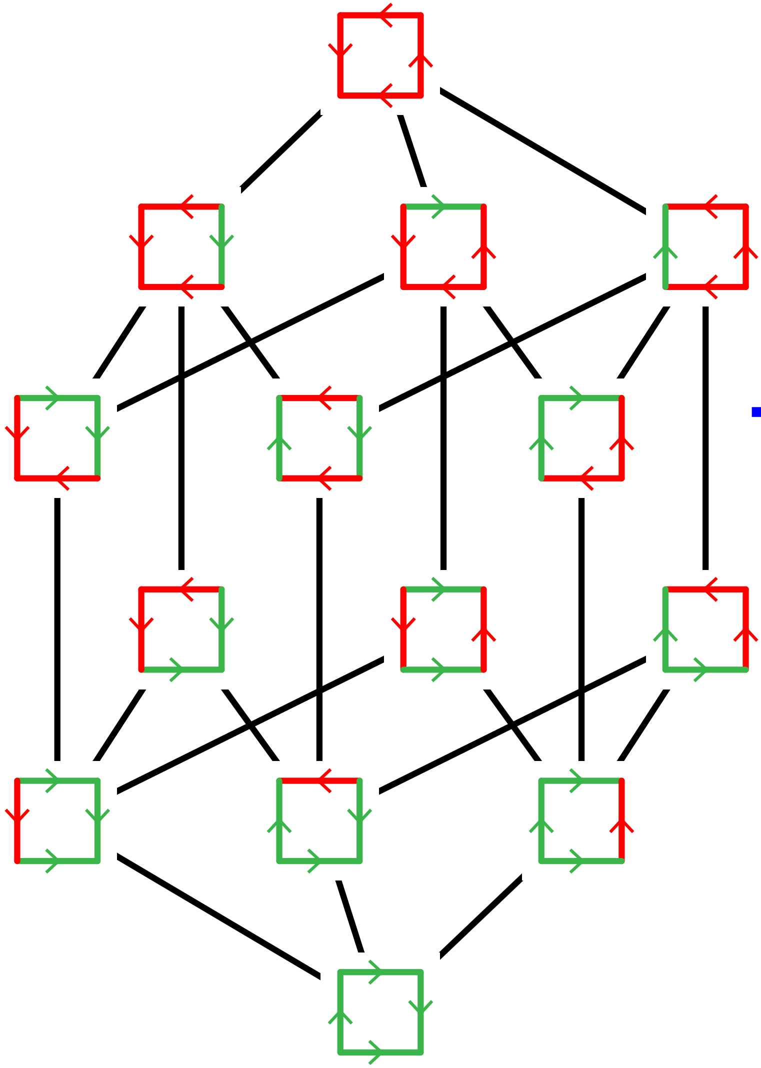

2. ACYCLIC REORIENTATION LATTICES AND THEIR QUOTIENTS

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arXiv:2111.12387

May 6, 2024
ISM Discovery School

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS?

Björner-Edelman-Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)
Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Lattice theory of the poset of regions* ('16)
Padrol–P.–Ritter, *Shard polytopes* ('23)
P., *Acyclic reorientation lattices and their lattice quotients* ('21$^+$)
Dana–Hanson–Thomas, *Shard polytopes via representation theory* (24$^+$)

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS?

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$

B distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$

inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C

poset of regions $\mathrm{PR}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

THM. The poset of regions $\mathrm{PR}(\mathcal{H}, B)$

- is never a lattice when B is not a simple region,
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement.

Björner-Edelman-Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)

THM. If $\mathrm{PR}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a lattice congruence of $\mathrm{PR}(\mathcal{H}, B)$, the cones obtained by glueing together the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan.

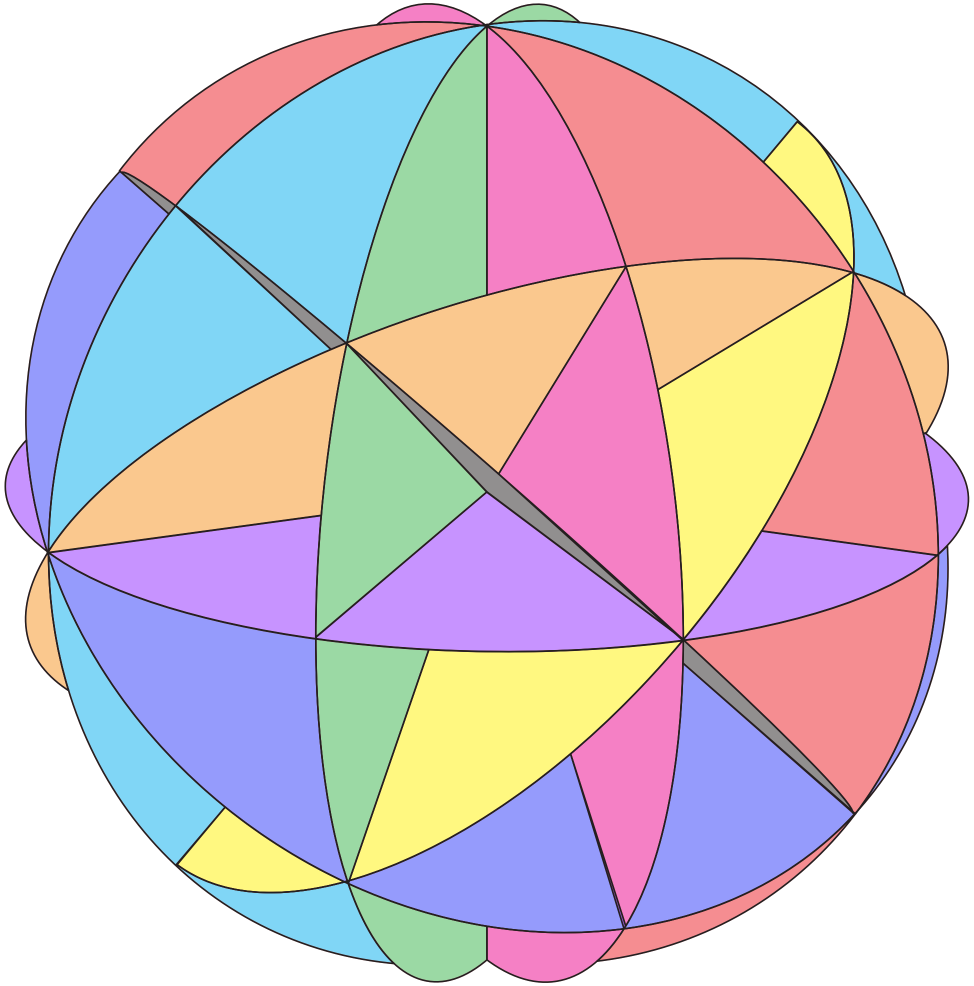
Reading, *Lattice congruences, fans and Hopf algebras* ('05)

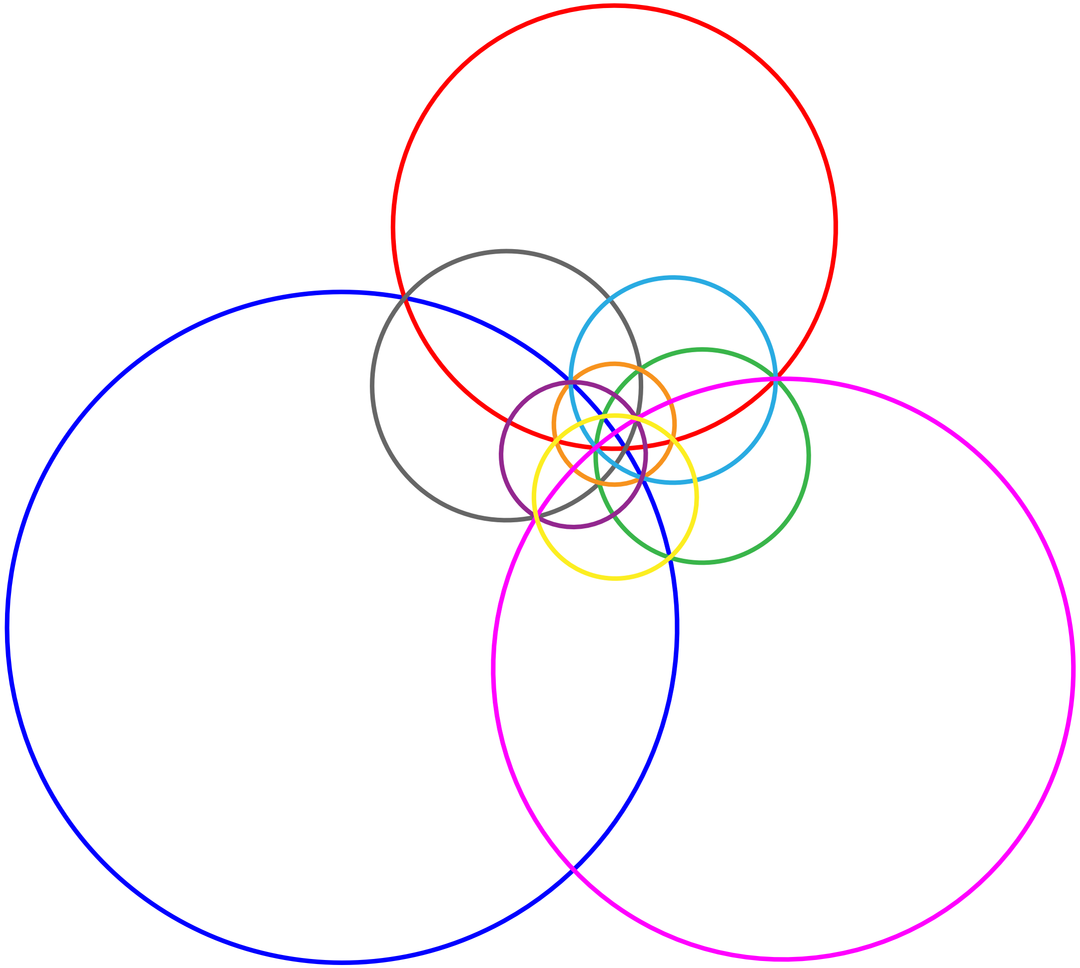
Is the quotient fan polytopal?

SHARDS FOR HYPERPLANE ARRANGEMENTS

shard = piece of hyperplane obtained after cutting all rank 2 subgroups

shard poset = (pre)poset of forcing relations among shards





Reading, *Lattice and order properties of the poset of regions in a hyperplane arrangement* ('03)

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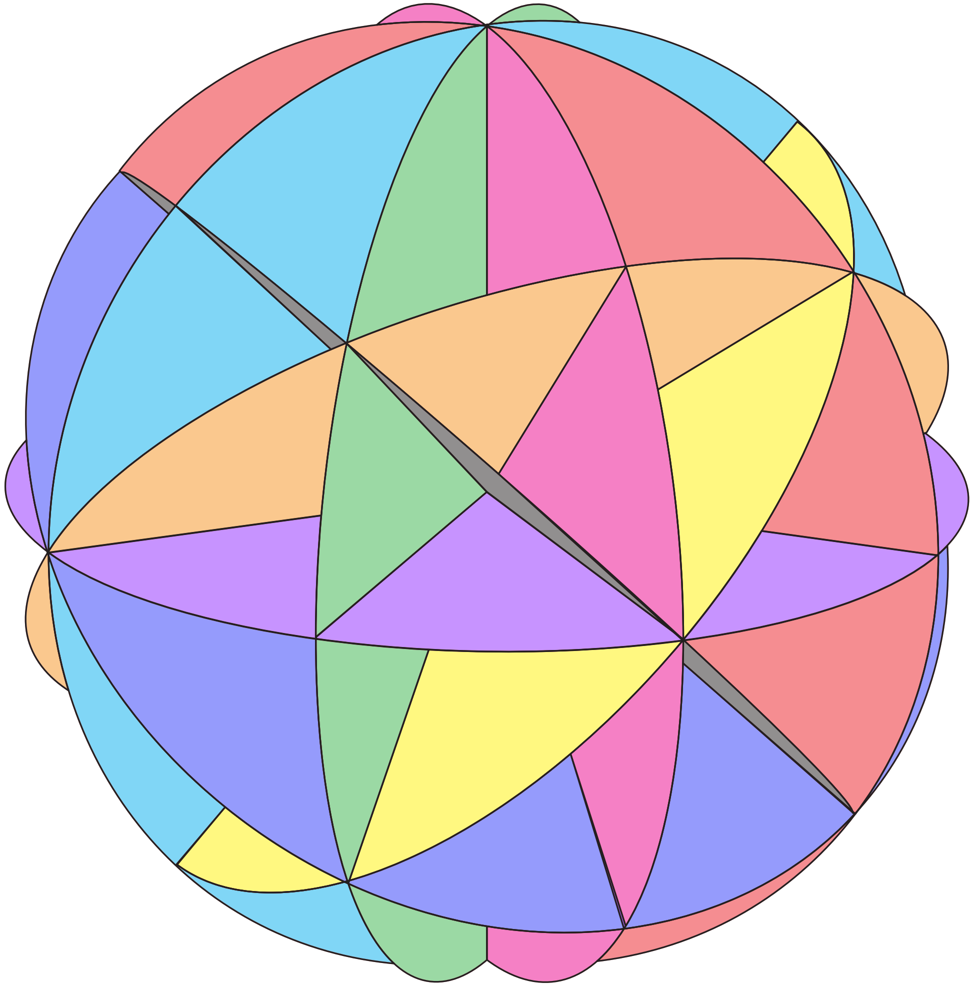

