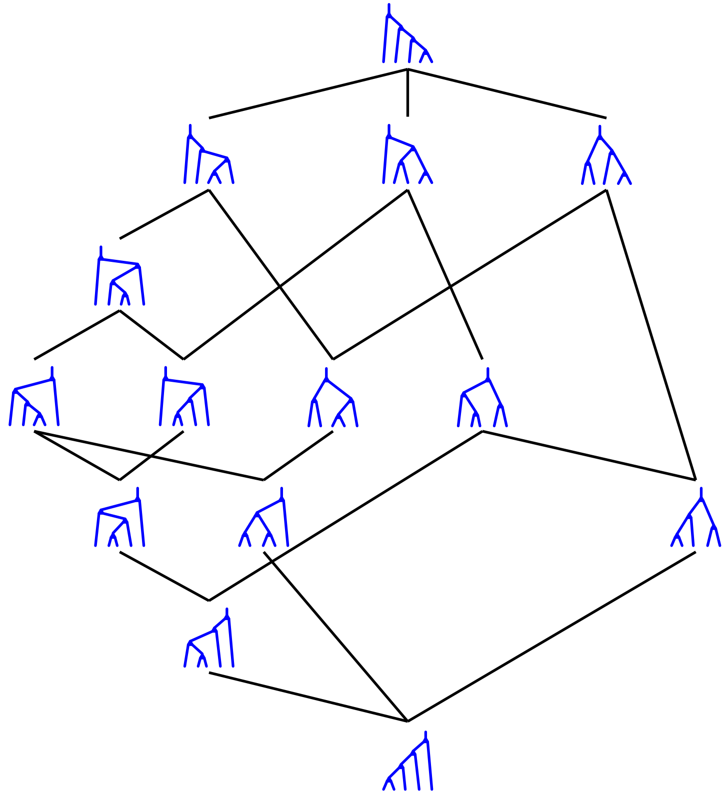

weak order $=$ permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice $=$ binary trees on $[n]$
ordered by paths of right rotations

sylvester congruence $=$ equivalence classes are sets of linear extensions of binary trees
$=$ equivalence classes are fibers of BST insertion
$=$ rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

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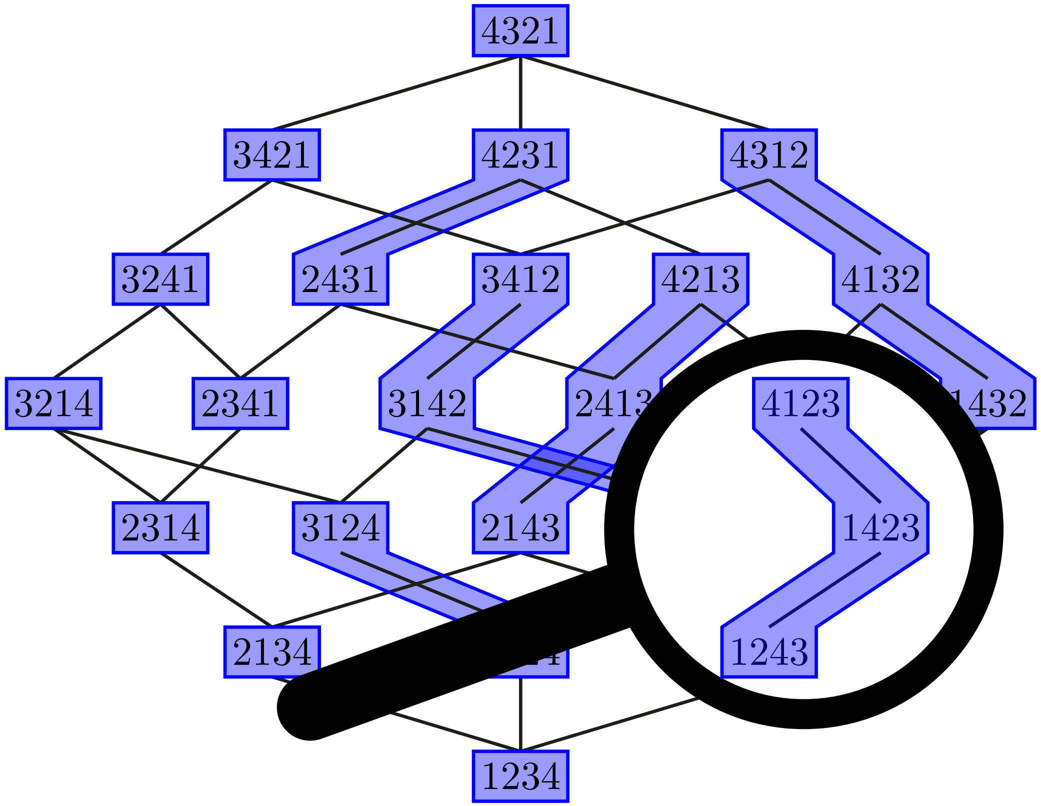

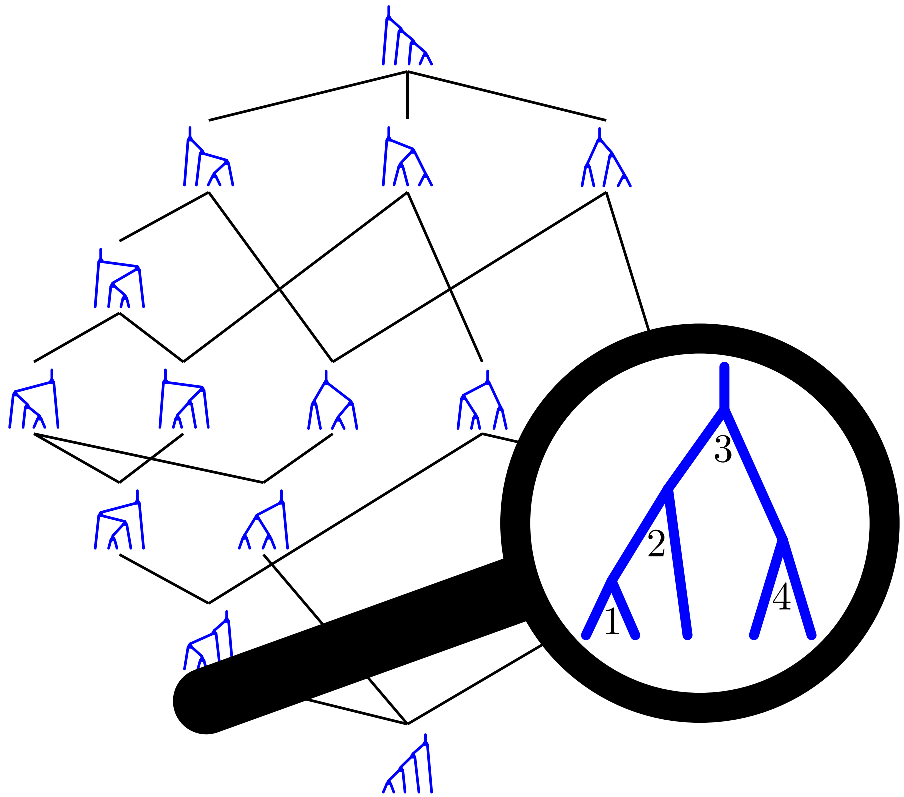

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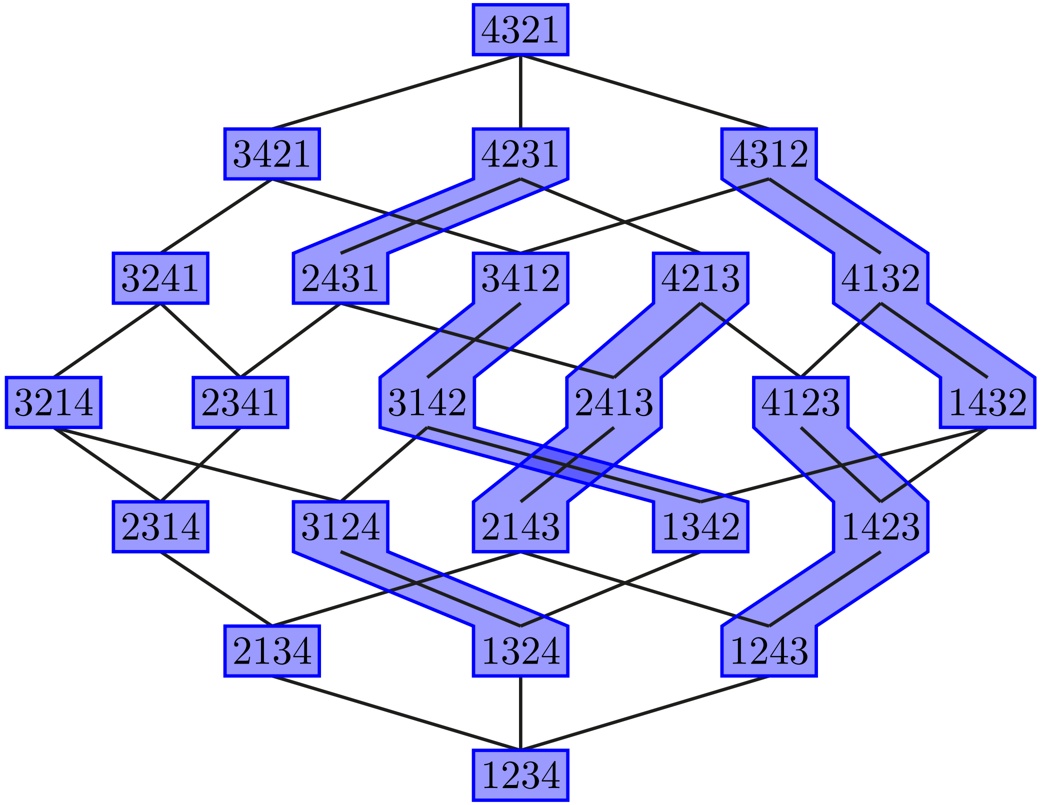

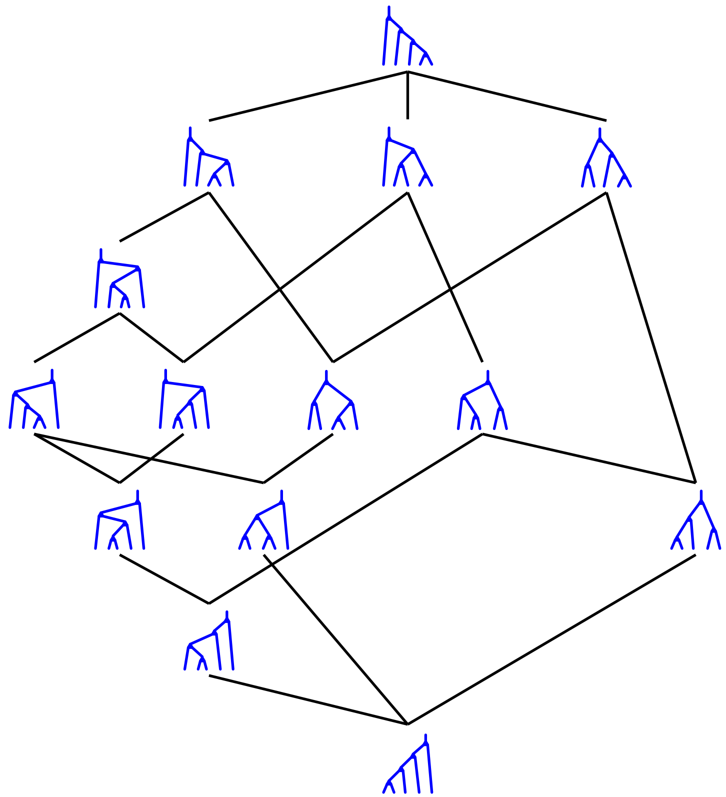

weak order = permutations of $[n]$
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Tamari lattice = binary trees on $[n]$
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lattice congruence = equivalence relation $\equiv$ which respects meets and joins

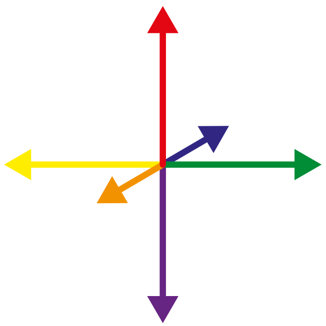
$x \equiv x'$ and $y \equiv y' \Longrightarrow x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$

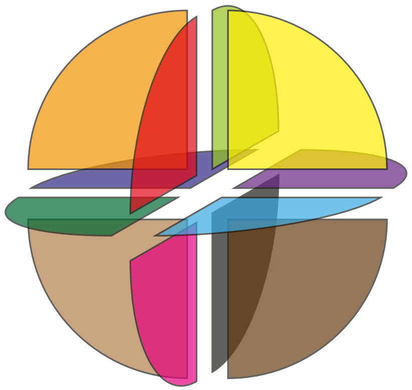
quotient lattice = lattice on classes with $X \leq Y \iff \exists\, x \in X,\ y \in Y, x \leq y$

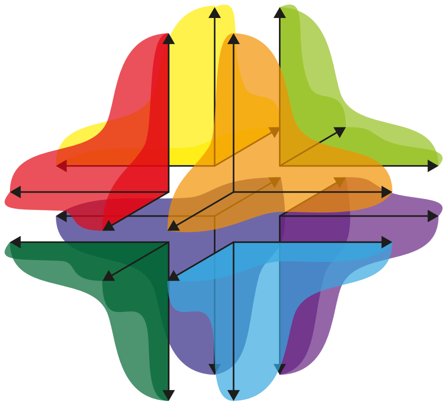
FANS: BRAID FAN AND SYLVESTER FAN

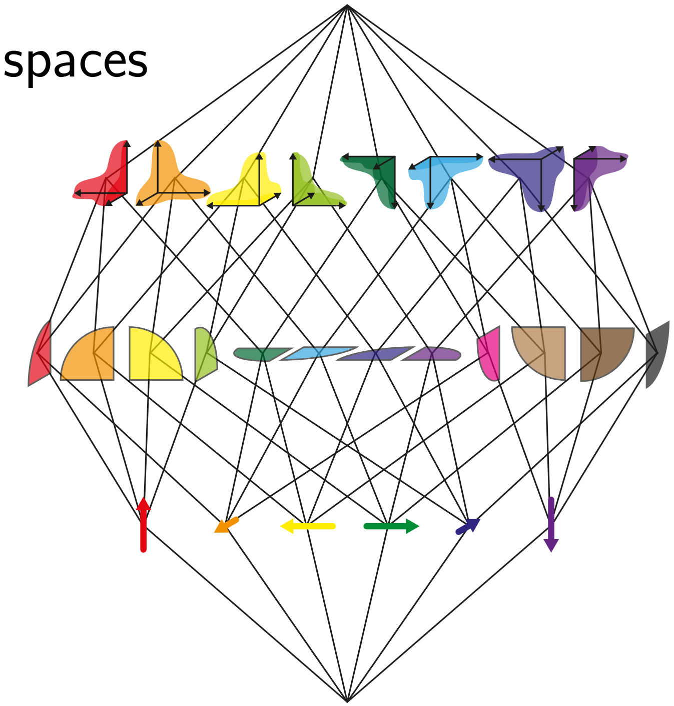
polyhedral cone = positive span of a finite set of vectors
= intersection of a finite set of linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face



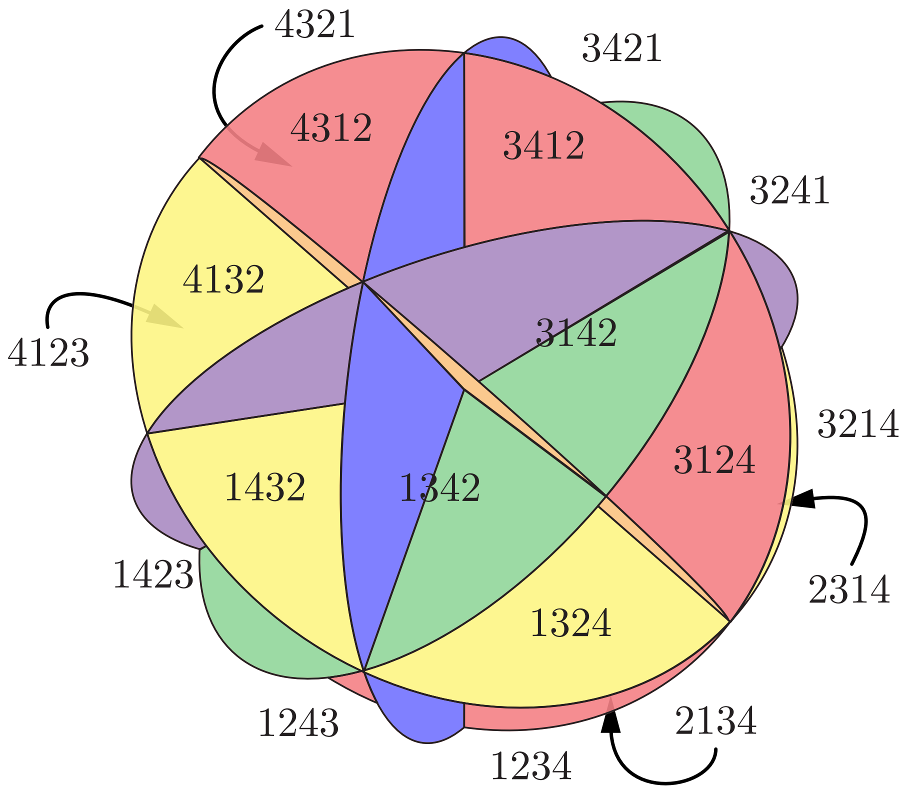




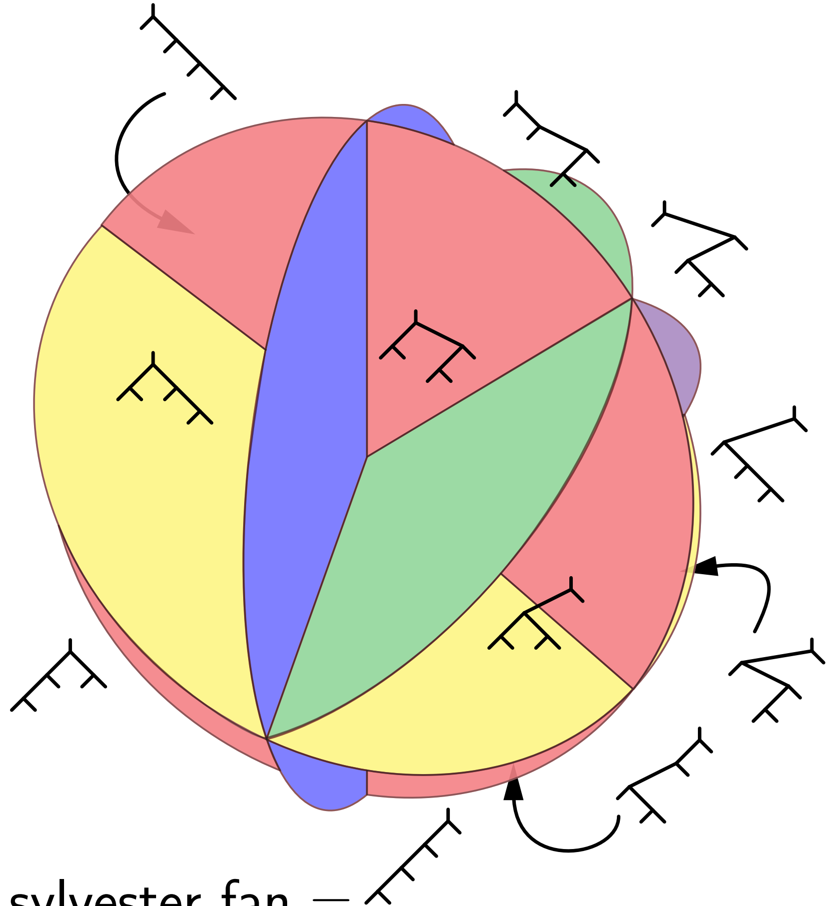


FANS: BRAID FAN AND SYLVESTER FAN

fan = collection of polyhedral cones closed by faces and intersecting along faces



braid fan =
$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \right\}$$



sylvester fan =
$$\mathbb{C}(T) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \}$$

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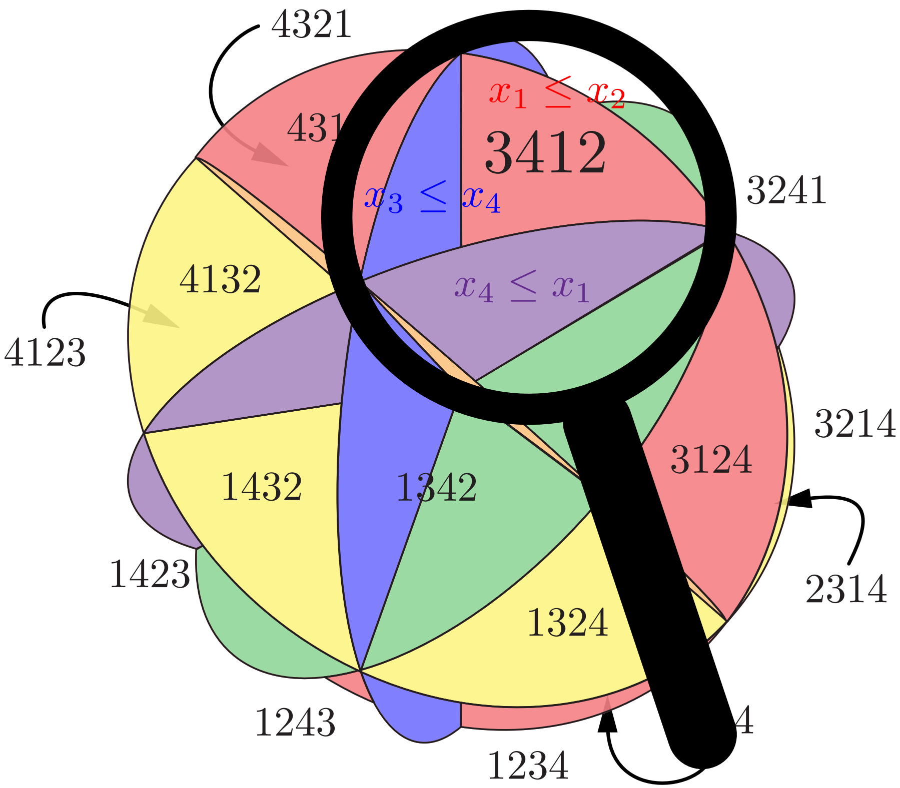

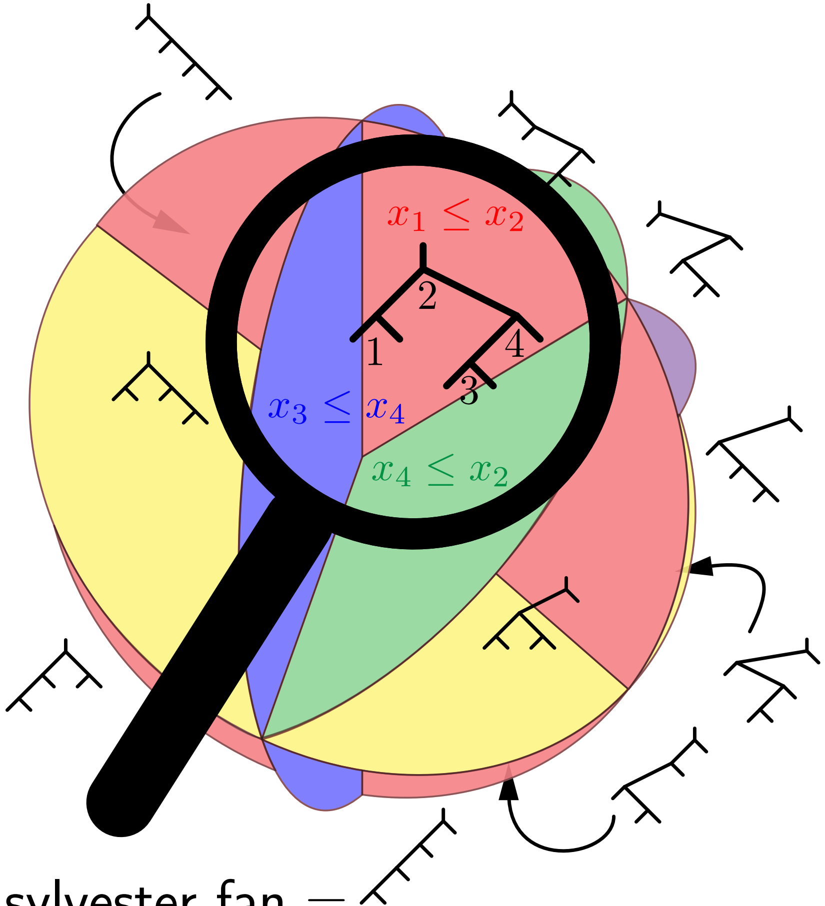

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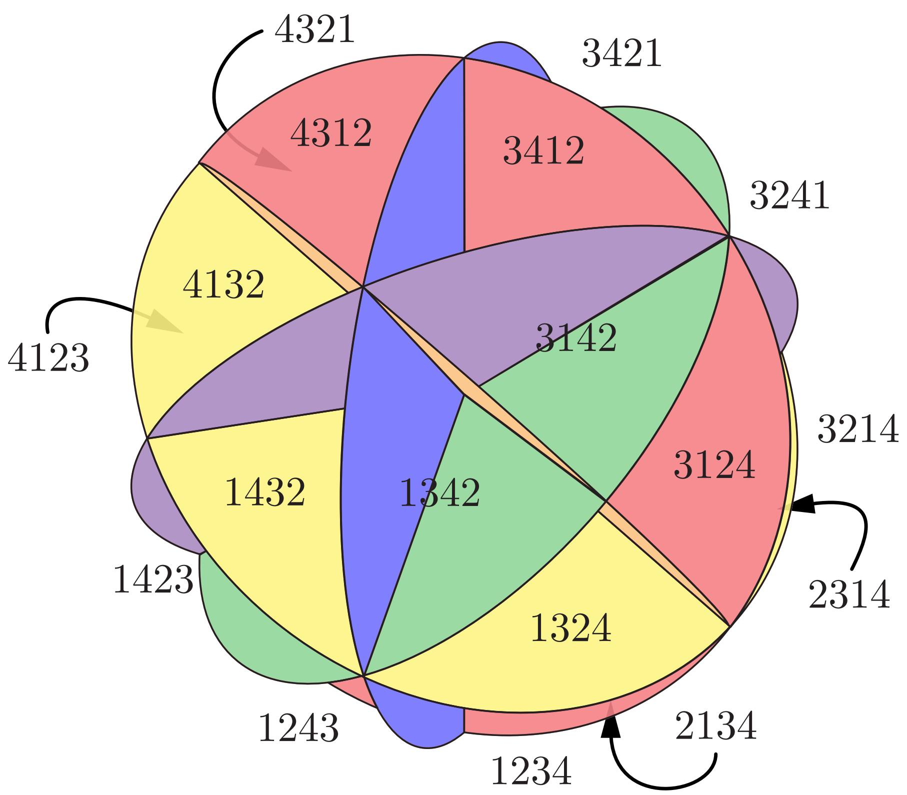

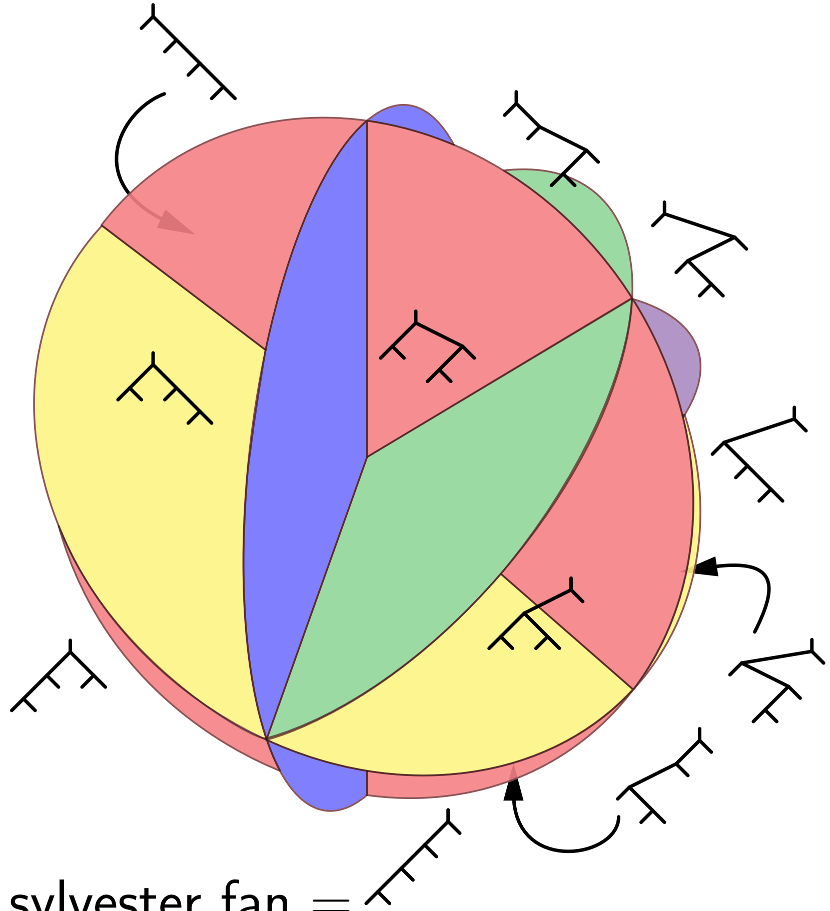

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$$\mathbb{C}(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T\}$$

quotient fan $=$ $\mathbb{C}(T)$ is obtained by glueing $\mathbb{C}(\sigma)$ for all linear extensions σ of T

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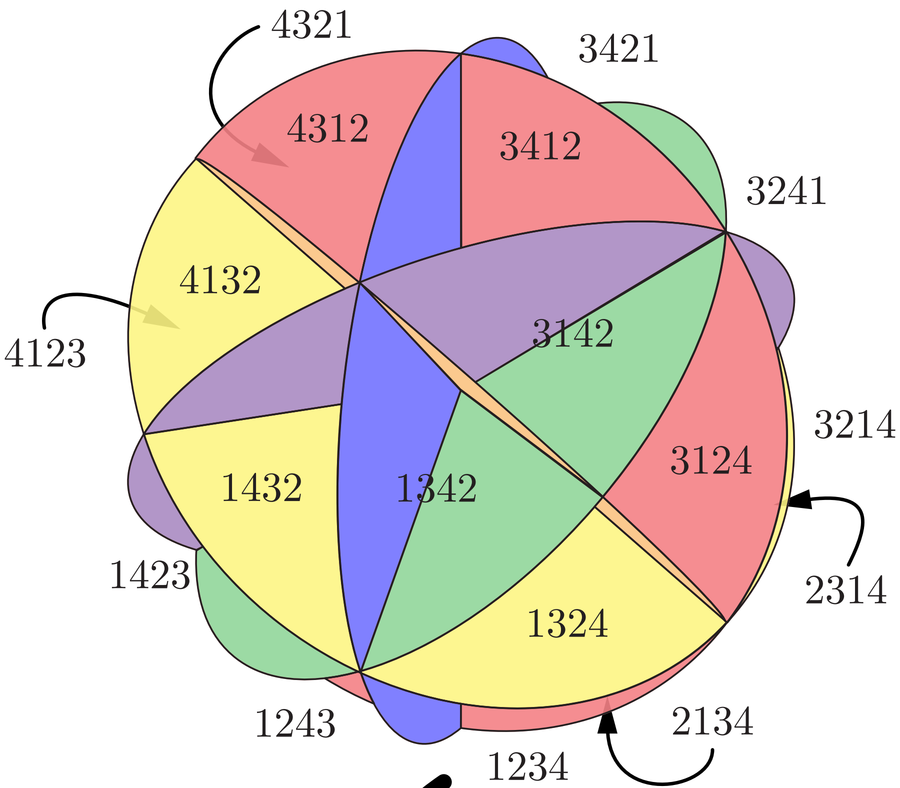
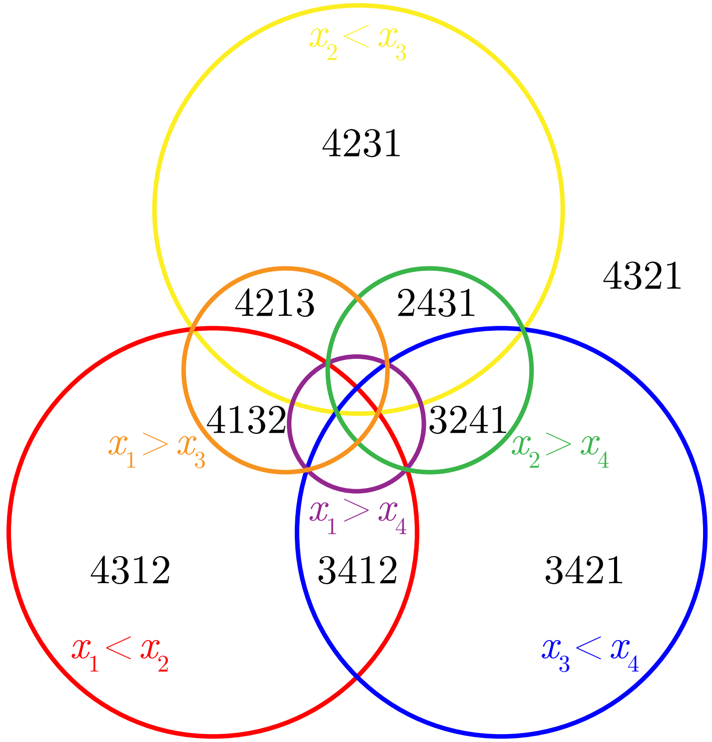
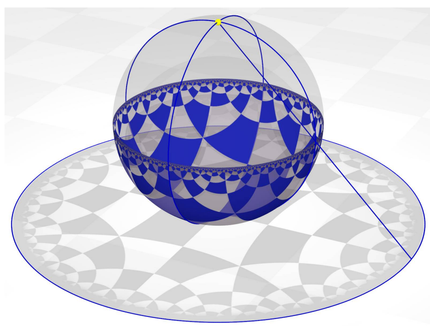




stereographic projection =



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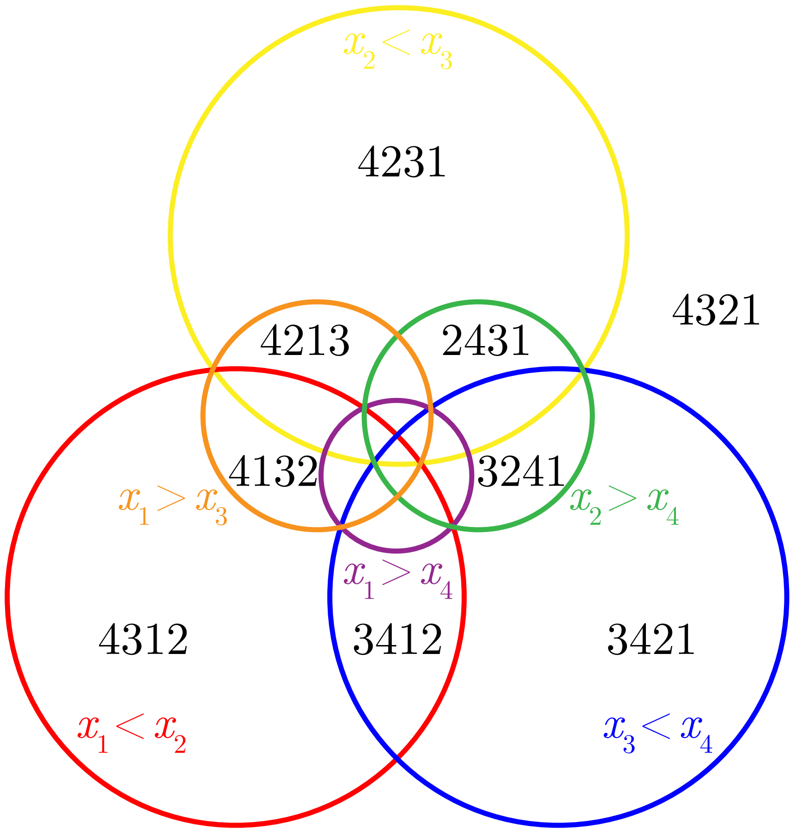

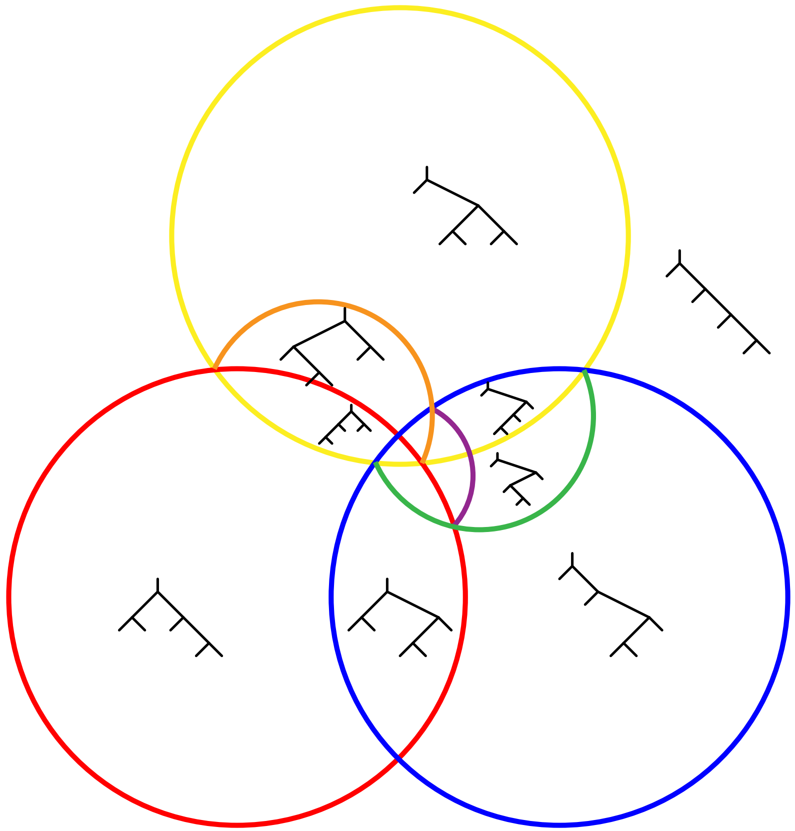

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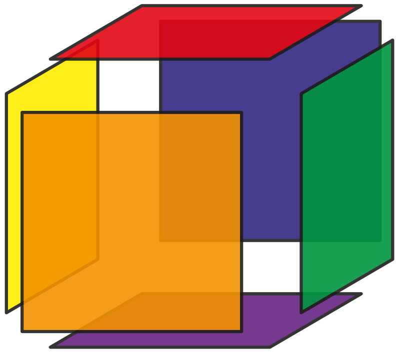
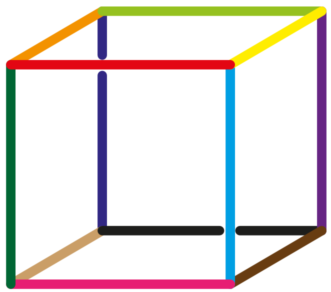
POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

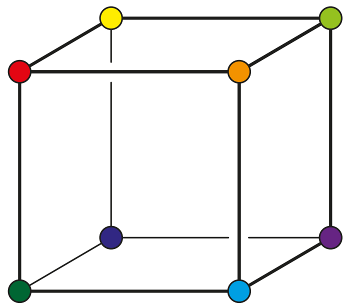
polytope = convex hull of a finite set of points
= bounded intersection of a finite set of affine half-spaces

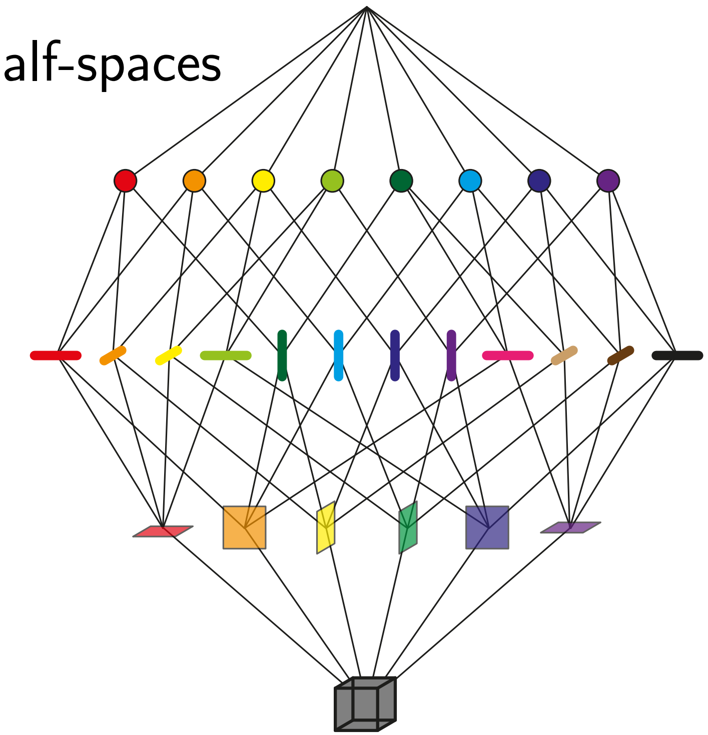
face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations









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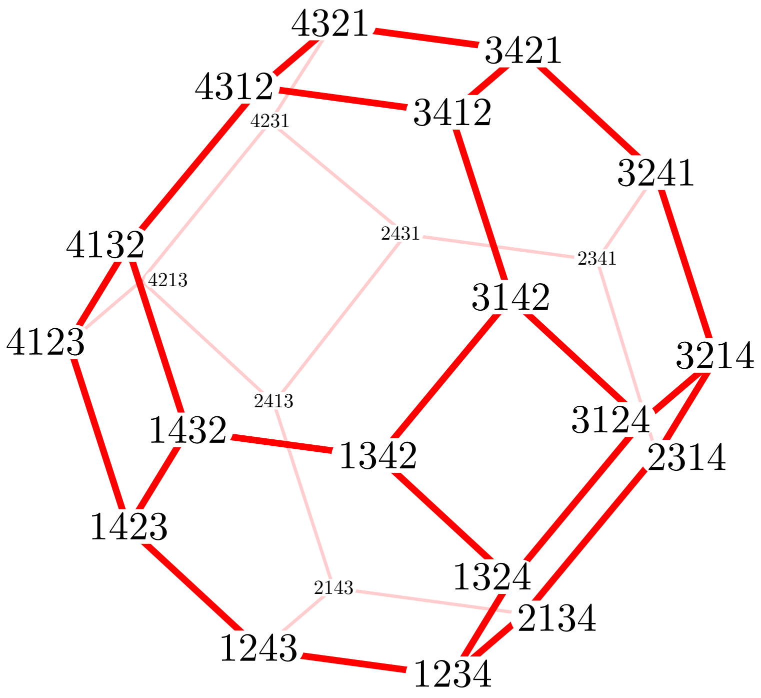

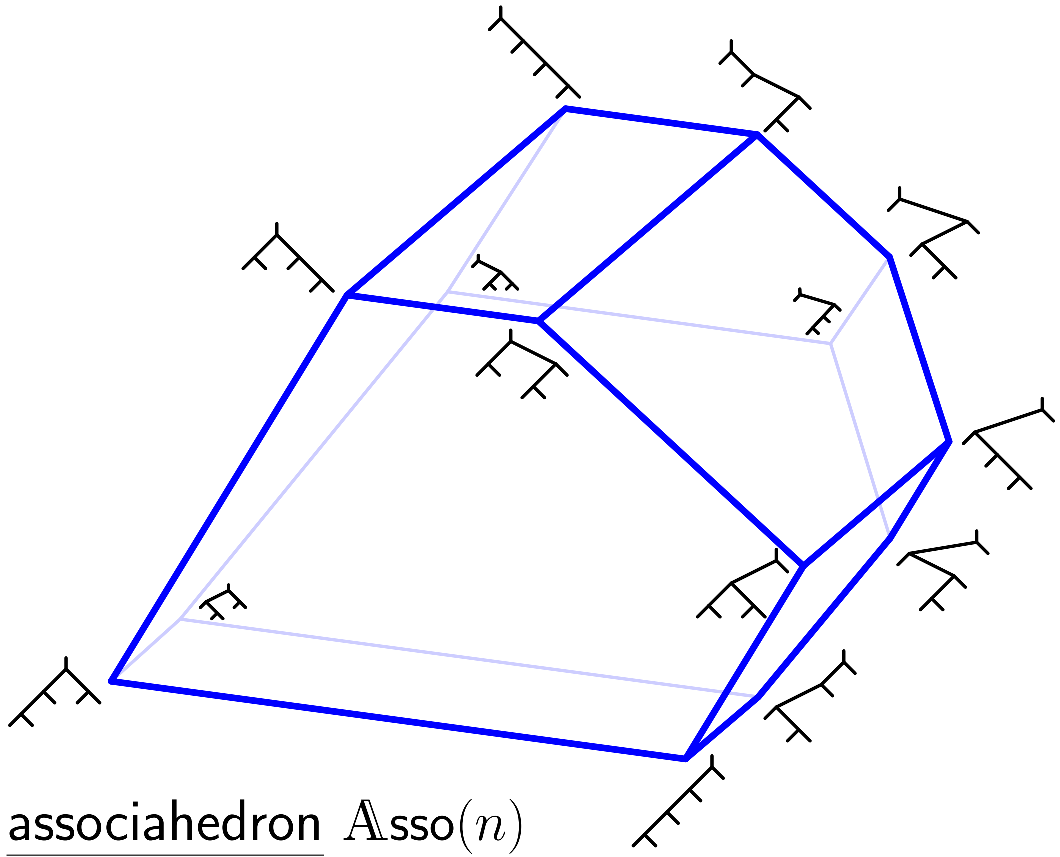

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$= \operatorname{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$

$= \mathbb{H} \cap \bigcap_{\varnothing \ne J \subsetneq [n]} \mathbb{H}_J$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \ge \binom{|J|+1}{2} \right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$= \operatorname{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree} \right\}$

$= \mathbb{H} \cap \bigcap_{1\le i<j\le n} \mathbb{H}_{[i,j]}$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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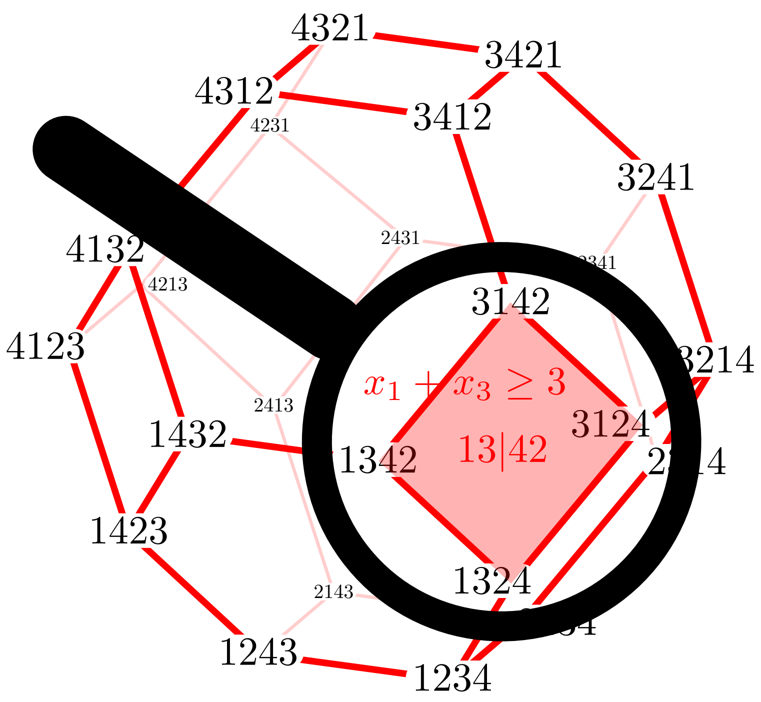

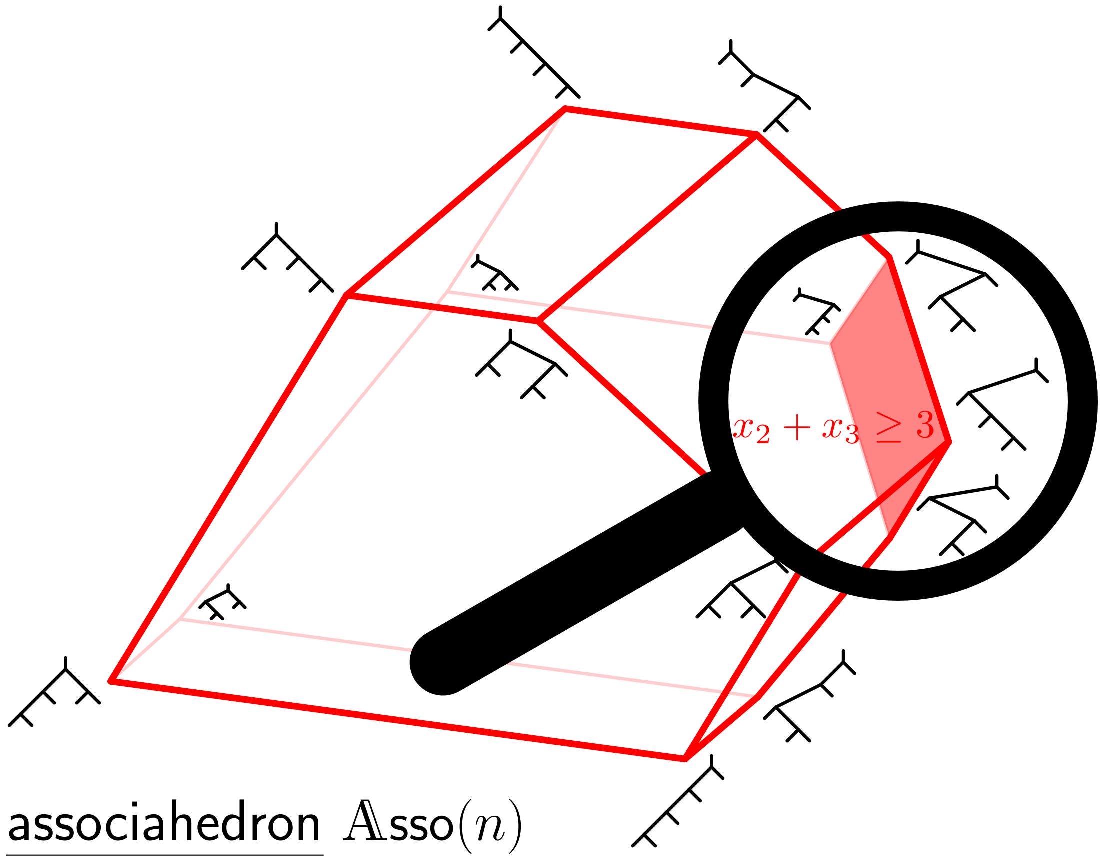

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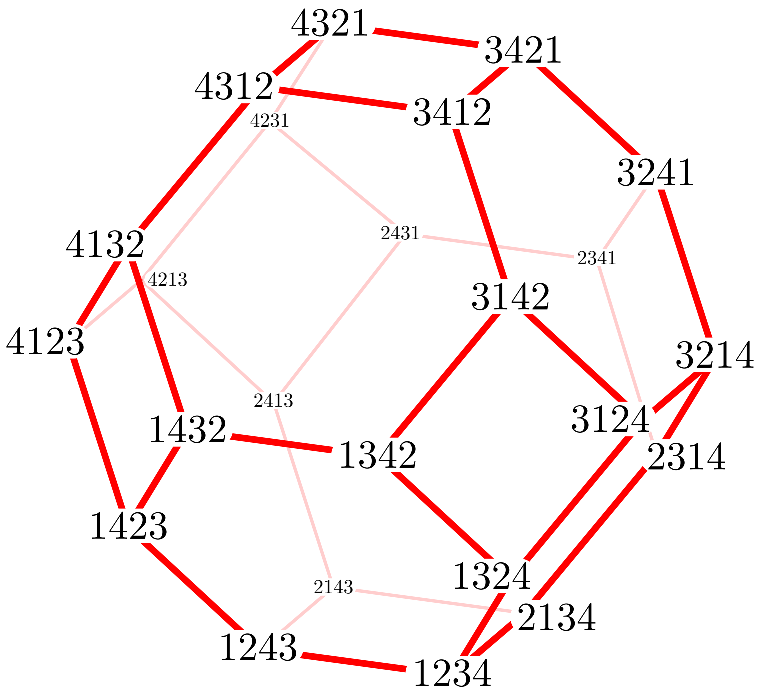

