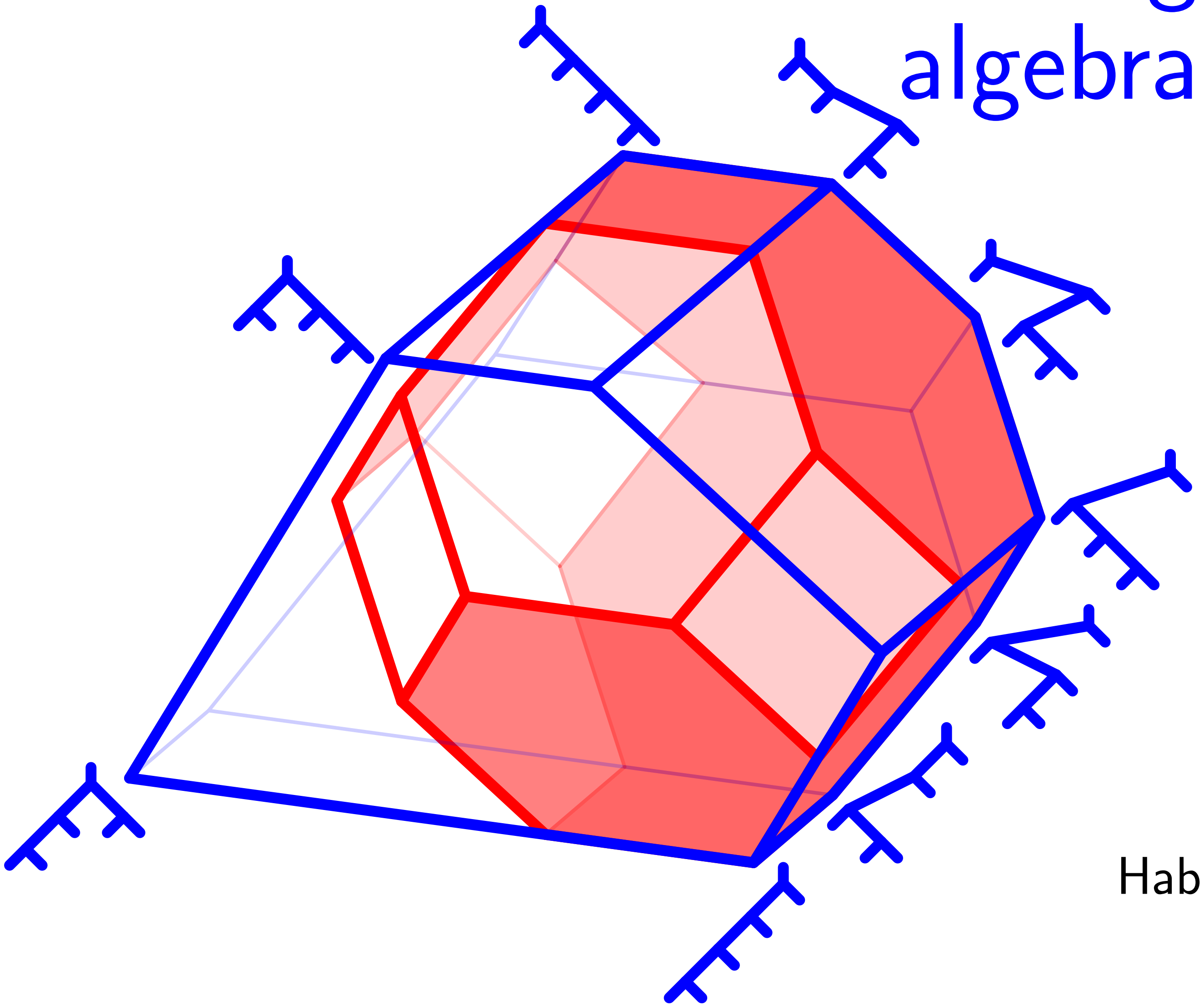
# From permutahedra to associahedra, a walk through geometric and algebraic combinatorics




V. PILAUD
(CNRS & École Polytechnique)

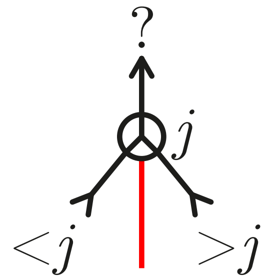

Habilitation à Diriger des Recherches
Friday July 10th, 2020

slides available at: http://www.lix.polytechnique.fr/~pilaud/documents/presentations/HDRDefense.pdf
manuscript available at: http://www.lix.polytechnique.fr/~pilaud/documents/reports/HDRVincentPilaud.pdf

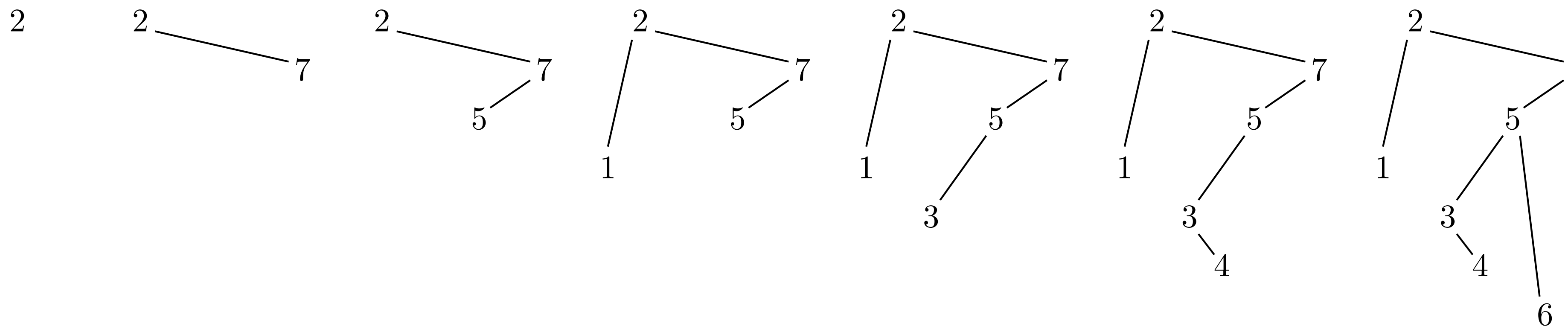
# THREE PERSPECTIVES ON BST INSERTION

# BINARY SEARCH TREE INSERTION

```
BSTinsert(T, x):
   if T = ∅ then return BST(x, ∅, ∅)
   if x < T.root then return BST(T.root, BSTinsert(T.left, x), T.right)
   if x > T.root then return BST(T.root, T.left, BSTinsert(T.right, x))
```




BST insertion of $2751346$:



# BINARY SEARCH TREE INSERTION

BSTinsert(T, x):
  if T $=\varnothing$ then return BST(x, $\varnothing$, $\varnothing$)
  if x $<$ T.root then return BST(T.root, BSTinsert(T.left, x), T.right)
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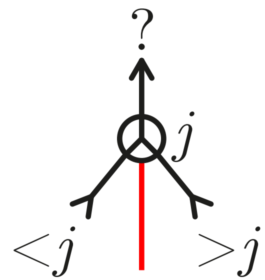


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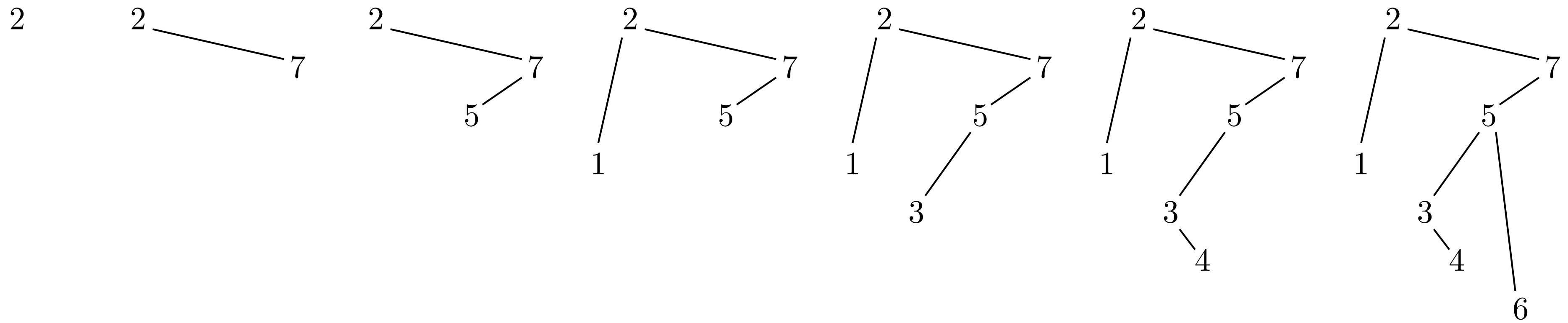


Three perspectives on BST insertion:

- lattice theory: weak order and Tamari lattice
- discrete geometry: permutahedra and associahedra
- Hopf algebras: Malvenuto–Reutenauer and Loday–Ronco algebras

# LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice = partially ordered set $L$ where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$
lattice congruence = equivalence relation on $L$ compatible with meets and joins

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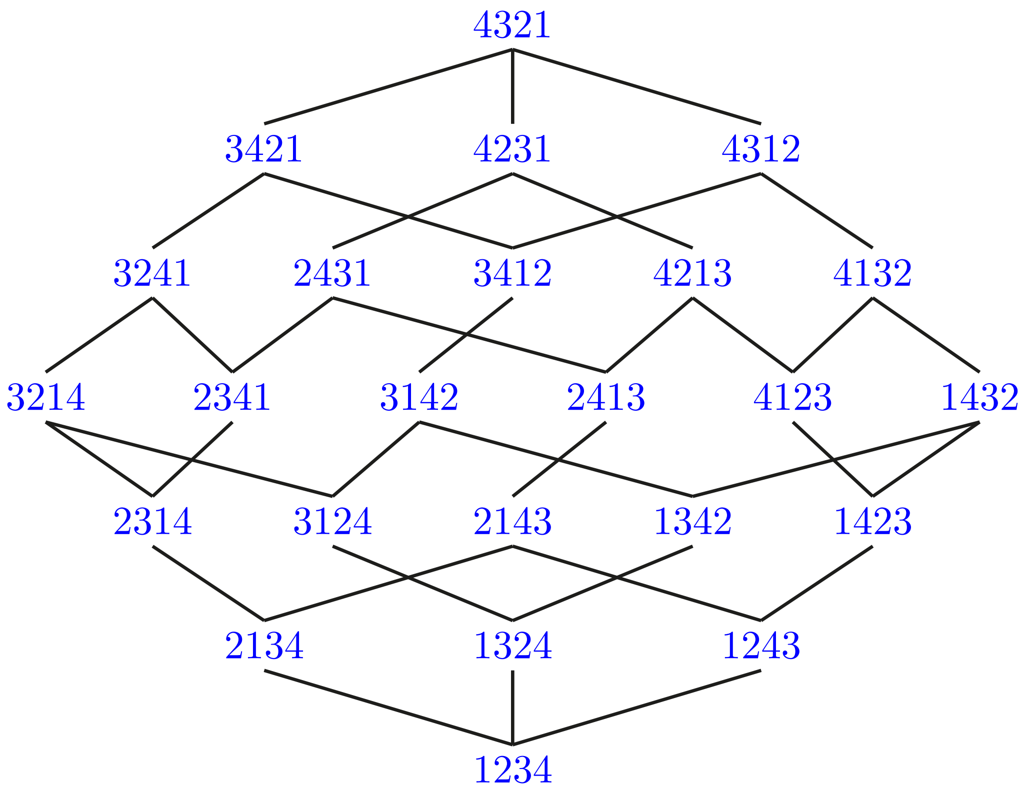


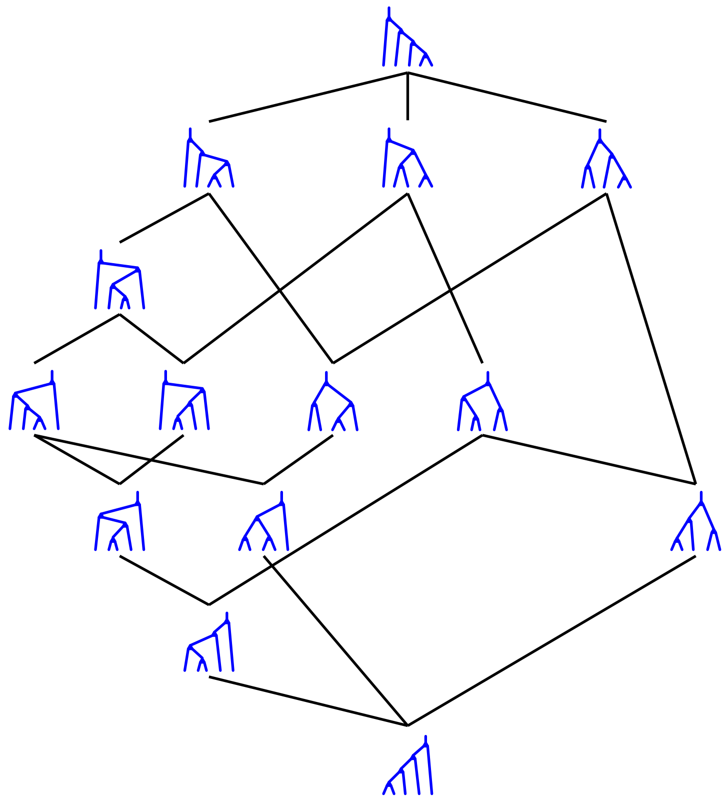

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ordered by inclusion of inversion sets