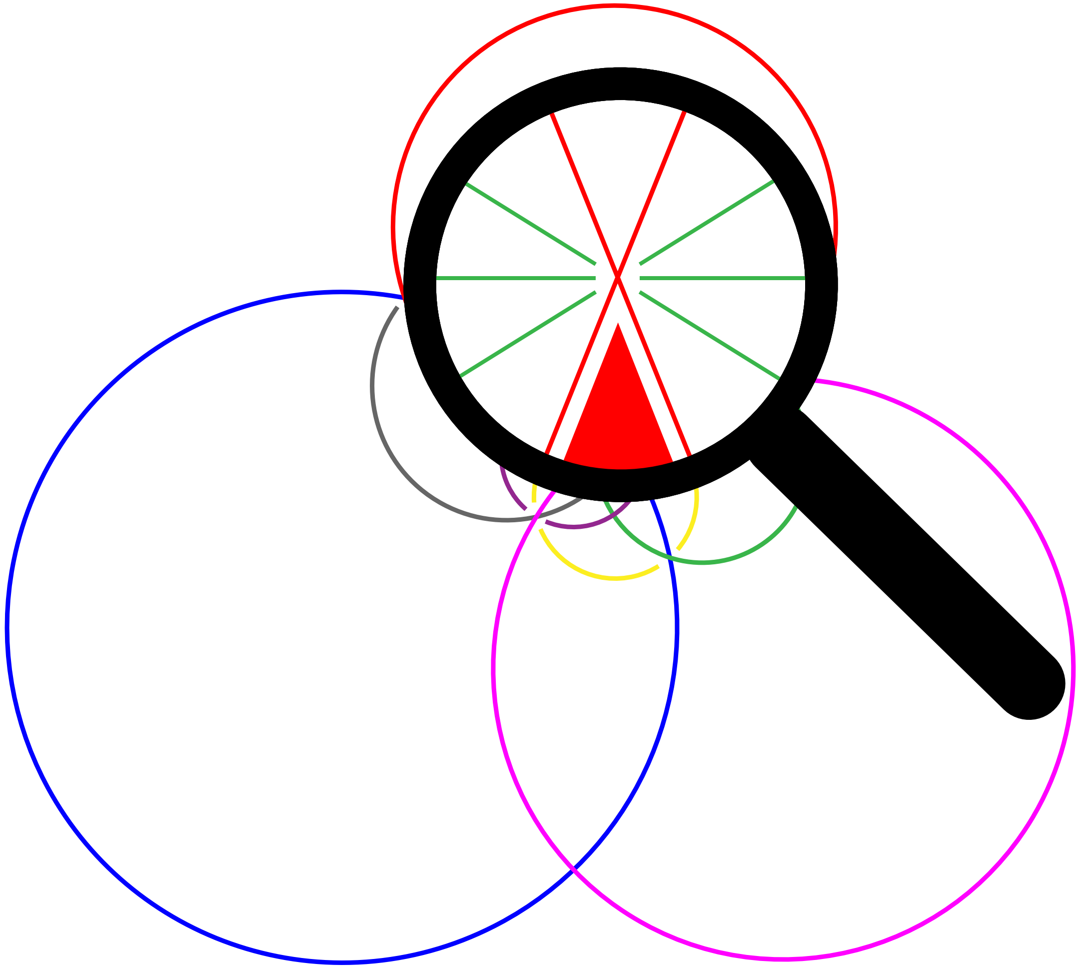

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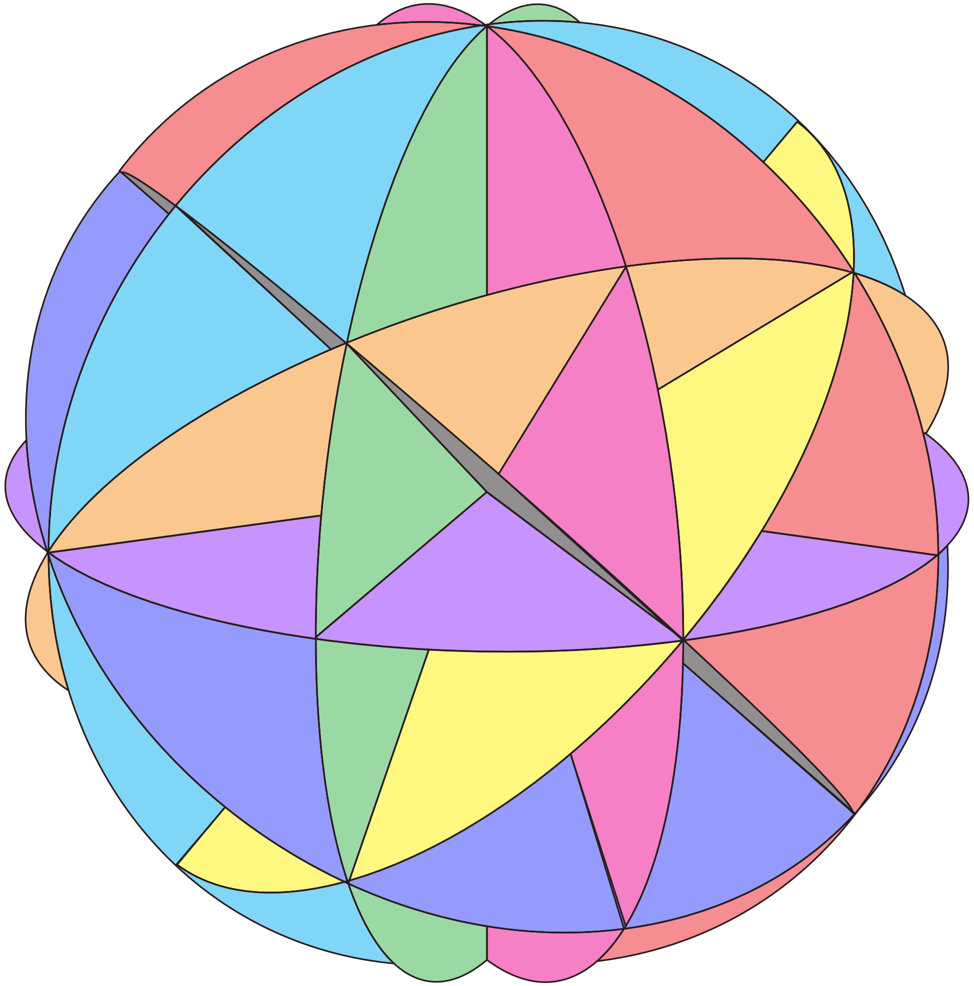

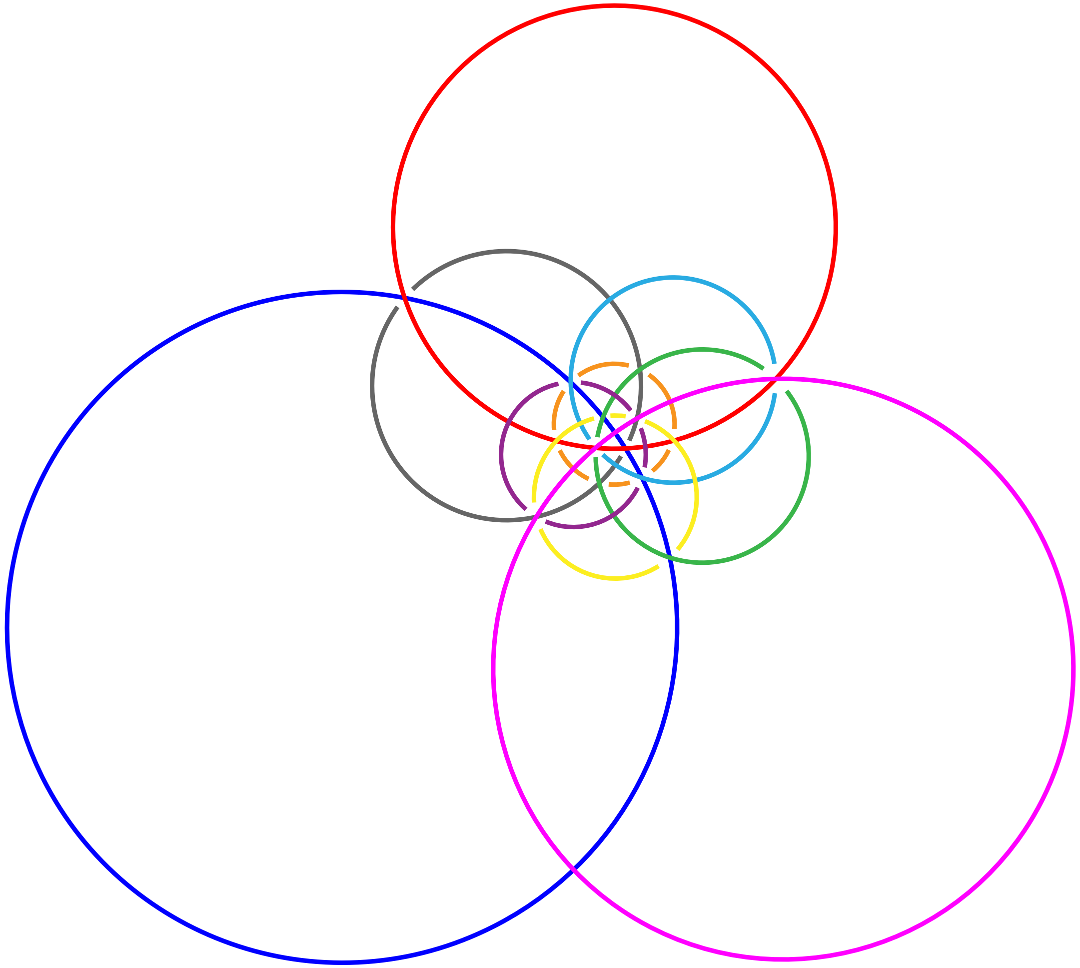

Reading, *Lattice and order properties of the poset of regions in a hyperplane arrangement* ('03)

SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

shard = piece of hyperplane obtained after cutting all rank 2 subgroups
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shard polytope for a shard Σ = polytope such that the union of walls of its normal fan
- contains the shard Σ,
- is contained in the union of the shards forcing Σ

Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

If any shard Σ admits a shard polytope $\mathbb{SP}(\Sigma)$, then

- for any lattice congruence $\equiv$ of $\mathrm{PR}(\mathcal{H}, B)$, the quotient fan $\mathcal{F}_\equiv$ is the normal of the Minkowski sum of the shard polytopes $\mathbb{SP}(\Sigma)$ for Σ in the shard ideal $\boldsymbol{\Sigma}_\equiv$
- if the arrangement $\mathcal{H}$ is simplicial, then the shard polytopes $\mathbb{SP}(\Sigma)$ form a basis for the type cone of the fan defined by $\mathcal{H}$ (up to translation)

Padrol–P.–Ritter, *Shard polytopes* ('23)

SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

Partial answers: Shard polytopes exist for
- type A and B Coxeter arrangements — Padrol–P.–Ritter, *Shard polytopes* ('23)
- all graphical arrangements whose poset of regions is a semidistributive lattice — P., *Acyclic reorientation lattices and their lattice quotients* ('21$^+$)
- Coxeter arrangements of simply laced types (A, D, E) — Dana–Hanson–Thomas, *Shard polytopes via representation theory* (24$^+$)

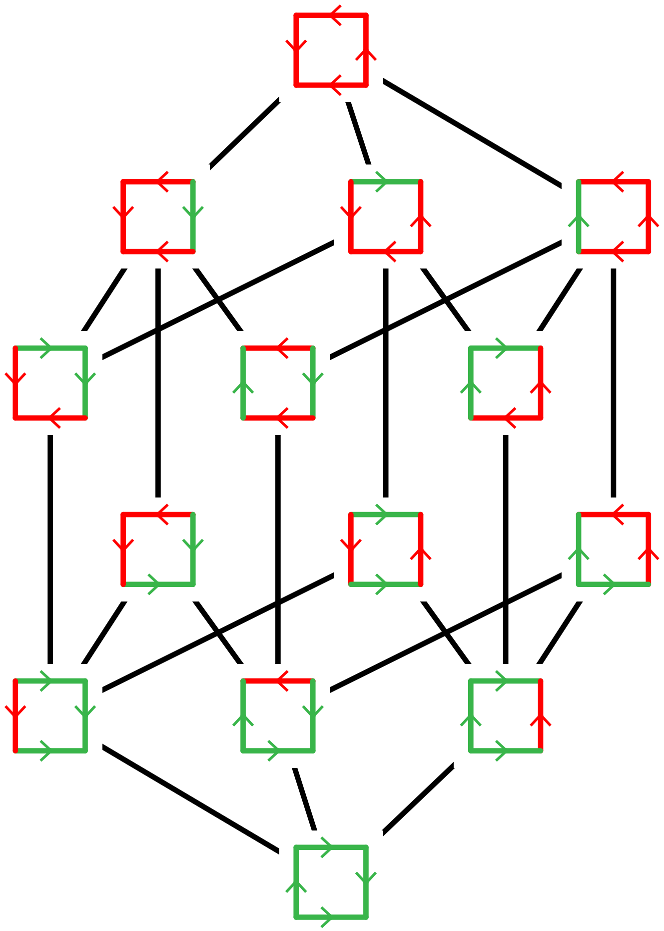
ACYCLIC REORIENTATION LATTICES

P., *Acyclic reorientation lattices and their lattice quotients* ('21^+)

ACYCLIC REORIENTATION POSETS

D directed acyclic graph

$\mathcal{AR}_D$ = all acyclic reorientations of D, ordered by inclusion of their sets of reversed arcs



minimal element D

maximal element $\bar{D}$

self-dual under reversing all arcs

cover relations = flipping a single arc

flippable arcs of E = transitive reduction of E

$= E \setminus \{(u,v) \in E \mid \exists$ directed path $u \rightsquigarrow v$ in $E\}$