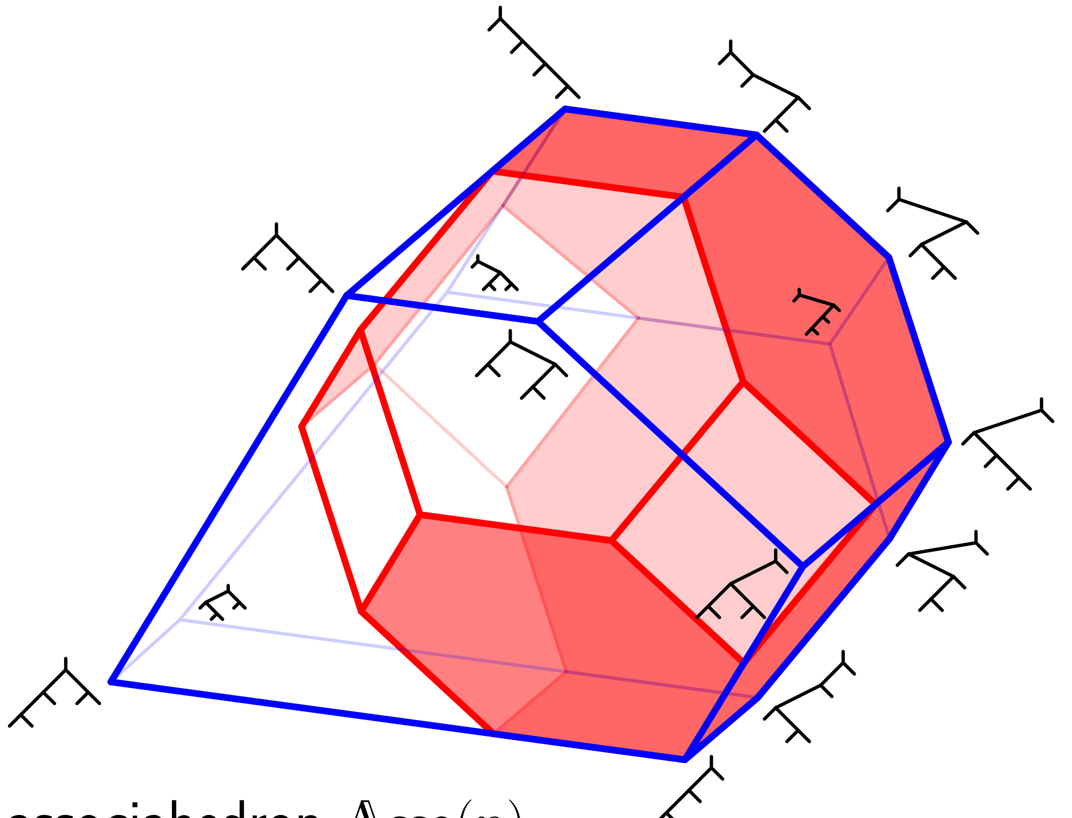

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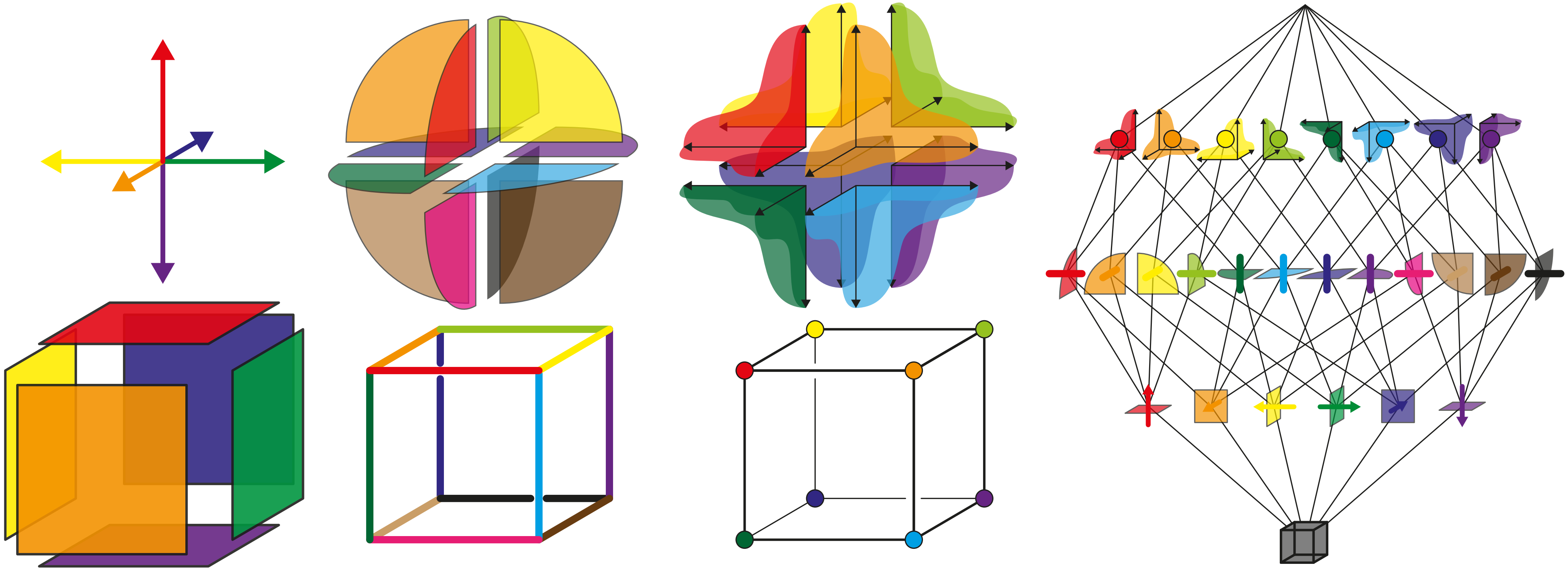
POLYWOOD

LATTICES – FANS – POLYTOPES



face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = $\{$ normal cone of $\mathbb{F}$ $\mid$ $\mathbb{F}$ face of $\mathbb{P}$ $\}$

LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{P}\mathrm{erm}(n)$

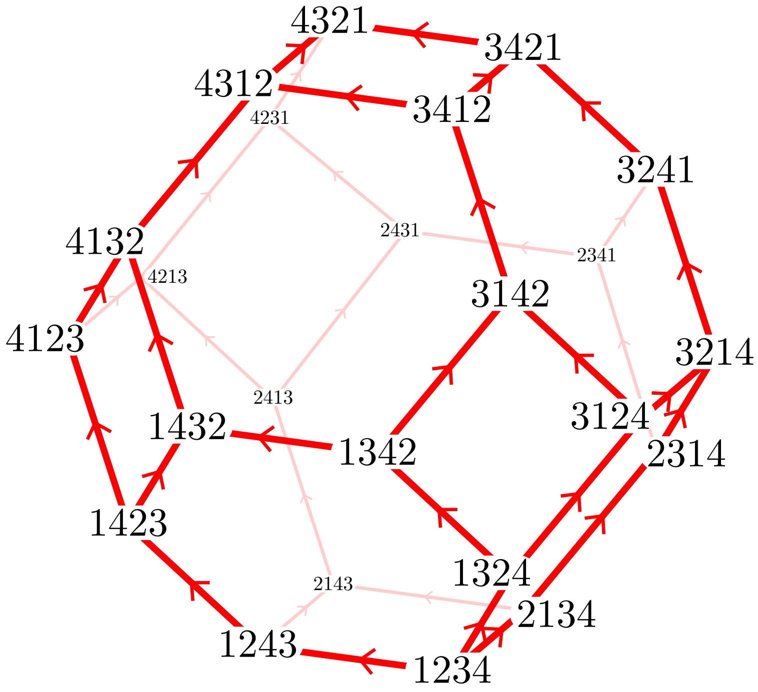
$\Longrightarrow$ braid fan

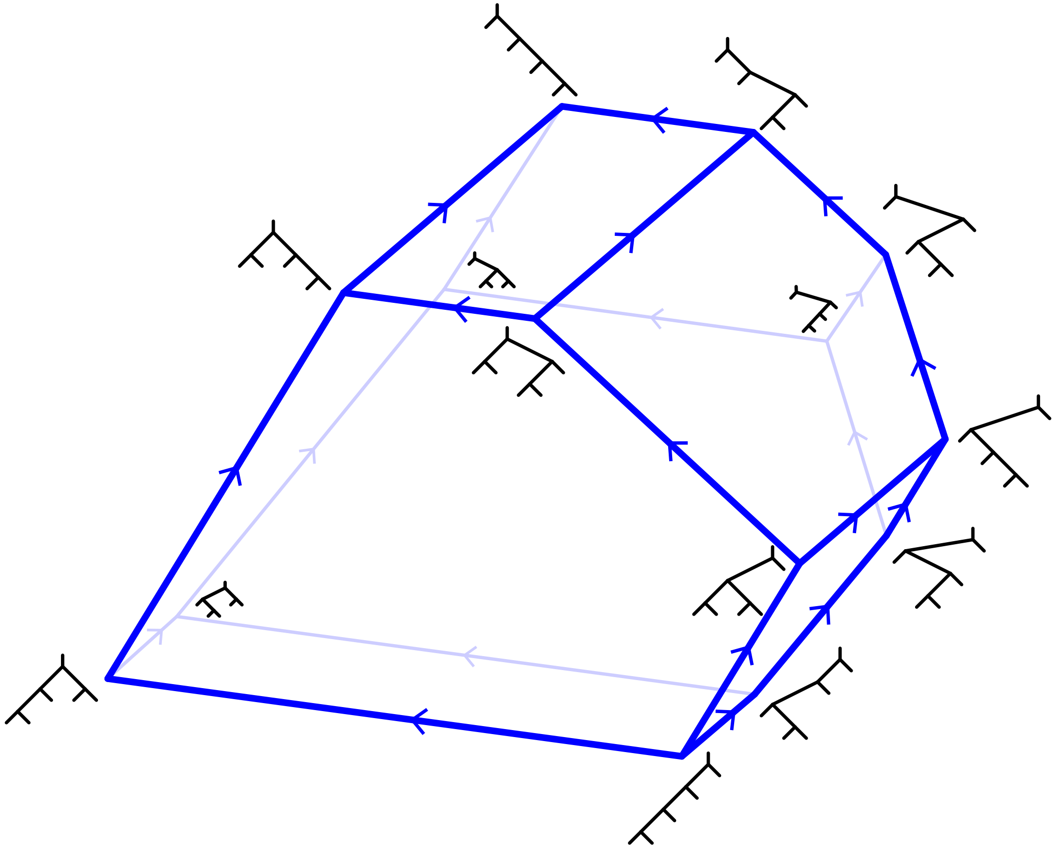
$\Longrightarrow$ weak order on permutations

associahedron $\mathbb{A}\mathrm{sso}(n)$

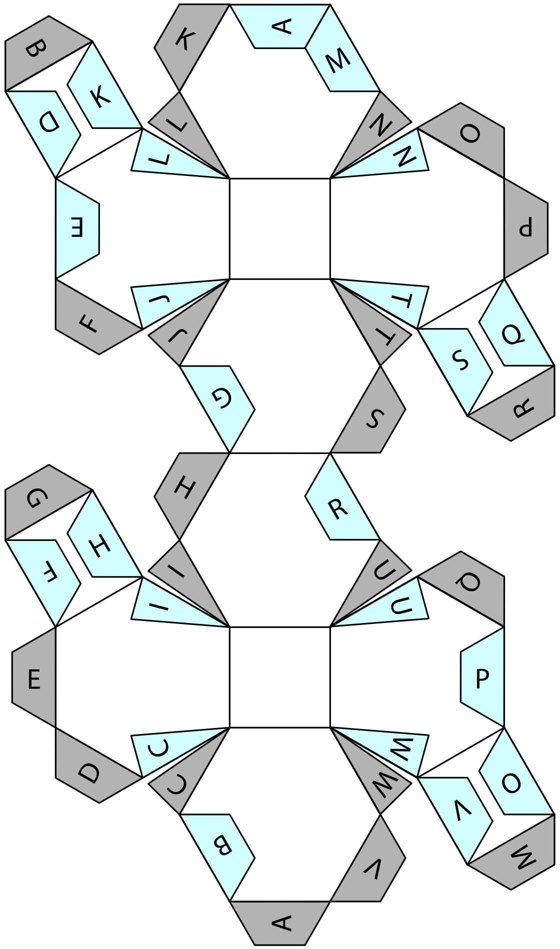
$\Longrightarrow$ Sylvester fan

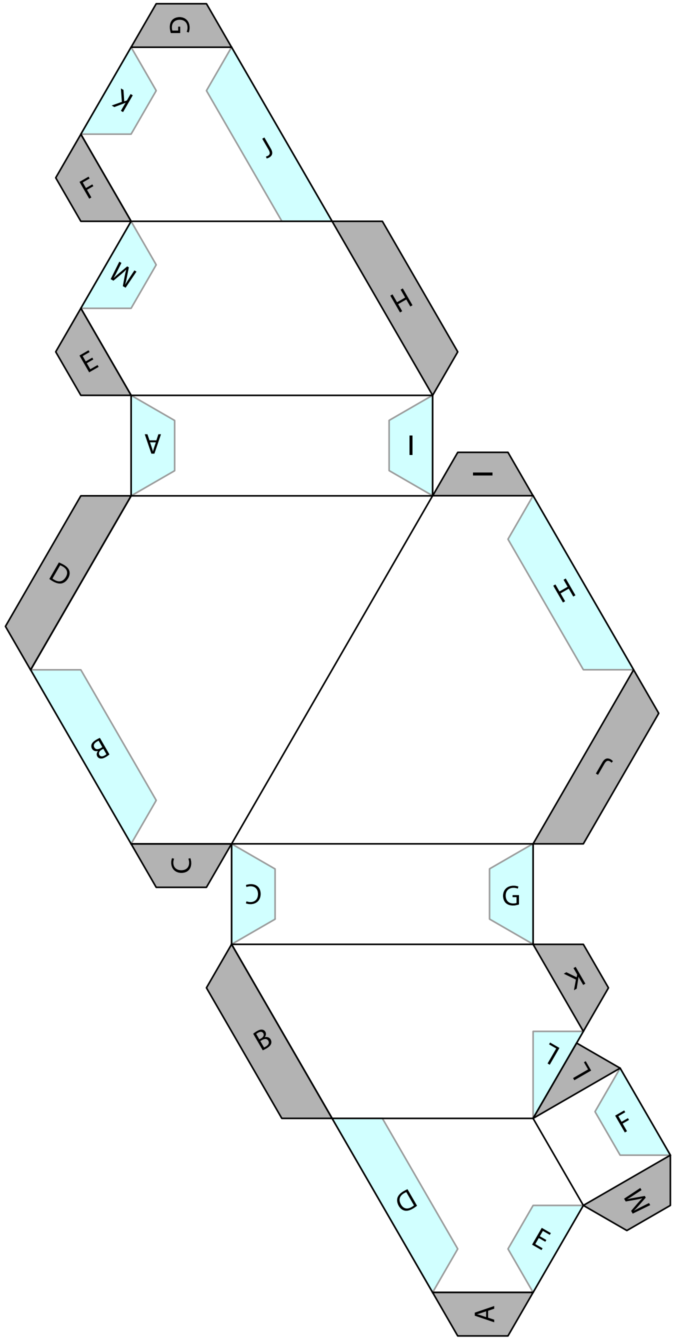
$\Longrightarrow$ Tamari lattice on binary trees





TAKE THEM HOME!





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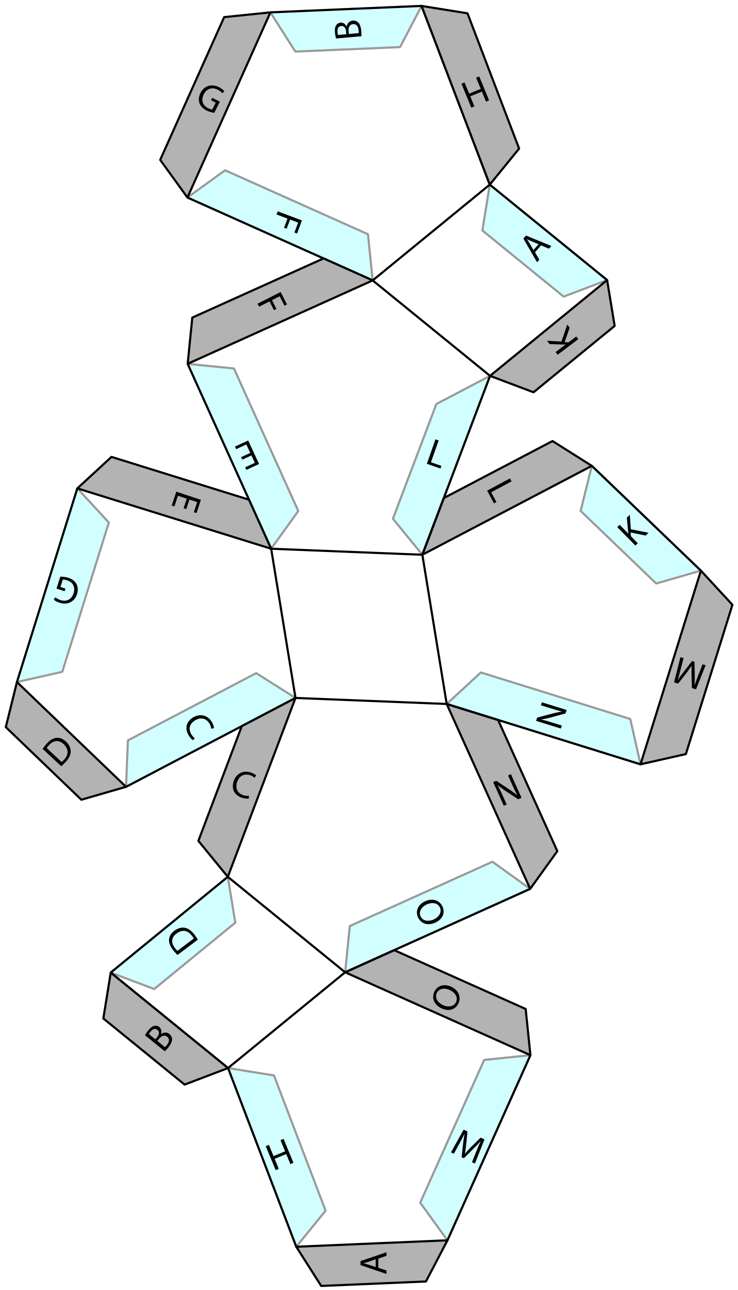
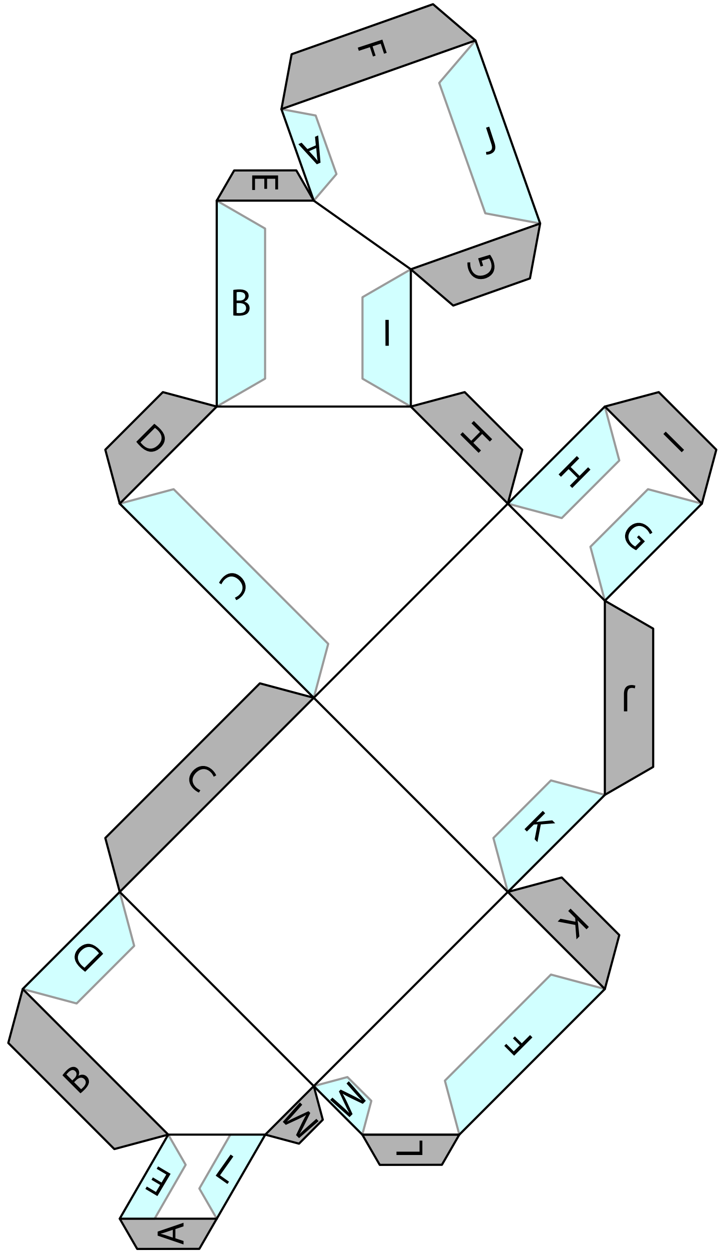




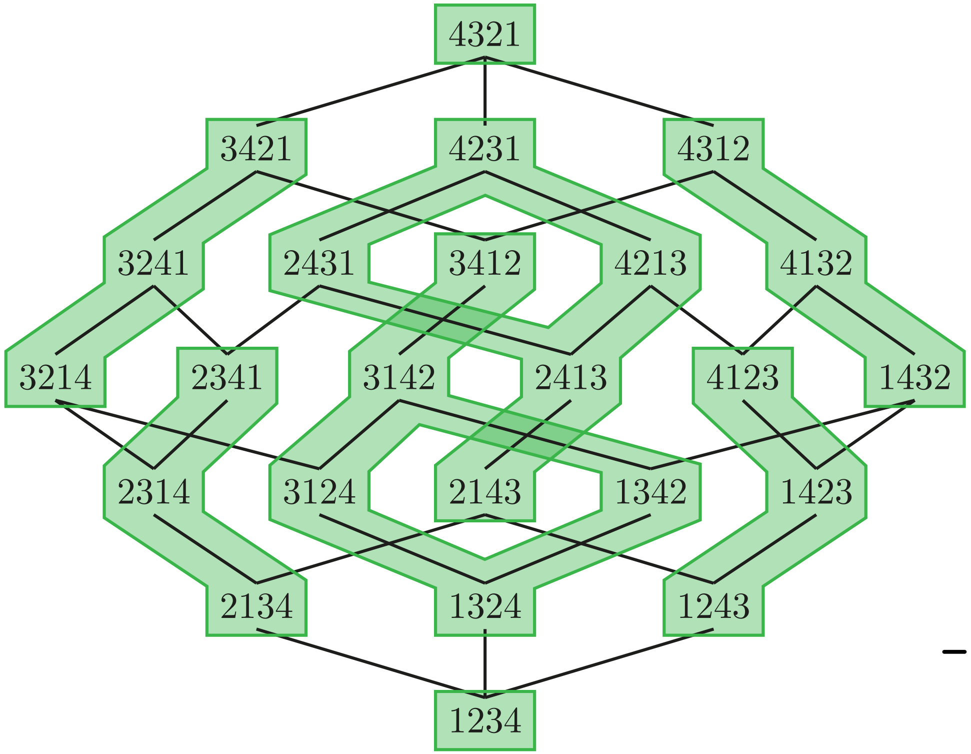
MY ZOO OF LATTICE QUOTIENTS

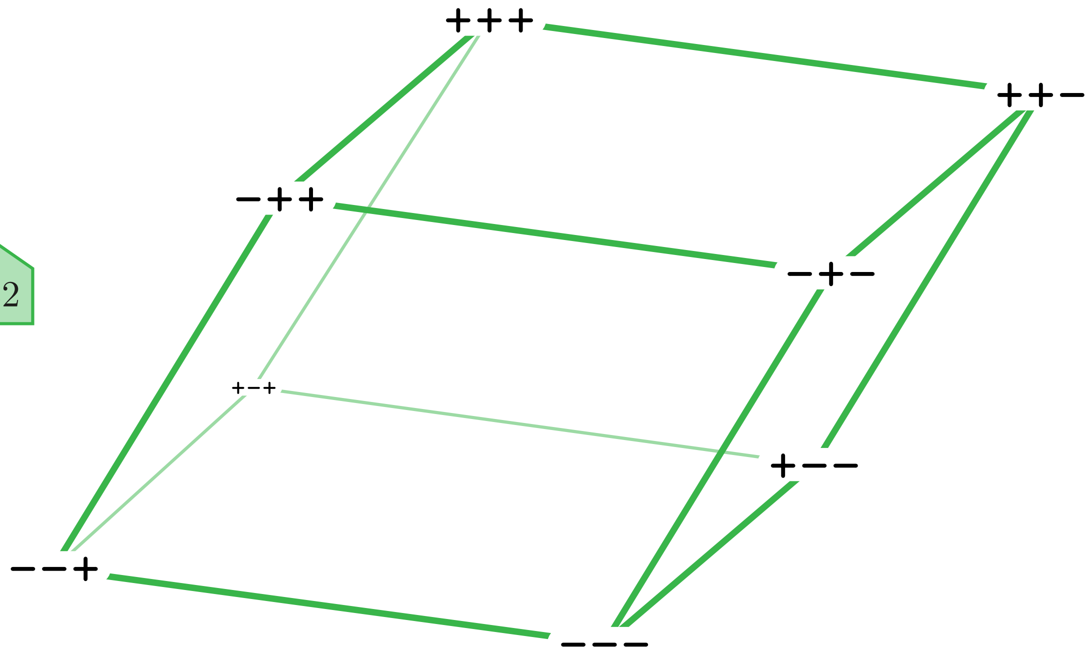
Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Finite Coxeter groups and the weak order* ('16)
Novelli–Reutenauer–Thibon, *Generalized descent patterns in permutations* ('11)
Hivert–Novelli–Thibon, *The algebra of binary search trees* ('05)
P., *Brick polytopes, lattice quotients, and Hopf algebras* ('18)
Chatel–P., *Cambrian algebras* ('17)
P.–Pons, *Permutrees* ('18)
Giraudo, *Algebraic and combinatorial structures on pairs of twin binary trees* ('12)
Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)
Reading, *Generic rectangulations* ('12)
Cardinal–P., *Rectangulotopes* ('24$^+$)

EXM 1: BOOLEAN LATTICE & CUBE

recoils of $\sigma \in \mathfrak{S}_n$ = positions $i \in [n-1]$ such that $\sigma^{-1}(i) > \sigma^{-1}(i+1)$

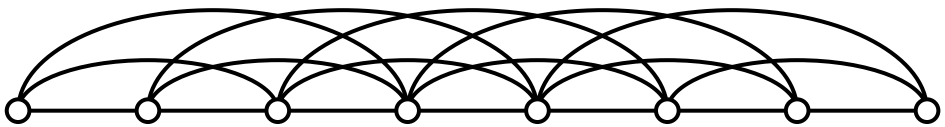
recoils congruence = "same recoils" = transitive closure of $UijV \equiv UjiV$ if $|i-j| > 1$





EXM 2: K-RECOIL SCHEME LATTICE & ZONOTOPES

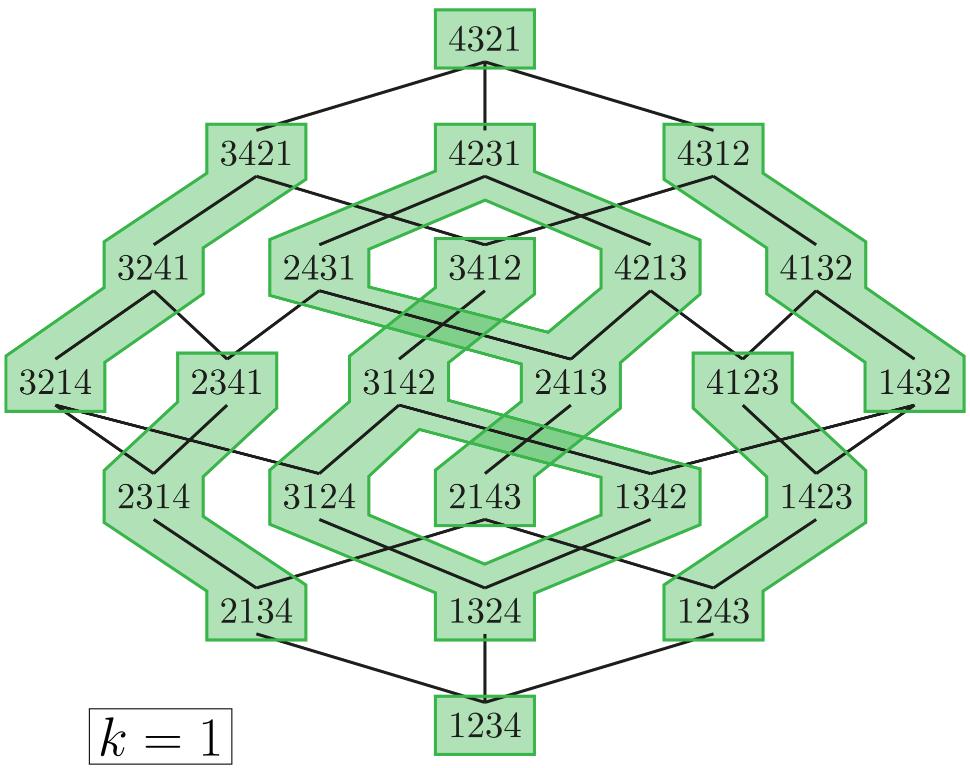
$\mathrm{G}^k(n)$ = graph with vertex set $[n]$ and edge set $\big\{\{i,j\} \in [n]^2 \;\big|\; i < j \le i+k\big\}$

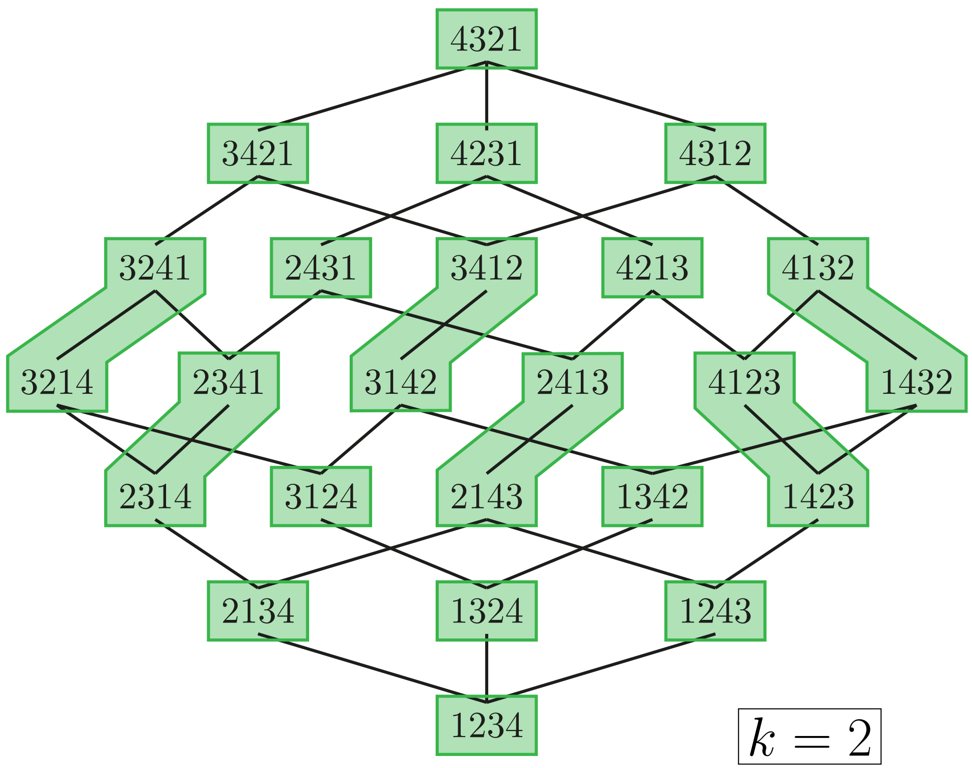


<u>k-recoils</u> of $\tau \in \mathfrak{S}_n$ = acyclic orientation of $\mathrm{G}^k(n)$ with edge $i \to j$ for all $i, j \in [n]$ such that $|i-j| \le k$ and $\tau^{-1}(i) < \tau^{-1}(j)$

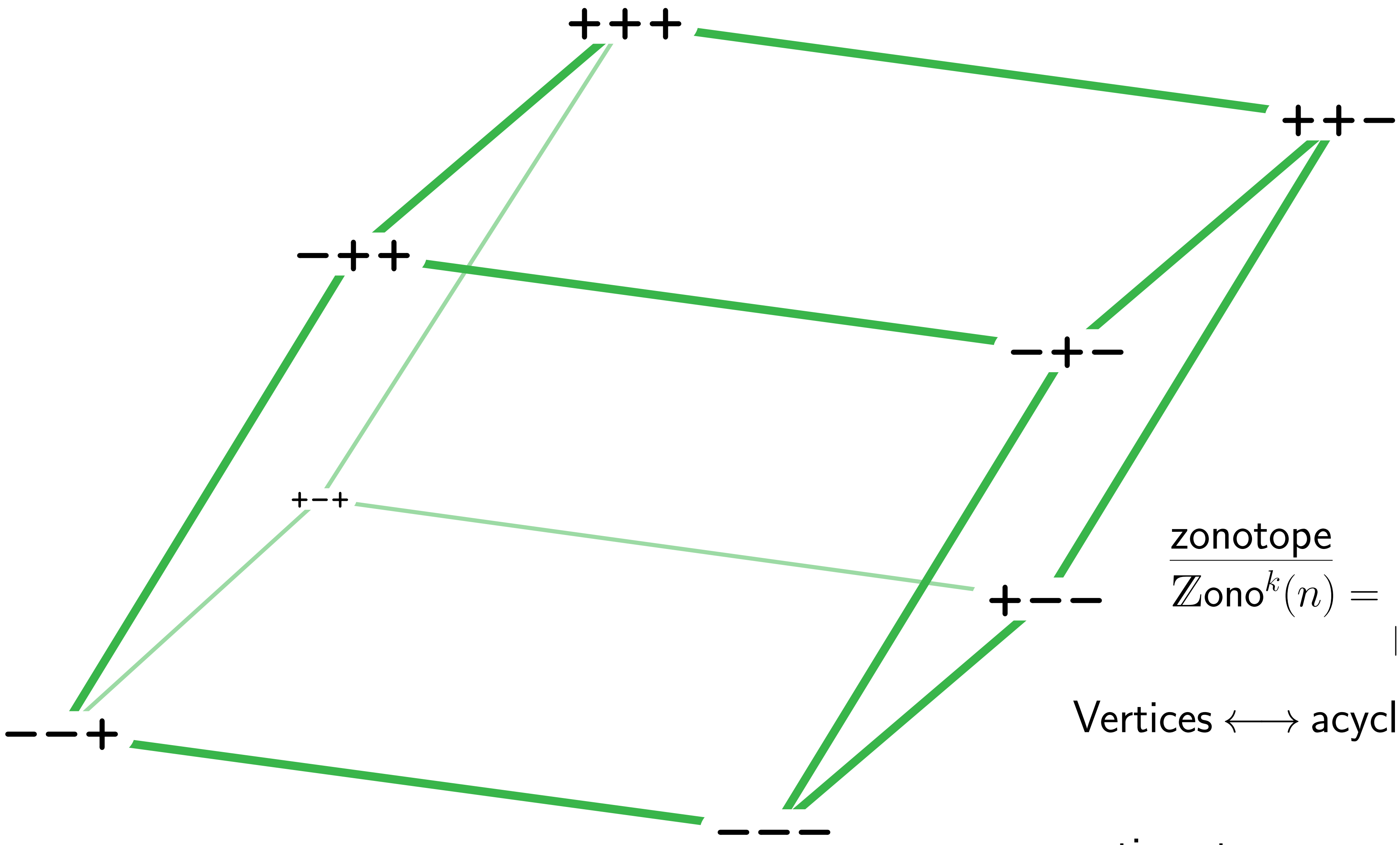
Novelli–Reutenauer–Thibon, *Generalized descent patterns in permutations and associated Hopf Algebras* ('11)

<u>k-recoils congruence</u> = "same k-recoils" = transitive closure of $UijV \equiv UjiV$ if $|i-j| > k$





EXM 2: K-RECOIL SCHEME LATTICE & ZONOTOPES



$\boxed{k = 1}$

zonotope

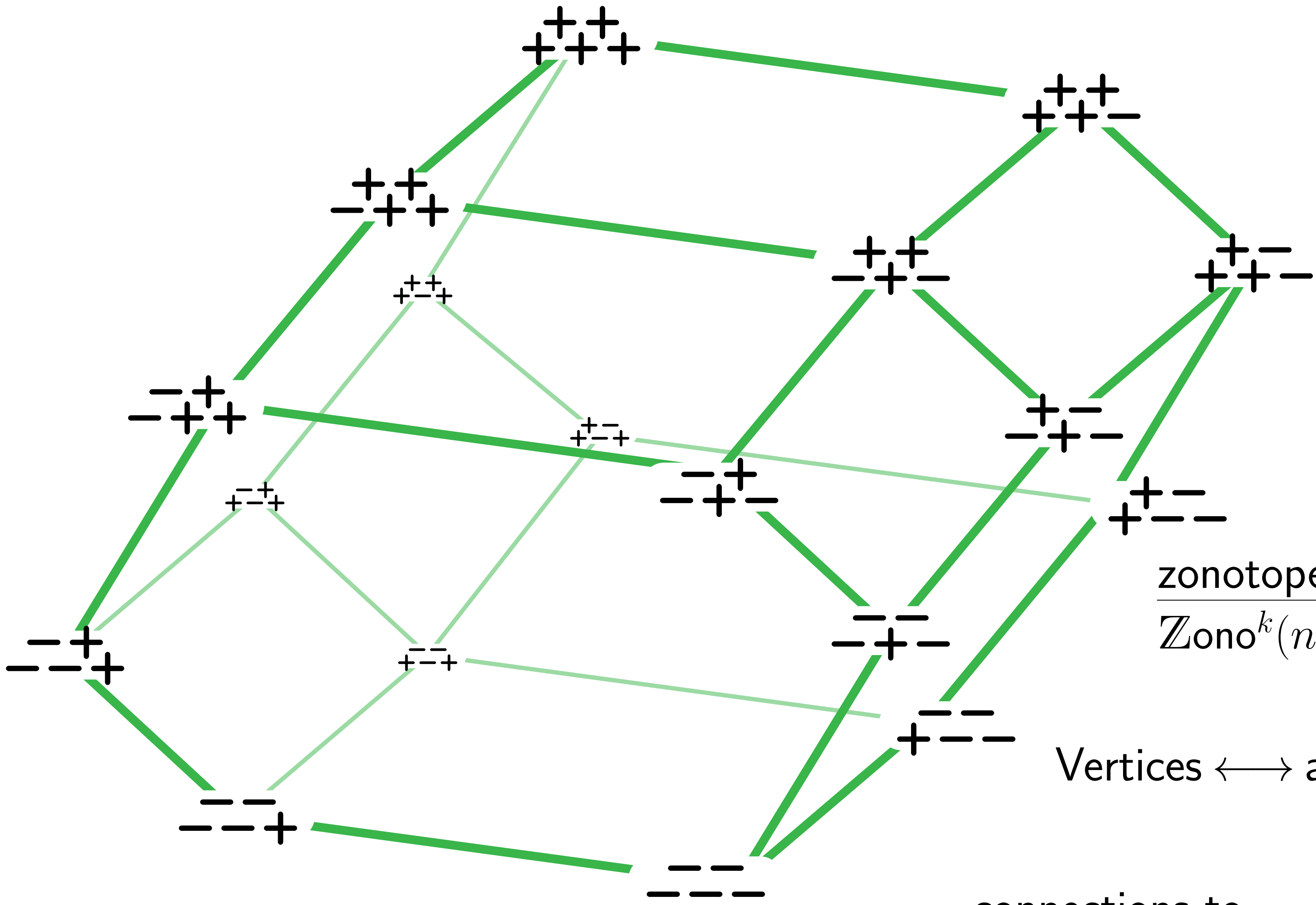
$$\mathbb{Z}\text{ono}^k(n) = \sum_{|i-j|\leq k} [\boldsymbol{e}_i, \boldsymbol{e}_j]$$

Vertices $\longleftrightarrow$ acyclic orientations of $\mathrm{G}^k(n)$

connections to

- matroids and oriented matroids
- hyperplane arrangements

EXM 2: K-RECOIL SCHEME LATTICE & ZONOTOPES



$\boxed{k = 2}$

zonotope

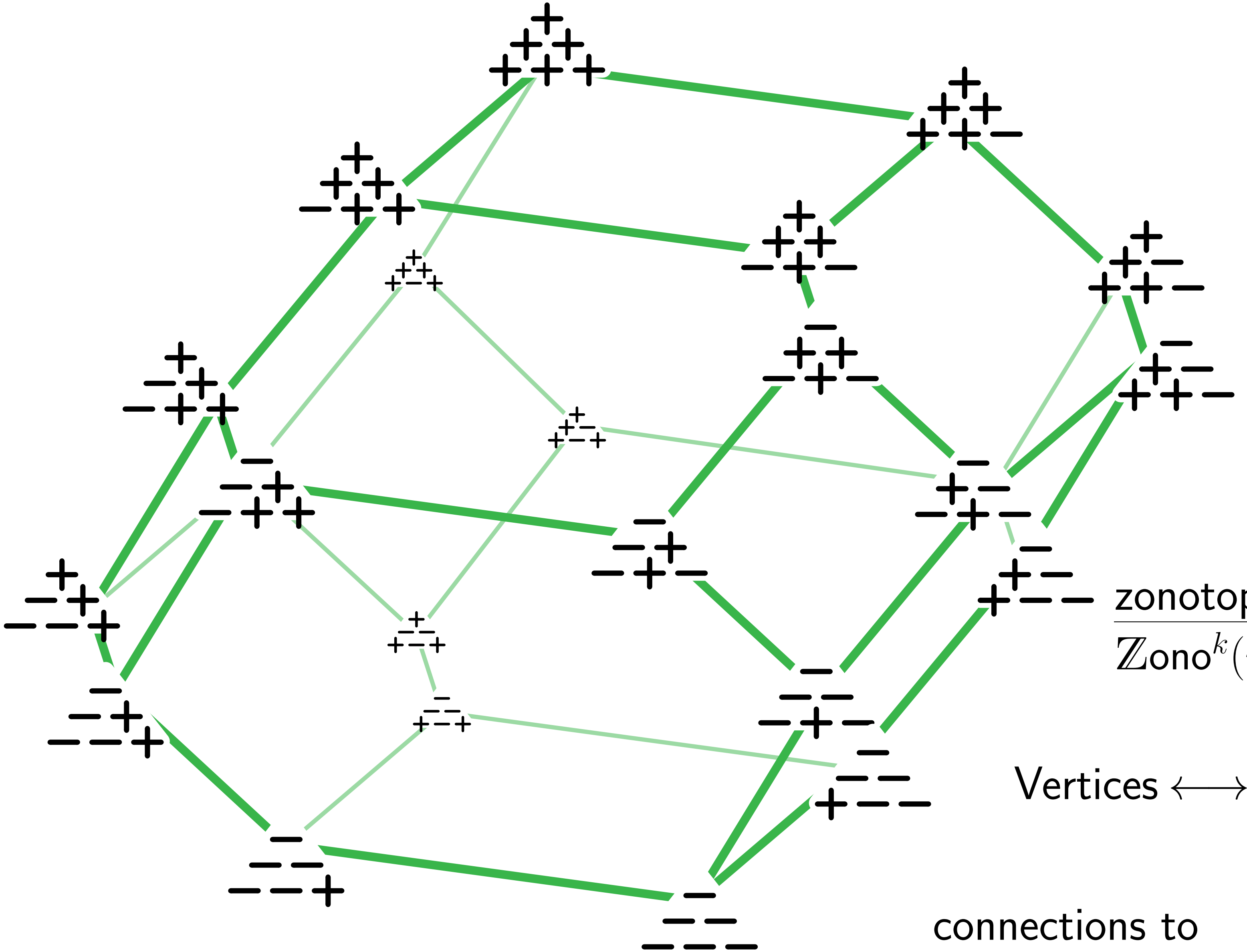
$$\mathbb{Z}\text{ono}^k(n) = \sum_{|i-j|\leq k} [\boldsymbol{e}_i, \boldsymbol{e}_j]$$

Vertices $\longleftrightarrow$ acyclic orientations of $\mathrm{G}^k(n)$

connections to

- matroids and oriented matroids
- hyperplane arrangements

EXM 2: K-RECOIL SCHEME LATTICE & ZONOTOPES



$k = 3$

zonotope

$$\mathbb{Z}\text{ono}^k(n) = \sum_{|i-j|\leq k} [\boldsymbol{e}_i, \boldsymbol{e}_j]$$

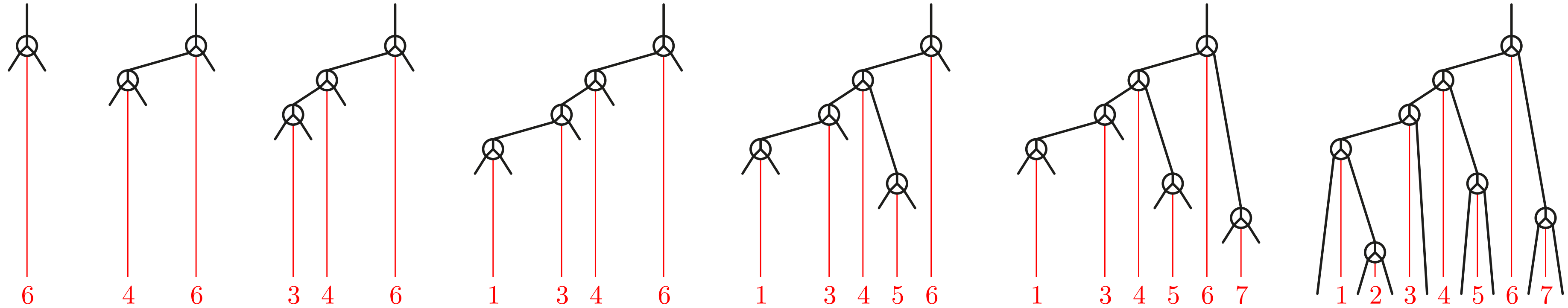
Vertices $\longleftrightarrow$ acyclic orientations of $\mathrm{G}^k(n)$

connections to

- matroids and oriented matroids
- hyperplane arrangements

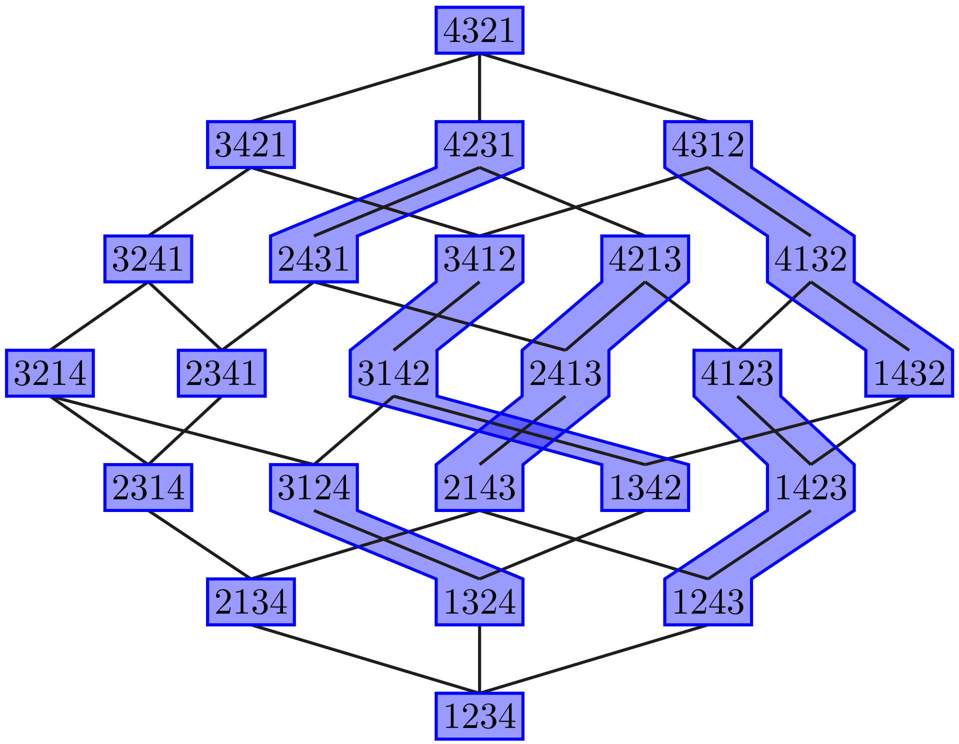
EXM 3: TAMARI LATTICE & LODAY'S ASSOCIAHEDRON

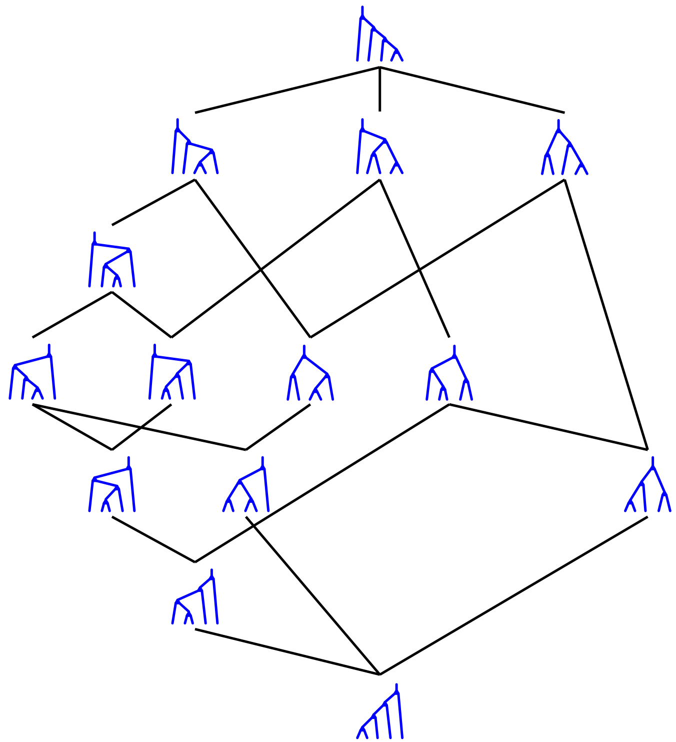
binary search tree insertion of 2751346



sylvester congruence = "same binary tree"