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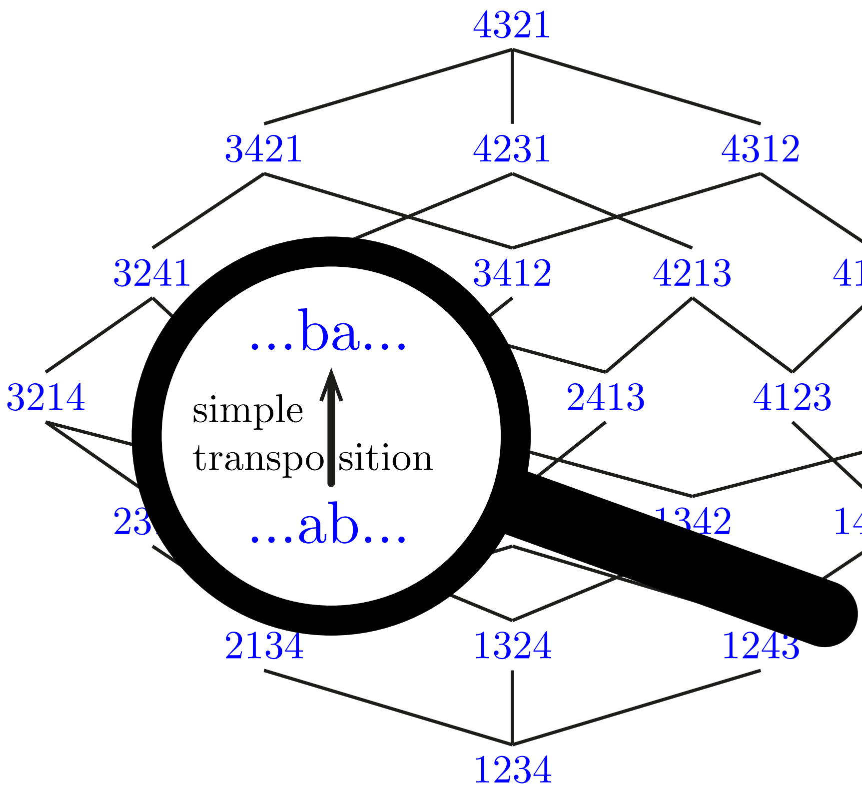


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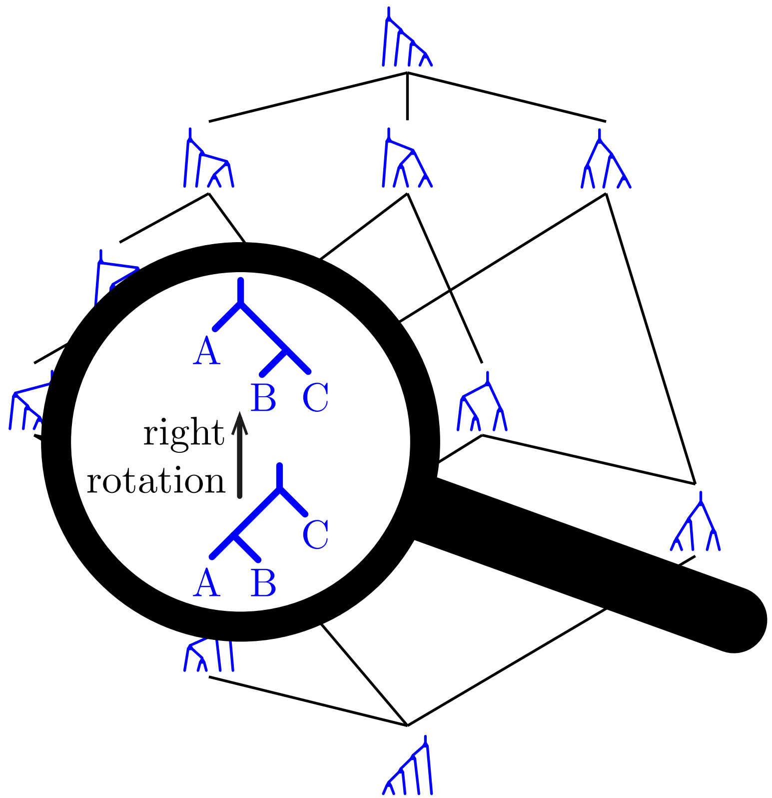


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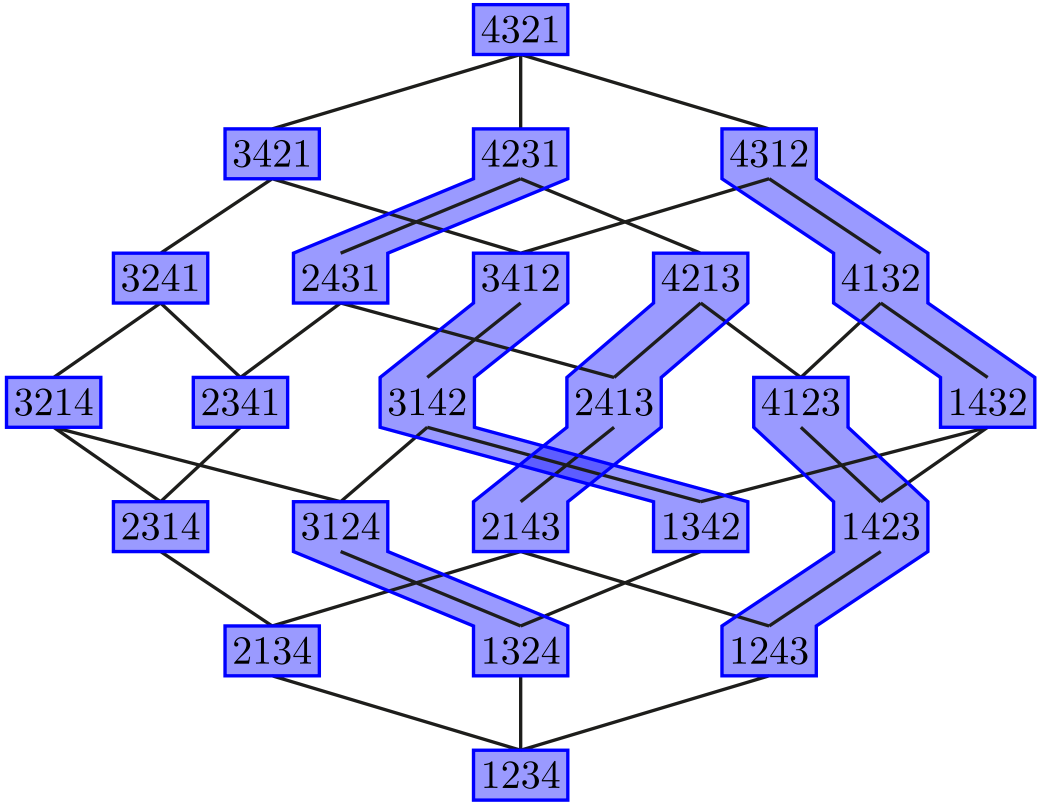


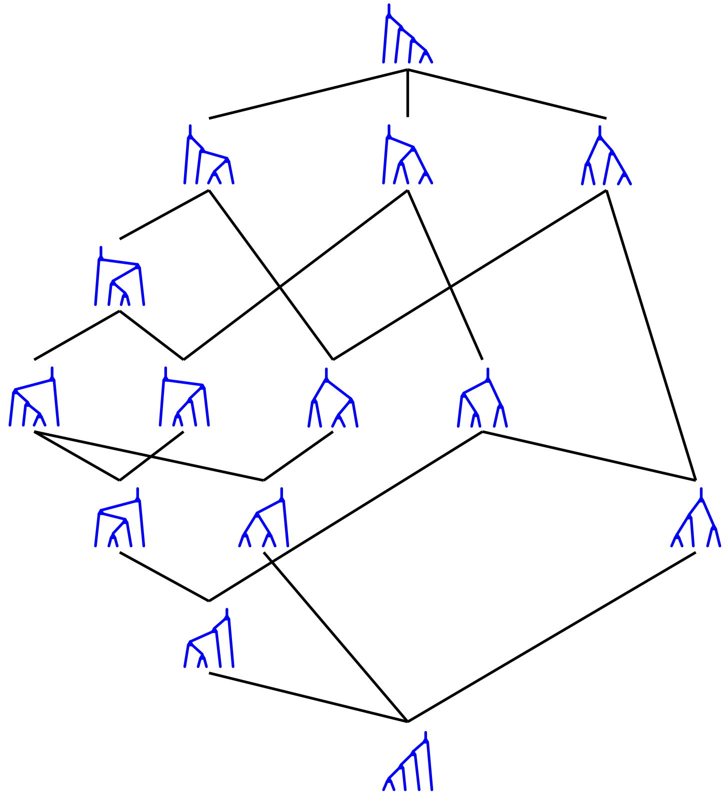

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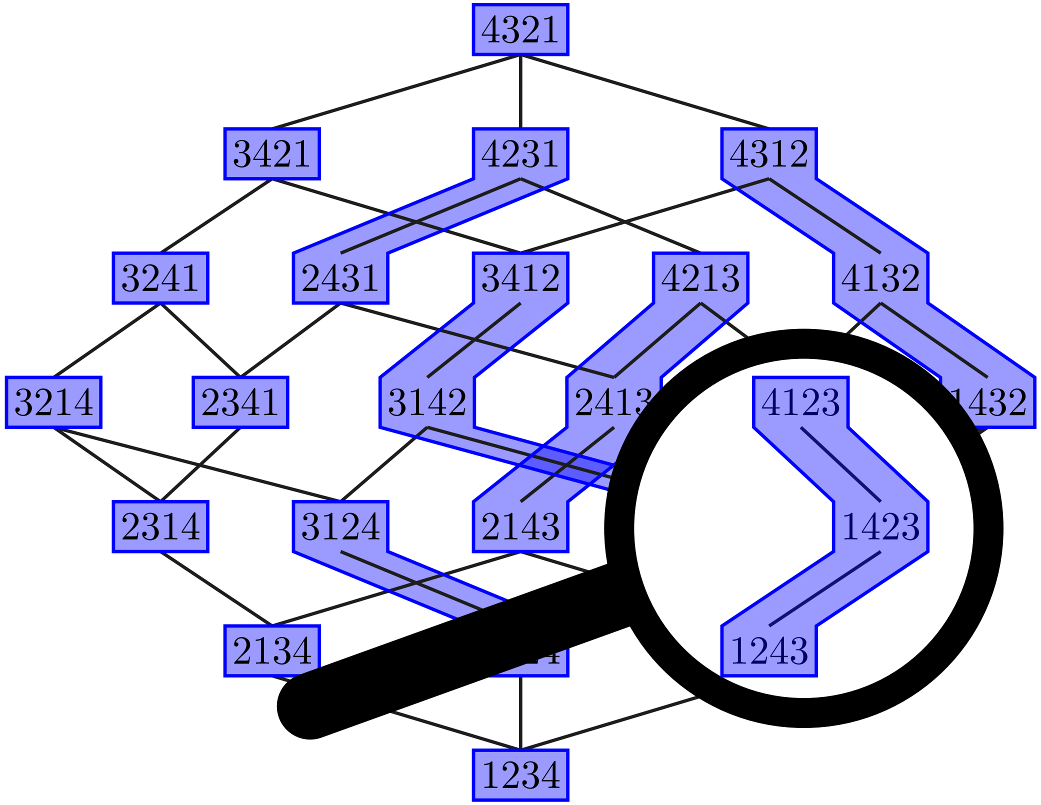


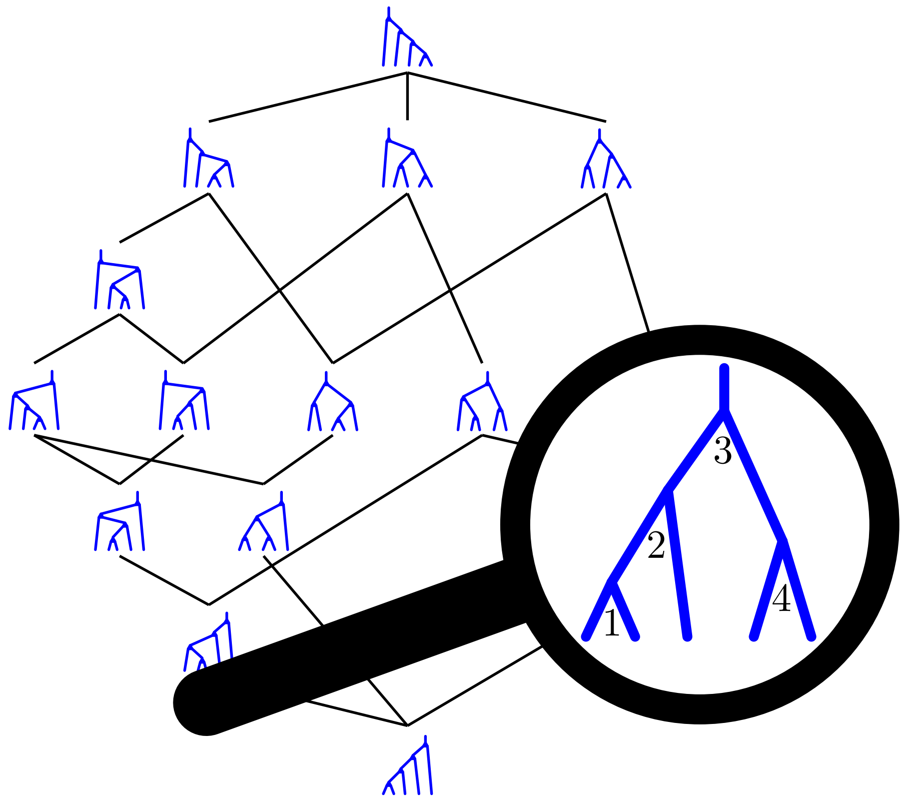


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fan = collection of polyhedral cones closed by faces and intersecting along faces