ACYCLIC REORIENTATION POSETS

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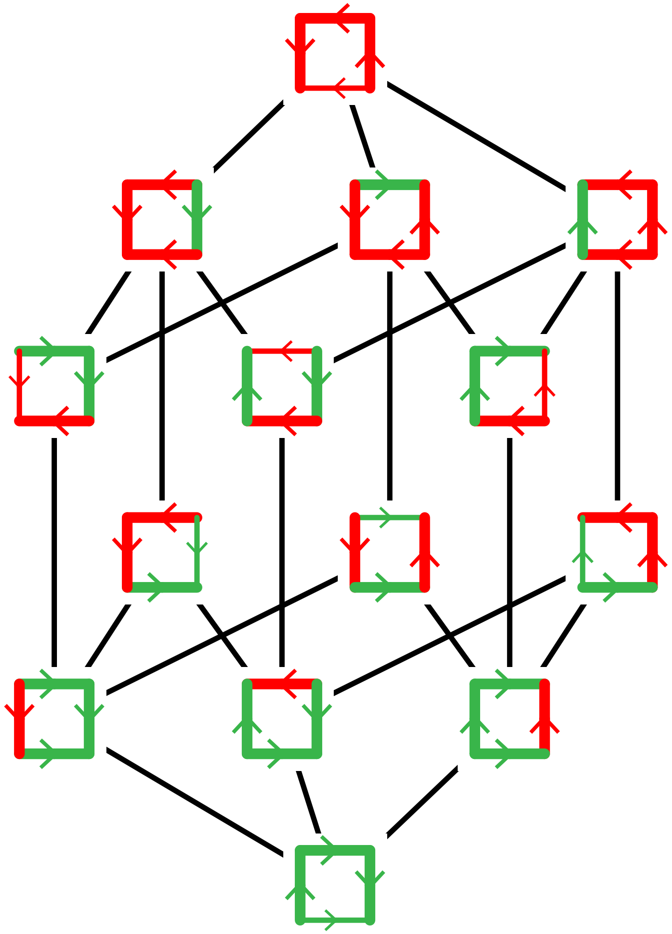

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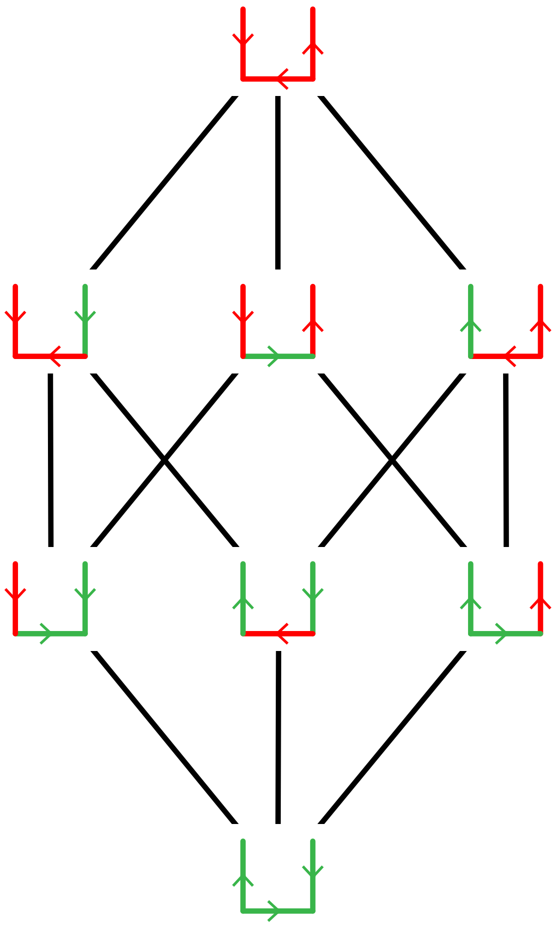
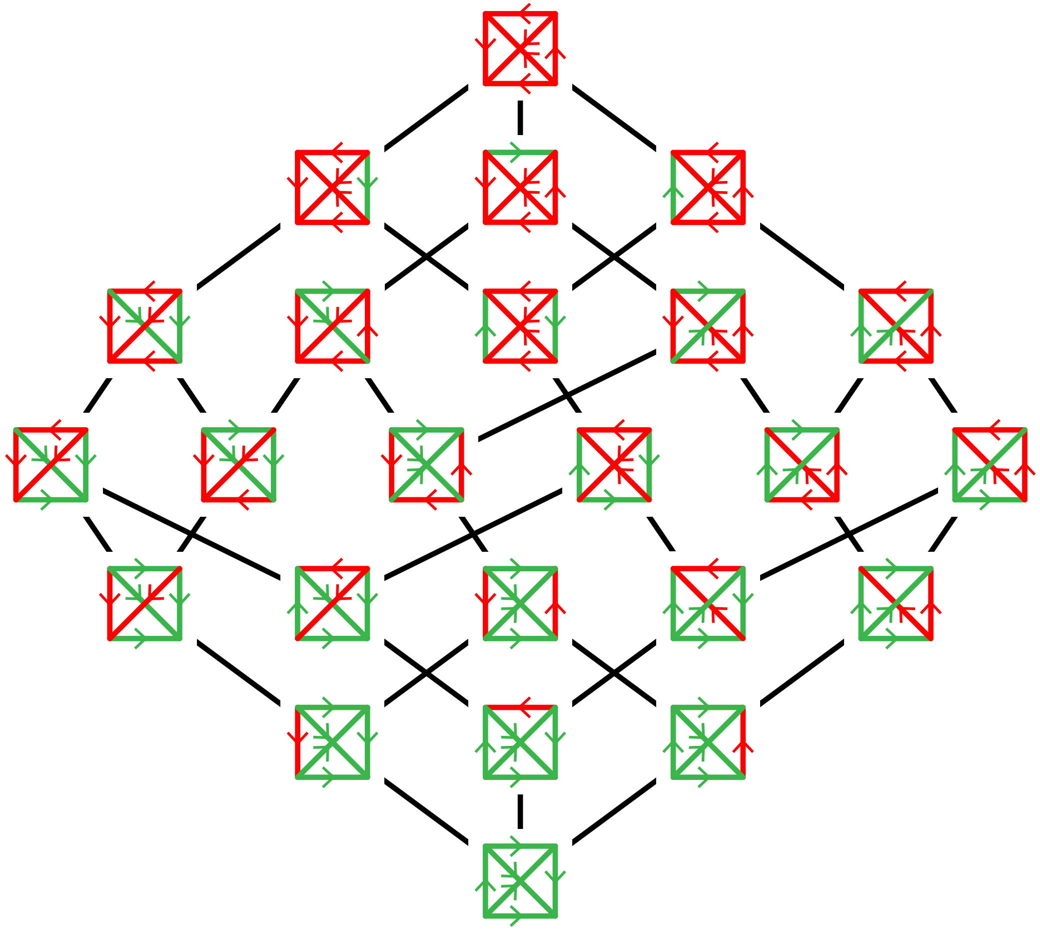
D forest



boolean lattice

D tournament

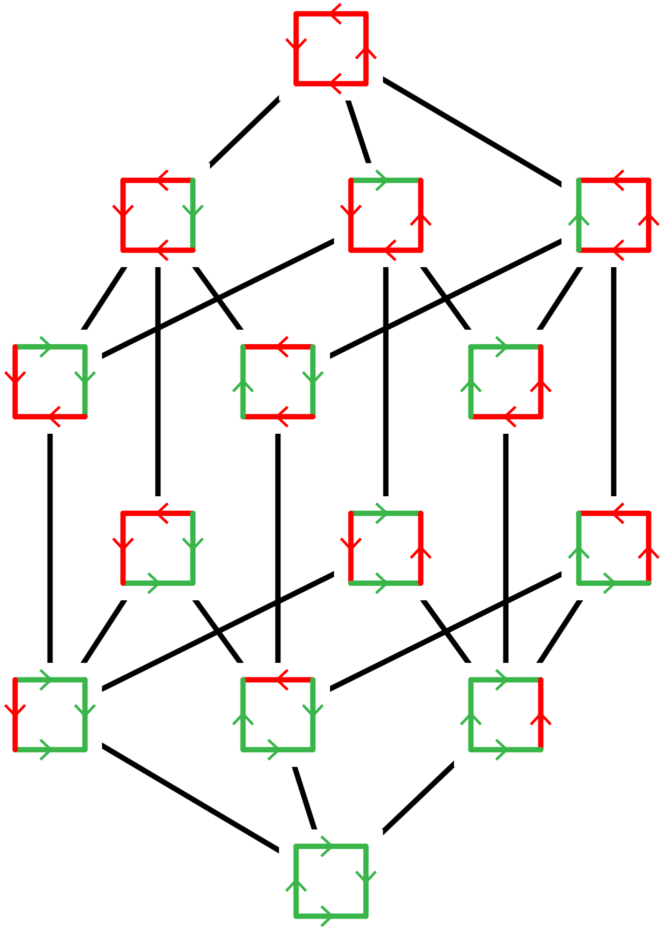


weak order

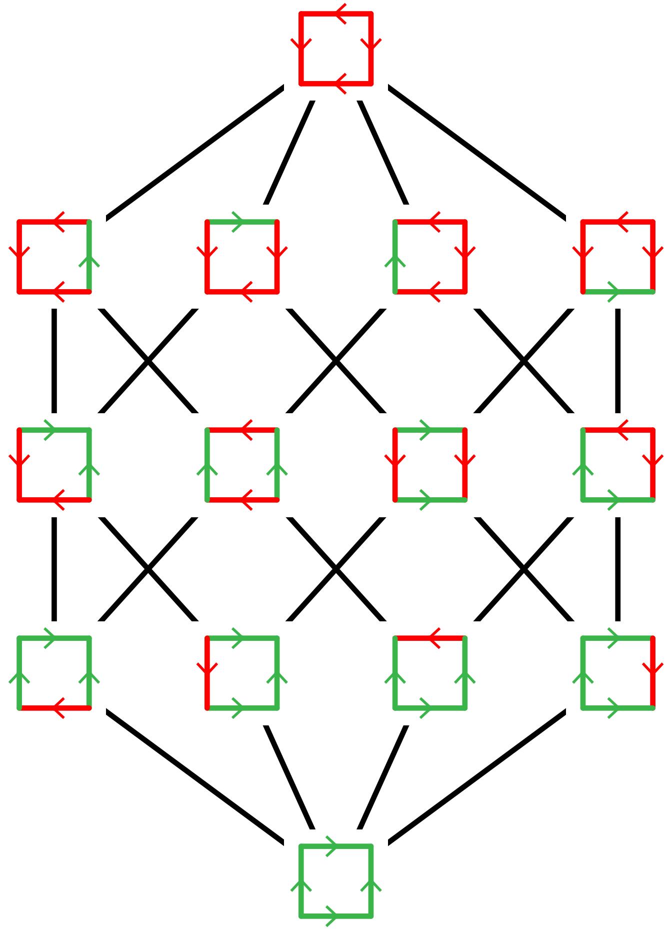
ACYCLIC REORIENTATION LATTICES

D vertebrate = transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate P. ('21$^+$)



lattice



not lattice

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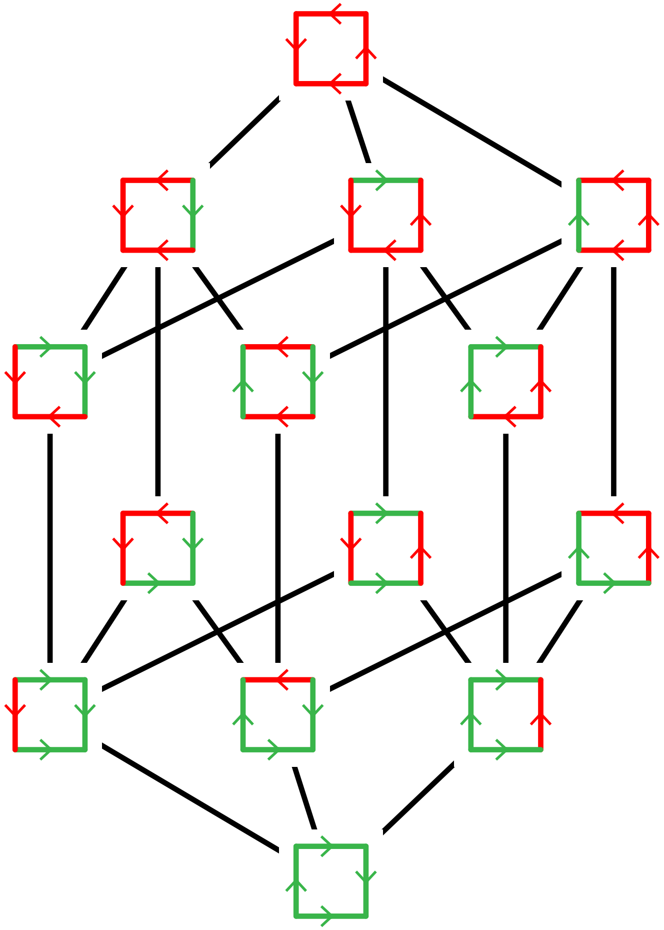
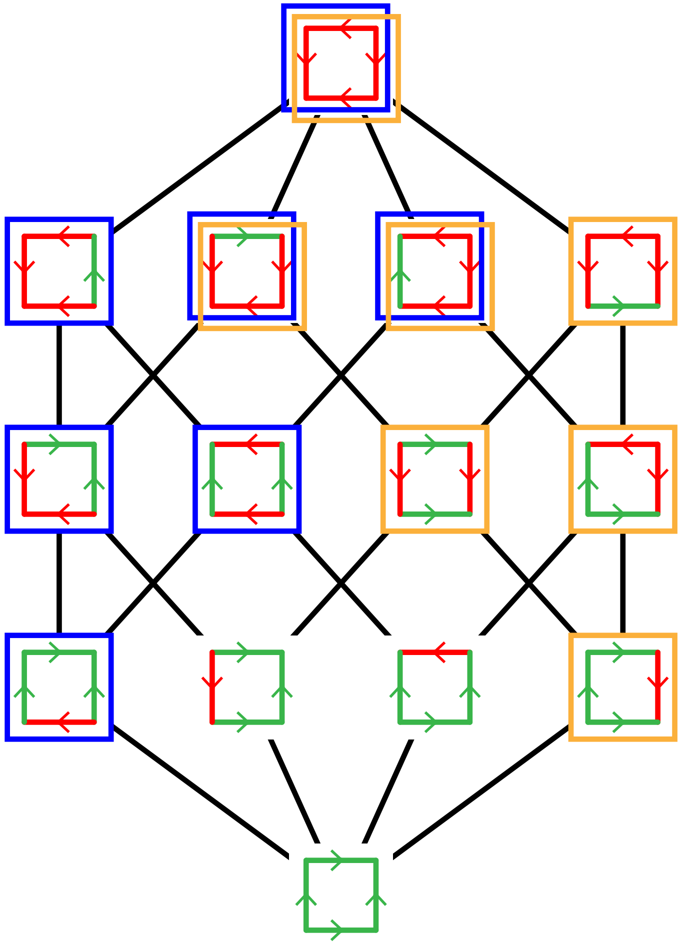