= transitive closure of $UacVbW \equiv UcaVbW$ where $a < b < c$





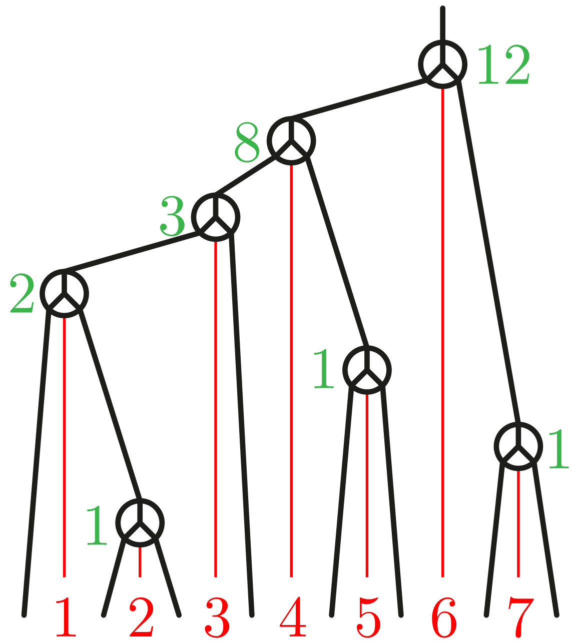
EXM 3: TAMARI LATTICE & LODAY'S ASSOCIAHEDRON

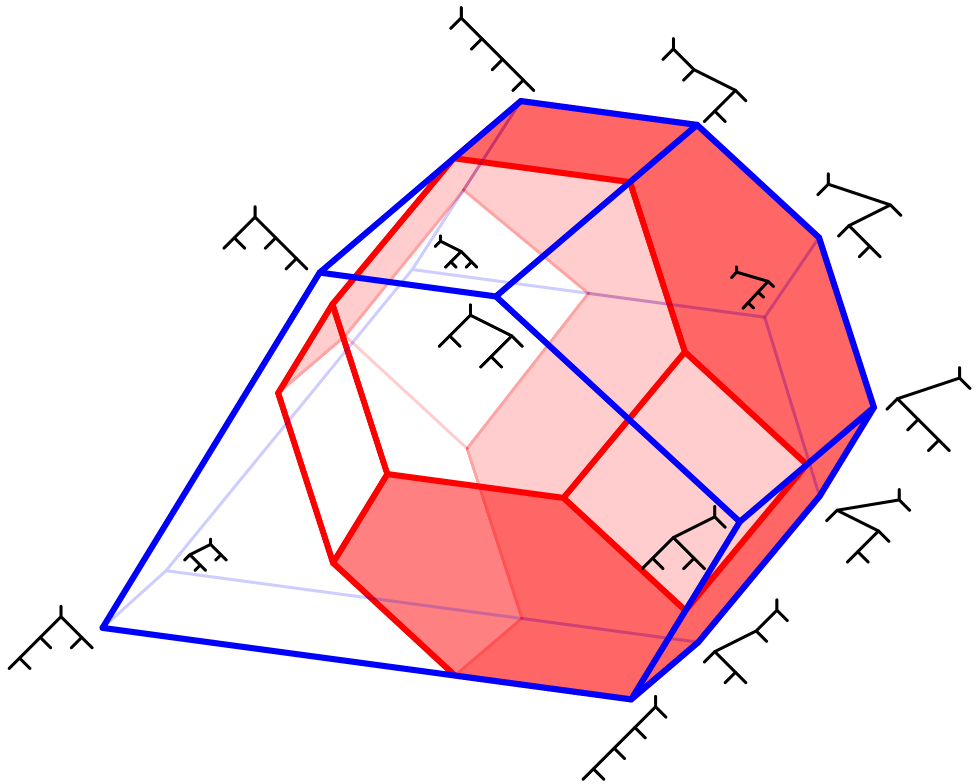
Loday's associahedron

$$\mathbb{A}\mathsf{sso}(n) = \operatorname{conv}\{\boldsymbol{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \boldsymbol{H}^{\ge}(i,j) = \sum_{1 \le i \le j \le n+1} \triangle_{[i,j]}$$

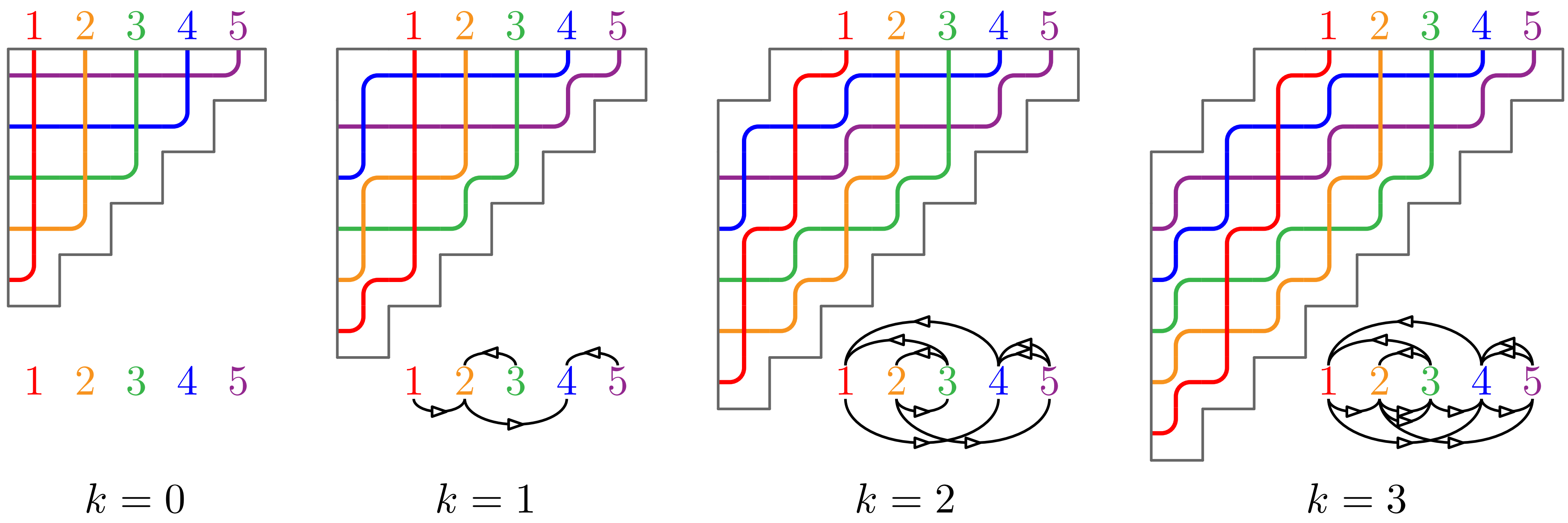
$$\boldsymbol{L}(\mathrm{T}) = \big[\ell(\mathrm{T},i) \cdot r(\mathrm{T},i)\big]_{i \in [n+1]} \qquad \boldsymbol{H}^{\ge}(i,j) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Shnider–Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)
Loday, *Realization of the Stasheff polytope* ('04)





EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

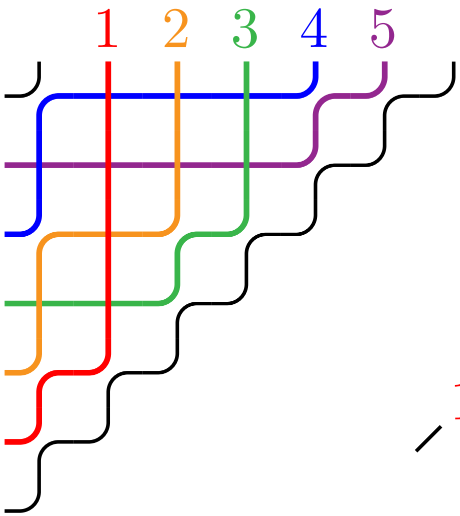


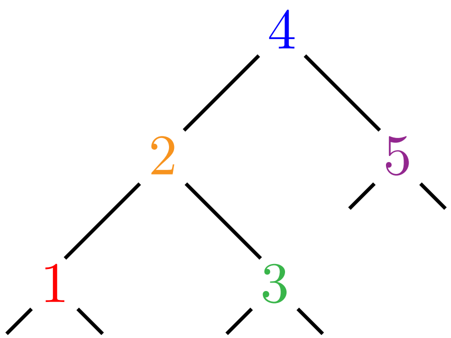
(k, n)-twist = pipe dream in the trapezoidal shape of height n and width k

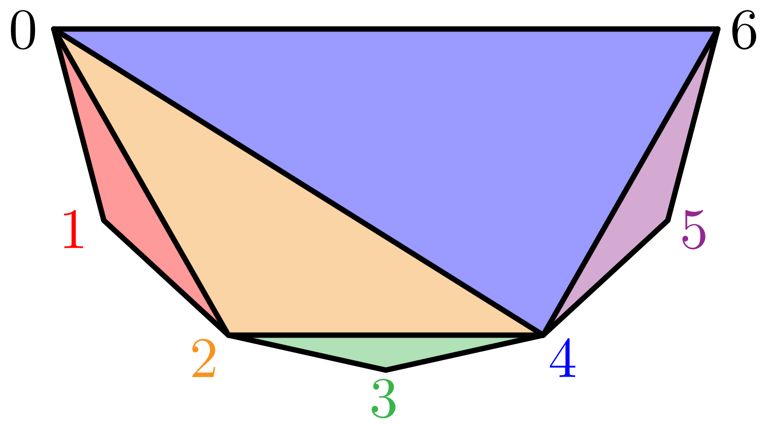
contact graph of a twist T = vertices are pipes of T and arcs are elbows of T

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i, j]$ of the $(n+2)$-gon
$(1, n)$-twist T	$\longleftrightarrow$	triangulation $\mathrm{T}^\star$ of the $(n+2)$-gon
pth relevant pipe of T	$\longleftrightarrow$	pth triangle of $\mathrm{T}^\star$
contact graph of T	$\longleftrightarrow$	dual binary tree of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$







Woo, *Catalan numbers and Schubert Polynomials for* $w = 1(n+1)\dots 2$. ('04)
Stump, *A new perspective on* k*-triangulations* ('11)
P. – Pocchiola, *Multitriangulations, pseudotriangulations and primitive sorting networks* ('12)

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i, j]$ of the $(n + 2)$-gon
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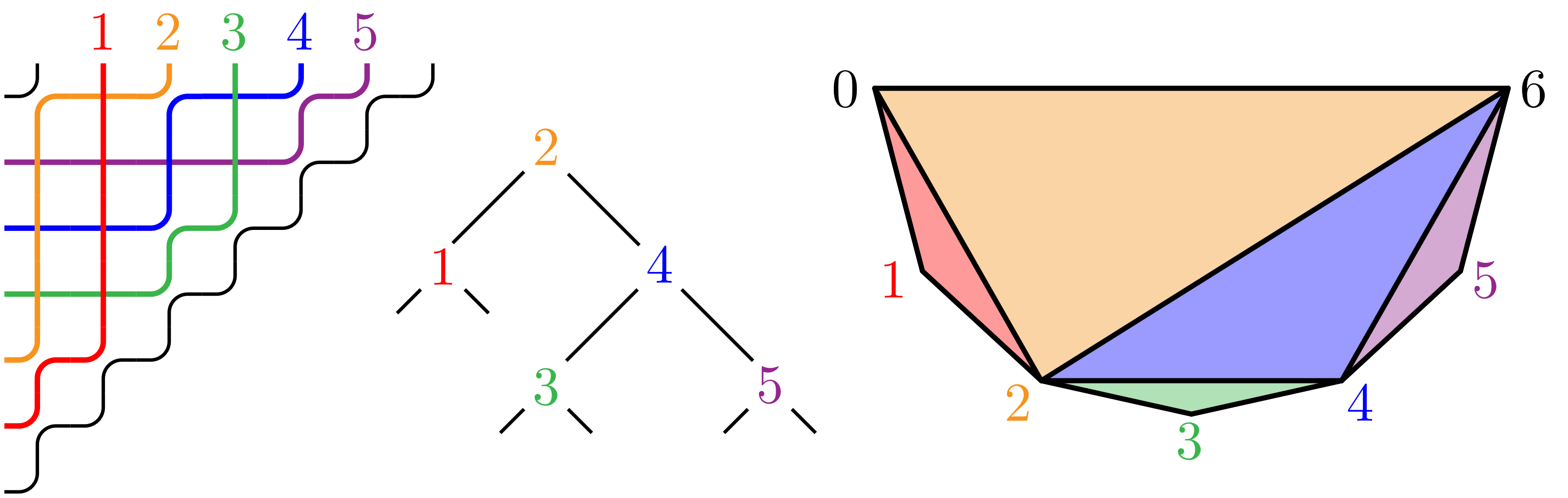
contact graph of T	$\longleftrightarrow$	dual binary tree of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$



Woo, *Catalan numbers and Schubert Polynomials for* $w = 1(n + 1) \dots 2$. ('04)
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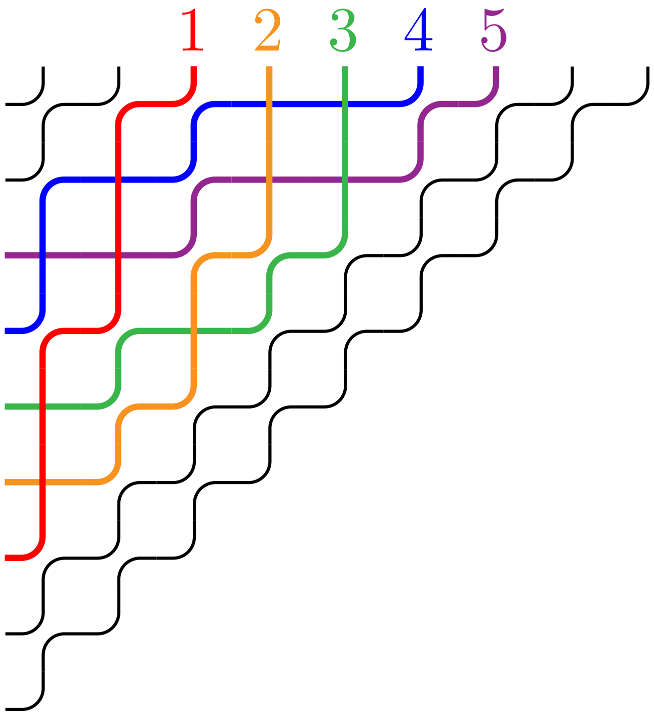
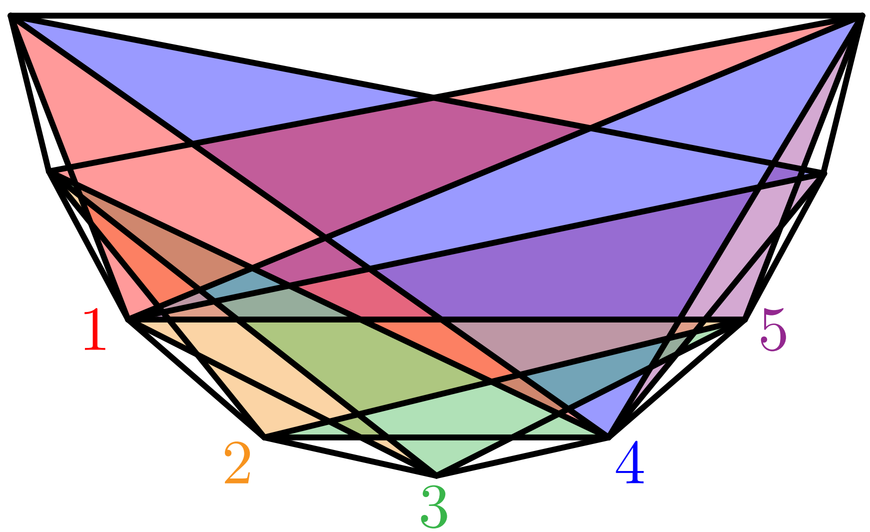
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i, j]$ of the $(n+2k)$-gon
(k, n)-twist T	$\longleftrightarrow$	k-triangulation $\mathrm{T}^\star$ of the $(n+2k)$-gon
pth relevant pipe of T	$\longleftrightarrow$	pth k-star of $\mathrm{T}^\star$
contact graph of T	$\longleftrightarrow$	dual graph of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$



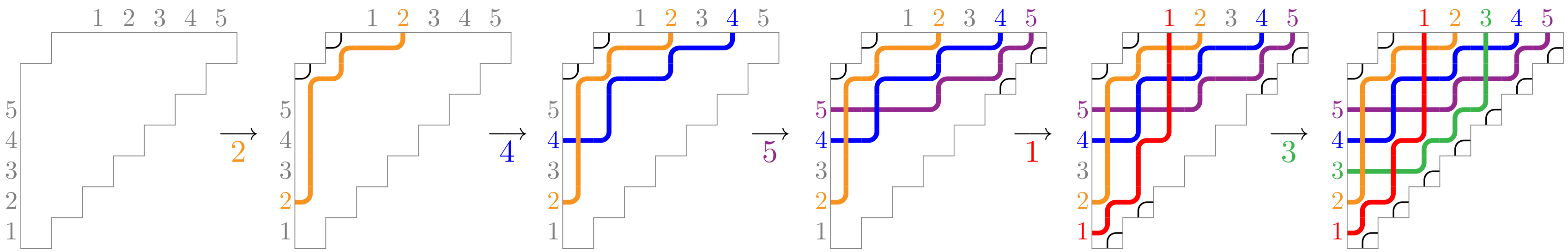


Stump, *A new perspective on k-triangulations* ('11)

P. – Pocchiola, *Multitriangulations, pseudotriangulations and primitive sorting networks* ('12)

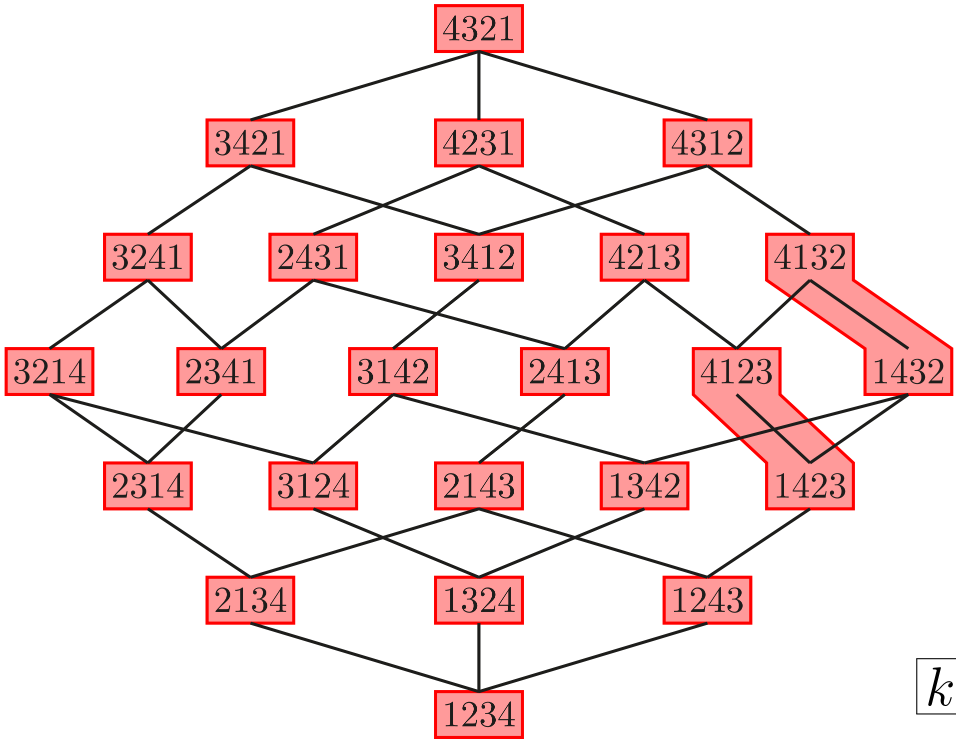
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

k-twist insertion of 31542 $\quad$ $k = 2$

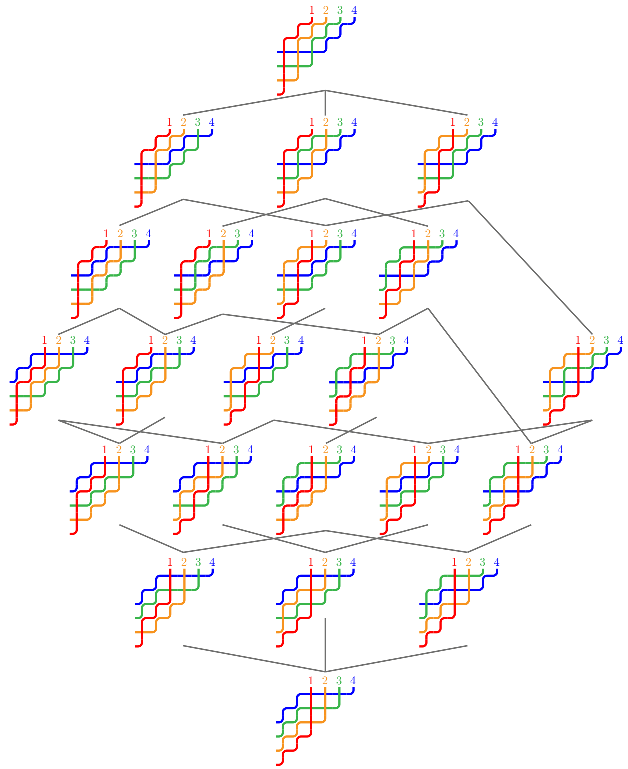


k-twist congruence = "same k-twist" $\quad$ P., *Brick polytopes, lattice quotients, and Hopf algebras* ('18)

= transitive closure of $UacV_1b_1V_2b_2 \dots V_kb_kW \equiv UcaV_1b_1V_2b_2 \dots V_kb_kW$
where $a < b_i < c$ for all $i \in [k]$



$k = 2$



EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

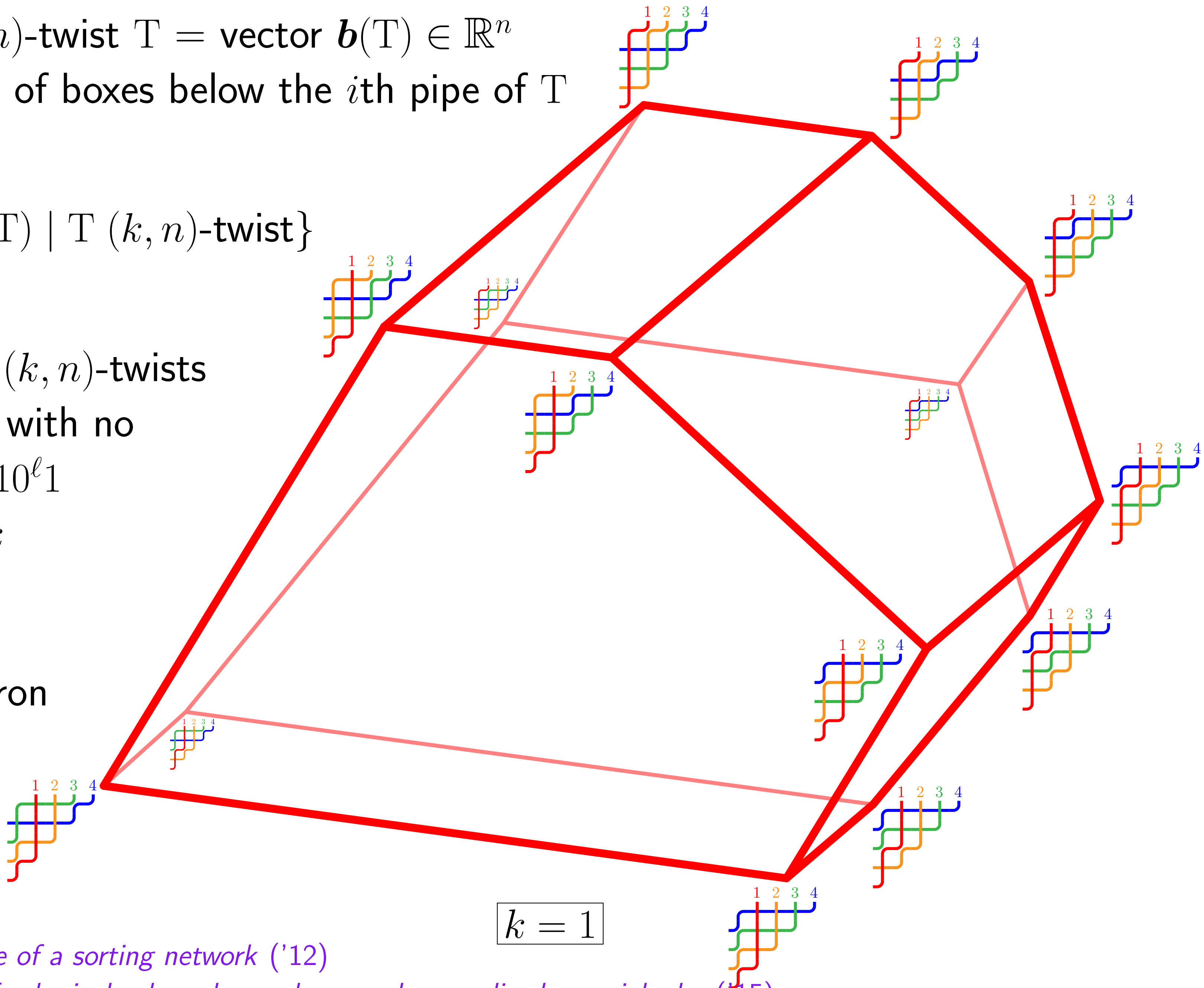
brick vector of a (k, n)-twist T = vector $\boldsymbol{b}(\mathrm{T}) \in \mathbb{R}^n$
with $\boldsymbol{b}(\mathrm{T})_i$ = number of boxes below the ith pipe of T

brick polytope
$\mathbb{B}\mathrm{rick}^k(n) = \mathrm{conv}\,\{\boldsymbol{b}(\mathrm{T}) \mid \mathrm{T}\ (k, n)\text{-twist}\}$

Vertices $\longleftrightarrow$ acyclic (k, n)-twists
Facets $\longleftrightarrow$ $0/1$-seqs with no subseqs $10^\ell 1$ for $\ell \geq k$

connections to

- Loday associahedron
- incidence cones of binary trees
- Tamari lattice



P.–Santos, *The brick polytope of a sorting network* ('12)
P.–Stump, *Brick polytopes of spherical subword complexes and generalized associahedra* ('15)

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

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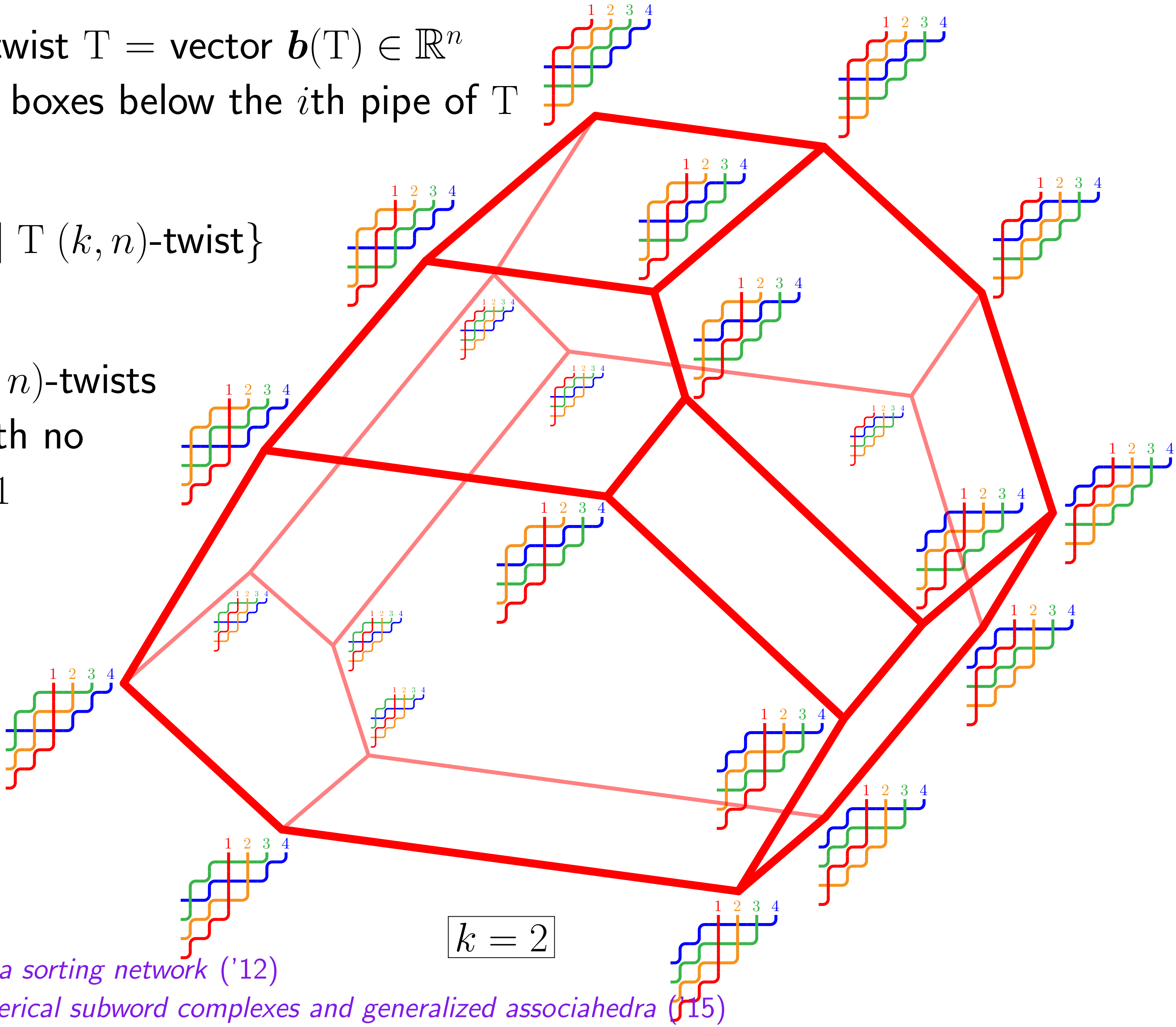
with $\boldsymbol{b}(\mathrm{T})_i$ = number of boxes below the ith pipe of T

brick polytope
$\mathrm{Brick}^k(n) = \mathrm{conv}\,\{\boldsymbol{b}(\mathrm{T}) \mid \mathrm{T}\ (k,n)\text{-twist}\}$

Vertices $\longleftrightarrow$ acyclic (k,n)-twists
Facets $\longleftrightarrow$ $0/1$-seqs with no subseqs $10^\ell 1$ for $\ell \geq k$

connections to

- Loday associahedron
- incidence cones of binary trees
- Tamari lattice



P.–Santos, *The brick polytope of a sorting network* ('12)
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brick polytope
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Facets $\longleftrightarrow$ $0/1$-seqs with no subseqs $10^\ell 1$ for $\ell \geq k$

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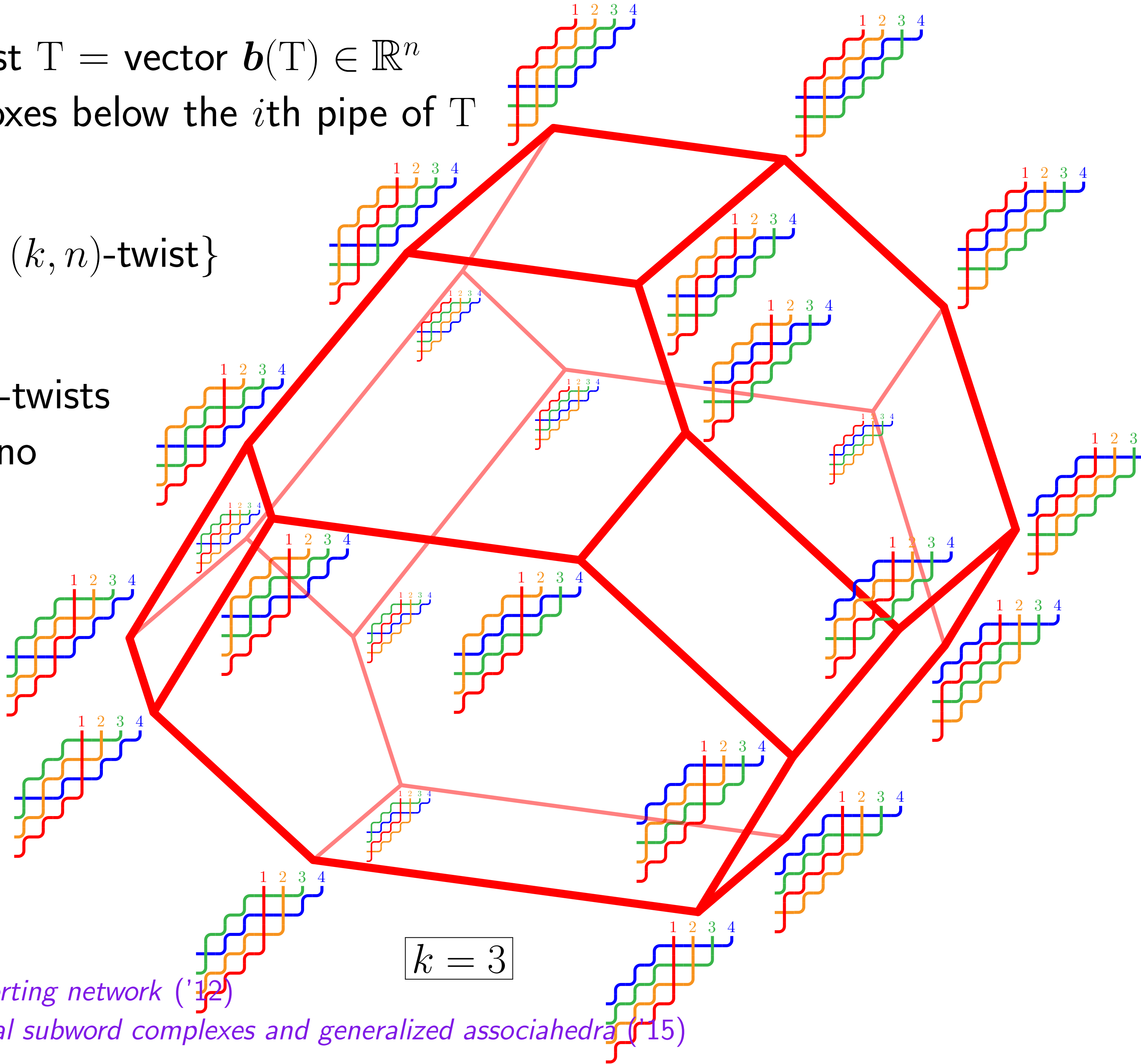

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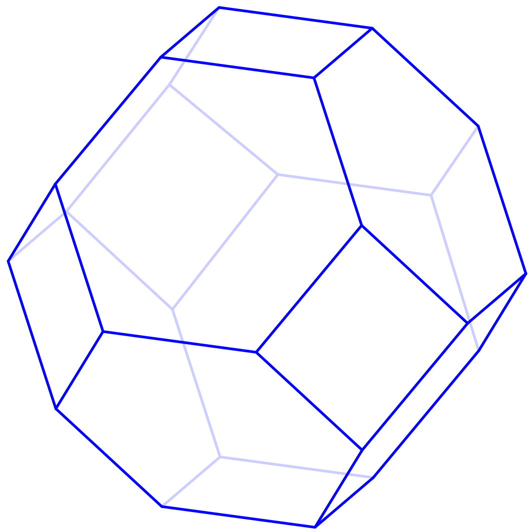
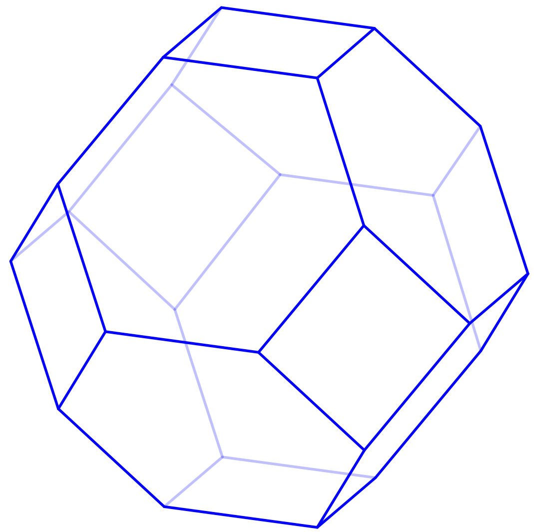
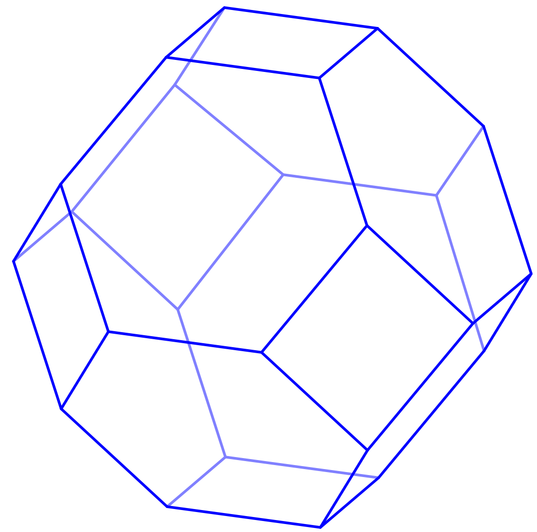
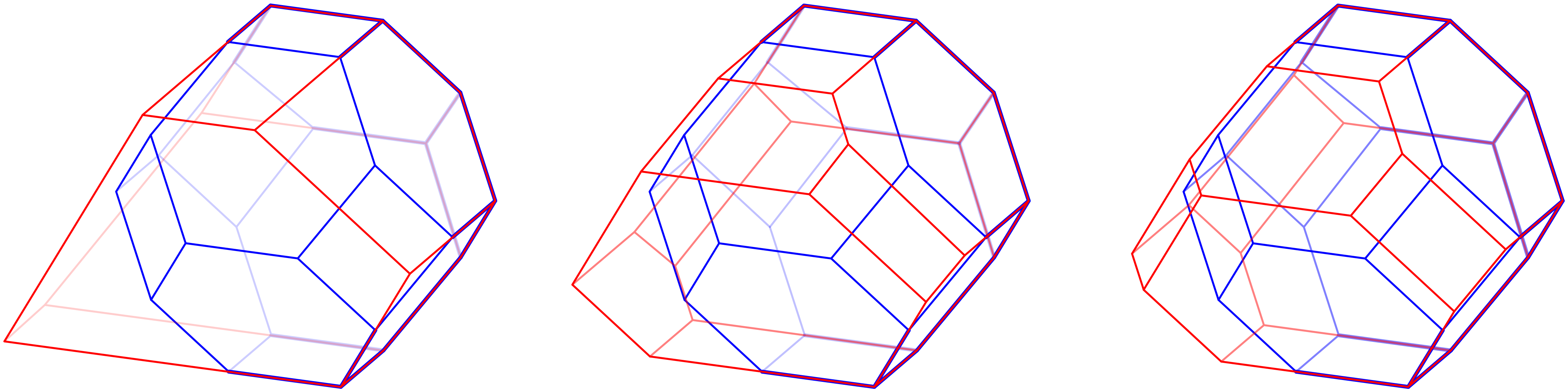
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES

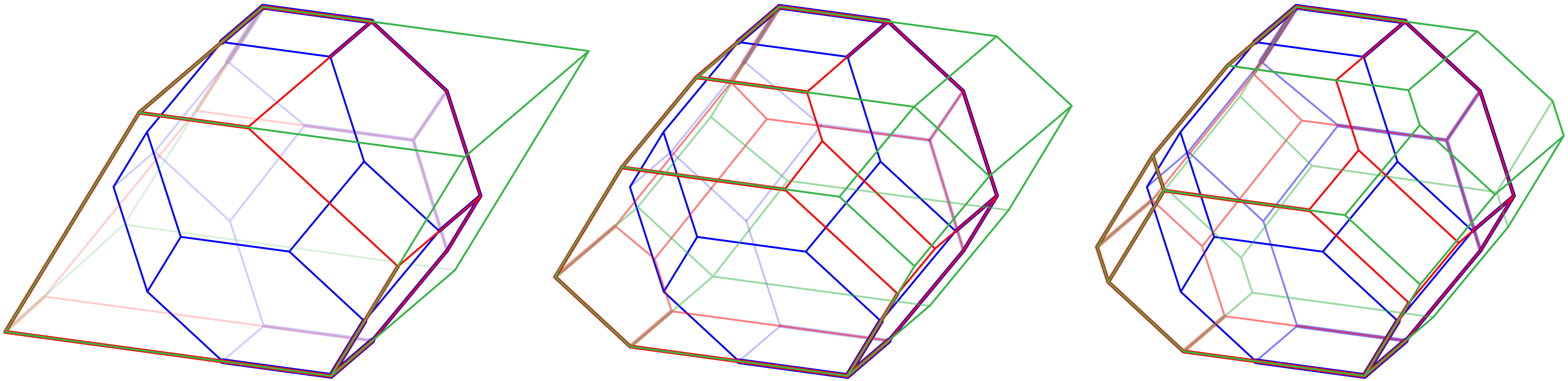
Permutahedron $\mathbb{P}\mathrm{erm}(n)$

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



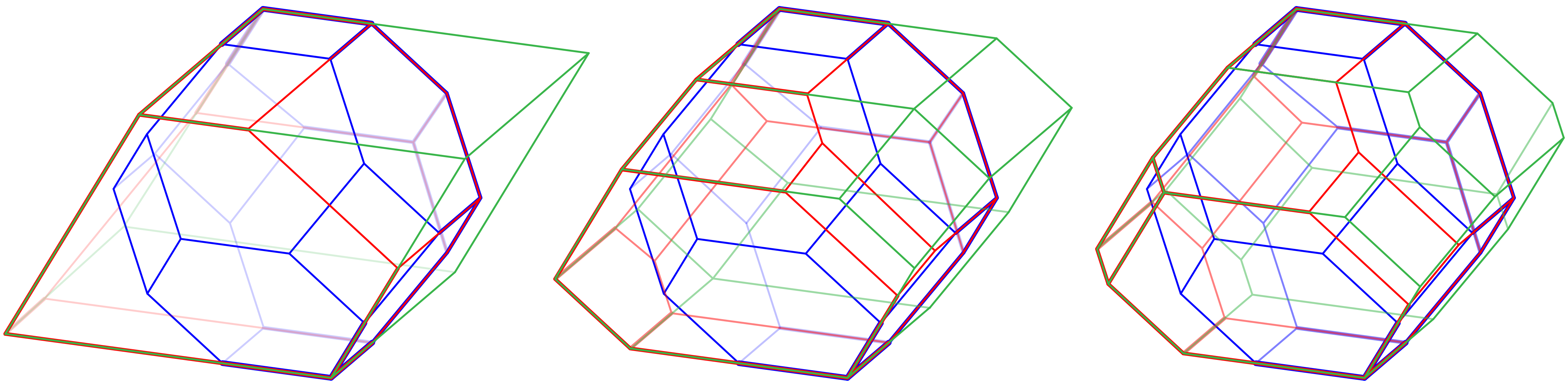
Permutahedron $\mathbb{P}\mathsf{erm}(n)$ $\subset$ Brick polytope $\mathbb{B}\mathsf{rick}^k(n)$

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



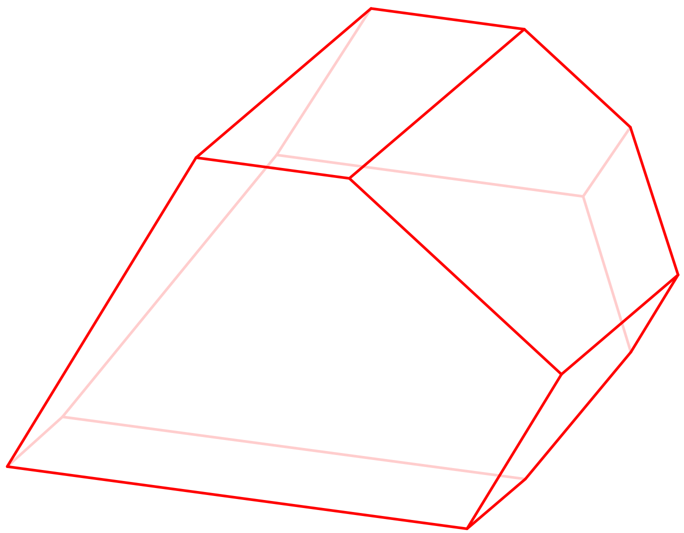
Permutahedron $\mathbb{P}\mathrm{erm}(n)$ $\subset$ Brick polytope $\mathbb{B}\mathrm{rick}^k(n)$ $\subset$ Zonotope $\mathbb{Z}\mathrm{ono}^k(n)$

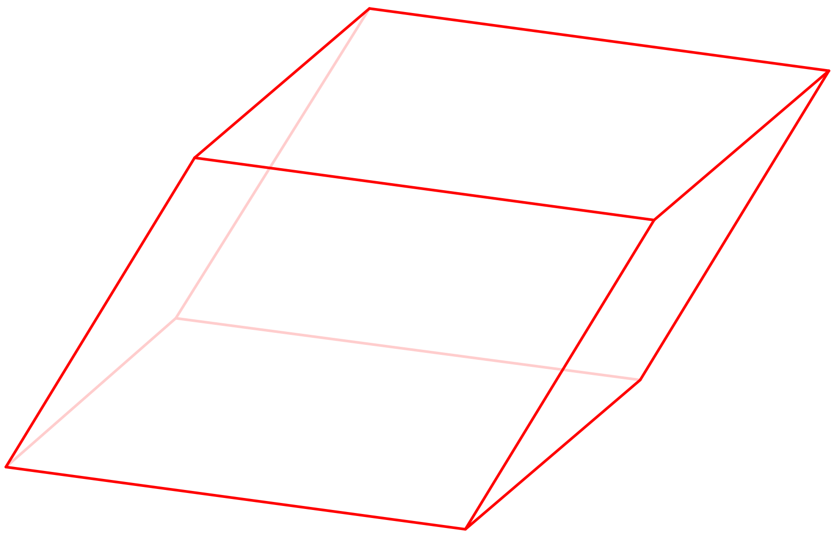
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



Permutahedron $\mathbb{P}\mathrm{erm}(n)$ $\subset$ Brick polytope $\mathbb{B}\mathrm{rick}^k(n)$ $\subset$ Zonotope $\mathbb{Z}\mathrm{ono}^k(n)$

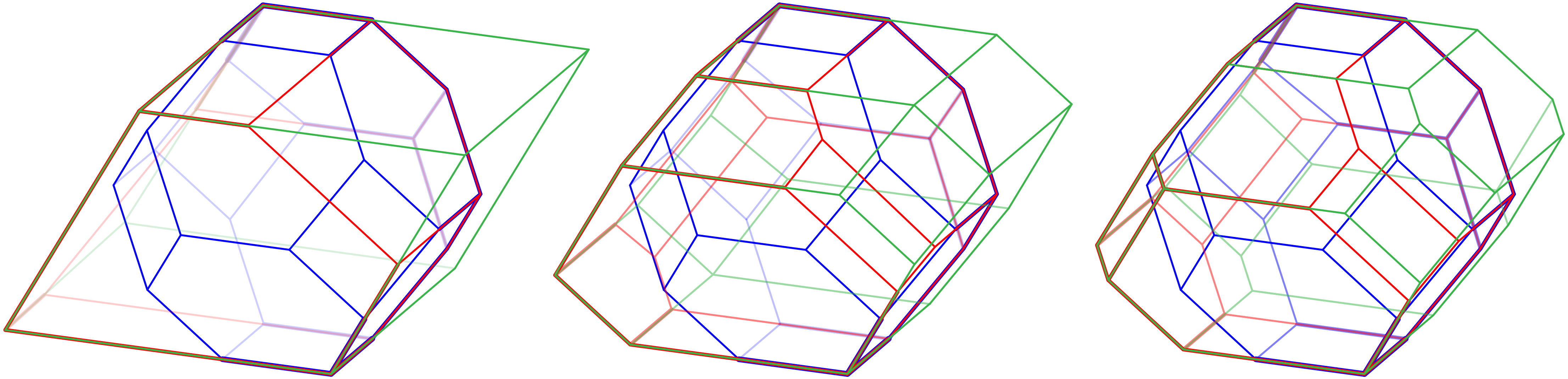
$\mathbb{B}\mathrm{rick}^1(n)$





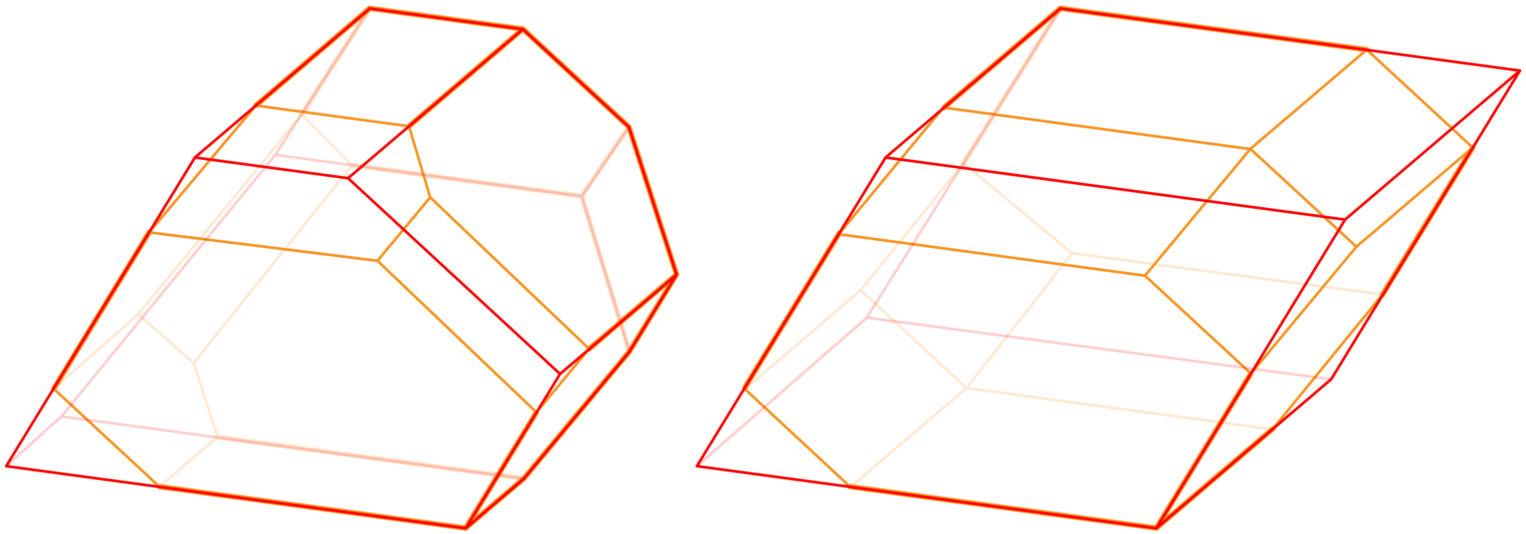
$\mathbb{Z}\mathrm{ono}^1(n)$

EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



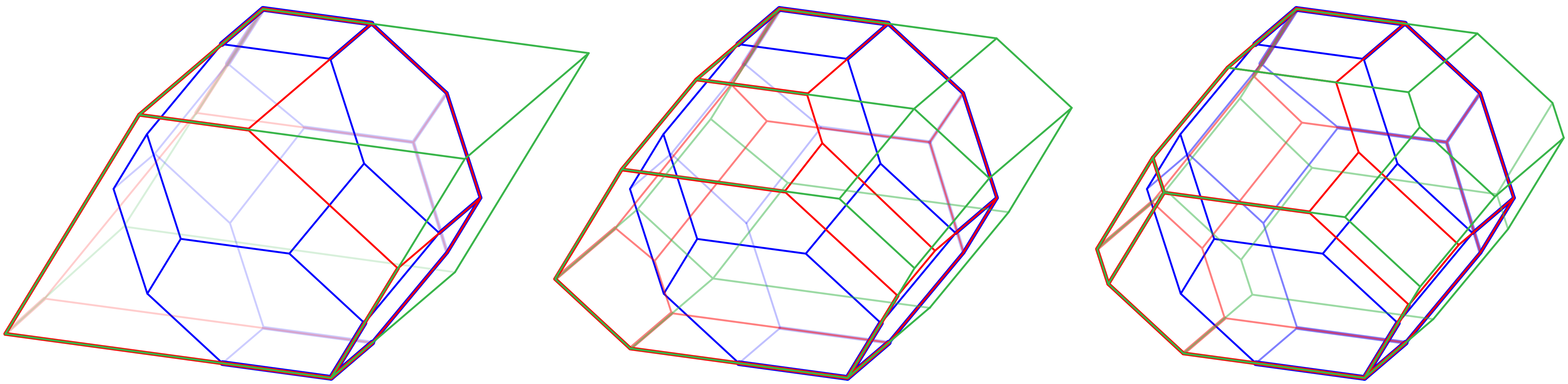
Permutahedron $\mathbb{P}\mathrm{erm}(n)$ $\subset$ Brick polytope $\mathbb{B}\mathrm{rick}^k(n)$ $\subset$ Zonotope $\mathbb{Z}\mathrm{ono}^k(n)$