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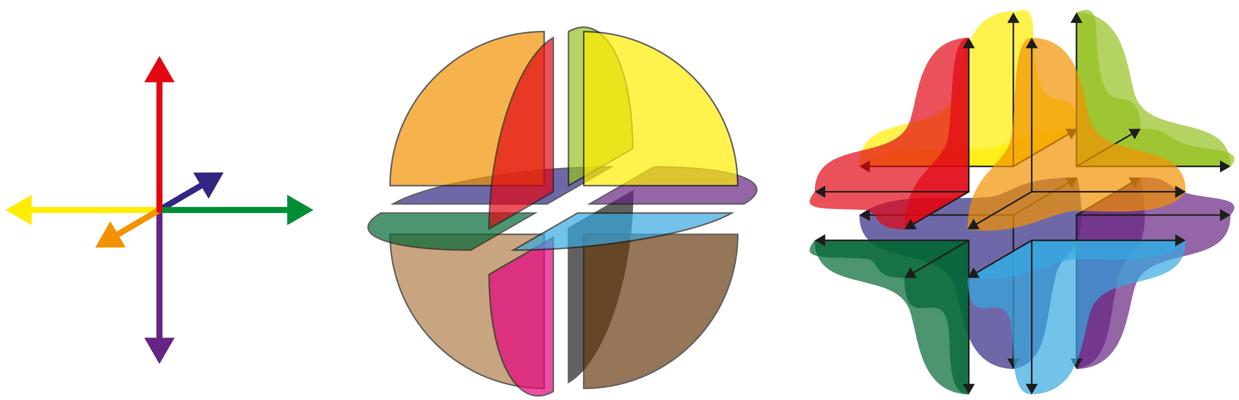
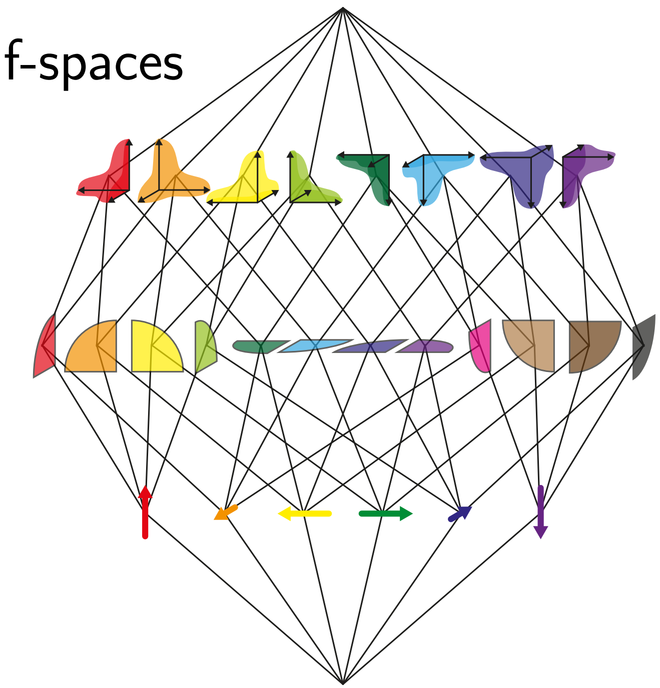
# POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

polyhedral cone = positive span of a finite set of $\mathbb{R}^n$
= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face





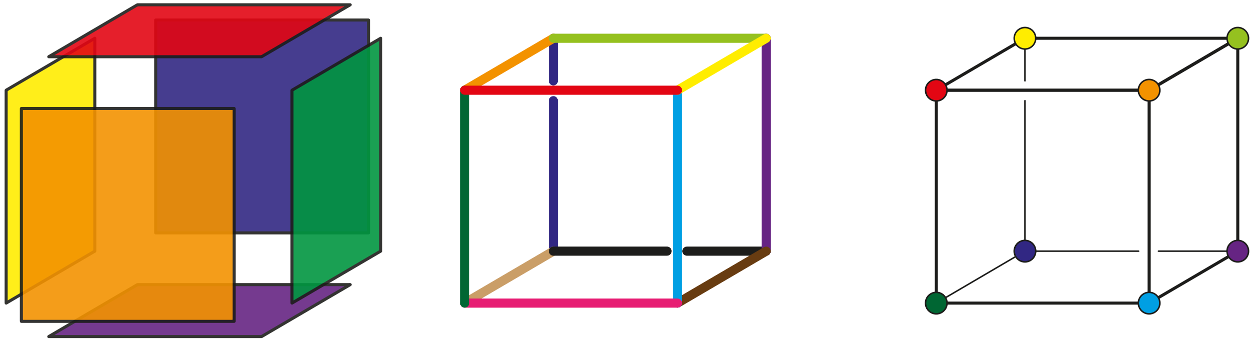
# POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

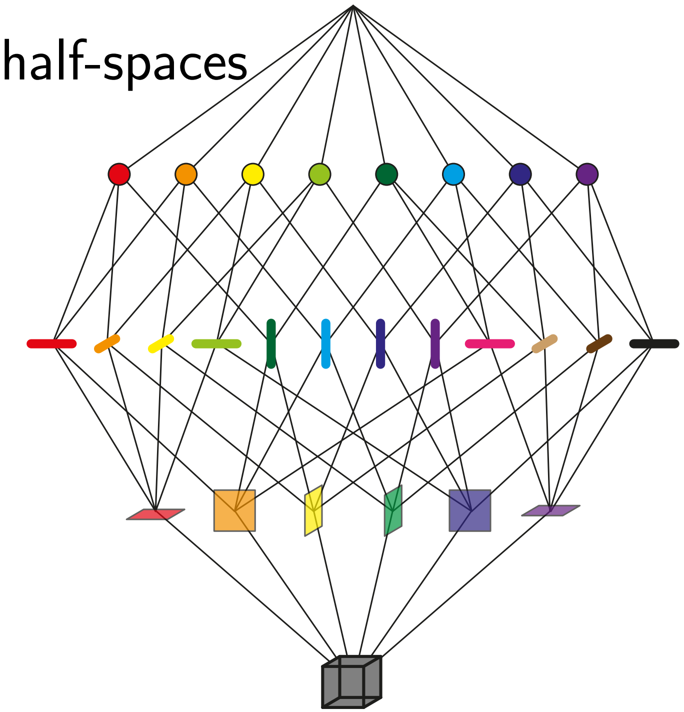
polytope $=$ convex hull of a finite set of $\mathbb{R}^n$

$=$ bounded intersection of finitely many affine half-spaces

face $=$ intersection with a supporting hyperplane

face lattice $=$ all the faces with their inclusion relations





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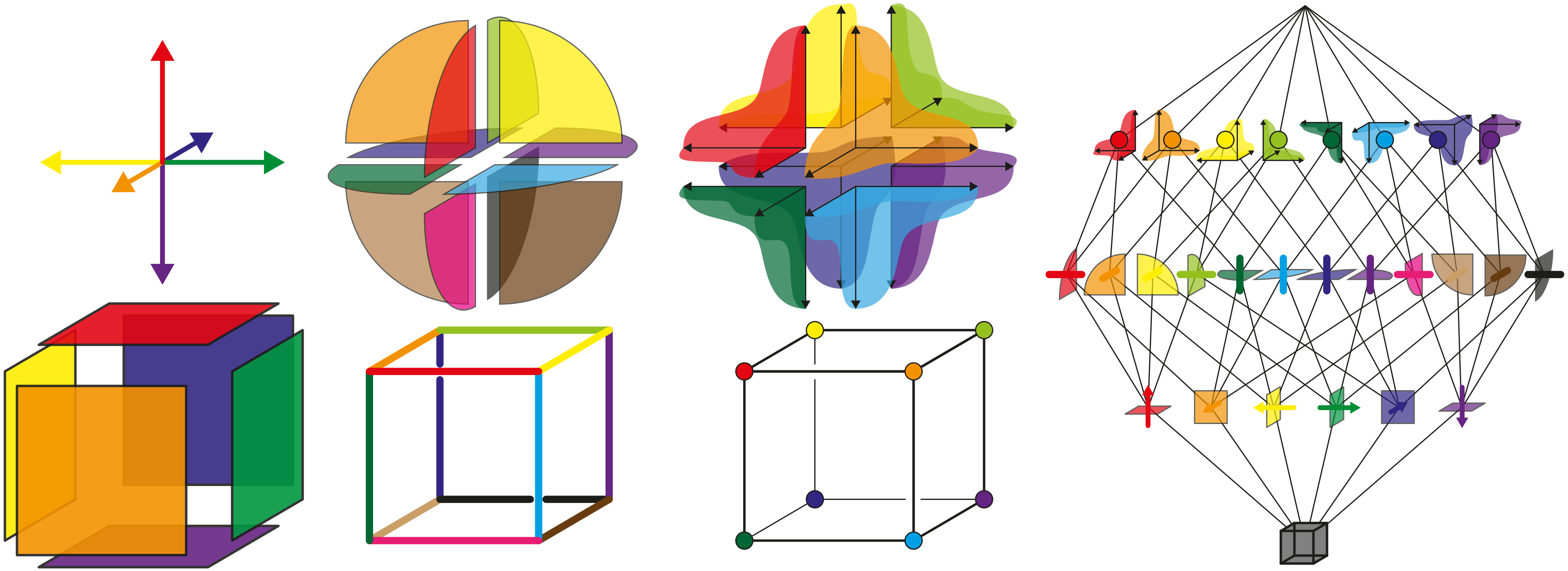

face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = { normal cone of $\mathbb{F}$ | $\mathbb{F}$ face of $\mathbb{P}$ }

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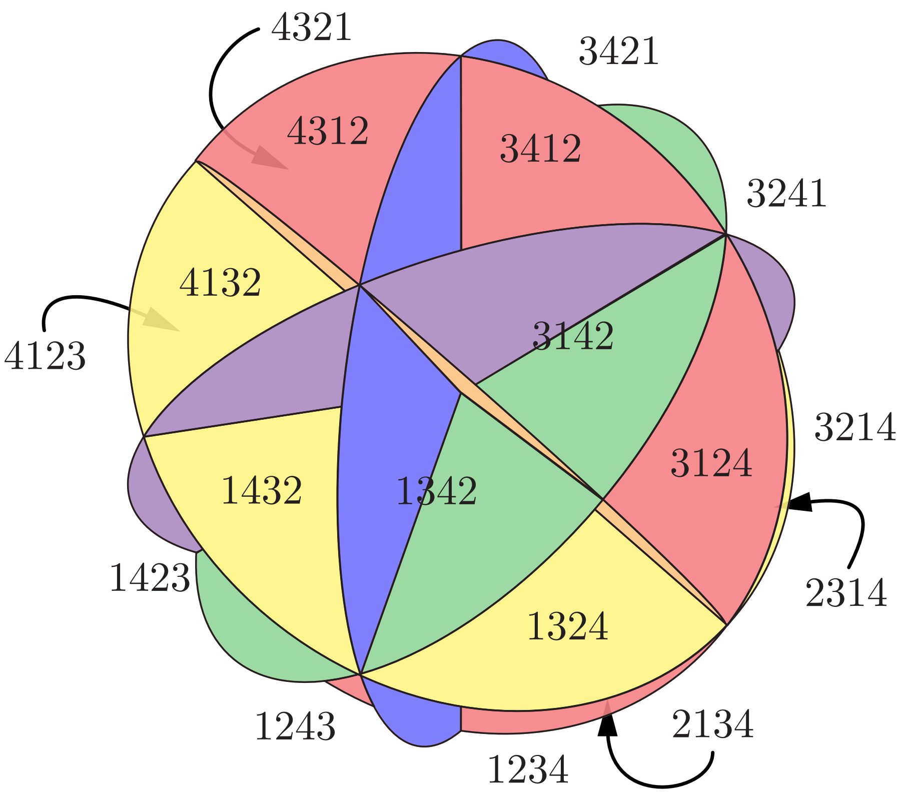


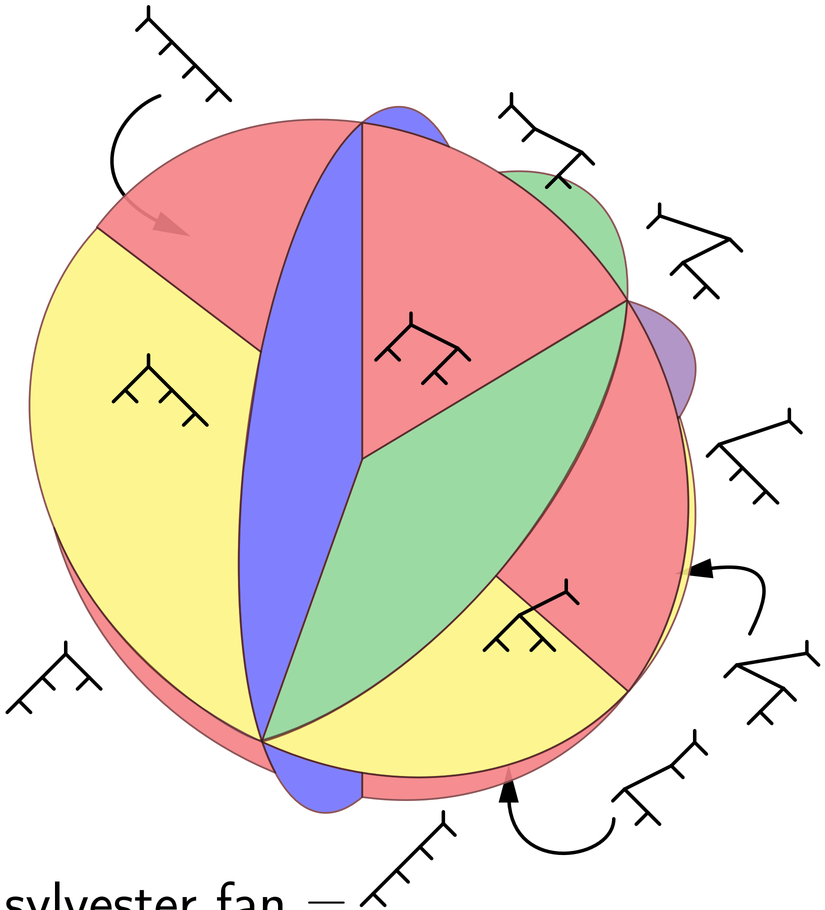

braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$

sylvester fan =

$$\mathbb{C}(T) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T \}$$

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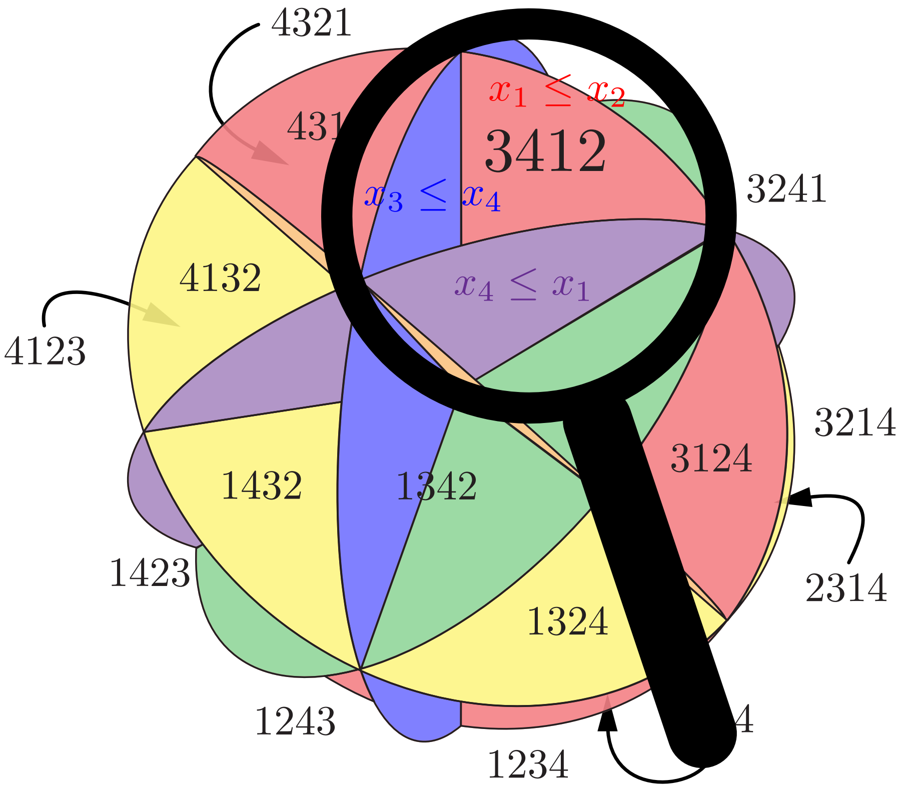