lattice



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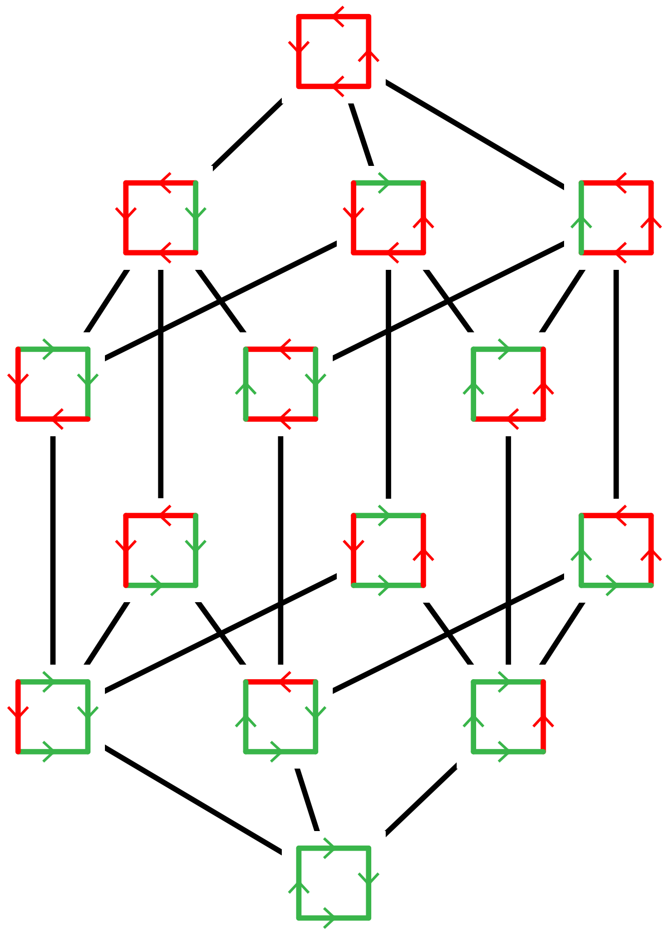


X subset of arcs of D is

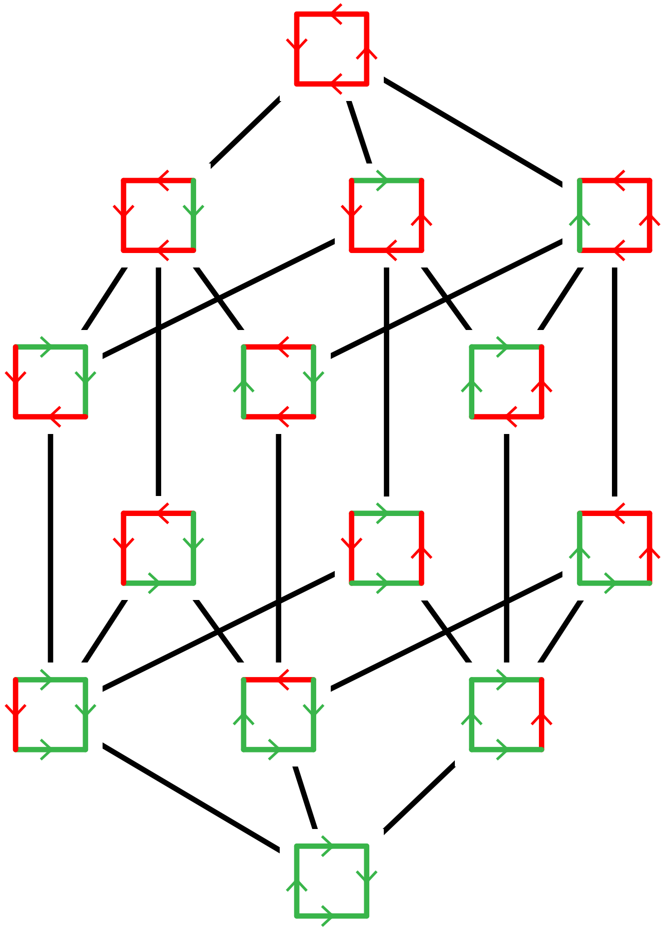
- closed if all arcs of D in the transitive closure of X also belong to X
- coclosed if its complement is closed
- biclosed if it is closed and coclosed

PROP. If D vertebrate, P. ('21$^+$)
X biclosed $\iff$ the reorientation of X is acyclic

ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

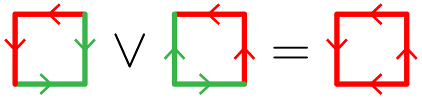
THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate P. ('21$^+$)

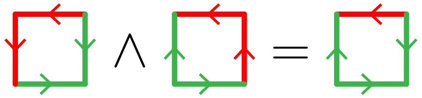


PROP. If D vertebrate, P. ('21$^+$)

$$\mathrm{bwd}(E_1 \vee \ldots \vee E_k) =$$
$$\text{transitive closure of } \mathrm{bwd}(E_1) \cup \cdots \cup \mathrm{bwd}(E_k)$$

$$\mathrm{fwd}(E_1 \wedge \ldots \wedge E_k) =$$
$$\text{transitive closure of } \mathrm{fwd}(E_1) \cup \cdots \cup \mathrm{fwd}(E_k)$$



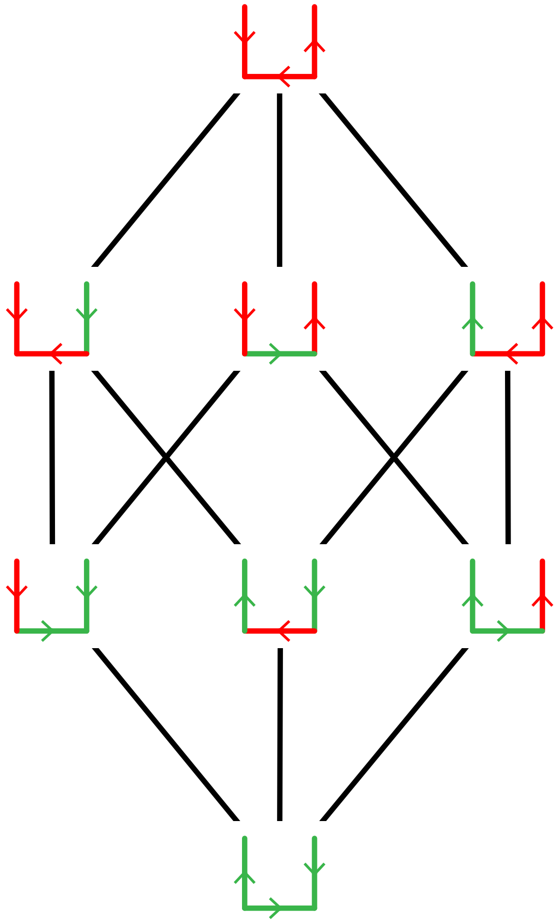


DISTRIBUTIVITY & SEMIDISTRIBUTIVITY

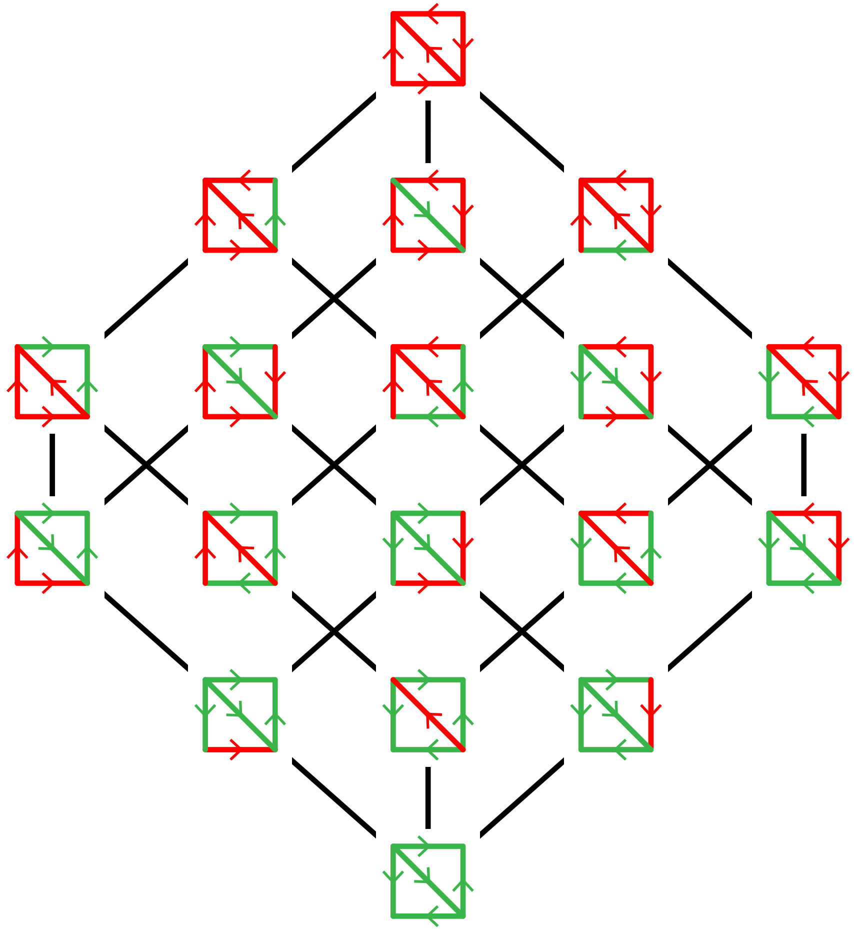
P., *Acyclic reorientation lattices and their lattice quotients* ('21$^+$)

DISTRIBUTIVE ACYCLIC REORIENTATION POSETS

THM. $\mathcal{AR}_D$ distributive lattice $\iff$ D forest $\iff$ $\mathcal{AR}_D$ boolean lattice P. ('21$^+$)