$\mathbb{B}\mathrm{rick}^1(n)$
$\cup$
$\mathbb{B}\mathrm{rick}^2(n)$



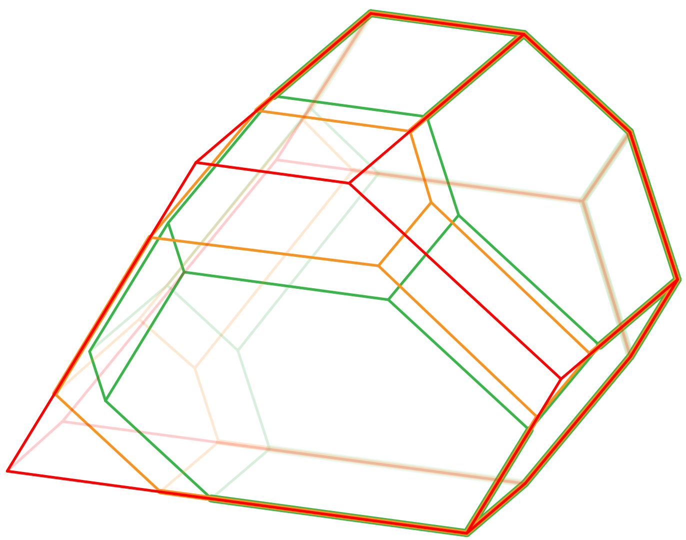
$\mathbb{Z}\mathrm{ono}^1(n)$
$\cup$
$\mathbb{Z}\mathrm{ono}^2(n)$

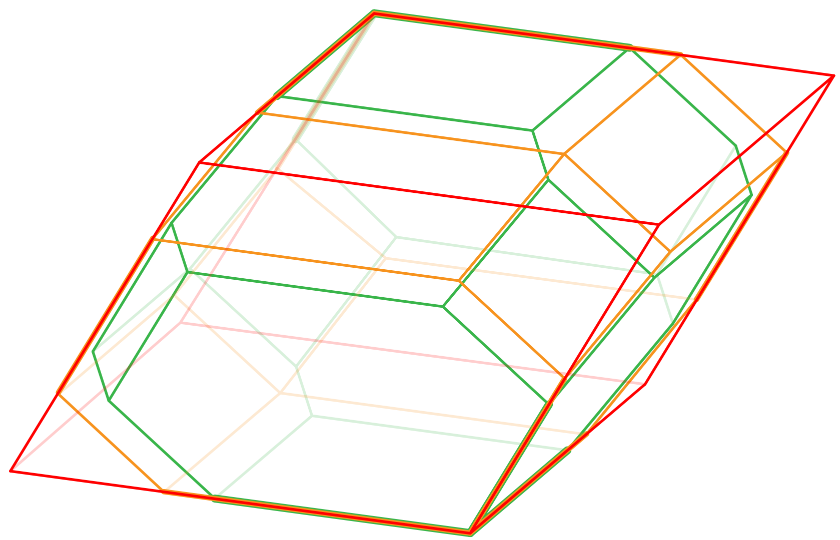
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



Permutahedron $\mathbb{P}\mathsf{erm}(n)$ $\subset$ Brick polytope $\mathbb{B}\mathsf{rick}^k(n)$ $\subset$ Zonotope $\mathbb{Z}\mathsf{ono}^k(n)$

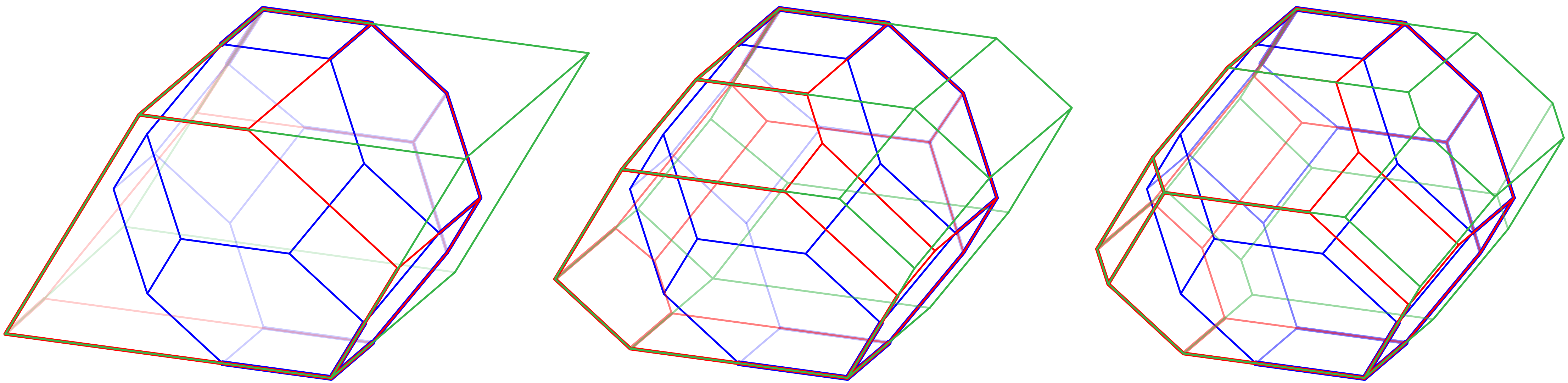
$\mathbb{B}\mathsf{rick}^1(n)$
$\cup$
$\mathbb{B}\mathsf{rick}^2(n)$
$\cup$
$\mathbb{B}\mathsf{rick}^3(n)$





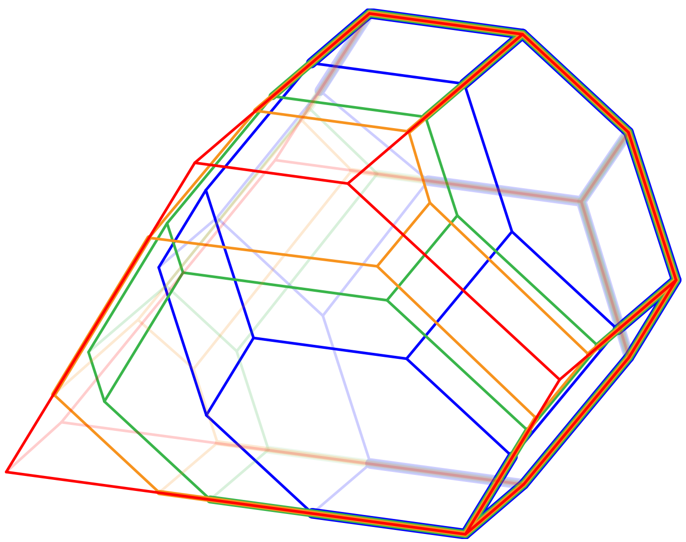
$\mathbb{Z}\mathsf{ono}^1(n)$
$\cup$
$\mathbb{Z}\mathsf{ono}^2(n)$
$\cup$
$\mathbb{Z}\mathsf{ono}^3(n)$

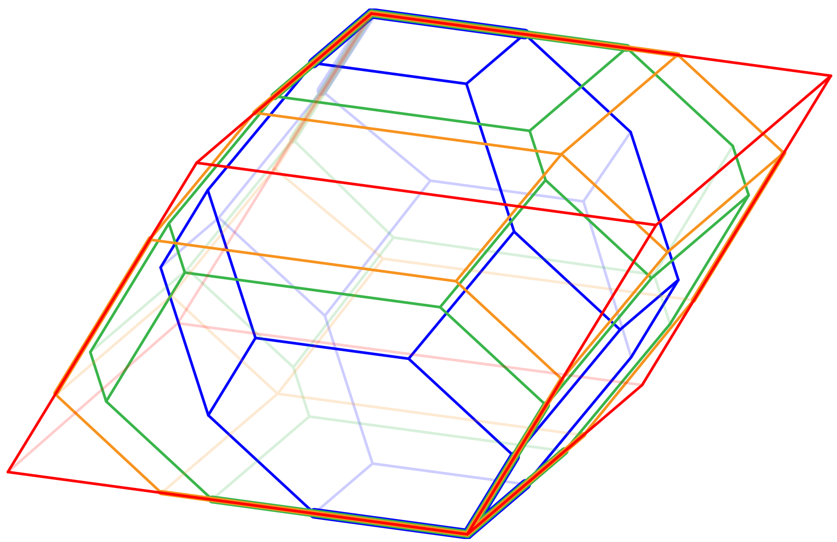
EXM 4: K-TWIST LATTICE & BRICK POLYTOPES



Permutahedron $\mathbb{P}\mathrm{erm}(n)$ $\subset$ Brick polytope $\mathbb{B}\mathrm{rick}^k(n)$ $\subset$ Zonotope $\mathbb{Z}\mathrm{ono}^k(n)$

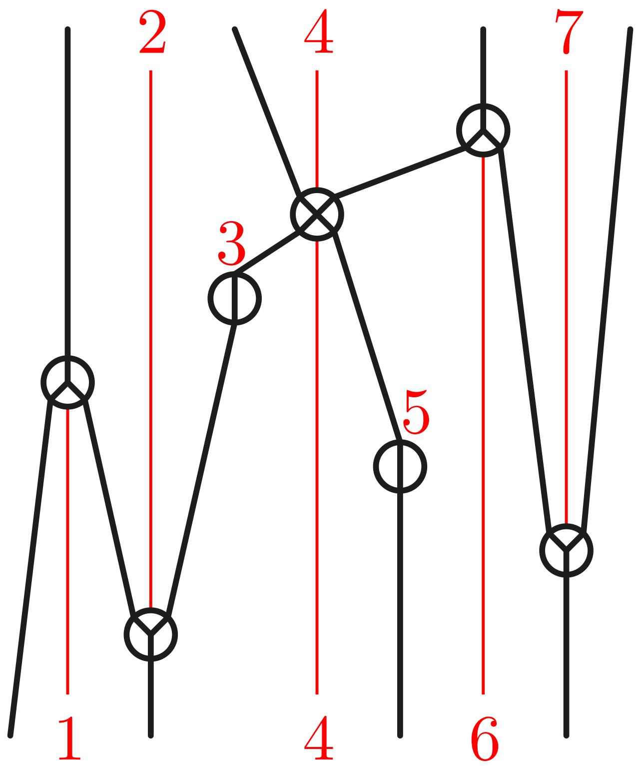
$\mathbb{B}\mathrm{rick}^1(n)$
$\cup$
$\mathbb{B}\mathrm{rick}^2(n)$
$\cup$
$\mathbb{B}\mathrm{rick}^3(n)$
$\cup$
$\vdots$
$\cup$
$\mathbb{P}\mathrm{erm}(n)$



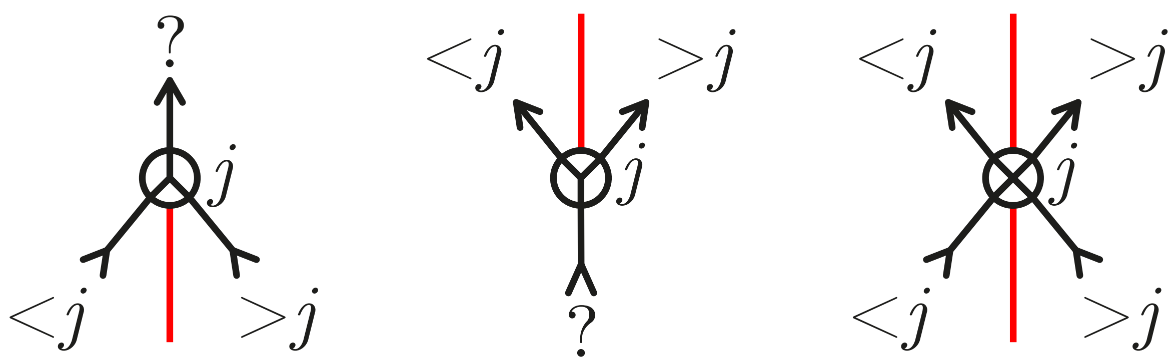


$\mathbb{Z}\mathrm{ono}^1(n)$
$\cup$
$\mathbb{Z}\mathrm{ono}^2(n)$
$\cup$
$\mathbb{Z}\mathrm{ono}^3(n)$
$\cup$
$\vdots$
$\cup$
$\mathbb{P}\mathrm{erm}(n)$

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA



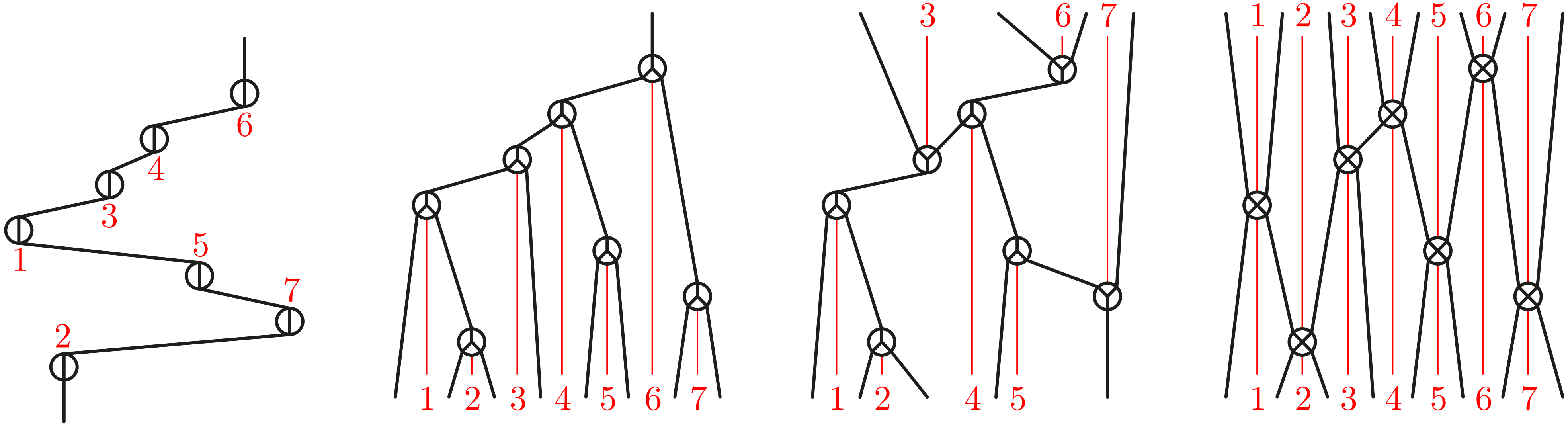
permutree = directed (bottom to top) and labeled (bijectively by $[n]$) tree such that



P.–Pons, *Permutrees* ('18)

Examples.

decoration		permutrees
⦶^{n}	$\longleftrightarrow$	permutations of $[n]$
☮^{n}	$\longleftrightarrow$	standard binary search trees
$\{$☮, Ⓨ$\}^n$	$\longleftrightarrow$	Cambrian trees
⊗^{n}	$\longleftrightarrow$	binary sequences

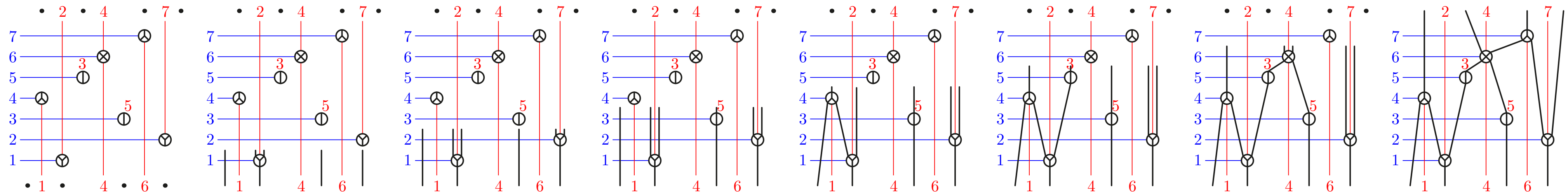


permutation | binary tree | Cambrian tree | binary sequence

P.–Pons, *Permutrees* ('18)

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

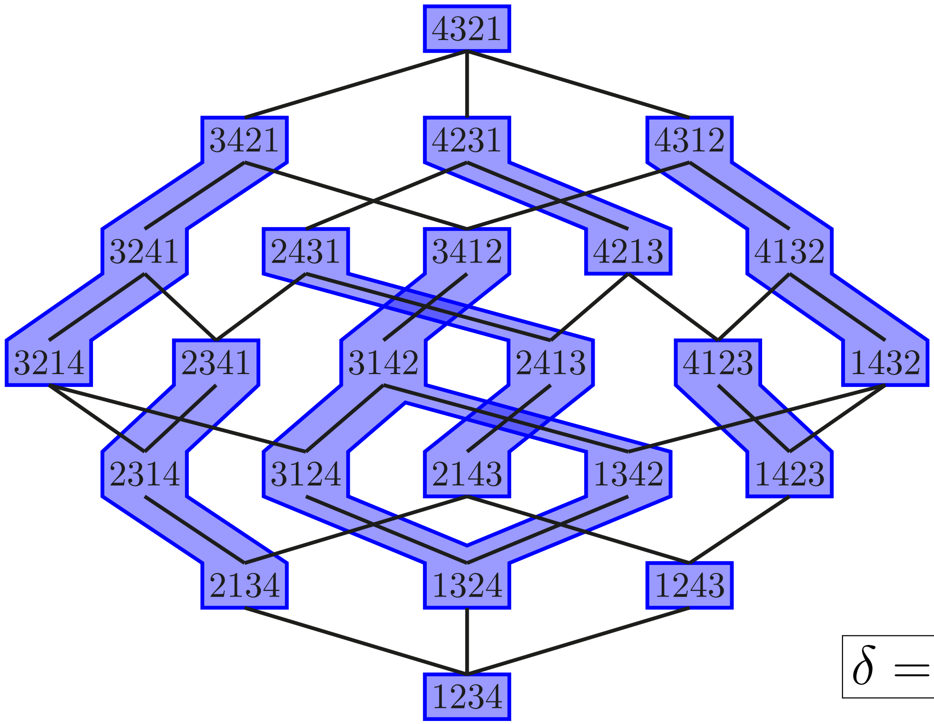
δ-permutree insertion of 2751346



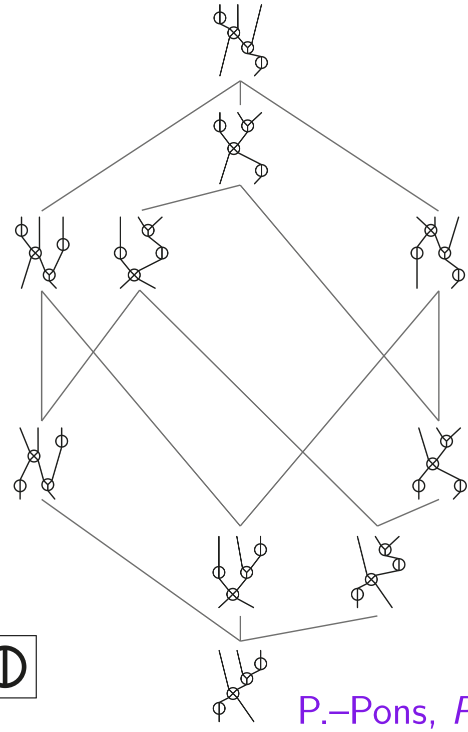
δ-permutree congruence $=$ "same δ-permutree"

$=$ tr. cl. $UacVbW \equiv UcaVbW$ where $a < b < c$ and $\delta_b \in \{☮, \otimes\}$

and $UbVacW \equiv UbVcaW$ where $a < b < c$ and $\delta_b \in \{Ⓨ, \otimes\}$



$\delta = \ominus\otimes Ⓨ \ominus$



P.–Pons, *Permutrees* ('18)

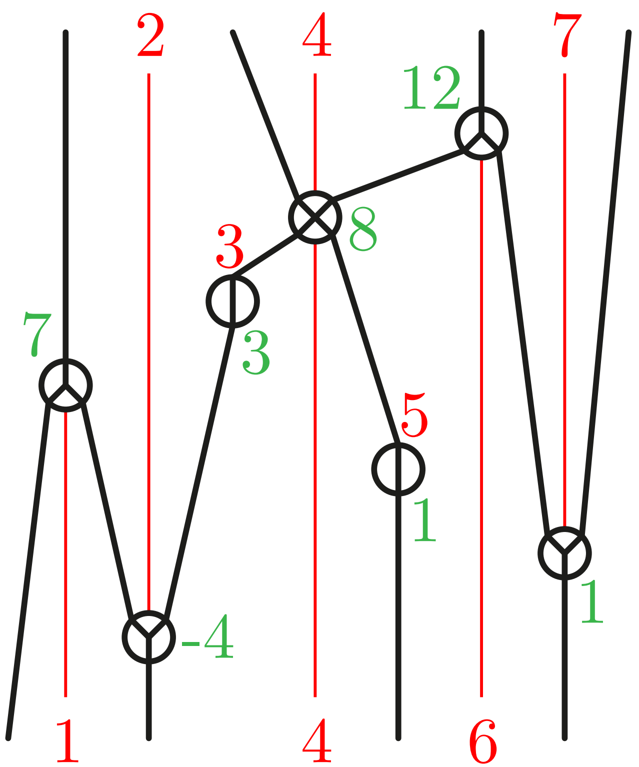
EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA

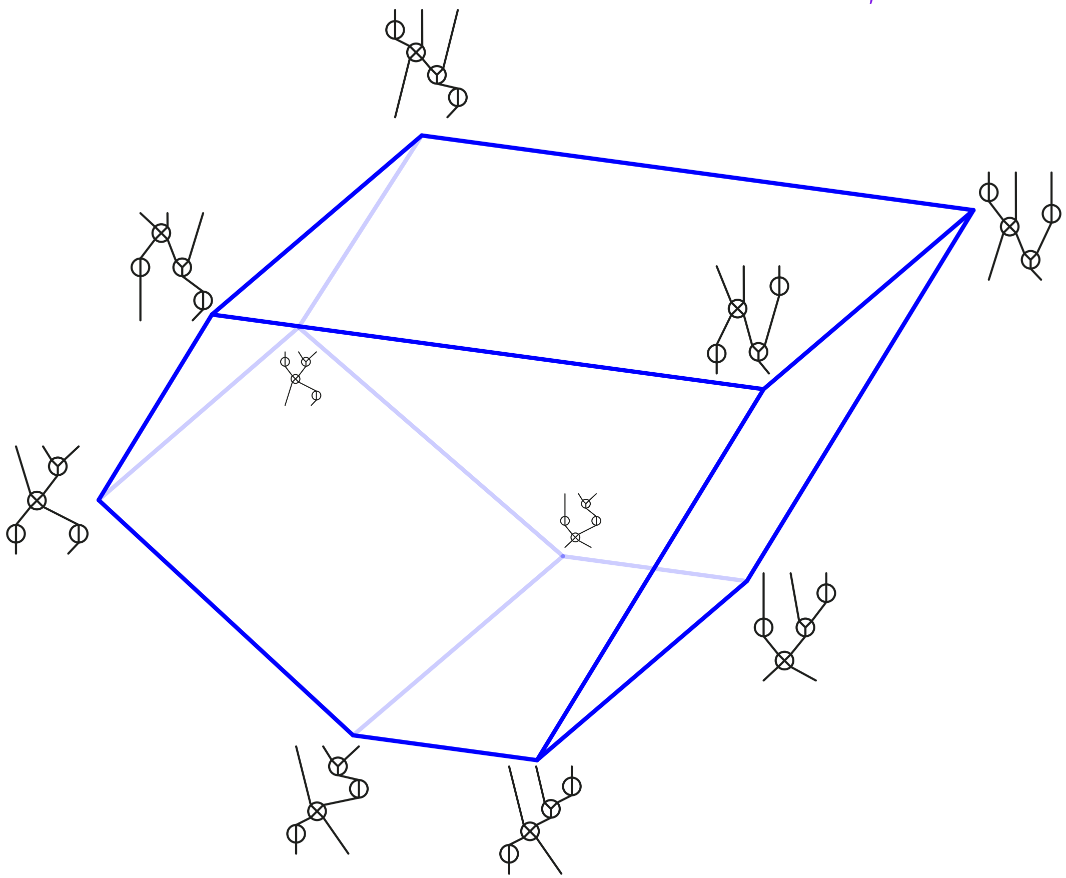
permutreehedron

$$\mathbb{PT}(\delta) = \operatorname{conv}\{\boldsymbol{L}(\mathrm{T}) \mid \mathrm{T}\ \delta\text{-permutree}\} = \mathbb{H} \cap \bigcap_{I\ \delta\text{-cut}} \boldsymbol{H}^{\geq}(I)$$

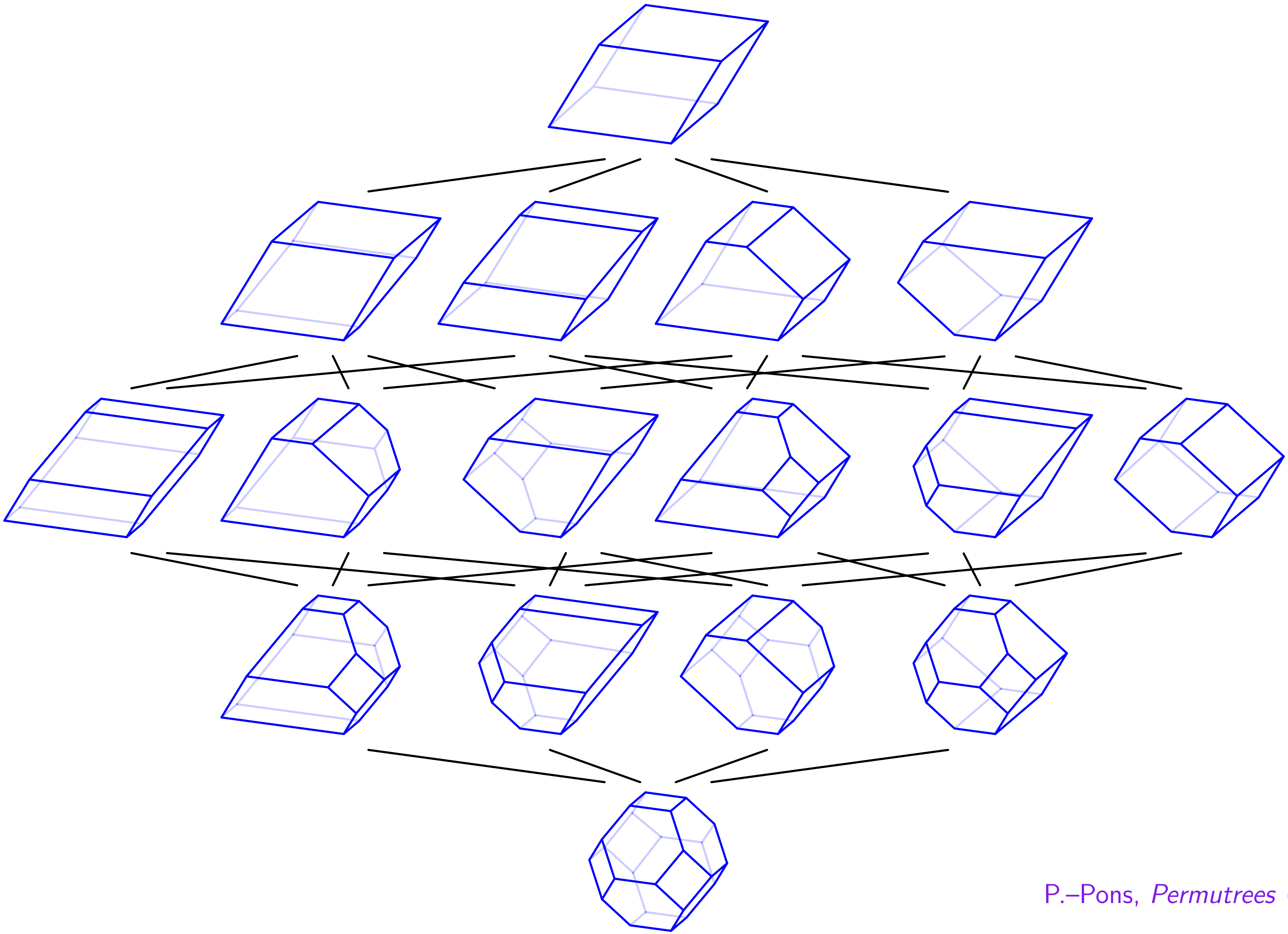
$$\boldsymbol{L}(\mathrm{T}) = \left[1 + d_i + \underline{\ell}_i \underline{r}_i - \overline{\ell}_i \overline{r}_i\right]_{i \in [n+1]} \qquad \boldsymbol{H}^{\geq}(I) = \left\{\boldsymbol{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \in I} x_i \geq \binom{|I|+1}{2}\right\}$$

P.–Pons, *Permutrees* ('18)



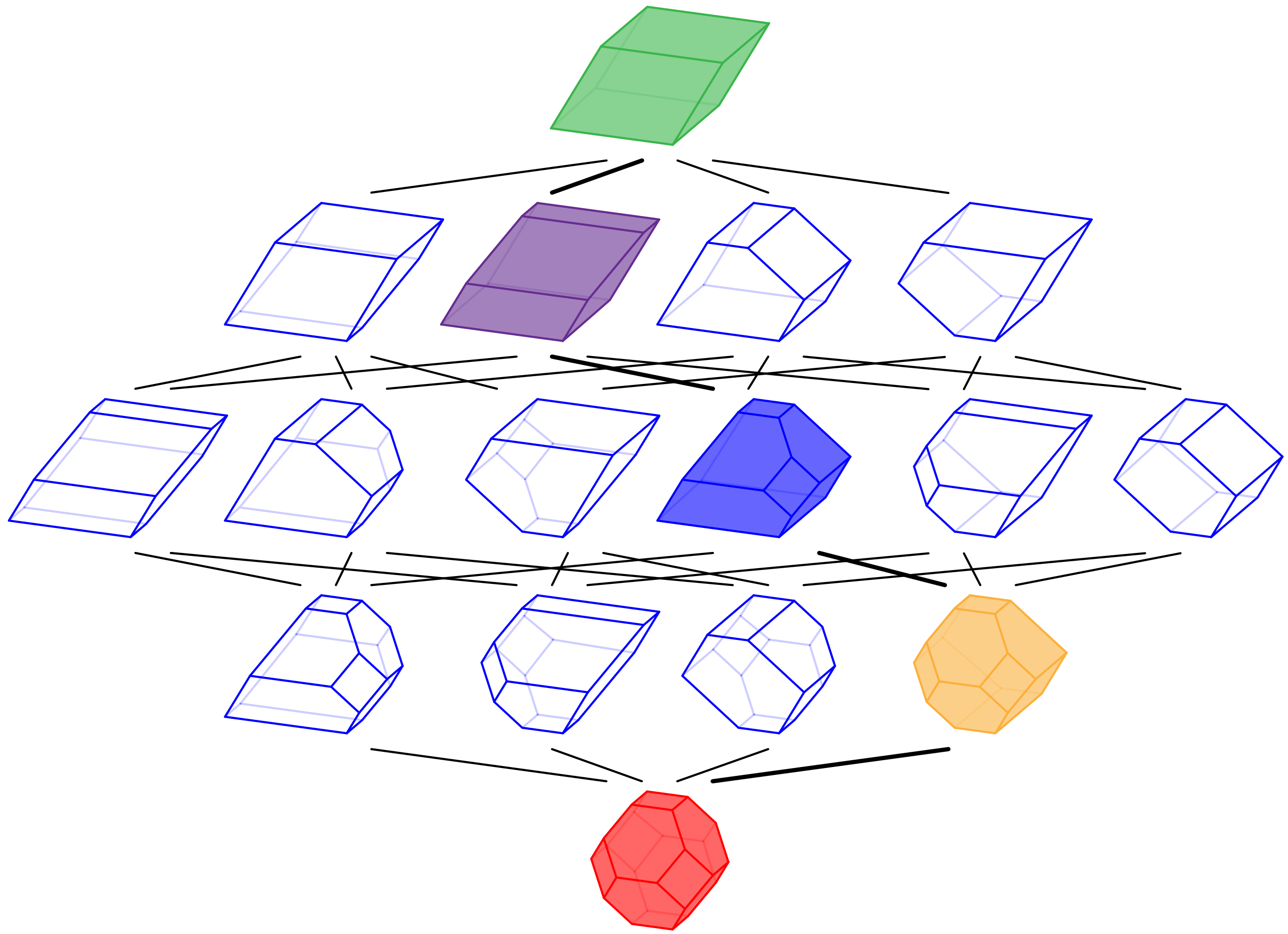


EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA



P.–Pons, *Permutrees* ('18)

EXM 5: PERMUTREE LATTICES AND PERMUTREEHEDRA



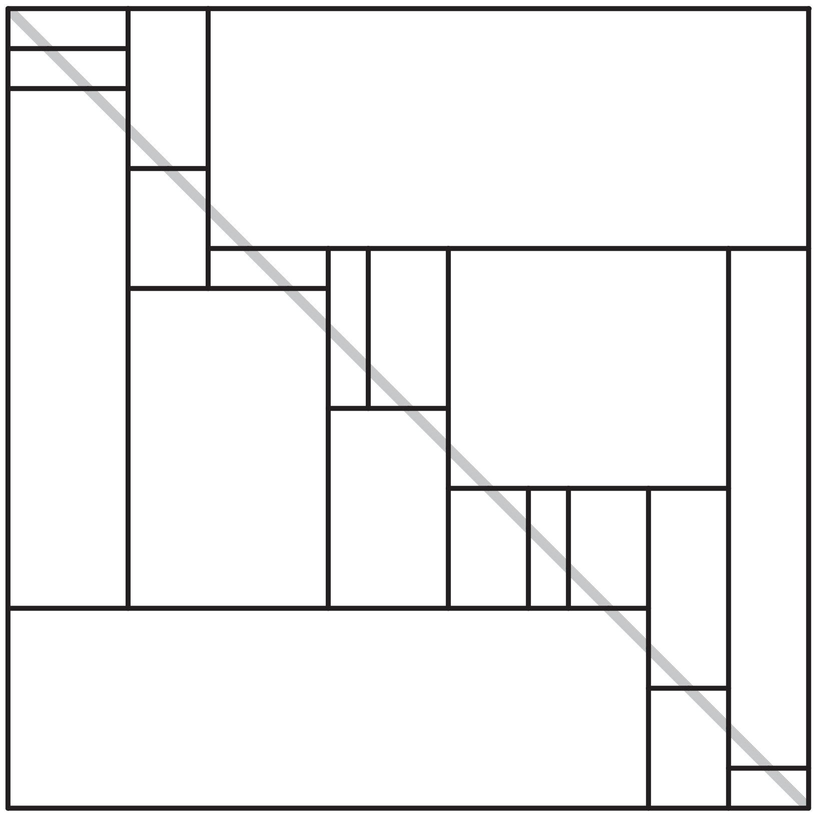
POLYWOOD

EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

Twin binary trees = pair of binary trees with opposite canopy
= (S, T) where S and T^{op} have a common linear extension

Giraudo, *Algebraic and combinatorial structures on pairs of twin binary trees* ('12)

in bijection with diagonal rectangulations



Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)

Baxter insertion = insert σ in a binary tree and σ^{op} in another binary tree

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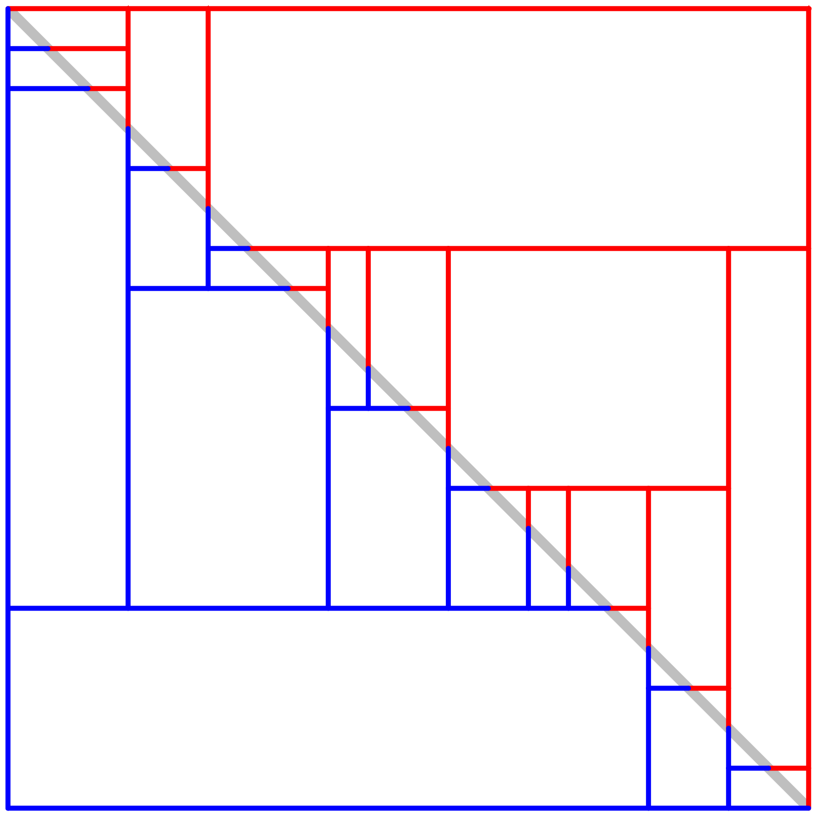
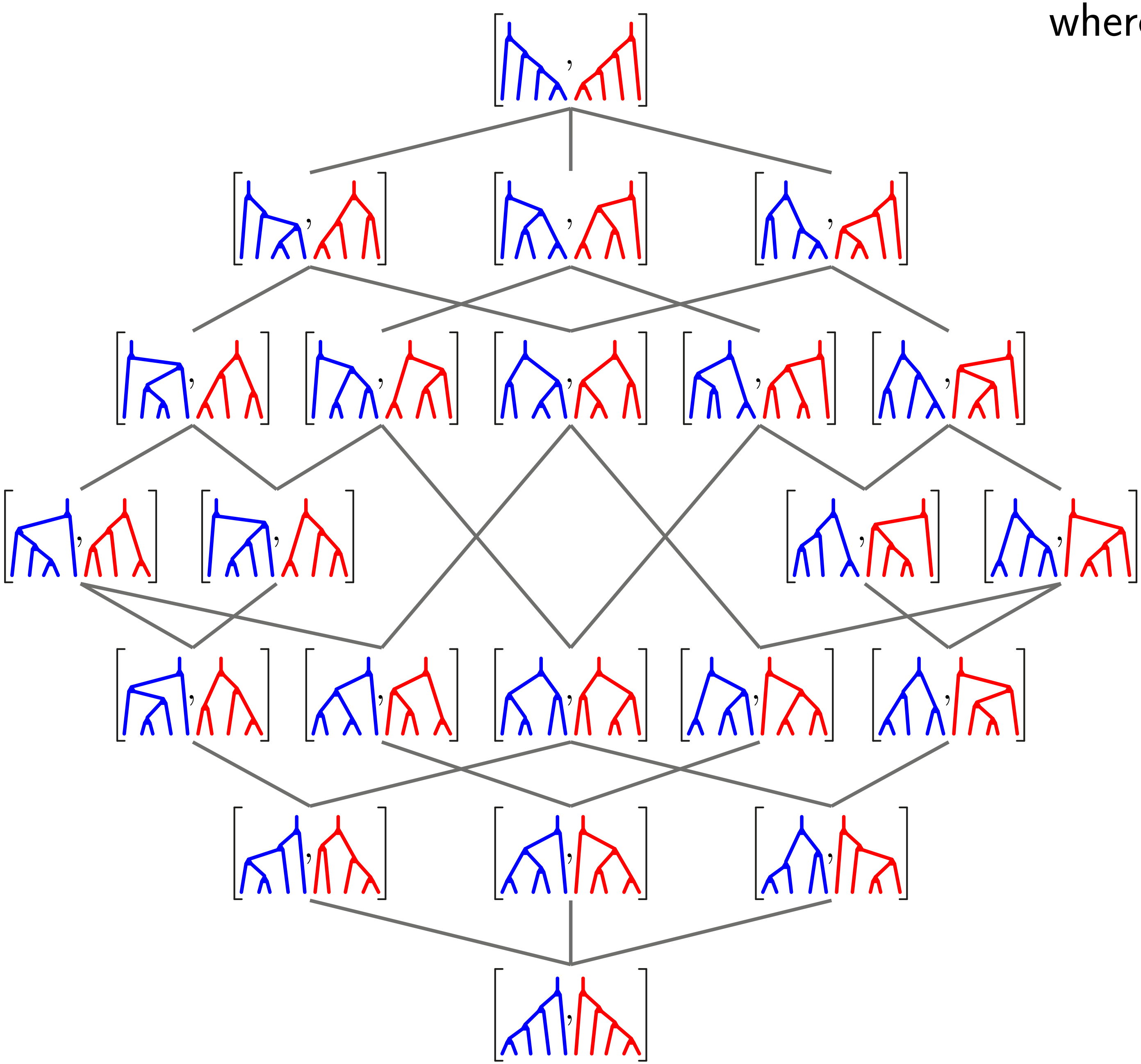


Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)

Baxter insertion = insert σ in a binary tree and σ^{op} in another binary tree

EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

Baxter congruence = "same twin binary trees" = tr. cl. of $UbVacWb'X \equiv UbVcaWb'X$ where $a < b, b' < c$



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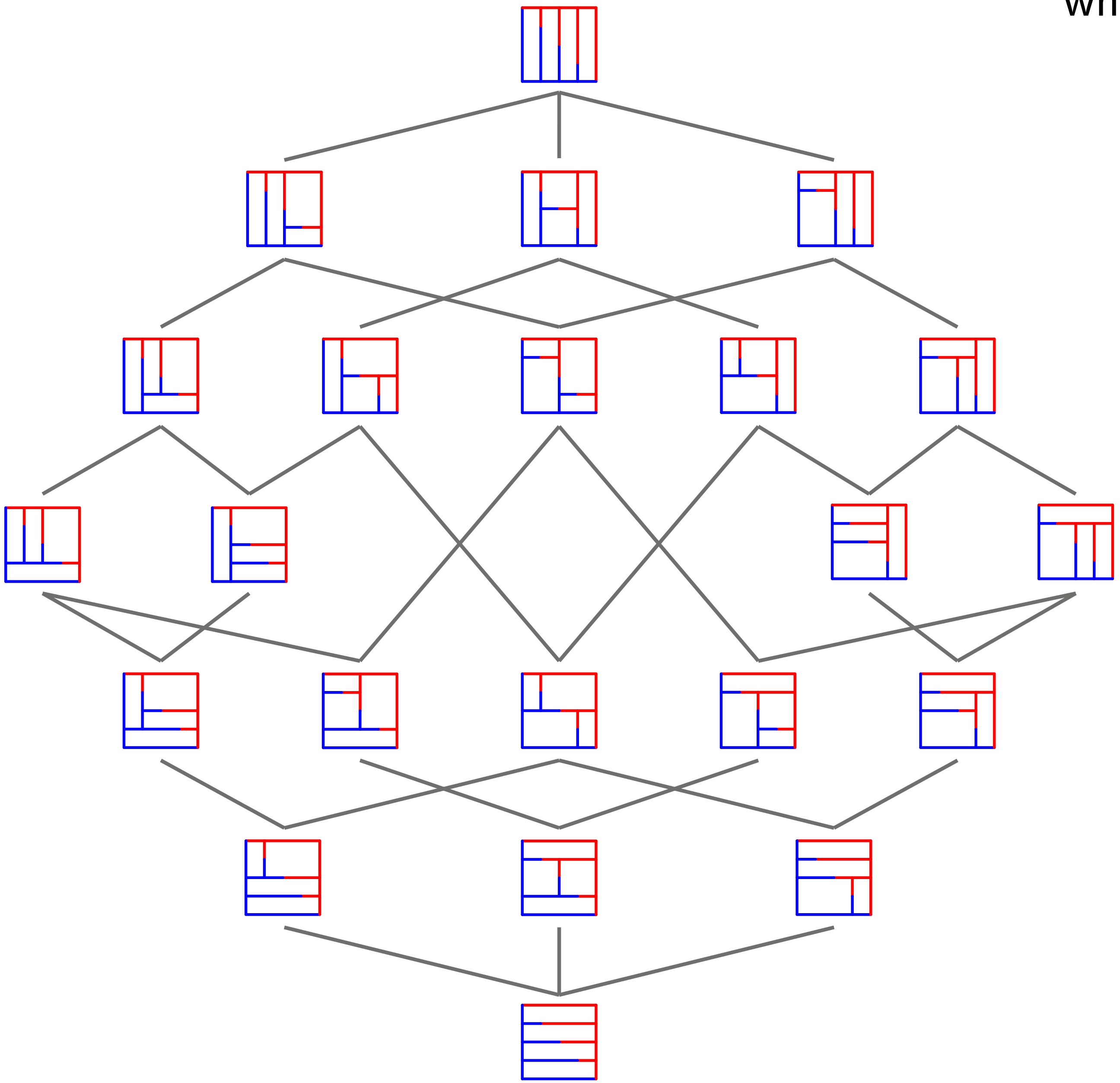
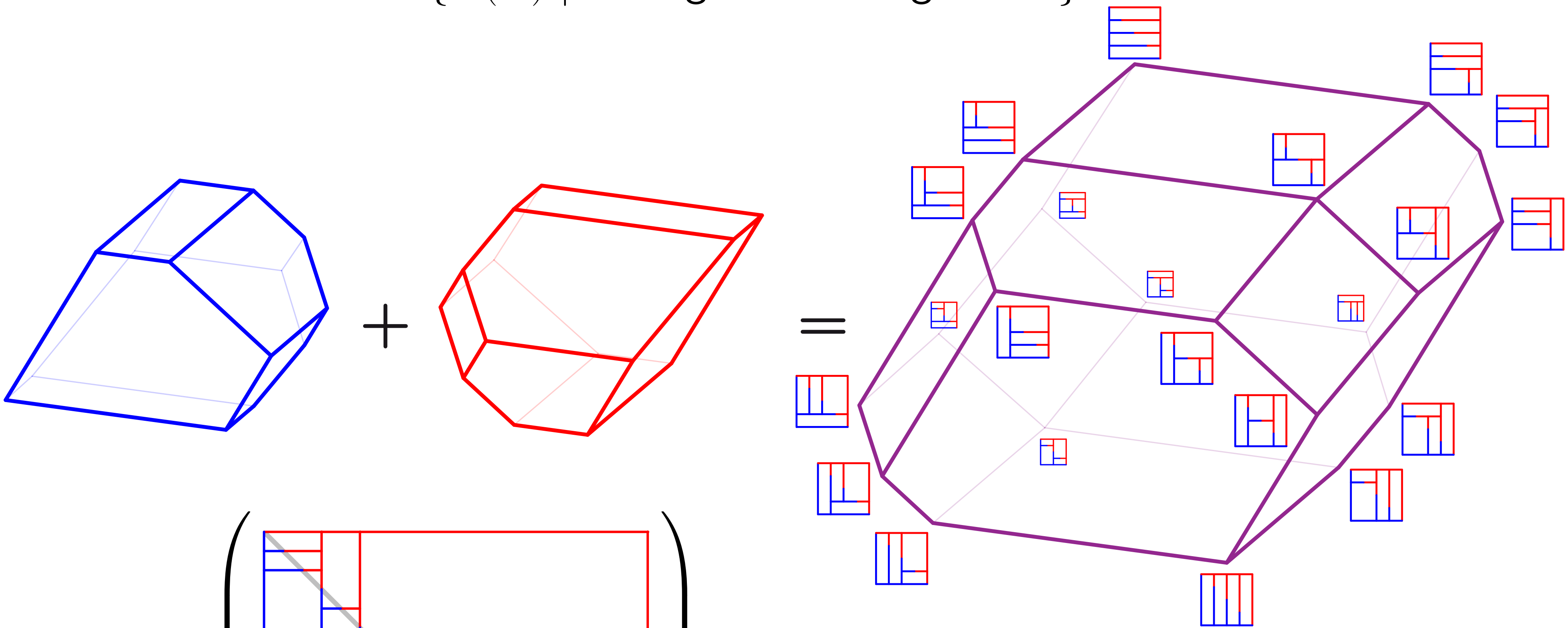
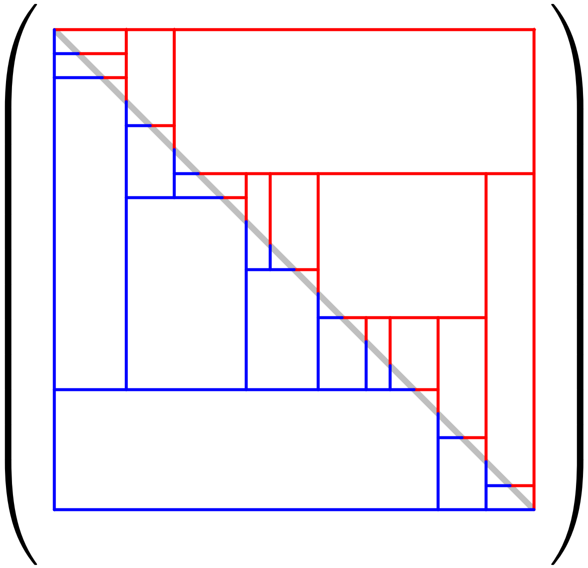


EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

Baxter polytope $=$ Minkowski sum of two opposite associahedra
$= \operatorname{conv}\{\boldsymbol{L}(R) \mid R \text{ diagonal rectangulation}\}$





$$\boldsymbol{L}\left(\quad\right) = \sum_{i\in[n]} (v_T^i \cdot h_T^i - v_S^i \cdot h_S^i) \cdot \boldsymbol{e}_i$$

Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)
Cardinal–P., *Rectangulotopes* ('24$^+$)

EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS

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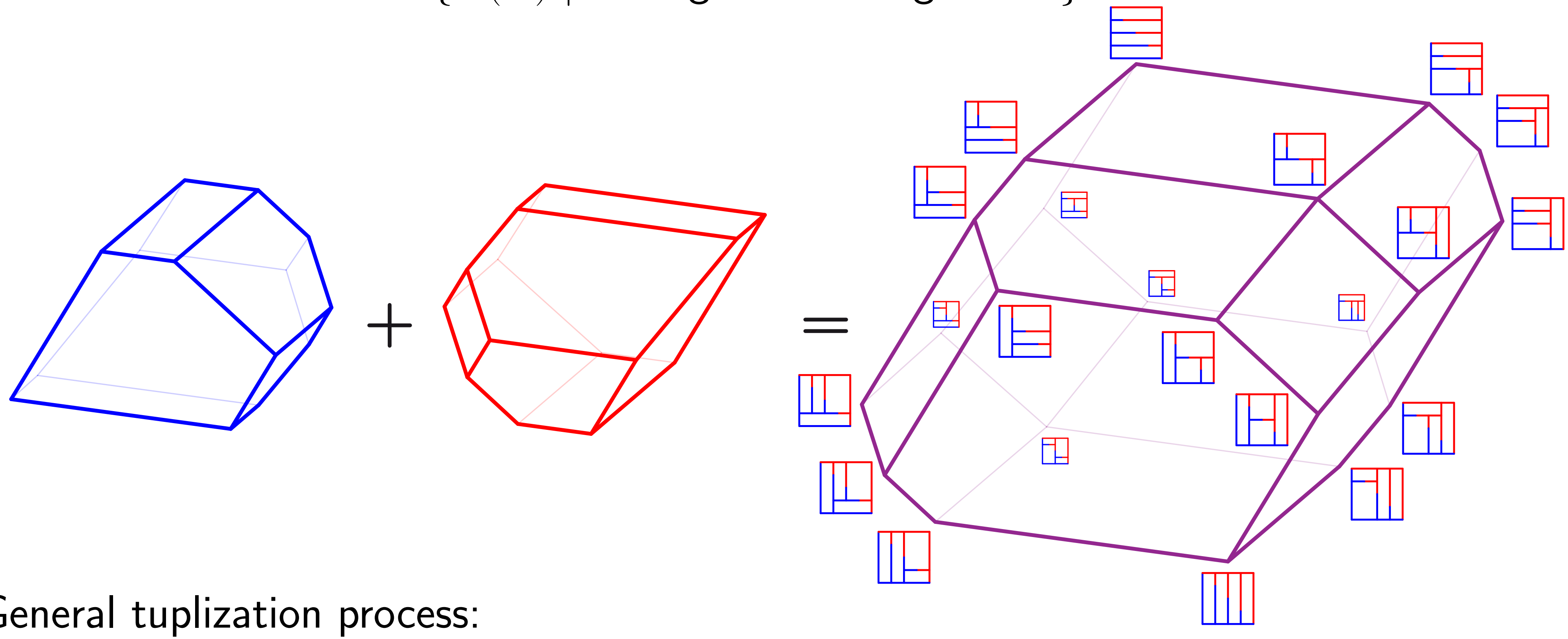
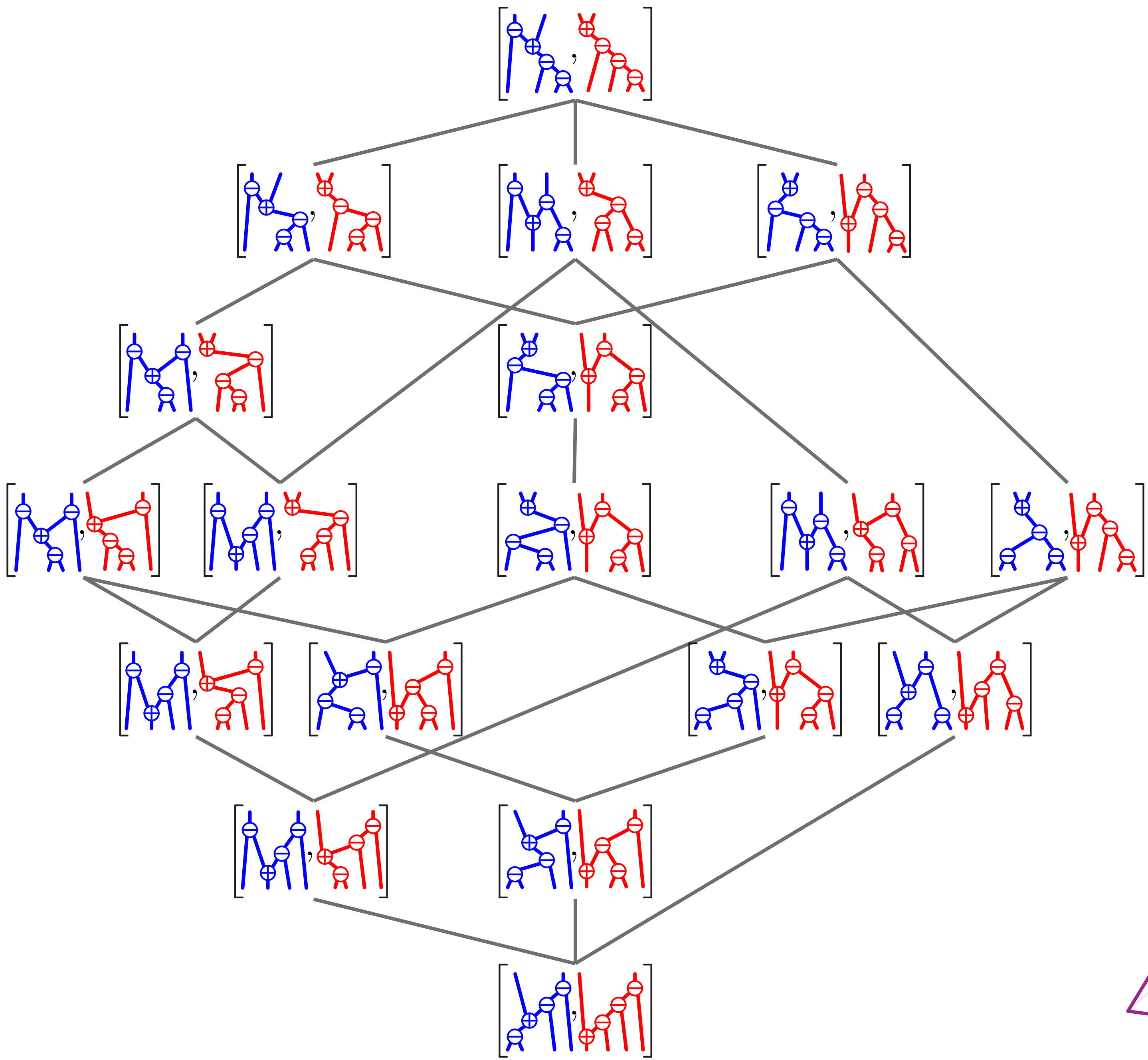


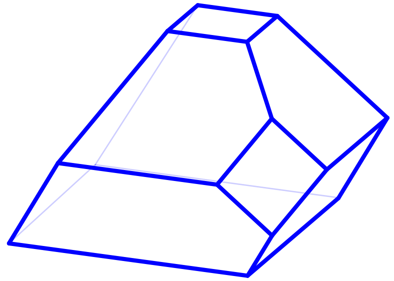
General tuplization process:

- tuples of objects representing classes
- intersection of lattice congruences
- Minkowski sum of polytopes

Law–Reading, *The Hopf algebra of diagonal rectangulations* ('12)
Cardinal–P., *Rectangulotopes* ('24^{+})

EXM 6: TWIN BINARY TREES AND DIAGONAL RECTANGULATIONS





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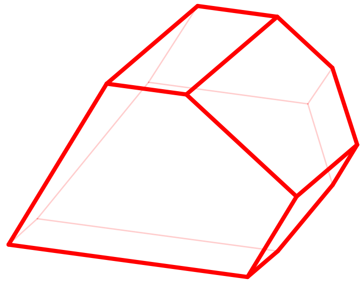

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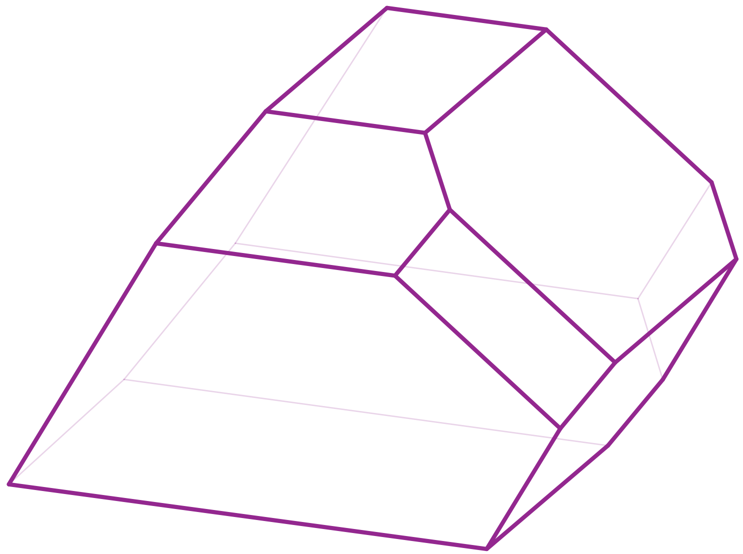
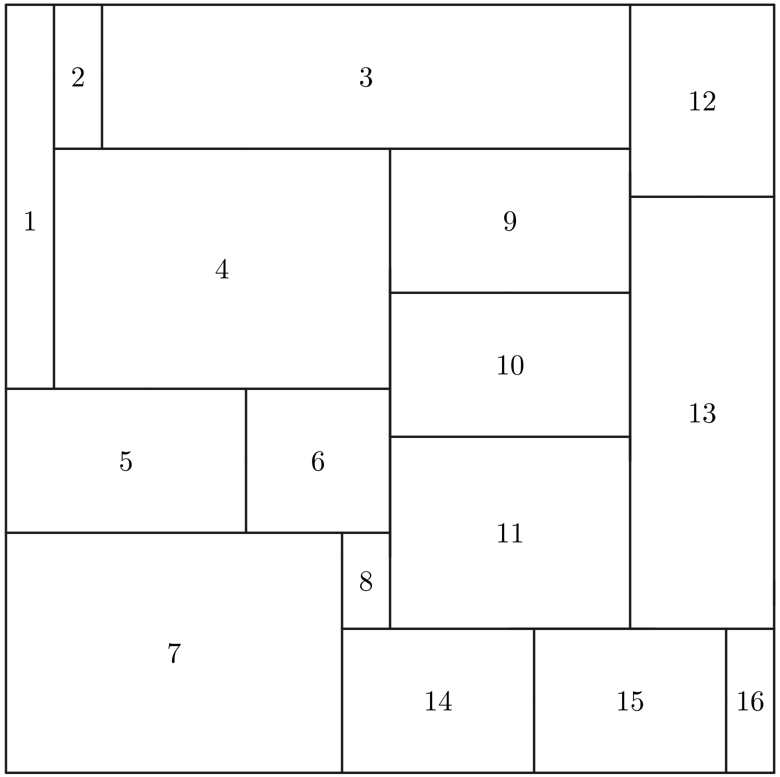


EXM 7: GENERIC RECTANGULATIONS

generic rectangulations = rectangulations of the square / wall slides



generic rectangulation congruence = tr. cl. of

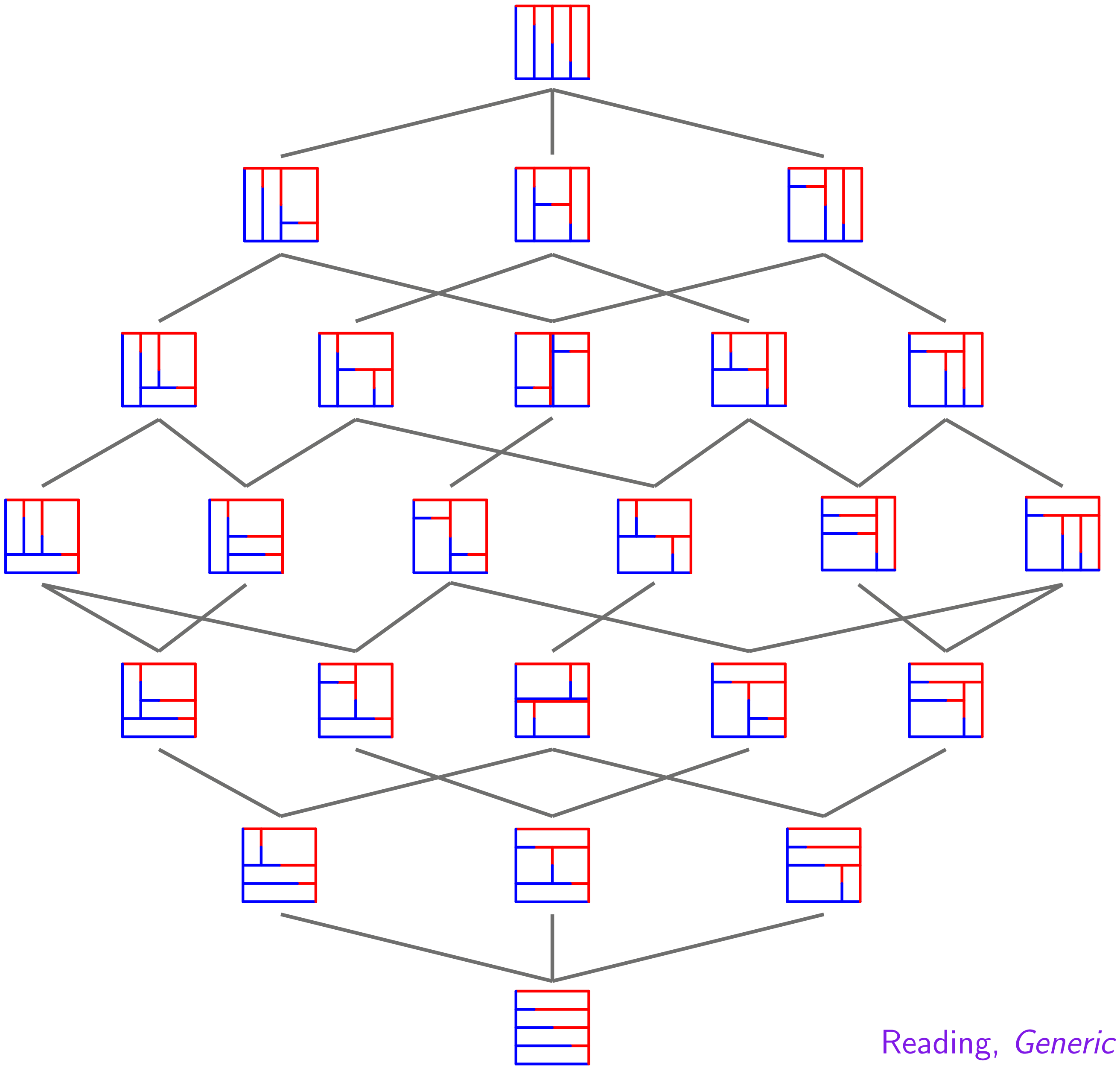
$$UbVdWaeXcY \equiv UbVdWeaXcY$$
$$UdVbWaeXcY \equiv UdVbWeaXcY$$
$$UcVaeWbXdY \equiv UcVeaWbXdY$$
$$UcVaeWdXbY \equiv UcVeaWdXbY$$

where $a < b < c < d < e$

Reading, *Generic rectangulations* ('12)

EXM 7: GENERIC RECTANGULATIONS

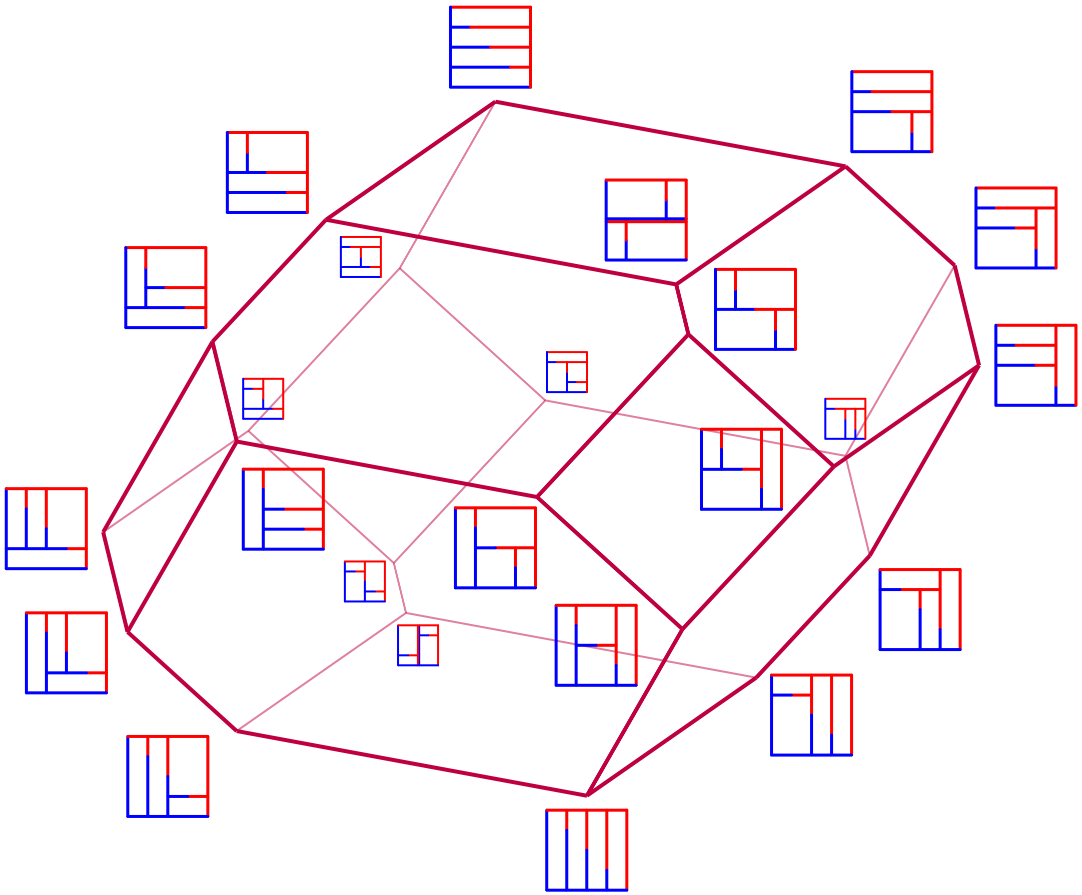
generic rectangulations = rectangulations of the square / wall slides

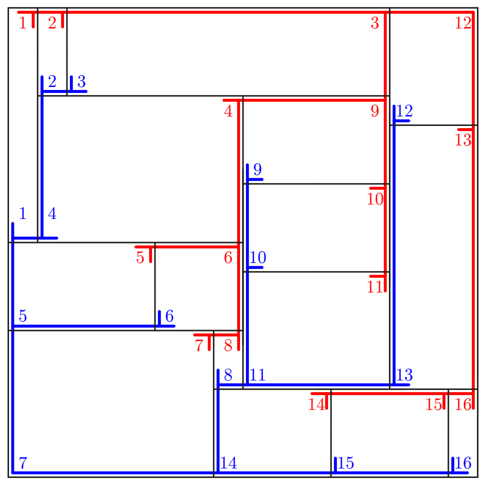


Reading, *Generic rectangulations* ('12)

EXM 7: GENERIC RECTANGULATIONS

generic rectangulations polytope $= \operatorname{conv}\{\boldsymbol{L}(R) \mid R \text{ generic rectangulation}\}$



$\boldsymbol{L}\Bigg($  $\Bigg)$

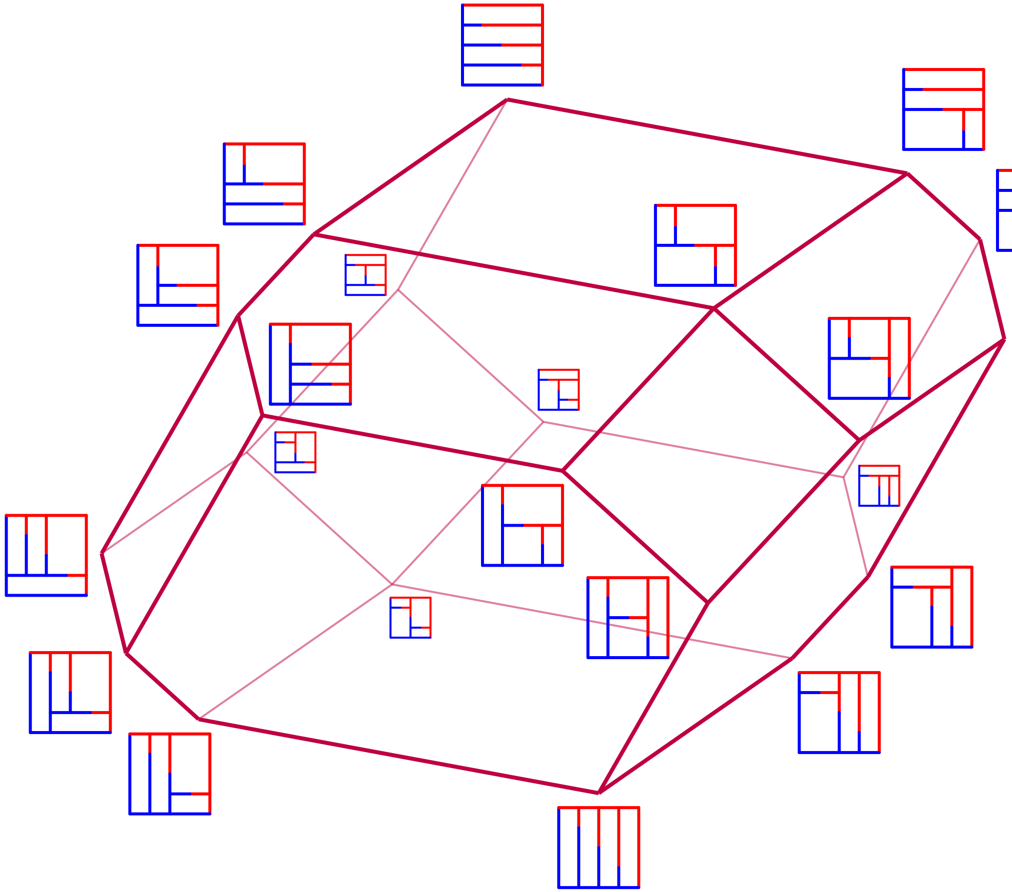
$$= \sum_{\substack{i<j \\ i \prec_R j}} v_S^i \cdot \left(H_S^i \cap H_T^j\right) \cdot v_T^j \cdot (\boldsymbol{e}_i - \boldsymbol{e}_j)$$

$$+ \sum_{\substack{i<j \\ i \succ_R j}} h_S^j \cdot \left(V_S^j \cap V_T^i\right) \cdot h_T^i \cdot (\boldsymbol{e}_j - \boldsymbol{e}_i)$$

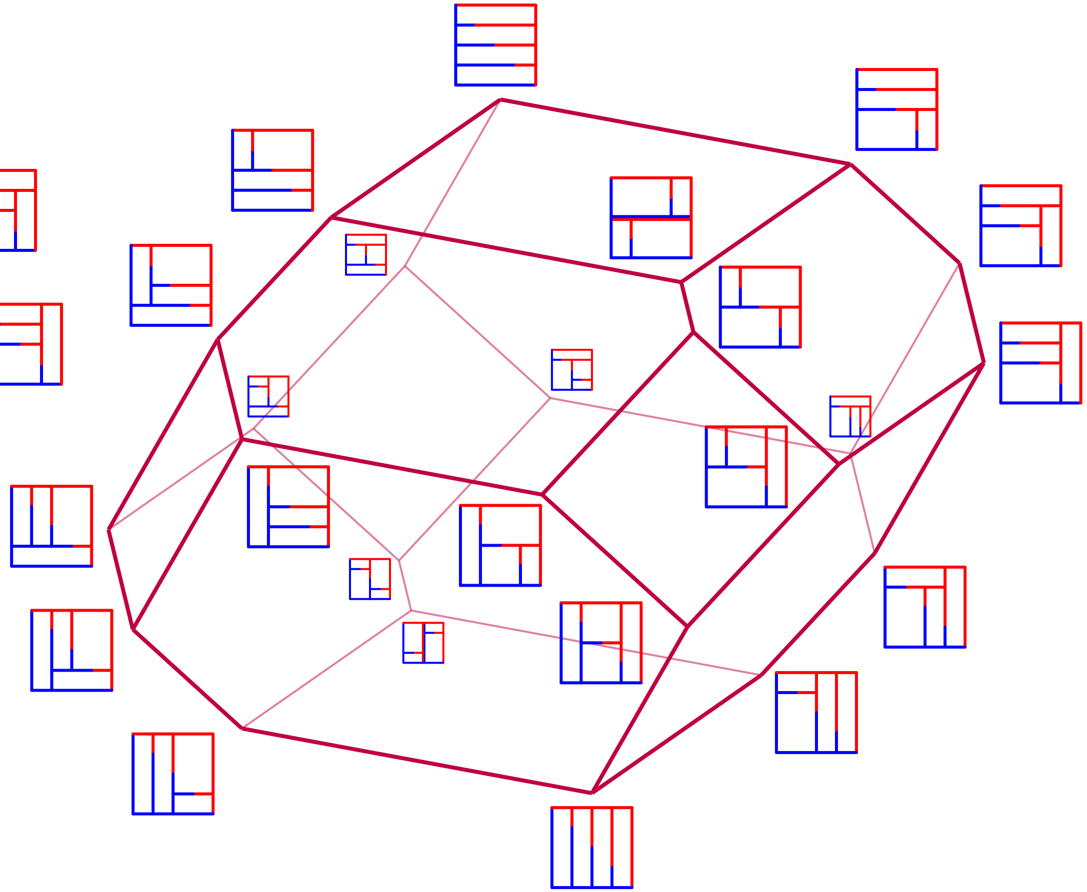
Cardinal–P., *Rectangulotopes* ('24^{+})

DEIAGONAL RECTANGULATIONS VS. GENERIC RECTANGULATIONS

diagonal rectangulations
twisted / cotwisted Baxter permutations
up and down arcs



generic rectangulations
2-clumped / 2-coclumped permutations
up, down, yin and yang arcs



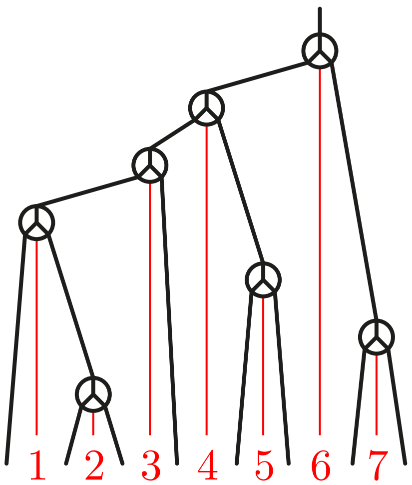
Cardinal–P., *Rectangulotopes* ('24$^+$)

SUMMARY OF CONGRUENCE ZOO

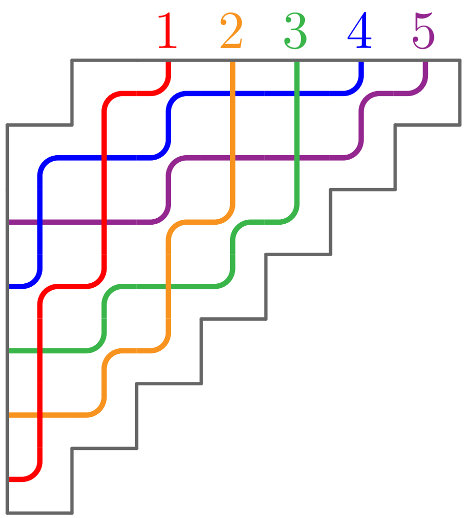
lattice congruence $=$ equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \quad \Longrightarrow \quad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

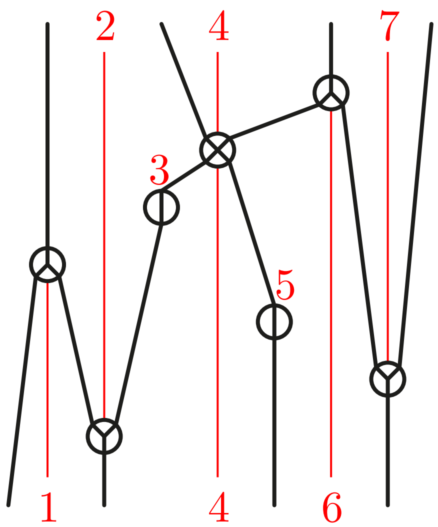
sylvester congruence	multiplization	Cambrianization	tuplization
$\cdots ac \cdots b \cdots$	$\cdots ac \cdots b_1 \cdots b_k \cdots$	left if $b \in \{$Ⓨ$, \otimes\}$	intersection
$\equiv \cdots ca \cdots b \cdots$	$\equiv \cdots ca \cdots b_1 \cdots b_k \cdots$	right if $b \in \{$☮$, \otimes\}$	of congruences
if $a < b < c$	if $a < b_i < c$		



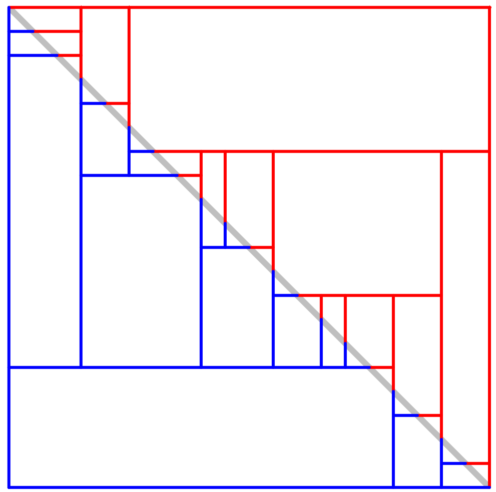
binary trees



multitriangulations



permutrees



diagonal rectangulations

LATTICE THEORY OF THE WEAK ORDER

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

CANONICAL JOIN REPRESENTATIONS

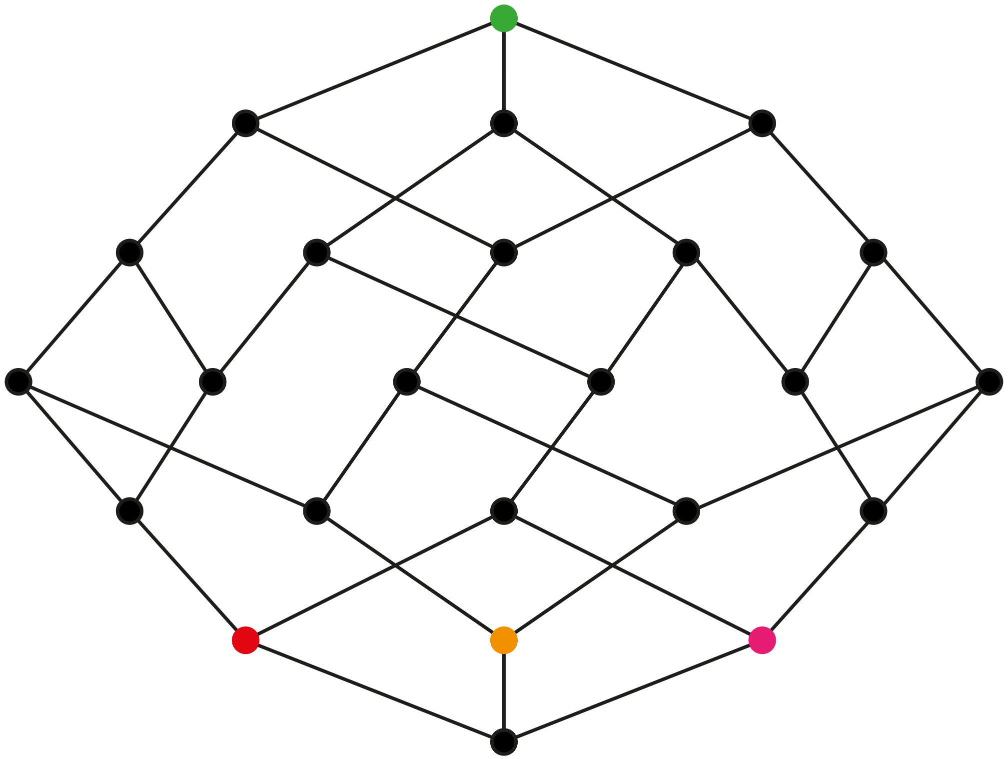
join representation of $y \in L$ = subset $J \subseteq L$ such that $y = \bigvee J$

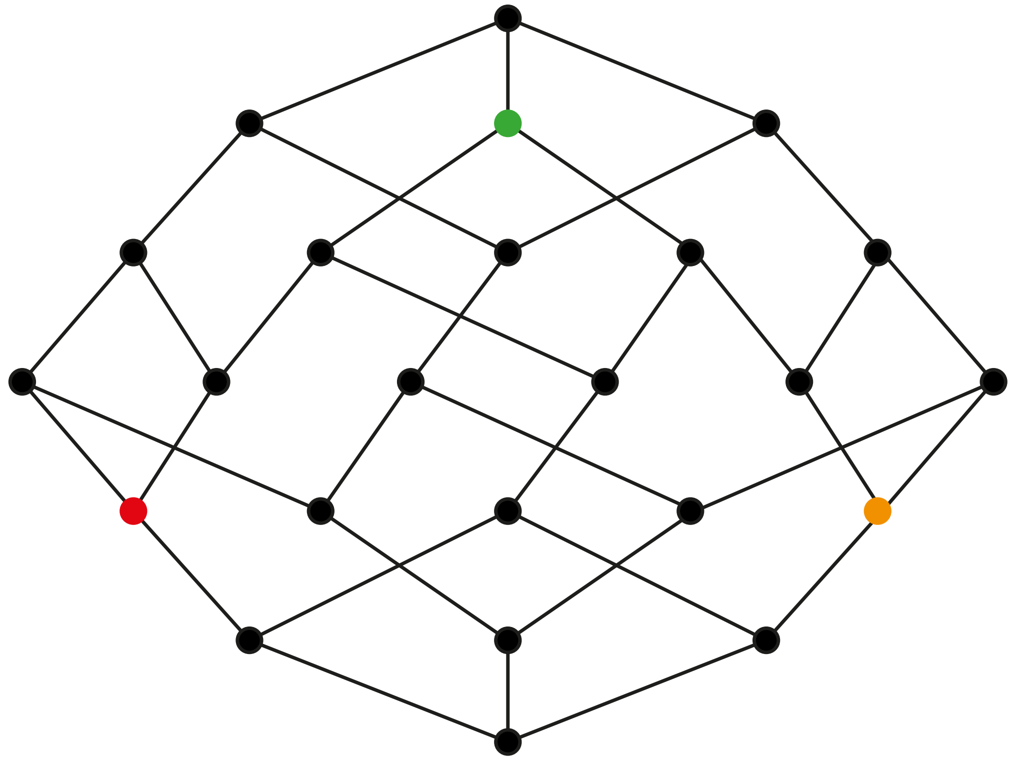
$y = \bigvee J$ irredundant if $\nexists\, J' \subsetneq J$ with $y = \bigvee J'$

ordered by containement of order ideals: $J \le J' \iff \forall\, z \in J,\ \exists\, z' \in J',\ z \le z'$

canonical join representation of y = minimal irredundant join representation of y

= lowest way to write y as a join





$\Longrightarrow$ a canonical join representation is an antichain of join irreducible elements of L

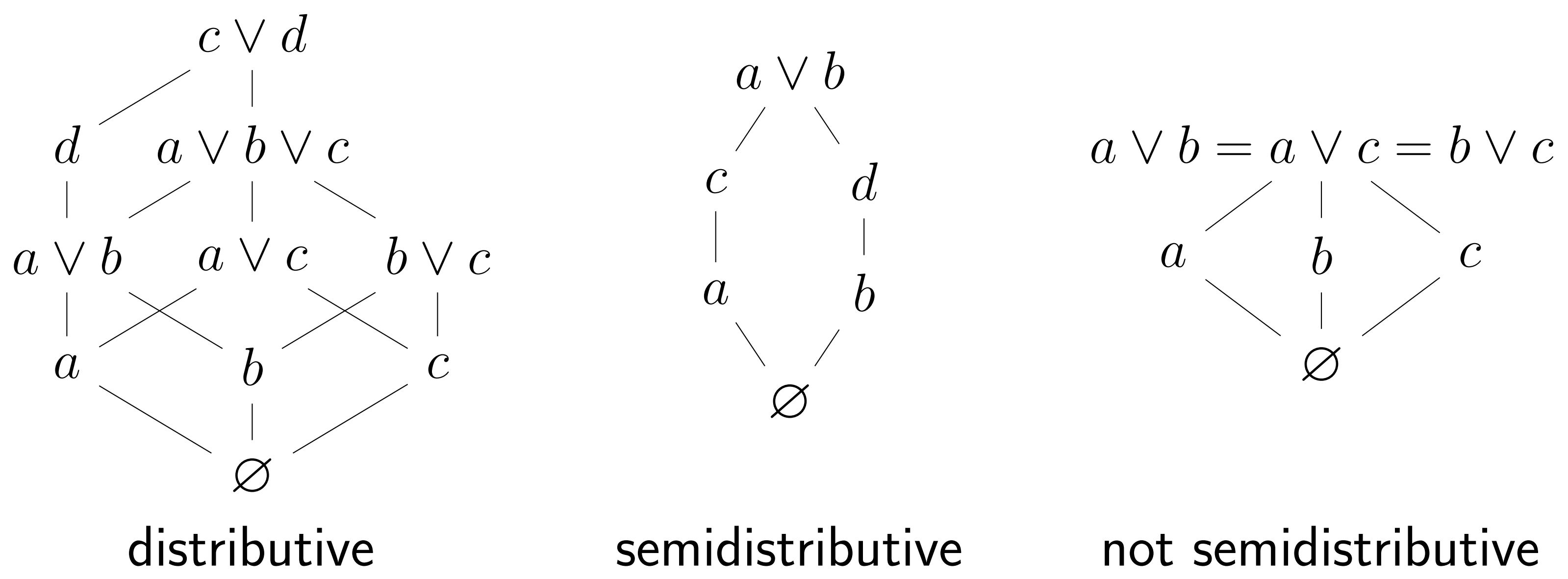
DISTRIBUTIVE & SEMIDISTRIBUTIVE LATTICES

$(L, \leq, \wedge, \vee)$ finite lattice is

- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

- semidistributive if both join and meet semidistributive



DISTRIBUTIVE & SEMIDISTRIBUTIVE LATTICES

$(L, \leq, \wedge, \vee)$ finite lattice is

- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

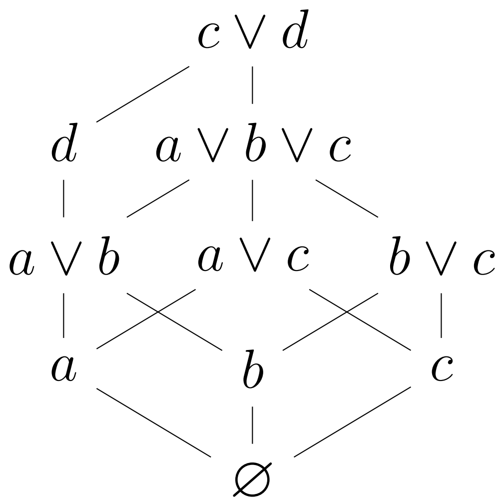
  $\Longrightarrow$ canonical join representations $=$ antichains of join irreducibles

  $\Longrightarrow$ $L \simeq$ inclusion poset of lower ideals of $JI(L)$

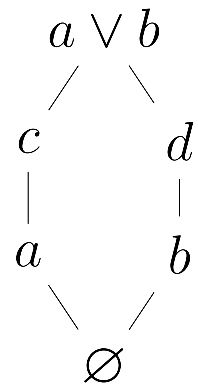
- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ admits the canonical join representation $y = \bigvee_{x \lessdot y} k_\vee(x, y)$ where $k_\vee(x, y)$ is the unique minimal element of $\{z \in L \mid x \vee z = y\}$

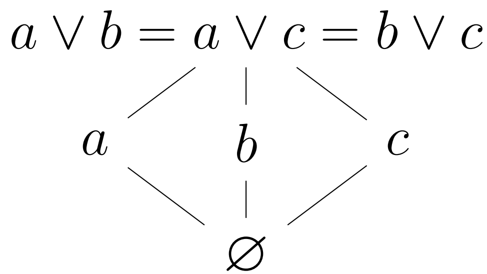
- semidistributive if both join and meet semidistributive



distributive



semidistributive

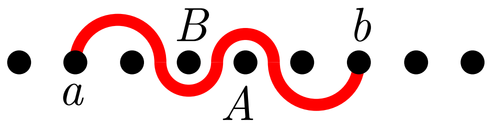


not semidistributive

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

draw all points (σ_i, i) and all segments from (σ_i, i) to $(\sigma_{i+1}, i+1)$ with $\sigma_i > \sigma_{i+1}$ and project down to an horizontal line allowing arcs to bend but not to cross or pass points

arc $=$
(a, b, A, B) with $1 \le a < b \le n$ and $A \sqcup B =]a, b[$

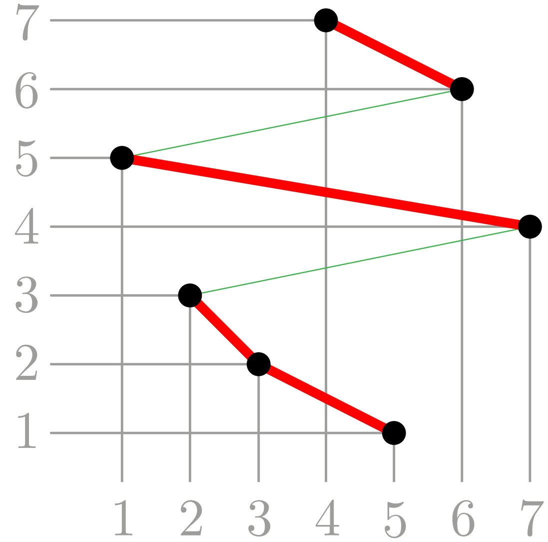


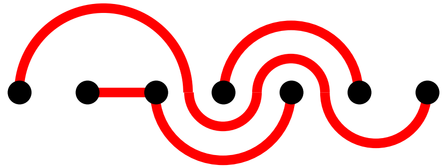
crossing arcs $=$
(a, b, A, B) and (a', b', A', B') such that there is $x \neq x'$ with

$$x \in (A \cup \{a, b\}) \cap (B' \cup \{a', b'\})$$
$$\text{and} \quad x' \in (B \cup \{a, b\}) \cap (A' \cup \{a', b'\})$$

noncrossing arc diagrams $=$
set of pairwise non-crossing arcs

permutation $\sigma = 5327164$

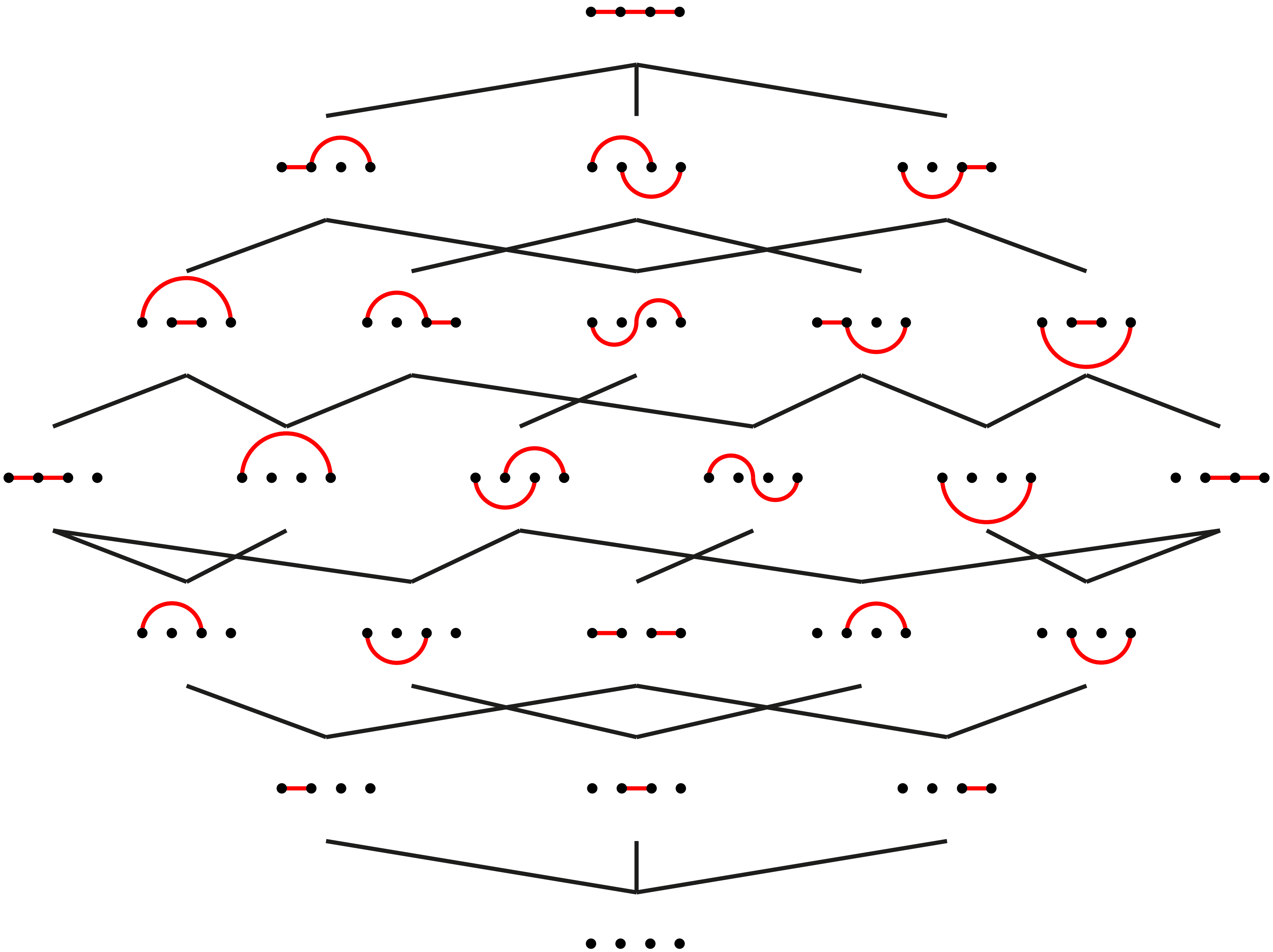




noncrossing arc diagram $\delta(\sigma)$

THM. δ is a bijection from permutations to noncrossing arc diagrams — Reading ('15)

CANONICAL JOIN REPRESENTATIONS & NONCROSSING ARC DIAGRAMS



THM. $\sigma = \bigvee_{\alpha \in \delta(\sigma)} \delta^{-1}(\{\alpha\})$ is the canonical join representation   Reading ('15)

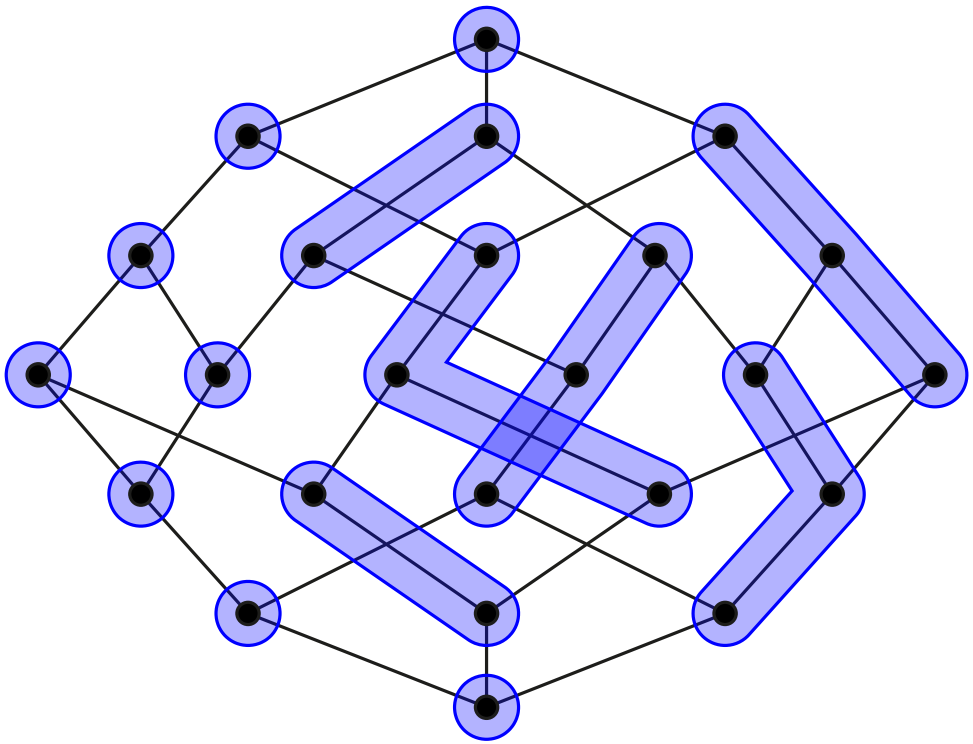
LATTICE CONGRUENCES & LATTICE QUOTIENTS

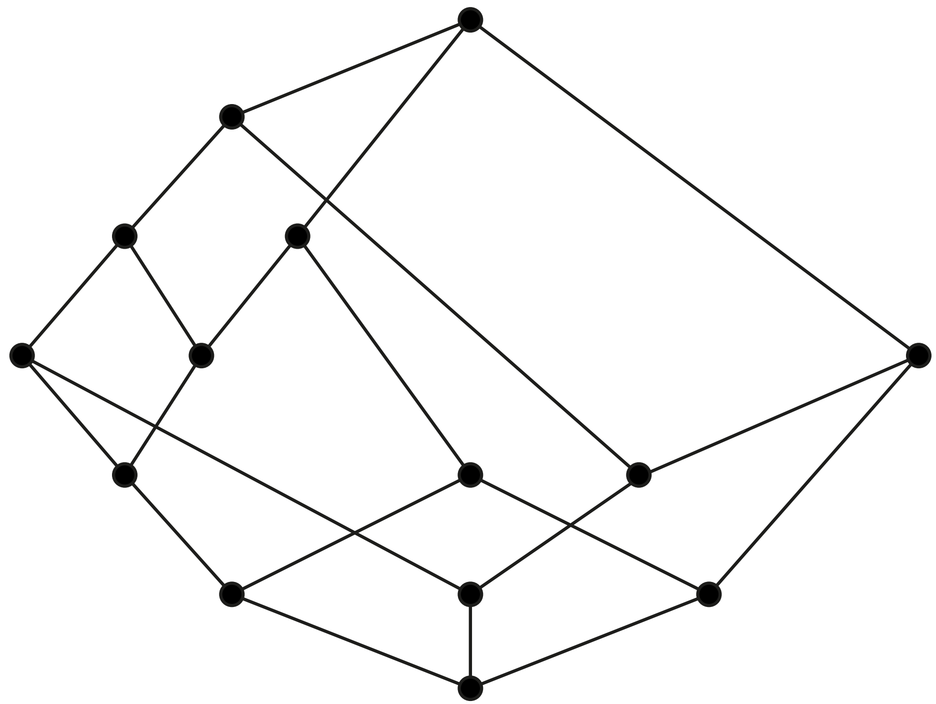
lattice congruence of L = equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \implies x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

lattice quotient of $L/\equiv$ = lattice on equivalence classes of L under $\equiv$ where