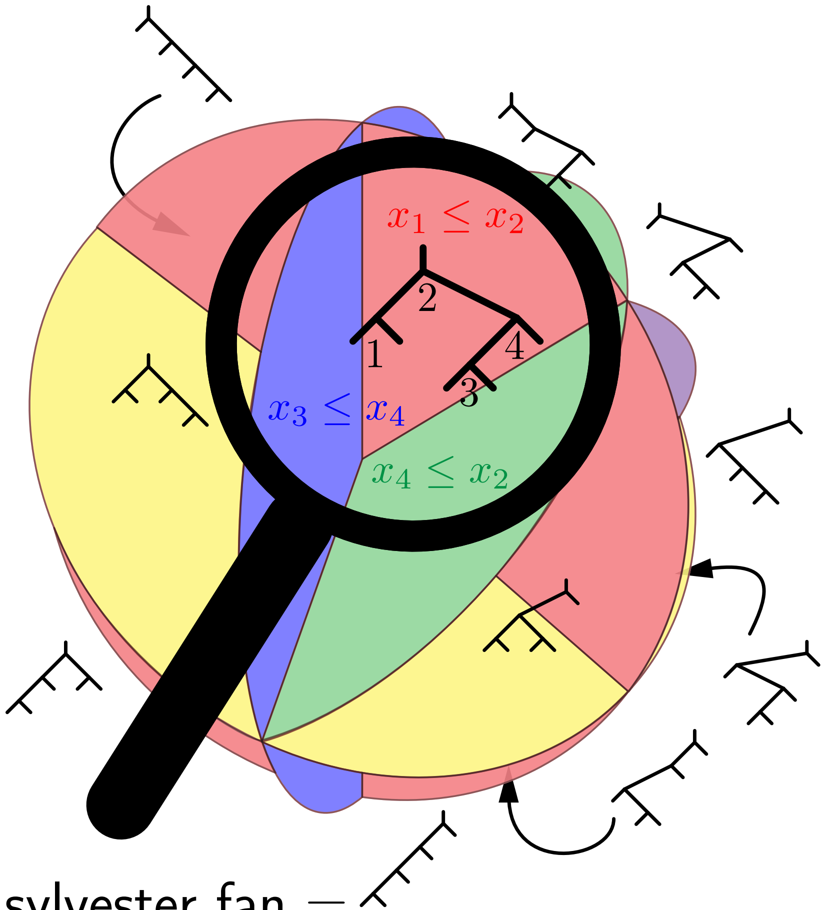


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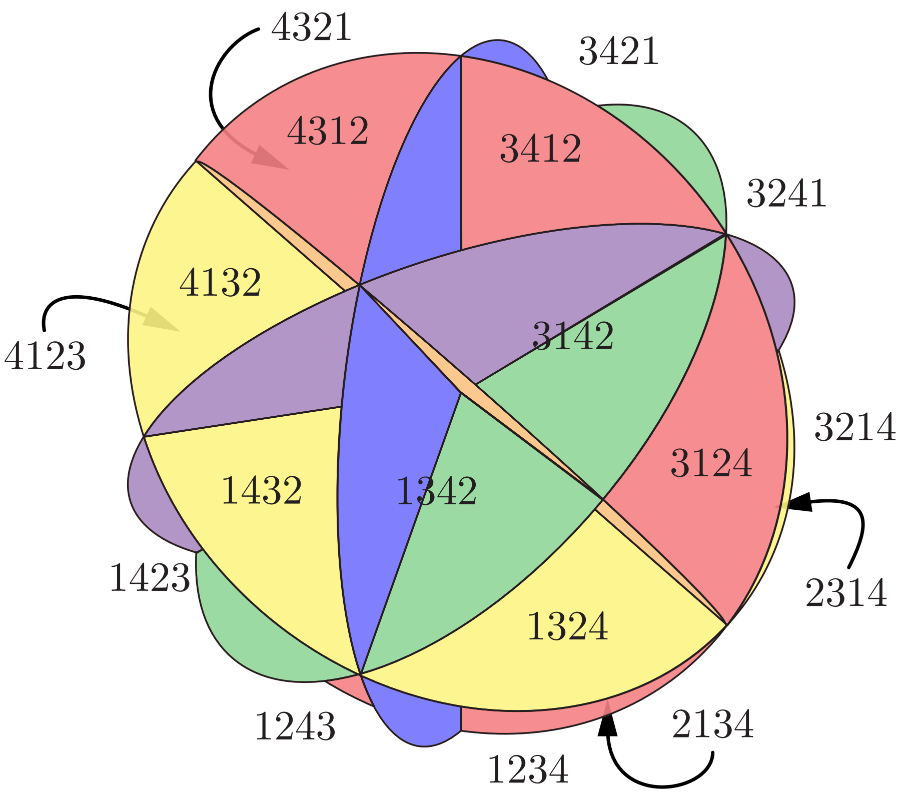


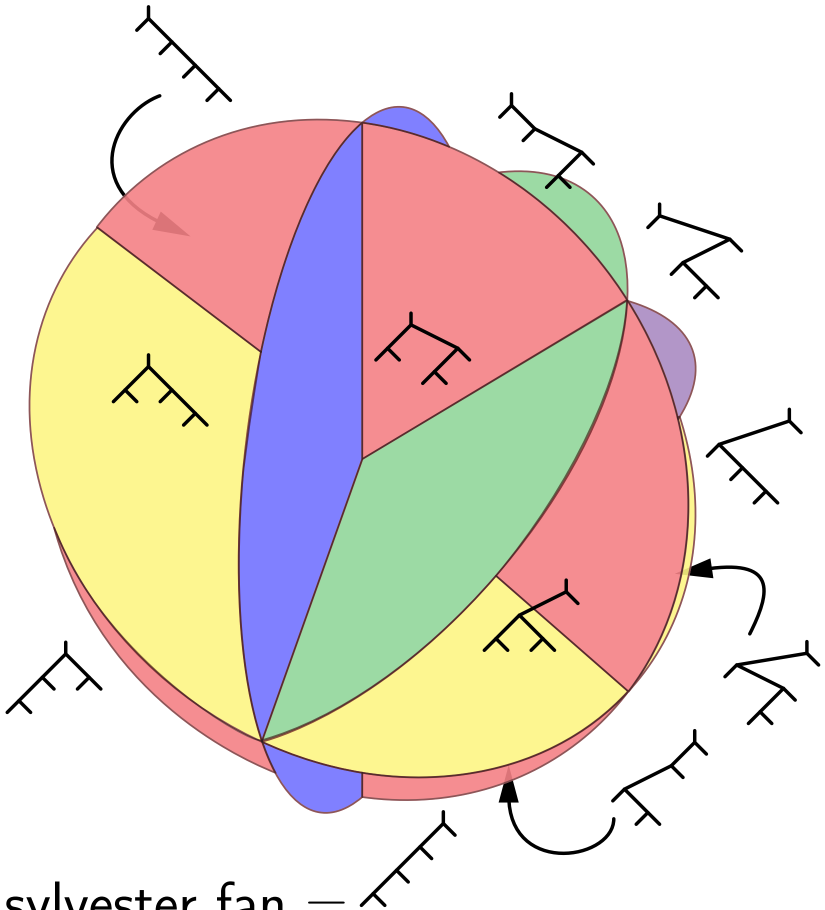

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quotient fan = $\mathbb{C}(T)$ obtained by glueing $\mathbb{C}(\sigma)$ for all $\sigma$ in the same BST insertion fiber

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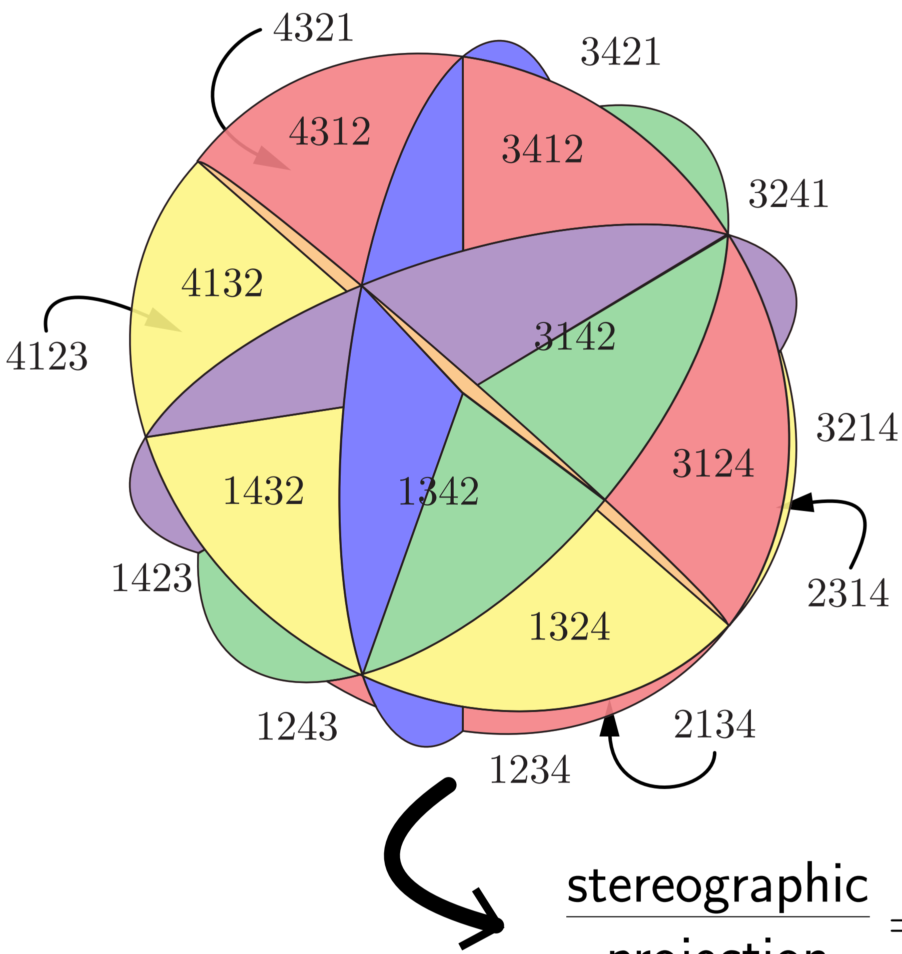
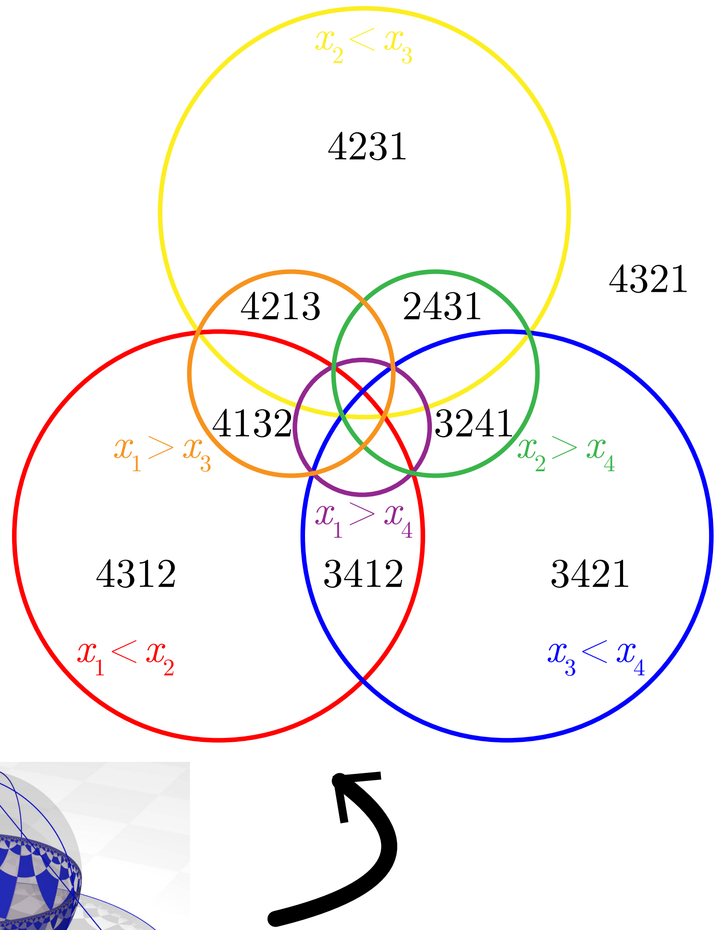
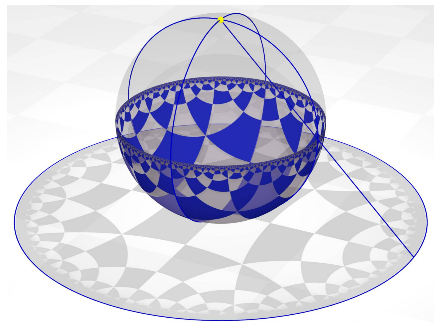