distributive



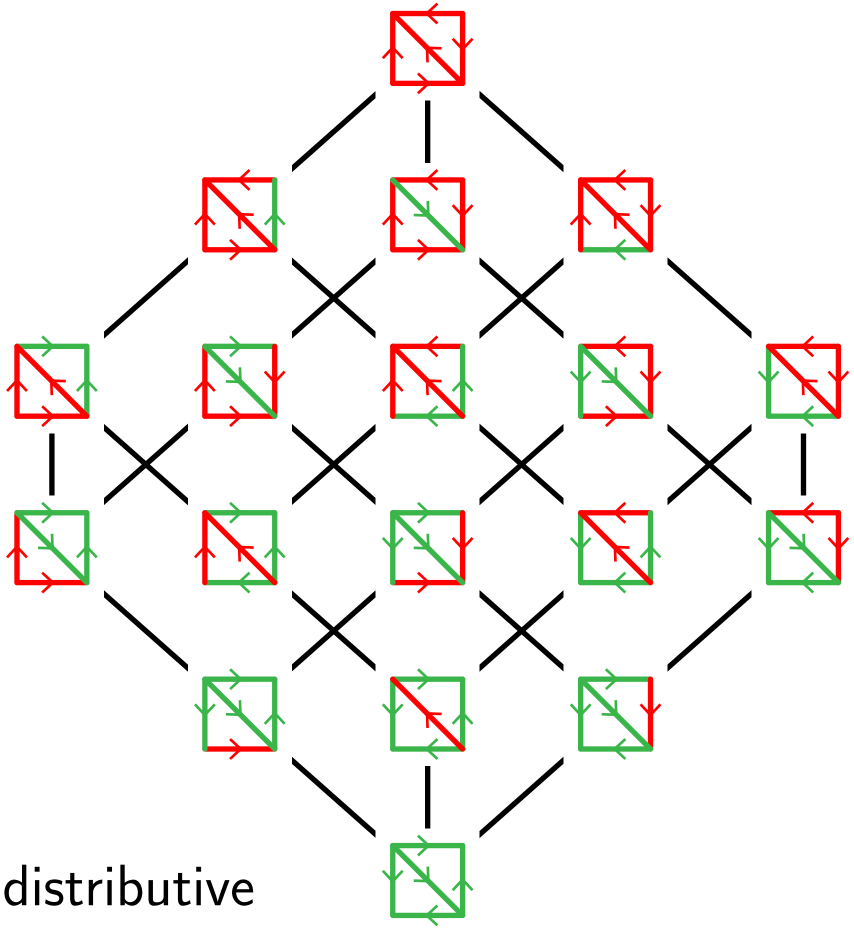
not distributive

SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

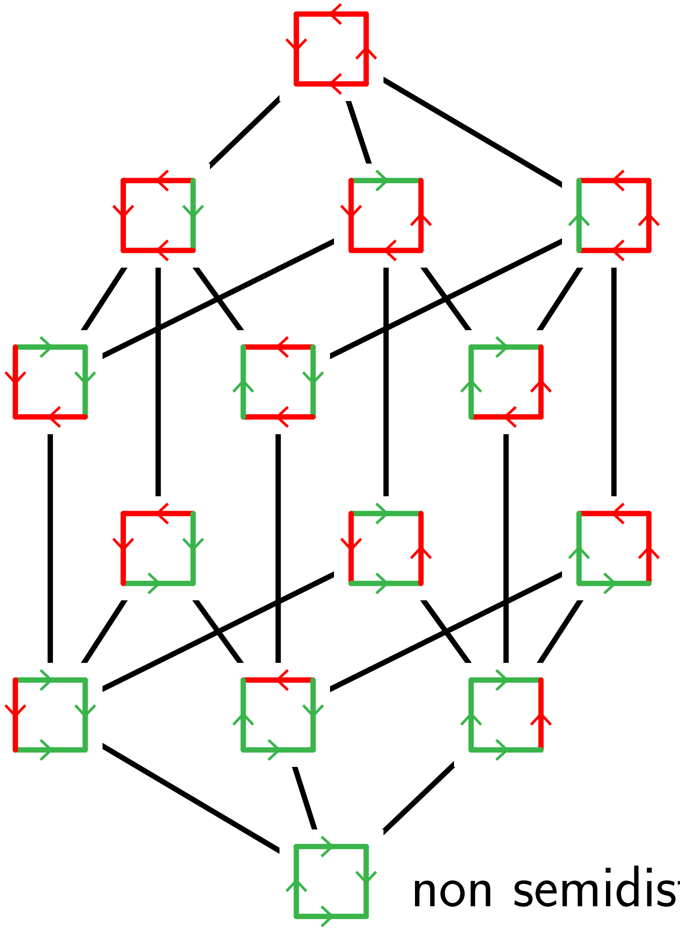
D skeletal $=$

- D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest
- D filled $=$ any directed path joining the endpoints of an arc in D induces a tournament

THM. $\mathcal{AR}_D$ semidistributive lattice $\iff$ D is skeletal — P. ('21$^+$)



semidistributive



non semidistributive

ROPES & NON-CROSSING ROPE DIAGRAMS

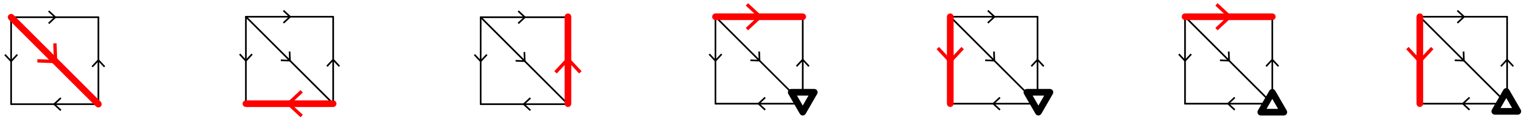
P., *Acyclic reorientation lattices and their lattice quotients* ('21$^+$)

ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \bigtriangledown, \bigtriangleup)$ where

- (u, v) is an arc of D
- $\bigtriangledown \sqcup \bigtriangleup$ partitions the transitive support of (u, v) minus $\{u, v\}$

ropes



ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \triangledown, \triangle)$ where

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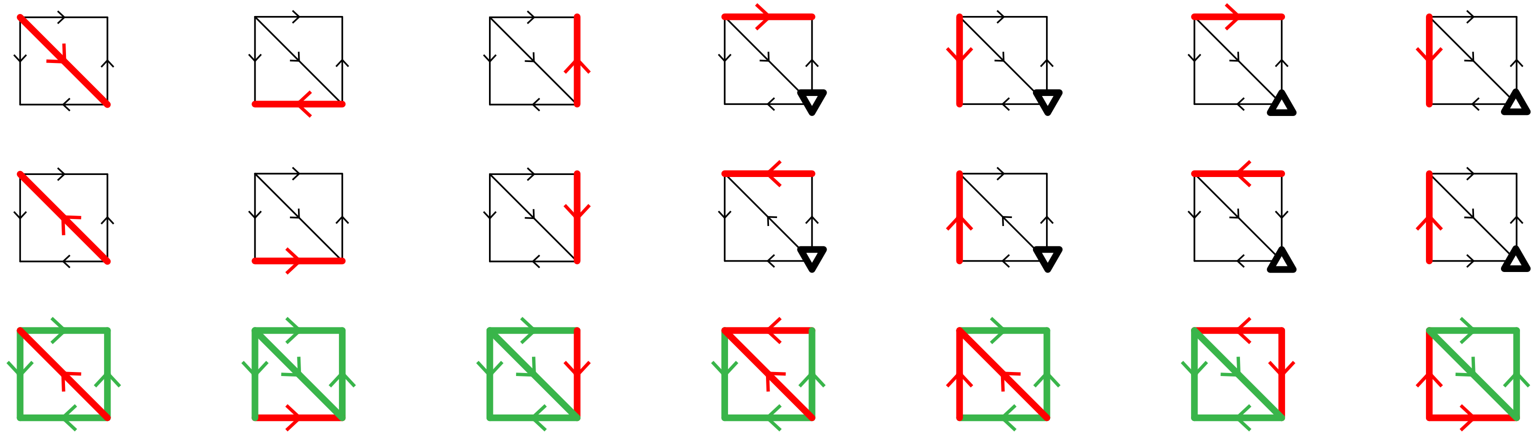
ropes

join irreducibles



THM. join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ ropes of D P. ('21$^+$)

ROPES & NON-CROSSING ROPE DIAGRAMS

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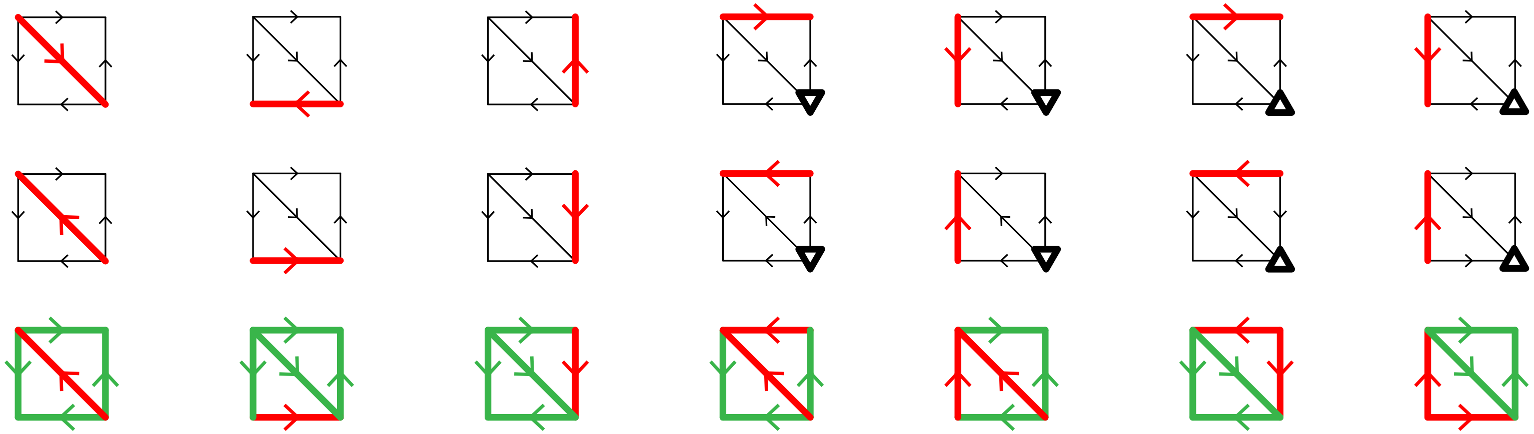
ropes

join irreducibles



THM. join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ ropes of D P. ('21$^+$)
canonical join representations of $\mathcal{AR}_D$ $\longleftrightarrow$ non-crossing rope diagrams of $\mathcal{AR}_D$

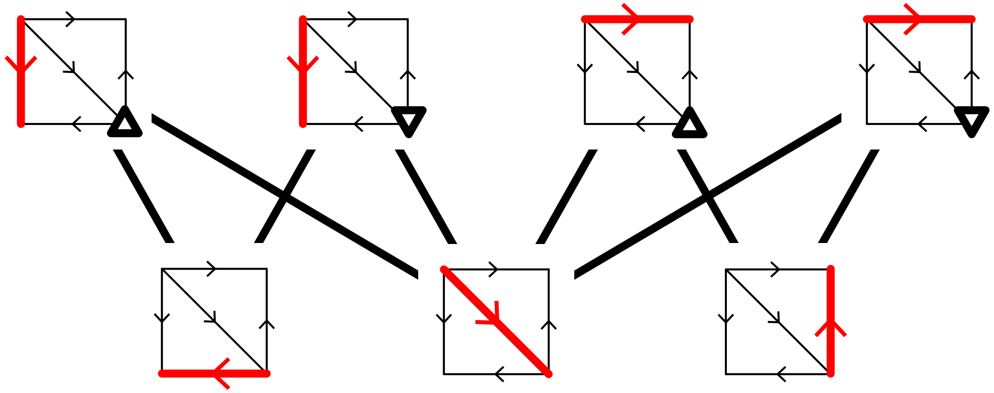
$(u, v, \bigtriangledown, \bigtriangleup)$ and $(u', v', \bigtriangledown', \bigtriangleup')$ are crossing if there are $w \neq w'$ such that
$w \in (\bigtriangledown \cup \{u, v\}) \cap (\bigtriangleup' \cup \{u', v'\})$ and $w' \in (\bigtriangleup \cup \{u, v\}) \cap (\bigtriangledown' \cup \{u', v'\})$

CONGRUENCES & QUOTIENTS

P., *Acyclic reorientation lattices and their lattice quotients* ('21$^+$)

SUBROPES ORDER

$(u, v, \bigtriangledown, \bigtriangleup)$ subrope of $(u', v', \bigtriangledown', \bigtriangleup')$ if $u, v \in \{u', v'\} \cup \bigtriangledown' \cup \bigtriangleup'$ and $\bigtriangledown \subseteq \bigtriangledown'$ and $\bigtriangleup \subseteq \bigtriangleup'$



PROP. congruence lattice of $\mathcal{AR}_D \simeq$ lower ideal lattice of subrope order P. ('21$^+$)

CORO. $\equiv$ lattice congruence of $\mathcal{AR}_D$

- E minimal in its $\equiv$-class $\iff$ $\delta(E) \subseteq \mathcal{R}_\equiv$
- quotient $\mathcal{AR}_D/\equiv \;\simeq\;$ subposet of $\mathcal{AR}_D$ induced by $\{E \in \mathcal{AR}_D \mid \delta(E) \subseteq \mathcal{R}_\equiv\}$

COHERENT CONGRUENCES

$(\mho, \Omega)$ = two of arbitrary subsets of V

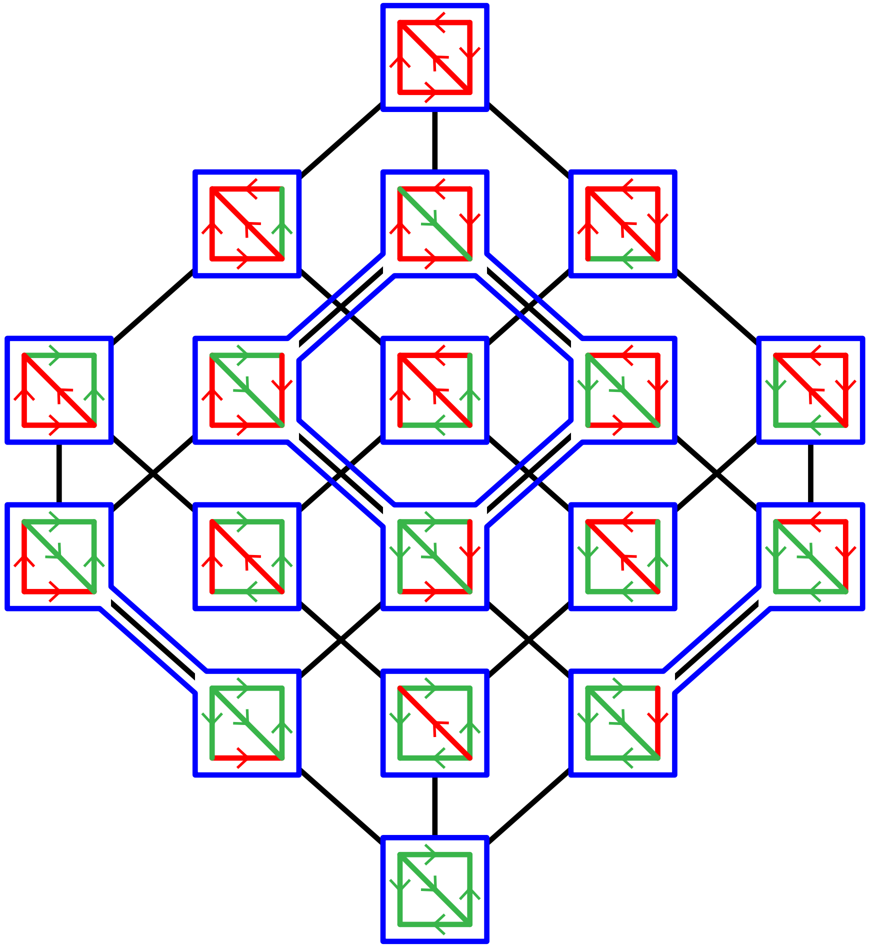
$\mathcal{R}_{(\mho,\Omega)}$ = lower ideal of ropes $(u, v, \triangledown, \vartriangle)$ of D such that $\triangledown \subseteq \mho$ and $\vartriangle \subseteq \Omega$