- $X \leq Y \iff \exists\, x \in X,\ y \in Y, \quad x \leq y$
- $X \wedge Y$ = equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ = equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$

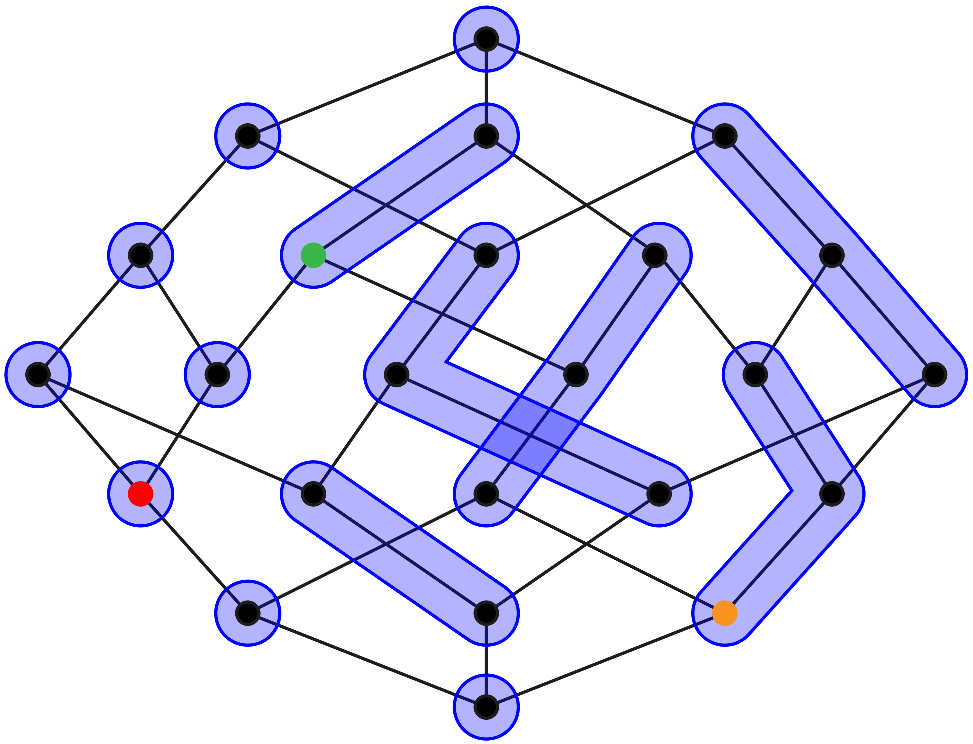


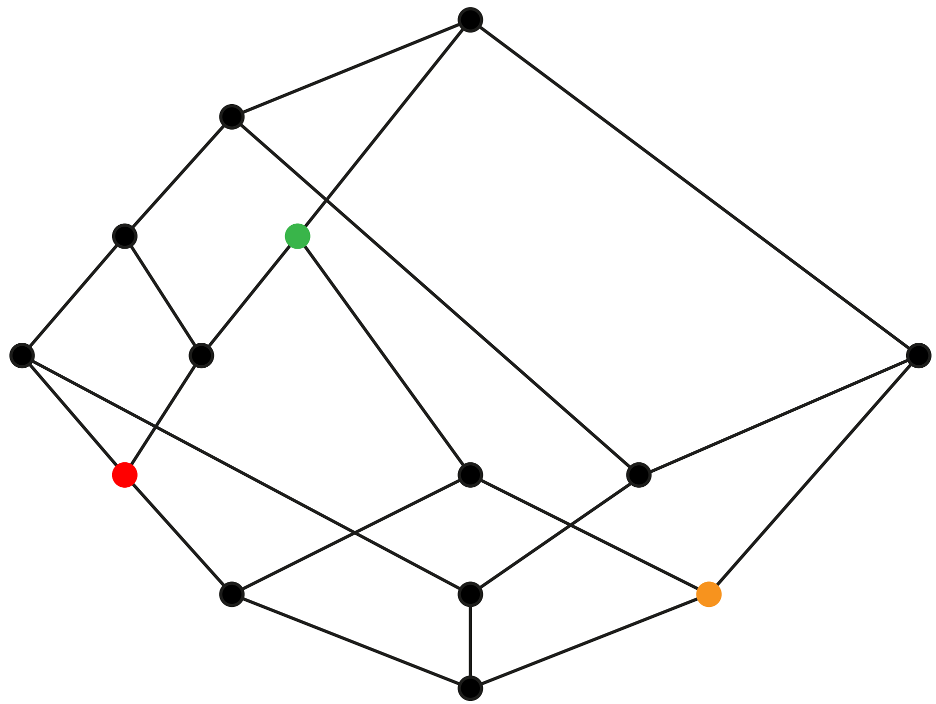


LATTICE QUOTIENTS & CANONICAL JOIN REPRESENTATIONS

$\equiv$ lattice congruence on L, then

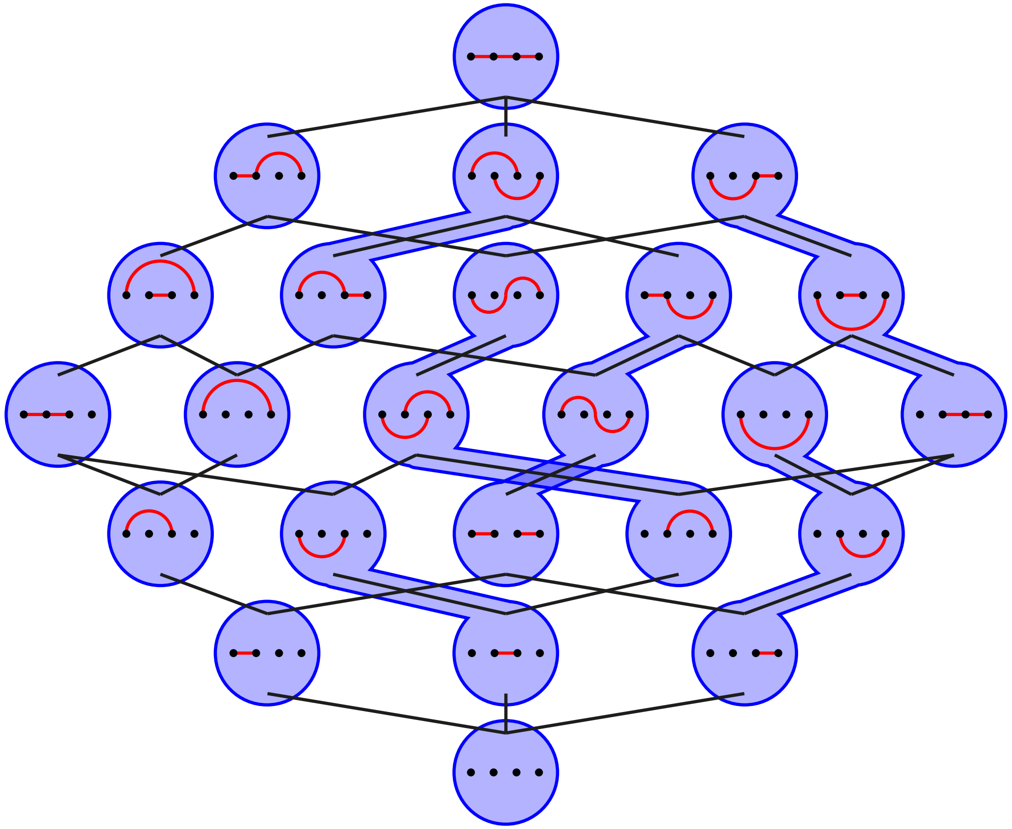
- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/\equiv$ is isomorphic (as poset) to the restriction of L to the elements x with $\pi_\downarrow(x) = x$
- $\pi_\downarrow(x) = x$ if and only if $\pi_\downarrow(j) = j$ for all canonical joinands j of x
- canonical join representations in $L/\equiv$ are canonical join representations in L that only involve join irreducibles j with $\pi_\downarrow(j) = j$

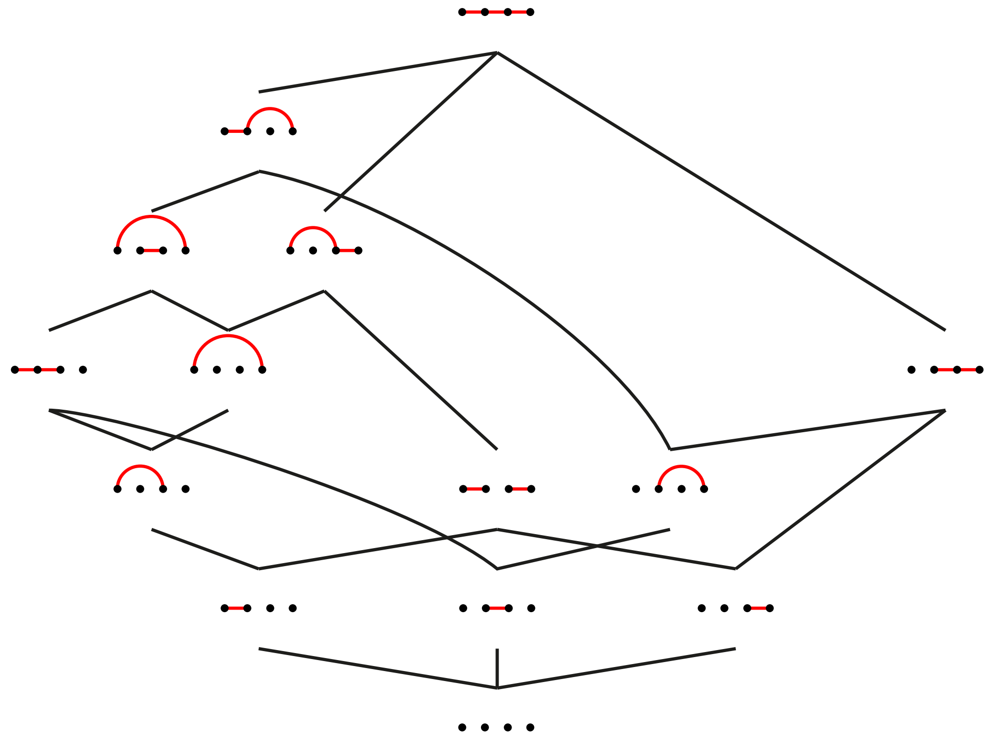




LATTICE QUOTIENTS OF THE WEAK ORDER

THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
$\mathcal{A}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$
$\mathfrak{S}_n/\equiv \; \simeq$ subposet induced by noncrossing arc diagrams with all arcs in $\mathcal{A}_\equiv$





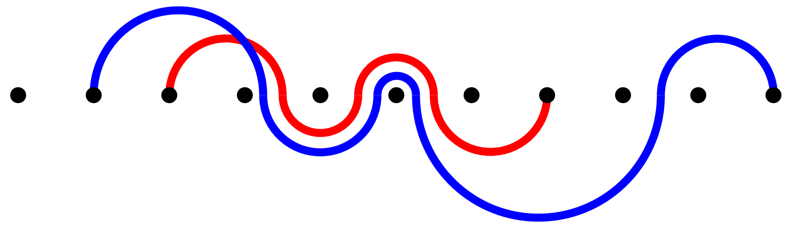
Reading ('15)

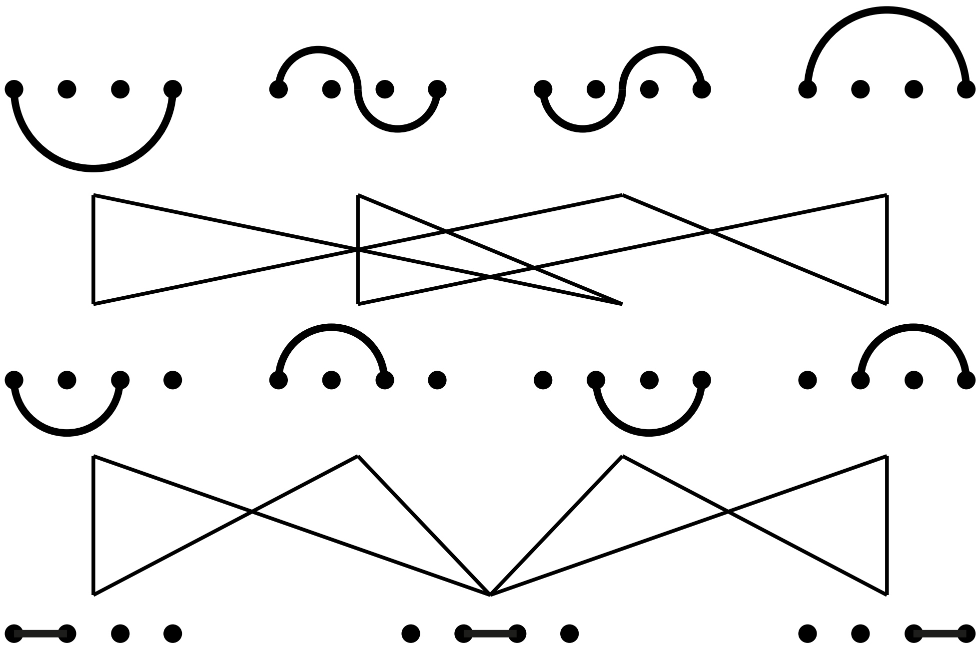
SUBARC ORDER

THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
$\mathcal{A}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$
$\mathfrak{S}_n/\equiv \; \simeq$ subposet induced by noncrossing arc diagrams with all arcs in $\mathcal{A}_\equiv$

THM. The following are equivalent for a set of arcs $\mathcal{A}$:

- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{A} = \mathcal{A}_\equiv$
- $\mathcal{A}$ is a lower ideal of the subarc order

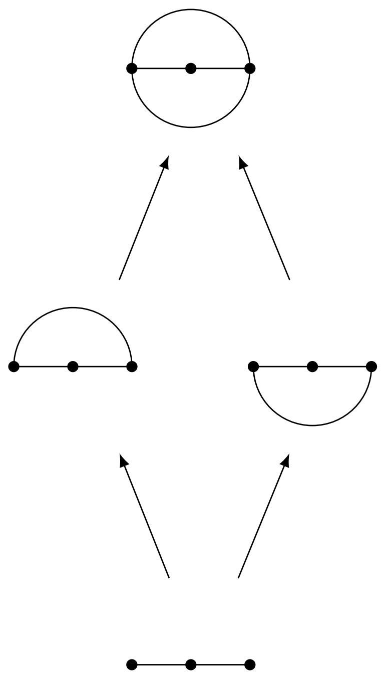




Reading ('15)

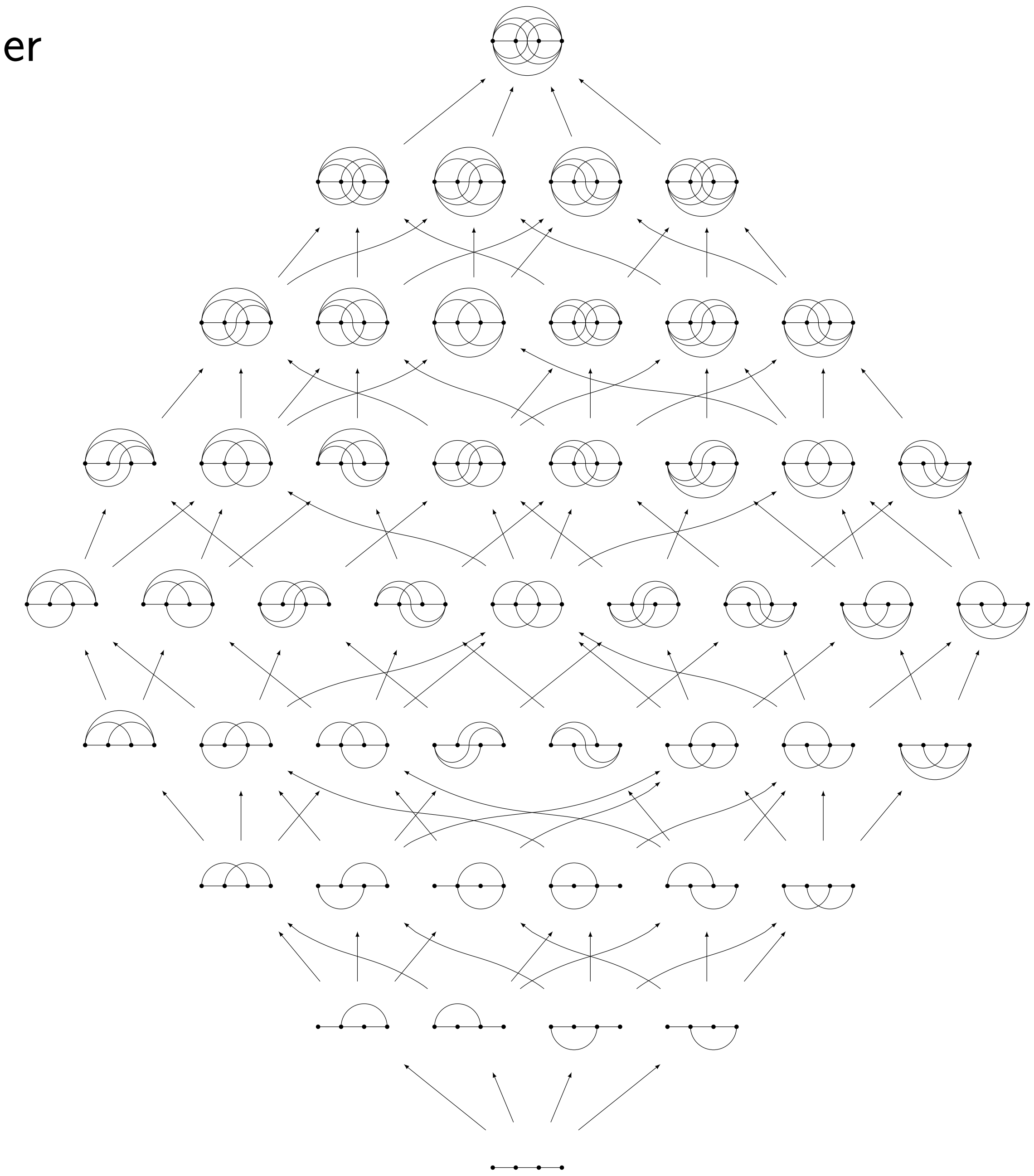
ARC IDEALS

arc ideal = lower ideal of the subarc order



essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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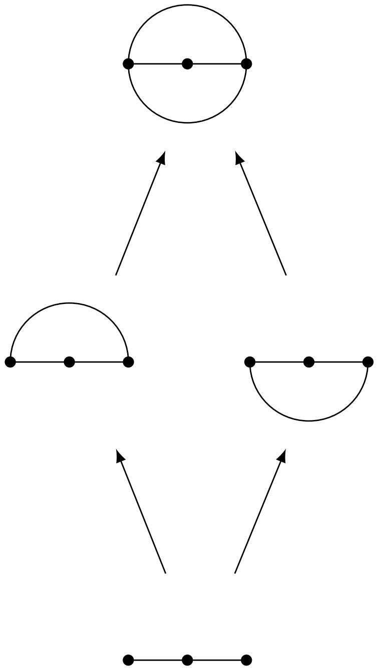

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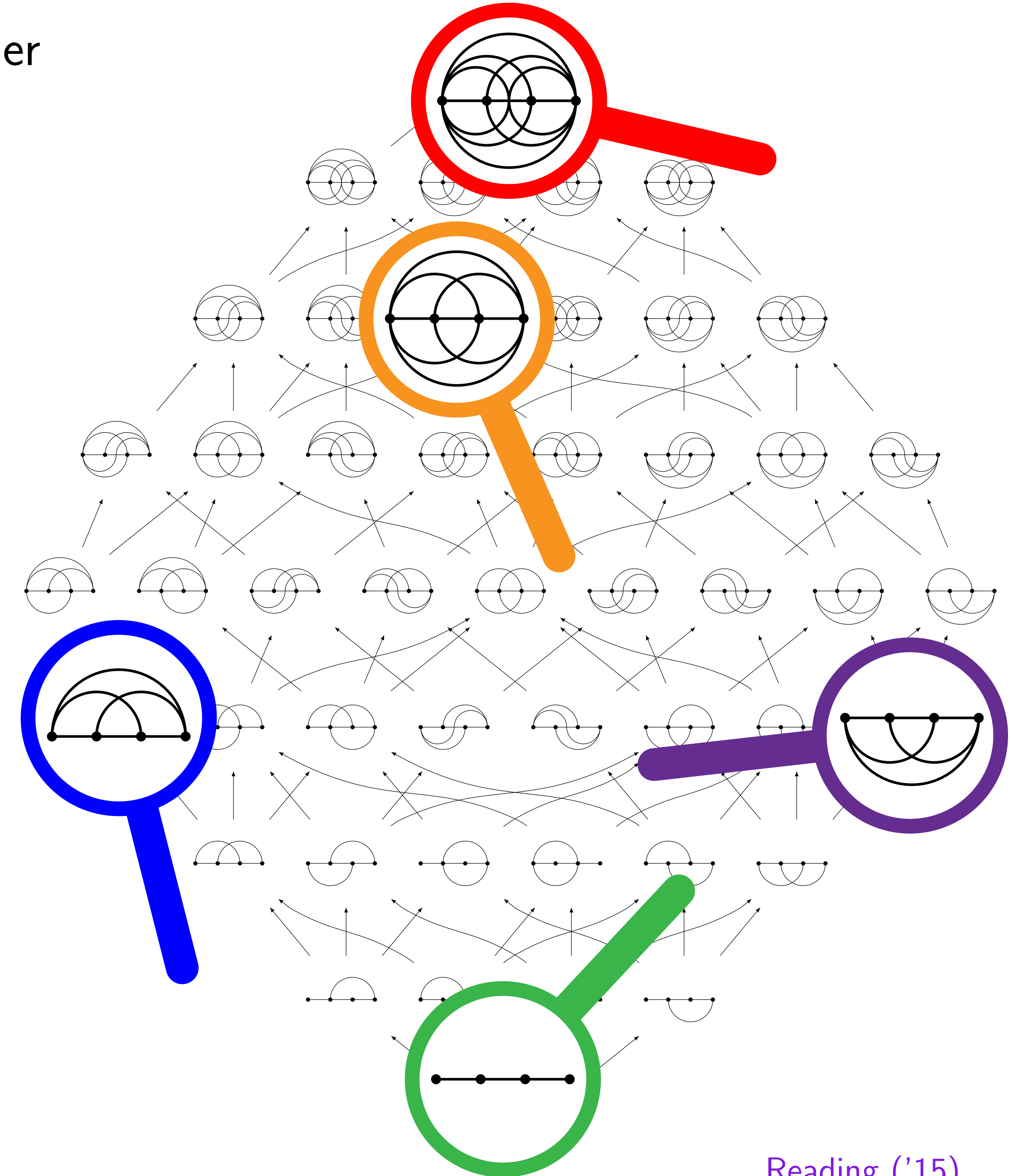


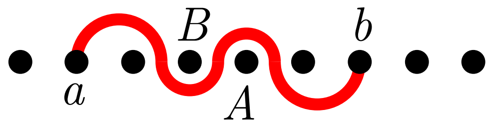
Reading ('15)

QUOTIENT FANS

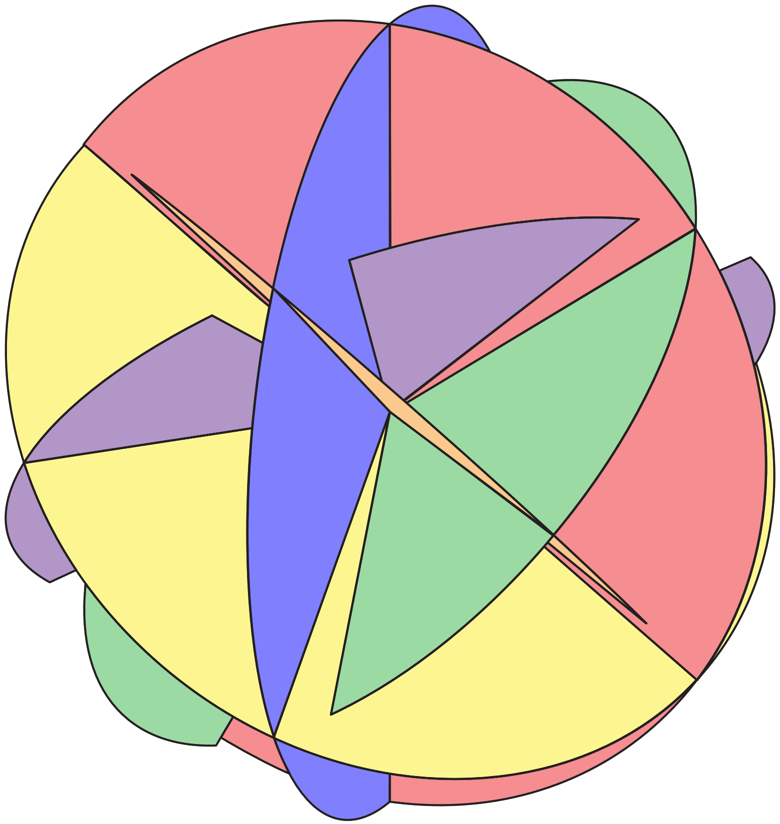
Reading, *Lattice congruences, fans and Hopf algebras* ('05)

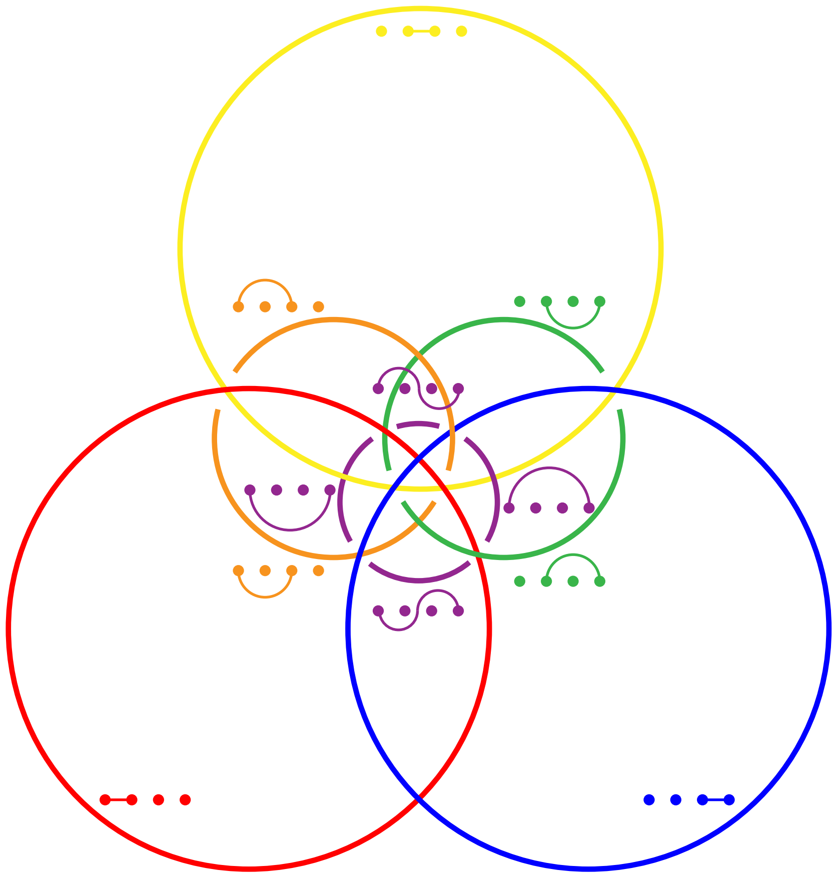
ARCS AND SHARDS

arc (a, b, A, B) with $1 \leq a < b \leq n$ and $A \sqcup B =]a, b[$



shard $\Sigma(a, b, A, B) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_{a'} \leq x_a = x_b \leq x_{b'}$ for all $a' \in A$ and $b' \in B \}$





ARCS AND SHARDS

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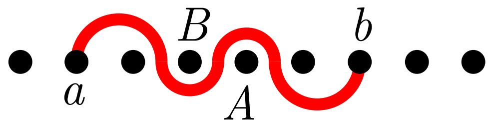
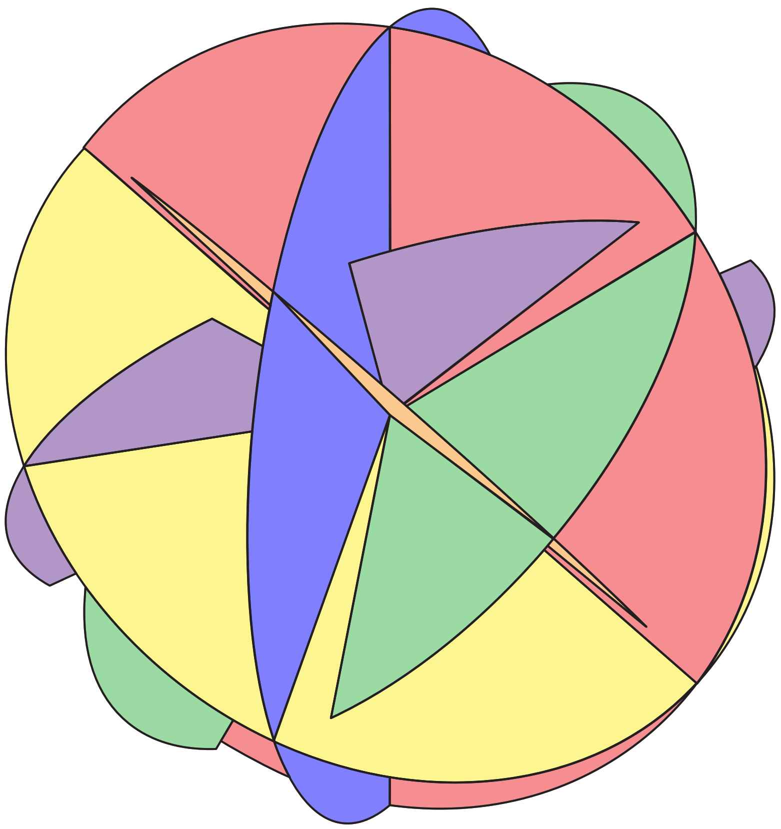
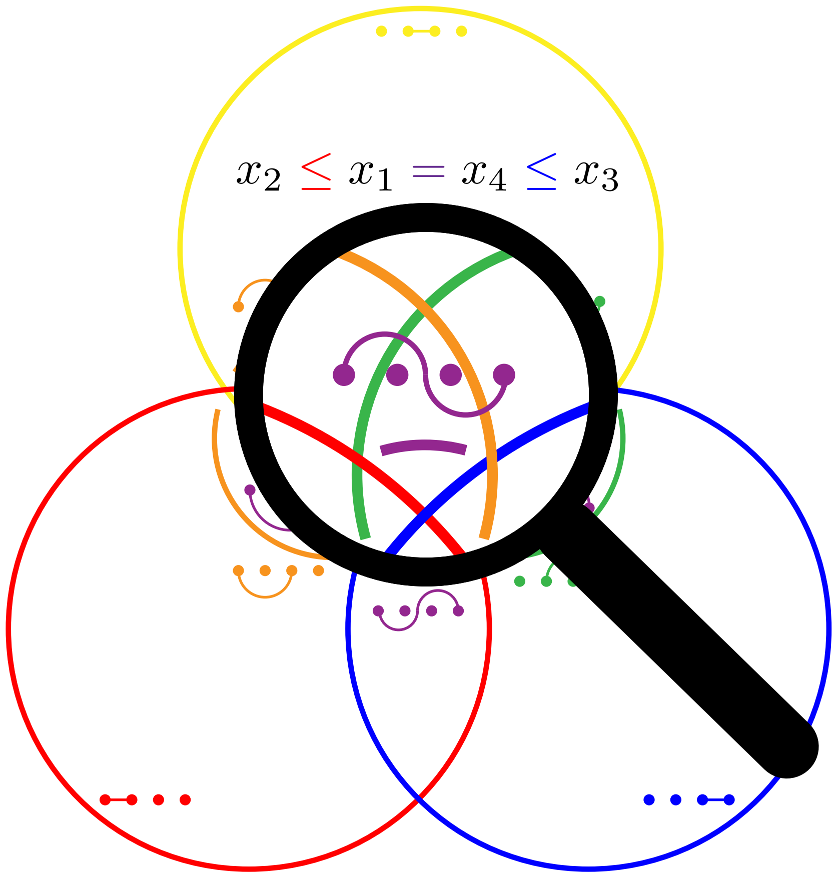


shard $\Sigma(a, b, A, B) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{a'} \leq x_a = x_b \leq x_{b'} \text{ for all } a' \in A \text{ and } b' \in B \right\}$



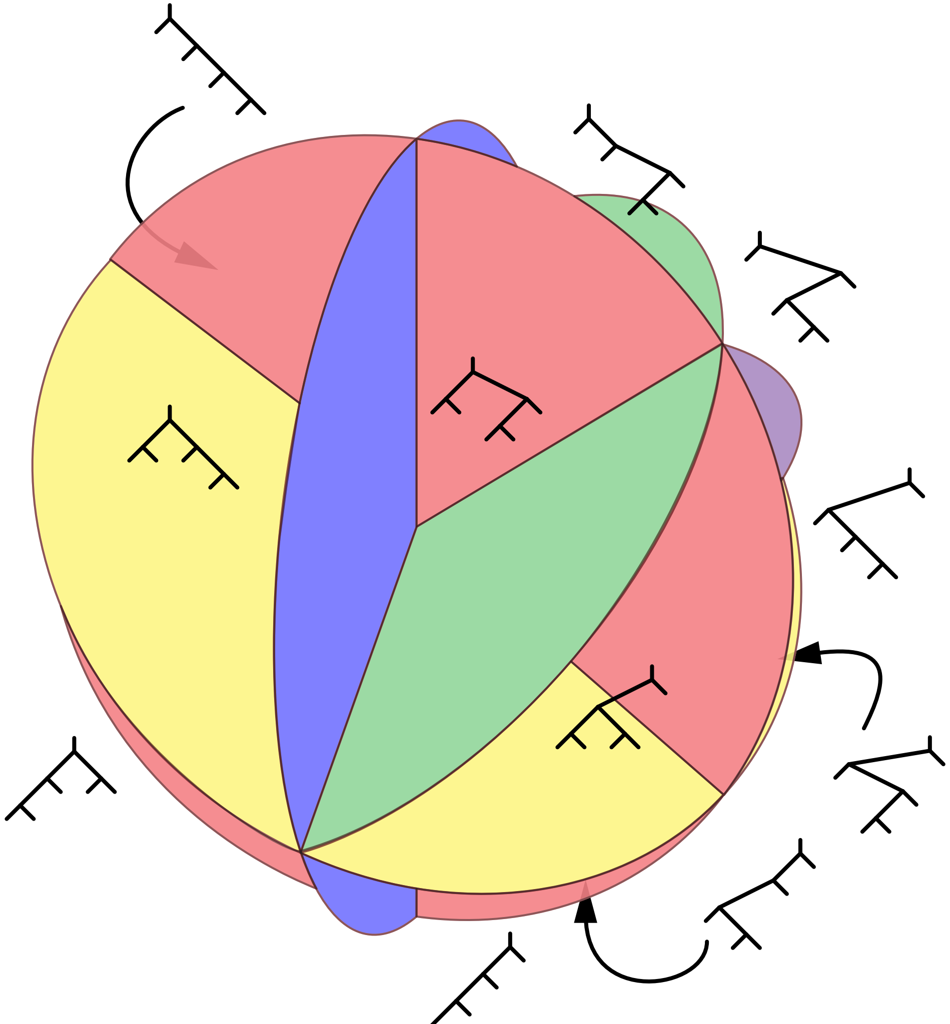


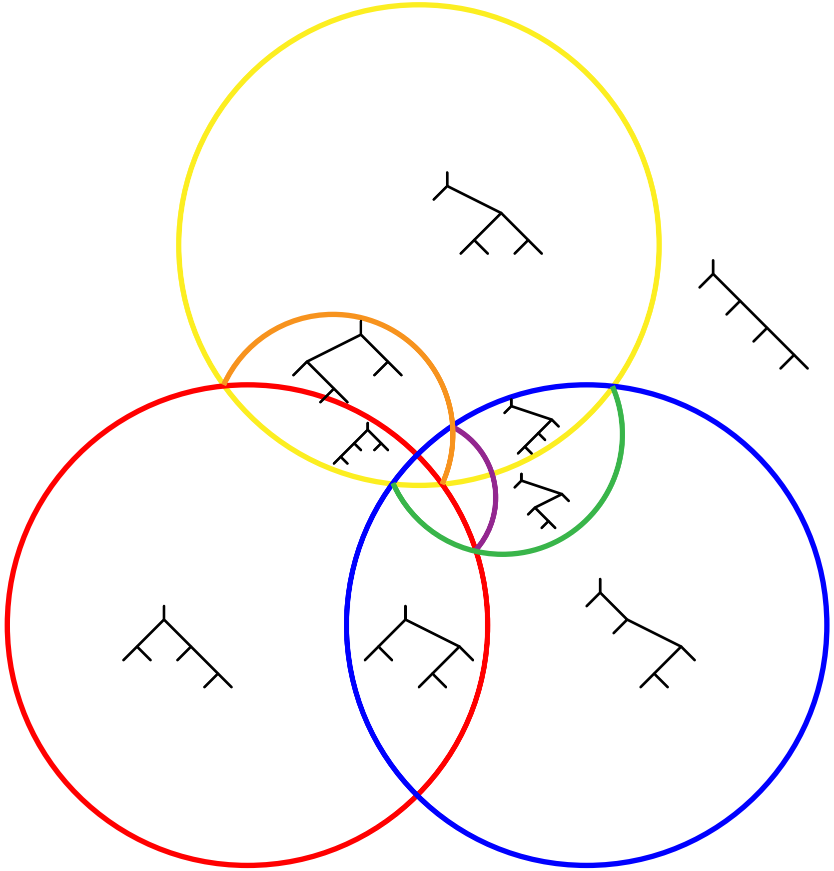
QUOTIENT FANS

quotient fan $\mathcal{F}_{\equiv}$ =

• the chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations σ in the same congruence class of $\equiv$

• the walls are given by the union of shards of $\equiv$ Reading ('05)





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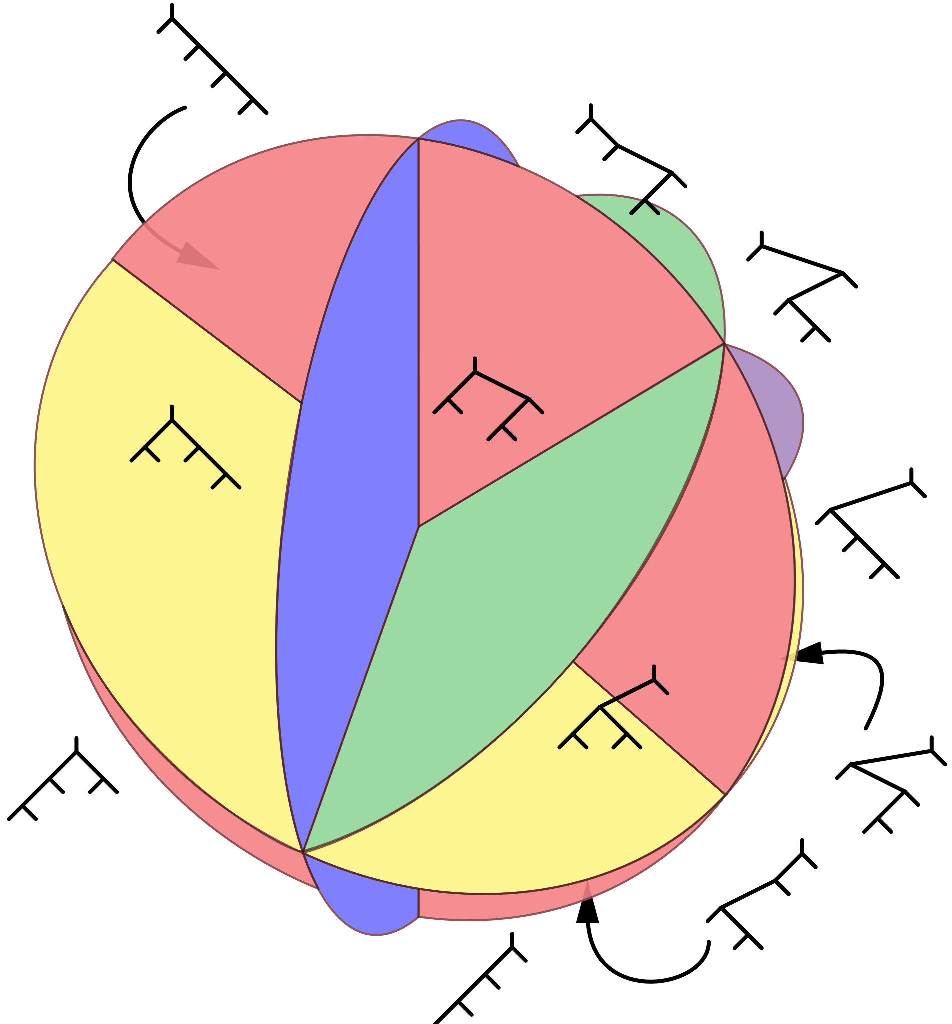
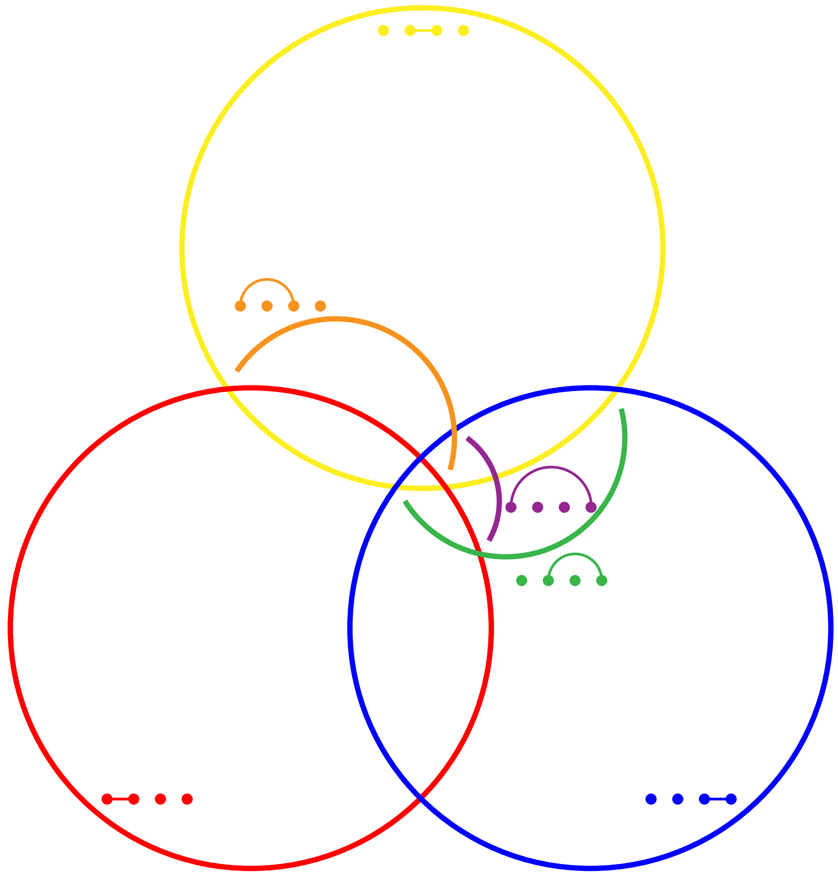
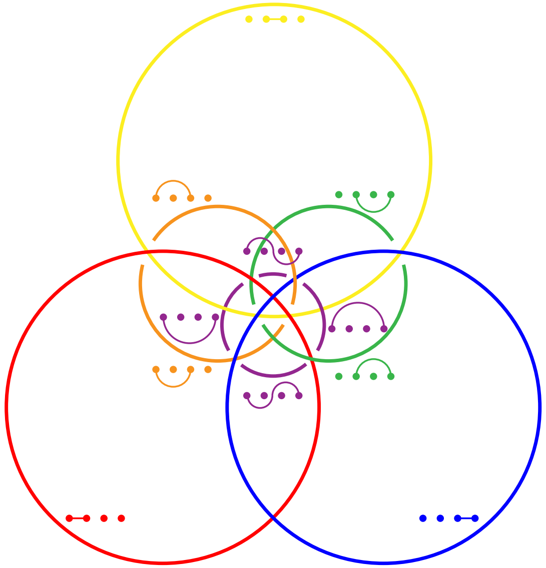
Reading ('05)

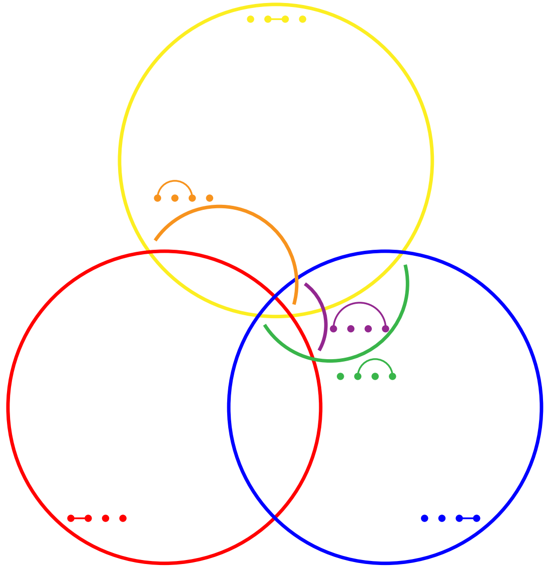




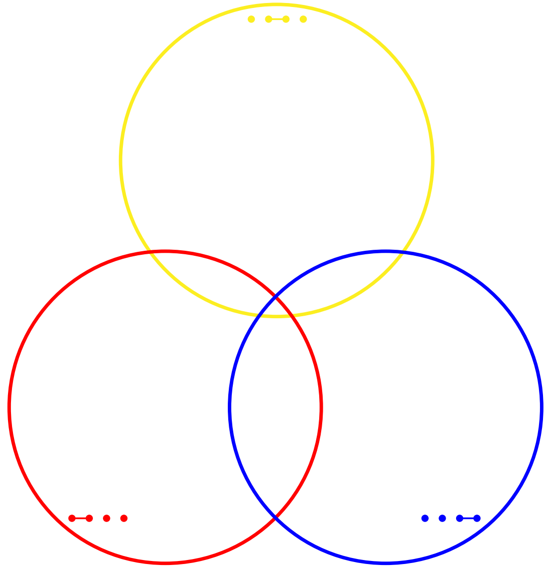
QUOTIENT FANS



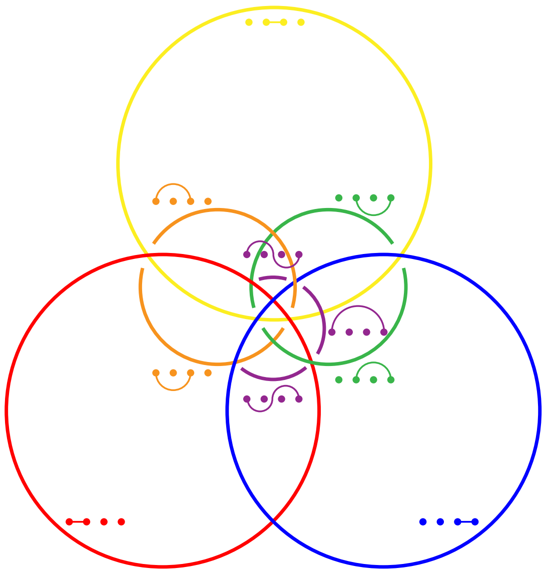
trivial



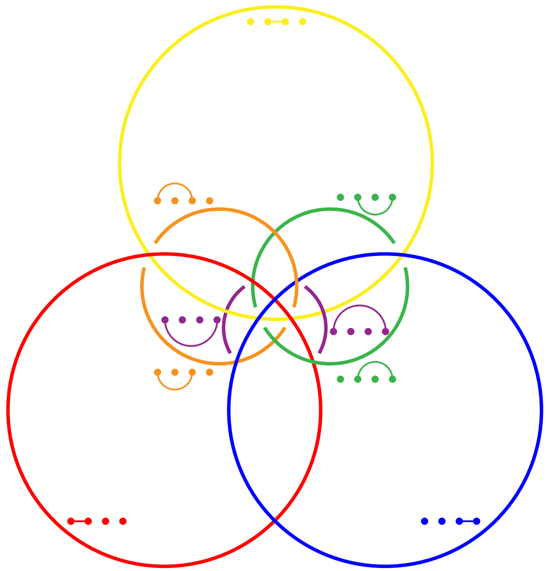
sylvester



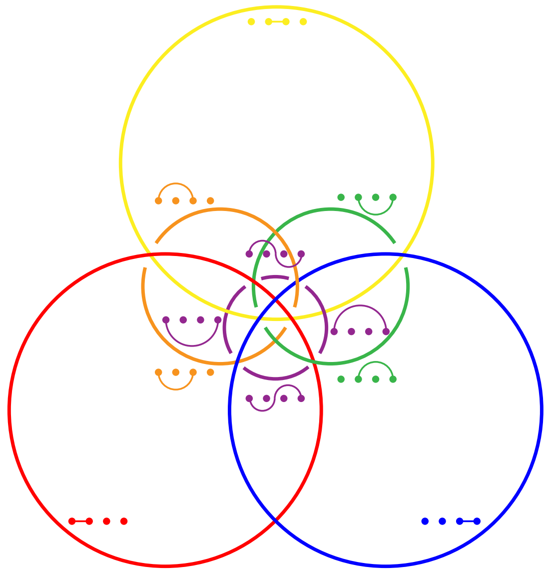
recoils



2-twists



diagonal rectangulations



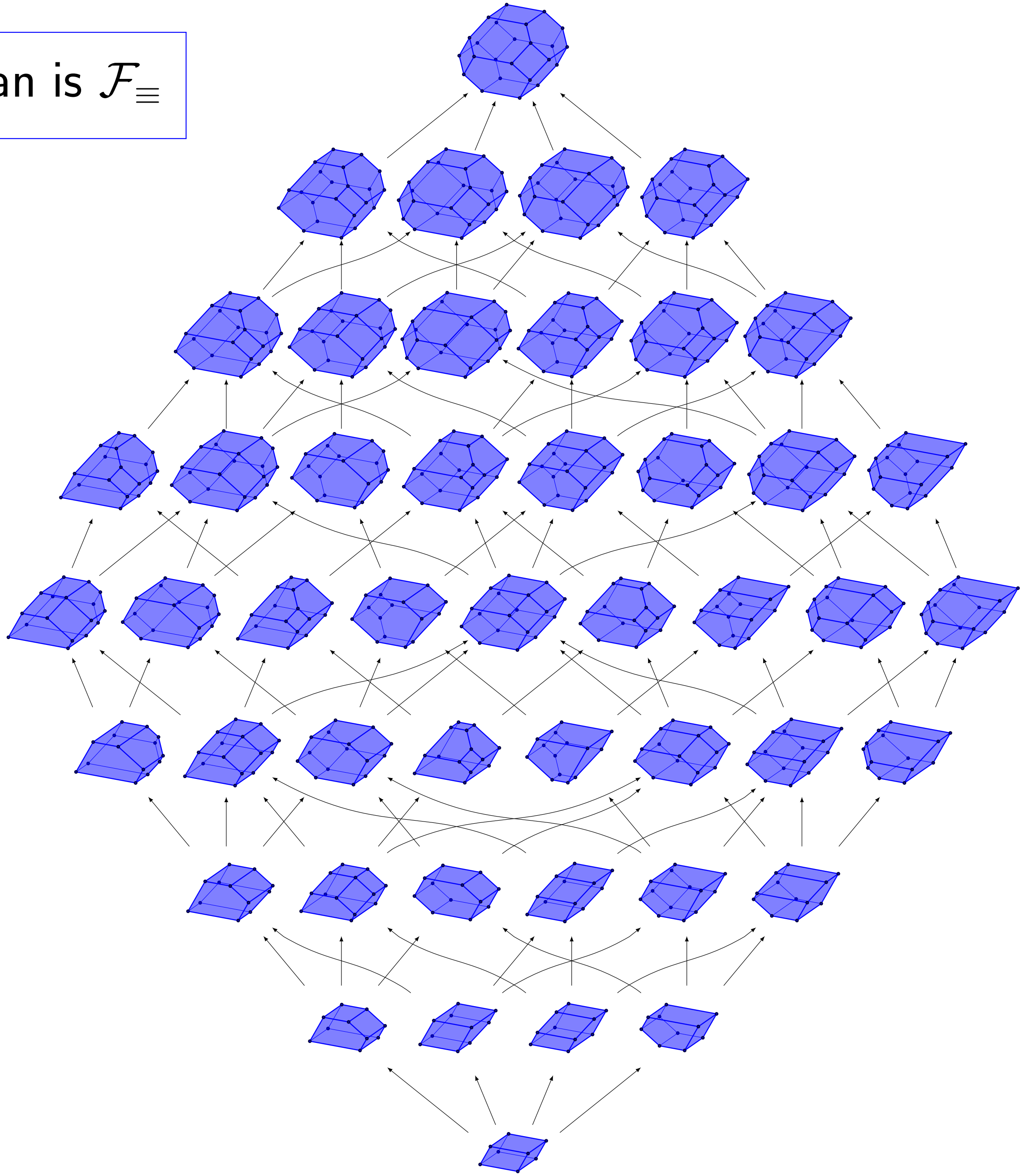
generic rectangulations

QUOTIENTOPES (BY LUCK)