stereographic projection =



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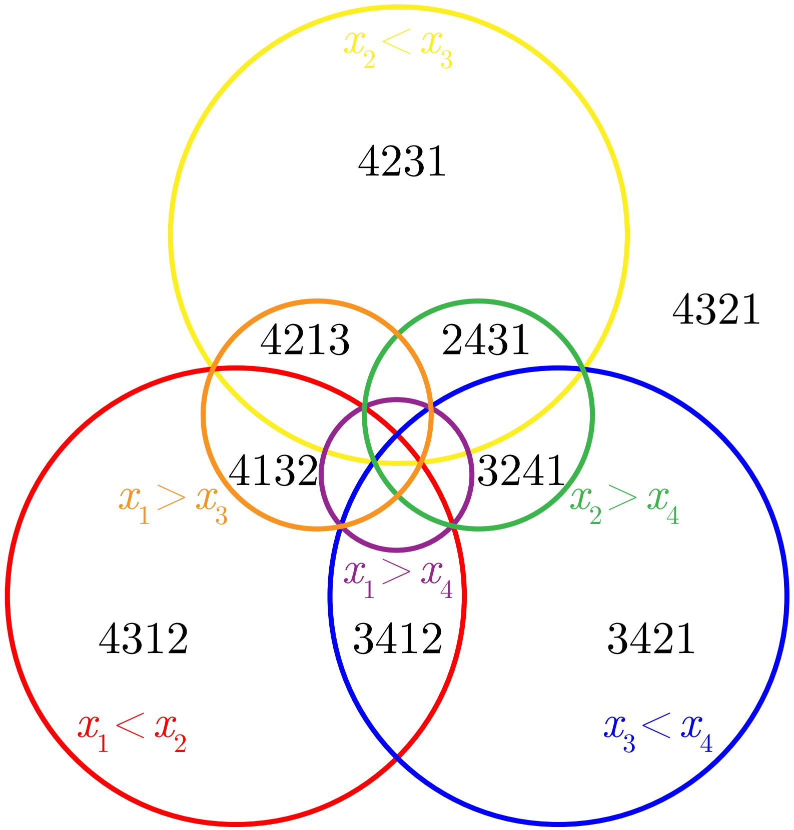


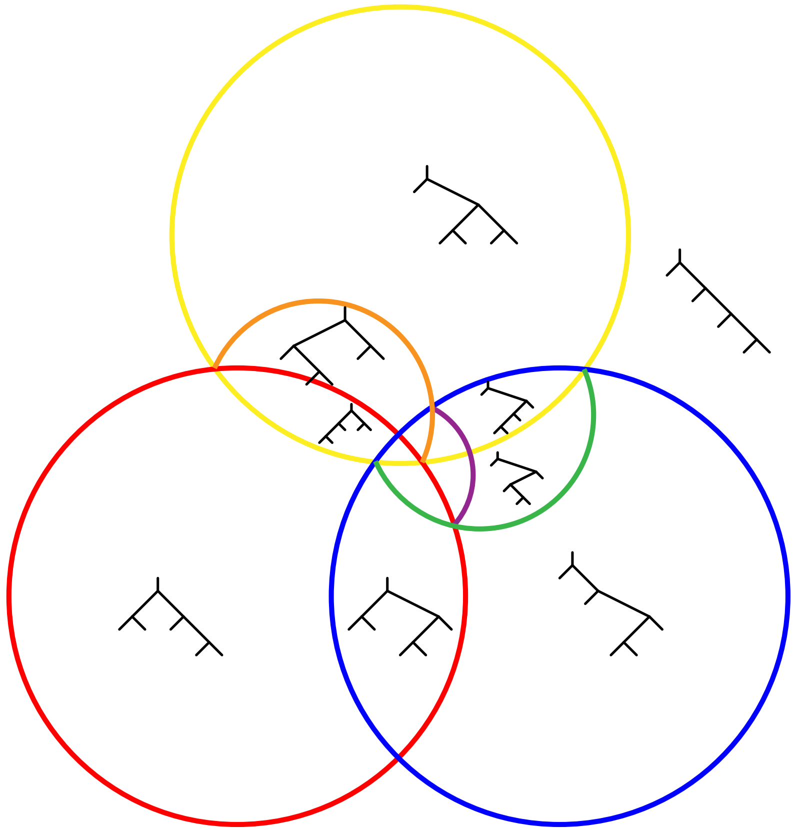

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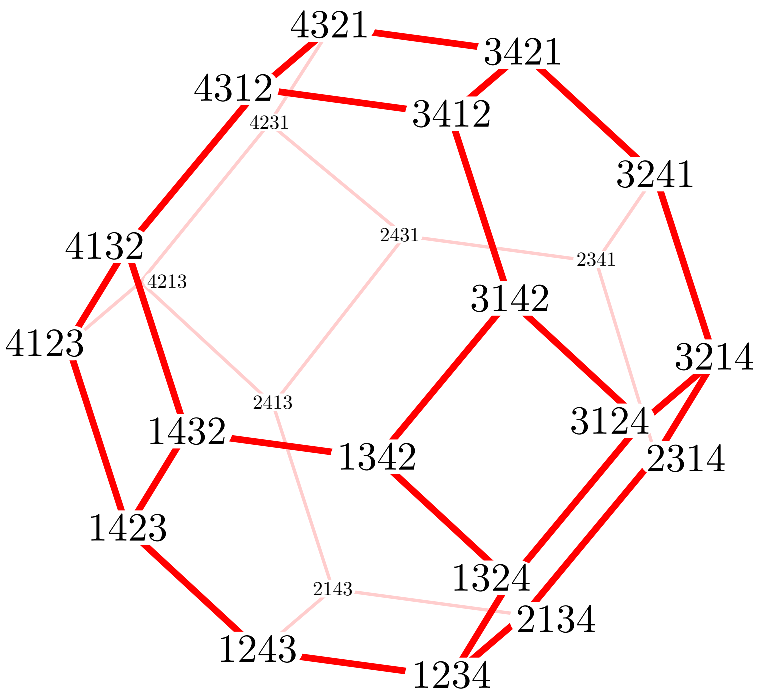


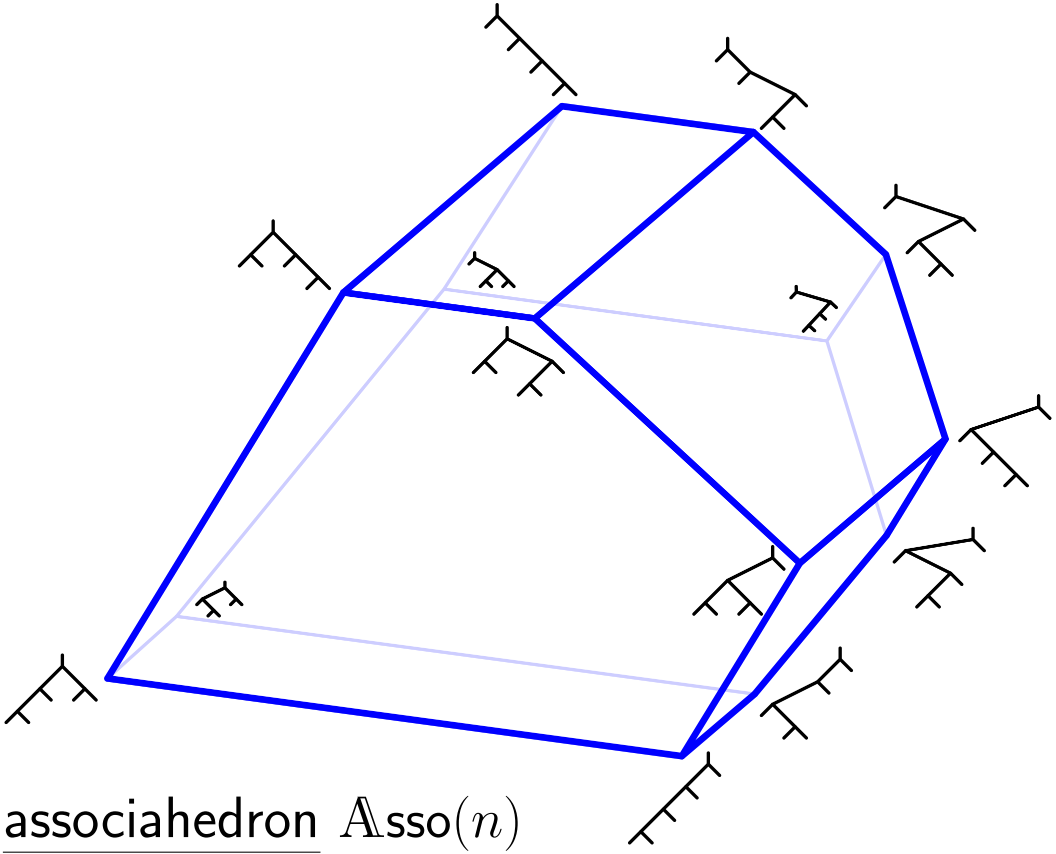

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$$= \mathrm{conv}\left\{[\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing\neq J\subsetneq[n]} \mathbb{H}_J$$

where $\mathbb{H}_J = \left\{\boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \geq \binom{|J|+1}{2}\right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

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Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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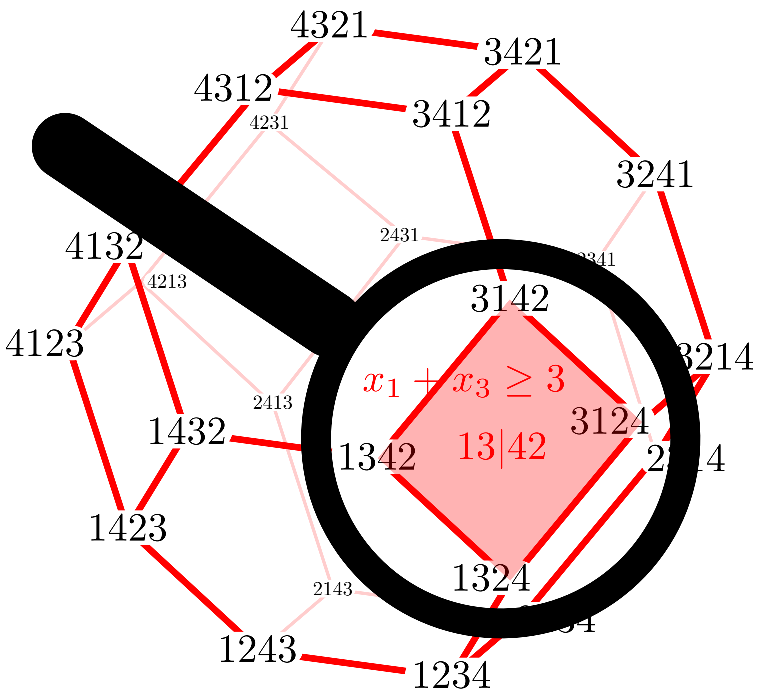


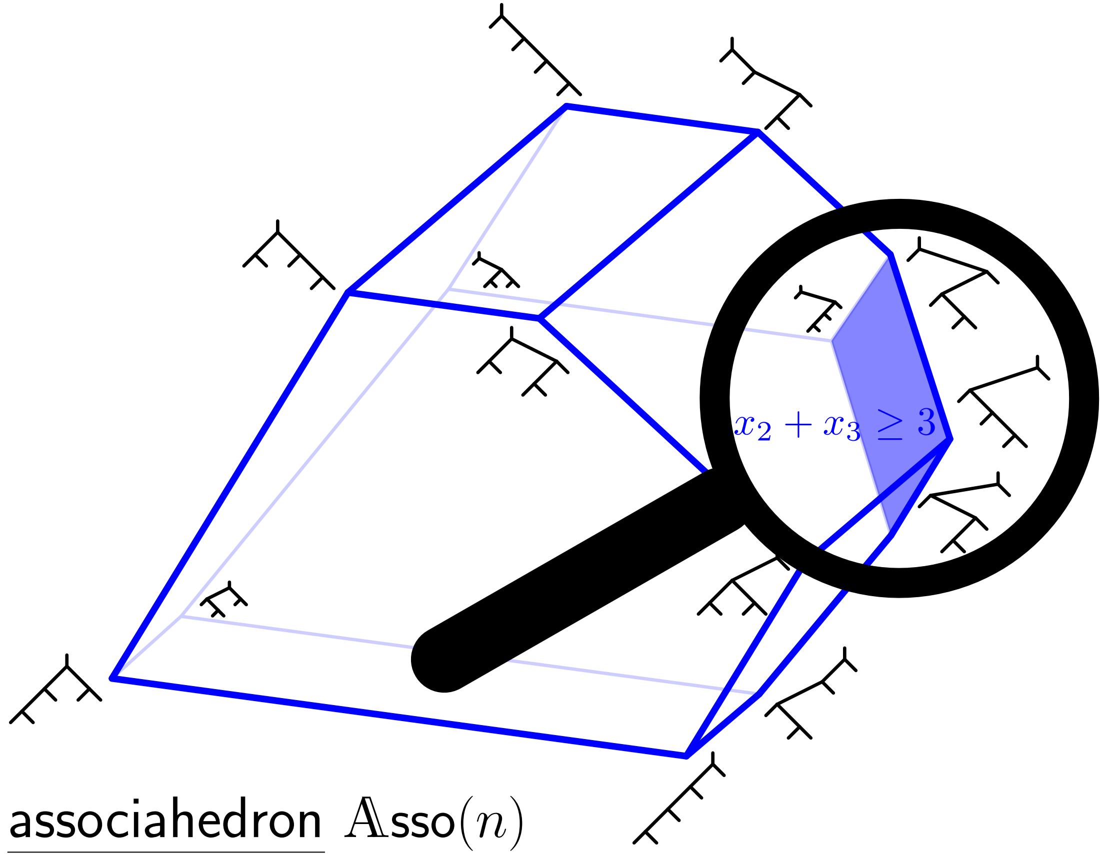


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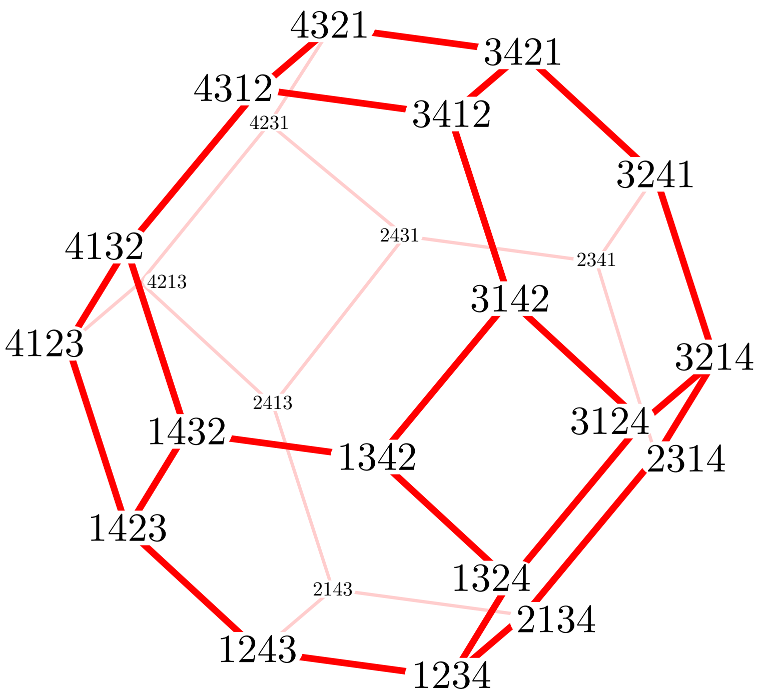