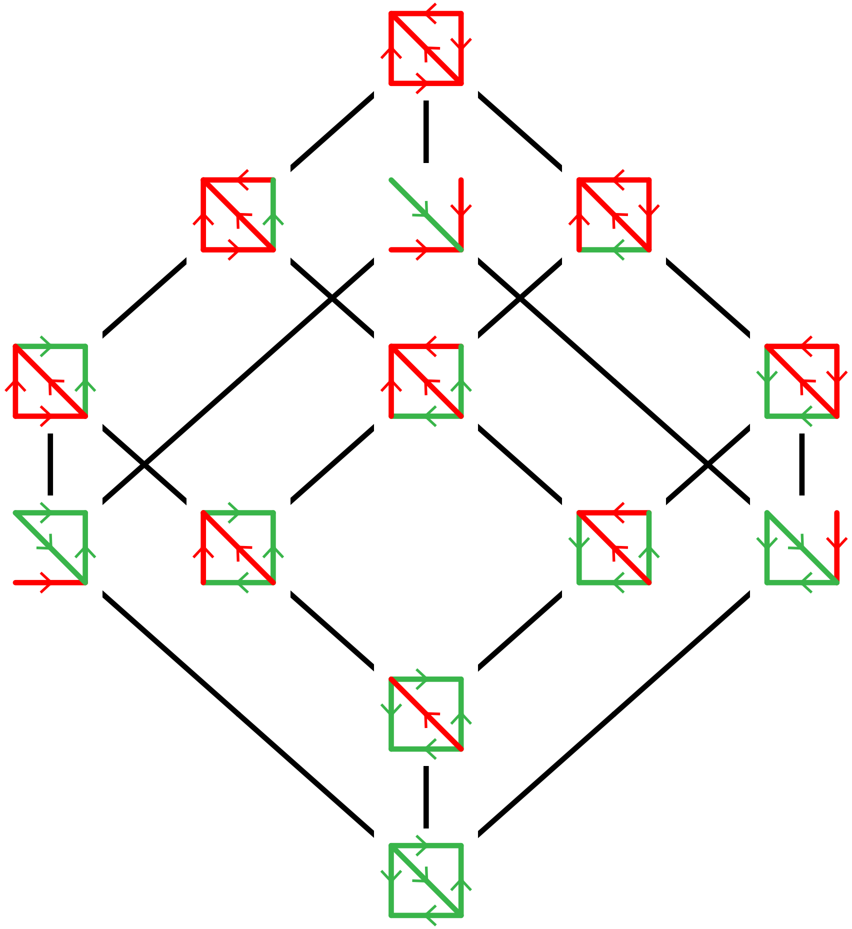
coherent congruence $\equiv_{(\mho,\Omega)}$ = congruence with subrope ideal $\mathcal{R}_{(\mho,\Omega)}$

P.–Pons, *Permutrees* ('18)

examples:

- sylvester congruence = subrope ideal contains only ropes $(u, v, \triangledown, \varnothing)$





COHERENT CONGRUENCES

$(\mho, \Omega)$ = two of arbitrary subsets of V

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coherent congruence $\equiv_{(\mho,\Omega)}$ = congruence with subrope ideal $\mathcal{R}_{(\mho,\Omega)}$

P.–Pons, *Permutrees* ('18)

examples:

- sylvester congruence = subrope ideal contains only ropes $(u, v, \nabla, \varnothing)$
- Cambrian congruences = when $\mho \sqcup \Omega = V$

Reading, *Cambrian lattices* ('06)

QUOTIENT FANS & QUOTIENTOPES

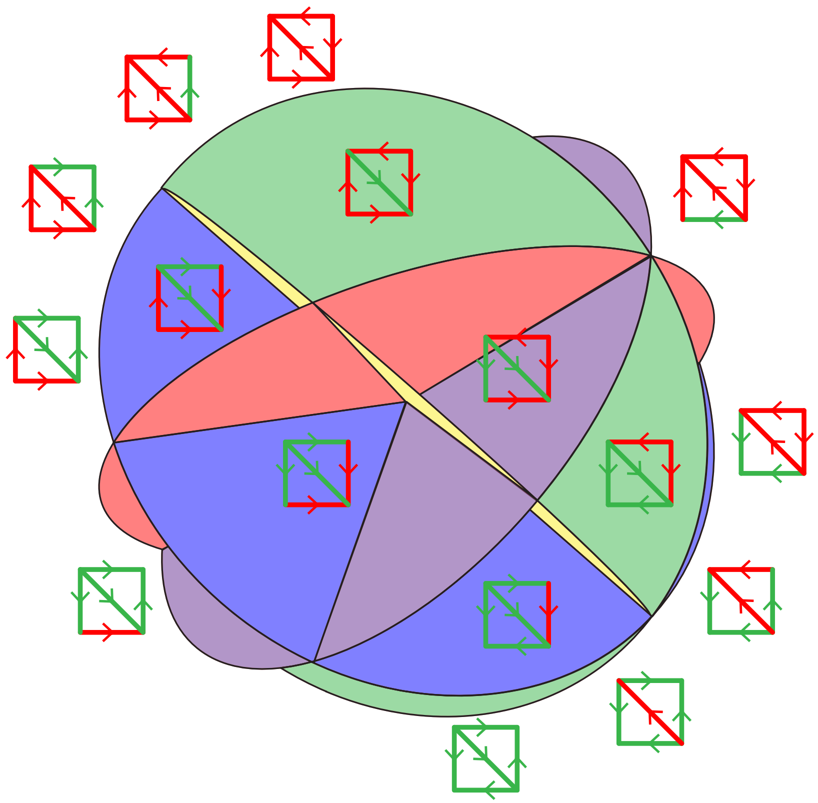
P., *Acyclic reorientation lattices and their lattice quotients* ('21$^+$)

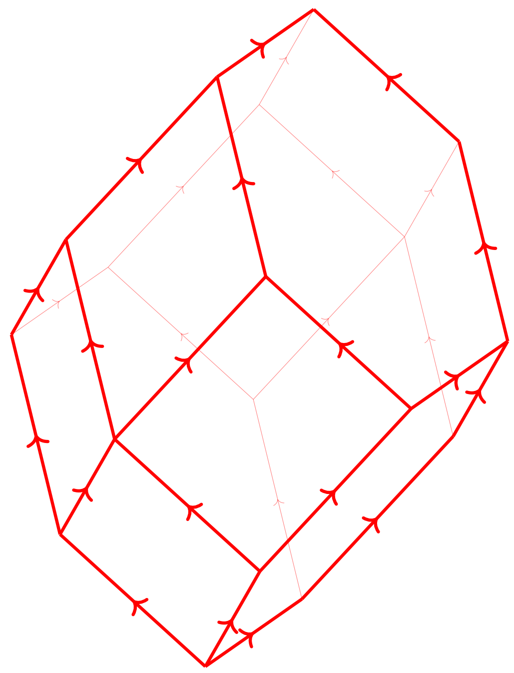
GRAPHICAL ARRANGEMENT & GRAPHICAL ZONOTOPE

D directed acyclic graph

graphical arrangement $\mathcal{H}_D$ = arrangement of hyperplanes $x_u = x_v$ for all arcs $(u, v) \in D$

graphical zonotope $\mathcal{Z}_D$ = Minkowski sum of $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all arcs $(u, v) \in D$

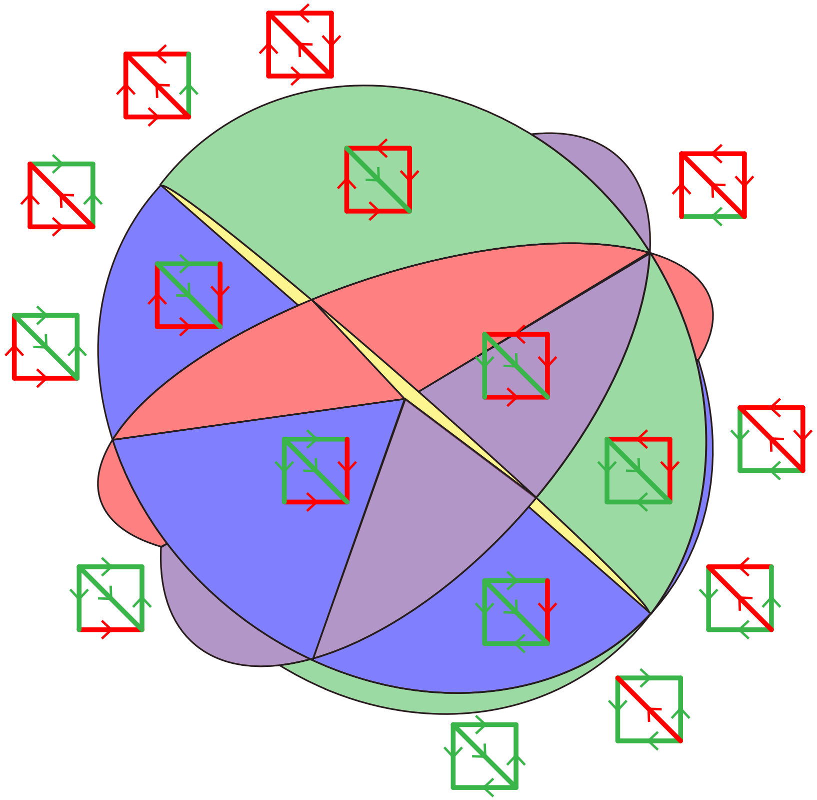


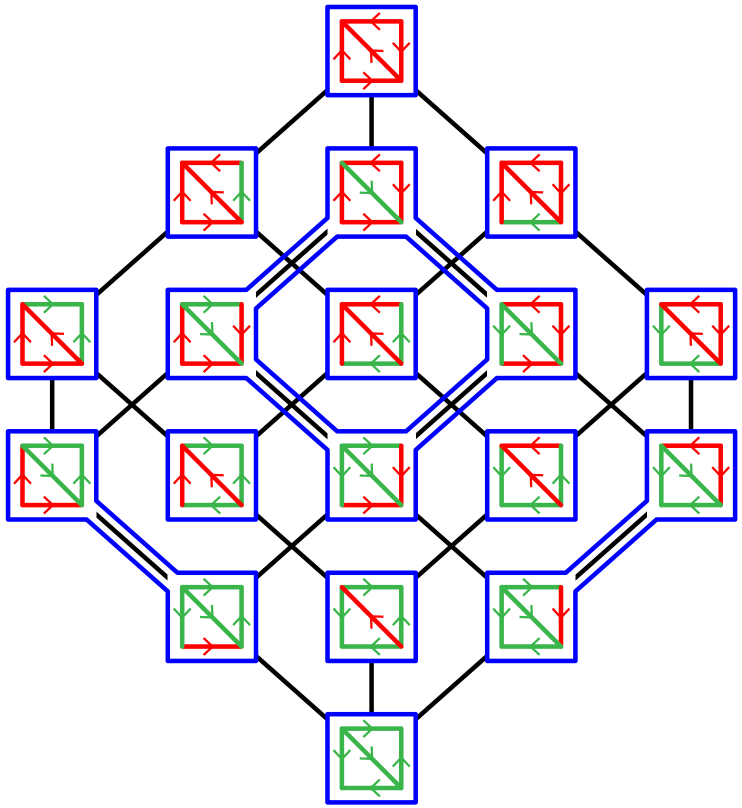


hyperplanes of $\mathcal{H}_D$	$\longleftrightarrow$	summands of $\mathcal{Z}_D$	$\longleftrightarrow$	arcs of D
regions of $\mathcal{H}_D$	$\longleftrightarrow$	vertices of $\mathcal{Z}_D$	$\longleftrightarrow$	acyclic reorientations of D
poset of regions of $\mathcal{H}_D$	$\longleftrightarrow$	oriented graph of $\mathcal{Z}_D$	$\longleftrightarrow$	acyclic reorientation poset of D

QUOTIENT FAN

THM. A lattice congruence $\equiv$ of $\mathcal{AR}_D$ defines a quotient fan $\mathcal{F}_\equiv$ where the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of $\mathcal{H}_D$ corresponding to acyclic reorientations in the same equivalence class of $\equiv$ P. ('21$^+$)





