LATTICE QUOTIENTS AND QUOTIENTOPES FOR SOME GENERALIZATIONS OF THE WEAK ORDER PROBLEM SESSION 4: GENTLE ASSOCIAHEDRA

Exercise 1. From a gentle quiver $\bar{Q}$, we construct its blossoming quiver $\bar{Q}^{❀}$ by completing each vertex v of $\bar{Q}$ with $2 - \mathrm{indeg}(v)$ incoming arrows and $2 - \mathrm{outdeg}(v)$ outgoing arrows. See Figure 1. Express the number of vertices $\#\bar{Q}_0^{❀}$ and arrows $\#\bar{Q}_1^{❀}$ of the blossoming quiver $\bar{Q}^{❀}$ in terms of that of $\bar{Q}$.

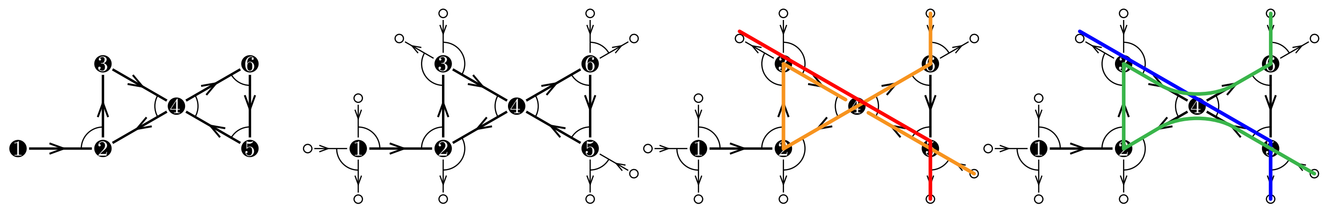


Figure 1. A gentle quiver (left), its blossoming quiver (middle left), two kissing walks (middle right), and two non-kissing walks (right).

Exercise 2. The non-kissing complex of a gentle quiver $\bar{Q}$ is the clique complex of the non-kissing relation on walks on the blossoming quiver $\bar{Q}^{❀}$. See Figure 1. The non-kissing lattice is the transitive closure of the increasing flip graph on non-kissing facets. Show that if the gentle quiver $\bar{Q}$ is an orientation of a path with no relation,

(1) the non-kissing complex of $\bar{Q}$ is isomorphic to the simplicial associahedron,
(2) the non-kissing lattice is a Cambrian lattice.

Exercise 3. Show that the non-kissing lattice of a quiver $\bar{Q}$ is anti-isomorphic to the non-kissing lattice of the reverse quiver $\bar{Q}^{\mathrm{rev}}$.