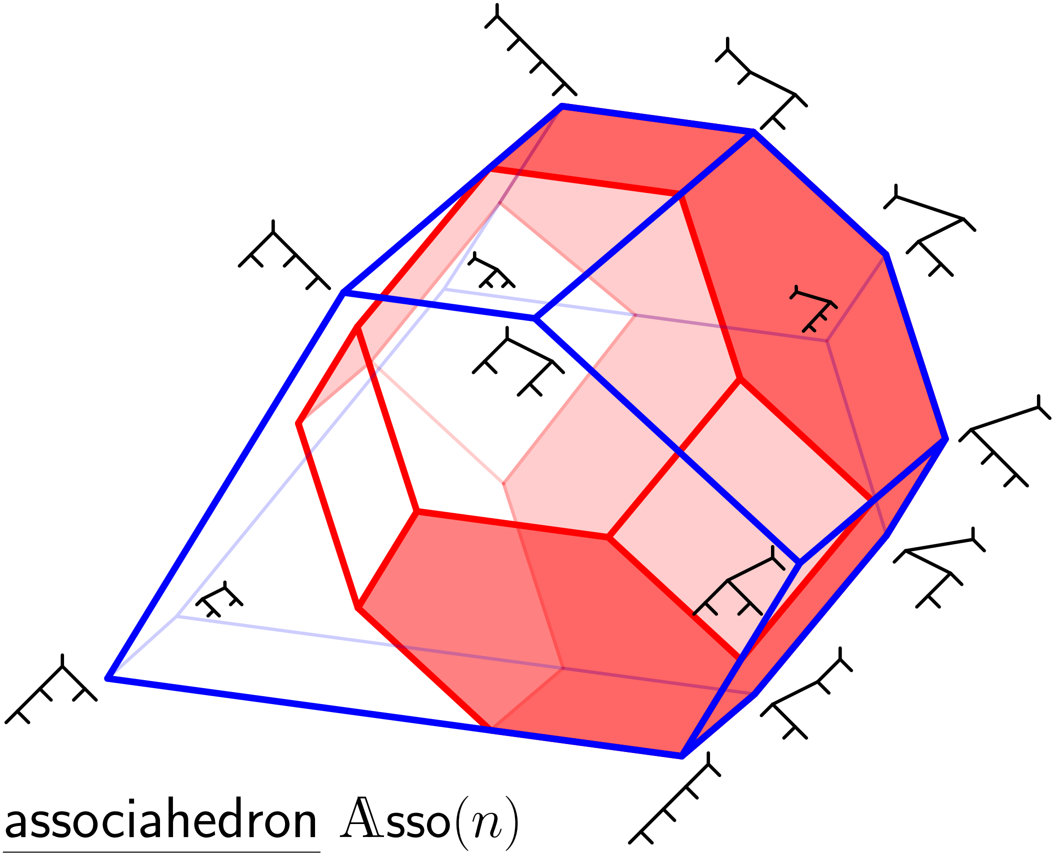

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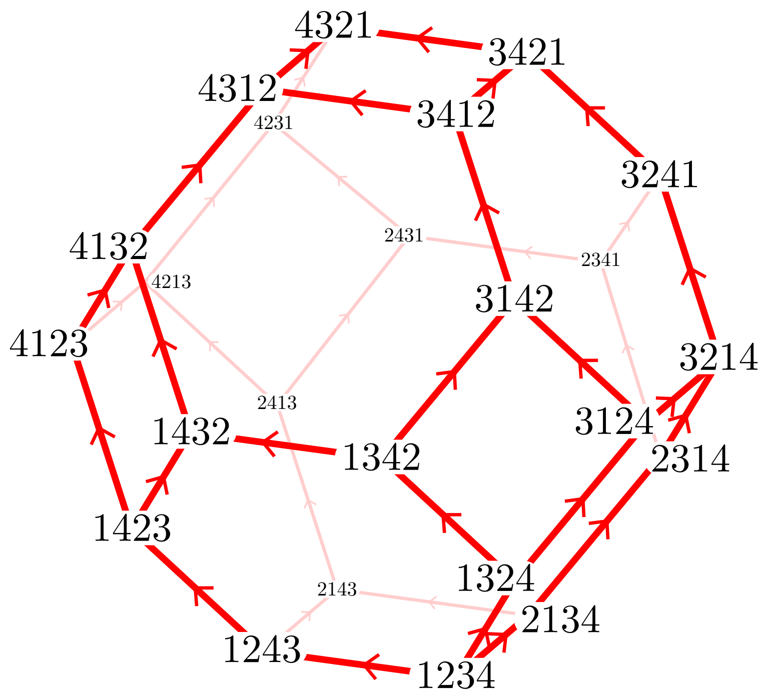
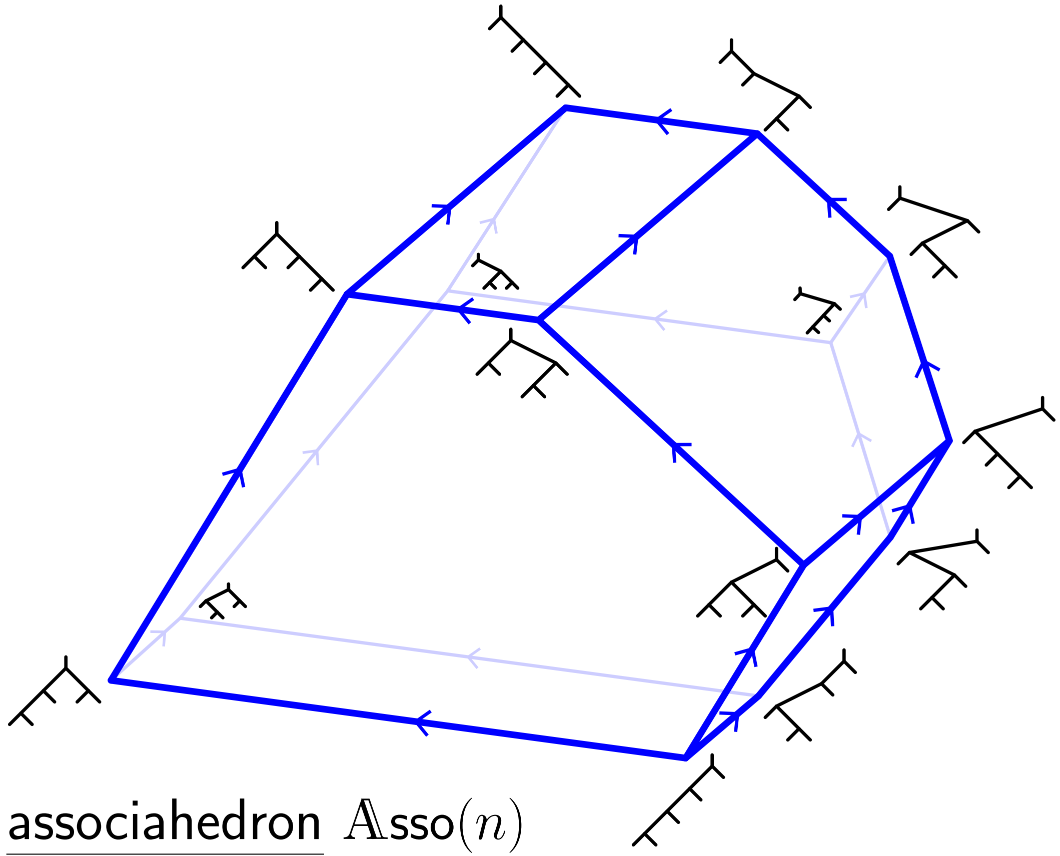



<u>permutahedron</u> $\mathbb{P}\text{erm}(n)$

$\Longrightarrow$ weak order on permutations



<u>associahedron</u> $\mathbb{A}\text{sso}(n)$

$\Longrightarrow$ Tamari lattice on binary trees

| | | | |
|---|---|---|---|
| Hasse diagram of | weak order = graph of | permutahedron oriented | $12\dots n \to n\dots 21$ |
| | Tamari lattice | associahedron | left $\to$ right comb |

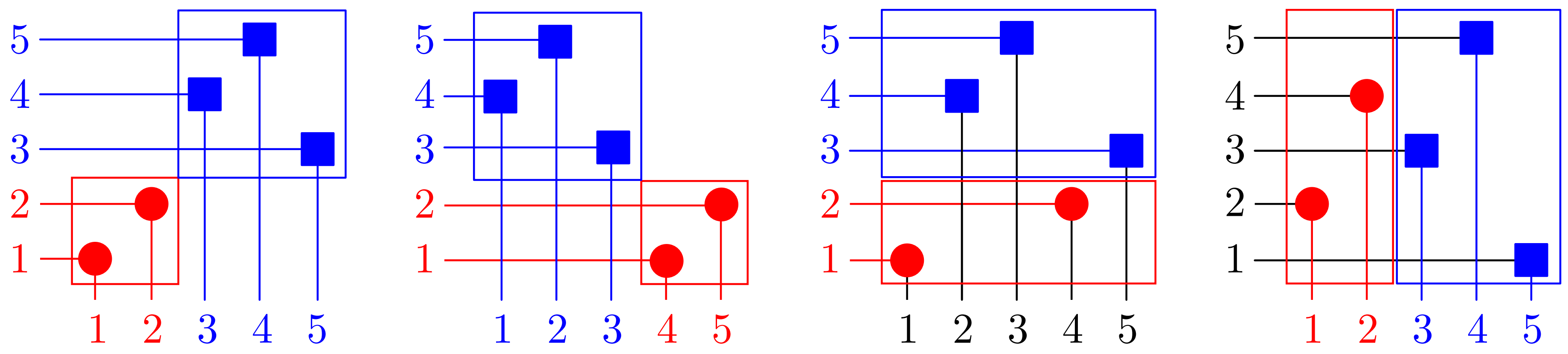
# HOPF ALGEBRAS: MALVENUTO–REUTENAUER AND LODAY–RONCO

product $=$ linear map $\cdot : V \otimes V \to V$ $=$ a tool to combine two elements (glue)
coproduct $=$ linear map $\triangle : V \to V \otimes V$ $=$ a tool to decompose an element (scisors)
Hopf algebra $= (V, \cdot, \triangle)$ such that $\triangle(a \cdot b) = \triangle(a) \cdot \triangle(b)$

Two operations on permutations:

shuffle $12 \sqcup\!\sqcup 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

convol. $12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$




$\rho\backslash\sigma$ $\rho/\sigma$ $\rho \sqcup\!\sqcup \sigma$ $\rho \star \sigma$

Weak order intervals: $\rho \sqcup\!\sqcup \sigma = \{\tau \in \mathfrak{S}_{p+q} \mid \rho\backslash\sigma \le \tau \le \rho/\sigma\}$

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| | Malvenuto–Reutenauer | $\supseteq$ | Loday–Ronco |
|---|---|---|---|
| vector space | $\langle\, \mathbb{F}_\sigma \mid \sigma$ permutation of any size $\rangle$ | | $\langle\, \mathbb{P}_T \mid T$ binary tree of any size $\rangle$ |
| product | $\mathbb{F}_\rho \cdot \mathbb{F}_\sigma = \sum_{\tau \in \rho \sqcup\!\sqcup \sigma} \mathbb{F}_\tau = \sum_{\rho \backslash \sigma \le \tau \le \rho / \sigma} \mathbb{F}_\tau$ | | $\mathbb{P}_R \cdot \mathbb{P}_S = \sum_{R \backslash S \le \tau \le R/S} \mathbb{P}_T$ |
| coproduct | $\triangle(\mathbb{F}_\tau) = \sum_{\tau \in \rho \star \sigma} \mathbb{F}_\rho \otimes \mathbb{F}_\sigma$ | | $\triangle(\mathbb{P}_T) = \sum_{R_1 \cdots R_k \| S \text{ cut of } T} \big(\prod_{i \in [k]} \mathbb{P}_{R_i}\big) \otimes \mathbb{P}_S$ |

# HOPF ALGEBRAS: MALVENUTO–REUTENAUER AND LODAY–RONCO

product $=$ linear map $\cdot : V \otimes V \to V$ $=$ a tool to combine two elements (glue)
coproduct $=$ linear map $\triangle : V \to V \otimes V$ $=$ a tool to decompose an element (scisors)
Hopf algebra $= (V, \cdot, \triangle)$ such that $\triangle(a \cdot b) = \triangle(a) \cdot \triangle(b)$

