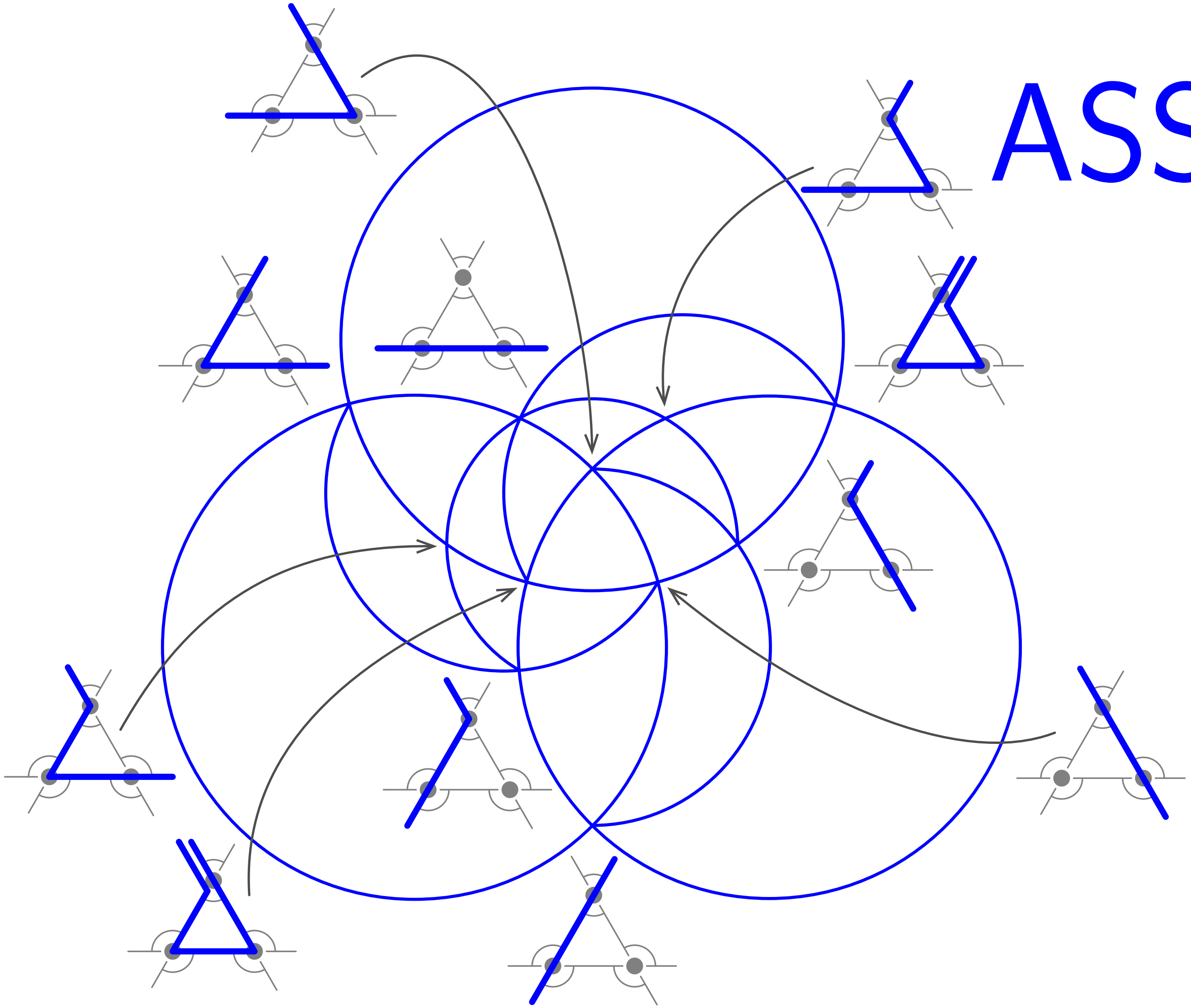


4. GENTLE ASSOCIAHEDRA

V. PILAUD
(Univ. Barcelona)

arXiv:1703.09953
T. MANNEVILLE

arXiv:1807.04730
arXiv:1707.07574
Y. PALU
P.-G. PLAMONDON

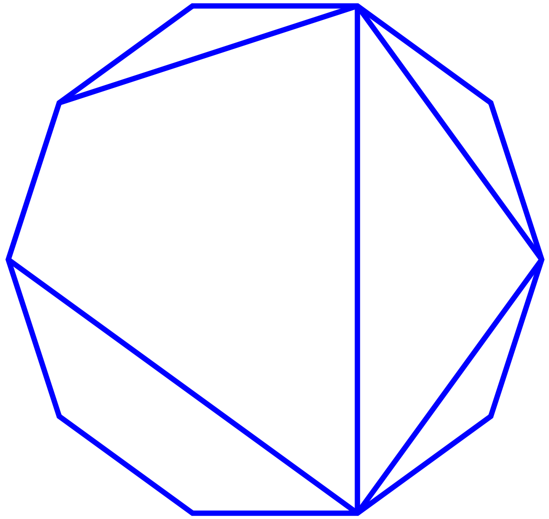
May 8, 2024
ISM Discovery School

MOTIVATION

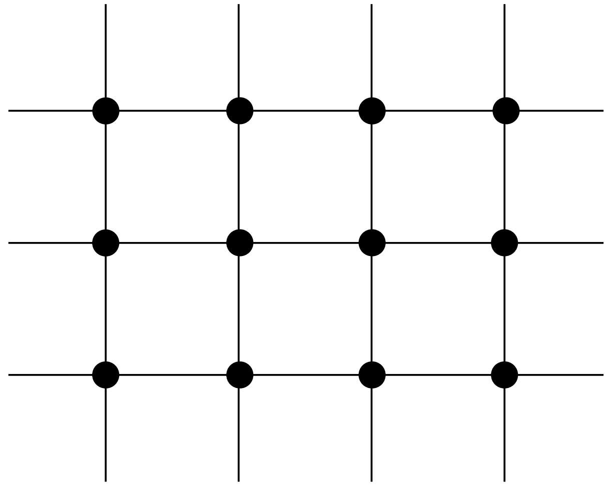
Baryshnikov, *On Stokes sets* ('01)
Chapoton, *Stokes posets and serpent nests* ('16)
Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('18)
Manneville–P., *Geometric realizations of the accordion complex* ('19)

Petersen–Pylyavskyy–Speyer, *A non-crossing standard monomial theory* ('10)
Santos–Stump-Welker, *Non-crossing sets and the Grassmann-associahedron* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)

TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

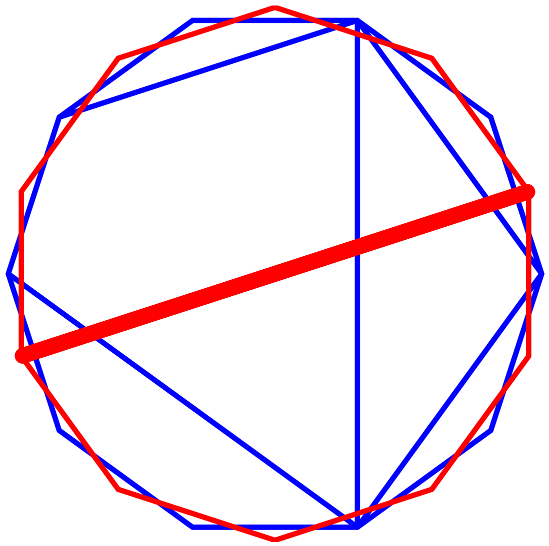


dissection

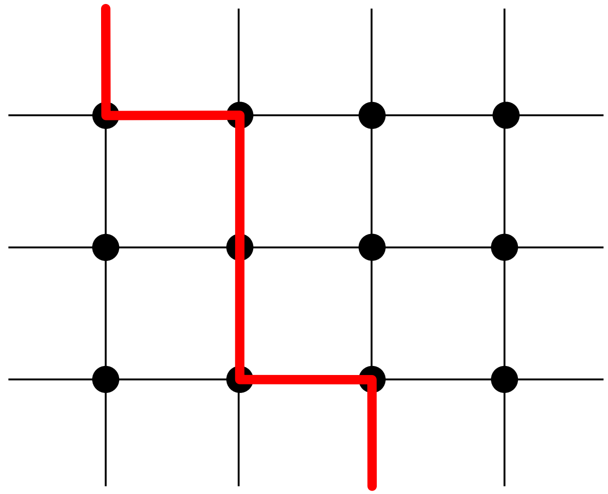


subset of $\mathbb{Z}^2$

TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

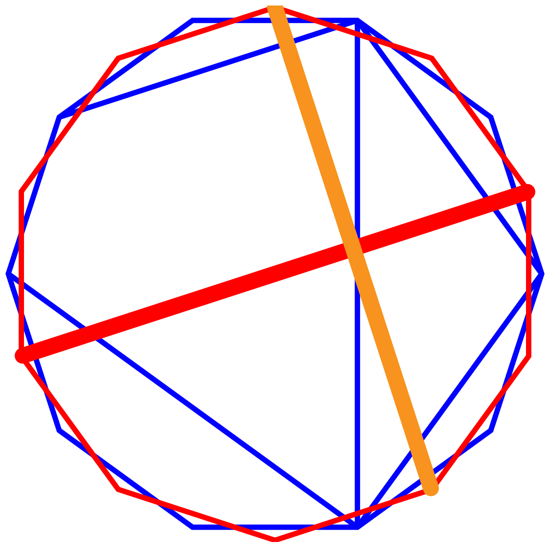


dissection
accordion

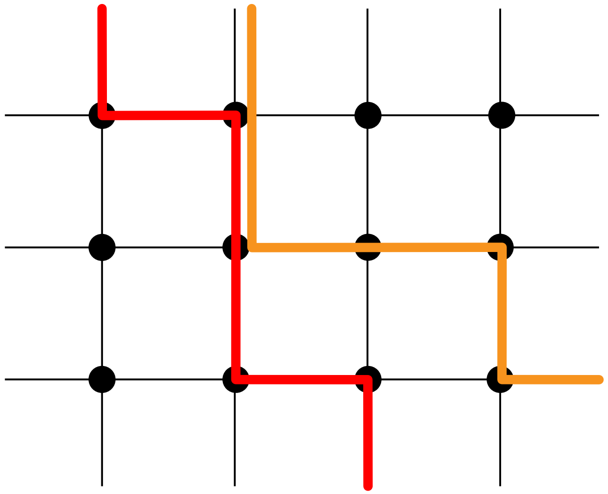


subset of $\mathbb{Z}^2$
monotone path

TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

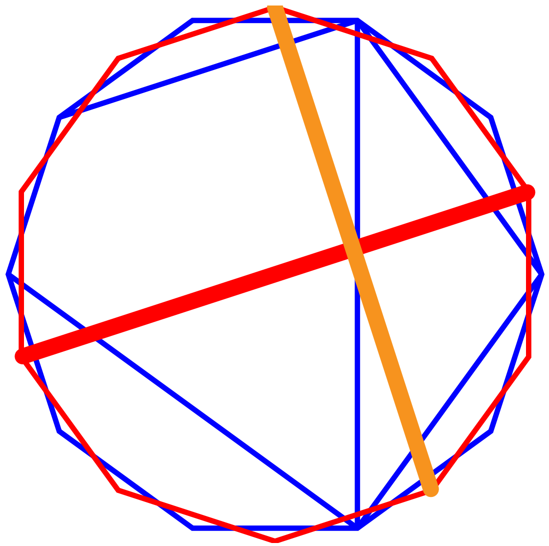


dissection
accordion
non-crossing complex

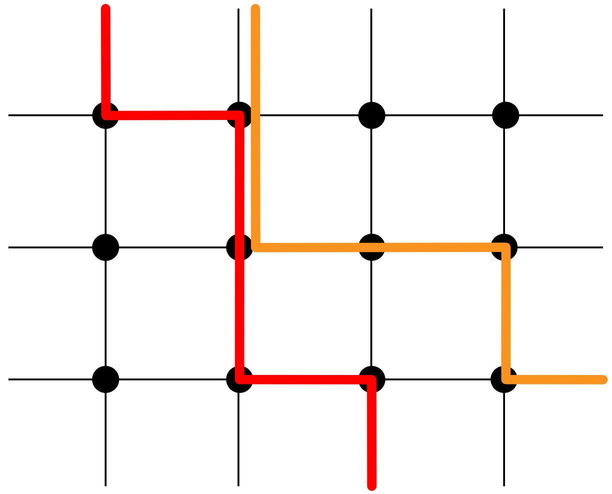


subset of $\mathbb{Z}^2$
monotone path
non-kissing complex

TWO GENERALIZATIONS OF THE ASSOCIAHEDRON



dissection
accordion
non-crossing complex



subset of $\mathbb{Z}^2$
monotone path
non-kissing complex

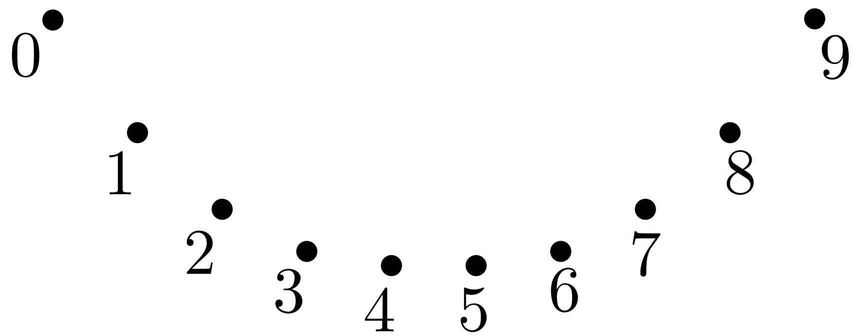
Baryshnikov, *On Stokes sets* ('01)
Chapoton, *Stokes posets and serpent nests* ('16)
Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('18)
Manneville–P., *Geometric realizations of the accordion complex* ('19)

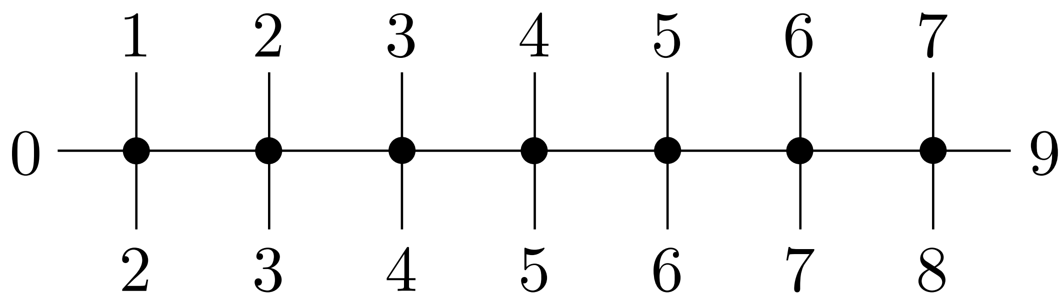
Petersen–Pylyavskyy–Speyer, *A non-crossing standard monomial theory* ('10)
Santos–Stump–Welker, *Non-crossing sets and the Grassmann-associahedron* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)
Garver–McConville, *Enumerative properties of grid-associahedra* ('17$^+$)

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

simplicial associahedron = simplicial complex with

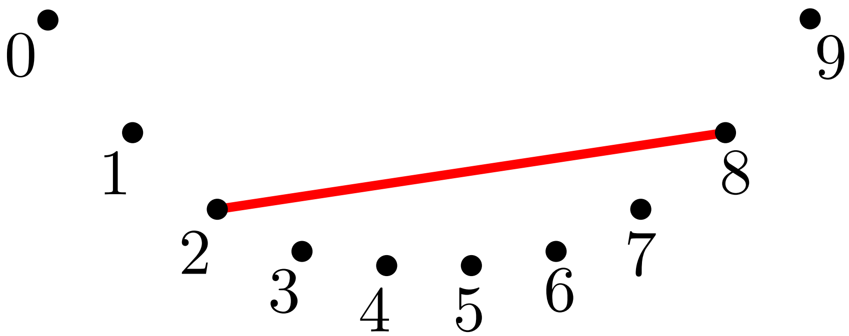
- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

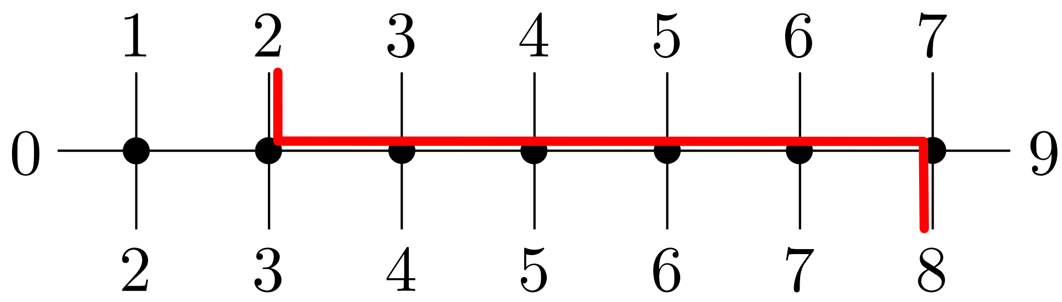




simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

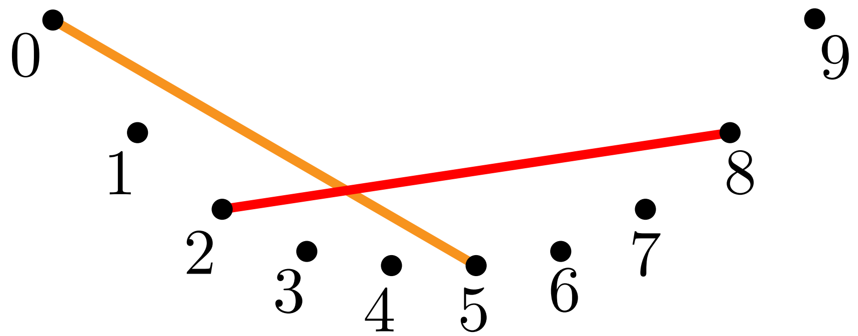


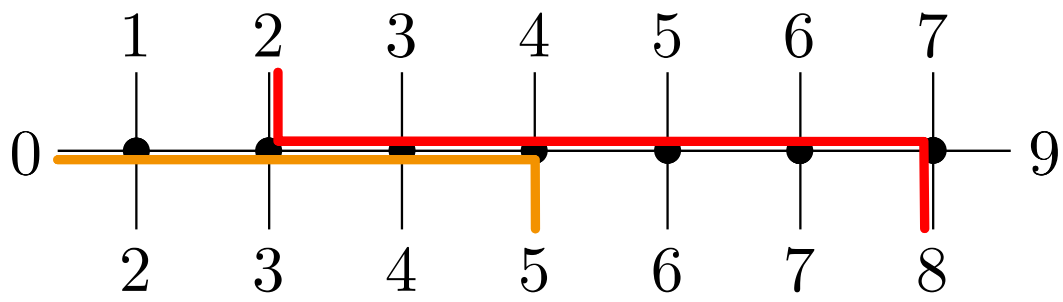


diagonal $\longleftrightarrow$ walk

simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon



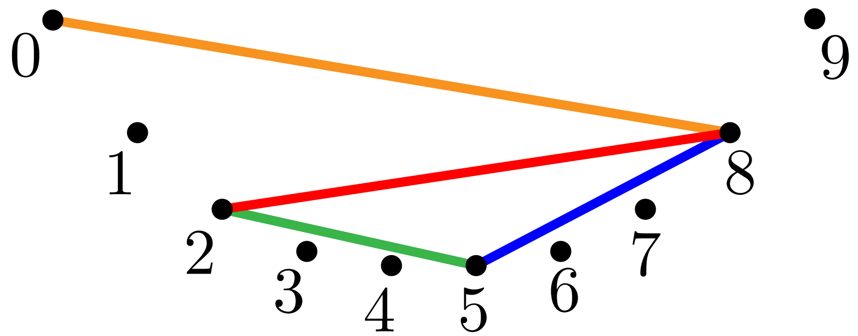


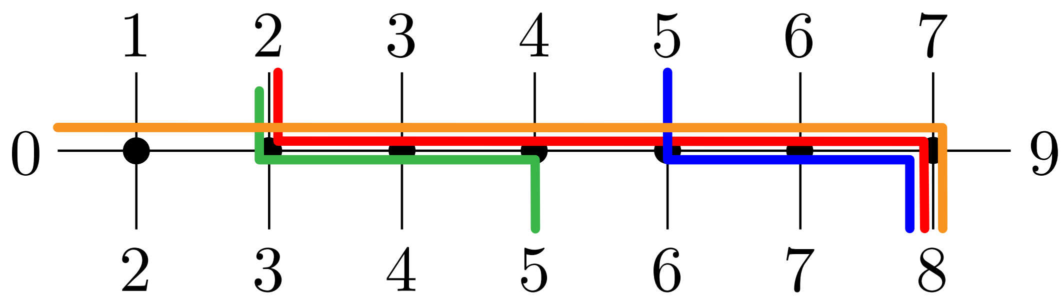
diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon



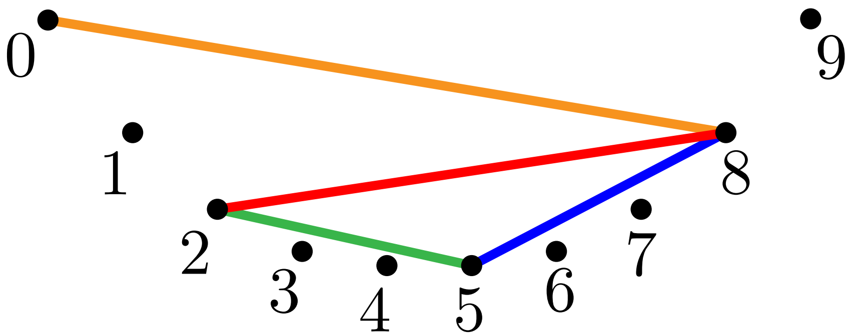


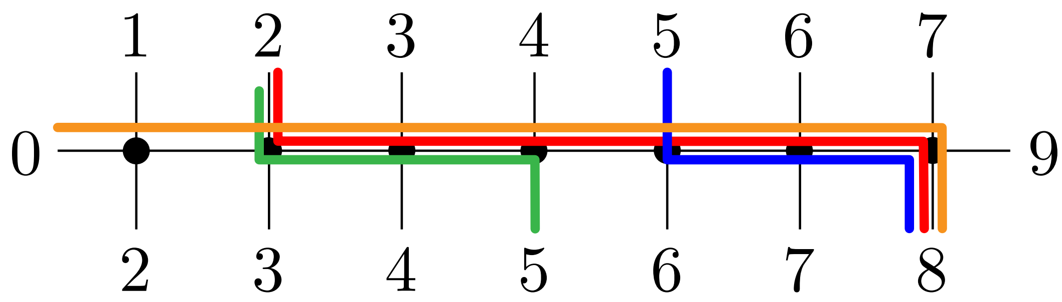
diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon





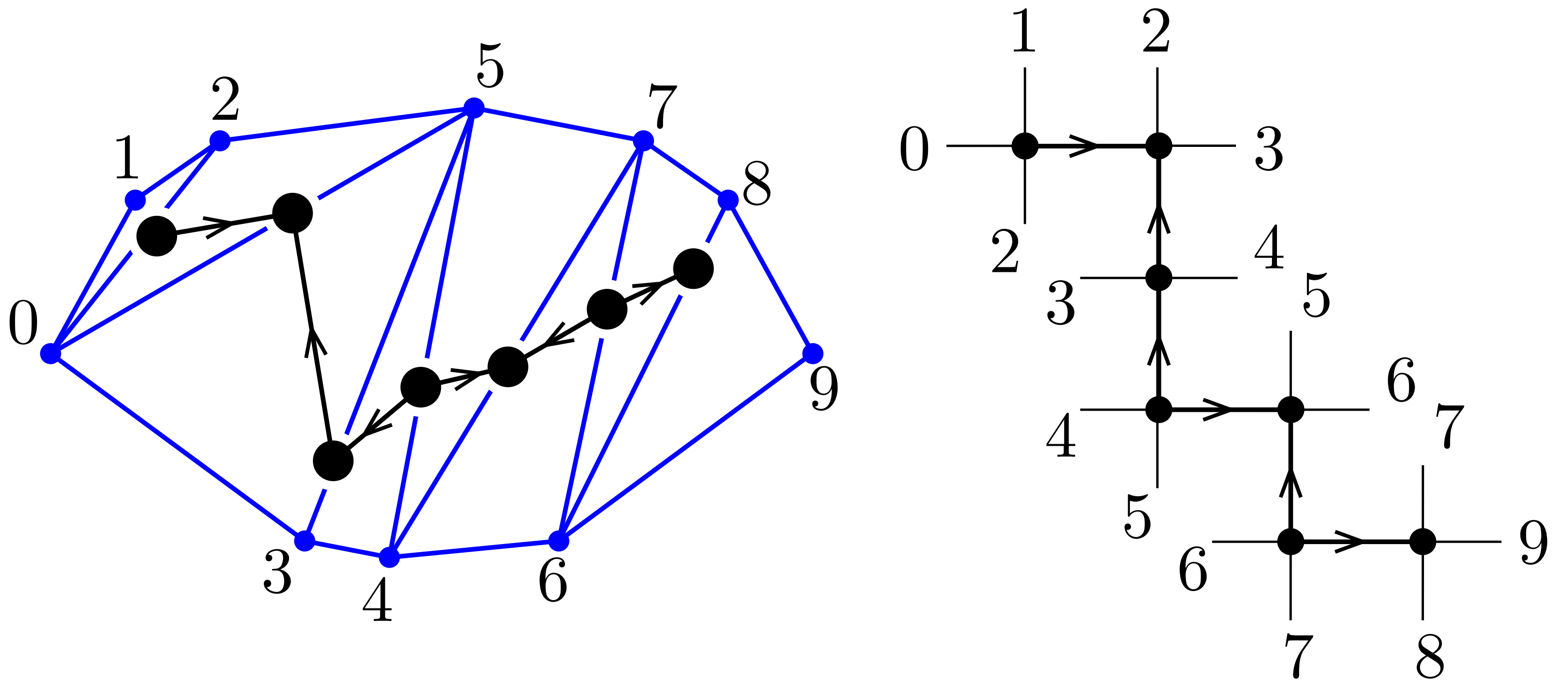
diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face
simplicial associahedron	$\longleftrightarrow$	non-kissing complex

McConville, *Lattice structures of grid Tamari orders* ('17)

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

simplicial associahedron = simplicial complex with

- vertices = internal diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon

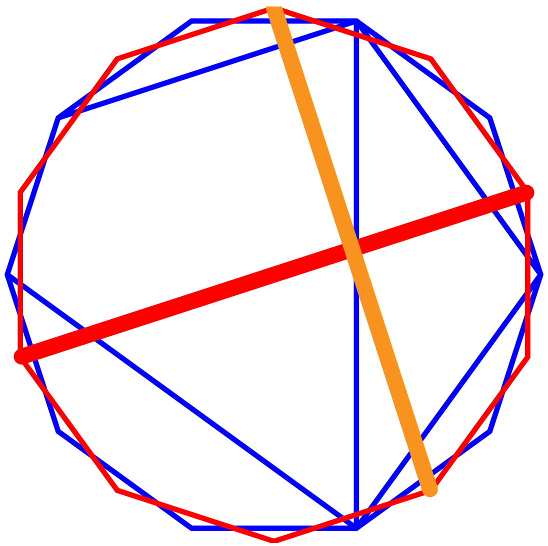


McConville, *Lattice structures of grid Tamari orders* ('17)

