LATTICE QUOTIENTS AND QUOTIENTOPES FOR SOME GENERALIZATIONS OF THE WEAK ORDER PROBLEM SESSION 2: ACYCLIC REORIENTATION LATTICES

1. Acyclic reorientation posets

Exercise 1. The acyclic reorientation poset of a directed acyclic graph D is the poset of acyclic reorientations of D ordered by inclusion of sets of reoriented arcs.

(1) What is the acyclic reorientation poset of an oriented forest? of a tournament (*i.e.* acyclic orientation of the complete graph)?
(2) Show that the cover relations of acyclic reorientation poset correspond to reorientations of single arcs. Describe the acyclic reorientations covering (resp. covered by) a given acyclic reorientation E.

Exercise 2. Let D be a directed acyclic graph. Show that if the acyclic reorientation poset of D is a lattice, then D is vertebrate, that is, the transitive reduction of any induced directed subgraph of D is a forest. (Note that the converse also holds, but is harder to show.)

2. Graphical arrangement

Exercise 3. Consider the graphical arrangement of a graph G on $[n]$, that is, the arrangement of the hyperplanes $\{\boldsymbol{x} \in \mathbb{R}_n \mid x_i = x_j\}$ for all edges $\{i,j\} \in G$.

(1) Show that the regions of this arrangement correspond to acyclic orientations of G.
(2) Show that the poset of regions of this arrangement with respect to the base region corresponding to an acyclic orientation D of G is isomorphic to the acyclic orientation poset of D.
(3) Show that the graphical arrangement of G is simplicial if and only if G is chordful (meaning that any cycle induces a clique).

Exercise 4. Describe the face lattice of the graphical zonotope $Z_G := \sum_{\{i,j\}\in G} \operatorname{conv}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}$.