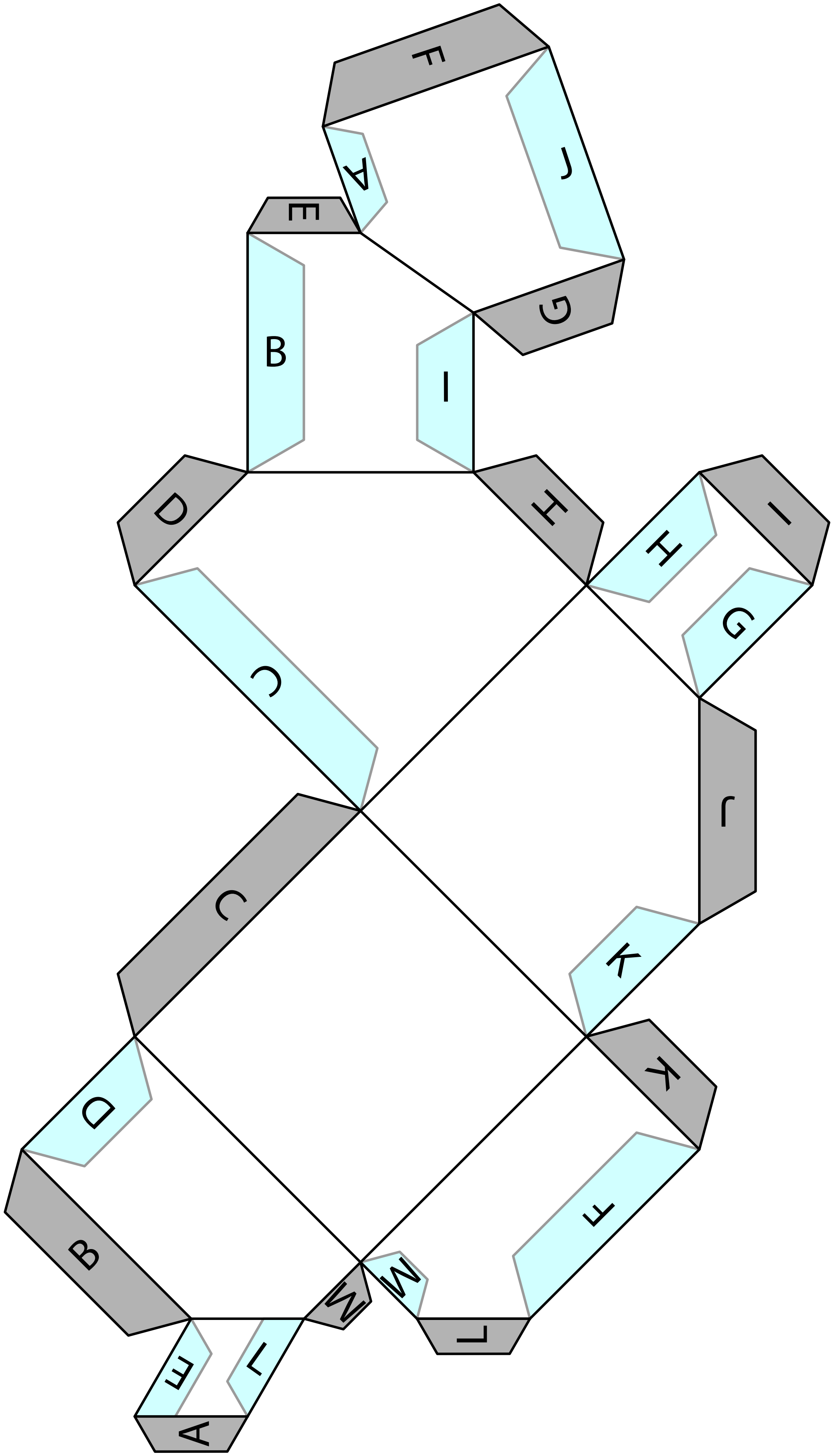
Exercise 4. Consider the gentle quiver $\overrightarrow{P}_n$ given by a path with $n-1$ arcs, oriented from left to right, and with no relation. Show that

- the biclosed sets of strings of $\overrightarrow{P}$ are in bijection with the inversion sets of permutations $[n]$,
- two biclosed sets if strings of $\overrightarrow{P}$ define the same facet of the non-kissing complex if and only if the corresponding permutations of $[n]$ define the same binary search tree.

Exercise 5. Define the multiplicity vector of a multiset $V := \{\{v_1, \dots, v_m\}\}$ as $\boldsymbol{m}_V := \sum_{i \in [m]} \boldsymbol{e}_{v_i}$. Recall that for a walk w in a non-kissing facet F,

- the $\boldsymbol{g}$-vector of w is $\boldsymbol{g}(w) := \boldsymbol{m}_{\mathrm{peaks}(w)} - \boldsymbol{m}_{\mathrm{deeps}(w)}$, where $\mathrm{peaks}(w)$ (resp. $\mathrm{deeps}(w)$) denotes the vertices along w where w has two outgoing (resp. incoming) arrows,
- the $\boldsymbol{c}$-vector of w in F is $\boldsymbol{c}(w \in F) := \varepsilon(w, F) \cdot \boldsymbol{m}_{\mathrm{ds}(w,F)}$, where $\mathrm{ds}(w, F)$ denotes the distinguished substring of w in F, and $\varepsilon(w, F) := 1$ if the two distinguished arrows of w point towards its ends, and $\varepsilon(w, F) := -1$ otherwise.

Show that the $\boldsymbol{g}$-vectors $\{\boldsymbol{g}(w) \mid w \in F\}$ and the $\boldsymbol{c}$-vectors $\{\boldsymbol{c}(w \in F) \mid w \in F\}$ form dual bases.



Chapoton–Fomin–Zelevinski's associahedron