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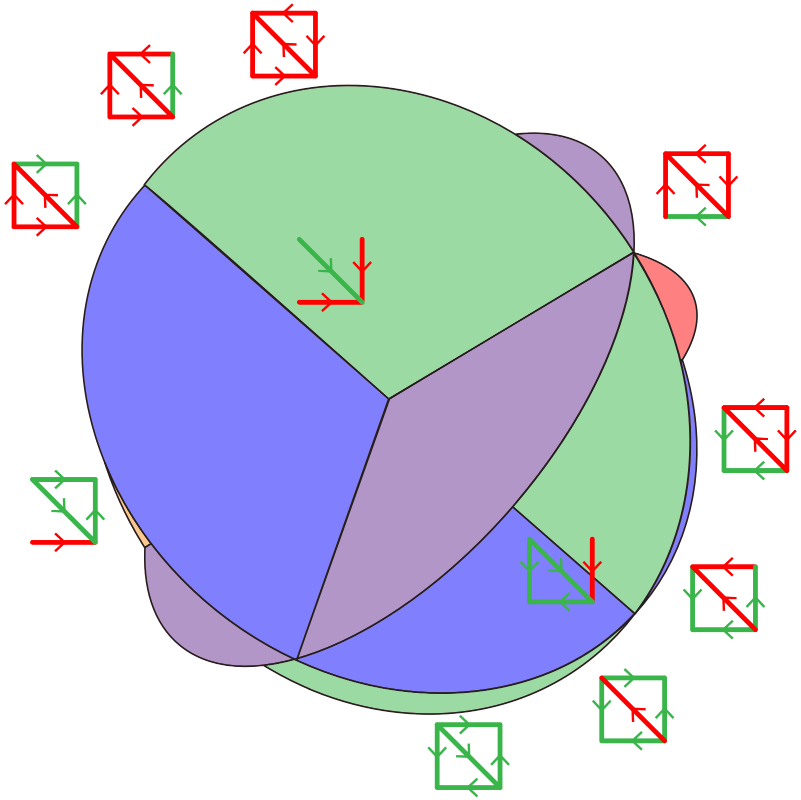
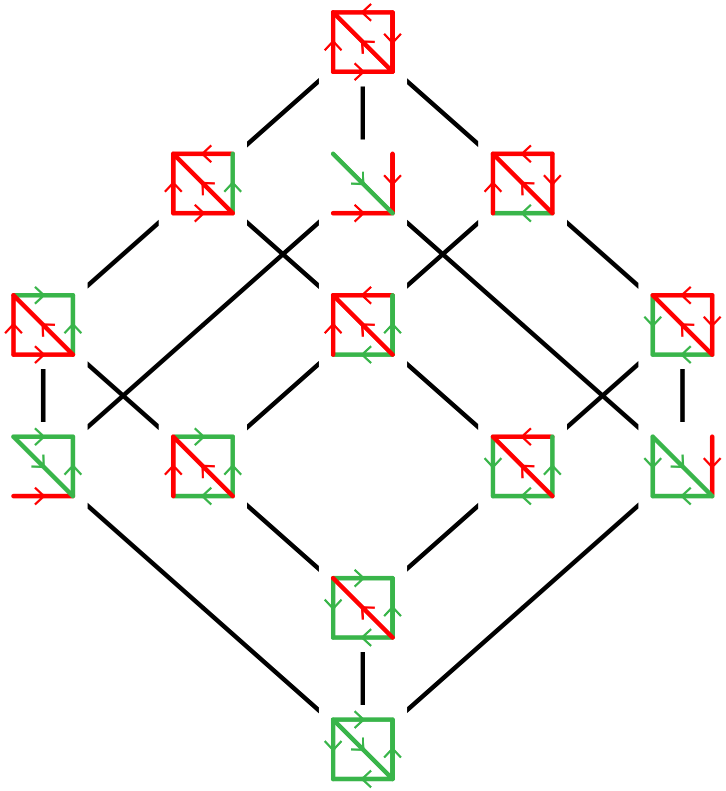
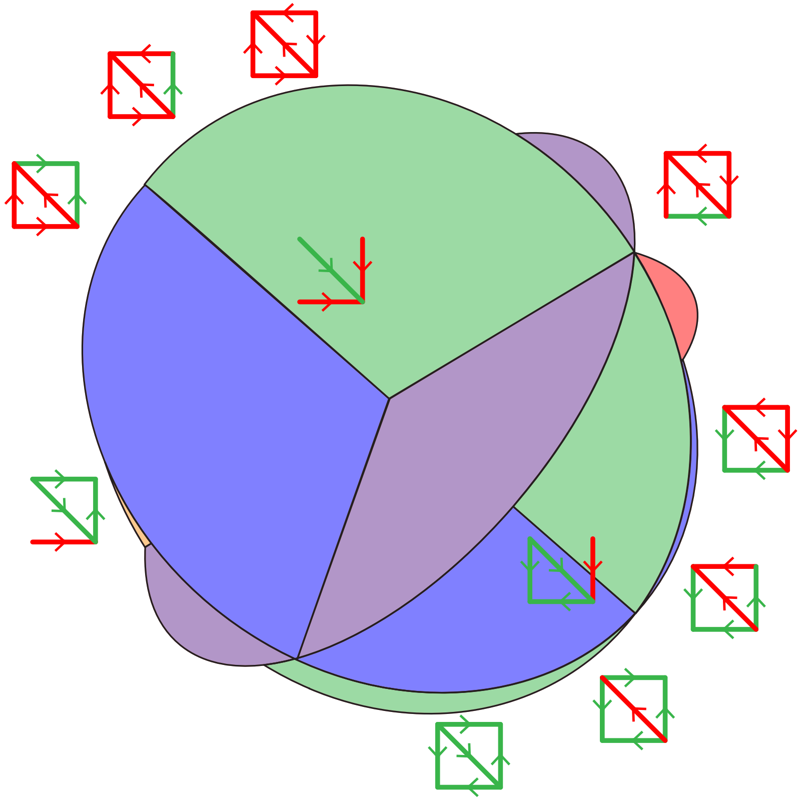


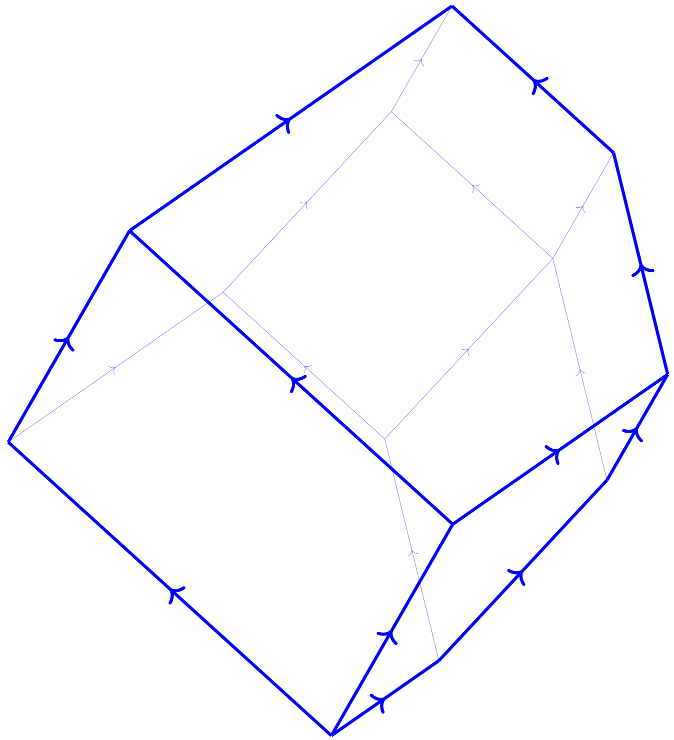


QUOTIENTOPES

THM. The quotient fan $\mathcal{F}_{\equiv}$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of
- a Minkowski sum of associahedra of Hohlweg – Lange, and
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter

P. ('21$^+$)





ρ-alternating matching = pair $(M_\triangledown, M_\vartriangle)$ with $M_\triangledown \subseteq \{u\} \cup \triangledown$ and $M_\vartriangle \subseteq \vartriangle \cup \{v\}$ s.t. $M_\triangledown$ and $M_\vartriangle$ are alternating along the transitive reduction of D

shard polytope of ρ = convex hull of signed charact. vectors of ρ-alternating matchings

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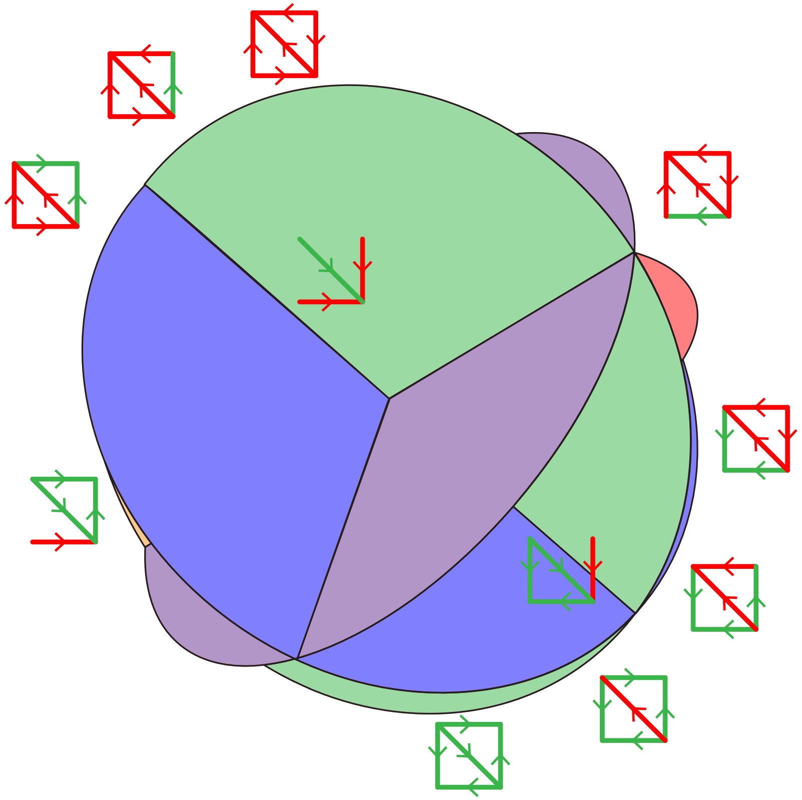
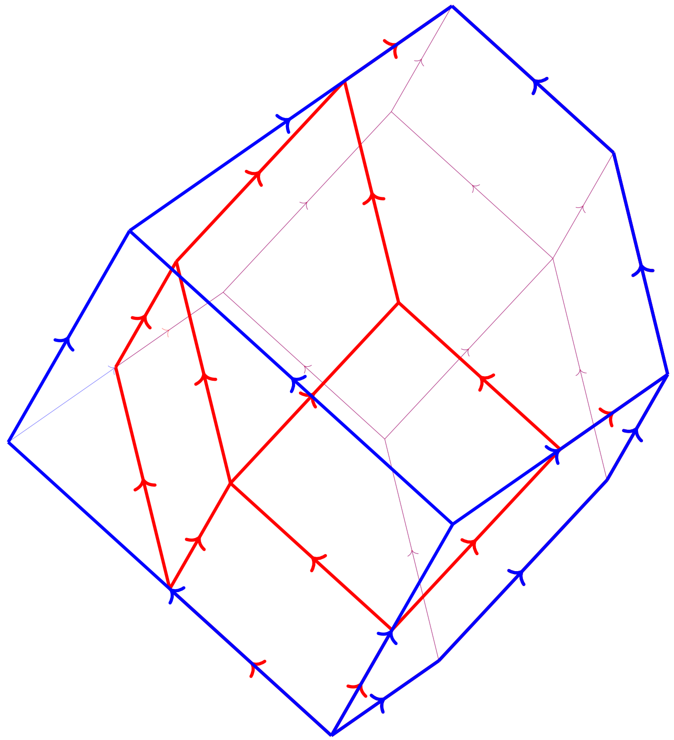
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter

P. ('21$^+$)



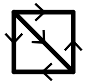
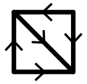


PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of D are facet defining inequalities of the graphical zonotope of D

