LATTICE QUOTIENTS AND QUOTIENTOPES
FOR SOME GENERALIZATIONS OF THE WEAK ORDER
PROBLEM SESSION 1: QUOTIENTOPES

1. Loday's associahedron

Exercise 1. In Loday's associahedron, the vertex corresponding to a binary tree T has i-th coordinate equal to the number of leaves in the left subtree times the number of leaves in the right subtree of the i-th node of T in inorder.

(1) Show that these vertices all lie in a common hyperplane.
(2) Compute the direction of the edge joining the vertices corresponding to two binary trees related by a rotation.

Exercise 2. The sylvester congruence $\equiv_{\text{sylv}}$ on permutations of $[n]$ is defined by

- $\sigma \equiv_{\text{sylv}} \tau$ if and only if σ and τ have the same insertion in a binary search tree,
- $\sigma \equiv_{\text{sylv}} \tau$ if and only if σ and τ are linear extensions of a common binary tree (labeled in inorder, and oriented towards its root),
- $\equiv_{\text{sylv}}$ is the transitive and symmetric closure of the rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ for some $a < b < c$,
- $\equiv_{\text{sylv}}$ is the congruence whose uncontracted arcs are the up arcs $(i, j,]i, j[, \varnothing)$.

Prove that these descriptions coincide. Describe the minimal and maximal permutations of the sylvester congruence classes.

Exercise 3. Loday's associahedron is also the Minkowski sum $\sum_{1 \le i \le j \le n} \triangle_{[i,j]}$ of the faces of the standard simplex $\triangle_{[n]} := \operatorname{conv}\{\boldsymbol{e}_i \mid i \in [n]\}$ corresponding to intervals of $[n]$.

(1) For $\varnothing \ne I \subseteq [n]$ and $\boldsymbol{v} \in \mathbb{R}^n$ generic, describe the vertex of $\triangle_I := \operatorname{conv}\{\boldsymbol{e}_i \mid i \in I\}$ maximizing the dot product with $\boldsymbol{v}$.
(2) Deduce a description of the vertex of Loday's associahedron maximizing the dot product with $\boldsymbol{v}$.
(3) Observe that your description only depends on the sylvester class of the permutation σ of $[n]$ such that $v_{\sigma(1)} \le \dots \le v_{\sigma(n)}$.
(4) Show Loday's formula for the coordinates of the vertices of this Minkowski sum.

2. Permutreehedra

Exercise 4. A permutree is a plane oriented tree whose nodes are bijectively labeled by $[n]$ such that each node i has one or two parents (resp. children), and if it has two, then $j < i$ for all nodes j in the left ancestor (resp. descendant) subtree of i, and $j > i$ for all nodes j in the right ancestor (resp. descendant) subtree of i. The decoration $\boldsymbol{\delta} \in \{\text{⦶}, \text{☮}, \text{Ⓨ}, \otimes\}^n$ of a permutree is defined such that $\boldsymbol{\delta}_i$ looks like the neighborhood of node i. The $\boldsymbol{\delta}$-permutree congruence $\equiv_{\boldsymbol{\delta}}$ on permutations of $[n]$ is defined by

- $\sigma \equiv_{\boldsymbol{\delta}} \tau$ if and only if σ and τ have the same insertion in a $\boldsymbol{\delta}$-permutree (see Figure 1),
- $\sigma \equiv_{\boldsymbol{\delta}} \tau$ if and only if σ and τ are linear extensions of a common $\boldsymbol{\delta}$-permutree,
- $\equiv_{\boldsymbol{\delta}}$ is the transitive and symmetric closure of the rewriting rules $UacVbW \equiv_{\boldsymbol{\delta}} UcaVbW$ if $\boldsymbol{\delta}_b \in \{\text{☮}, \otimes\}$ and $UbVacW \equiv_{\boldsymbol{\delta}} UbVcaW$ if $\boldsymbol{\delta}_b \in \{\text{Ⓨ}, \otimes\}$, for some letters $a < b < c$ and some words U, V, W,
- $\equiv_{\boldsymbol{\delta}}$ is the congruence whose uncontracted arcs are the arcs which do not pass above the points j with $\boldsymbol{\delta}_j \in \{\text{Ⓨ}, \otimes\}$ nor below the points j with $\boldsymbol{\delta}_j \in \{\text{☮}, \otimes\}$.

Prove that these descriptions coincide. Describe the minimal and maximal permutations of the $\boldsymbol{\delta}$-permutree congruence classes.