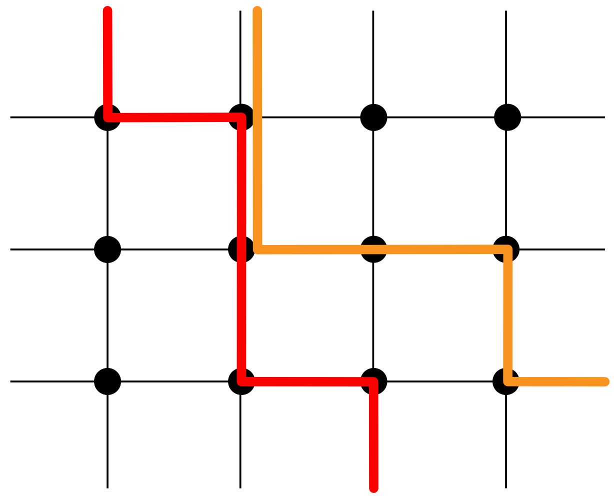
FIRST HALF OF THE TALK

Show that non-crossing and non-kissing complexes coincide

To this end, generalize both:



non-crossing complex
to dissections of surfaces

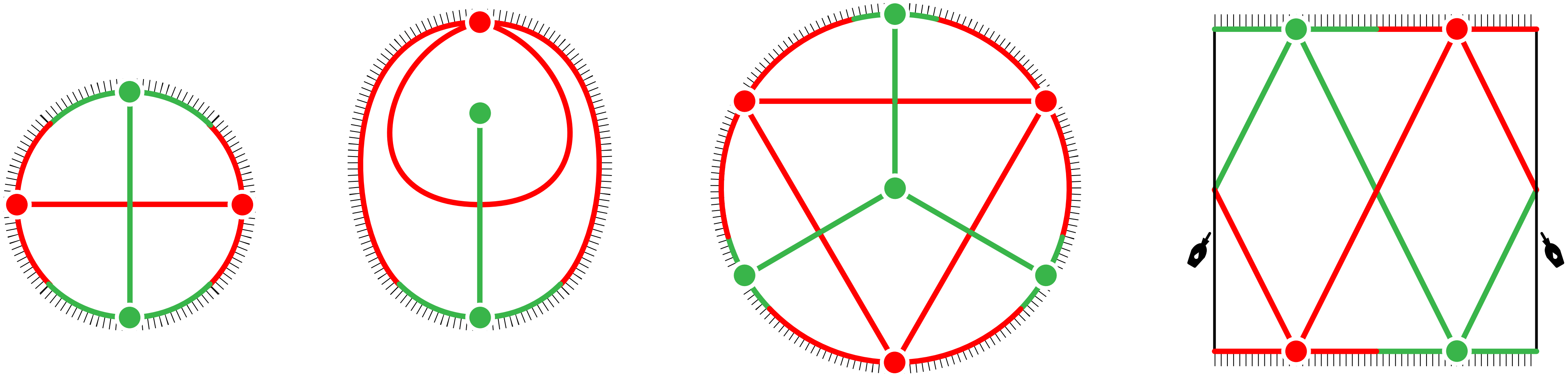


non-kissing complex
to gentle quivers

Palu–P.–Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras* ('19)

NON-CROSSING COMPLEX

Palu–P.–Plamondon,
Non-kissing and non-crossing complexes for locally gentle algebras ('19)

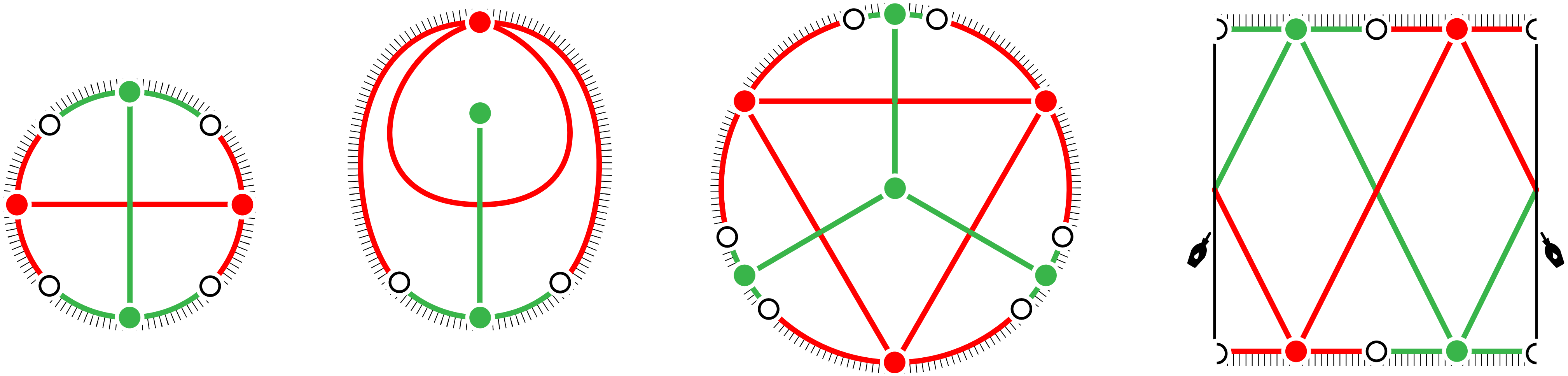


$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

DUAL DISSECTIONS



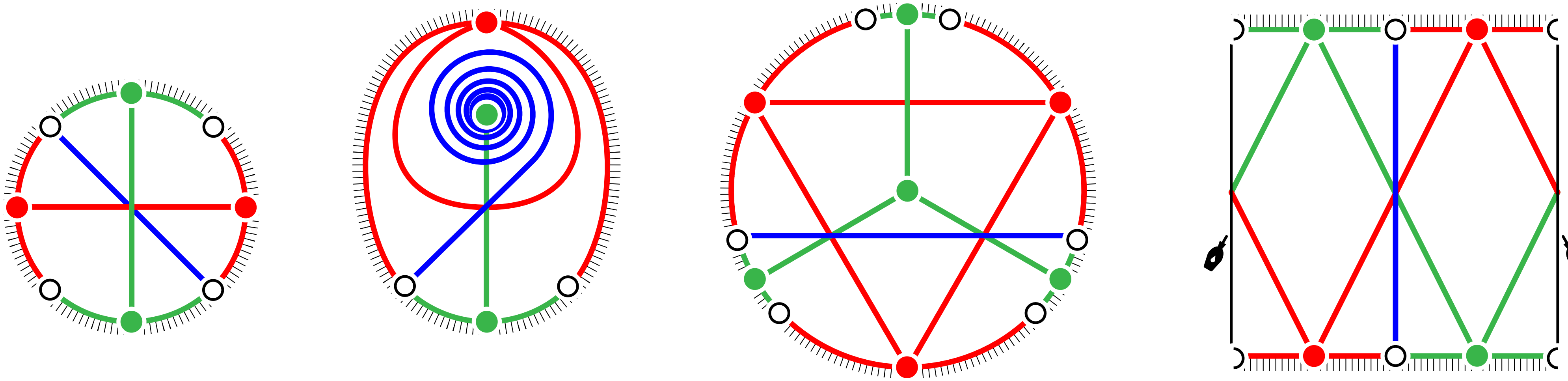
$\mathcal{S}$ = orientable surface with or without boundaries

V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

blossom vertices = white vertices, alternating with $\mathrm{V} \cup \mathrm{V}^*$ along the boundary of $\mathcal{S}$

DUAL DISSECTIONS



$\mathcal{S}$ = orientable surface with or without boundaries

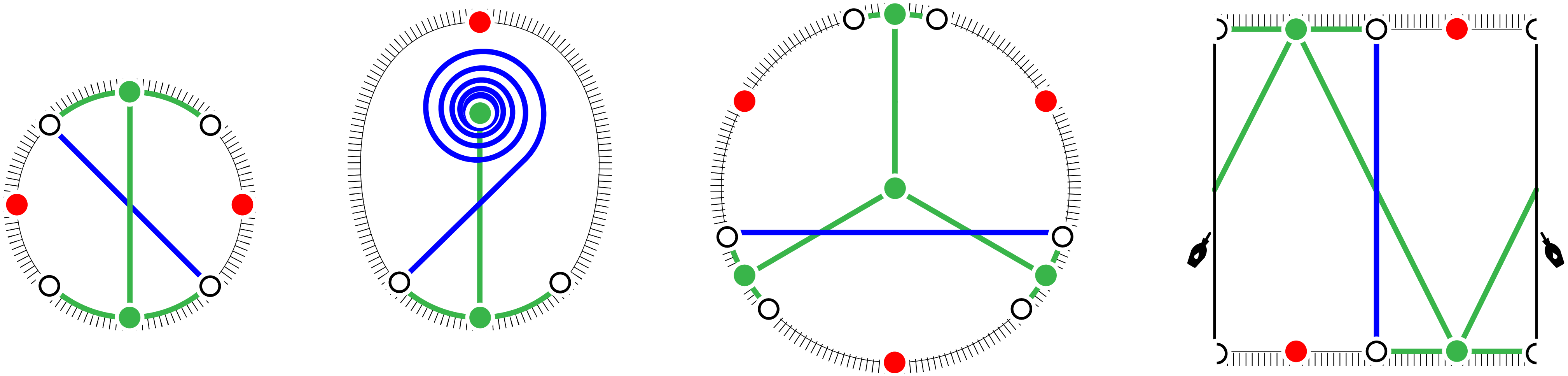
V and V^* two families of marked points

D and D^* two dual dissections of $\mathcal{S}$

blossom vertices = white vertices, alternating with $\mathrm{V} \cup \mathrm{V}^*$ along the boundary of $\mathcal{S}$

B-curve = curve which at each endpoint either reaches a blossom point or infinitely circles around a puncture of $\mathcal{S}$

ACCORDIONS



D-accordion $=$ B-curve α such that whenever α meets a face f of D,

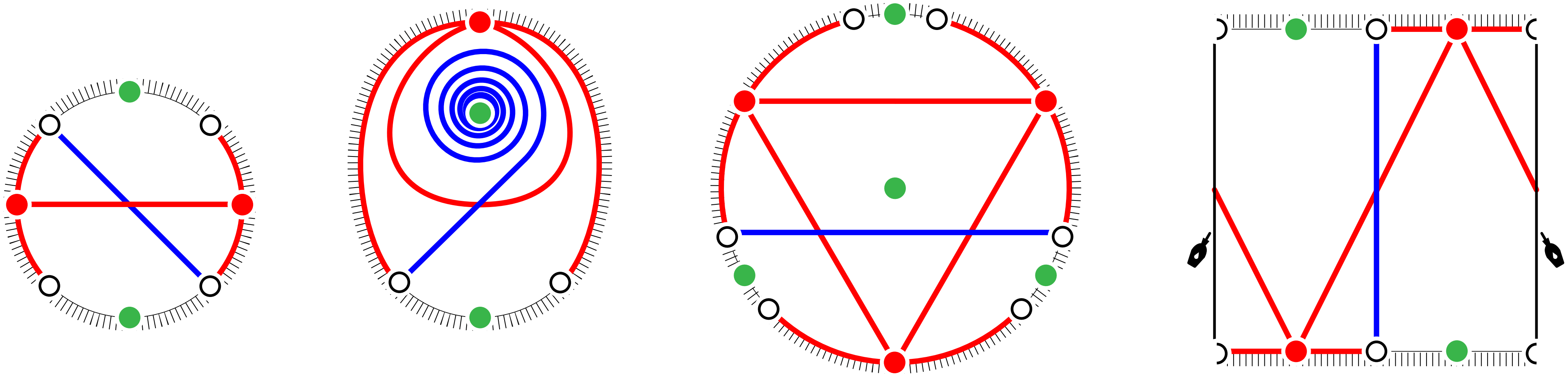
(i) it enters crossing an edge a of f and leaves crossing an edge b of f

(ii) the two edges a and b of f crossed by α are consecutive along the boundary of f,

(iii) α, a and b bound a disk inside f that does not contain f^*.

D-accordion complex $=$ simplicial complex of pairwise non-crossing sets of D-accordions

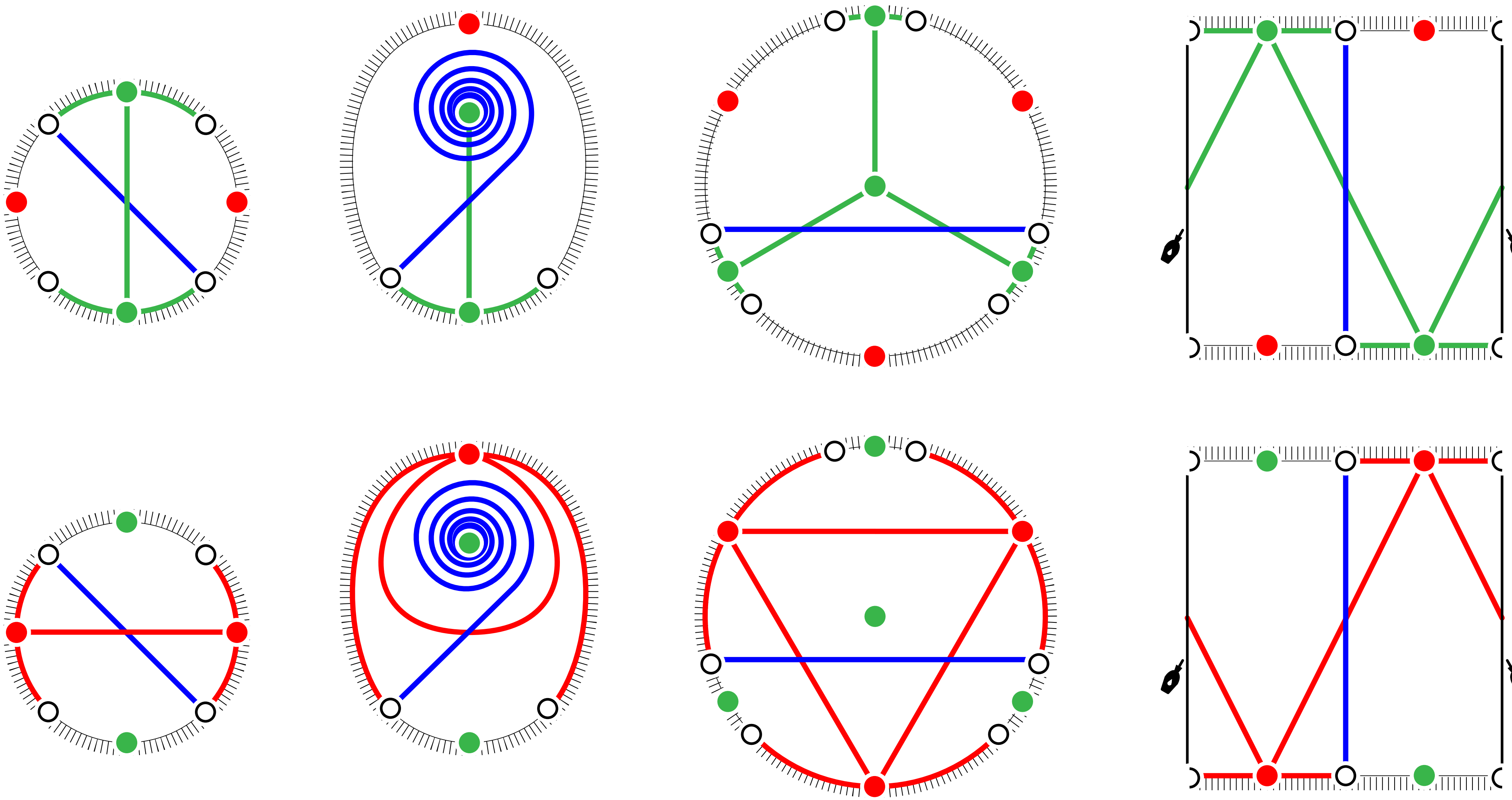
SLALOMS



D^*-slalom $=$ B-curve α of $\bar{S}$ such that, whenever α crosses an edge a^* of D^* contained in two faces f^*, g^* of D^*, the marked points f and g lie on opposite sides of α in the union of f^* and g^* glued along a^*.

D^*-slalom complex $=$ simplicial complex of pairwise non-crossing sets of D^*-slaloms

D-ACCORDIONS = D*-SLALOMS

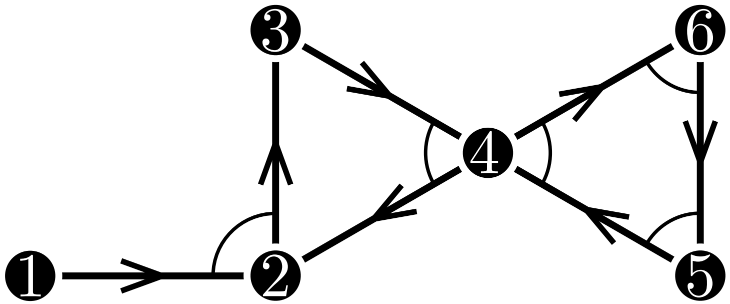


$(\mathrm{D}, \mathrm{D}^*)$-non-crossing complex = D-accordion complex = D^*-slalom complex

NON-KISSING COMPLEX

Brüstle–Douville–Mousavand–Thomas–Yıldırım,
On the combinatorics of gentle algebras ('20)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

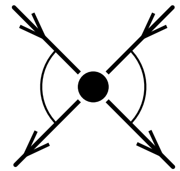
GENTLE QUIVERS AND STRINGS



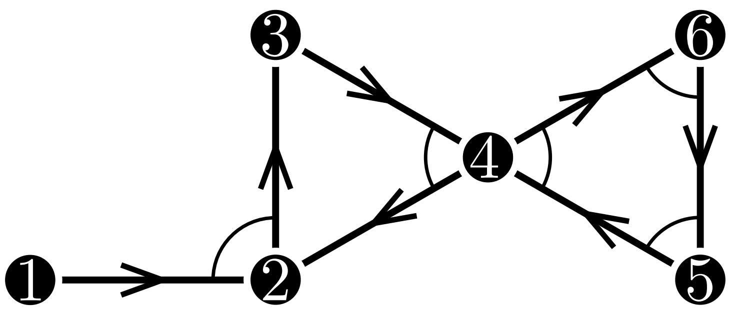
gentle quiver $\bar{Q} =$

- quiver $Q =$ oriented graph (Q_0, Q_1, s, t)
- relations $I =$ forbid certain paths

where

- forbidden paths all of length 2
- locally at each vertex, subgraph of 

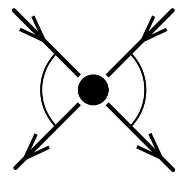
GENTLE QUIVERS AND STRINGS

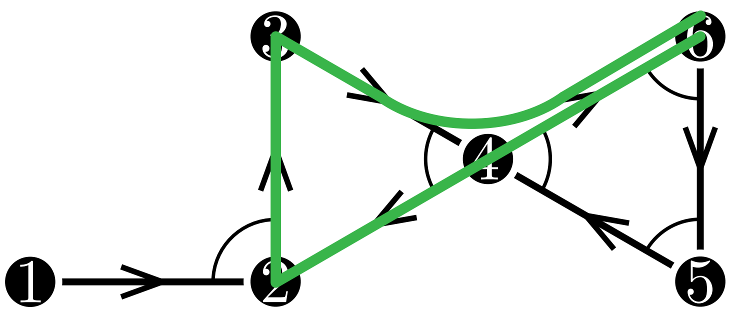


gentle quiver $\bar{Q}$ =

- quiver Q = oriented graph (Q_0, Q_1, s, t)
- relations I = forbid certain paths

where

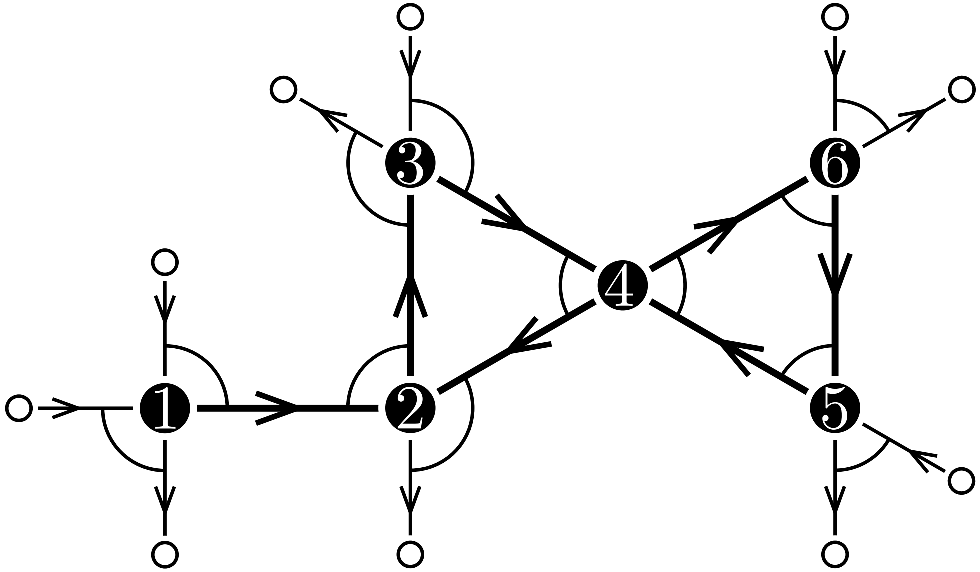
- forbidden paths all of length 2
- locally at each vertex, subgraph of 



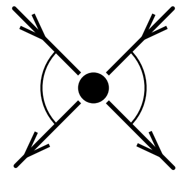
string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$ with $\alpha_k \in Q_1$, $\varepsilon_k \in \{-1, 1\}$ such that

- $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$
- contains no factor π or π^{-1} for any path $\pi \in I$
- contains no $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for any arrow $\alpha \in Q_1$

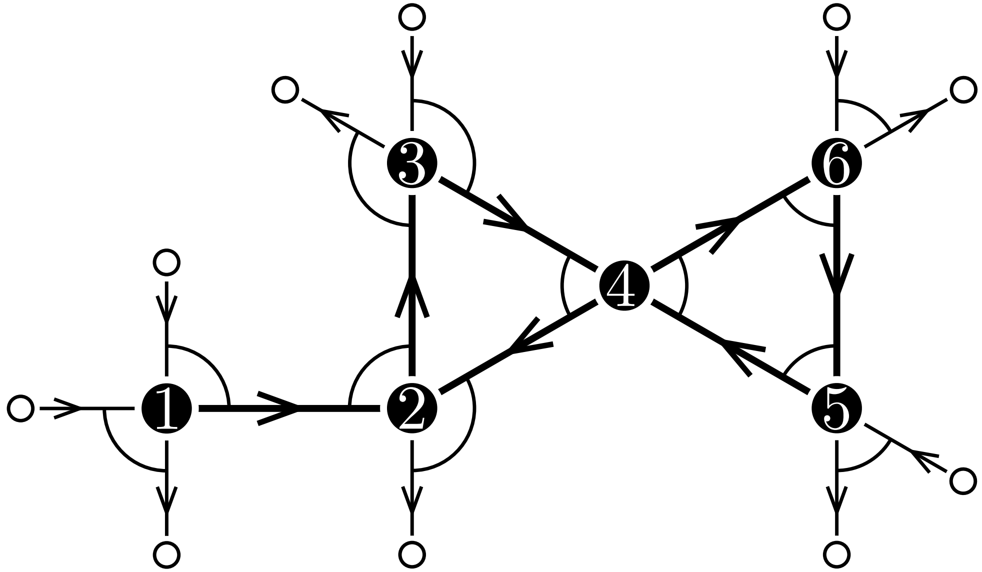
BLOSSOMING QUIVERS AND WALKS

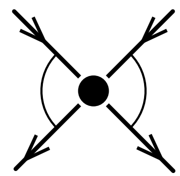


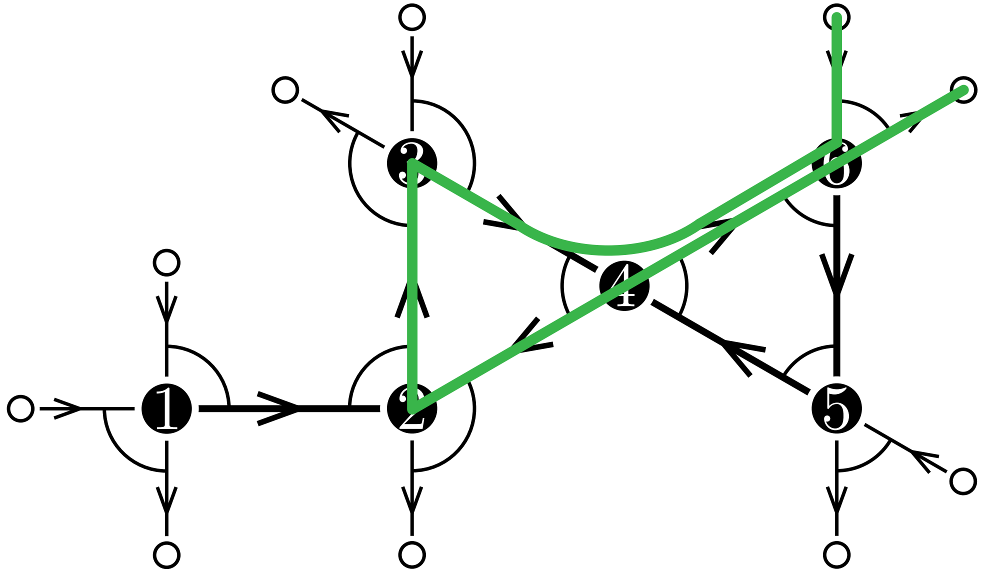
<u>blossoming quiver</u> $\bar{Q}^{❀} =$