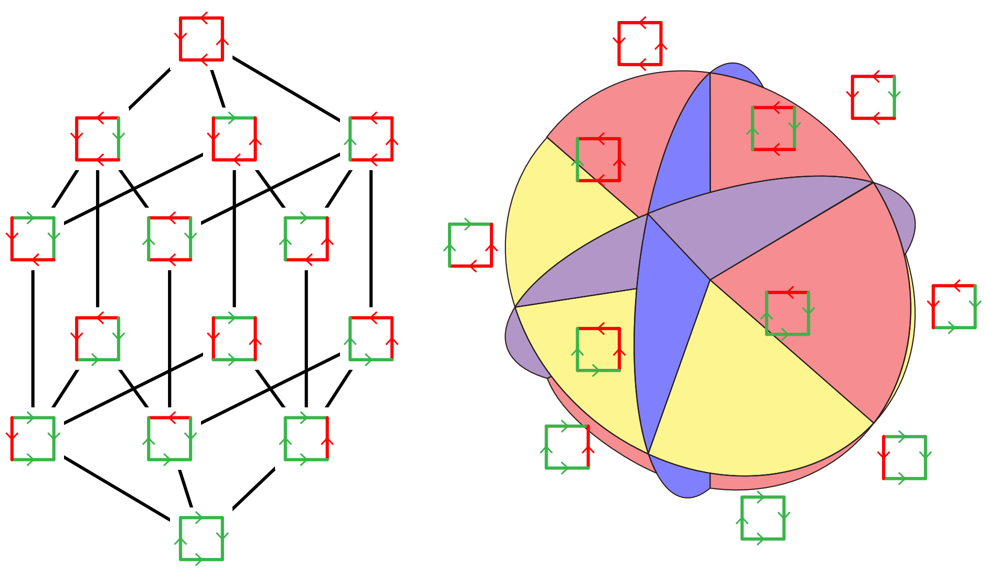
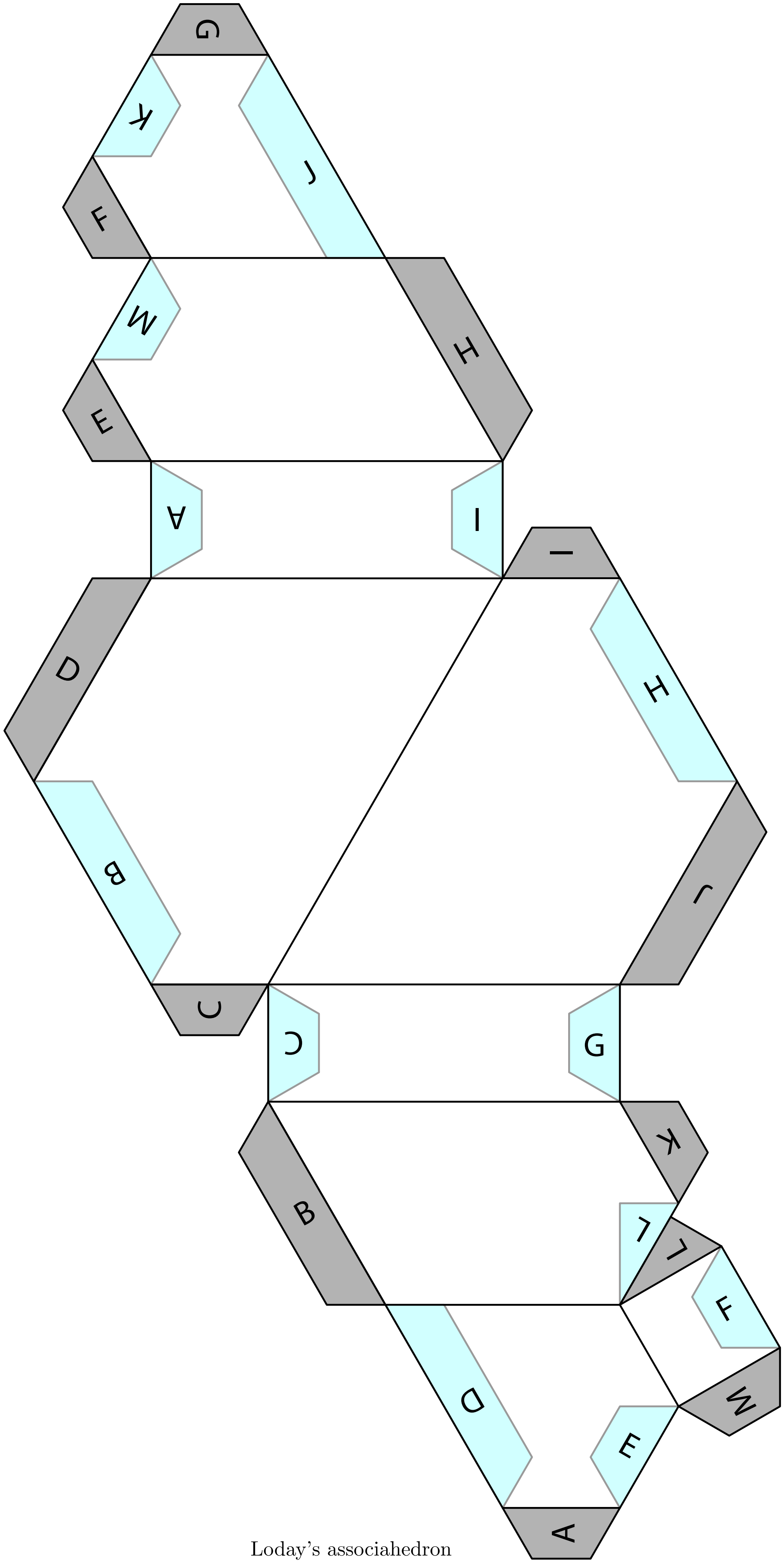


Figure 1. The acyclic reorientation lattice of a directed acyclic graph and graphical arrangement of the corresponding undirected graph.



Loday’s associahedron