LATTICE QUOTIENTS AND QUOTIENTOPES
FOR SOME GENERALIZATIONS OF THE WEAK ORDER
PROBLEM SESSION 3: S-WEAK ORDER

Exercise 1. For a n-tuple $\boldsymbol{s}$ of non-negative integers, an $\boldsymbol{s}$-tree is a plane tree where node i has $s_i + 1$ children which are either leaves or nodes $j > i$.

(1) Give a product formula for the number of $\boldsymbol{s}$-trees.
(2) Show that the $\boldsymbol{s}$-trees are also counted by

$$\sum_{\boldsymbol{j}} \prod_{i \in [n-1]} \binom{s_i + 1}{j_i} \Big(1 - i + \sum_{k \in [i]} j_k\Big),$$

where the first sum ranges over all $\boldsymbol{j} := (j_1, \dots, j_{n-1})$ with $j_i \geq 0$ and $j_1 + \dots + j_i \geq i$ for all $i \in [n-1]$, and $j_1 + \dots + j_{n-1} = n - 1$. (These are called Lidskii type formulas, see the course of Martha Yip, and the paper *Realizing the $\boldsymbol{s}$-permutahedron via flow polytopes* by Rafael González D'Léon, Alejandro Morales, Eva Philippe, Daniel Tamayo Jiménez, and Martha Yip.)

Exercise 2. Assume that $\boldsymbol{s}$ is a strict composition, that is, $s_j \neq 0$ for all $j \in [n]$. Describe a bijection between the $\boldsymbol{s}$-trees and the permutations of $1^{s_1} 2^{s_2} \cdots n^{s_n}$ avoiding the pattern 212.

Exercise 3. Consider the insertion algorithm in $\boldsymbol{s}$-bushes described in the course (see Figure 1).

(1) Show that the closure $\bar{\mathbb{F}}_{\mathsf{B}}$ of the fiber of an $\boldsymbol{s}$-bush B is a polyhedron.
(2) Show that the equations defining the affine span of $\bar{\mathbb{F}}_{\mathsf{B}}$ correspond to the holes of B, that is, the pairs $1 \leq i < j \leq n$ of nodes of B such that j has two incoming edges with greatest common ancester i
(3) For an $\boldsymbol{s}$-tree T, show that the upper (resp. lower) facets of $\bar{\mathbb{F}}_{\mathsf{T}}$ (in the usual direction $\boldsymbol{\omega} := (n, \dots, 1) - (1, \dots, n)$) correspond to the ascents (resp. descents) of T, that is, the pairs $1 \leq i \leq j$ such that
   - i is the greatest ancestor of j such that the increasing path from i to j in T takes the leftmost (resp. rightmost) outgoing edge at each node, except at node i,
   - either $s_j = 0$ or the leftmost (resp. rightmost) edge of j is a leaf.

(Note that there is also a similar description of the upper and lower facets of $\bar{\mathbb{F}}_{\mathsf{B}}$ for any $\boldsymbol{s}$-bush B, but it is more complicated.)