add blossoms to complete each vertex to 

BLOSSOMING QUIVERS AND WALKS

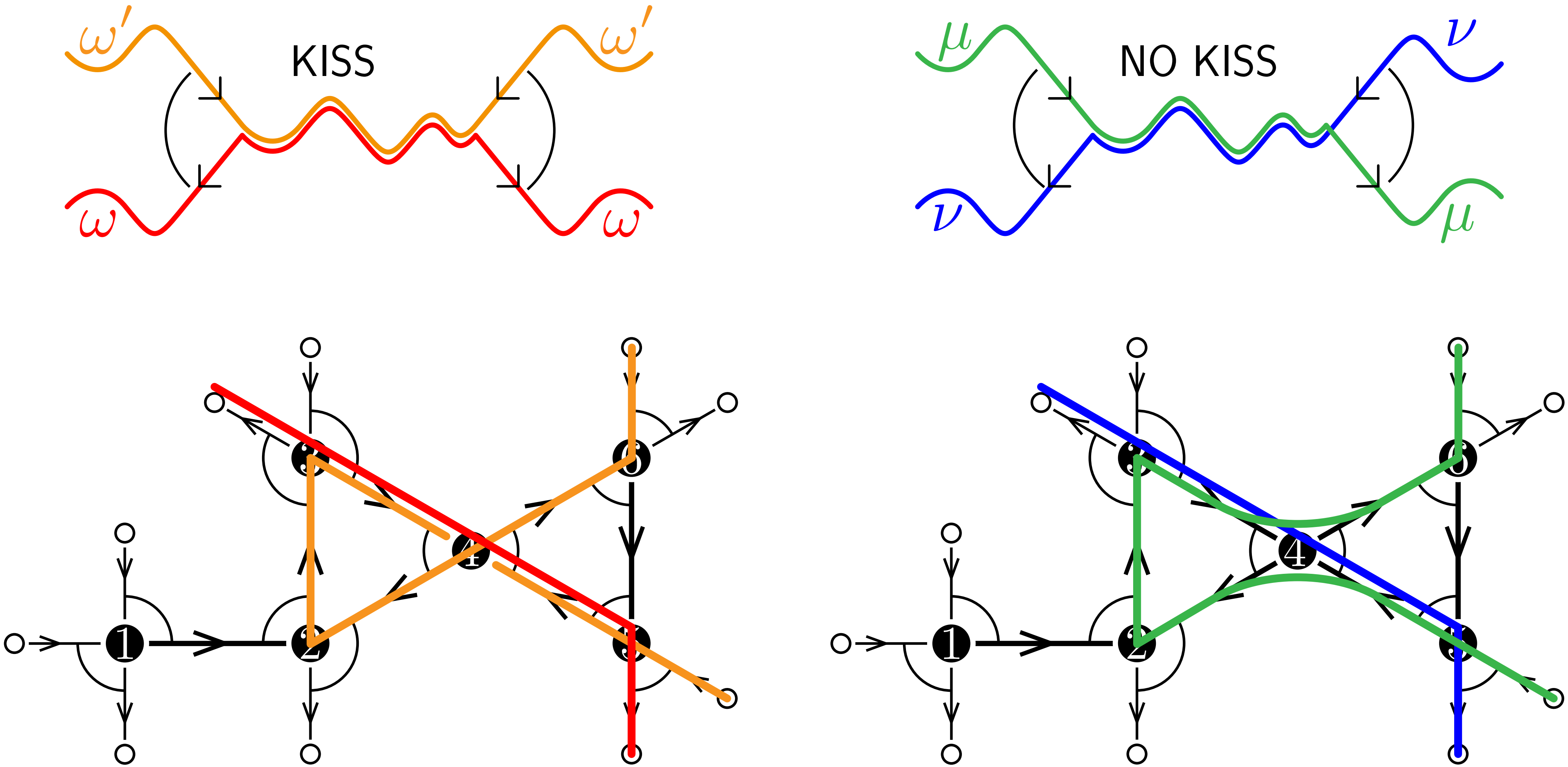


blossoming quiver $\bar{Q}^{❀} =$
add blossoms to complete each vertex to 

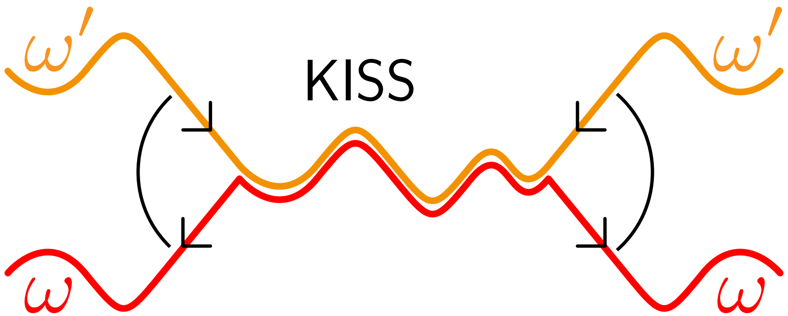


walk $\omega =$ maximal string in $\bar{Q}^{❀}$
from blossoms to blossoms

KISSING

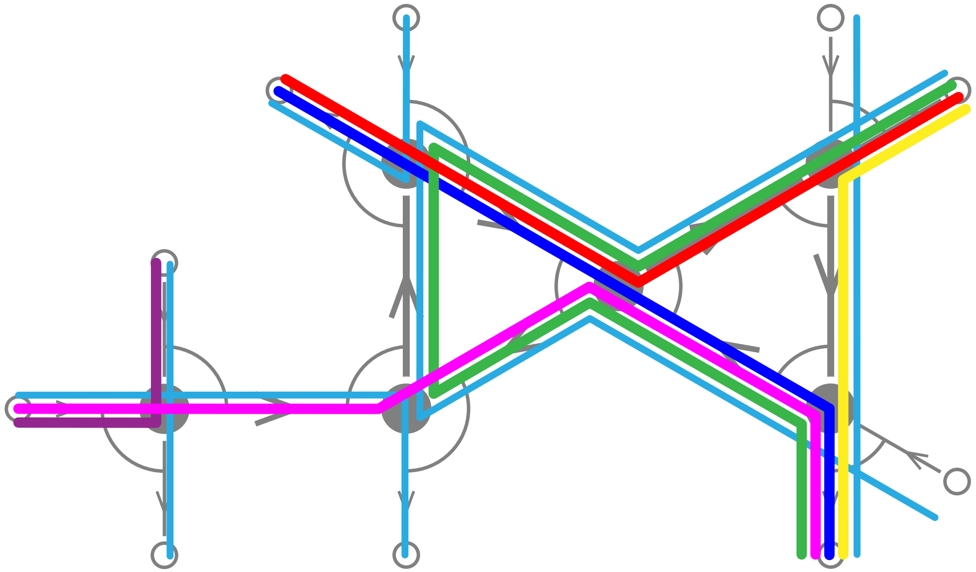


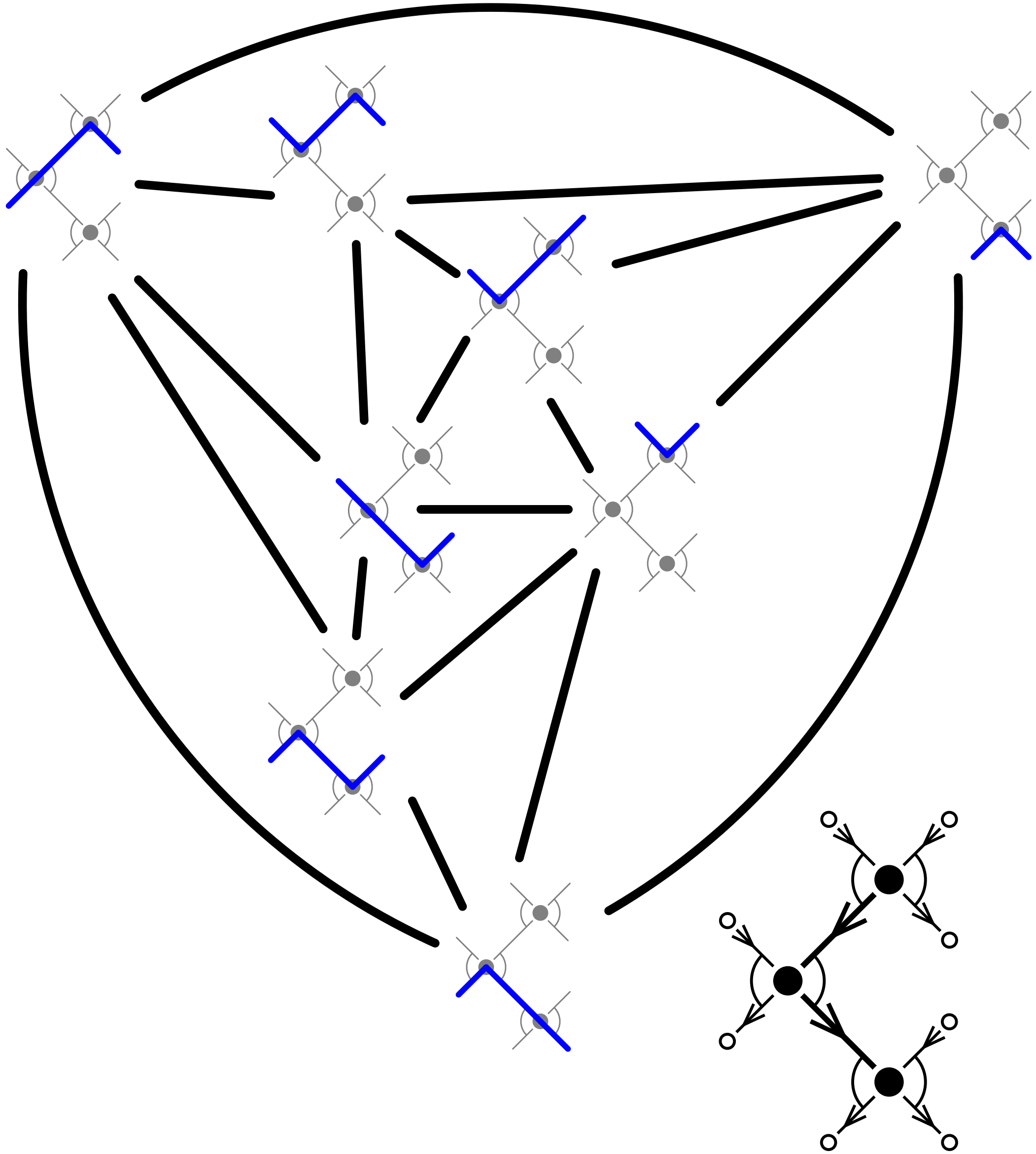
NON-KISSING COMPLEX



[reduced] non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices = [bending] walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing [bending] walks in $\bar{Q}^{❀}$





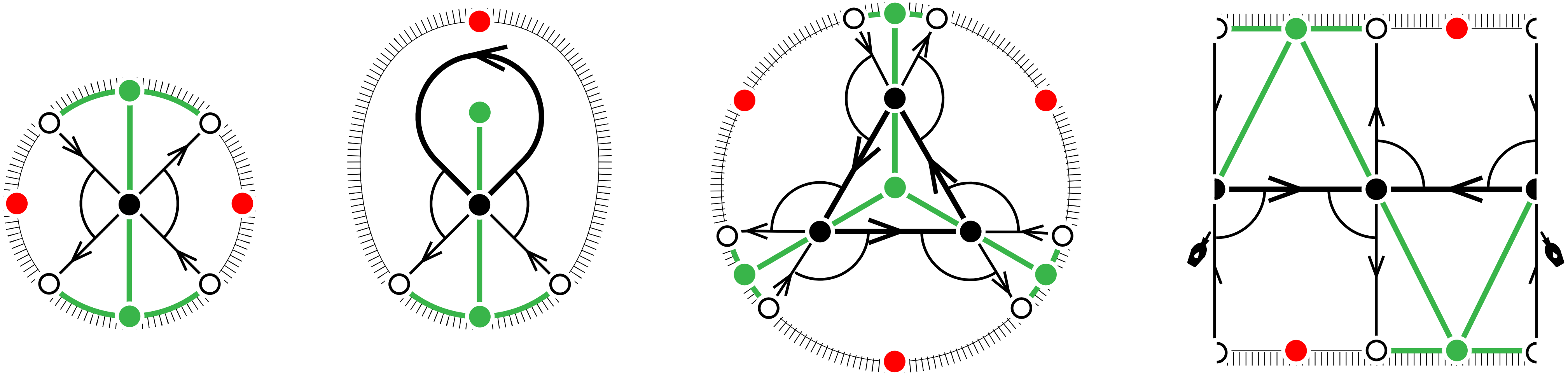
NON-CROSSING VS NON-KISSING

Palu–P.–Plamondon,
Non-kissing and non-crossing complexes for locally gentle algebras ('19)

QUIVER OF A DISSECTION

quiver $\bar{Q}_{\mathrm{D}}$ of a dissection =

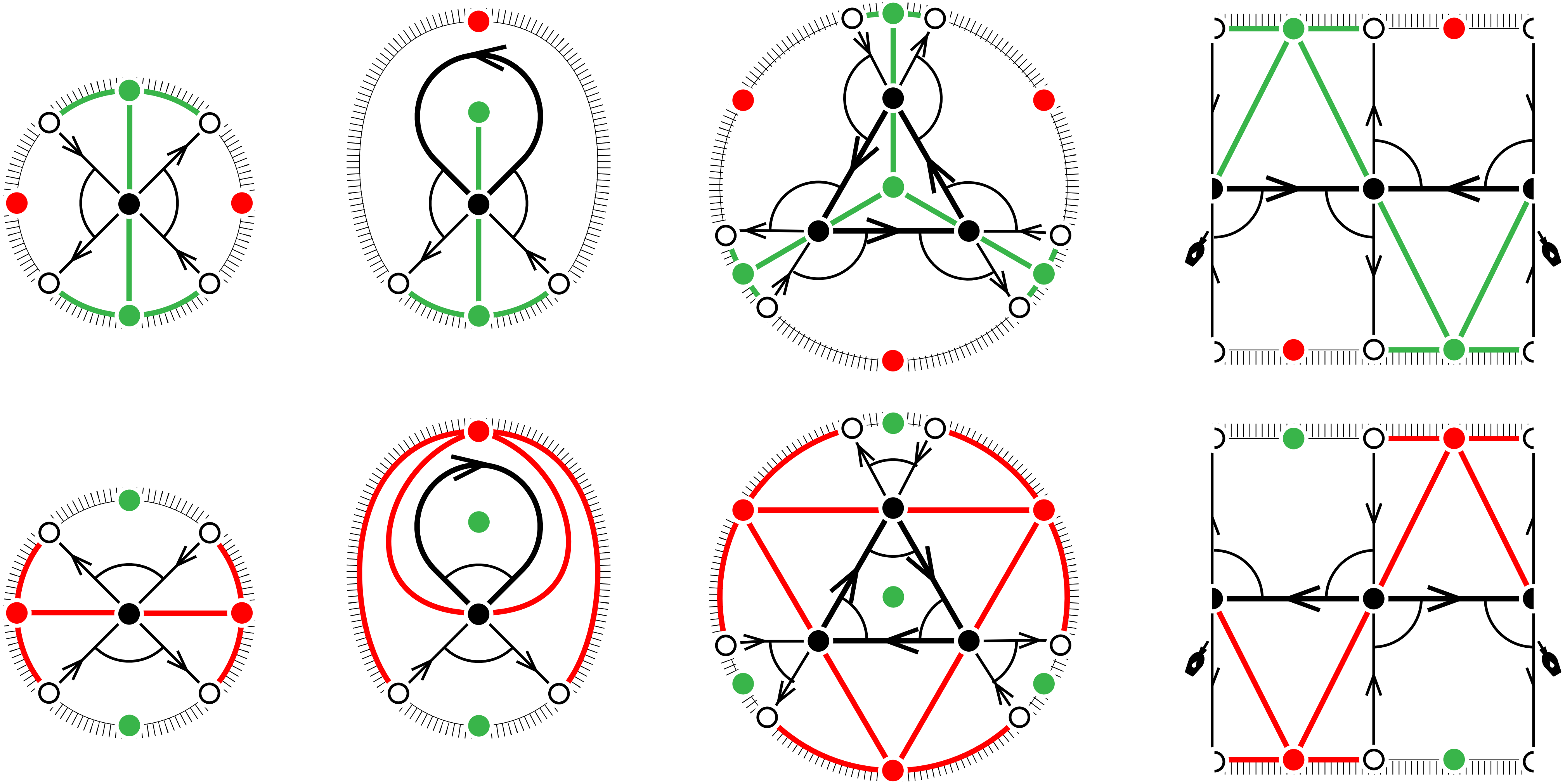
- vertices = edges of D (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of D
- relations = three consecutive edges around a face of D



QUIVER OF A DISSECTION

quiver $\bar{Q}_{\mathrm{D}}$ of a dissection =

- vertices = edges of D (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of D
- relations = three consecutive edges around a face of D



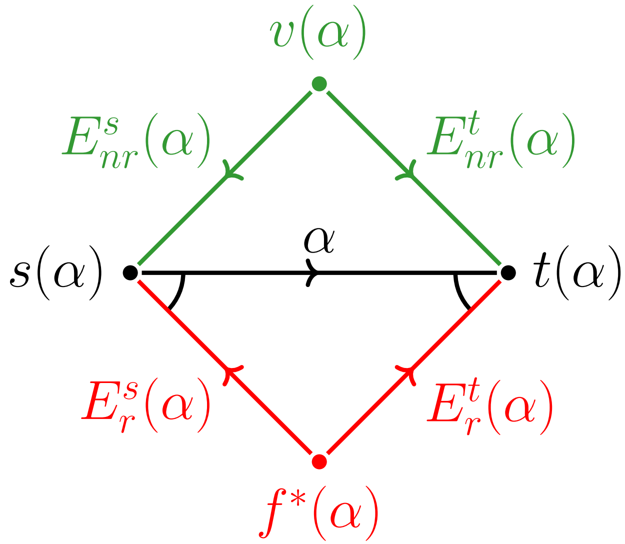
SURFACE OF A GENTLE QUIVER

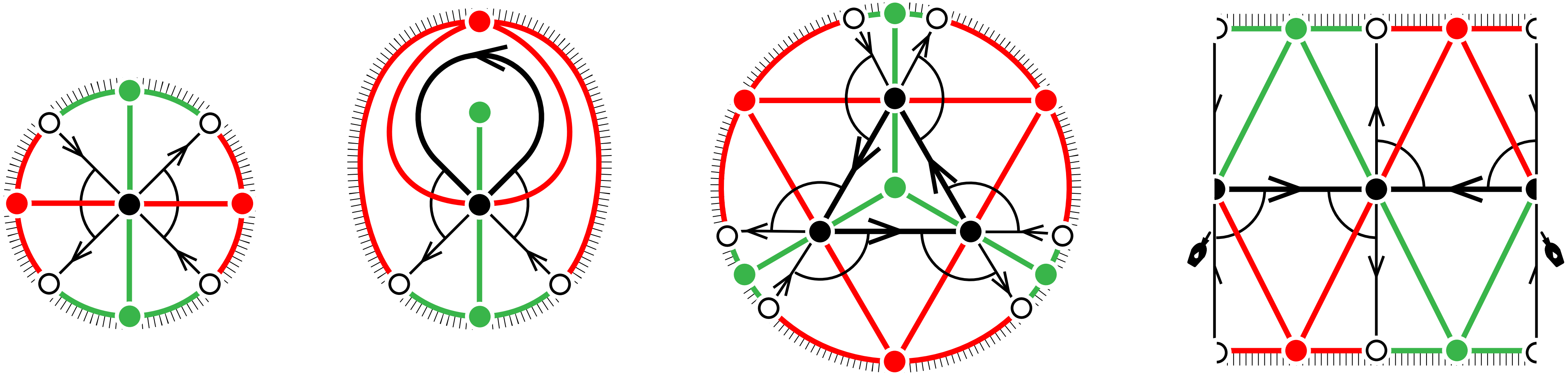
surface $\mathcal{S}_{\bar{Q}}$ of quiver $\bar{Q}$ = surface obtained from the blossoming quiver $\bar{Q}^{❀}$ as follows:

(i) for each arrow $\alpha \in Q_1^{❀}$, consider a lozenge

(ii) for any $\alpha, \beta \in Q_1^{❀}$ with $t(\alpha) = s(\beta)$, proceed to the following identifications:

- if $\alpha\beta \in I$, then glue $E_r^t(\alpha)$ with $E_r^s(\beta)$,
- if $\alpha\beta \notin I$, then glue $E_{nr}^t(\alpha)$ with $E_{nr}^s(\beta)$.

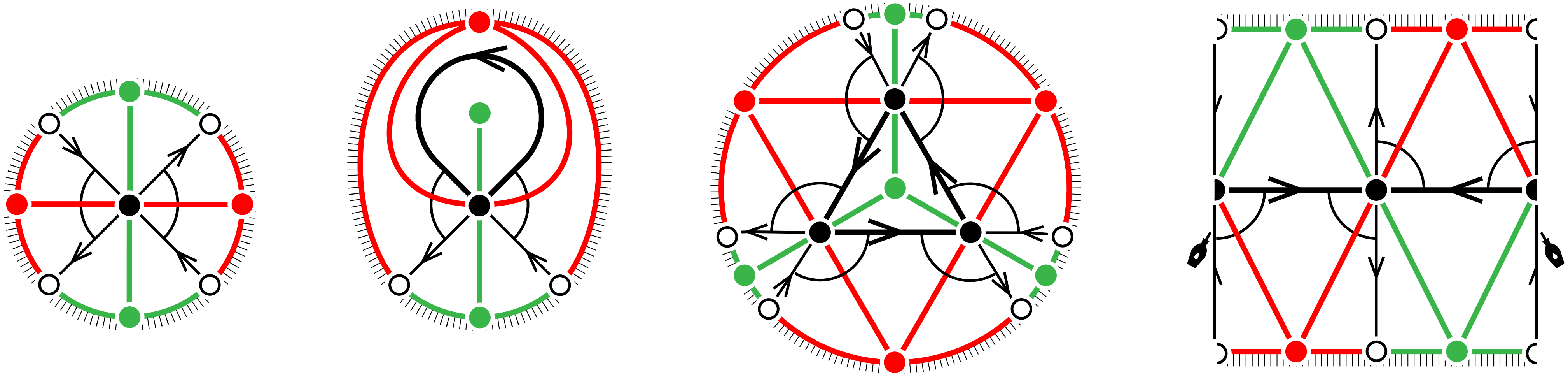




NON-CROSSING VS NON-KISSING

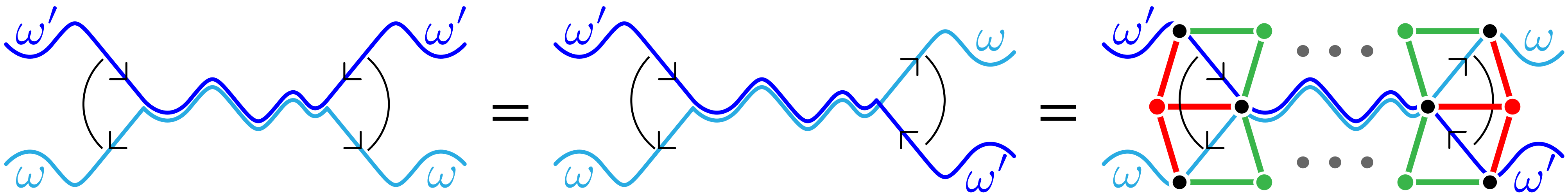
PROP. The two previous constructions are inverse to each other and define bijections:

pairs of dual dissections on a surface $\longleftrightarrow$ gentle quivers



PROP. It defines isomorphisms between:

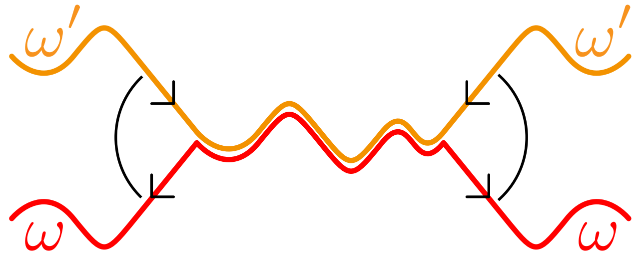
non-crossing complex of dissections $\longleftrightarrow$ non-kissing complex of gentle quiver



SECOND HALF OF THE TALK

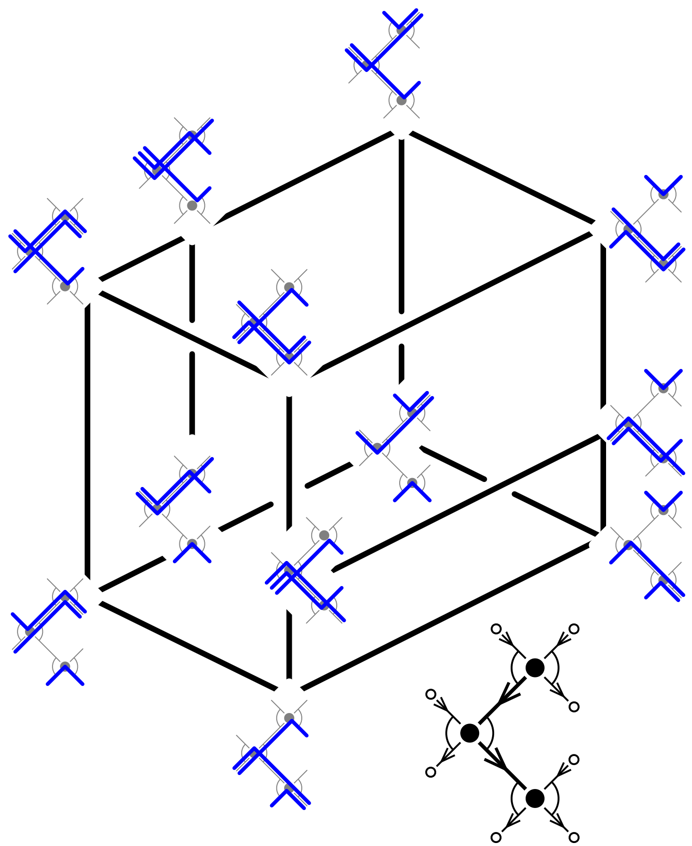
non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices = walks in $\bar{Q}^{\circledast}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing walks in $\bar{Q}^{\circledast}$