Two operations on permutations:

shuffle $12 ⧢ 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

convol. $12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$

| | Malvenuto–Reutenauer | $\supseteq$ | Loday–Ronco |
|---|---|---|---|
| vector space | $\langle\, \mathbb{F}_\sigma \mid \sigma$ permutation of any size $\rangle$ | | $\langle\, \mathbb{P}_T \mid T$ binary tree of any size $\rangle$ |
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| coproduct | $\triangle(\mathbb{F}_\tau) = \sum_{\tau \in \rho \star \sigma} \mathbb{F}_\rho \otimes \mathbb{F}_\sigma$ | | $\triangle(\mathbb{P}_T) = \sum_{R_1 \cdots R_k \Vert S \text{ cut of } T} \Big( \prod_{i \in [k]} \mathbb{P}_{R_i} \Big) \otimes \mathbb{P}_S$ |

Hopf subalgebra $=$ define $\mathbb{P}_T = \sum_\tau \mathbb{F}_\tau$ over all permutations $\tau$ in the BST fiber of $T$

# A WALK THROUGH THE MANUSCRIPT

Objective: Explore further the interactions

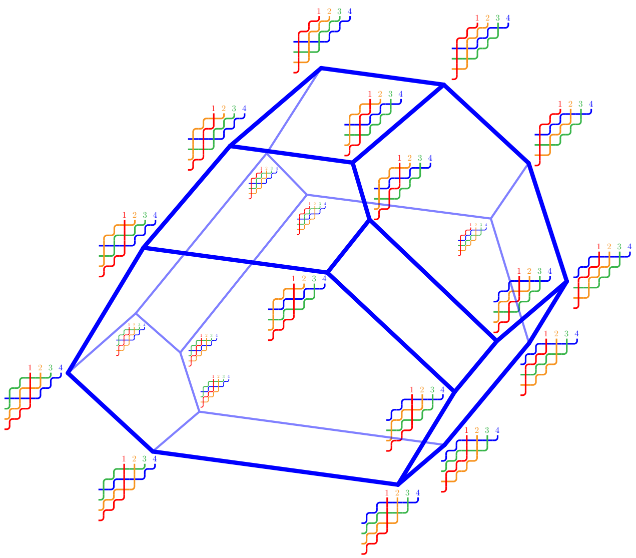
| | combinatorics | geometry | algebra |
|---|---|---|---|
| permutations | weak order | permutahedron $\mathbb{P}\mathsf{erm}(n)$ | MR Hopf algebra |
| binary trees | Tamari lattice | associahedron $\mathbb{A}\mathsf{sso}(n)$ | LR Hopf algebra |
| binary sequences | boolean lattice | parallelepiped $\mathbb{P}\mathsf{ara}(n)$ | recoil Hopf algebra |

# PART I. LATTICE CONGRUENCES, POLYTOPES AND HOPF ALGEBRAS

Objective: Explore further the interactions

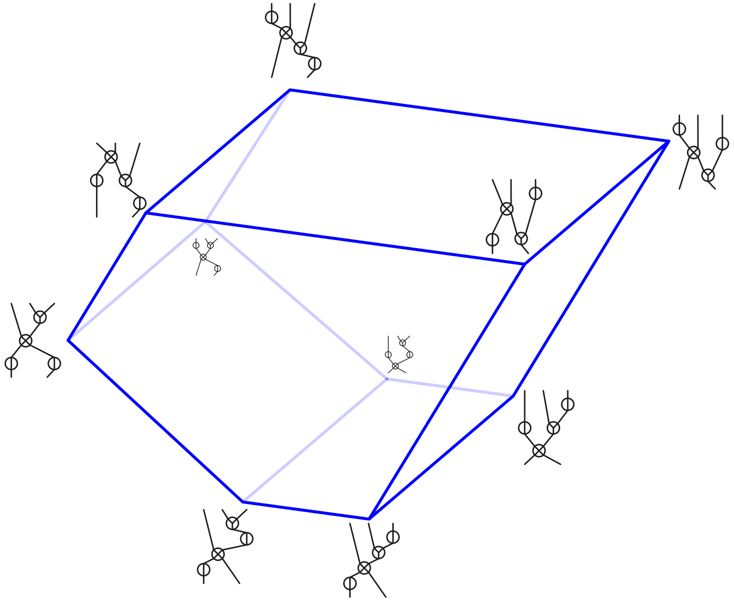
| | combinatorics | geometry | algebra |
|---|---|---|---|
| permutations | weak order | permutahedron $\mathbb{P}\mathrm{erm}(n)$ | MR Hopf algebra |
| binary trees | Tamari lattice | associahedron $\mathbb{A}\mathrm{sso}(n)$ | LR Hopf algebra |
| binary sequences | boolean lattice | parallelepiped $\mathbb{P}\mathrm{ara}(n)$ | recoil Hopf algebra |

## Chap 2. Brick polytopes



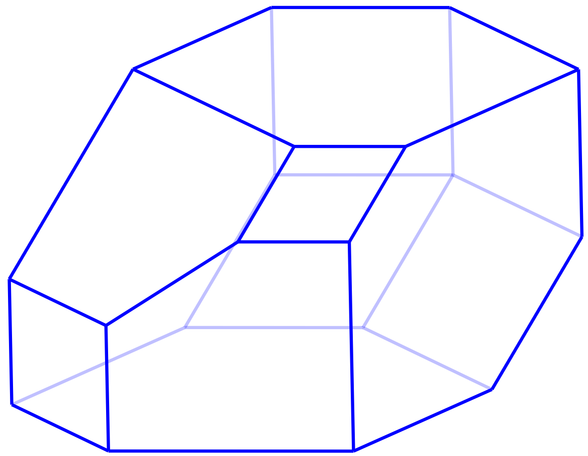
P.–Santos ('12)
P. ('16)

## Chap 3. Permutreehedra



P.–Pons ('18)
Albertin–P.–Ritter ($20^+$)

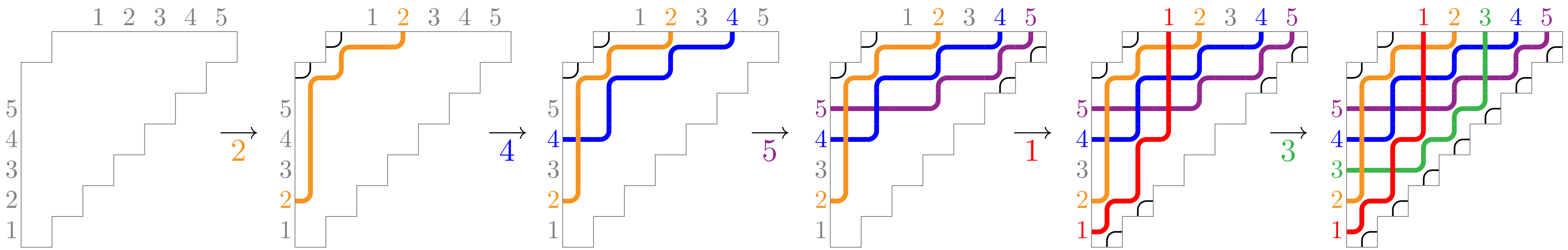
## Chap 4. Quotientopes



P.–Santos ('19) P. ('19)
Padrol–P.–Ritter ('$20^+$)

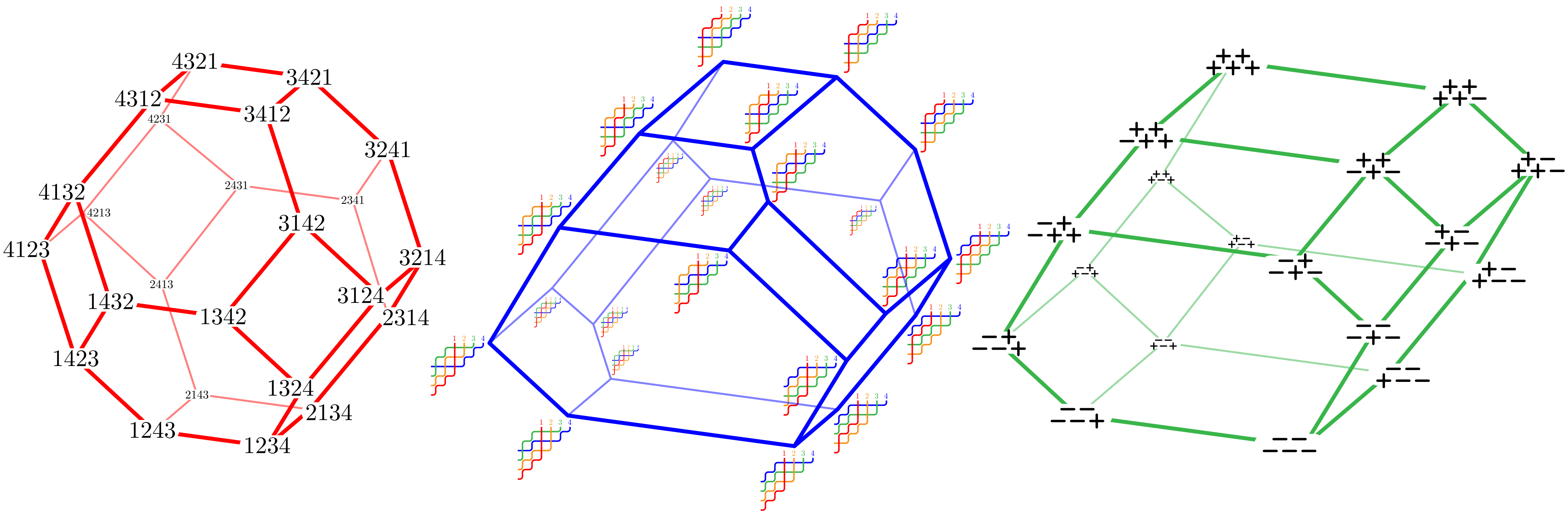
$(k,n)$-twist $=$ pipe dream that sends $i \mapsto \begin{cases} i \text{ if } k+1 \le i \le k+n, \\ n+2k+1-i \text{ otherwise.} \end{cases}$

$k$-twist insertion $=$ permutations of $[n] \longrightarrow$ acyclic $(k,n)$-twists

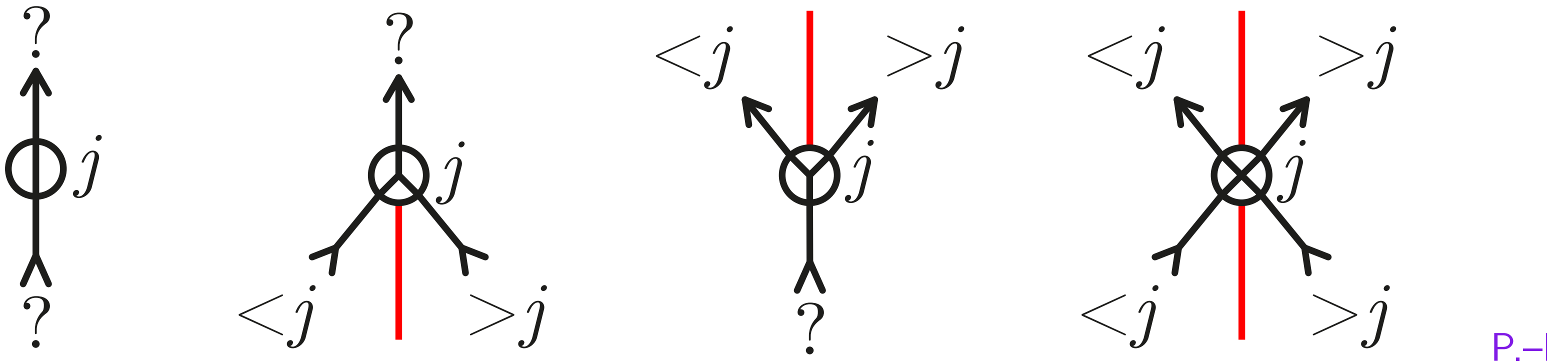



$k$-twist cong. $=$ fibers $k$-twist insertion

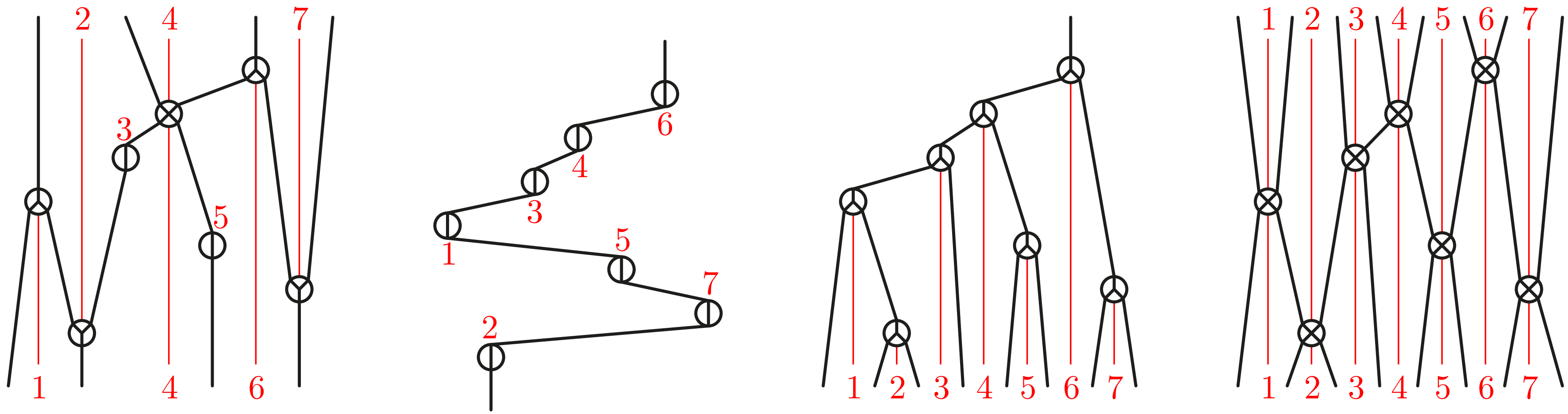
$= $ rewriting rule $UacV_1b_1 \dots V_kb_kW \equiv_k UcaV_1b_1 \dots V_kb_kW$ with $a < b_i < c$