P.–Santos, *Quotientopes* ('19)

QUOTIENTOPES

quotientope = polytope whose normal fan is $\mathcal{F}_{\equiv}$



P.–Santos, *Quotientopes* ('19)

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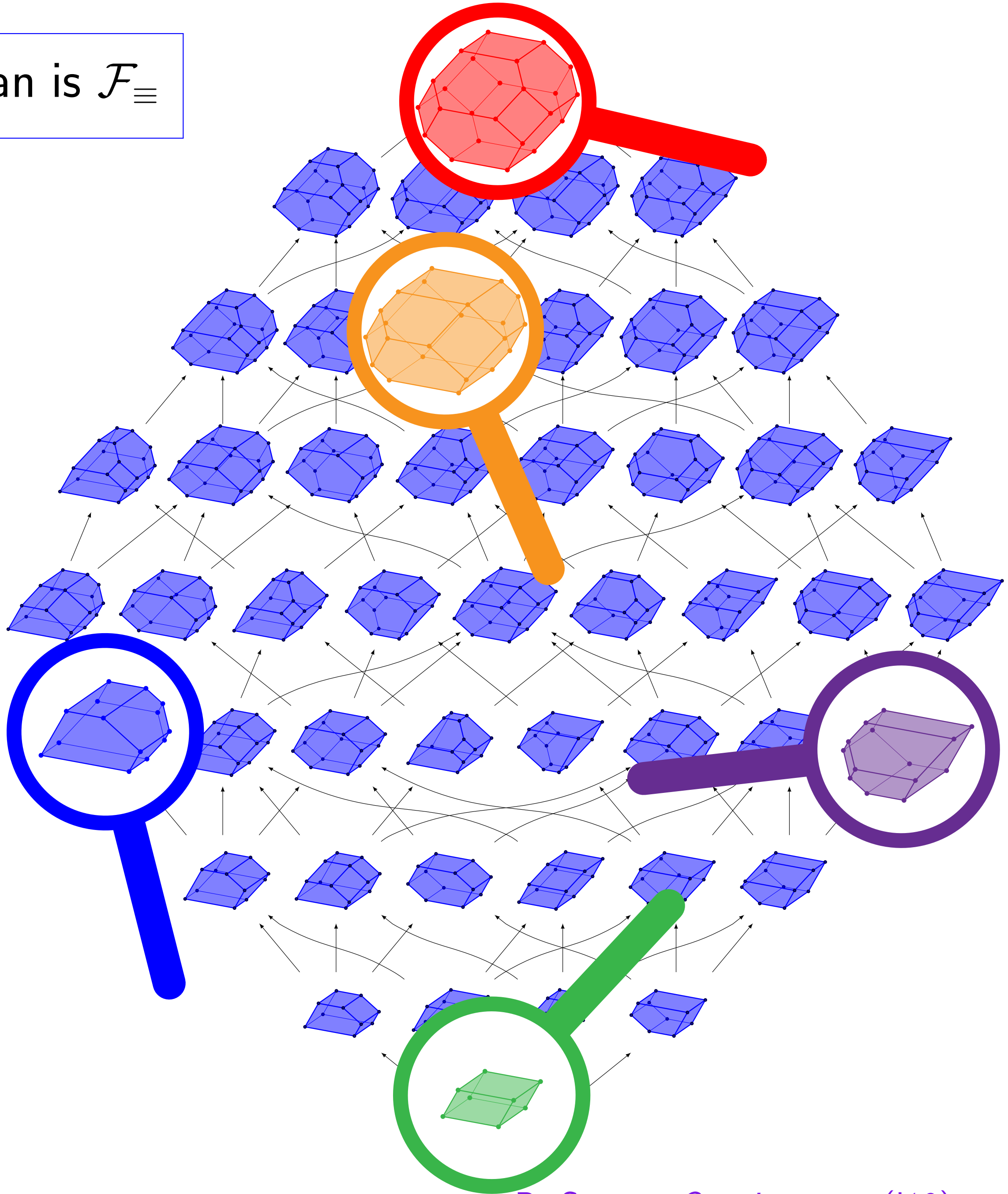

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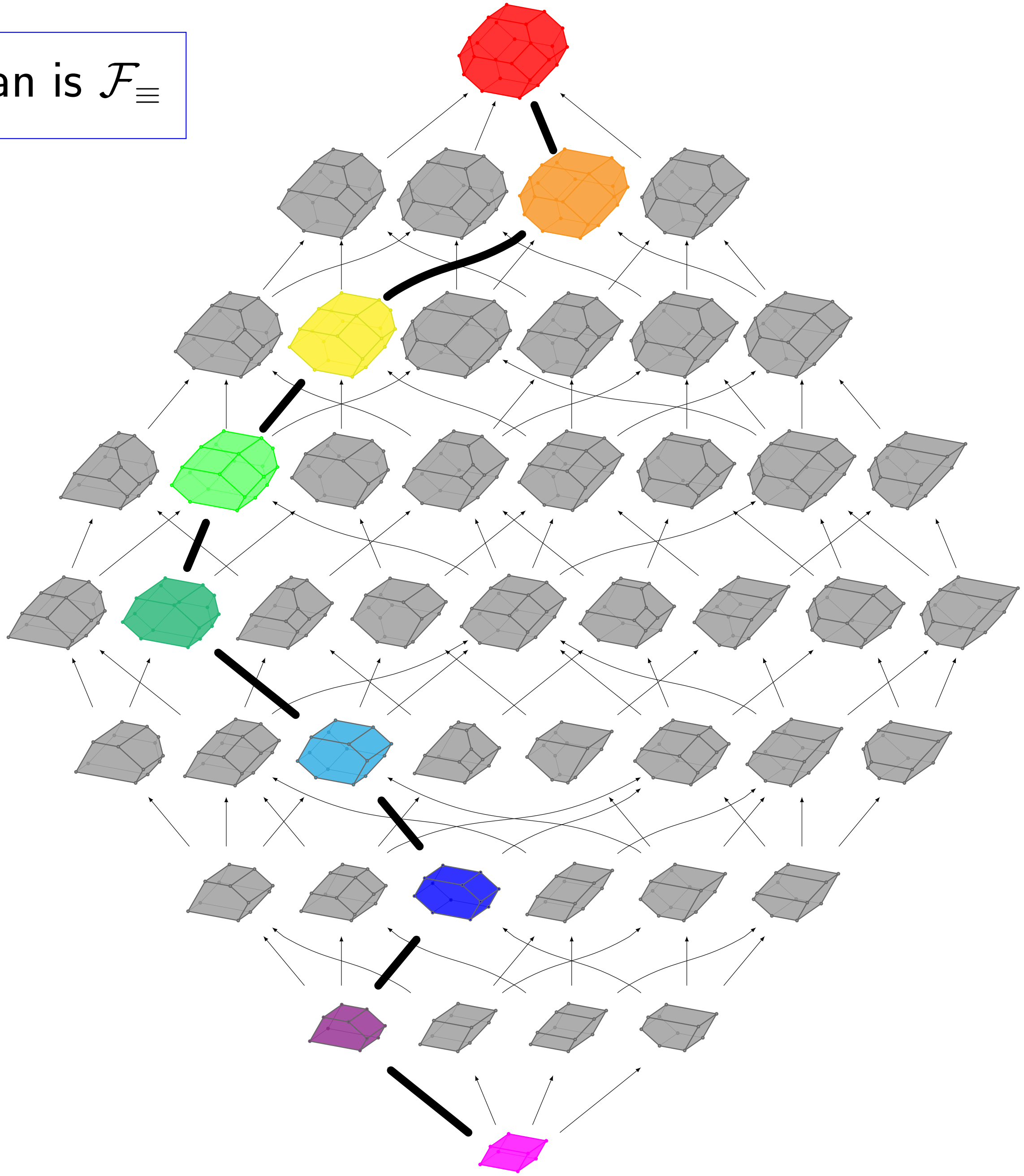
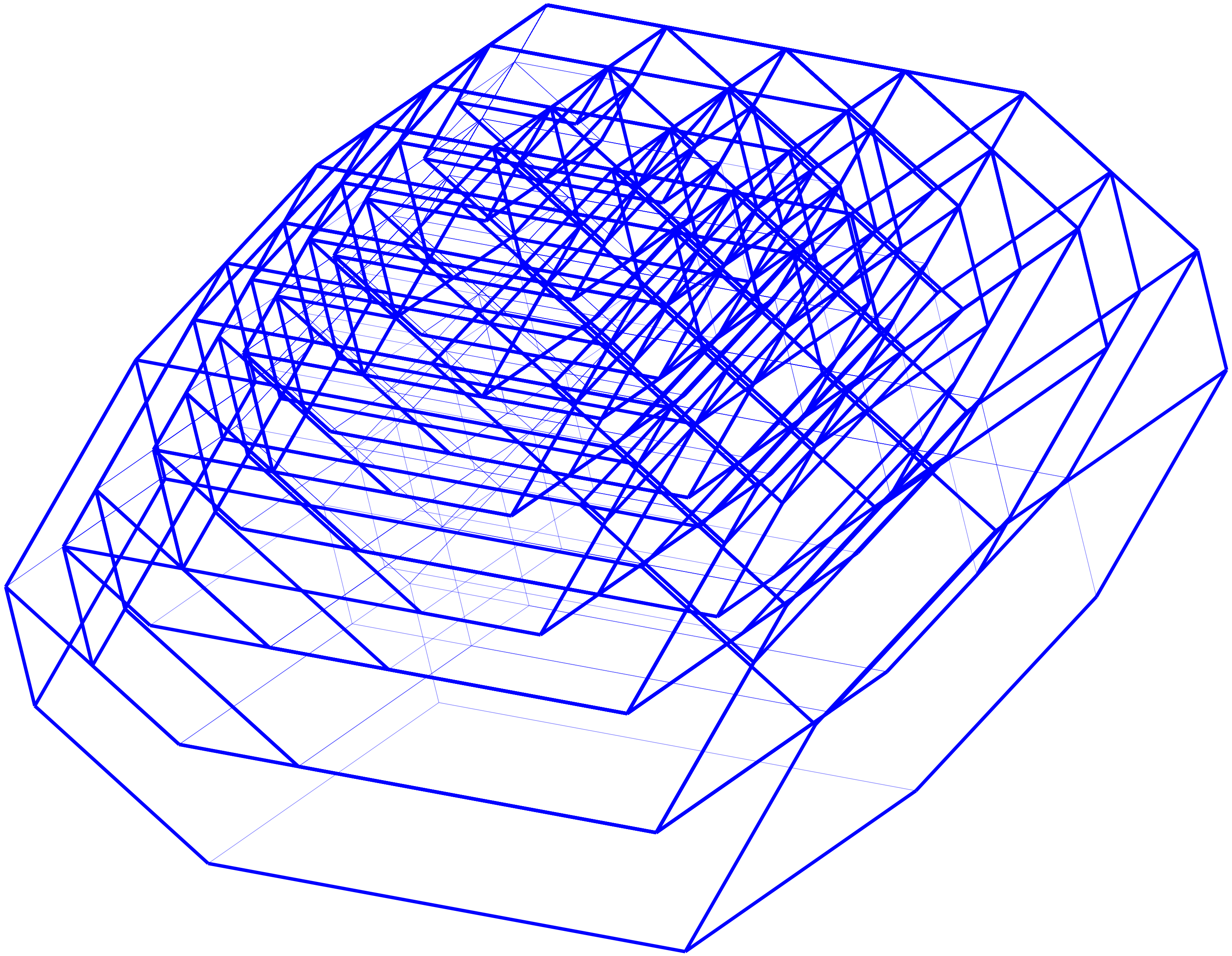


P.–Santos, *Quotientopes* ('19)

3D PRINTING PROJECT!

print the common refinement of all 47 (essential) 3-dimensional quotientopes



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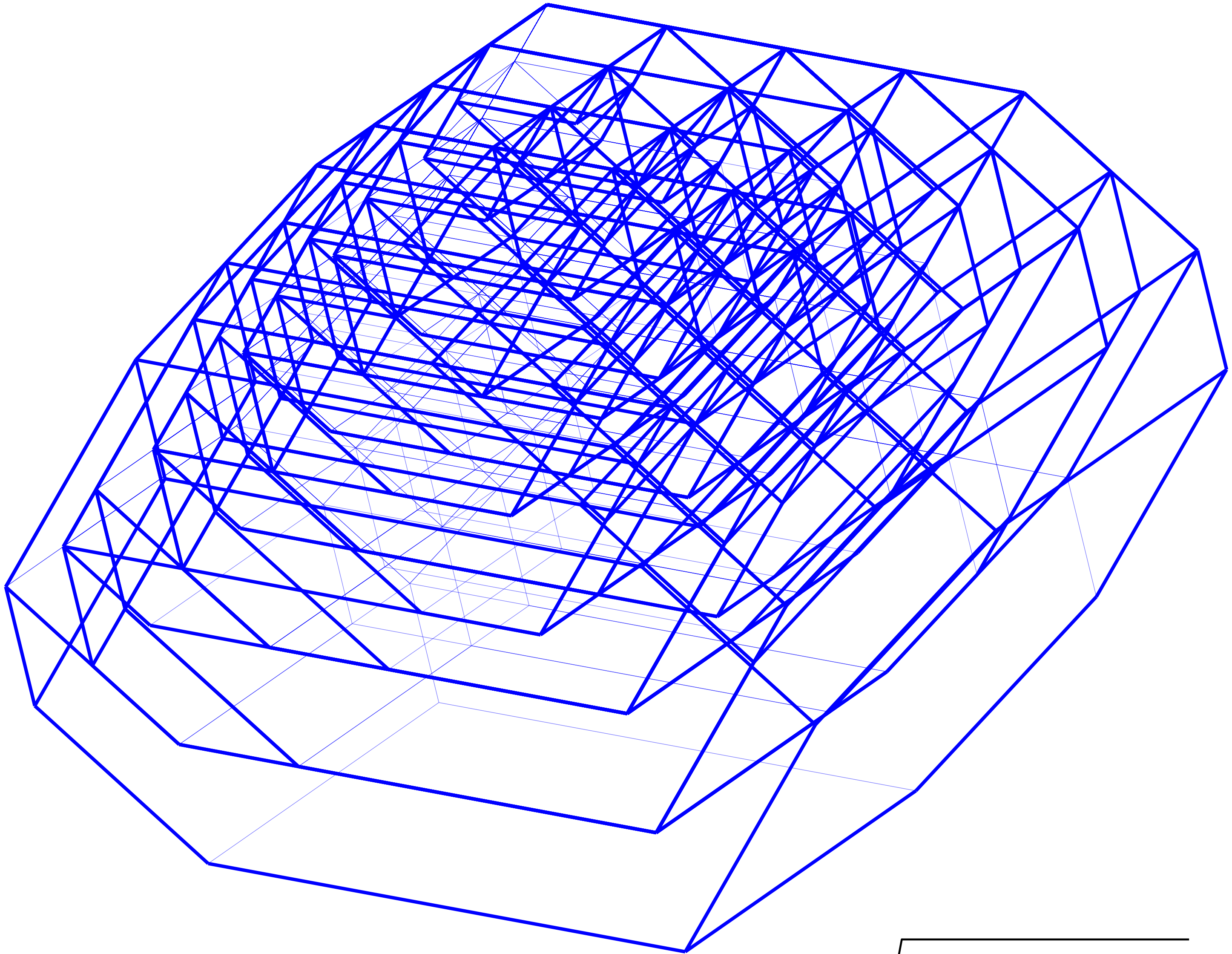


2287 polytopes to print

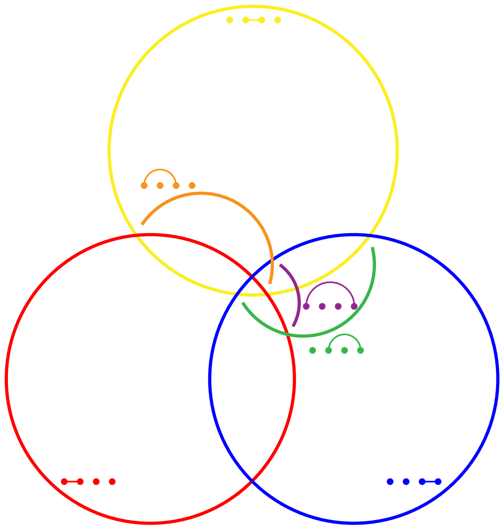
$$\sqrt[3]{\max \frac{\mathrm{vol}(\mathbb{P}\mathsf{erm})}{\mathrm{vol}(\mathbb{P}\mathsf{iece})}} = \sqrt[3]{3888} \simeq 16$$

QUOTIENTOPES (BY MINKOWSKI SUMS)

Padrol–P.–Ritter, *Shard polytopes* ('23)

INTERSECTIONS OF CONGRUENCES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
then the quotient fan $\mathcal{F}_{\equiv}$ is the common refinement of the quotient fans $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$



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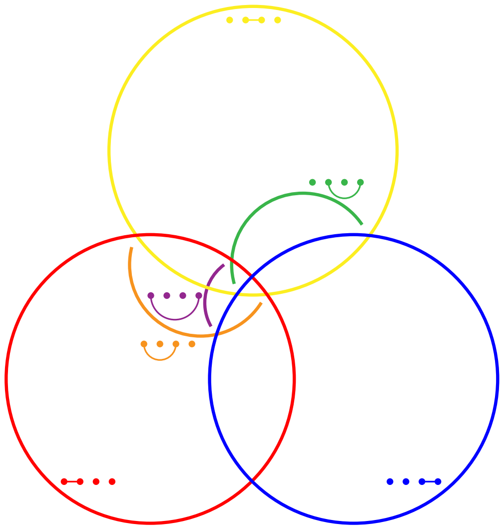
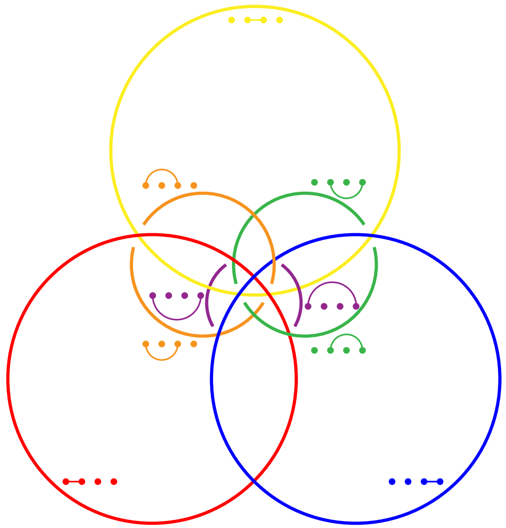


$=$



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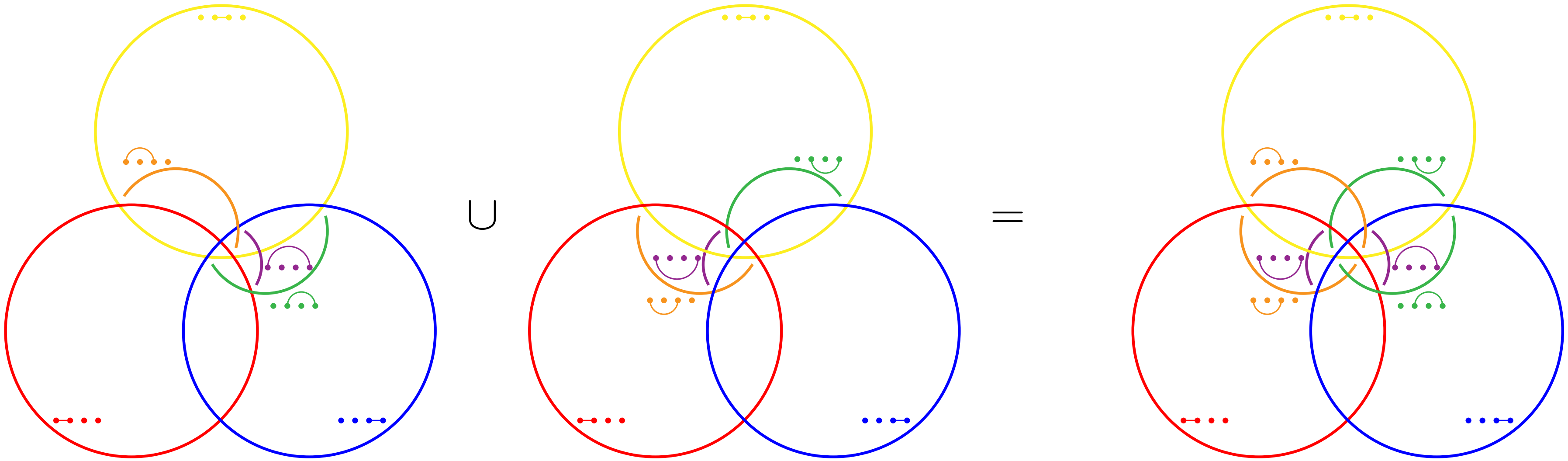
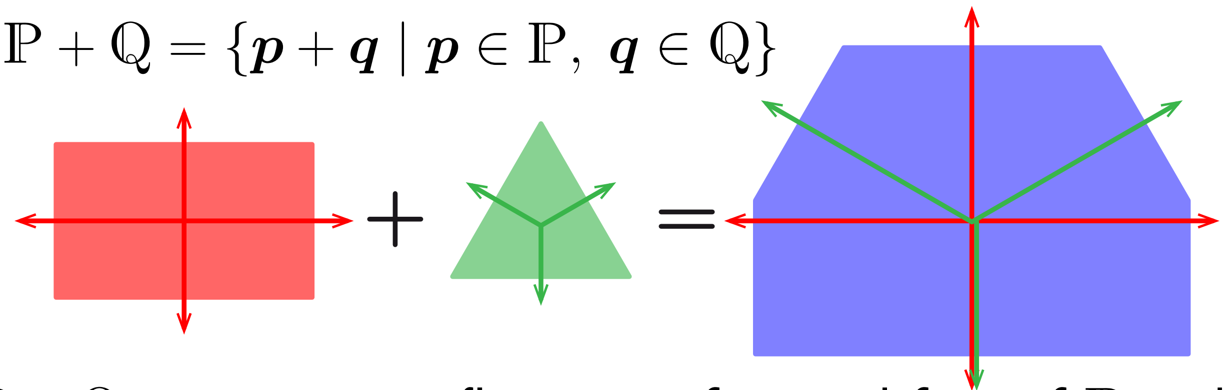


Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$



Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$

INTERSECTIONS OF CONGRUENCES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
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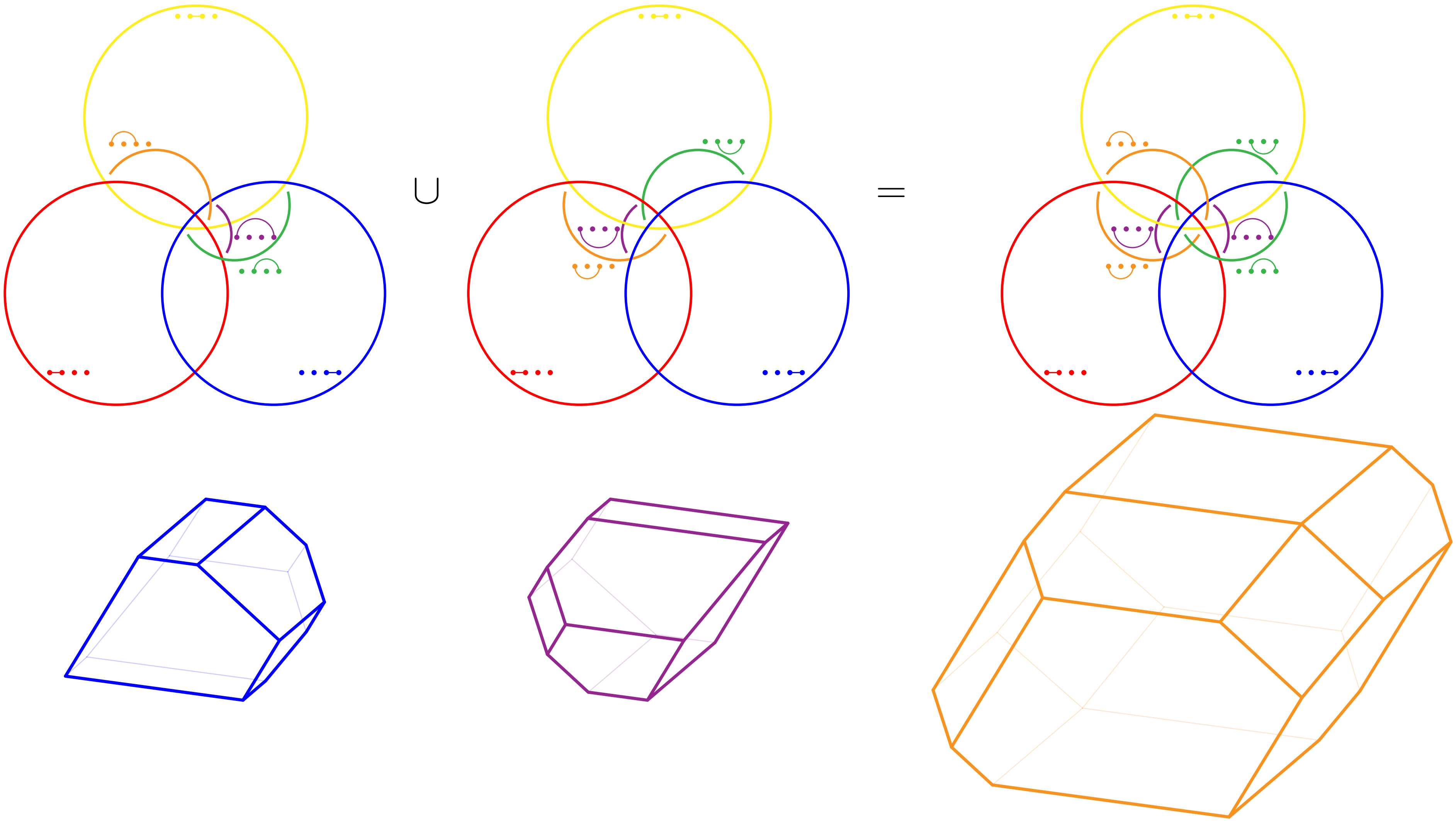

MINKOWSKI SUMS OF ASSOCIAHEDRA

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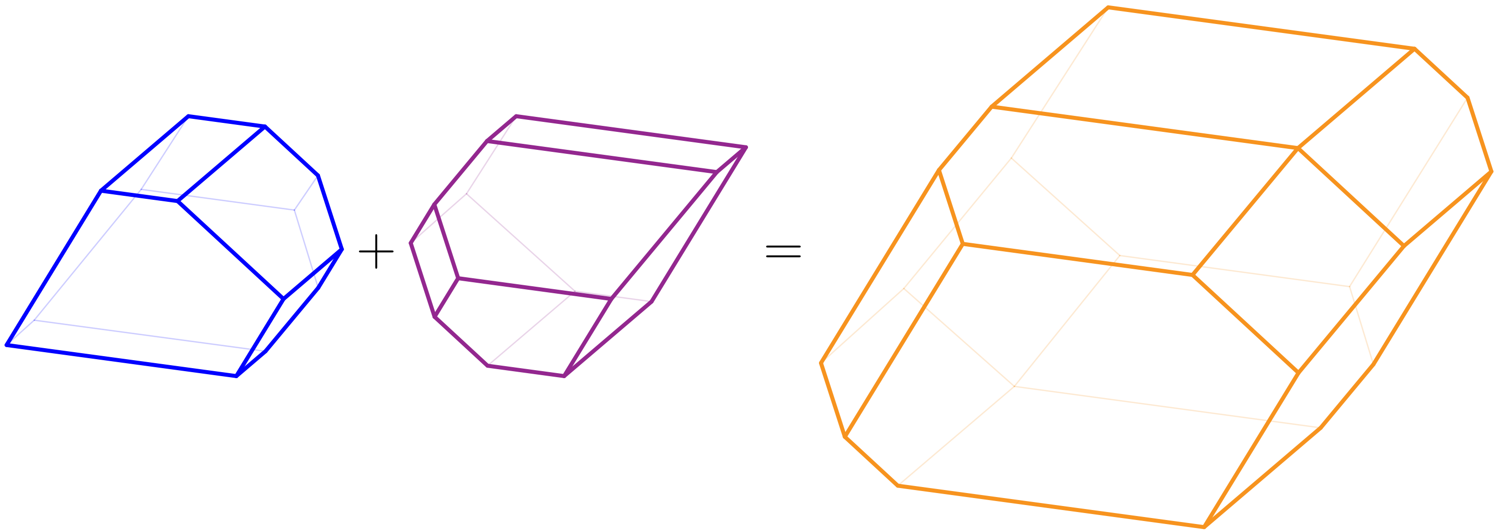
Principal arc ideals are Cambrian congruences

Any quotient fan is realized by a Minkowski sum of (low dim.) associahedra



Padrol–P.–Ritter, *Shard polytopes* ('23)

SHARD POLYTOPES

Padrol–P.–Ritter, *Shard polytopes* ('23)

SHARD POLYTOPES

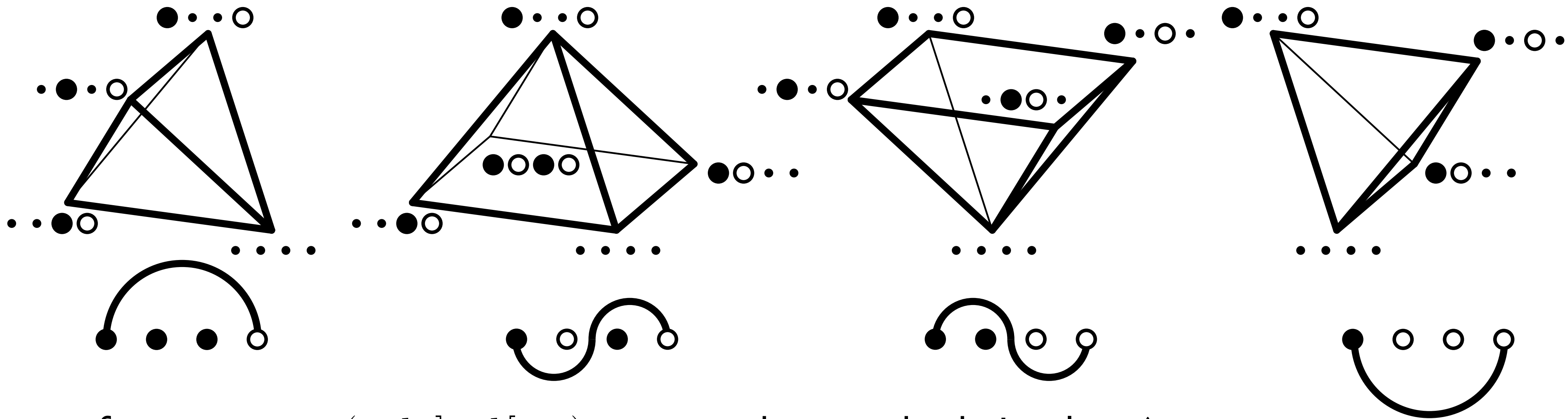
for an arc $\alpha = (a, b, A, B)$, define

- α-matching $=$ sequence $a \le a_1 < b_1 < \cdots < a_k < b_k \le b$ where $\begin{cases} a_i \in \{a\} \cup A \\ b_i \in B \cup \{b\} \end{cases}$
- characteristic vector $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$

shard polytope $\mathbb{SP}(\alpha)$

$$= \operatorname{conv}\left\{\chi(M) \;\middle|\; M \ \alpha\text{-matching}\right\}$$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{cl} x_j = 0 & \text{for all } j \in [n] \smallsetminus [a,b] \\ 0 \le x_{a'} \le 1 & \text{for all } a' \in \{a\} \cup A \\ -1 \le x_{b'} \le 0 & \text{for all } b' \in B \cup \{b\} \\ 0 \le \sum_{i \le j} x_i \le 1 & \text{for all } j \in [n] \end{array} \right\}$$

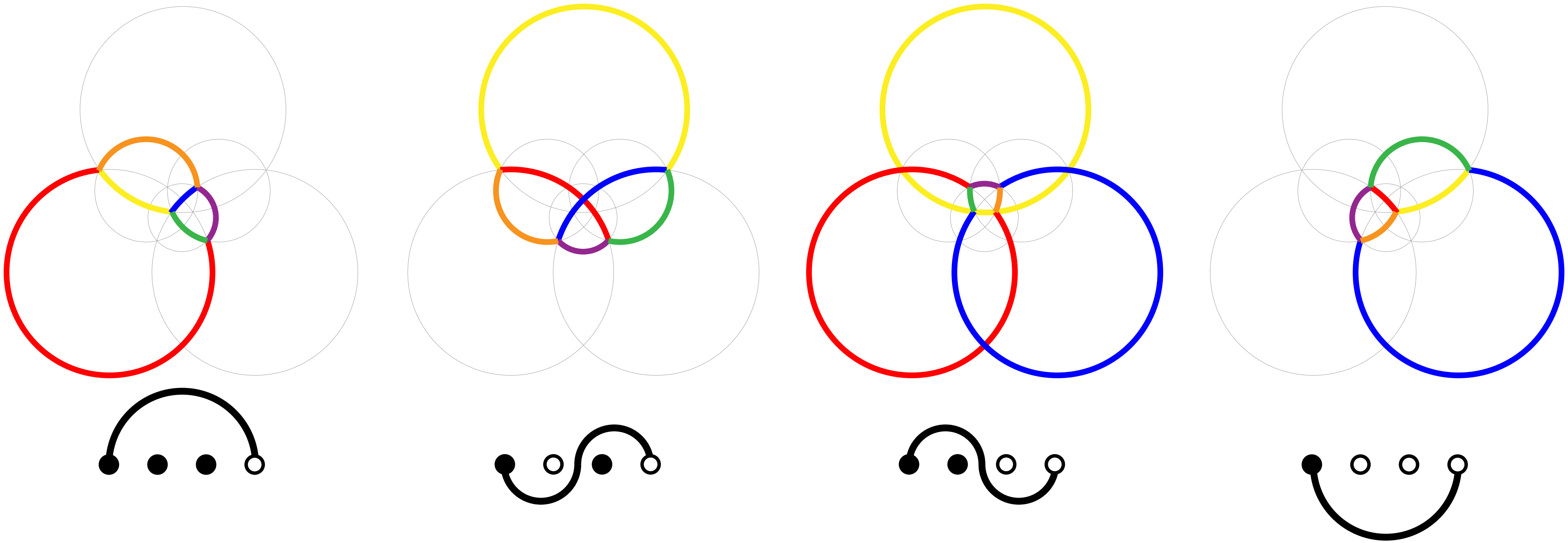


exm: for an up arc $(a, b,]a, b[, \varnothing)$, we get the standard simplex $\triangle_{[a,b]} - \boldsymbol{e}_b$

shard polytope $\mathbb{SP}(\alpha) = \operatorname{conv}\{\chi(M) \mid M \ \alpha\text{-matching}\}$

The union of the walls of the normal fan of the shard polytope $\mathbb{SP}(\alpha)$

- contains the shard $\Sigma(\alpha)$,
- is contained in the union of the shards $\Sigma(\alpha')$ for α' subarc of α



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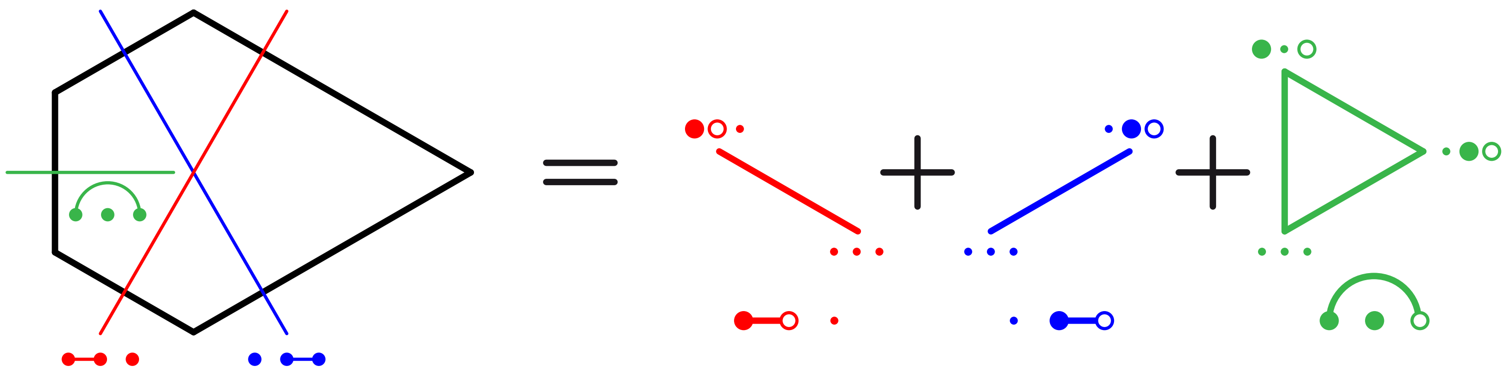
For any lattice congruence $\equiv$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the Minkowski sum of the shard polytopes $\mathbb{SP}(\alpha)$ for $\alpha \in \mathcal{A}_\equiv$ Padrol–P.–Ritter, *Shard polytopes* ('23)



SHARD POLYTOPES

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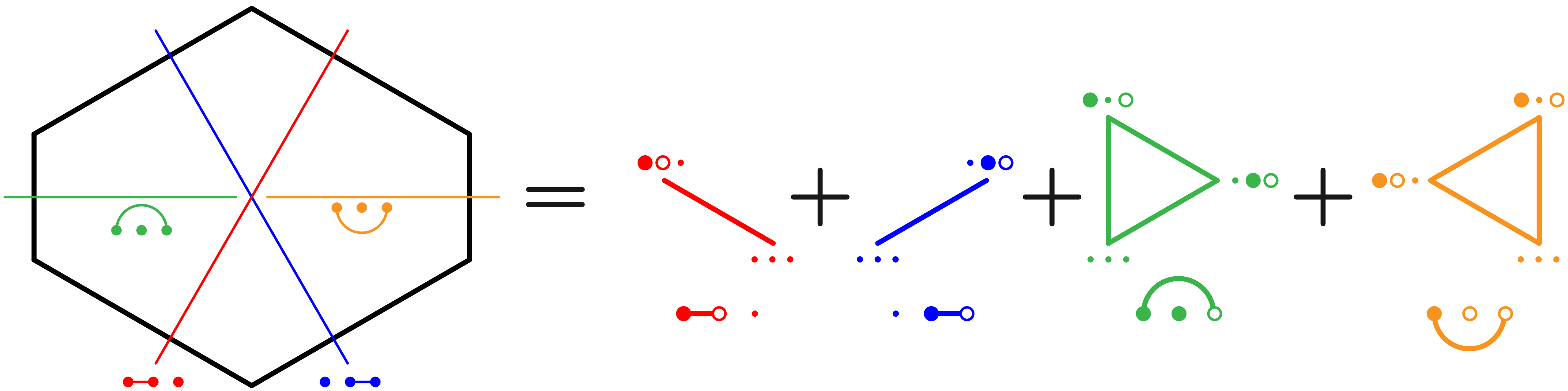


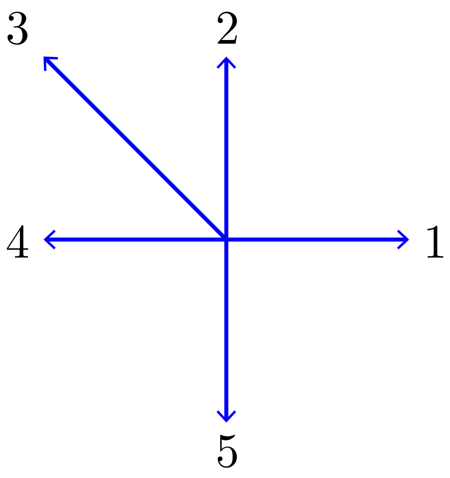
SHARD POLYTOPES AND TYPE CONES

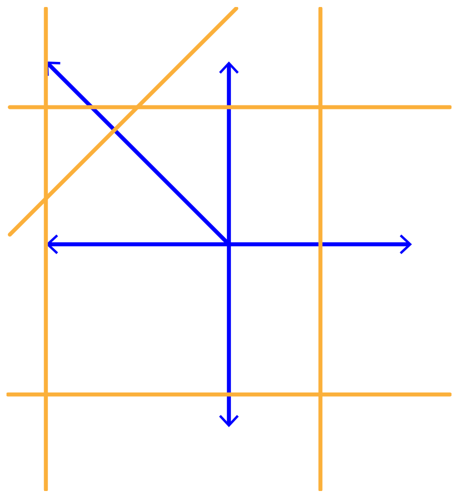
CHOOSING RIGHT-HAND-SIDES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

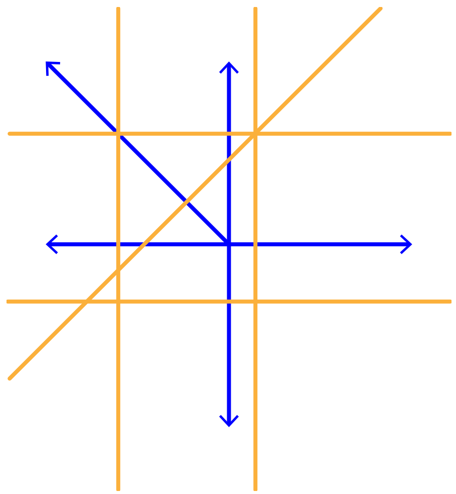
$\boldsymbol{G} = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\}$

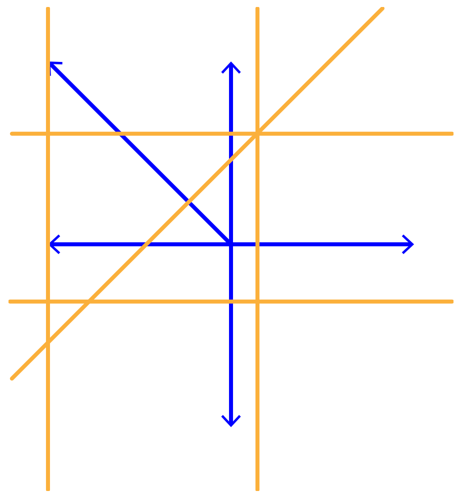




A



B



C

CHOOSING RIGHT-HAND-SIDES

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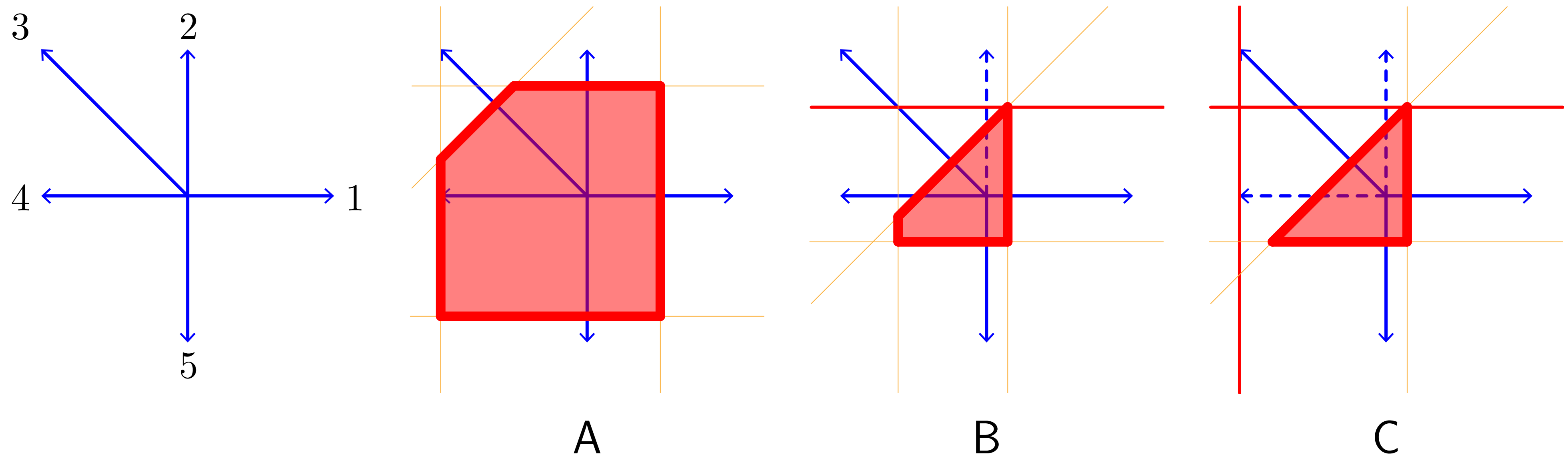

CHOOSING RIGHT-HAND-SIDES

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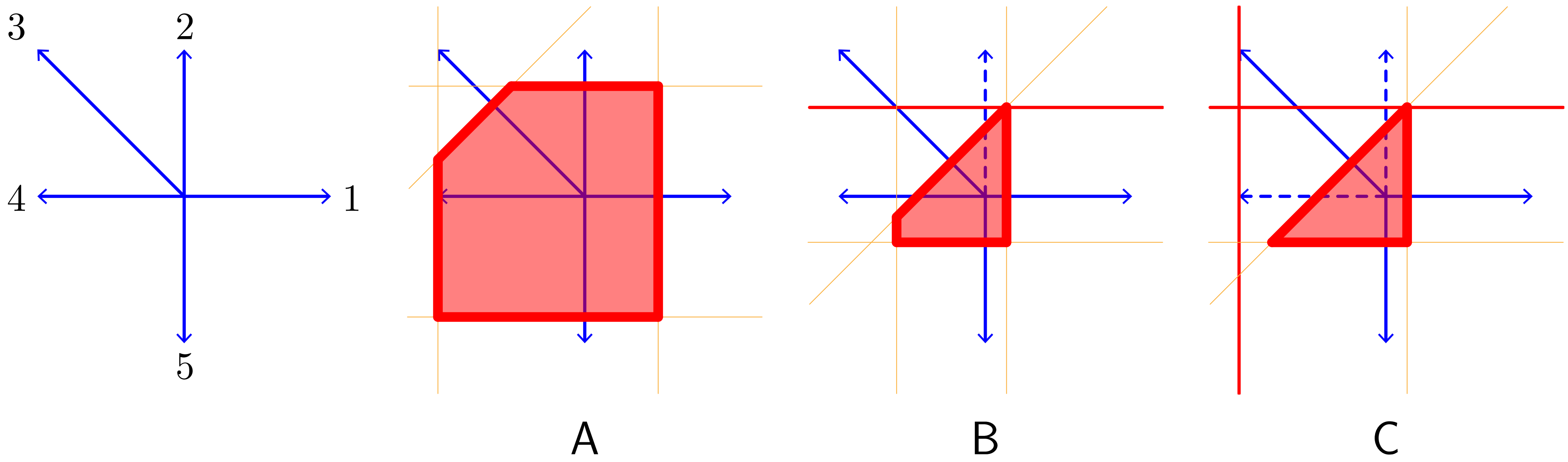

When is $\mathcal{F}$ the normal fan of $\mathbb{P}_{\boldsymbol{h}}$?