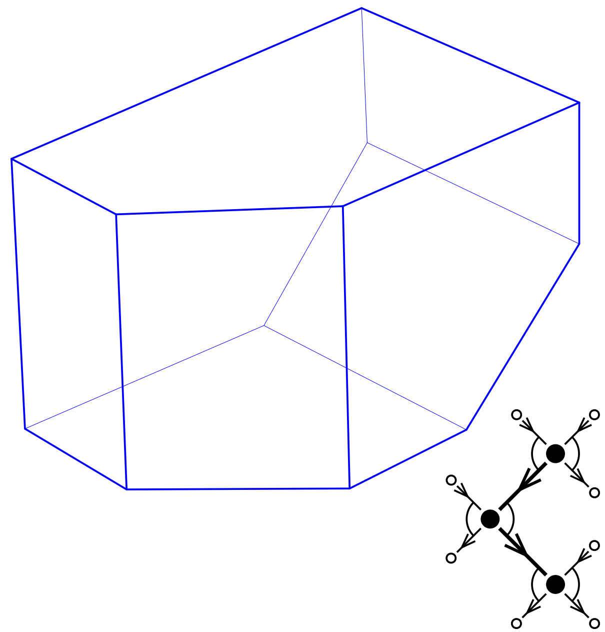


... generalizing the associahedron

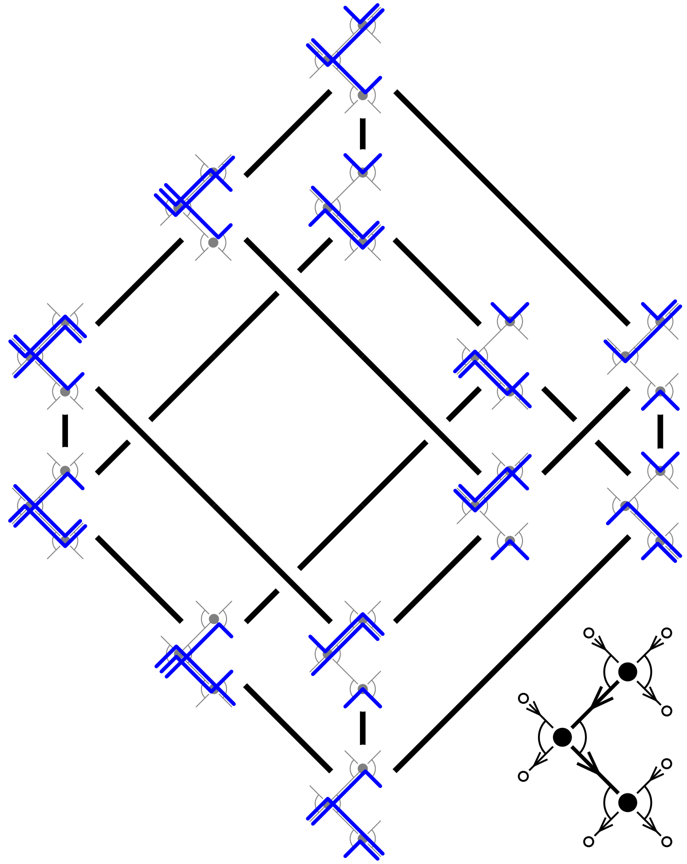
Flip graph



Associahedron

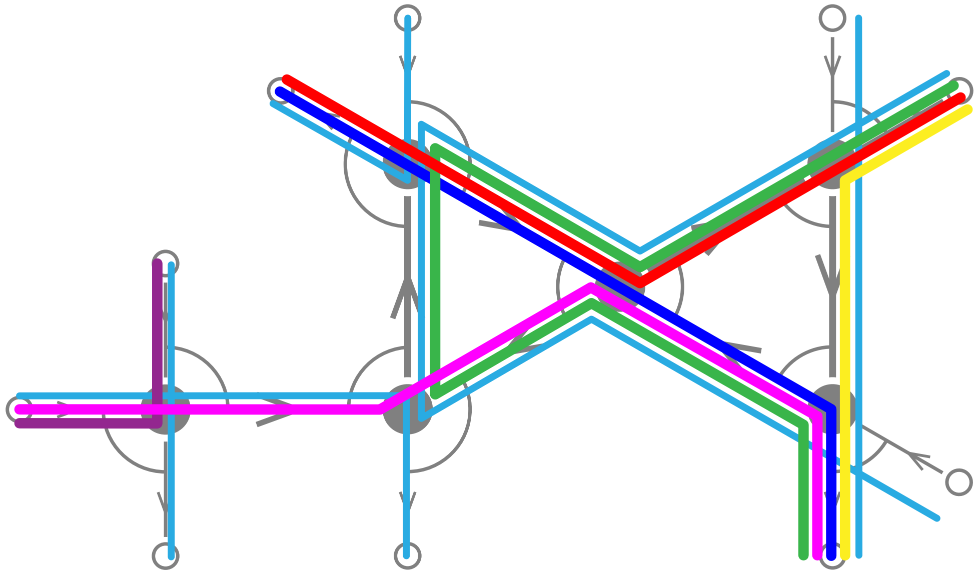


Tamari lattice



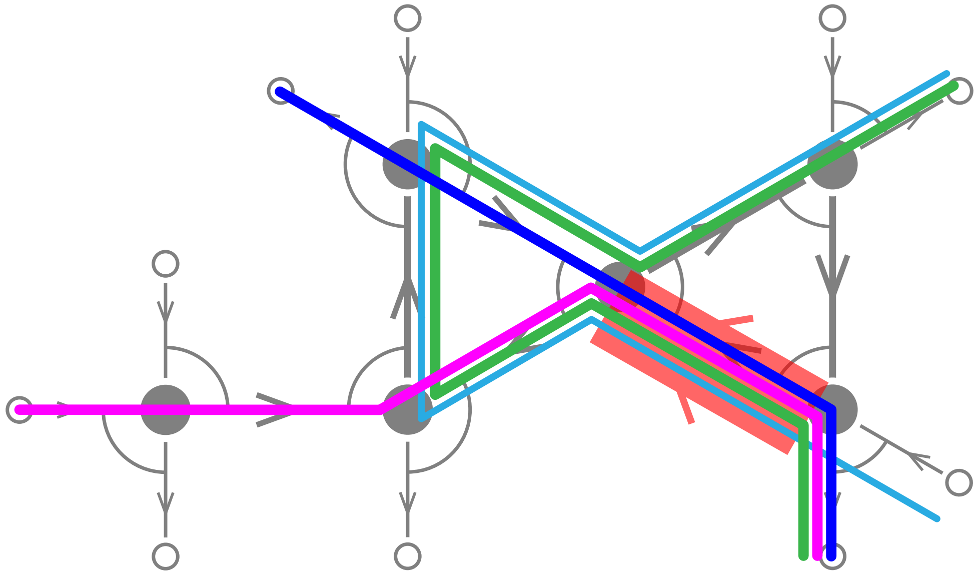
DISTINGUISHED ARROWS AND FLIPS

McConville, *Lattice structures of grid Tamari orders* ('17)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)



F face of $\mathcal{NK}(\bar{Q})$

DISTINGUISHED WALKS, ARROWS AND STRINGS

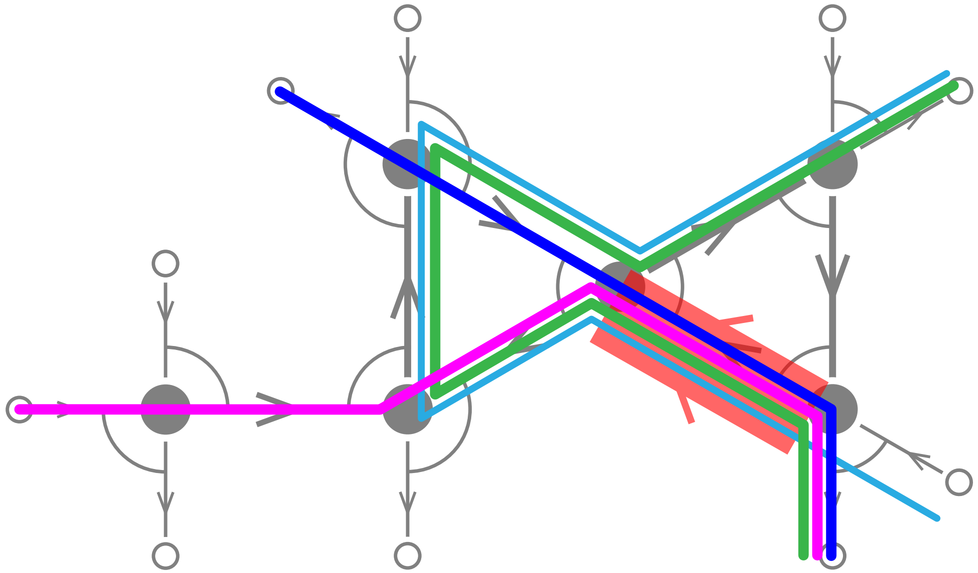


F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$

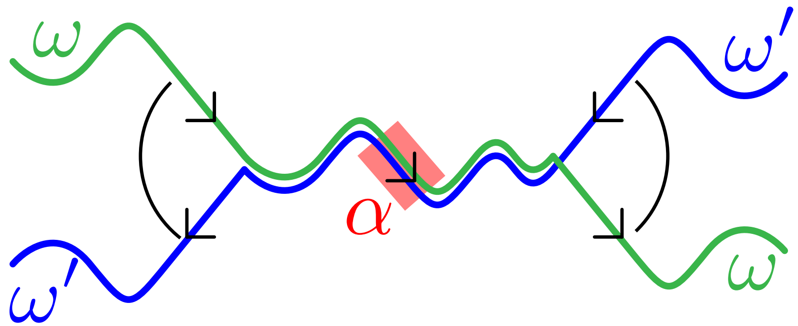
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

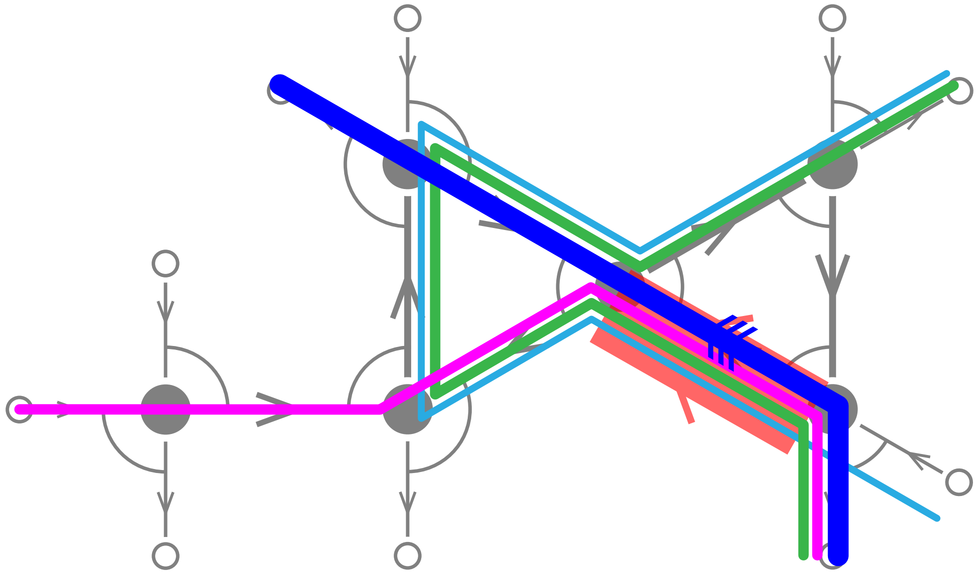
$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

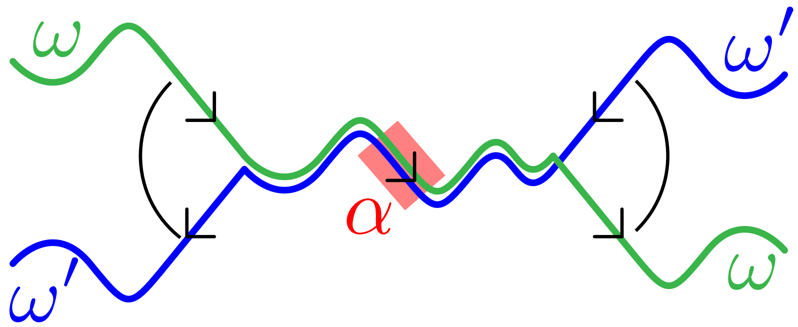
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$

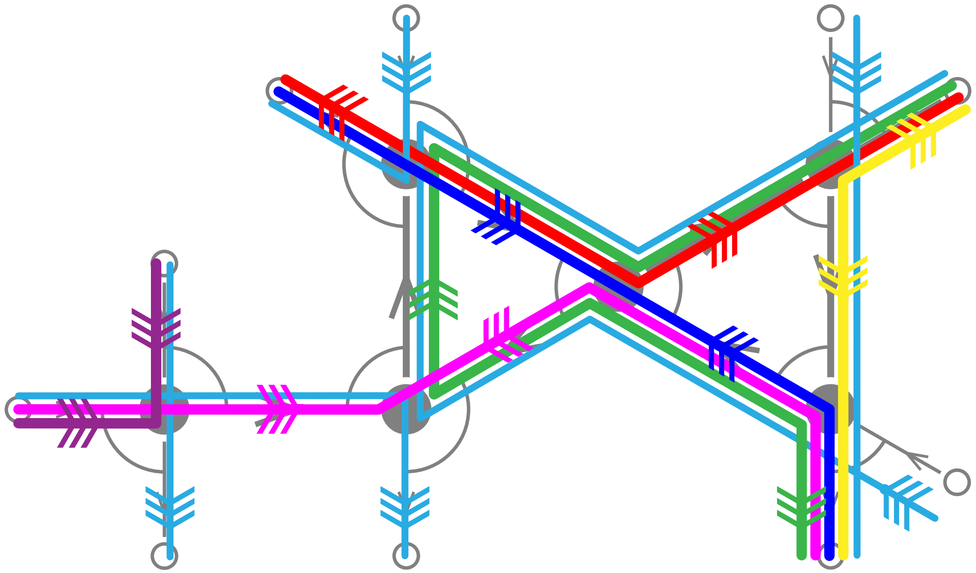


$\omega \prec_\alpha \omega'$ countercurrent order at α

distinguished walk at α in F = $\text{dw}(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in F = $\text{da}(\omega, F) = \{\alpha \in Q_1 \mid \omega = \text{dw}(\alpha, F)\}$

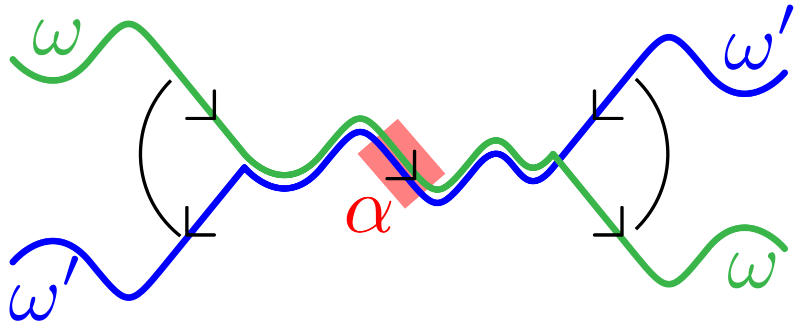
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

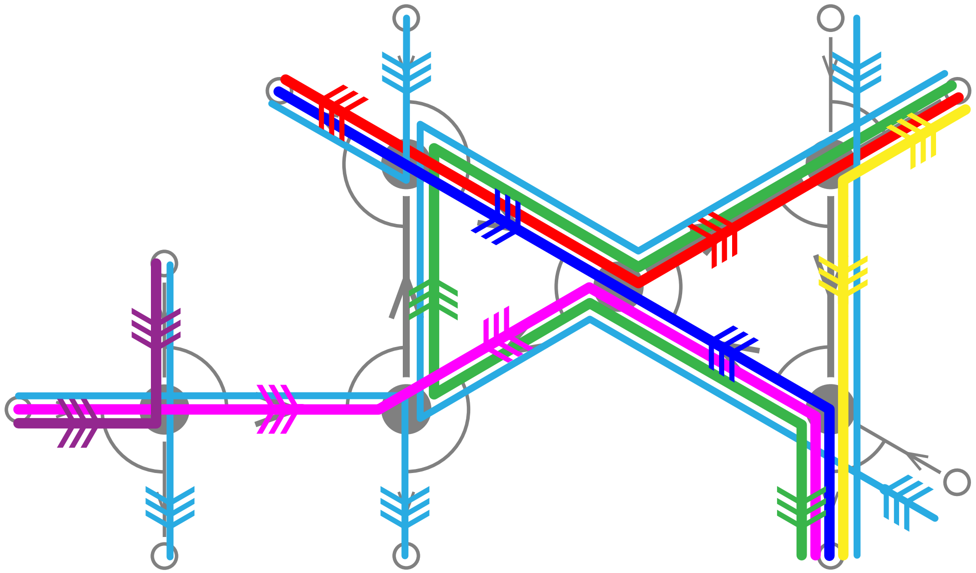
distinguished walk at α in F $= \mathrm{dw}(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in F $= \mathrm{da}(\omega, F) = \{\alpha \in Q_1 \mid \omega = \mathrm{dw}(\alpha, F)\}$

PROP. For any facet $F \in \mathcal{NK}(\bar{Q})$,

- each bending walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

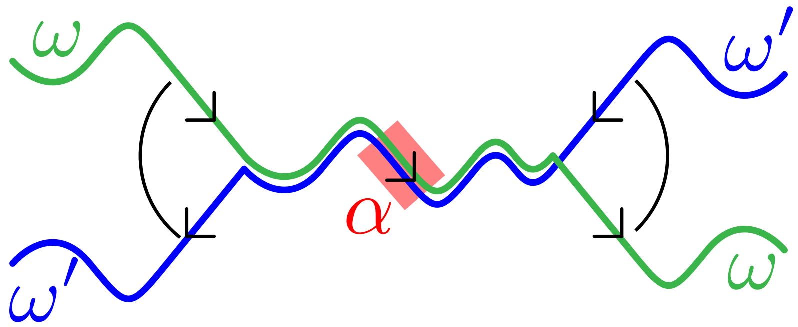
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$



$\omega \prec_\alpha \omega'$ countercurrent order at α

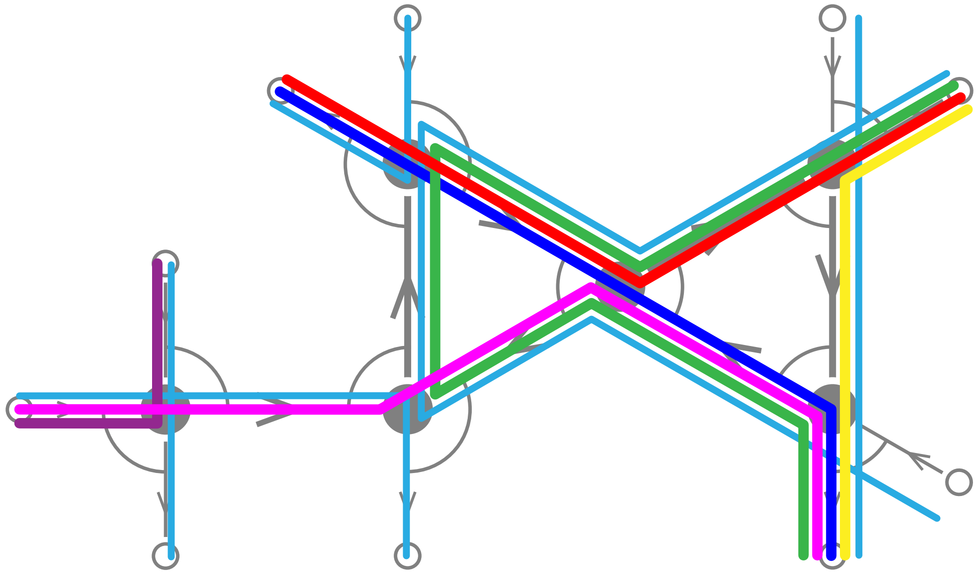
distinguished walk at α in $F = \text{dw}(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

distinguished arrows of ω in $F = \text{da}(\omega, F) = \{\alpha \in Q_1 \mid \omega = \text{dw}(\alpha, F)\}$

PROP. For any facet $F \in \mathcal{NK}(\bar{Q})$,

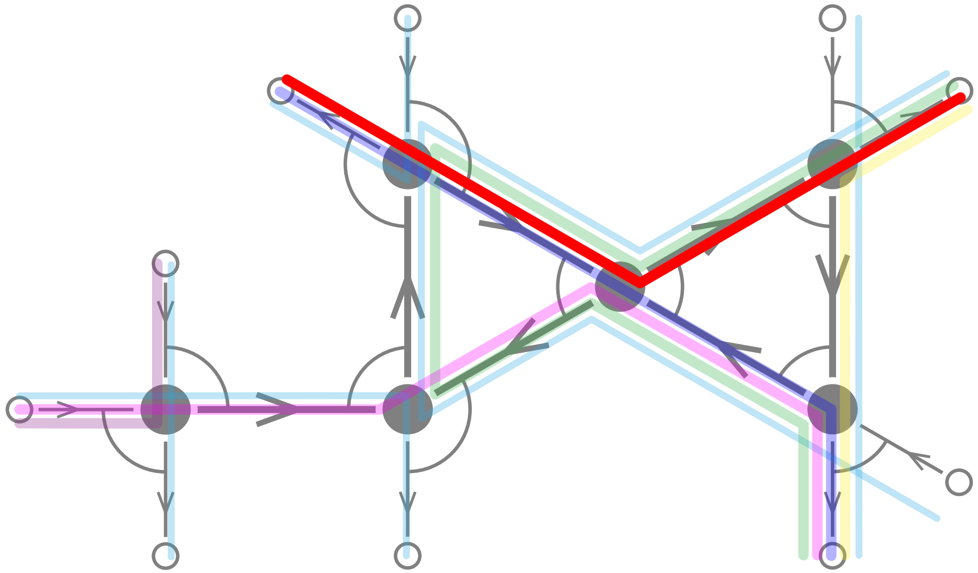
- each bending walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

CORO. $\mathcal{NK}(\bar{Q})$ is pure of dimension $|Q_0|$.

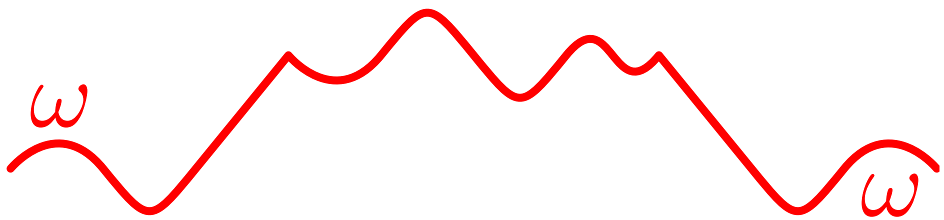


F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

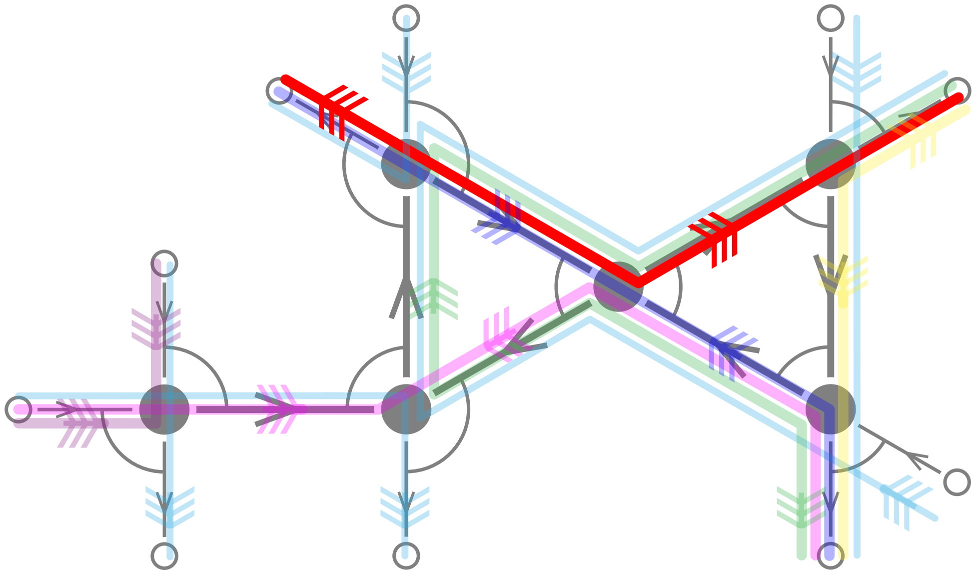
FLIPS



F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$\omega \in F$ we want to "flip"



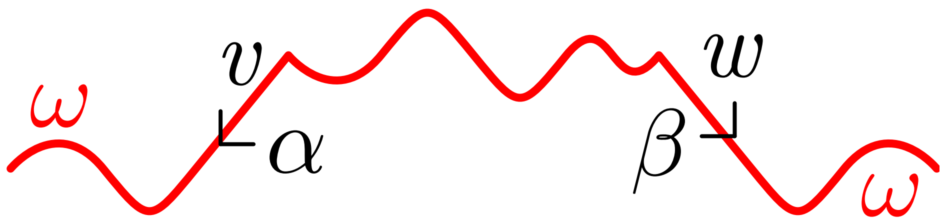
FLIPS



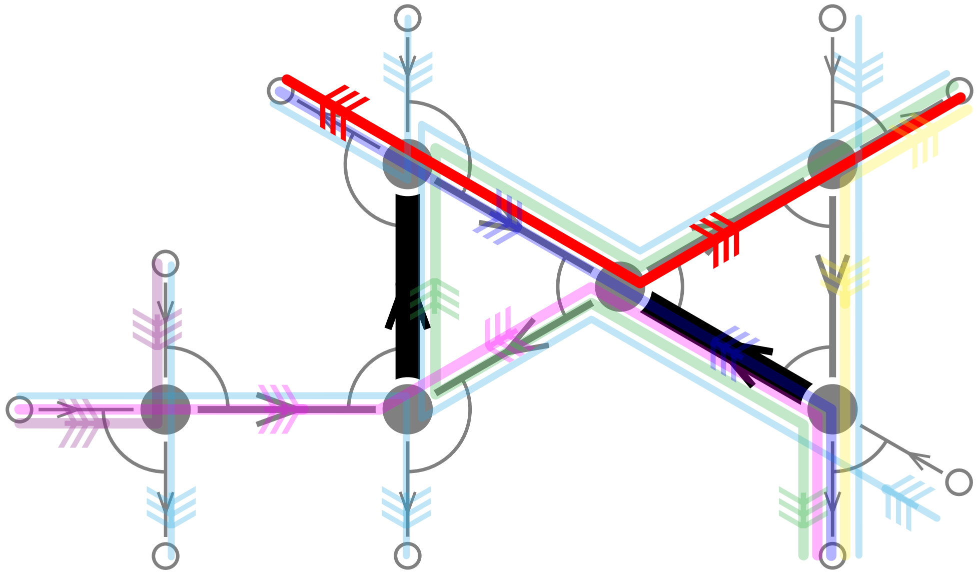
F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$



FLIPS

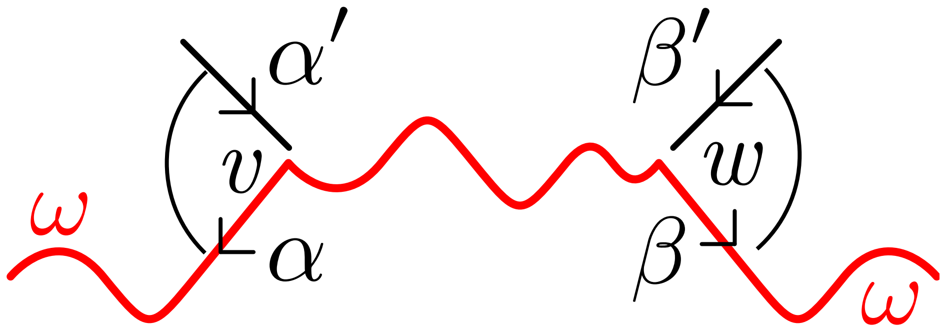


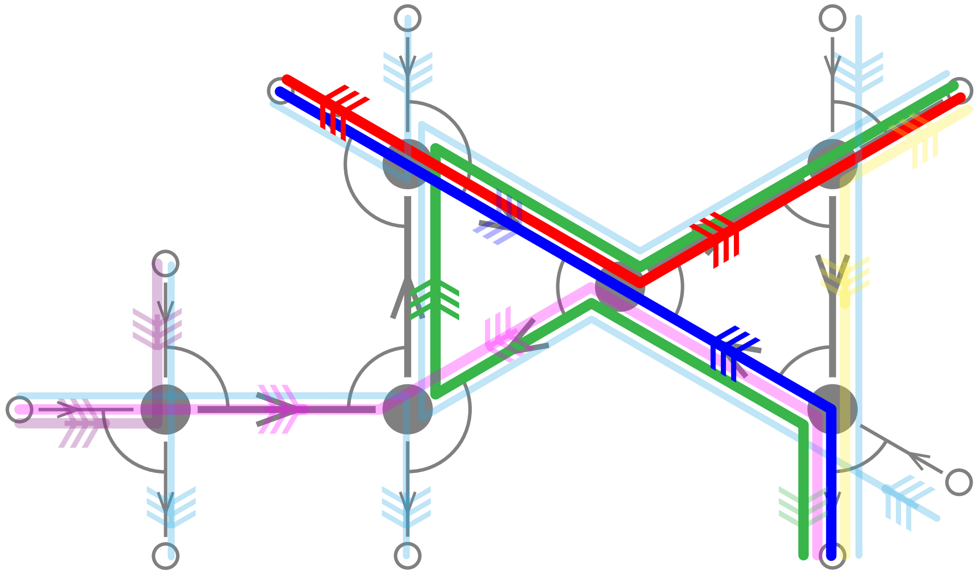
F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$