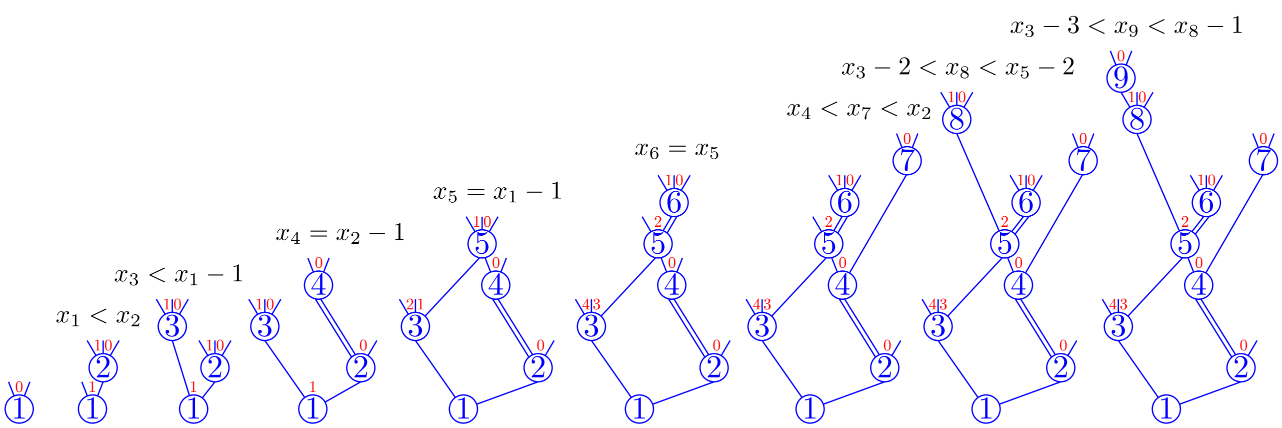
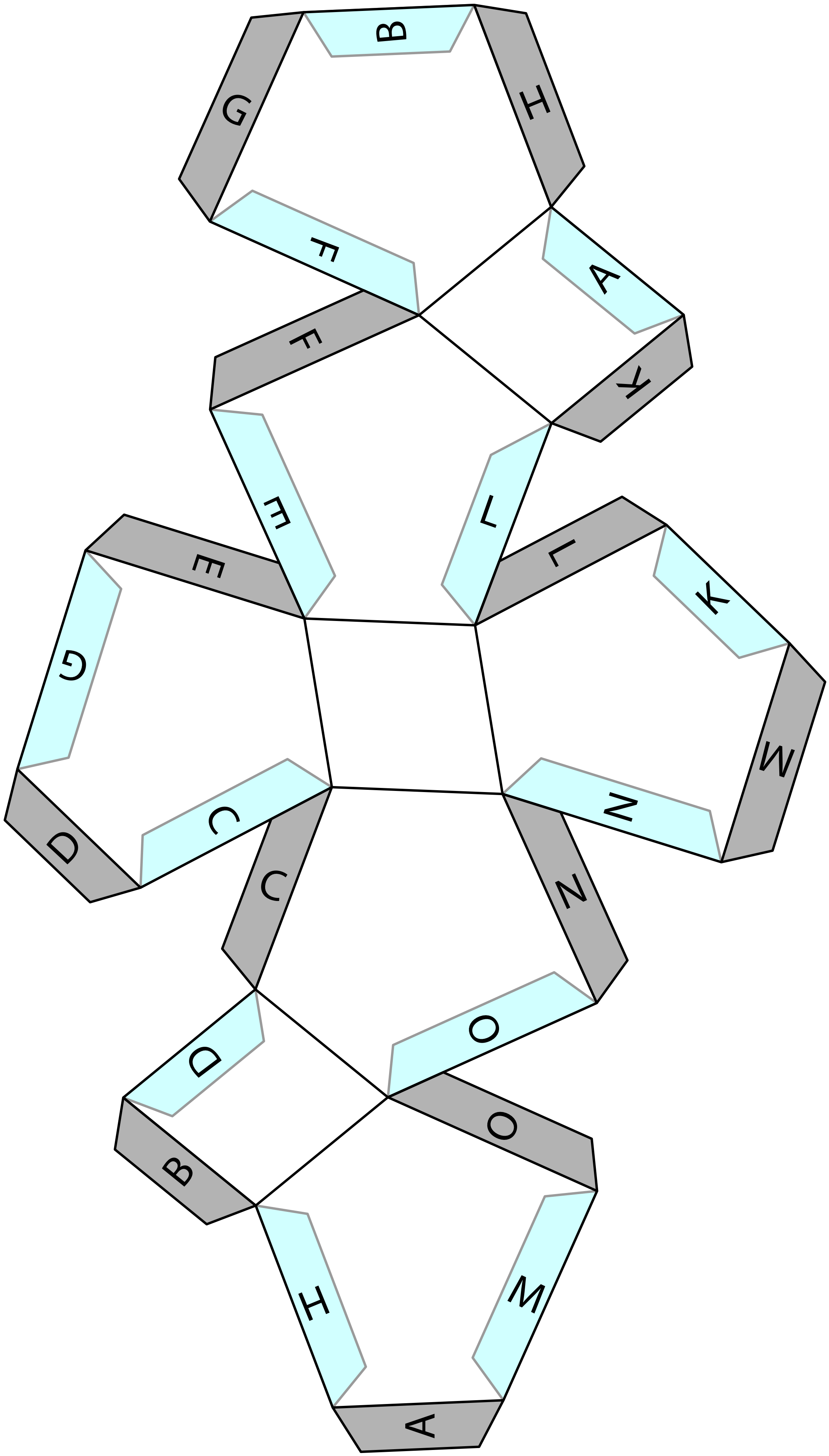


Figure 1. The insertion algorithm in $\boldsymbol{s}$-bushes.
Here $\boldsymbol{s} = (1, 2, 2, 0, 2, 2, 1, 2, 1)$ and $\boldsymbol{x} = (5, 6, 3, 5, 4, 4, 5.5, 1.5, .25)$.



associahedron as secondary polytope