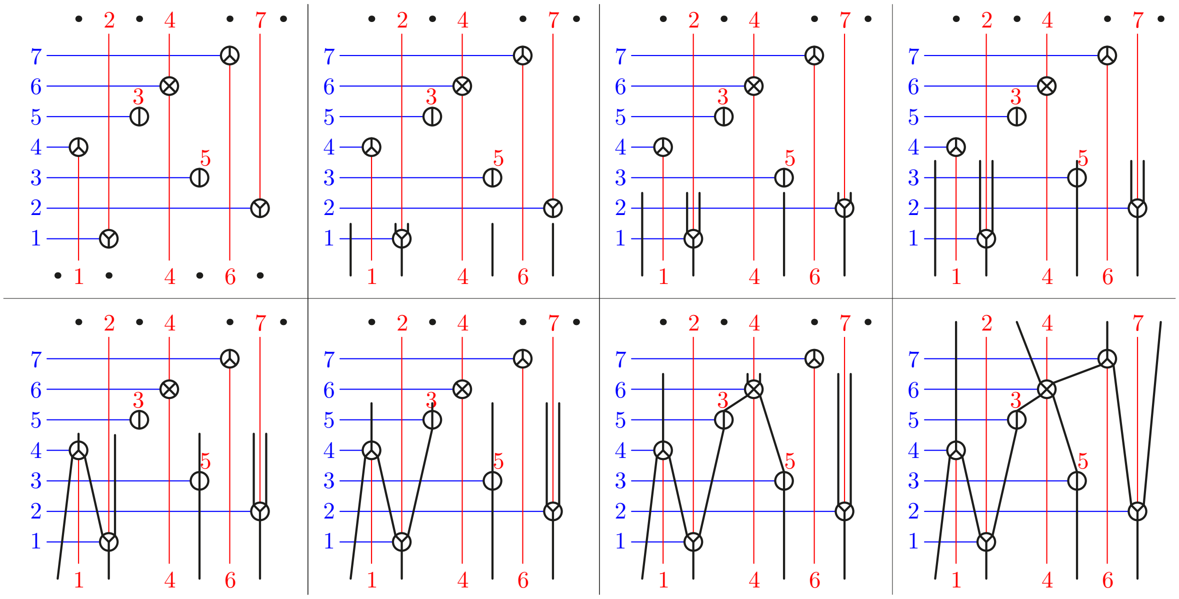


FIGURE 1. The insertion algorithm in $\boldsymbol{\delta}$-permutrees.

Exercise 5. Fix a decoration $\boldsymbol{\delta} \in \{$⦶, ☮, Ⓨ, ⊗$\}^n$. Consider all permutations of $[m]$ with $m \le n$ avoiding $ac \cdots b$ with $a < b < c$ and $\boldsymbol{\delta}_b \in \{$☮, ⊗$\}$ and $b \cdots ac$ with $a < b < c$ and $\boldsymbol{\delta}_b \in \{$Ⓨ, ⊗$\}$. Construct the generating tree $T_{\boldsymbol{\delta}}$ on these permutations, where the parent of a permutation is obtained by deleting its largest letter. Call free gap (resp. banned gap) in a permutation σ of $[m]$ (with $m \le n$) any position where one can add $m+1$ to get a child of σ. See Figure 2.

(1) Show that the number $\mathbf{C}(\boldsymbol{\delta}, g)$ of $\boldsymbol{\delta}$-permutrees with g free gaps is given by

$$\mathbf{C}(\boldsymbol{\delta}, g) = \begin{cases} \mathbf{1}_{g>2} \cdot (g-1) \cdot \mathbf{C}(\boldsymbol{\delta}', g-1) & \text{if } \boldsymbol{\delta}_n = \text{⦶} \\ \mathbf{1}_{g\ge 2} \cdot \displaystyle\sum_{g' \ge g-1} \mathbf{C}(\boldsymbol{\delta}', g') & \text{if } \boldsymbol{\delta}_n = \text{Ⓨ or ☮} \\ \mathbf{1}_{g=2} \cdot \displaystyle\sum_{g' \ge 2} g' \cdot \mathbf{C}(\boldsymbol{\delta}', g') & \text{if } \boldsymbol{\delta}_n = \text{⊗} \end{cases}$$

where $\boldsymbol{\delta}' := \boldsymbol{\delta}_1 \dots \boldsymbol{\delta}_{n-1}$.