F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

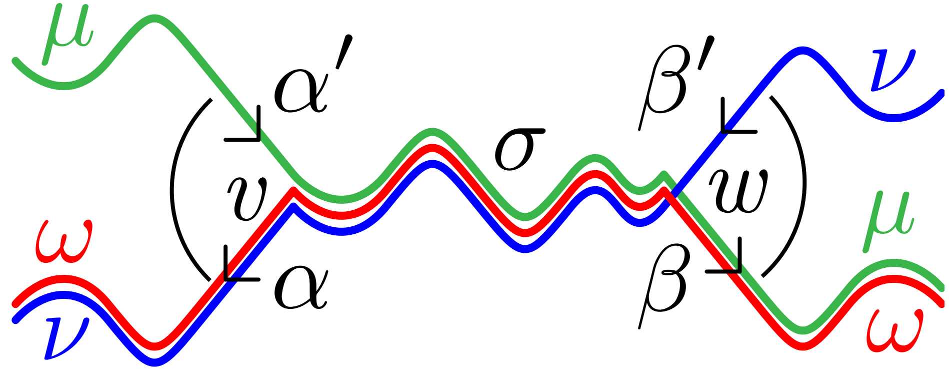
$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

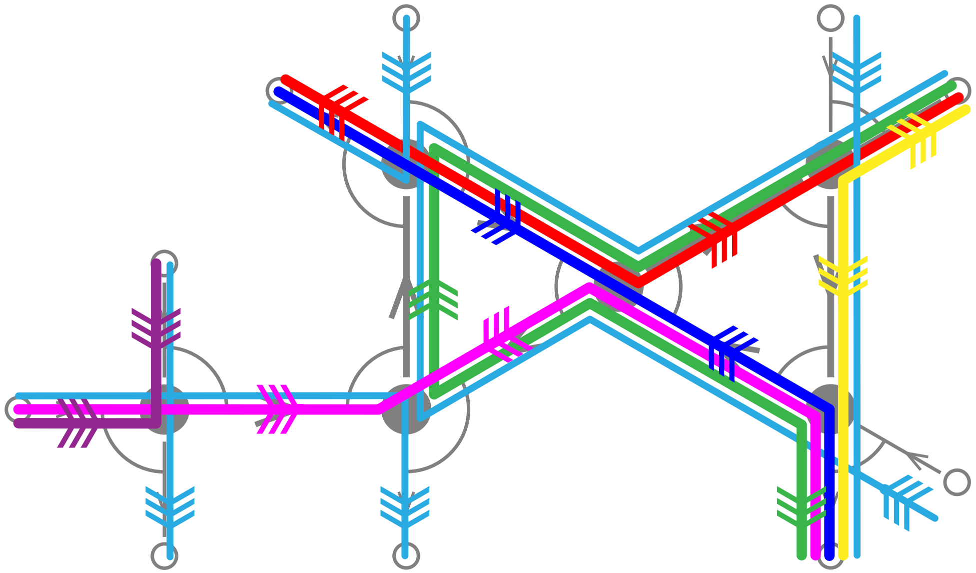
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

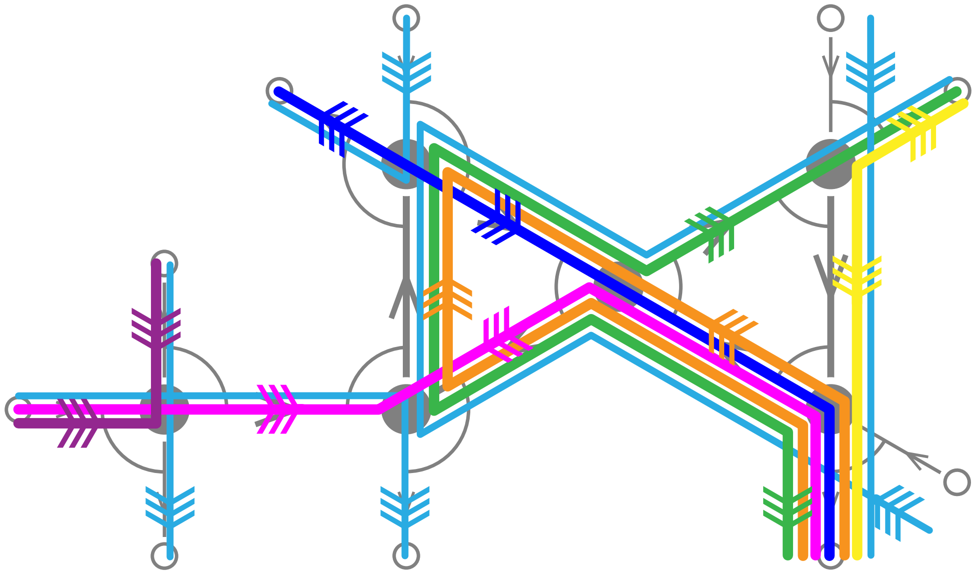
$\mu = \mathsf{dw}(\alpha', F)$ and $\nu = \mathsf{dw}(\beta', F)$

$\omega = \nu[\cdot, v]\, \sigma\, \mu[w, \cdot]$



FLIPS





F facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

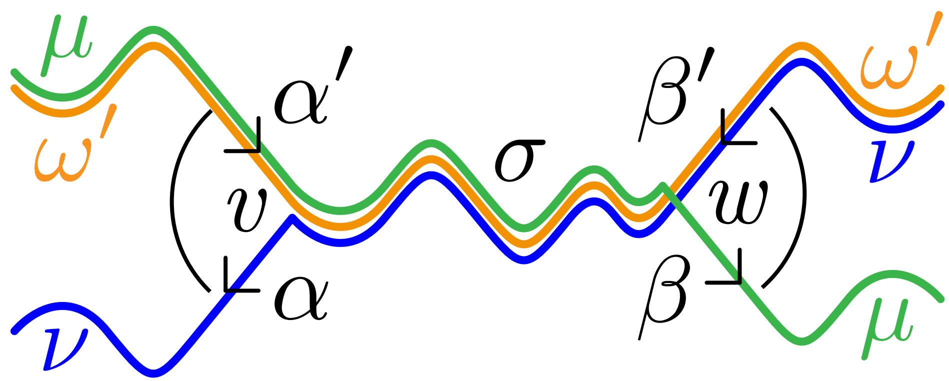
$\{\alpha, \beta\} = \mathsf{da}(\omega, F)$

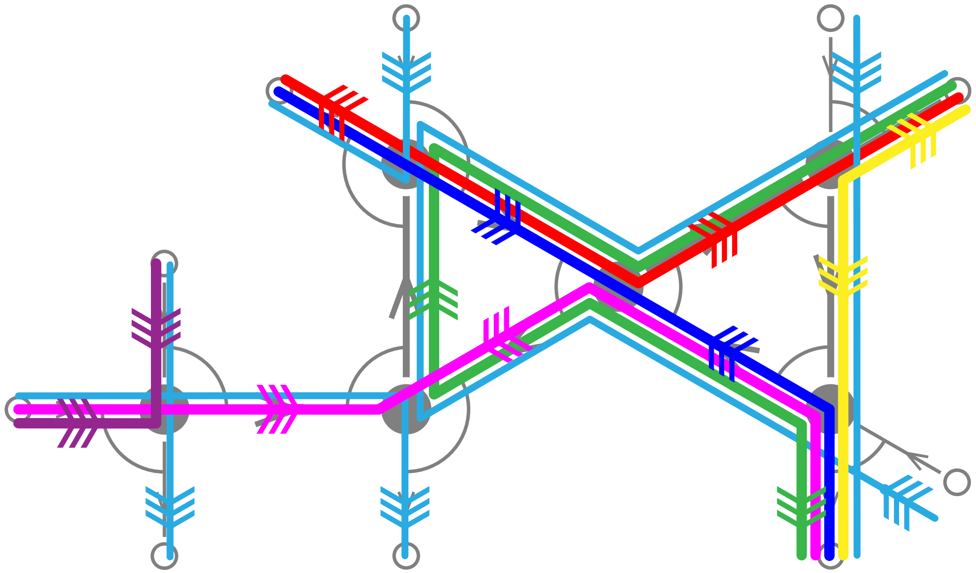
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = \mathsf{dw}(\alpha', F)$ and $\nu = \mathsf{dw}(\beta', F)$

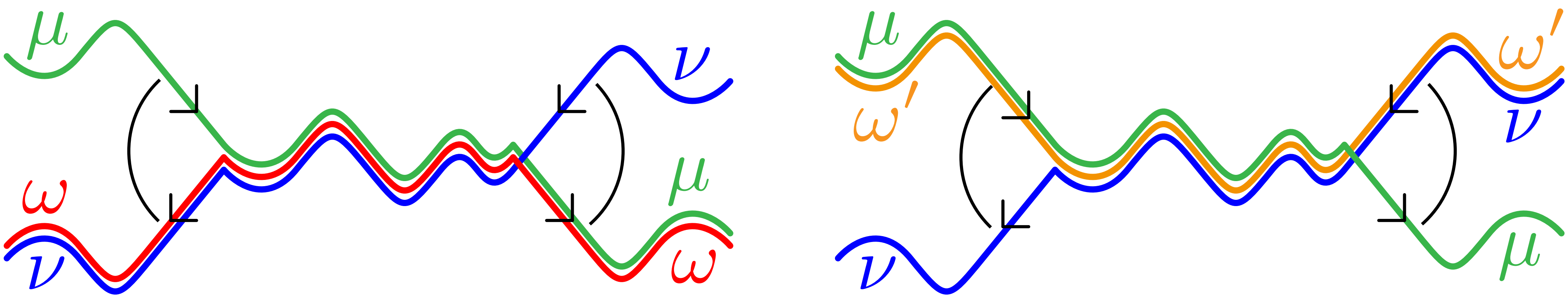
$\omega = \nu[\cdot, v]\,\sigma\,\mu[w, \cdot]$

$\omega' = \mu[\cdot, v]\,\sigma\,\nu[w, \cdot]$





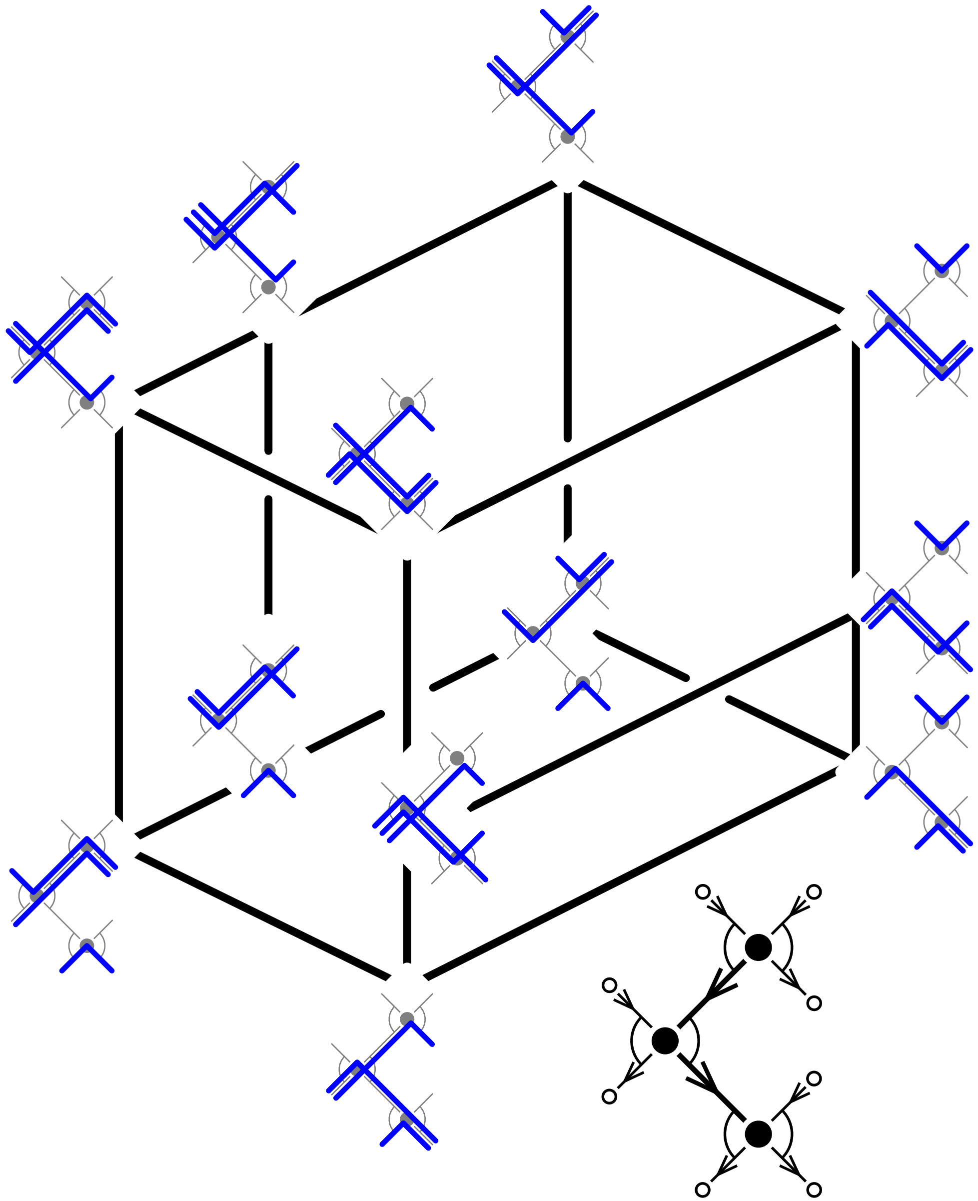
PROP. ω' kisses ω but no other walk of F. Moreover, ω' is the only such walk.



FLIPS

flip graph =

- vertices = non-kissing facets
- edges = flips



GENTLE ASSOCIAHEDRA

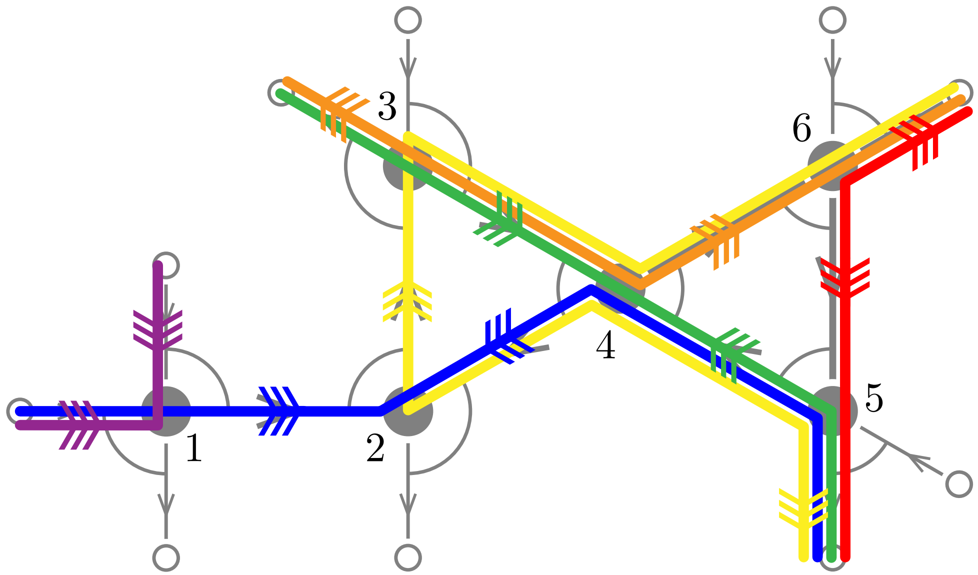
Manneville–P., *Geometric realizations of the accordion complex* ('19)
Hohlweg–P.–Stella, Polytopal realizations of finite type g-vector fans ('18)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

G-VECTORS & C-VECTORS

<u>multiplicity vector</u> $\mathbf{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of $Q_0 \;=\; \sum_{i \in [m]} \mathbf{e}_{v_i} \;\in\; \mathbb{R}^{Q_0}$

<u>g-vector</u> $\mathbf{g}(\omega)$ of a walk $\omega \;=\; \mathbf{m}_{\text{peaks}(\omega)} - \mathbf{m}_{\text{deeps}(\omega)}$

<u>c-vector</u> $\mathbf{c}(\omega \in F)$ of a walk ω in a non-kissing facet $F \;=\; \varepsilon(\omega, F)\, \mathbf{m}_{\text{ds}(\omega, F)}$



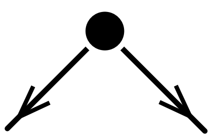
$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

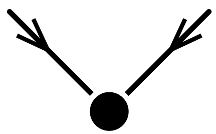
$\mathbf{g}(F)$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

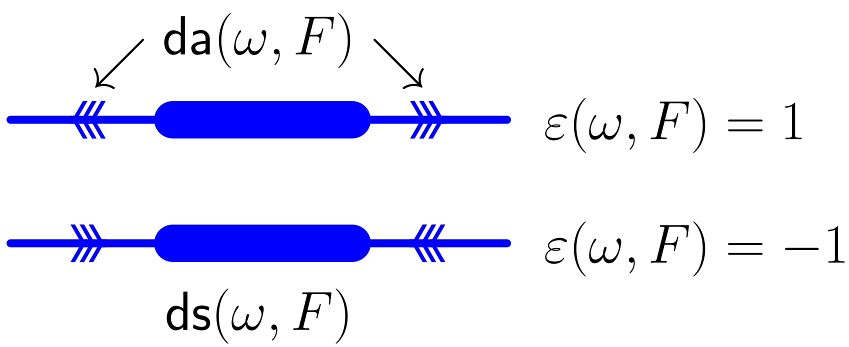
$\mathbf{c}(F)$

peak





deep

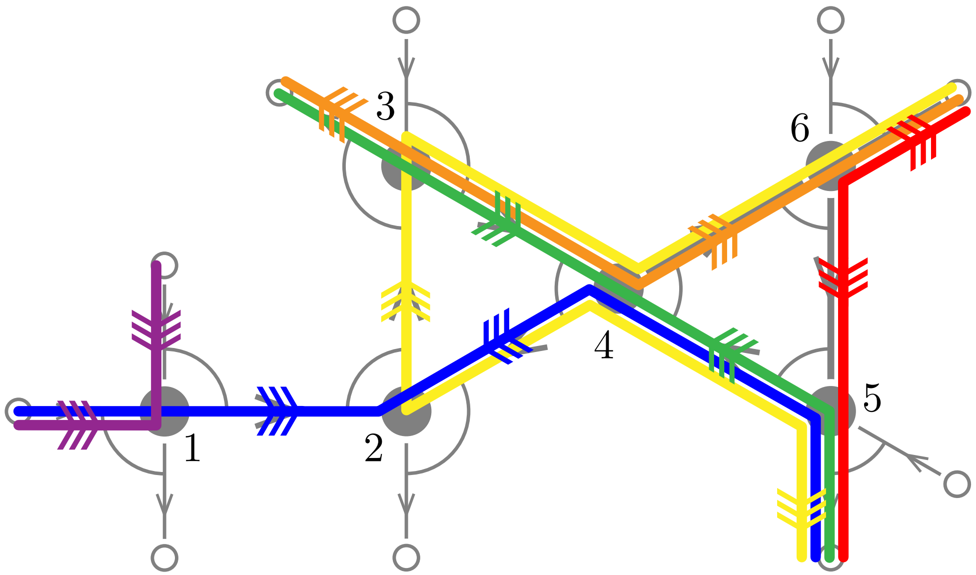


G-VECTORS & C-VECTORS

<ins>multiplicity vector</ins> $\mathbf{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of $Q_0 \quad = \quad \sum_{i \in [m]} \mathbf{e}_{v_i} \; \in \; \mathbb{R}^{Q_0}$

<ins>g-vector</ins> $\mathbf{g}(\omega)$ of a walk $\omega \quad = \quad \mathbf{m}_{\mathsf{peaks}(\omega)} - \mathbf{m}_{\mathsf{deeps}(\omega)}$

<ins>c-vector</ins> $\mathbf{c}(\omega \in F)$ of a walk ω in a non-kissing facet $F \quad = \quad \varepsilon(\omega, F)\, \mathbf{m}_{\mathsf{ds}(\omega, F)}$



$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{g}(F)$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{c}(F)$

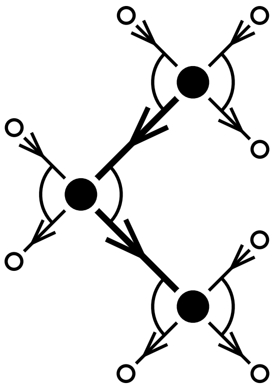
PROP. For any non-kissing facet F, the sets of vectors

$$\mathbf{g}(F) := \{\mathbf{g}(\omega) \mid \omega \in F\} \qquad \text{and} \qquad \mathbf{c}(F) := \{\mathbf{c}(\omega \in F) \mid \omega \in F\}$$

form dual bases.

Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

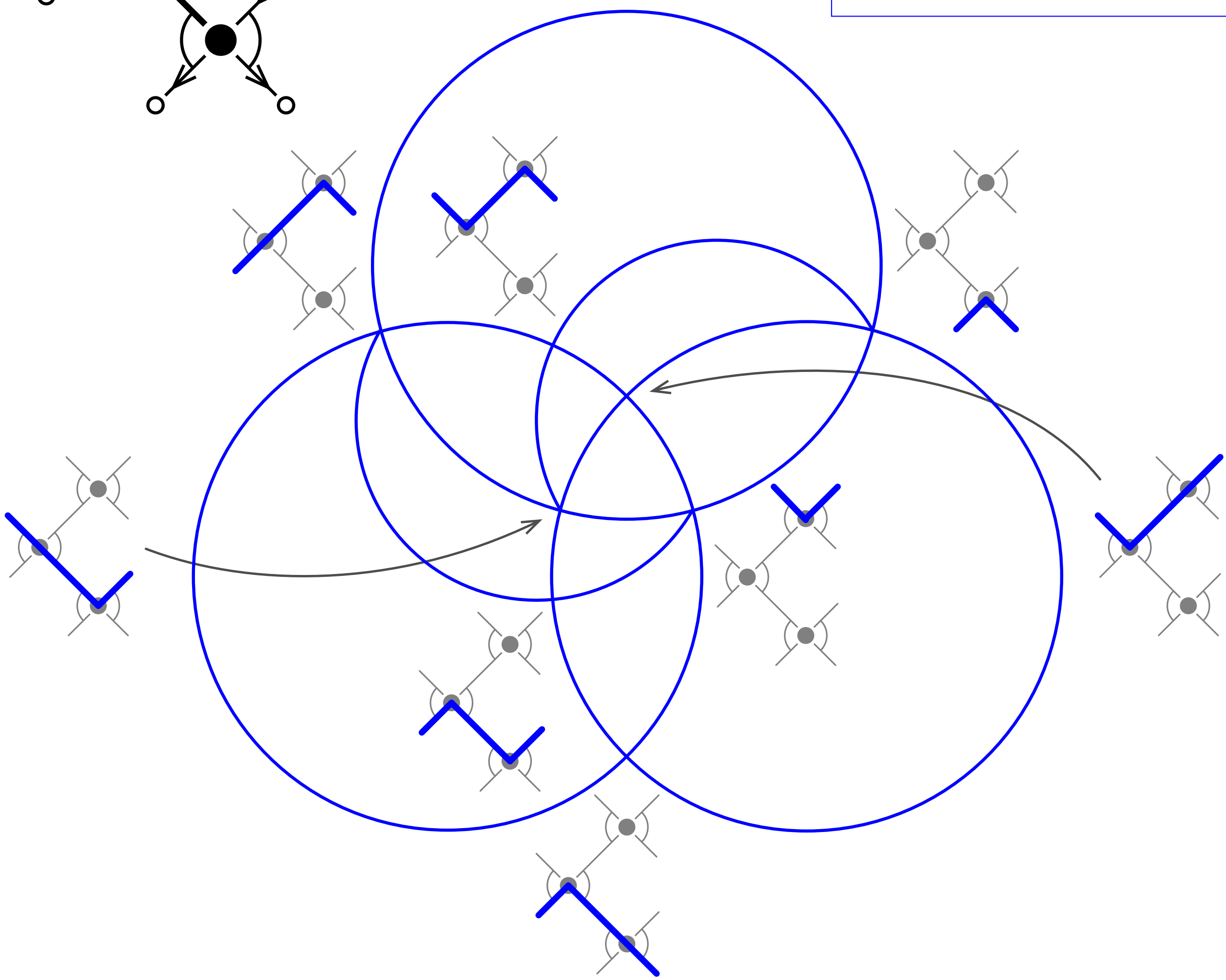
$\mathbf{G}$-VECTOR FAN



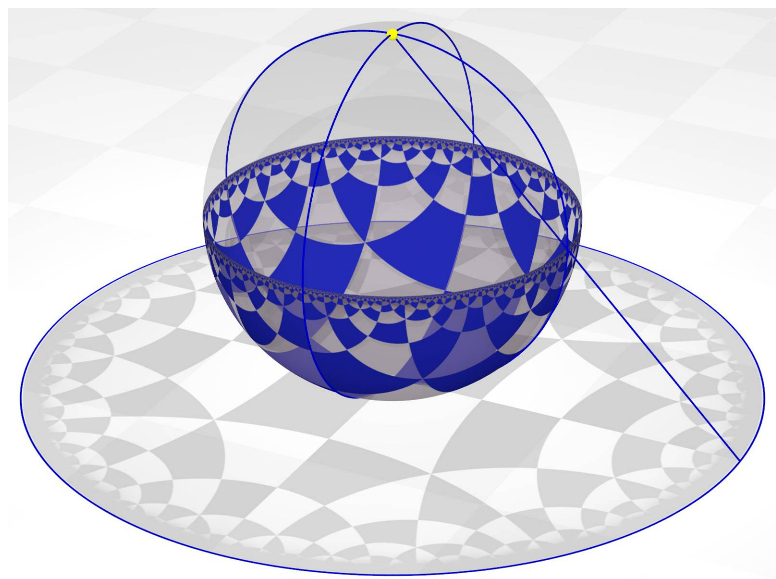
THM. For any gentle quiver $\bar{Q}$, the collection of cones

$$\mathcal{F}^{\mathbf{g}}(\bar{Q}) := \left\{ \mathbb{R}_{\geq 0}\mathbf{g}(F) \;\middle|\; F \in \mathcal{NK}(\bar{Q}) \right\}$$

forms a compl. simpl. fan, called g-vector fan of $\bar{Q}$.



stereographic projection
from $(1, 1, 1)$



NON-KISSING ASSOCIAHEDRON

kissing number $\mathsf{kn}(\omega) = \sum_{\omega'}$ number of times ω and ω' kiss

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{NK}(\bar{Q})$, the two sets of $\mathbb{R}^{Q_0}$ given by

(i) the convex hull of the points

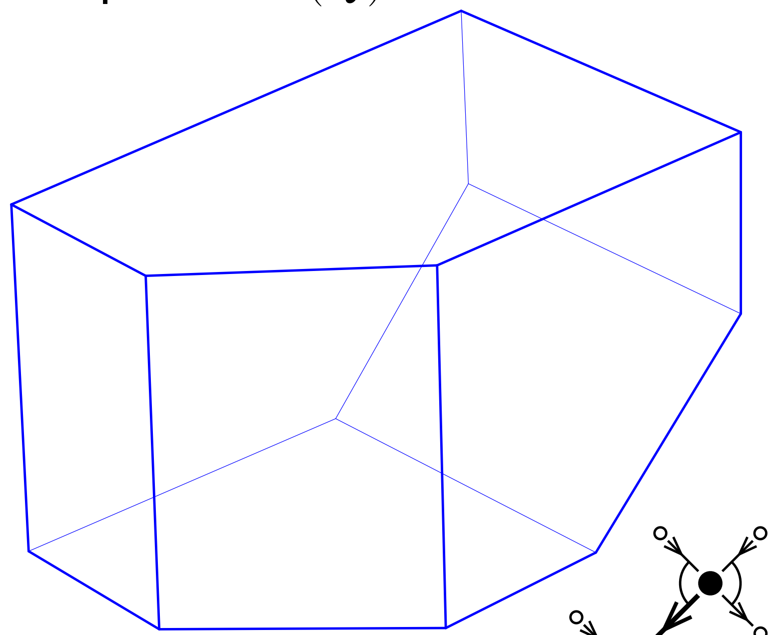
$$\mathbf{p}(F) := \sum_{\omega \in F} \mathsf{kn}(\omega)\, \mathbf{c}(\omega \in F),$$

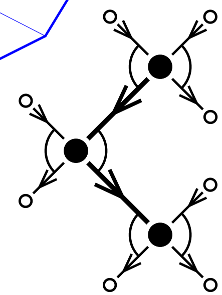
for all non-kissing facets $F \in \mathcal{NK}(\bar{Q})$,

(ii) the intersection of the halfspaces

$$\mathbf{H}^{\geq}(\omega) := \left\{ \mathbf{x} \in \mathbb{R}^{Q_0} \;\middle|\; \langle\, \mathbf{g}(\omega) \mid \mathbf{x} \,\rangle \leq \mathsf{kn}(\omega) \right\}.$$

for all walks ω of $\bar{Q}$,





define the same polytope, whose normal fan is the g-vector fan $\mathcal{F}^{\mathrm{g}}$. We call it the $\bar{Q}$-associahedron and denote it by Asso.