P. ('21$^+$)

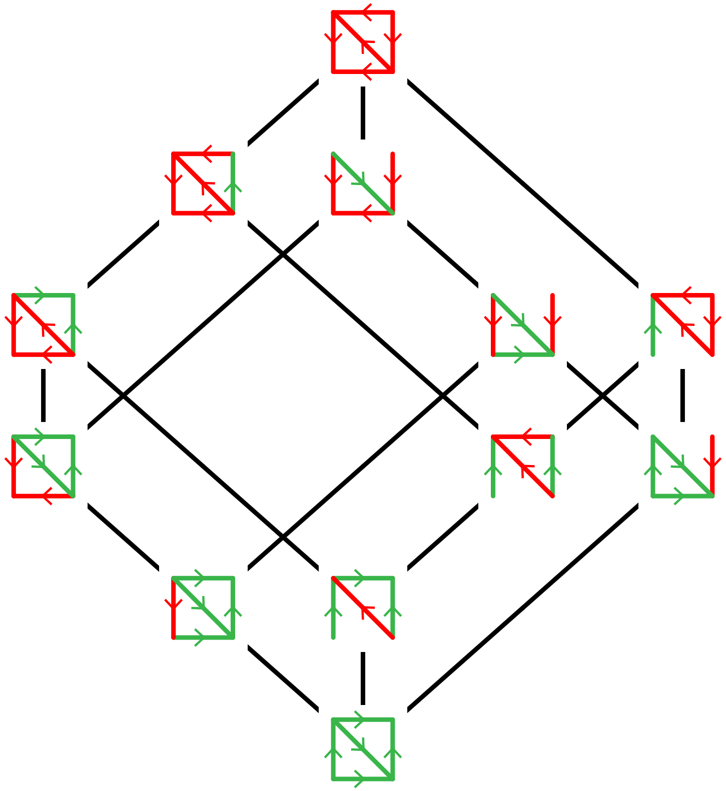
SOME OPEN PROBLEMS

SIMPLE ASSOCIAHEDRA

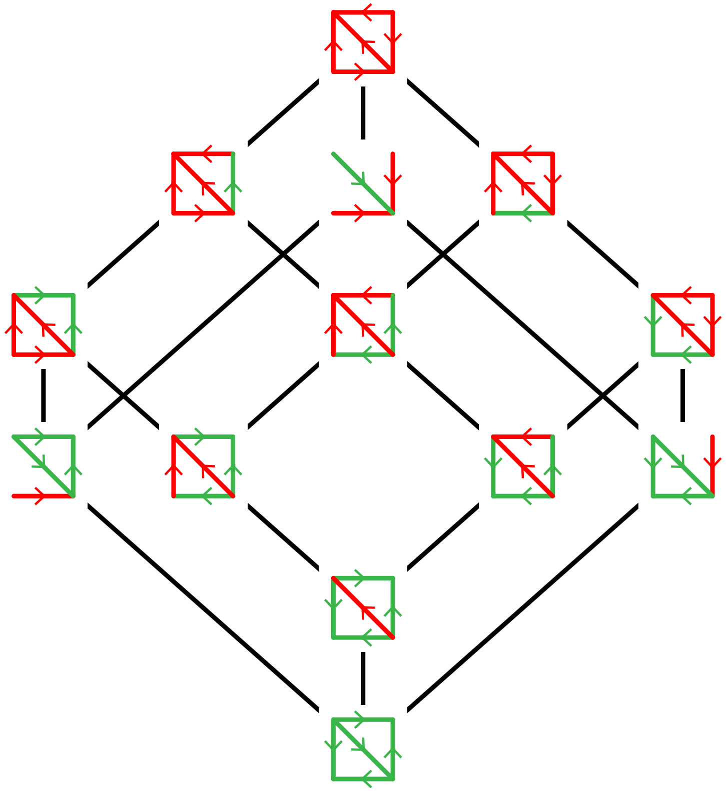
CONJ. D has no induced subgraph isomorphic to  or 

$\iff$ the Hasse diagram of the D-Tamari lattice is regular

$\iff$ the D-associahedron is a simple polytope

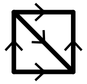


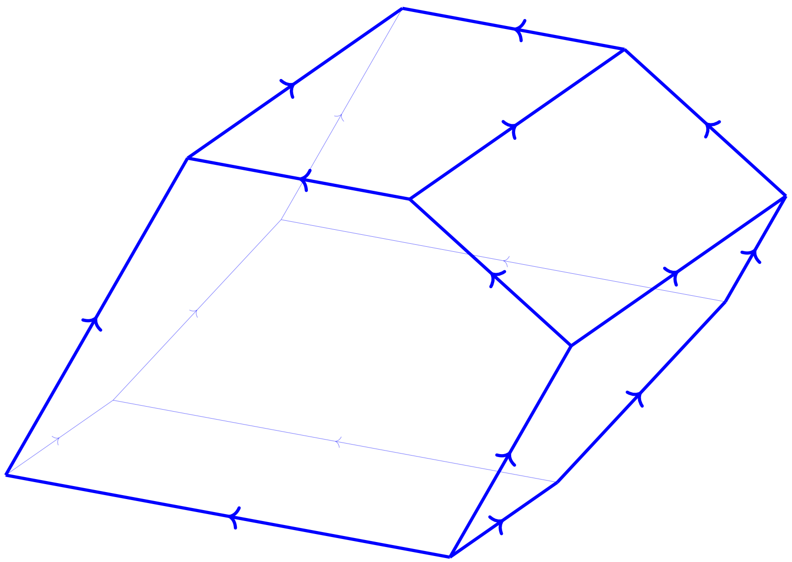
regular

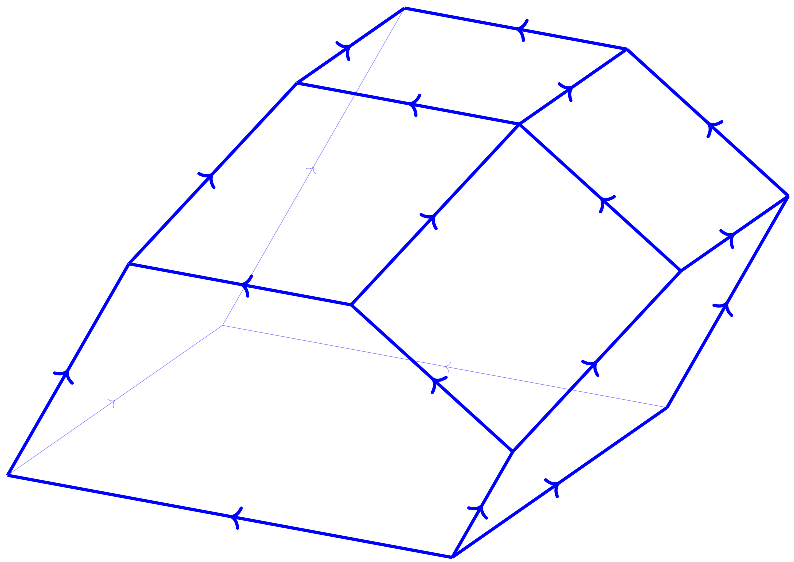


non regular

ISOMORPHIC CAMBRIAN ASSOCIAHEDRA

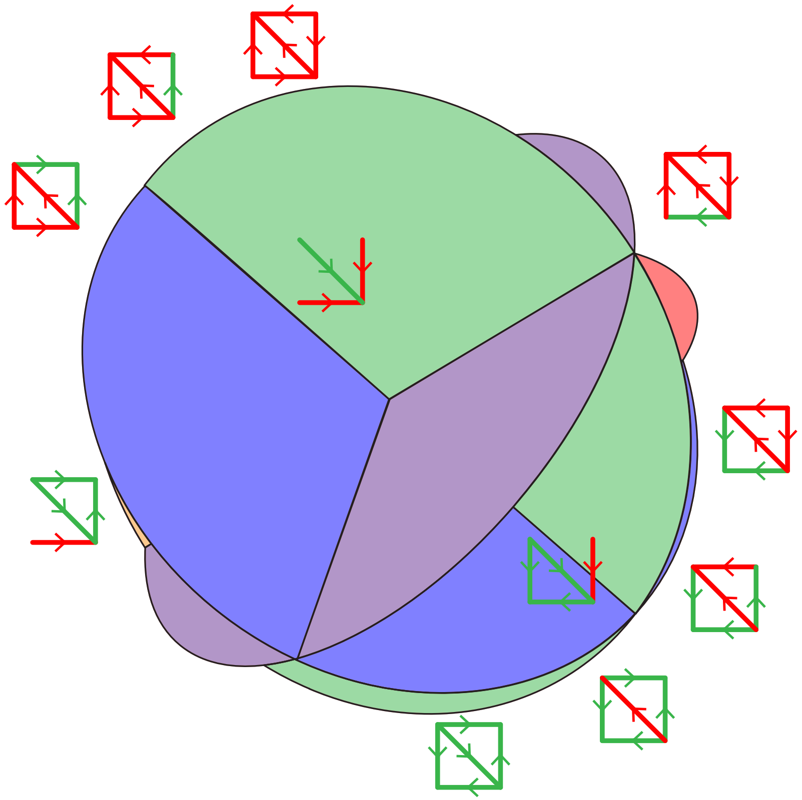
CONJ. D has no induced subgraph isomorphic to 
$\iff$ all Cambrian associahedra of D have the same number of vertices
$\iff$ all Cambrian associahedra of D have isomorphic 1-skeleta
$\iff$ all Cambrian associahedra of D have isomorphic face lattices

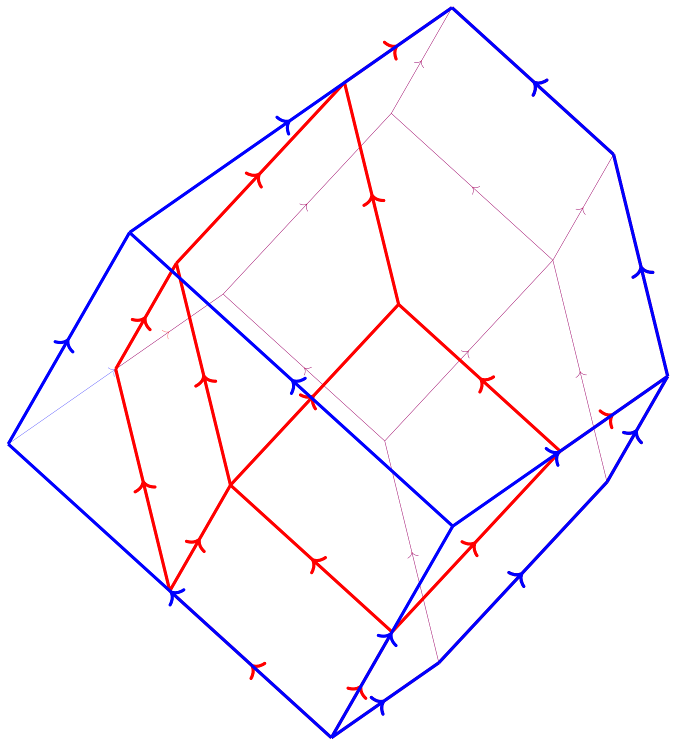




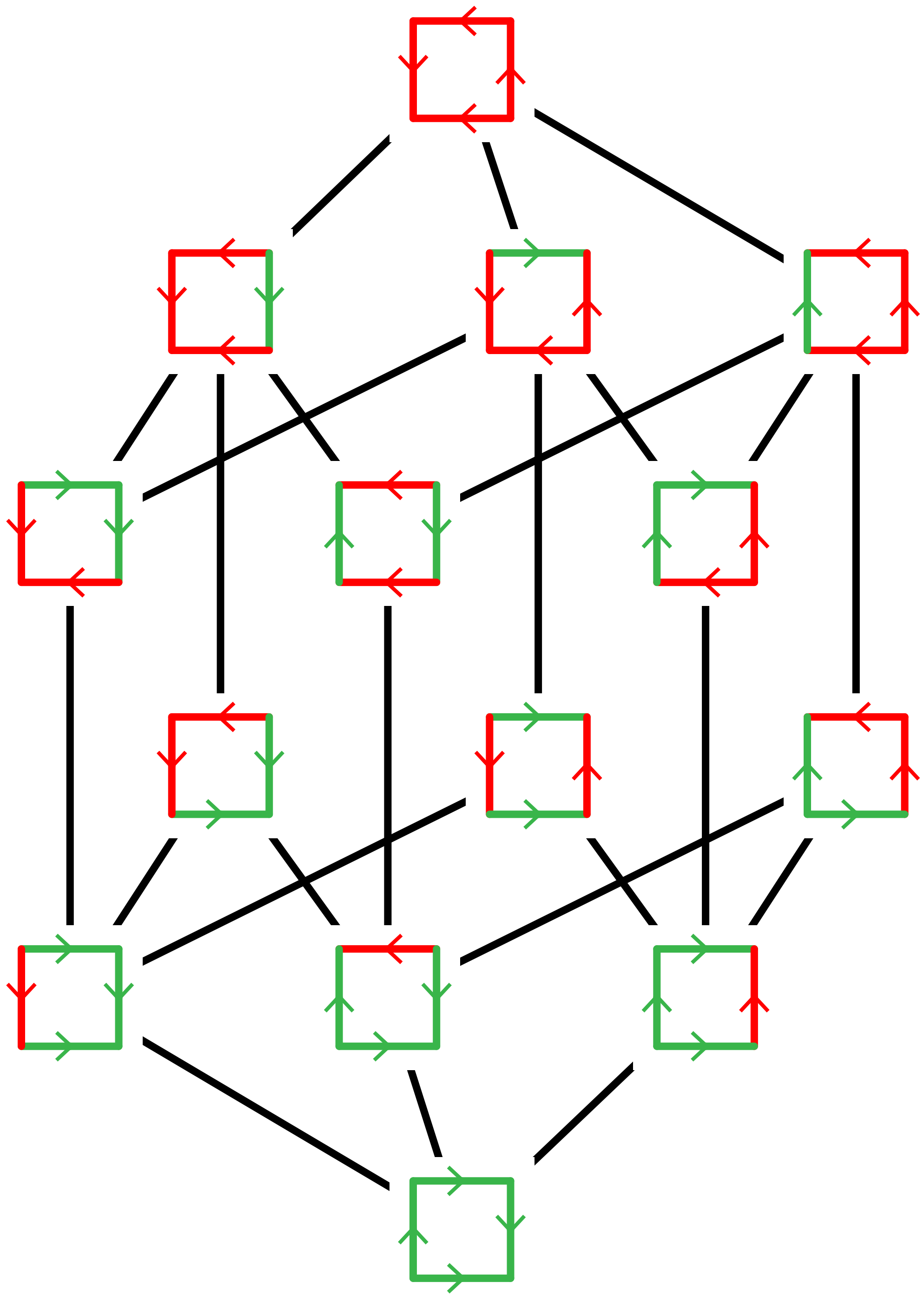
REMOVAHEDRA

PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of D are facet defining inequalities of the graphical zonotope of D P. ('21$^+$)





CONJ. For any $\mho, \Omega \subseteq V$, the quotient fan $\mathcal{F}_{(\mho,\Omega)}$ is the normal fan of the polytope obtained by deleting inequalities of the graphical zonotope of D



THANKS