(2) Deduce that the number of $\boldsymbol{\delta}$- and $\boldsymbol{\delta}'$-permutrees coincide if $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ only differ by changing some Ⓨ entries into ☮ entries, and *vice versa*.

(3) Check the formula for $\boldsymbol{\delta} =$ ⦶^{n}, $\boldsymbol{\delta} =$ ☮^{n} and $\boldsymbol{\delta} =$ ⊗^{n}.

3. Rectangulotopes

Exercise 6. Consider a binary tree T, with nodes labeled in inorder by $[n]$.

(1) Prove that there is a directed path between any two consecutive nodes.

(2) Show that the following are equivalent for any $1 \le j < n$:
- the $(j+1)$st leaf of T is a right leaf,
- there is an oriented path from its jth node to its $(j+1)$st node,
- the jth node of T has an empty right subtree,
- the $(j+1)$st node of T has a non-empty left subtree,
- the cone corresponding to T is located in the halfspace $x_j \le x_{j+1}$.

The binary sequence of left/right leaves of T is called the canopy of T.

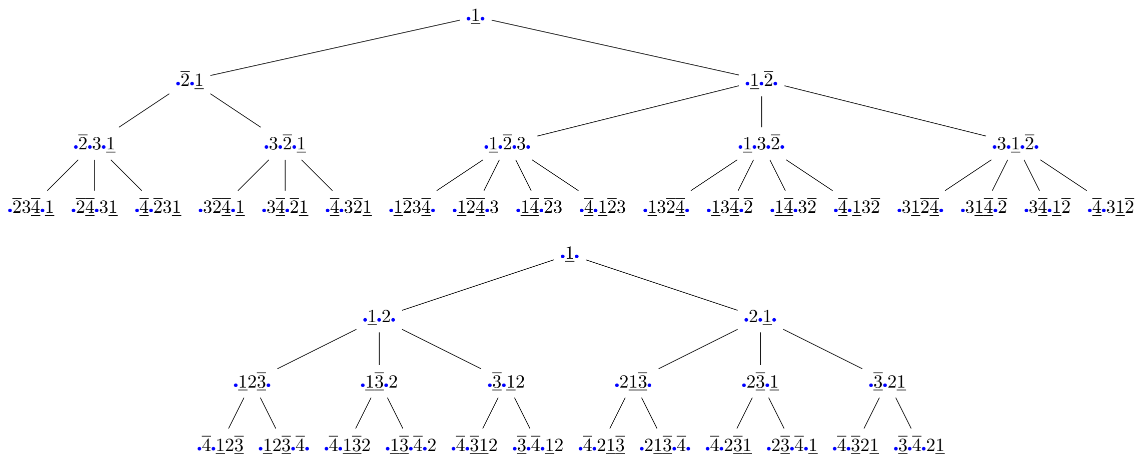


FIGURE 2. The generating tree of $\boldsymbol{\delta}$-permutrees for $\boldsymbol{\delta} =$ ☮ⓎⒾ⊗ (up) and $\boldsymbol{\delta} =$ ☮Ⓘ⊗Ⓨ (down).

Exercise 7. Let T and T' be two binary trees with nodes labeled in inorder, and orient T from its leaves to its root and T' from its root to its leaves.

(1) Show that T and T' have a common linear extension if and only if they have opposite canopies.
(2) Show that pairs of binary trees with opposite canopies are in bijection with diagonal rectangulations (that is, rectangulations of the unit square, up to wall slides, such that each rectangle meets the north-west to southeast diagonal). These sets are counted by Baxter numbers $B_n = \binom{n+1}{1}^{-1}\binom{n+1}{2}^{-1}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$.

Exercise 8. Consider the Baxter congruence of the weak order, whose uncontracted arcs are all up arcs $(i, j,]i, j[, \varnothing)$, and all down arcs $(i, j, \varnothing,]i, j[)$.

(1) Show that the Baxter congruence has a congruence class for each diagonal rectangulation.
(2) Show that the Hasse diagram of the quotient of the weak order by the Baxter congruence is the Minkowski sum of two opposite associahedra.
(3) Give a Loday type formula for the coordinates of the vertex of this polytope corresponding to a given diagonal rectangulation.