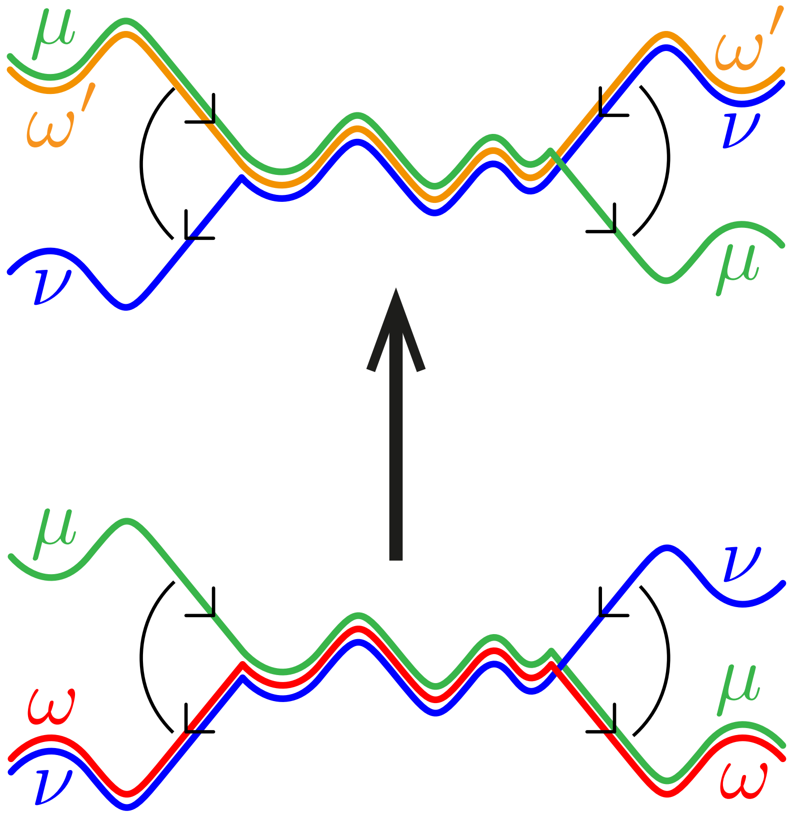
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

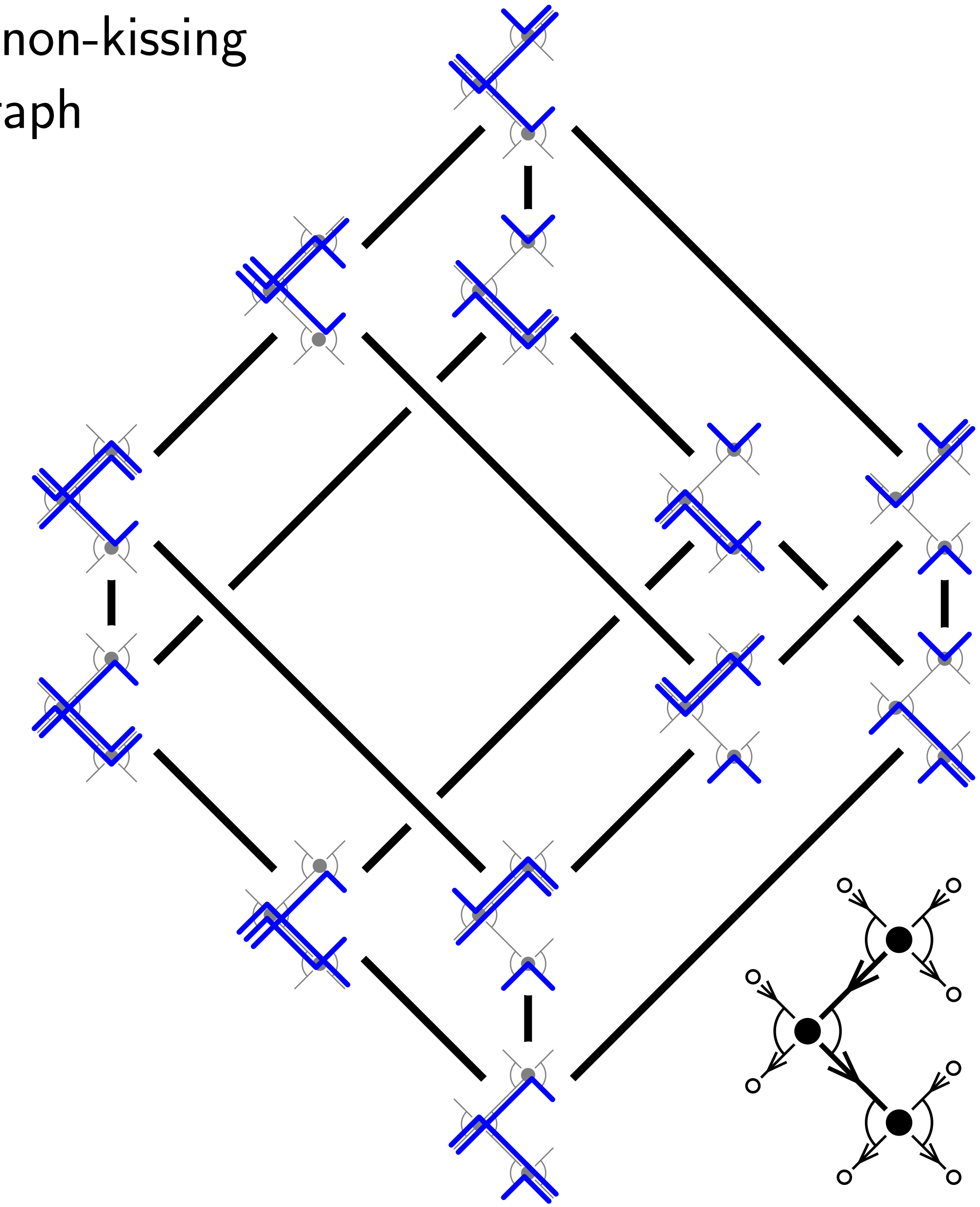
NON-KISSING LATTICE

McConville, *Lattice structures of grid Tamari orders* ('17)
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

NON-KISSING LATTICE

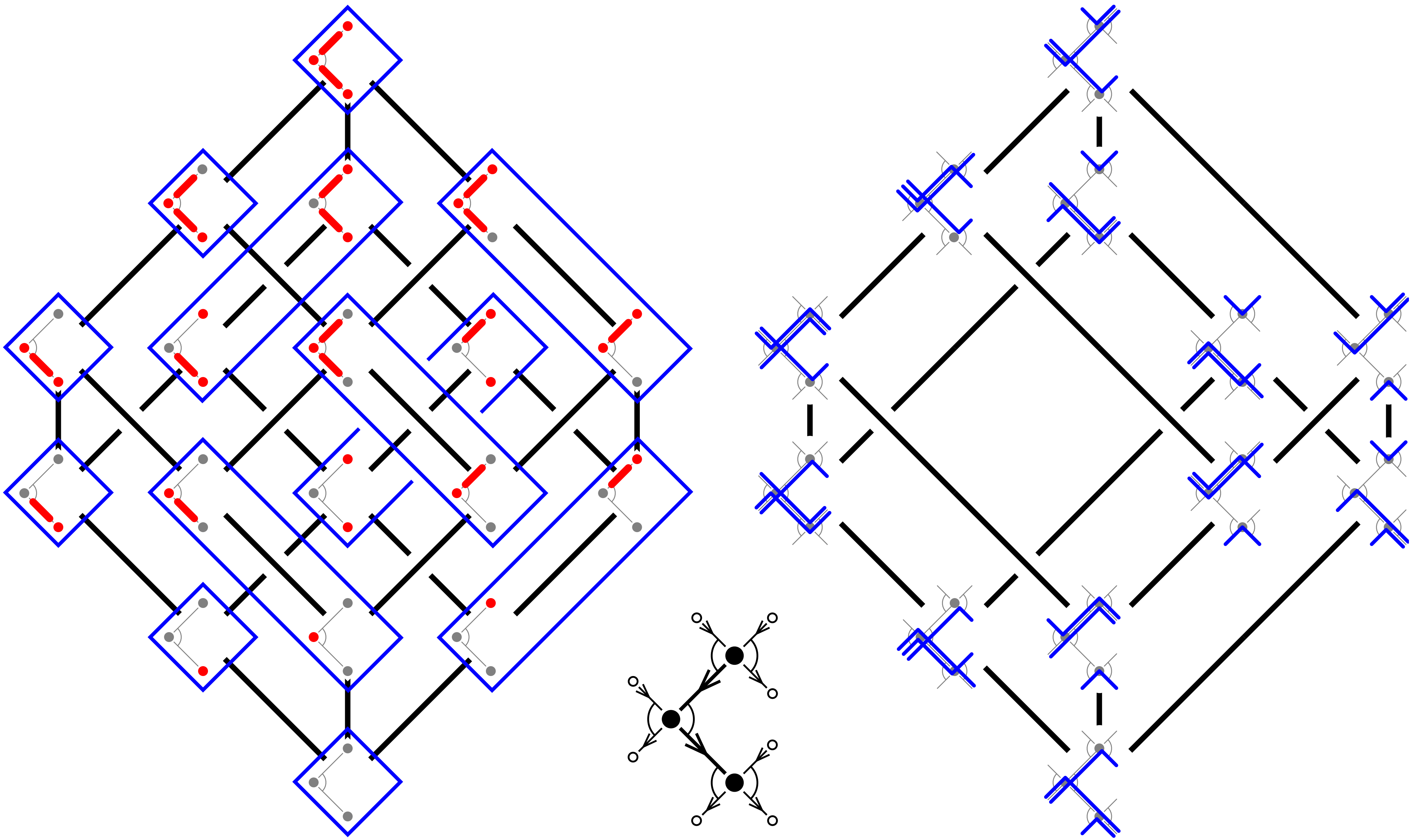
THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{NK}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.





Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

NON-KISSING LATTICE



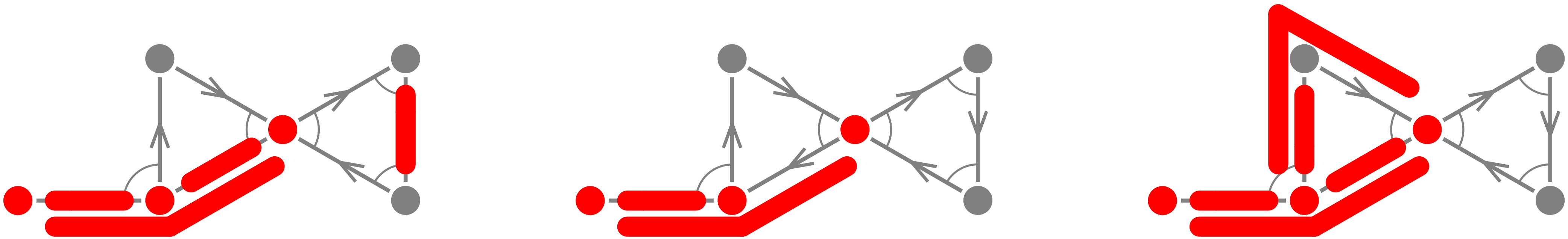
BICLOSED SETS OF STRINGS

σ, τ oriented strings

concatenation $\sigma \circ \tau = \{ \sigma\alpha\tau \mid \alpha \in Q_1 \text{ and } \sigma\alpha\tau \text{ string of } \bar{Q} \}$

closure $S^{\mathrm{cl}} = \bigcup_{\substack{\ell \in \mathbb{N} \\ \sigma_1, \dots, \sigma_\ell \in S}} \sigma_1 \circ \dots \circ \sigma_\ell =$ all strings obtained by concatenation of some strings of S

closed $\iff S^{\mathrm{cl}} = S$ coclosed $\iff \bar{S}^{\mathrm{cl}} = \bar{S}$ biclosed $=$ closed and coclosed



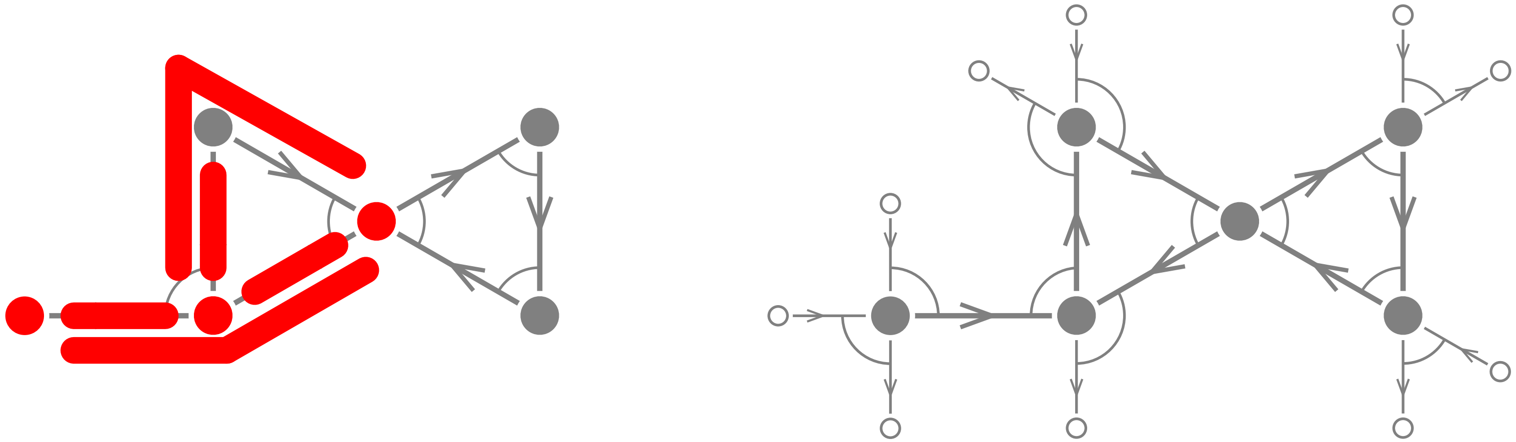
THM. For any gentle quiver $\bar{Q}$ such that $\mathcal{NK}(\bar{Q})$ is finite, the inclusion poset on biclosed sets of strings of $\bar{Q}$ is a congruence-uniform lattice.

McConville, *Lattice structures of grid Tamari orders* ('17)

Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('18)

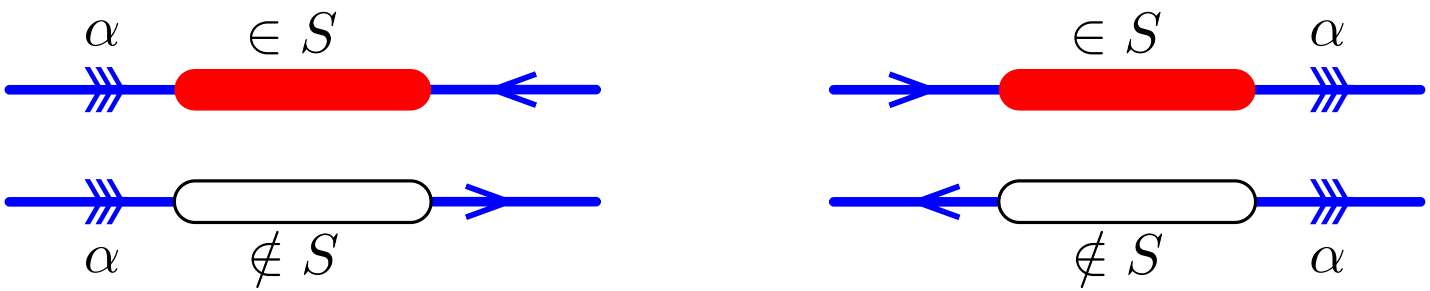
Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

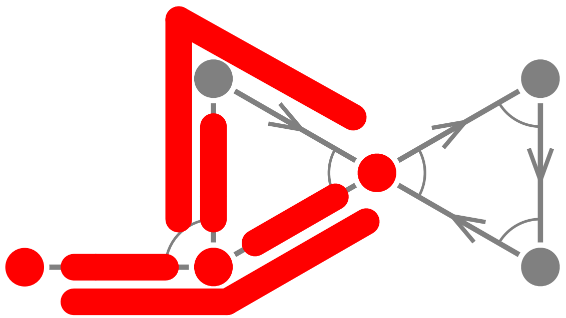
$\omega(\alpha, S)$ = walk constructed with the local rules:

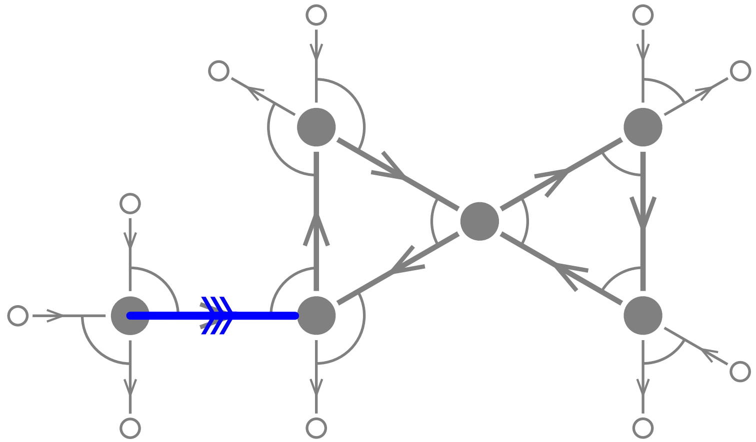


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

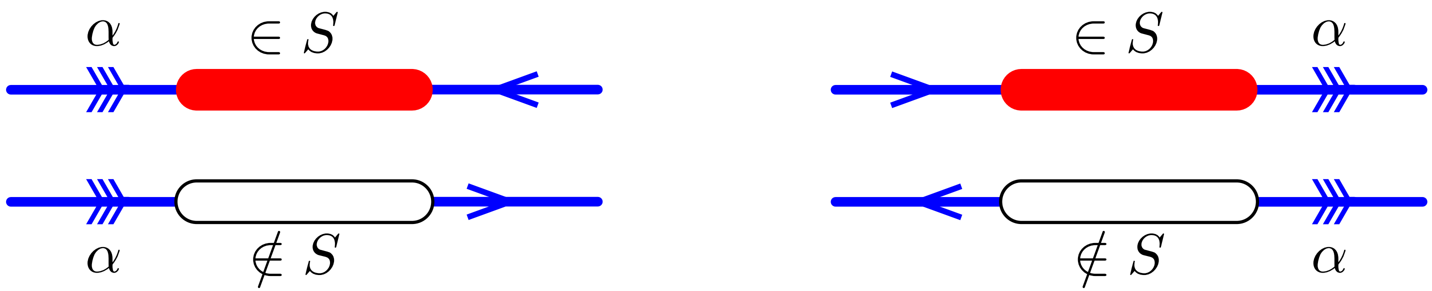
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

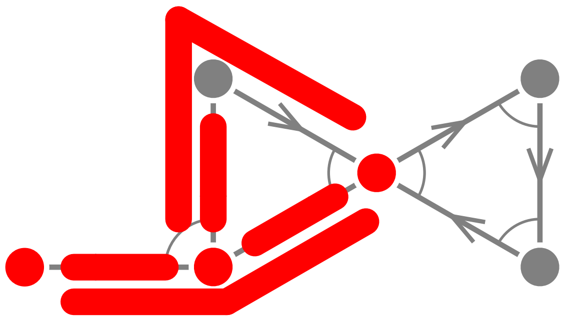
$\omega(\alpha, S)$ = walk constructed with the local rules:

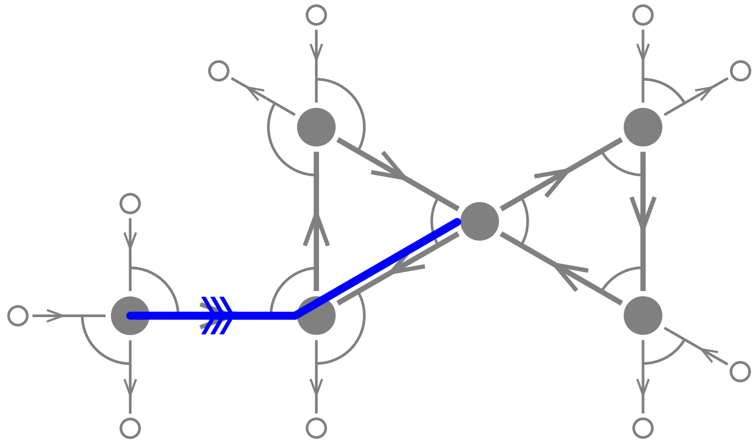


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

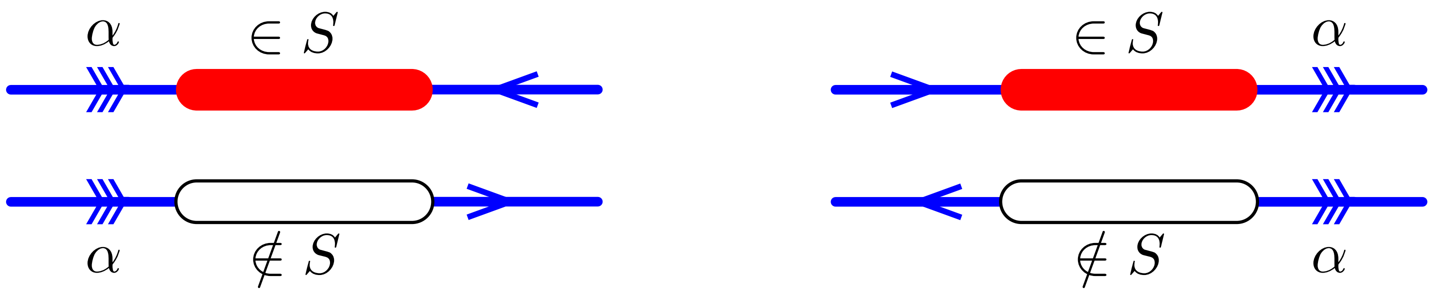
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

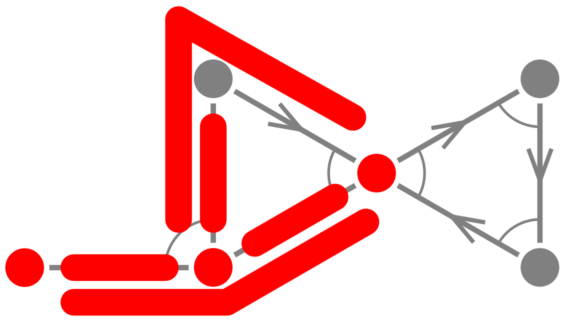
$\omega(\alpha, S) =$ walk constructed with the local rules:

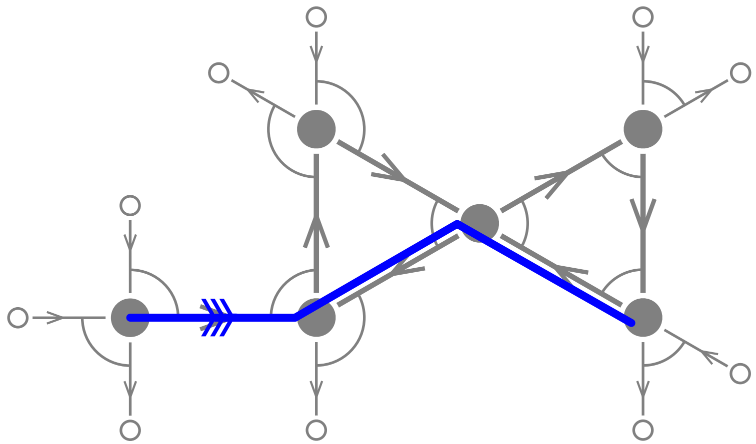


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

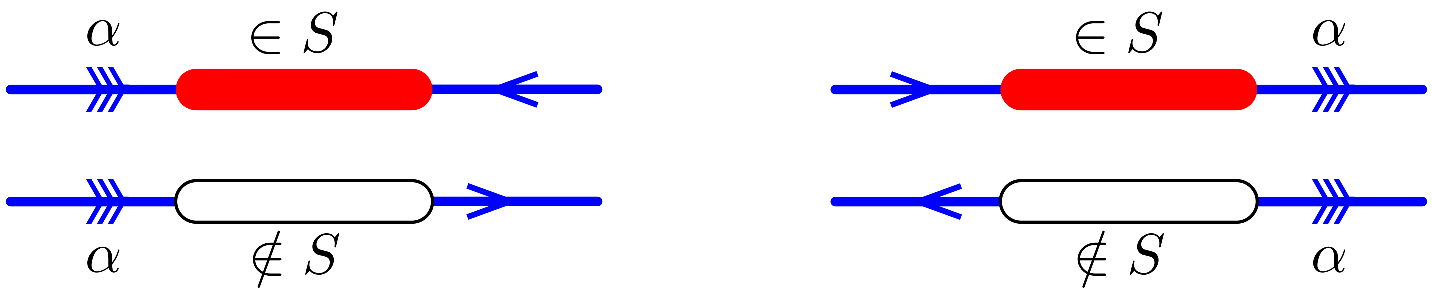
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