WALL-CROSSING INEQUALITIES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

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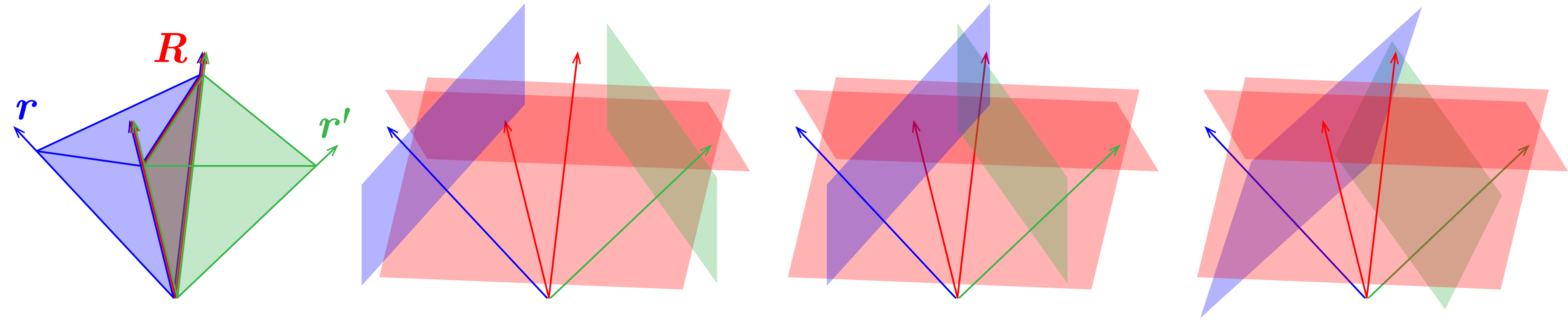
for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$



WALL-CROSSING INEQUALITIES

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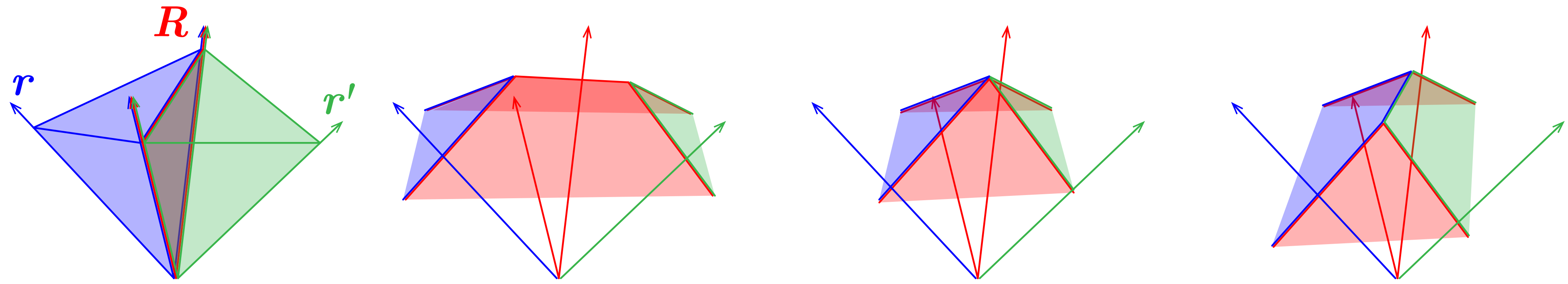

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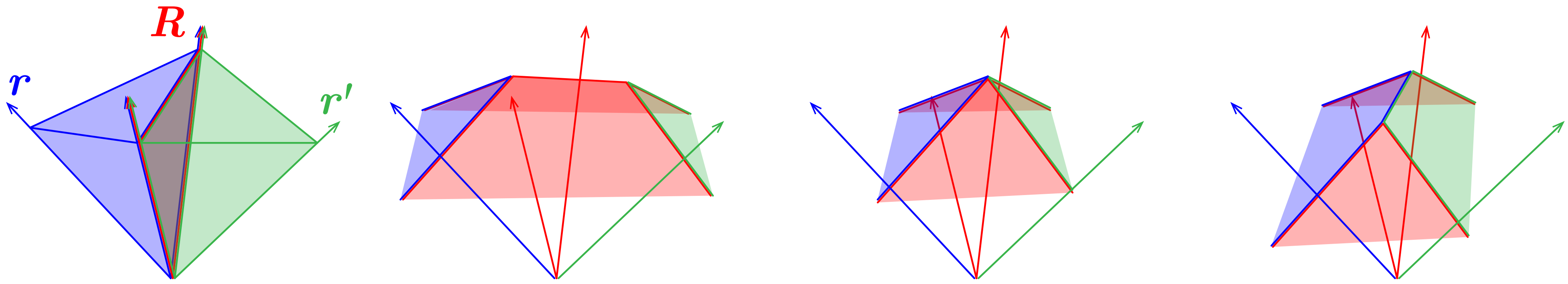

wall-crossing inequality for a wall $\boldsymbol{R}$ = $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, h_{\boldsymbol{s}} > 0$ where

- $\boldsymbol{r}, \boldsymbol{r'}$ = rays such that $\boldsymbol{R} \cup \{\boldsymbol{r}\}$ and $\boldsymbol{R} \cup \{\boldsymbol{r'}\}$ are chambers of $\mathcal{F}$
- $\alpha_{\boldsymbol{R},\boldsymbol{s}}$ = coeff. of unique linear dependence $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, \boldsymbol{s} = 0$ with $\alpha_{\boldsymbol{R},\boldsymbol{r}} + \alpha_{\boldsymbol{R},\boldsymbol{r'}} = 2$

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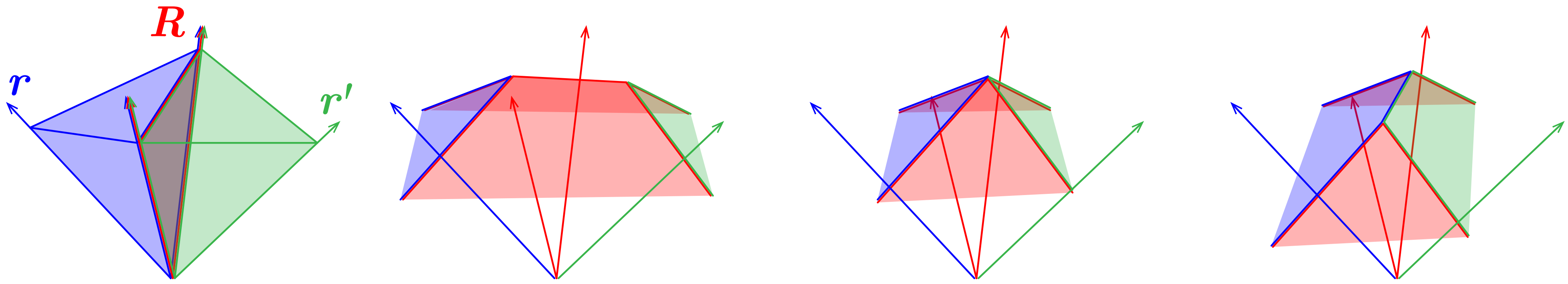

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$\mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}} \iff \boldsymbol{h}$ satisfies all wall-crossing inequalities of $\mathcal{F}$

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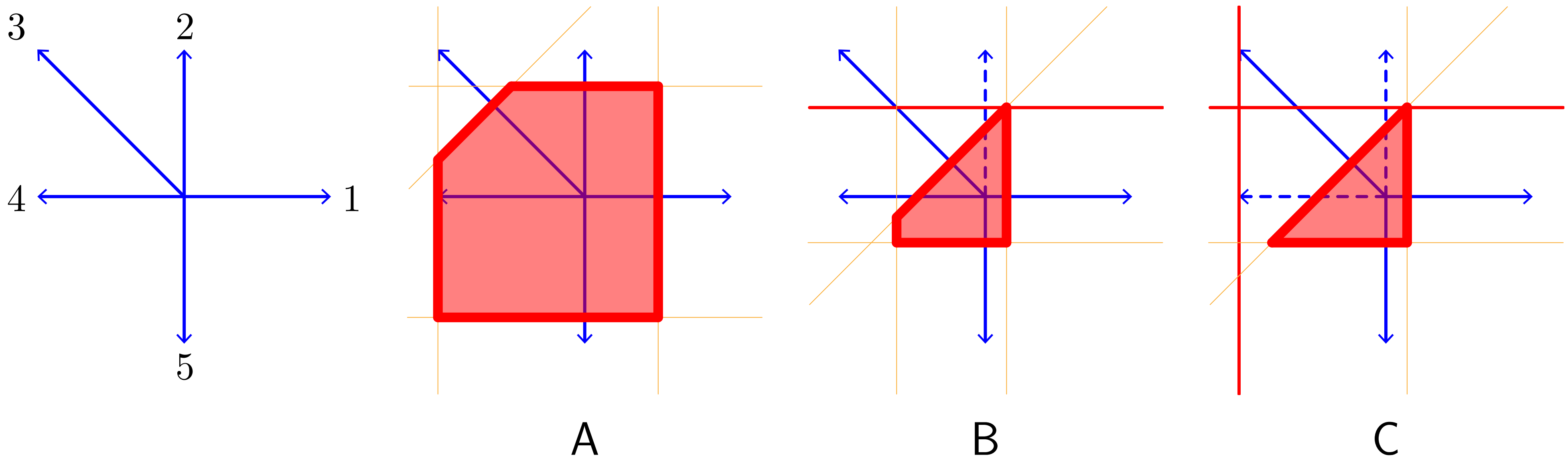


wall-crossing inequalities:

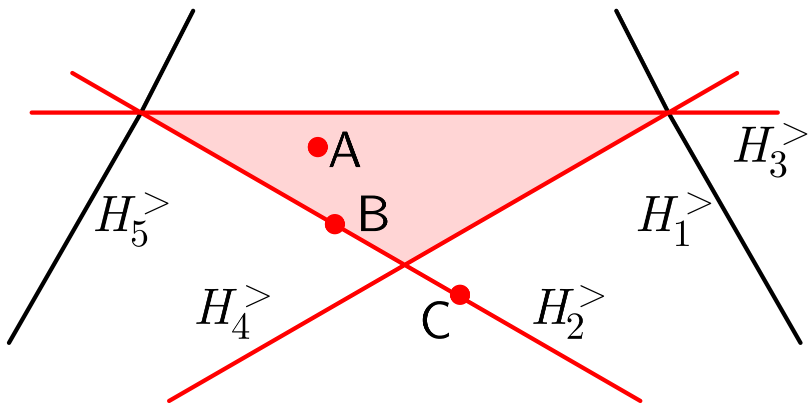
wall 1 : $h_2 + h_5 > 0$

wall 2 : $h_1 + h_3 > h_2$

wall 3 : $h_2 + h_4 > h_3$

wall 4 : $h_3 + h_5 > h_4$

wall 5 : $h_1 + h_4 > 0$



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<u>type cone</u> $\mathbb{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ McMullen ('73)

$= \{\boldsymbol{h} \in \mathbb{R}^N \mid \mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}\}$

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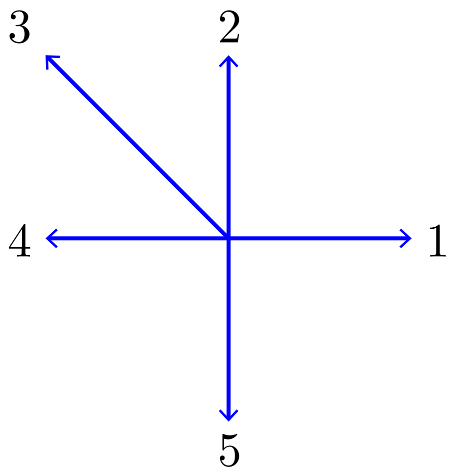

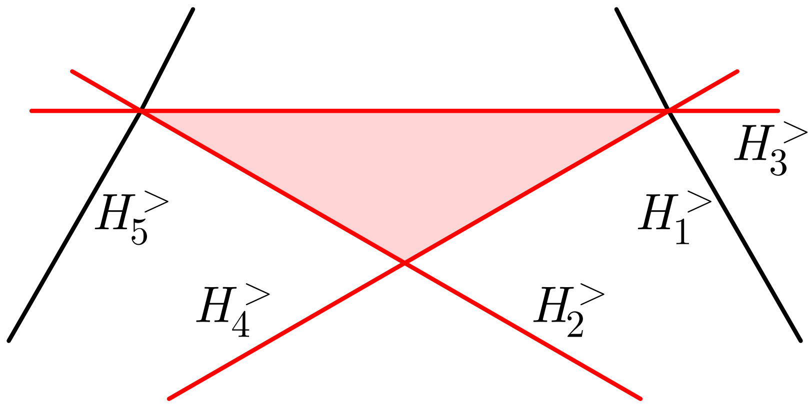

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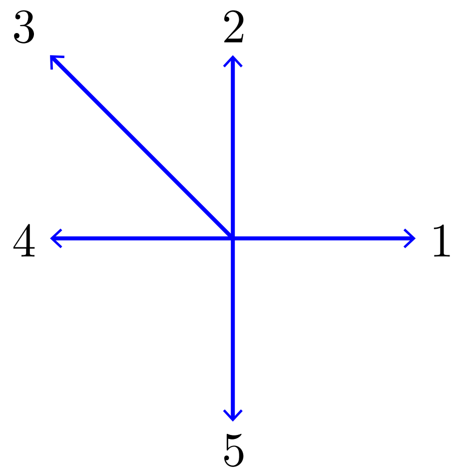

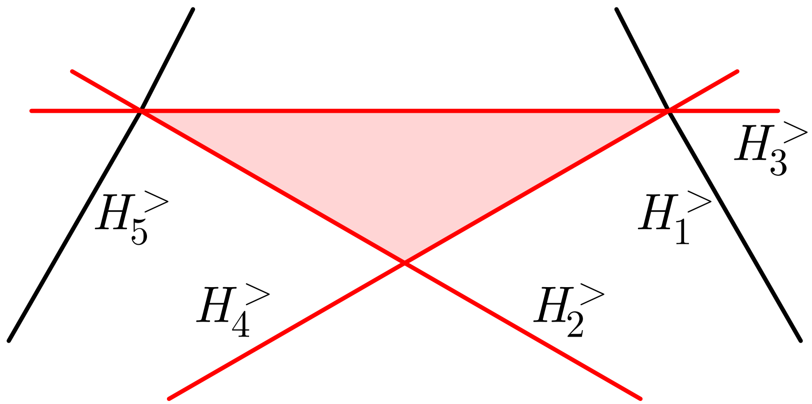

some properties of $\mathbb{TC}(\mathcal{F})$:

- $\mathbb{TC}(\mathcal{F})$ is an open cone (dilations preserve normal fans)
- $\mathbb{TC}(\mathcal{F})$ has lineality space $\boldsymbol{G}\,\mathbb{R}^n$ (translations preserve normal fans)
- dimension of $\mathbb{TC}(\mathcal{F})/\boldsymbol{G}\,\mathbb{R}^n = N - n$

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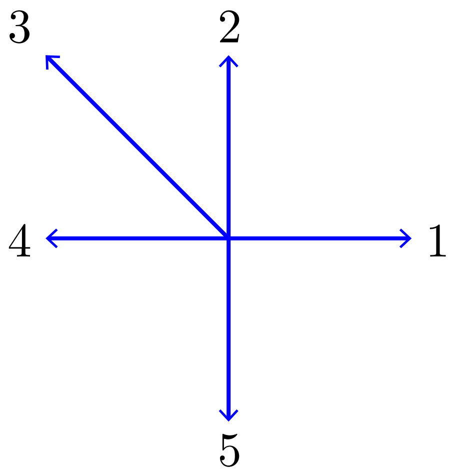
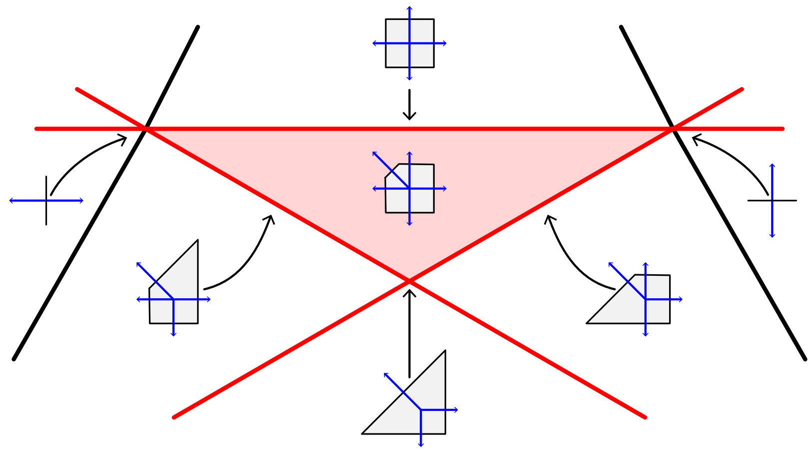




some properties of $\mathbb{TC}(\mathcal{F})$:

- closure of $\mathbb{TC}(\mathcal{F})$ = polytopes whose normal fan coarsens $\mathcal{F}$ = deformation cone
- Minkowski sums $\longleftrightarrow$ positive linear combinations

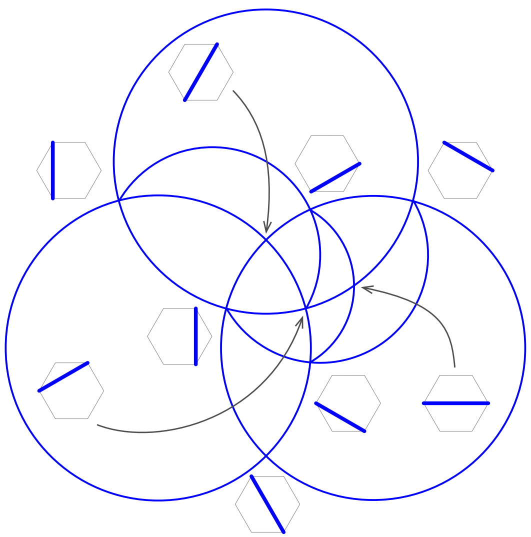
SIMPLICIAL TYPE CONE

Assume that the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial
$\boldsymbol{K} = (N-n)\times N$-matrix whose rows are inner normal vectors of the facets of $\mathbb{TC}(\mathcal{F}(\delta))$
All polytopal realizations of $\mathcal{F}$ are affinely equivalent to

$$\mathbb{R}_{\boldsymbol{\ell}} = \left\{ \boldsymbol{z} \in \mathbb{R}^N \;\middle|\; \boldsymbol{K}\boldsymbol{z} = \boldsymbol{\ell} \text{ and } \boldsymbol{z} \geq 0 \right\}$$

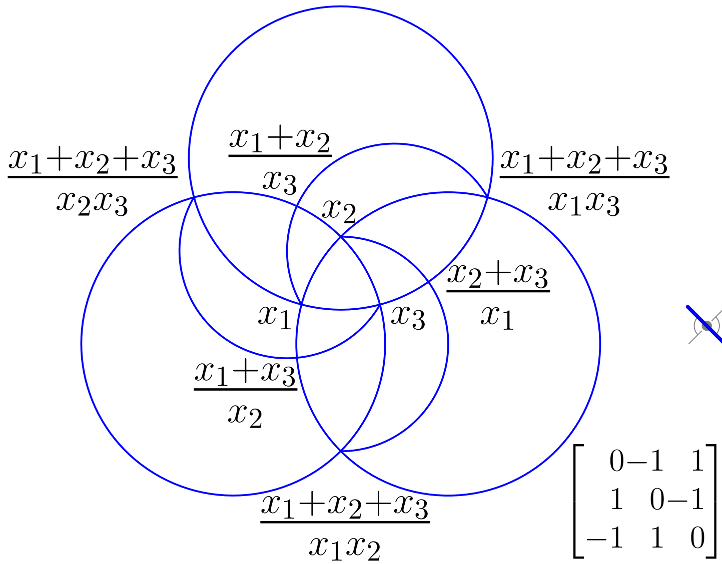
for any positive vector $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$ Padrol–Palu–P.–Plamondon ('19$^+$)

Fundamental exms: $\boldsymbol{g}$-vector fans of cluster-like complexes



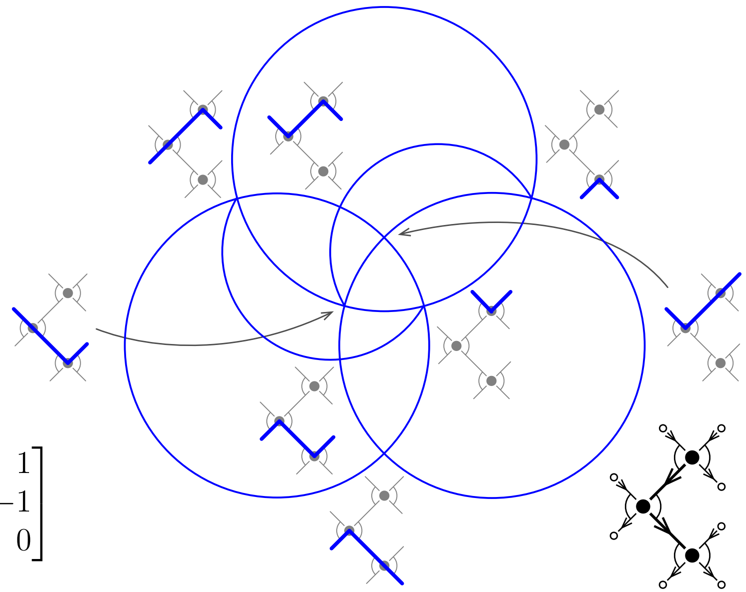
sylvester fans

Arkani-Hamed–Bai–He–Yan ('18)



finite type $\boldsymbol{g}$-vector fans
wrt any seed (acyclic or not)

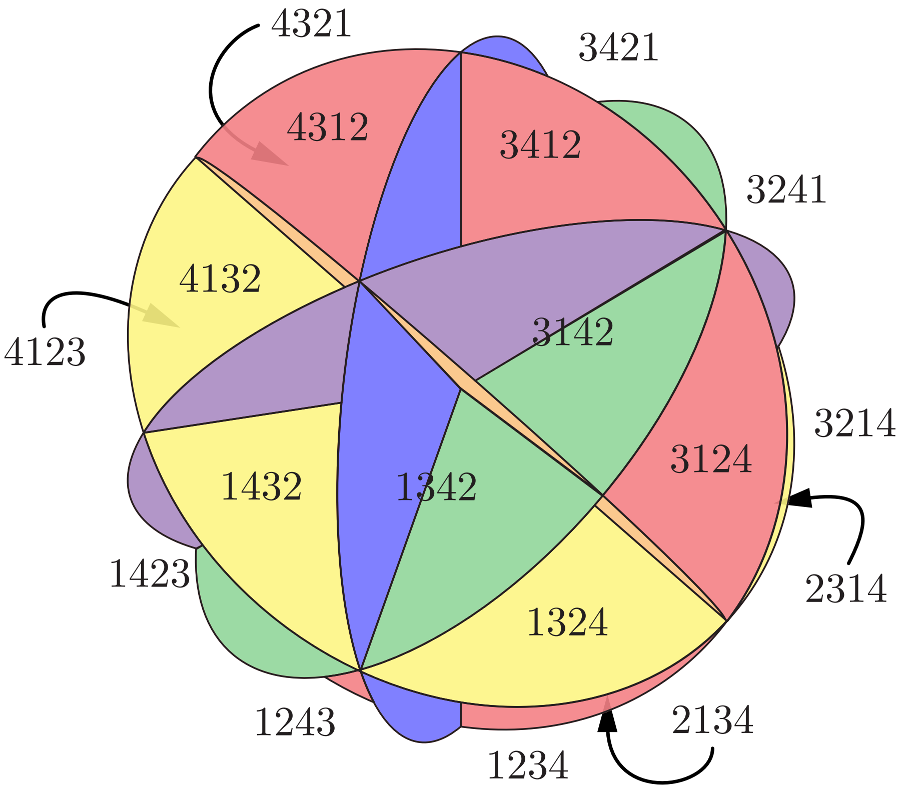
BMCLDMTY ('18$^+$)

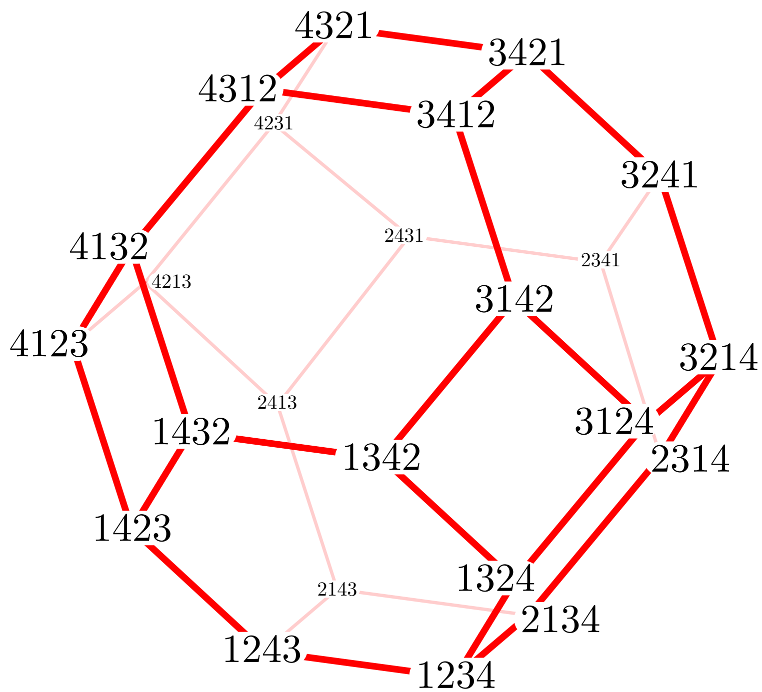


finite gentle fans
for brick and 2-acyclic quivers

Palu–P.–Plamondon ('18)

SUBMODULAR FUNCTIONS





closed type cone of braid fan = {deformed permutahedra} = {submodular functions}

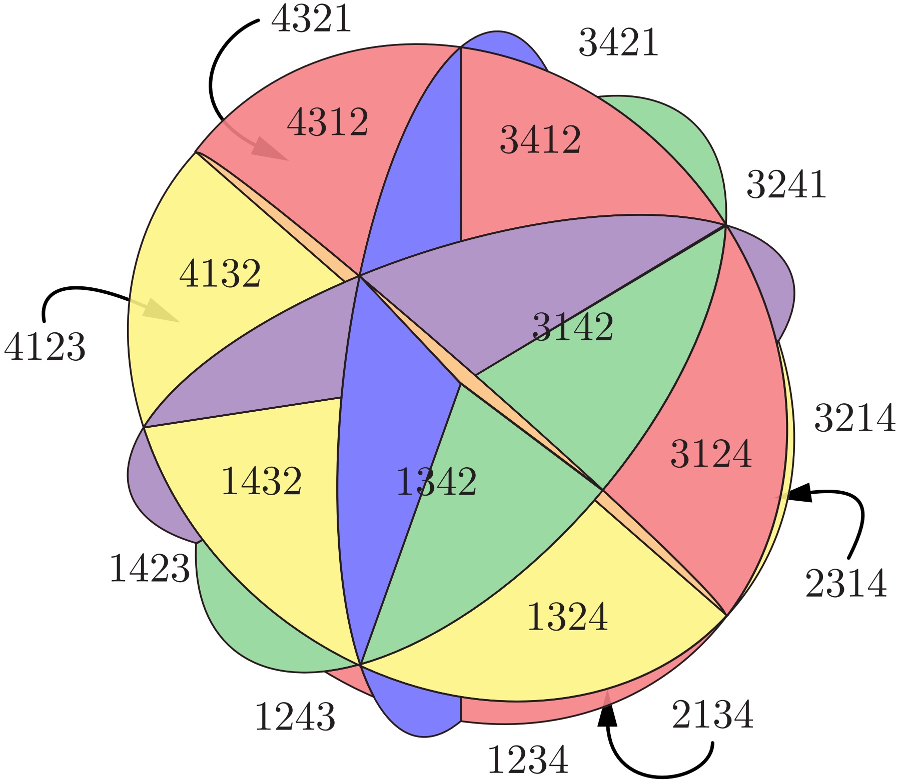
deformed permutahedron = polytope whose normal fan coarsens the braid fan

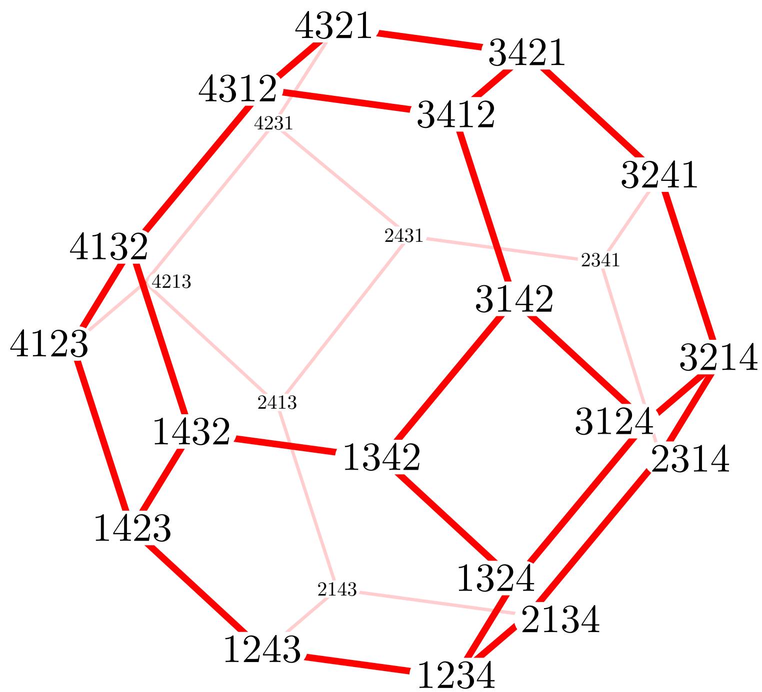
$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \langle\, \mathbb{1} \mid \boldsymbol{x} \,\rangle = z_{[n]} \text{ and } \langle\, \mathbb{1}_R \mid \boldsymbol{x} \,\rangle \ge z_R \text{ for all } R \subseteq [n] \right\}$$

for some vector $\boldsymbol{z} \in \mathbb{R}^{2^{[n]}}$ such that $z_R + z_S \le z_{R\cup S} + z_{R\cap S}$ and $z_\varnothing = 0$

Postnikov ('09) Postnikov–Reiner–Williams ('08)

SUBMODULAR FUNCTIONS





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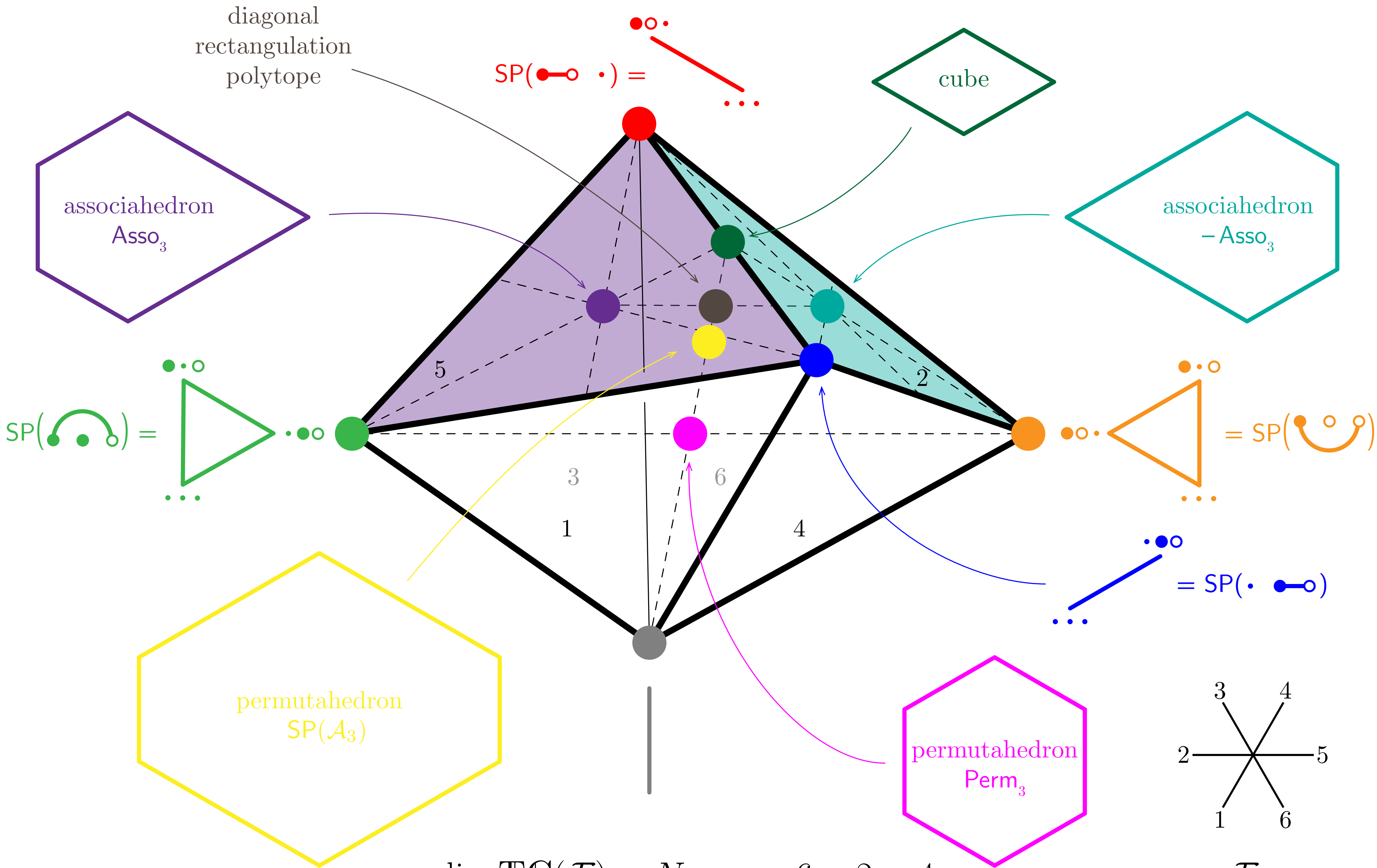
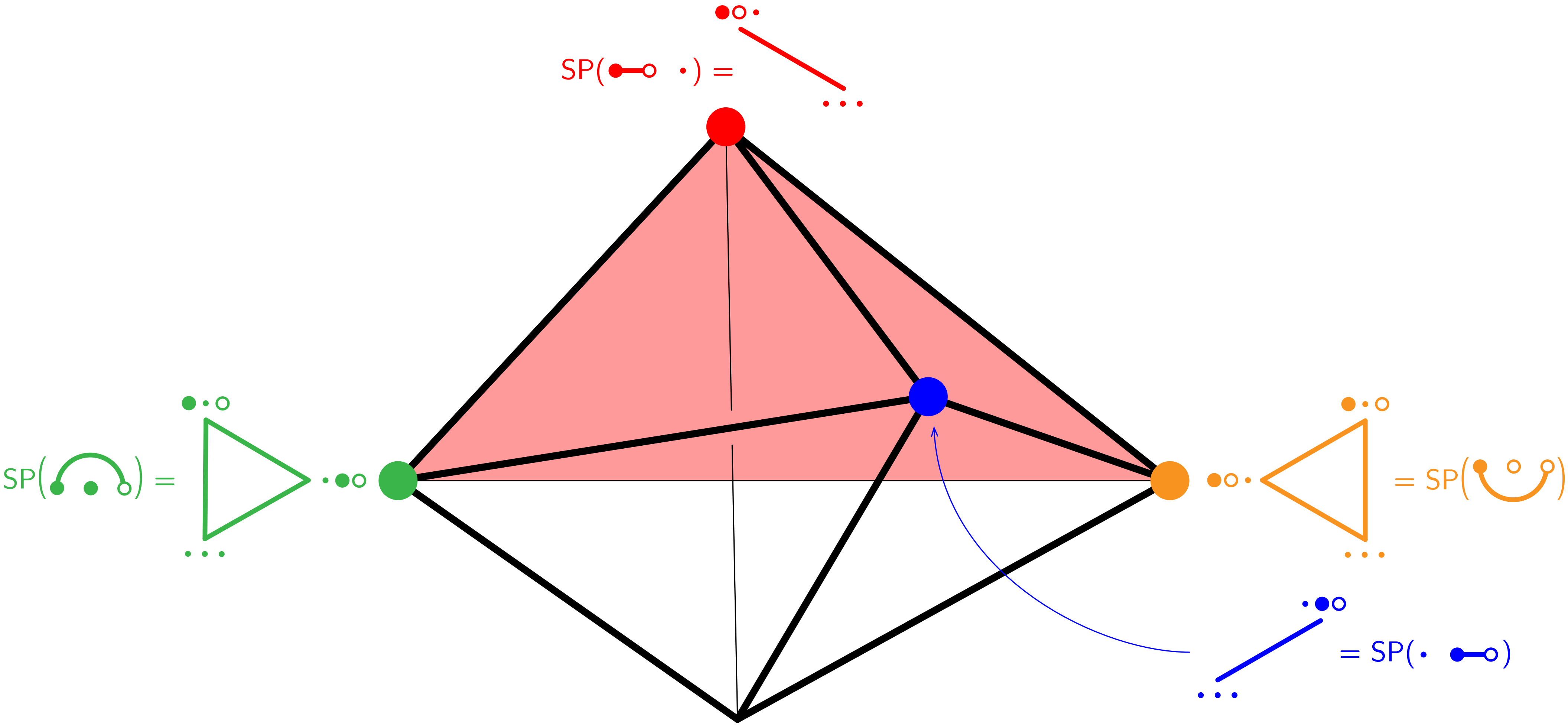
where $\mathcal{J} = \{J \subseteq [n] \mid |J| \geq 2\}$ Postnikov ('09) Postnikov–Reiner–Williams ('08)

SUBMODULAR FUNCTIONS



$$\dim \mathbb{TC}(\mathcal{F}) = N - n = 6 - 2 = 4$$

SUBMODULAR FUNCTIONS



SUBMODULAR FUNCTIONS

all quotientopes of PS ('18)
are Minkowski sums of
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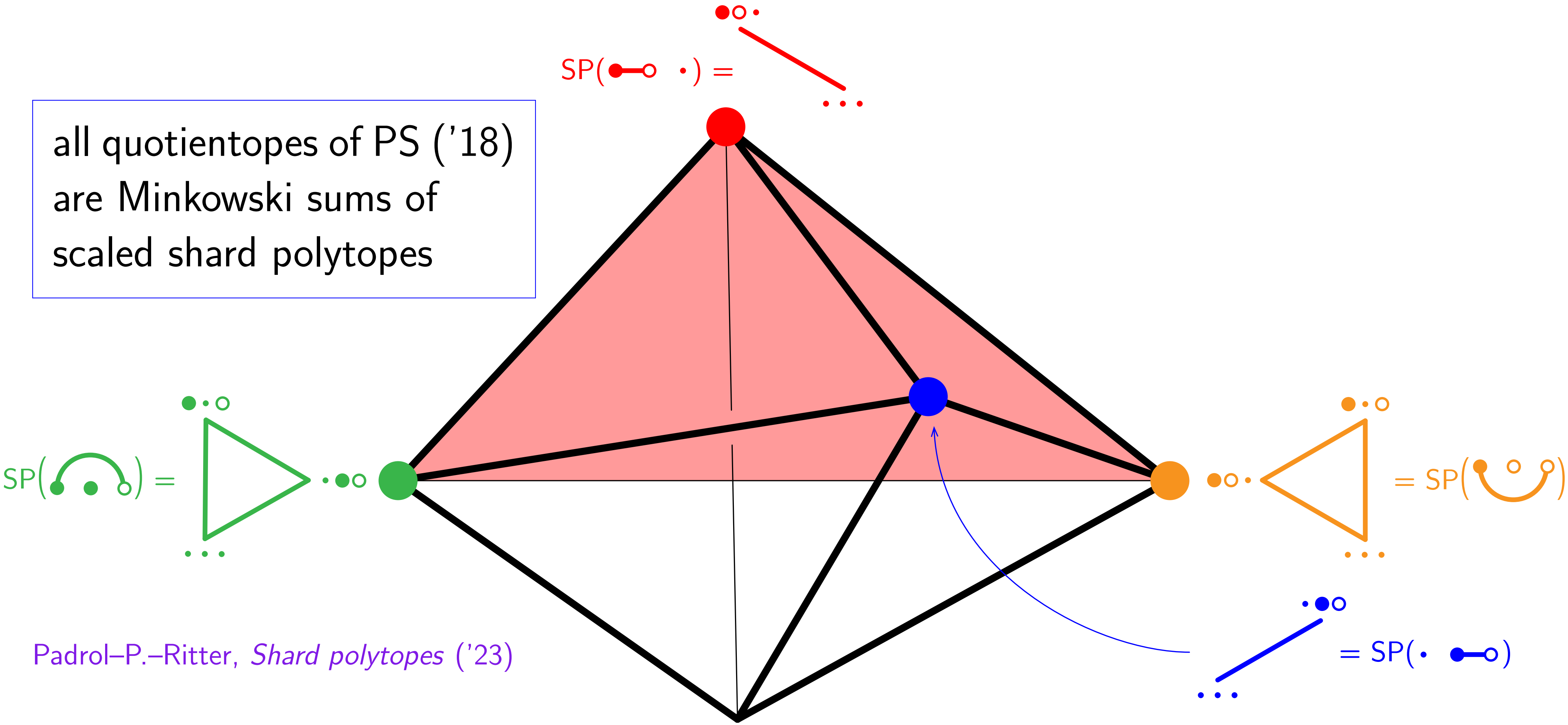

Padrol–P.–Ritter, *Shard polytopes* ('23)

SUBMODULAR FUNCTIONS

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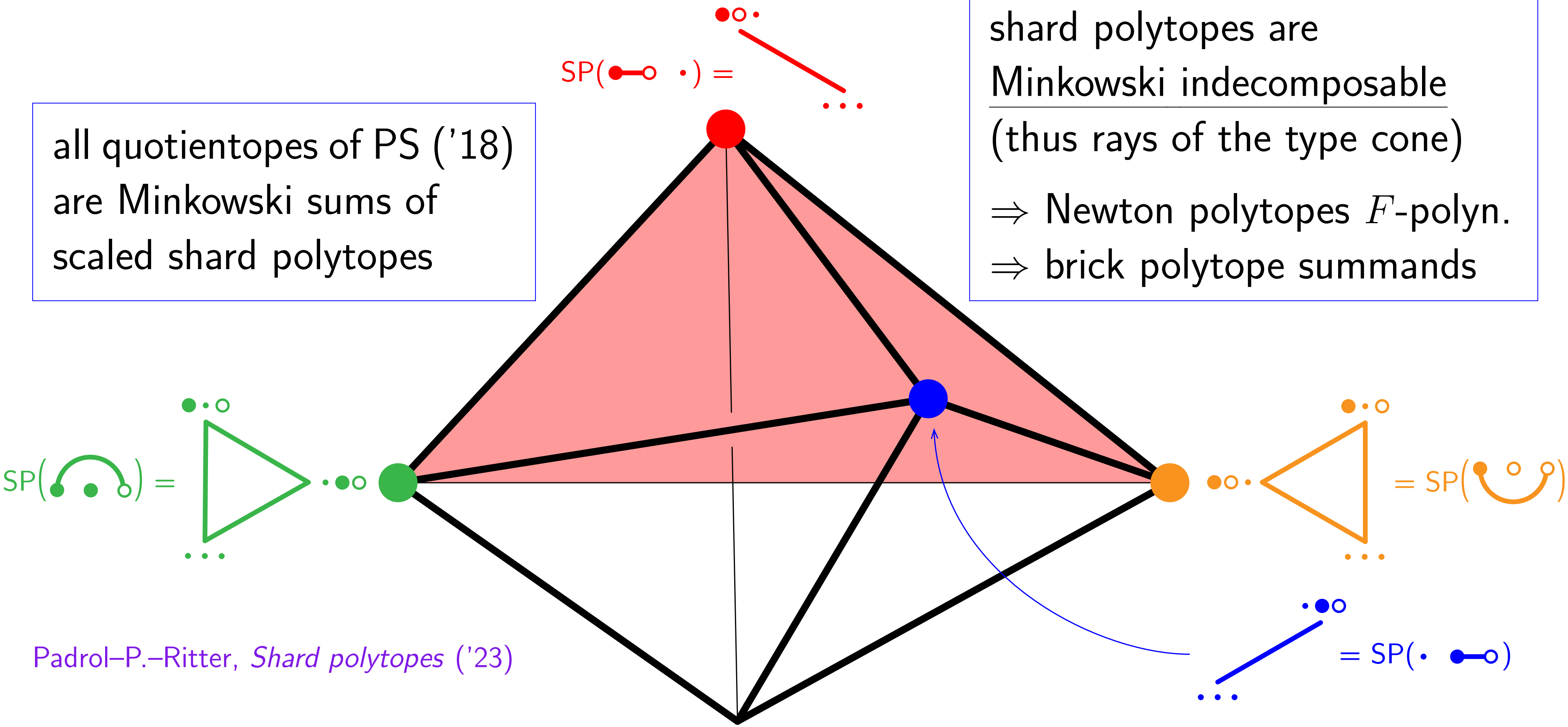
⇒ Newton polytopes F-polyn.

⇒ brick polytope summands



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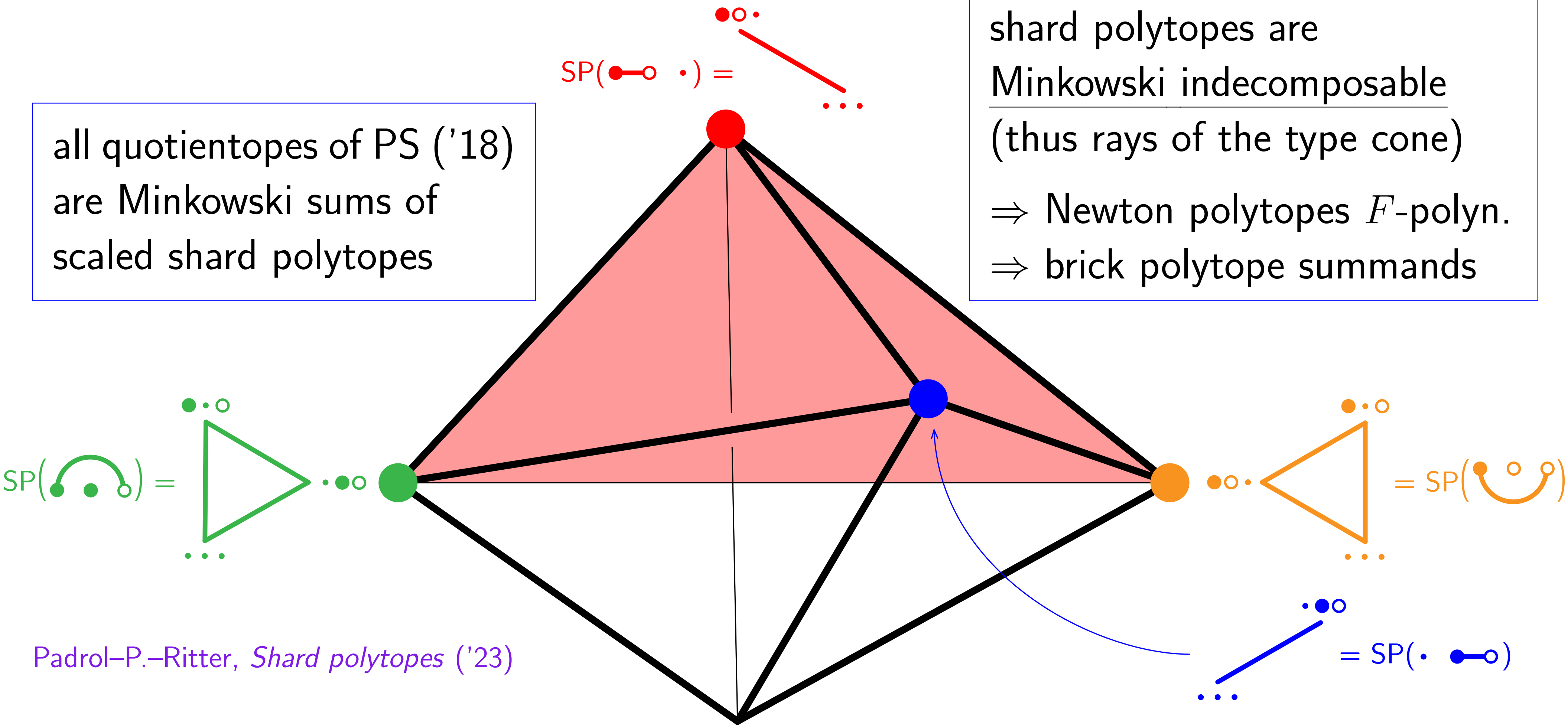
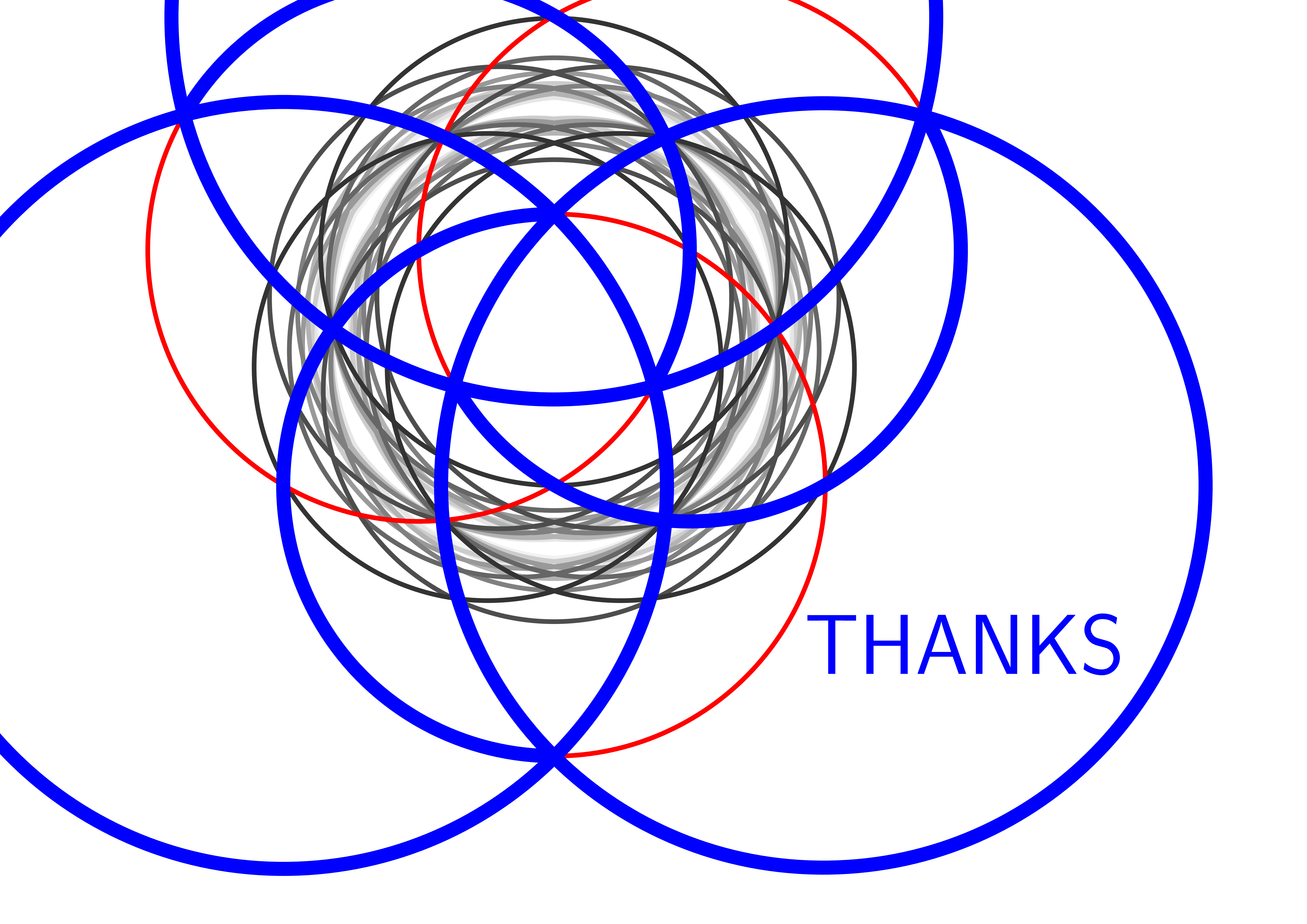


Padrol–P.–Ritter, *Shard polytopes* ('23)

Any deformed permutahedron is a Minkowski sum and difference of shard polytopes

$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \sum_{J \in \mathcal{J}} y_J \, \triangle_J = \sum_{I \in \mathcal{J}} s_I \, \mathbb{SP}(\alpha_I)$$

with explicit (combinatorial) exchange matrices between the parameters $\boldsymbol{s}$, $\boldsymbol{y}$ and $\boldsymbol{z}$


THANKS