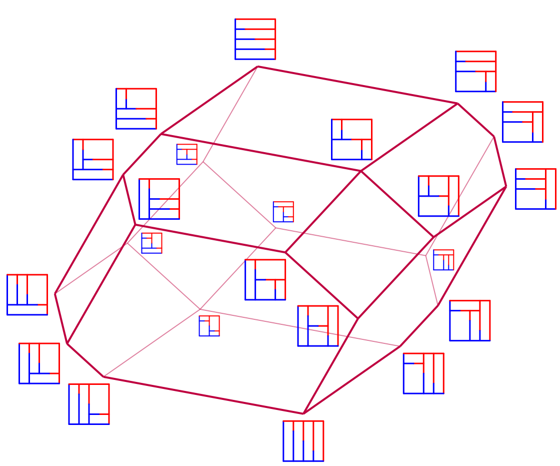
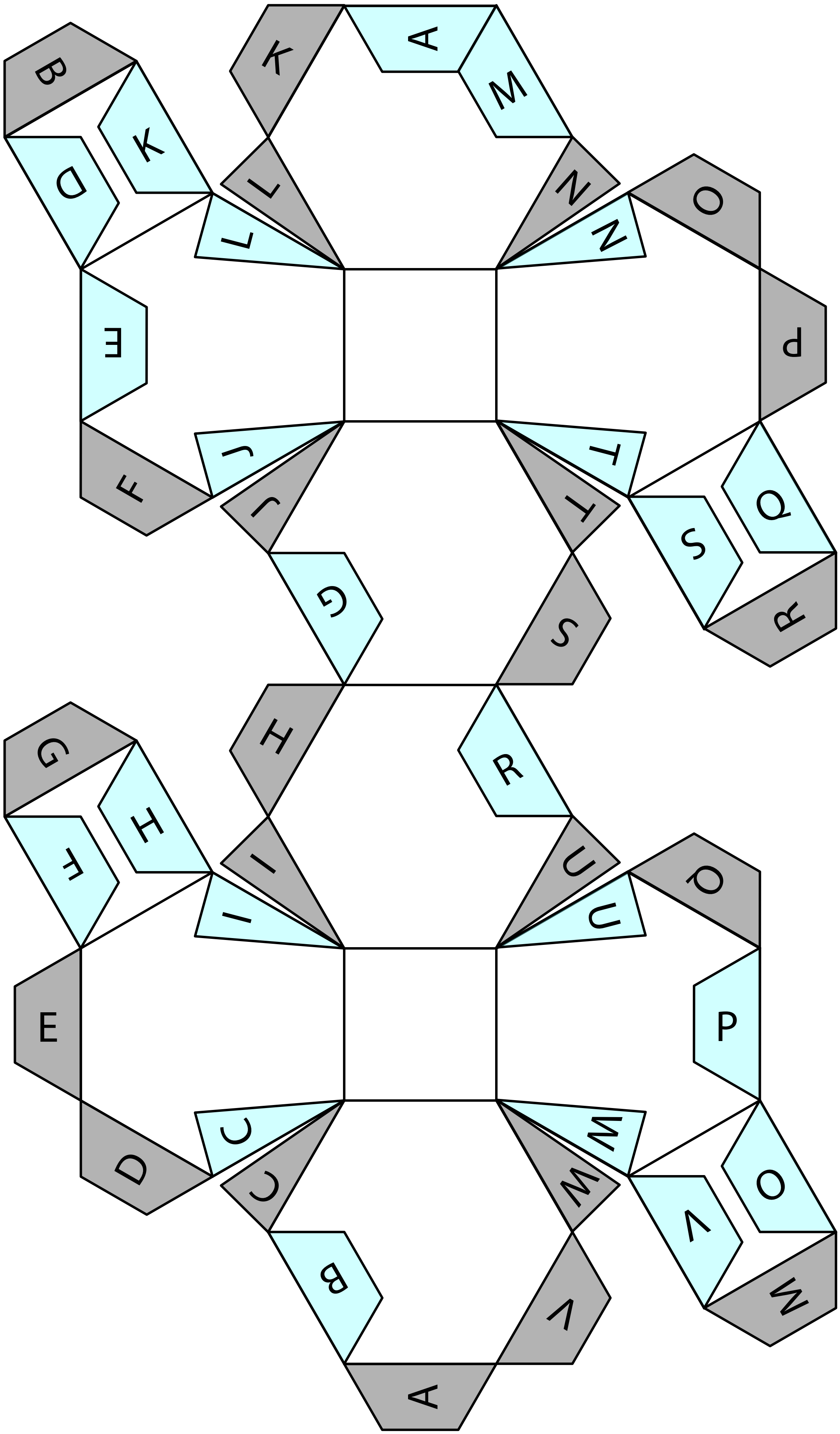


FIGURE 3. The diagonal rectangulation polytope.



permutahedron