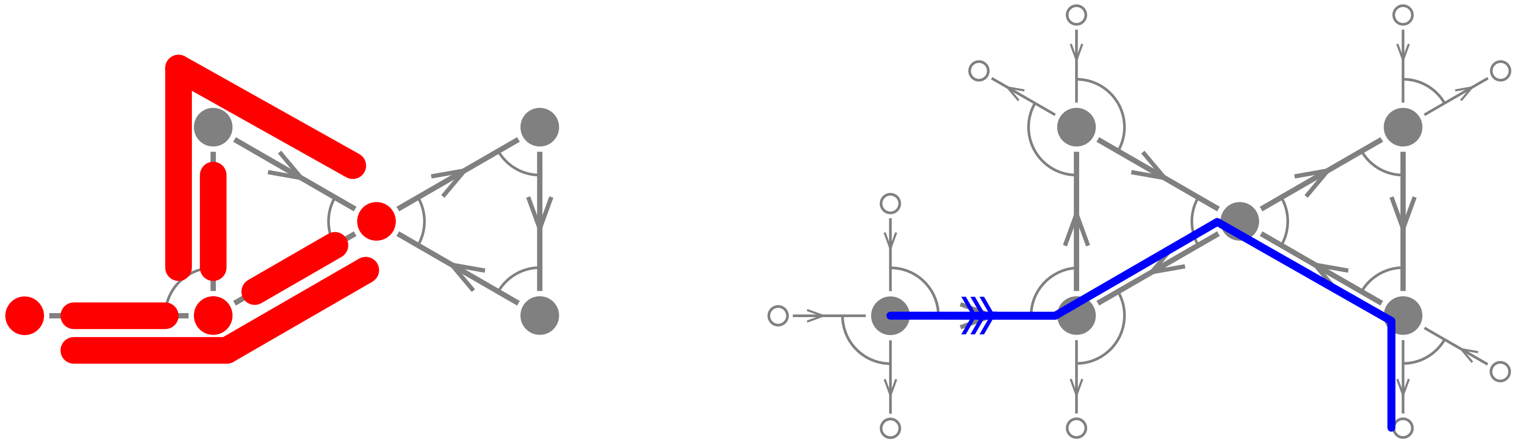
$\omega(\alpha, S)$ = walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

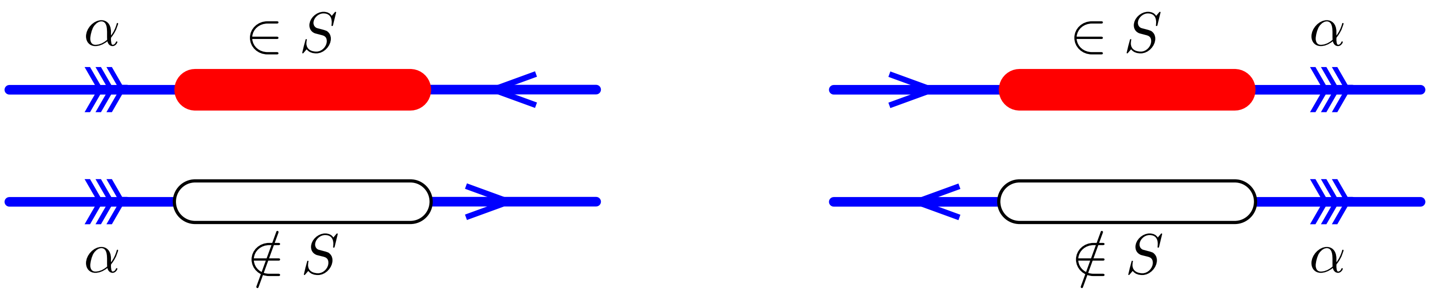
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

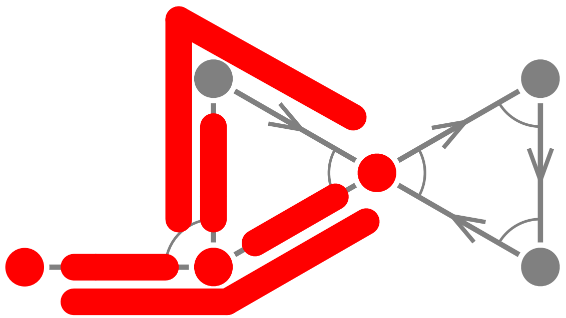
$\omega(\alpha, S) =$ walk constructed with the local rules:

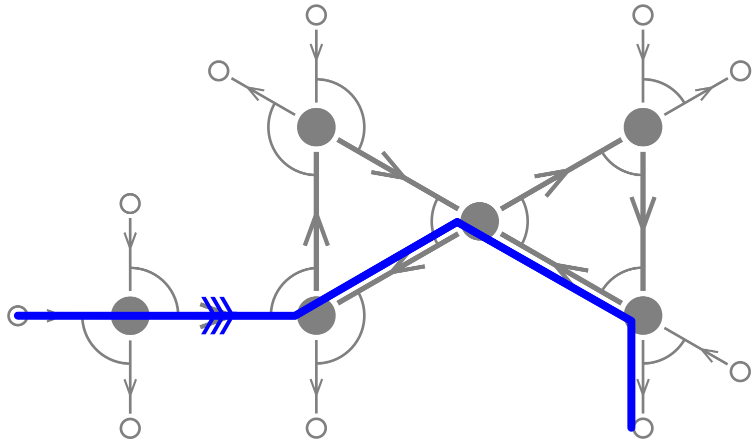


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

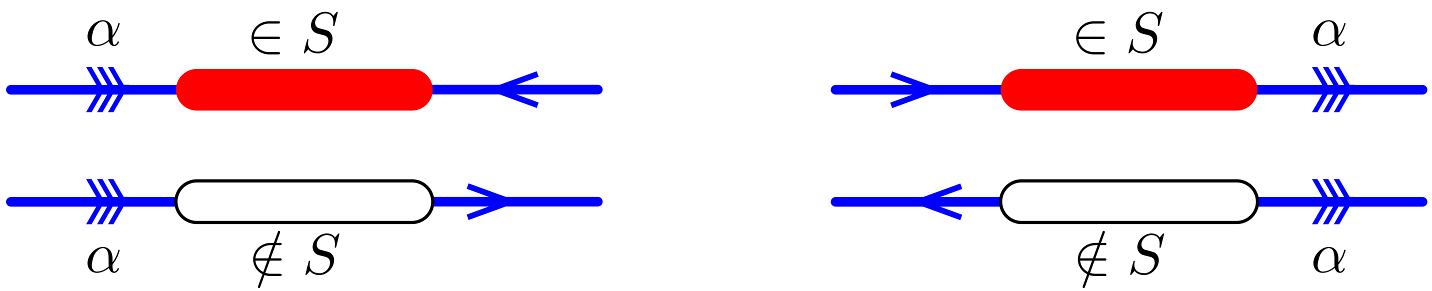
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

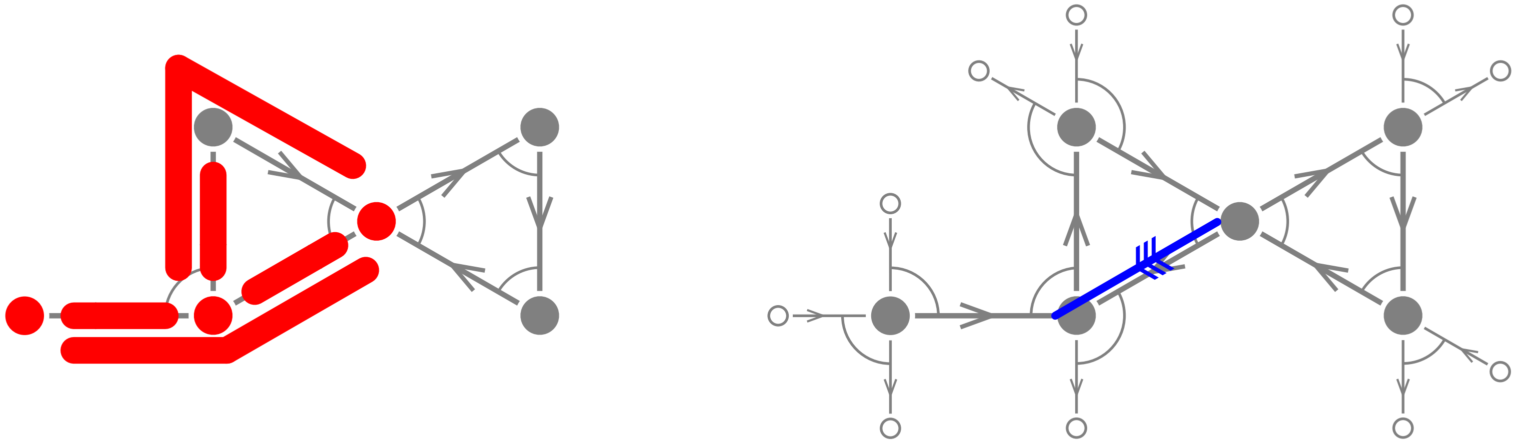
$\omega(\alpha, S)$ = walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

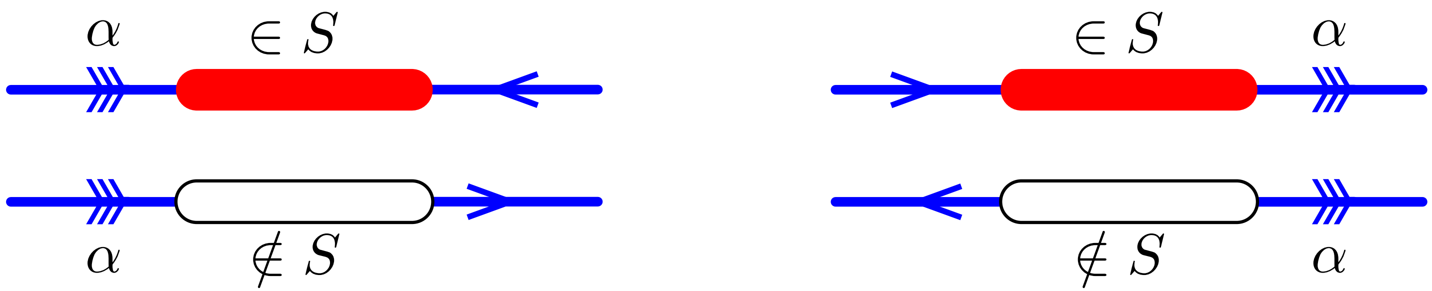
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



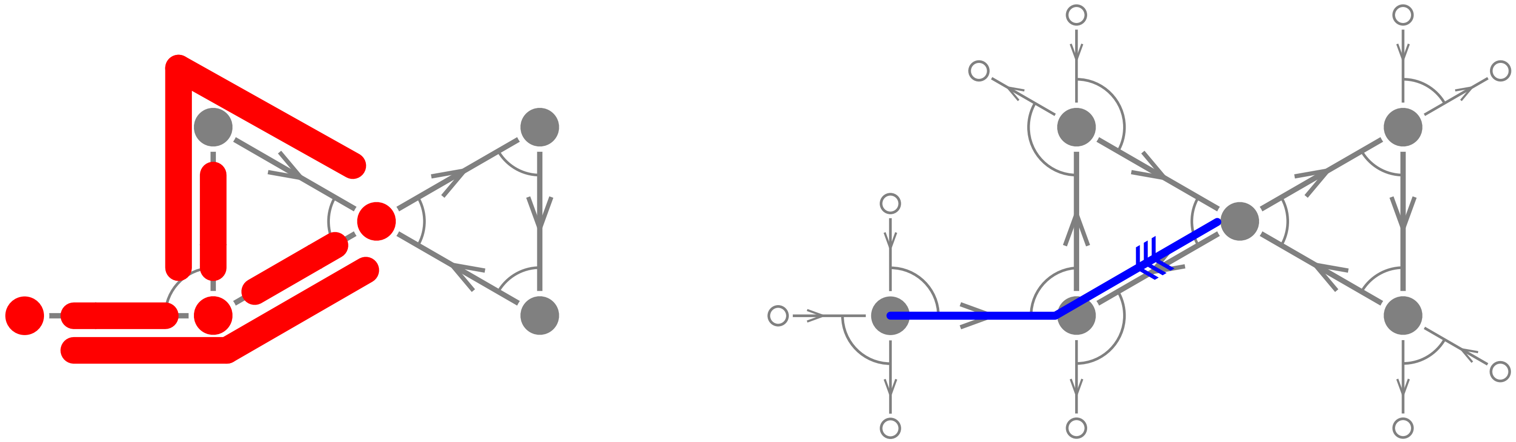
S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S) =$ walk constructed with the local rules:



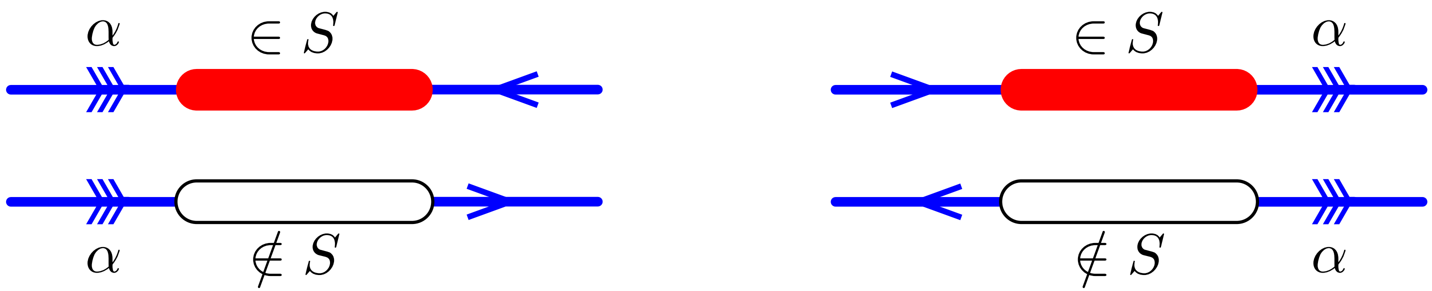
McConville, *Lattice structures of grid Tamari orders* ('17)

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

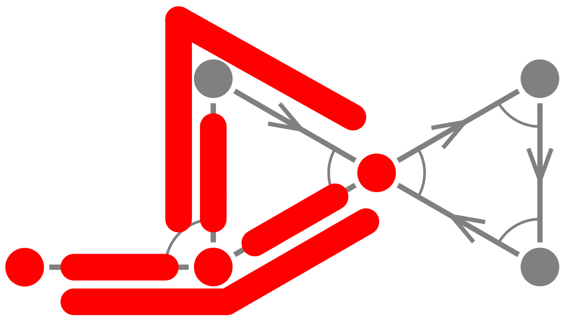
$\omega(\alpha, S) =$ walk constructed with the local rules:

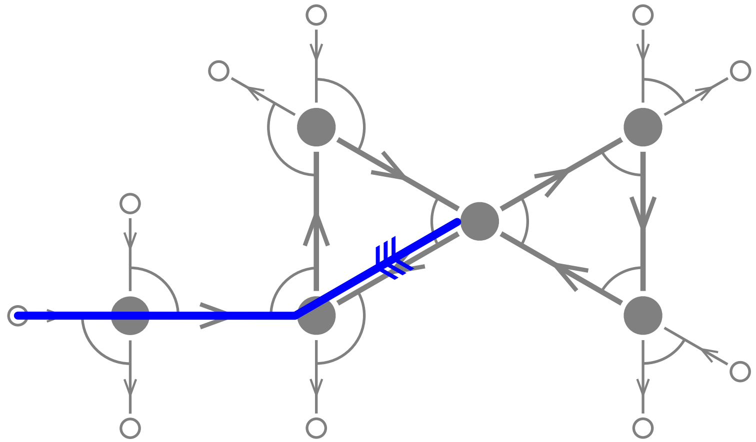


McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

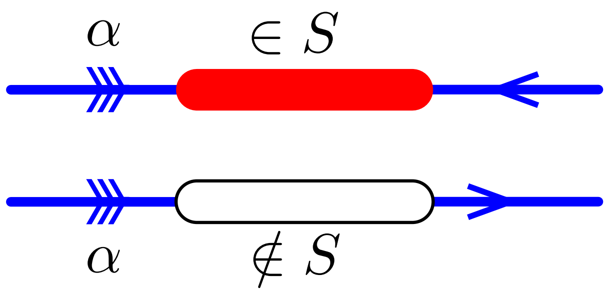
Surjection from biclosed sets of strings to non-kissing facets

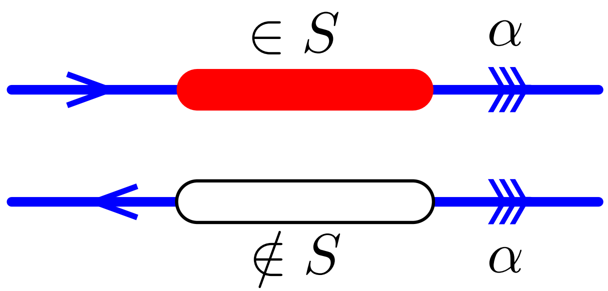




S biclosed, $\alpha \in Q_1$

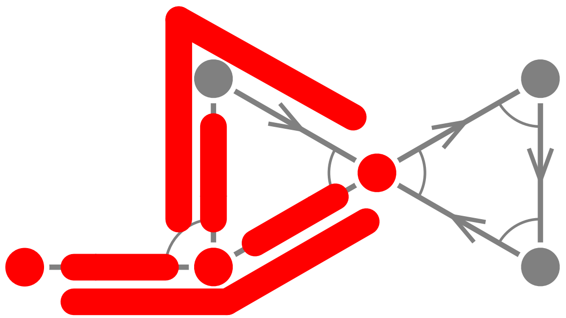
$\omega(\alpha, S) =$ walk constructed with the local rules:

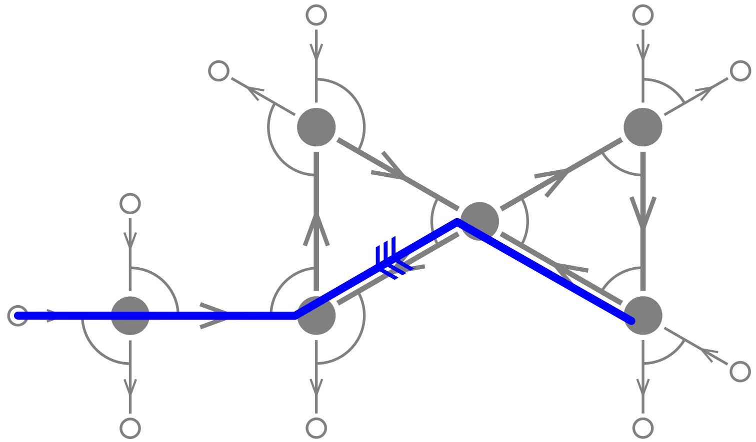




McConville, *Lattice structures of grid Tamari orders* ('17)

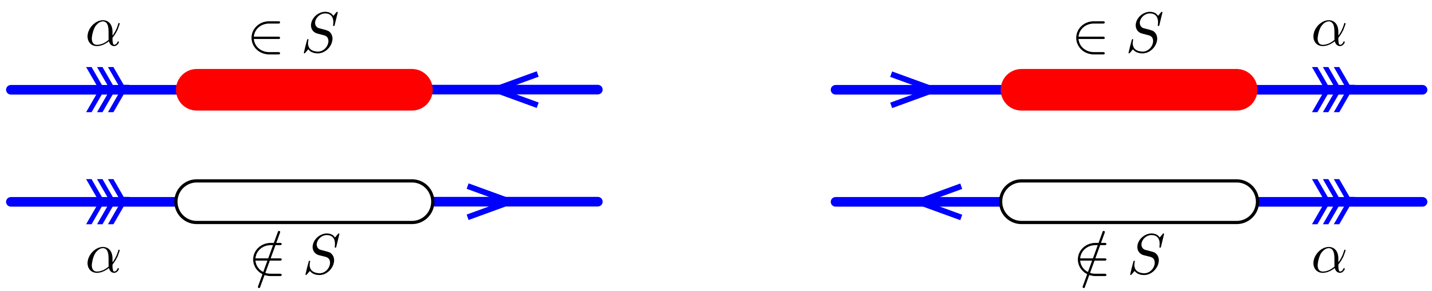
Surjection from biclosed sets of strings to non-kissing facets





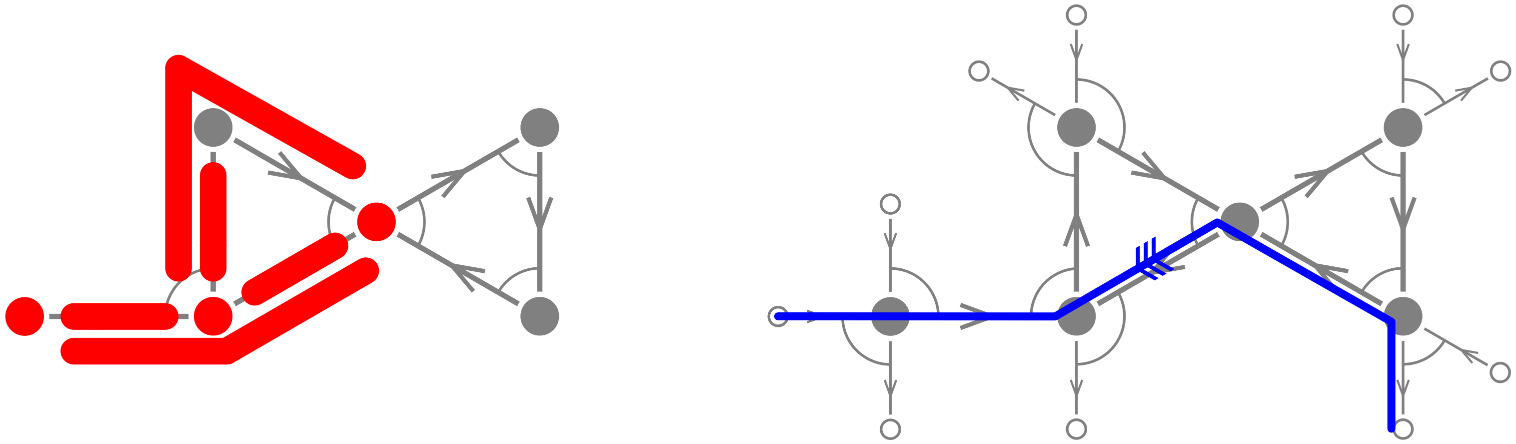
S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:



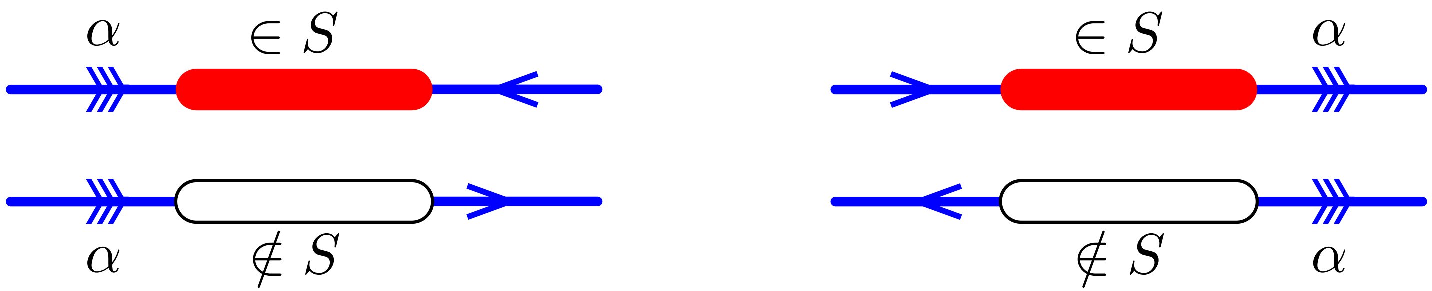
McConville, *Lattice structures of grid Tamari orders* ('17)

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

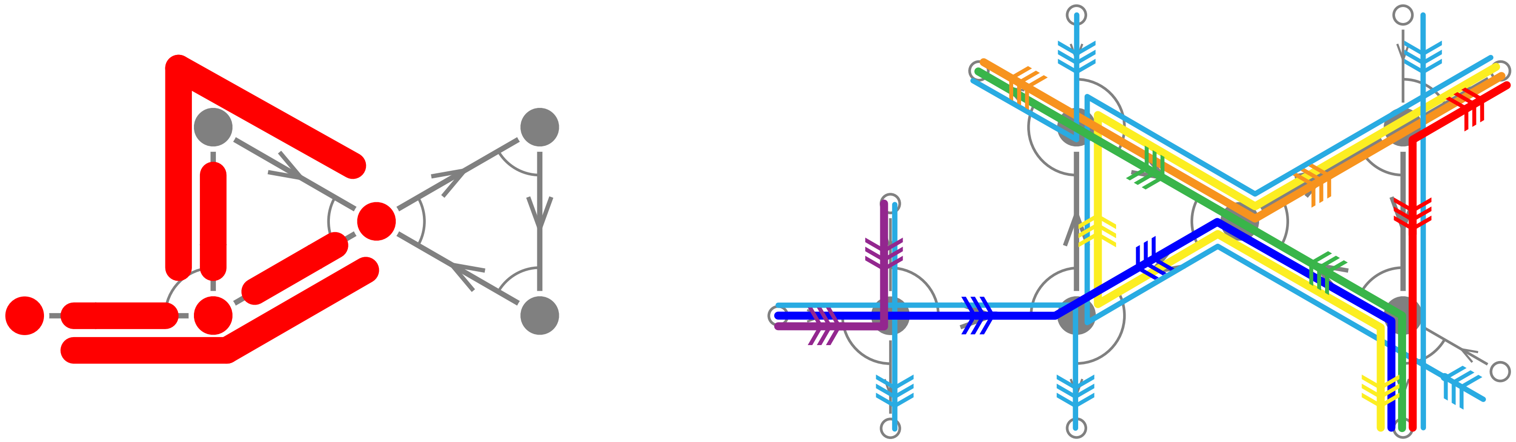
$\omega(\alpha, S)$ = walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets

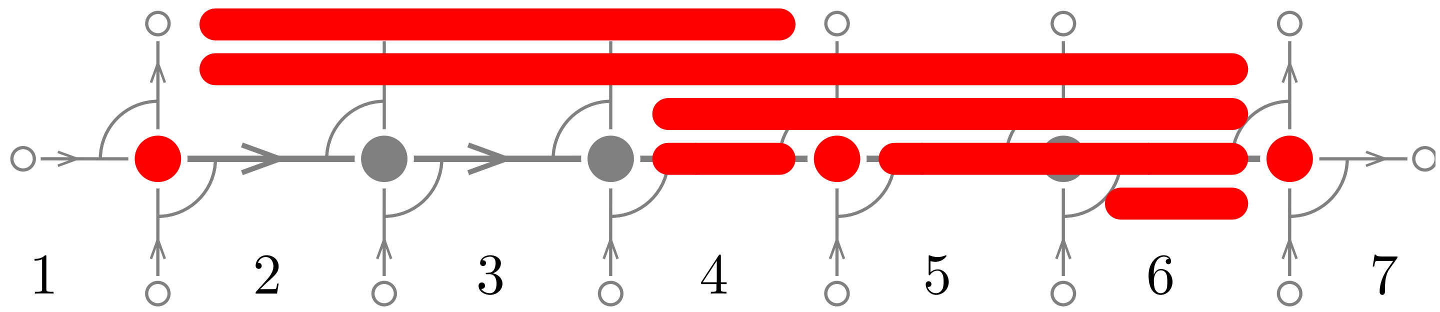


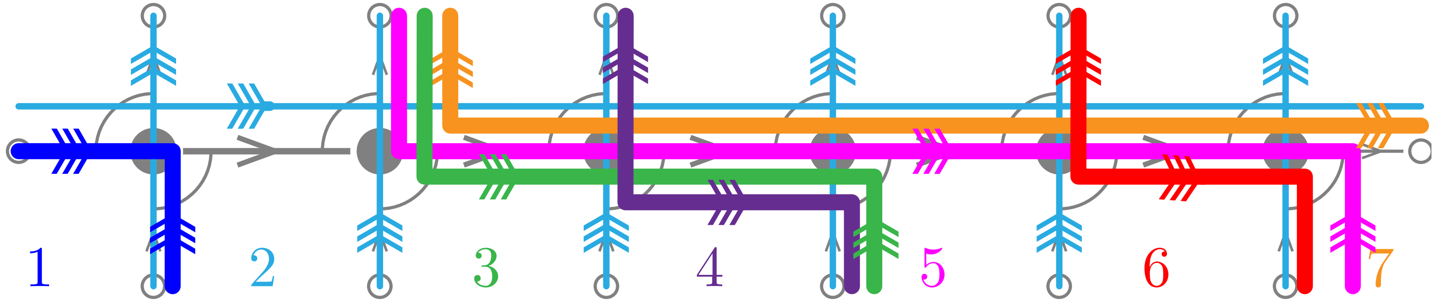
PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

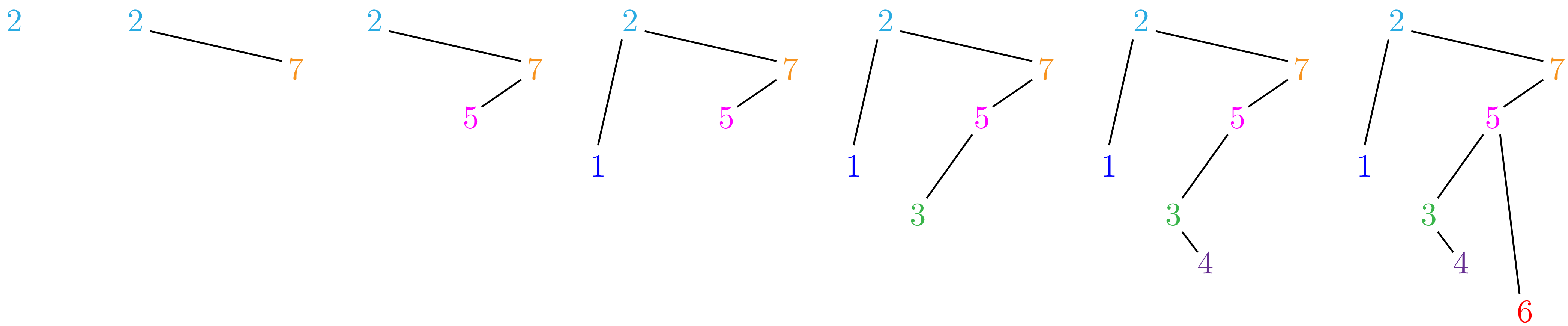
McConville, *Lattice structures of grid Tamari orders* ('17)

EXM: BINARY SEARCH TREE INSERTION AGAIN

inversion set of 2751346

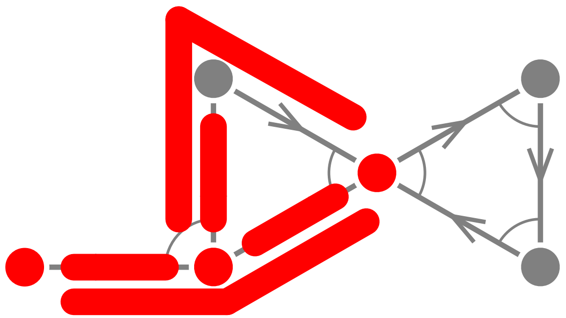


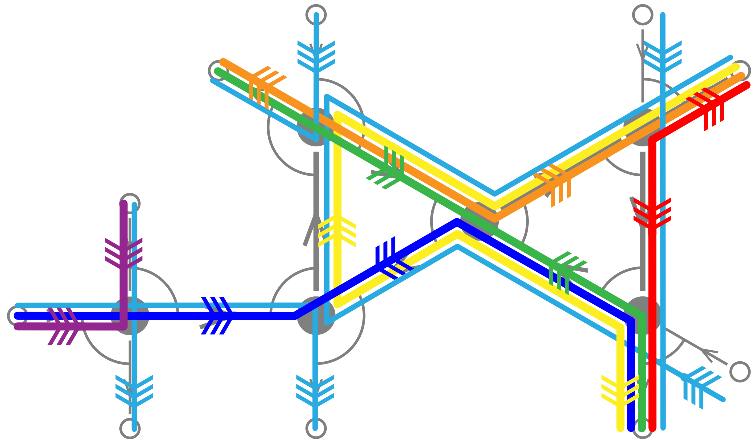




NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



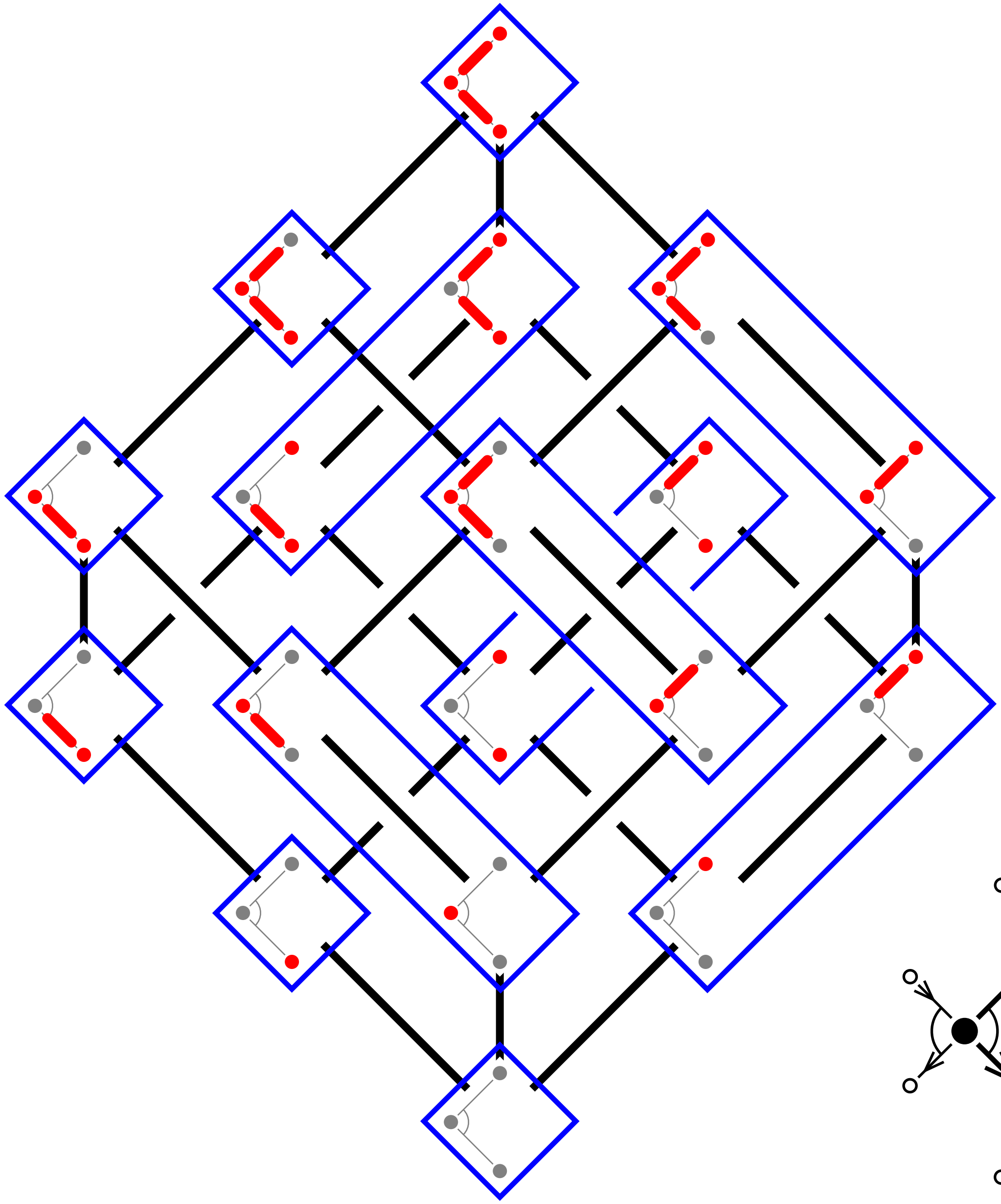


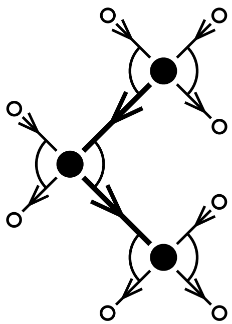
PROP. $\eta(S) := \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

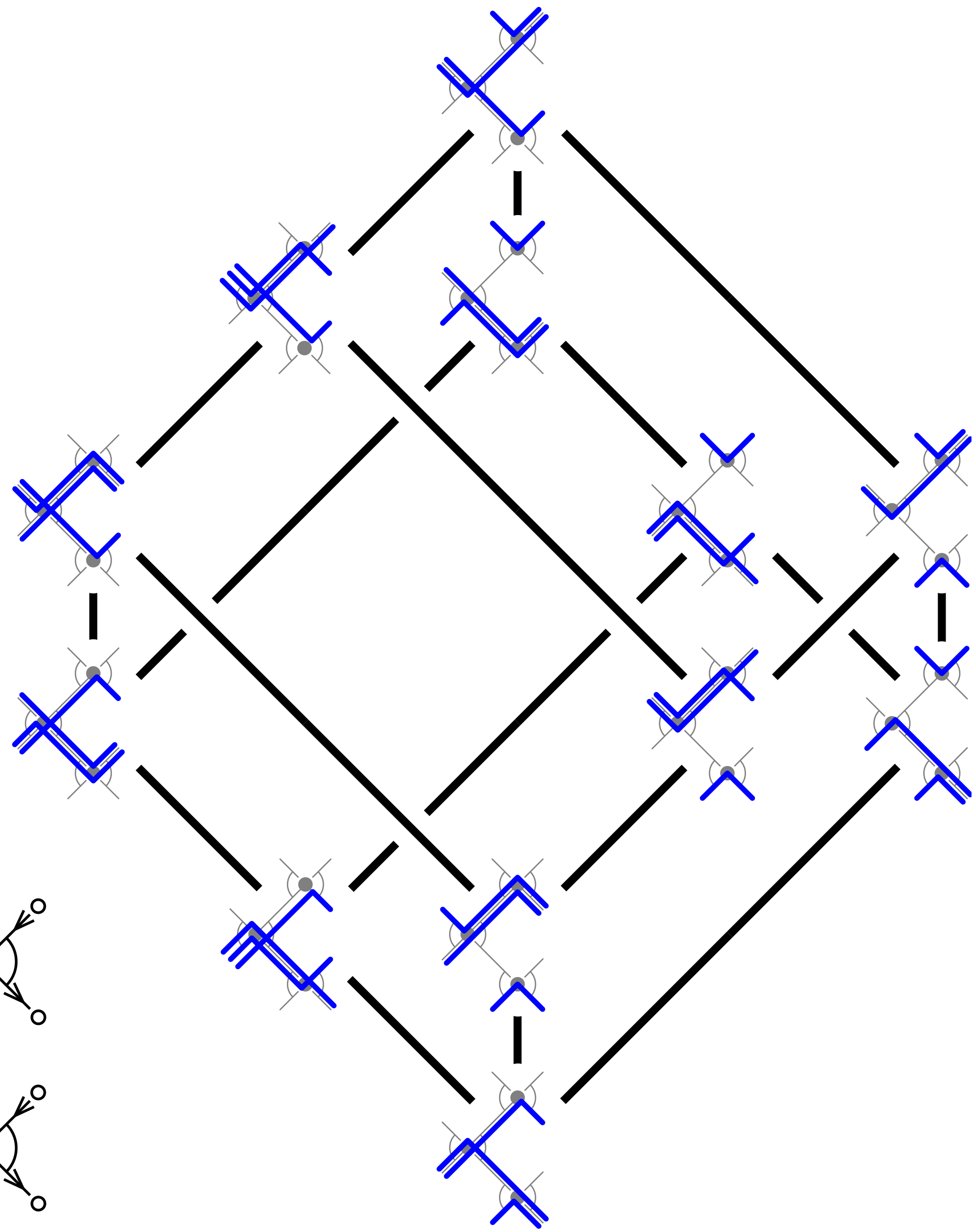
THM. The map η defines a lattice morphism from biclosed sets to non-kissing facets.

McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING LATTICE







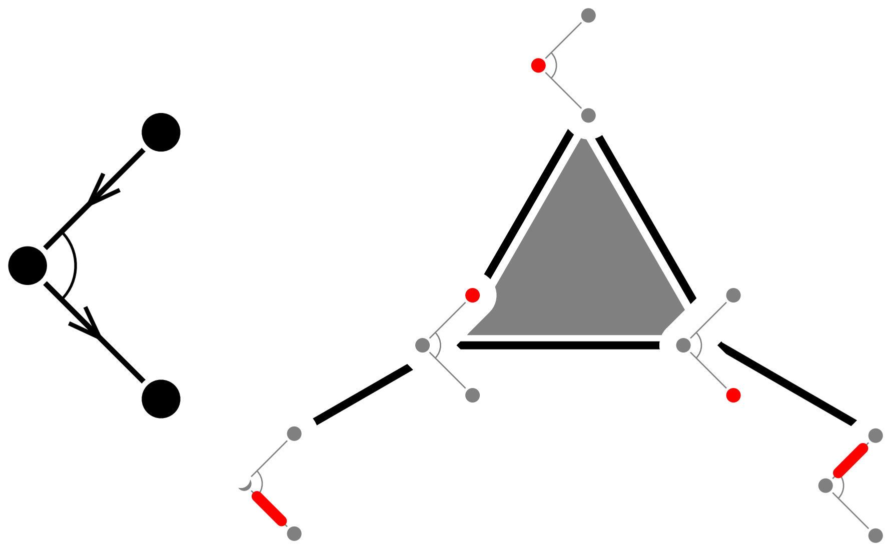
NON-KISSING LATTICE

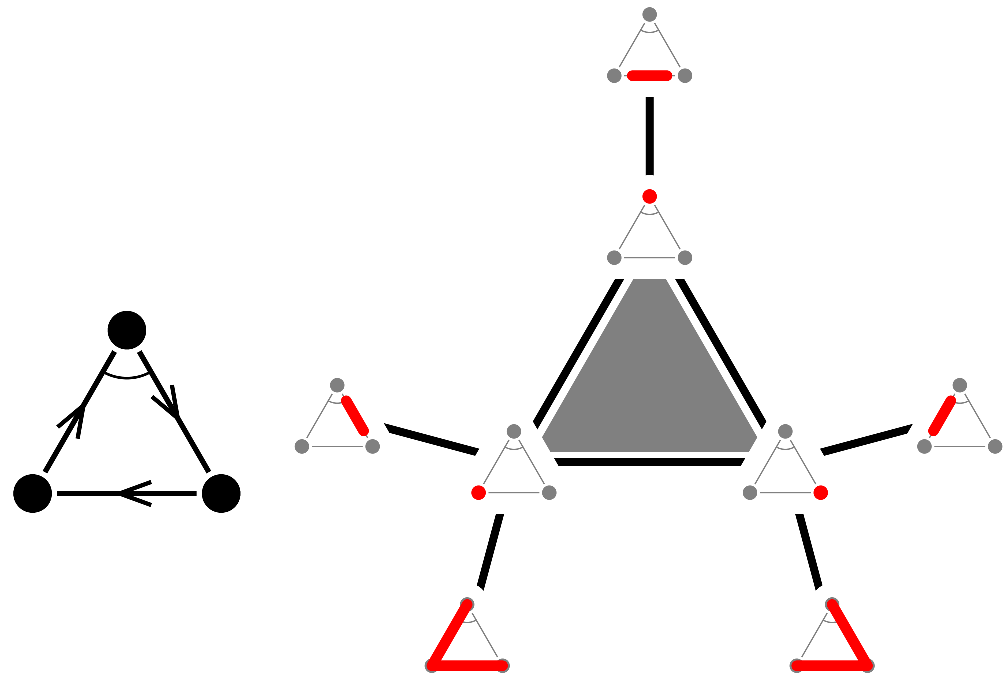
THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{NK}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('21)

Much more nice combinatorics:

- join-irreducible elements of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ are in bijection with distinguishable strings
- canonical join complex of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is a generalization of non-crossing partitions

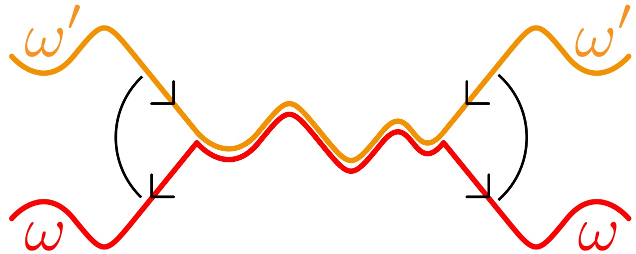




SUMMARY

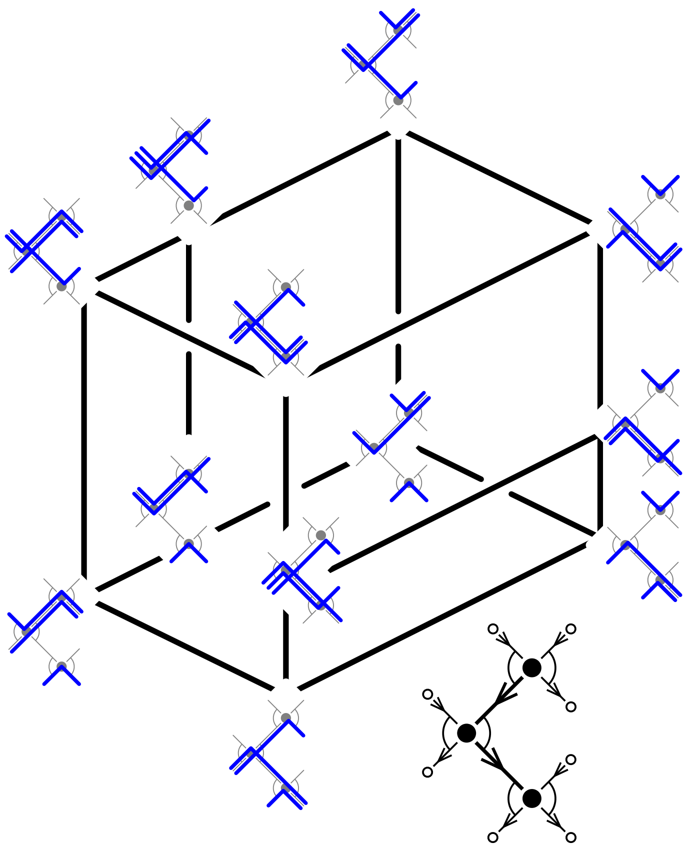
non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices $=$ walks in $\bar{Q}^{❀}$ (that are not self-kissing)
- faces $=$ collections of pairwise non-kissing walks in $\bar{Q}^{❀}$

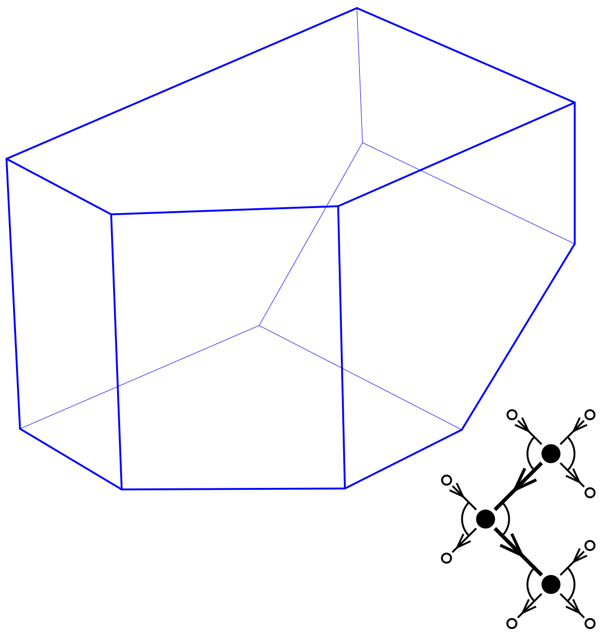


... generalizing the associahedron

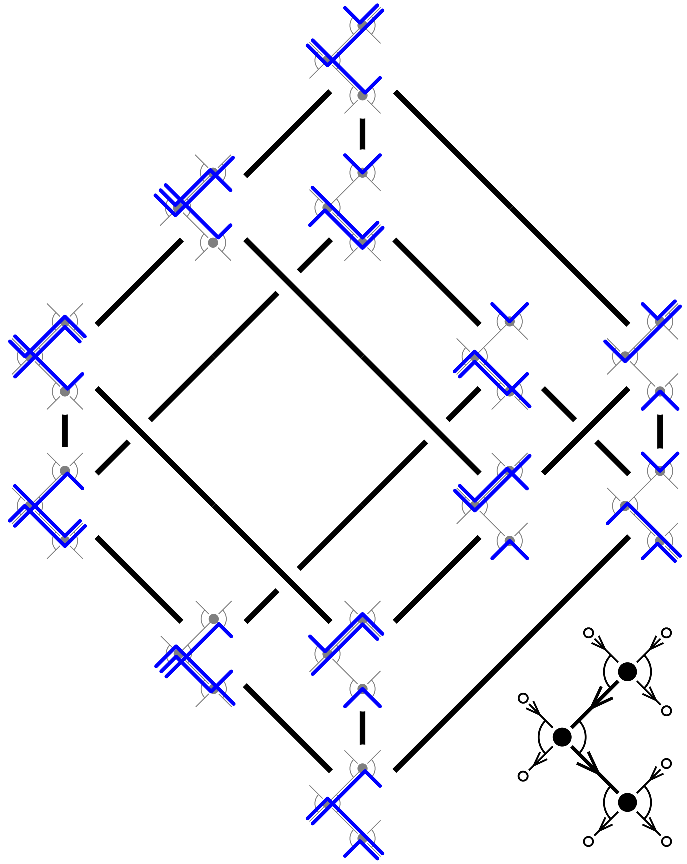
Flip graph

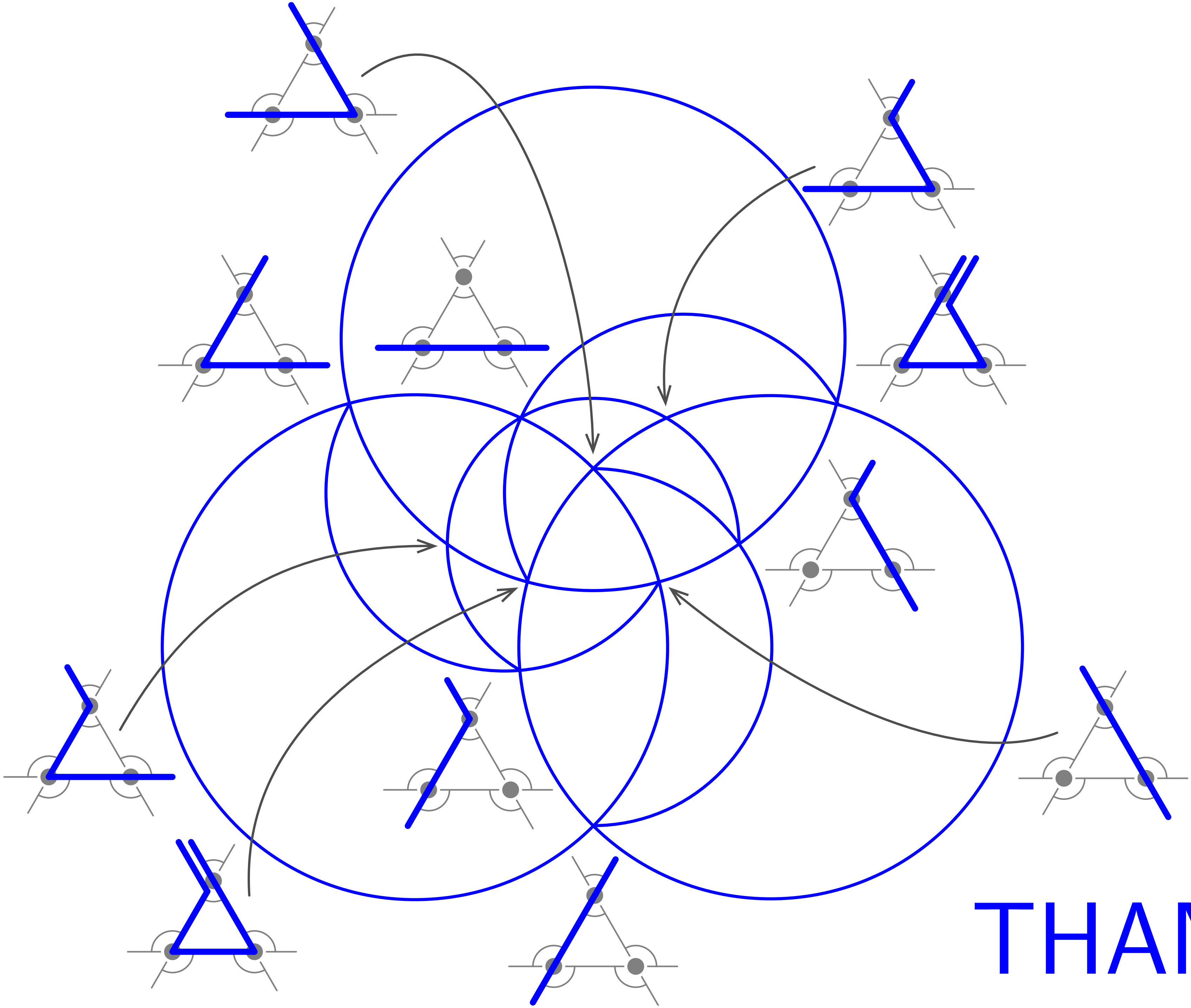


Associahedron



Tamari lattice





THANKS