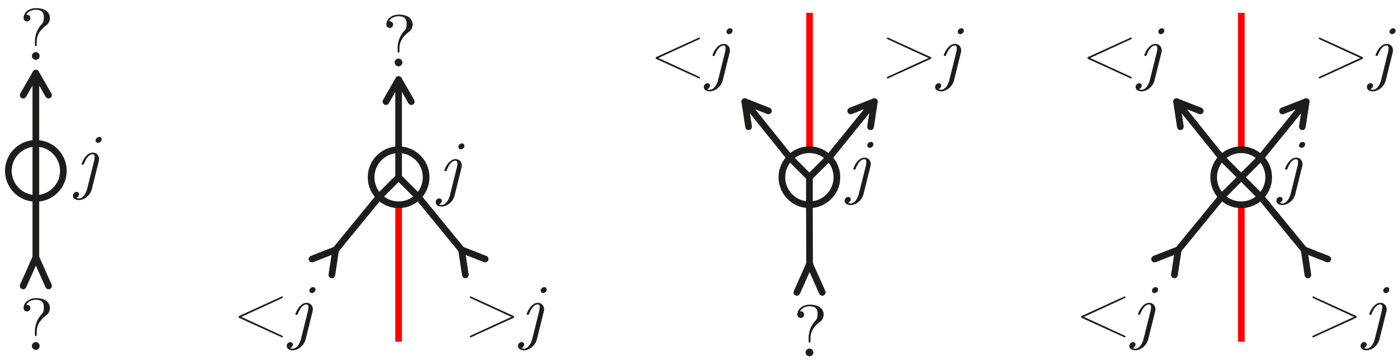
permutree $=$ directed (bottom to top) and labeled (bijectively by $[n]$) tree such that




P.–Pons ('18)

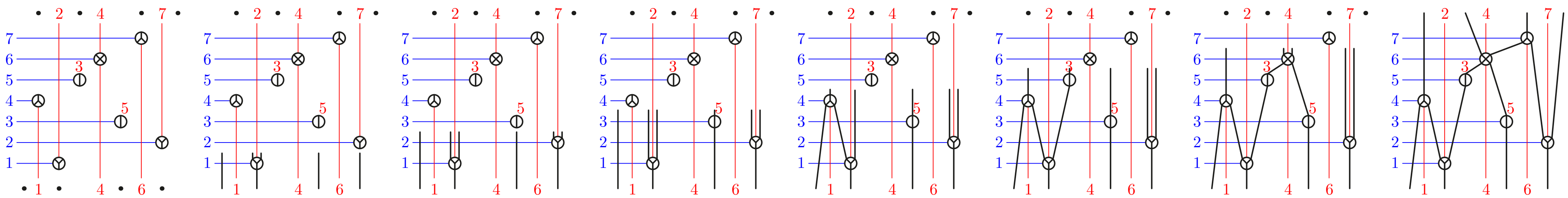



generic permutation binary tree binary sequence permutree $=$ directed (bottom to top) and labeled (bijectively by $[n]$) tree such that




P.–Pons ('18)

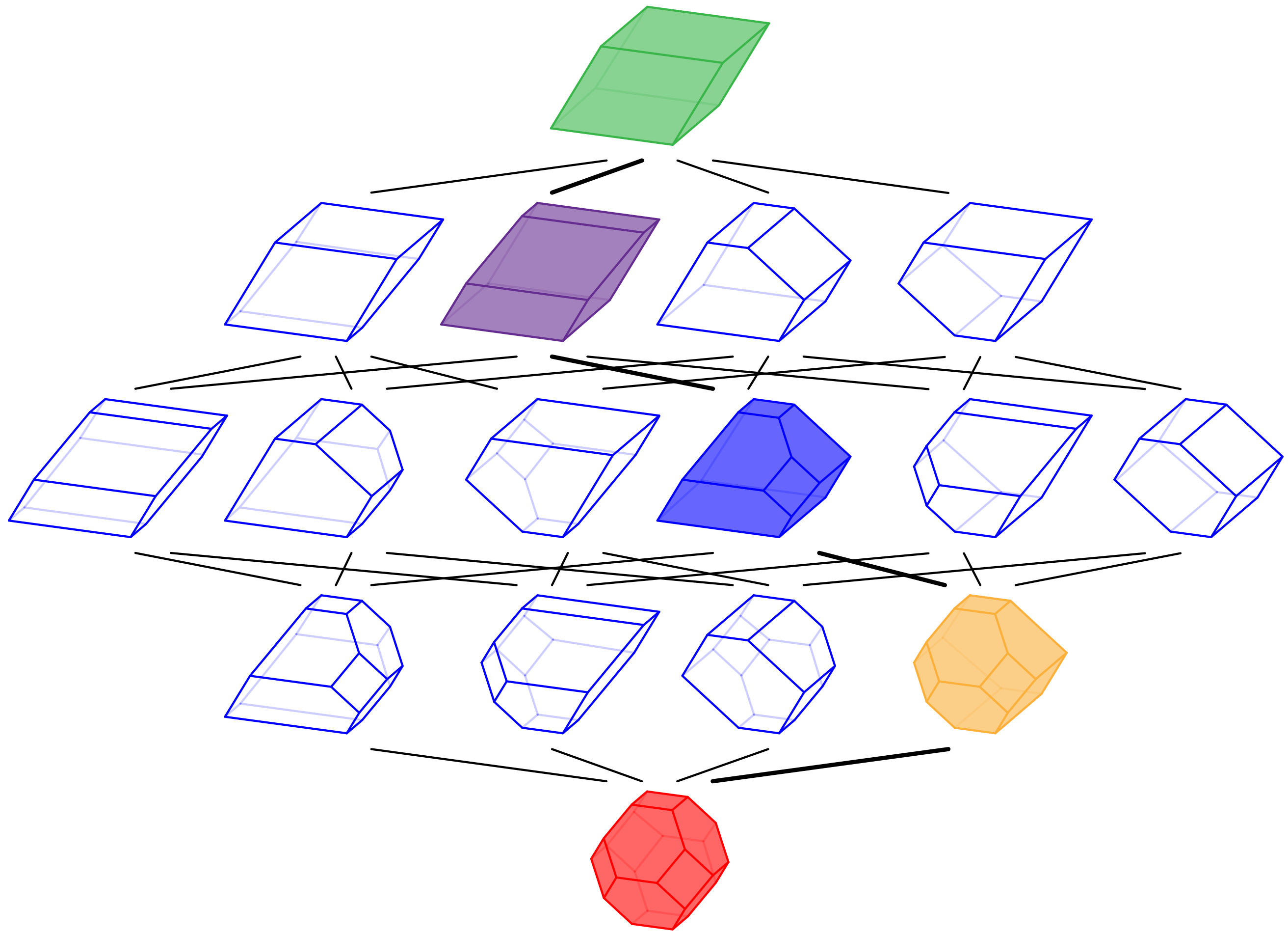
$\delta$-permutree insertion of $2751346$



$\delta$-permutree cong. $=$ fibers $\delta$-permutree insertion

$=$ rewriting rules $UacVbW \equiv_\delta UcaVbW$ if $\delta_b \in \{☮, \otimes\}$

$UbVacW \equiv_\delta UbVcaW$ if $\delta_b \in \{Ⓨ, \otimes\}$

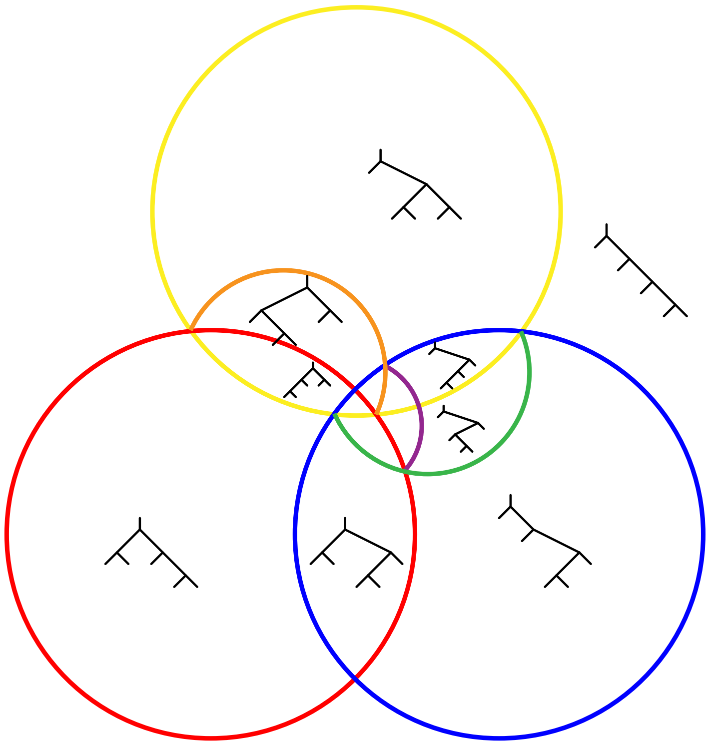


P.–Pons ('18)

lattice congruence = equivalence relation on $L$ compatible with meets and joins:

$$x \equiv x' \text{ and } y \equiv y' \text{ implies } x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

quotient fan $\mathcal{F}_{\equiv}$ = chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations $\sigma$ in the same congruence class of $\equiv$ Reading ('05)



# CHAP 4. QUOTIENTOPES

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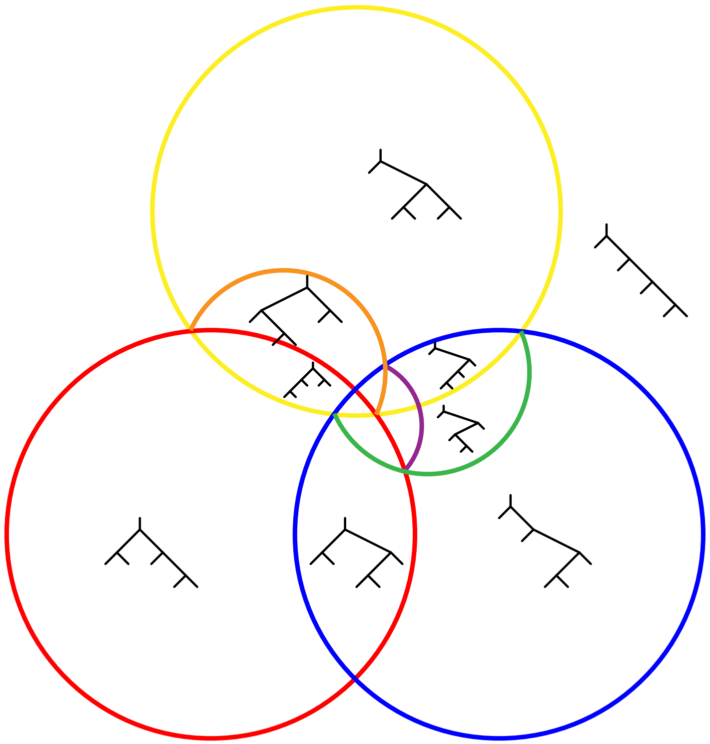
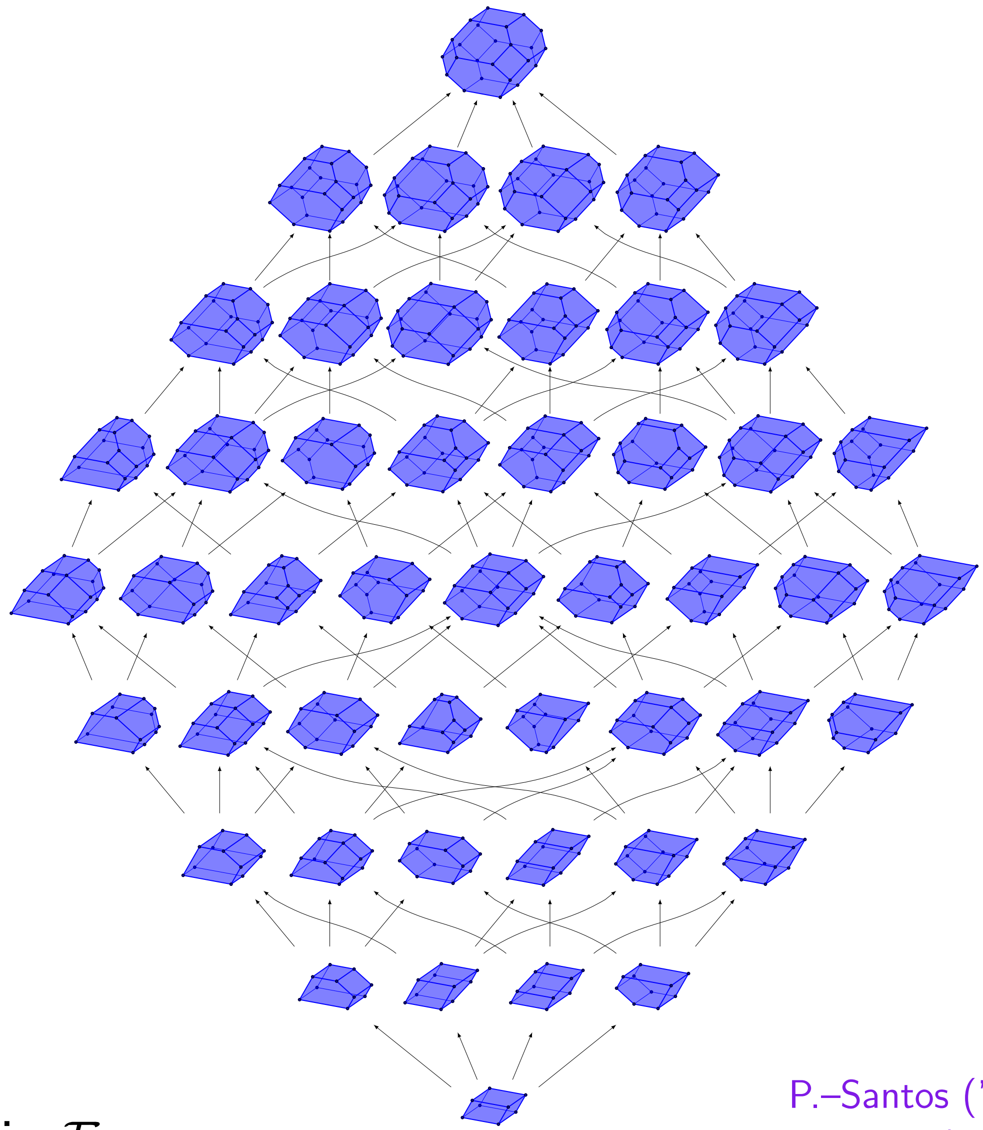




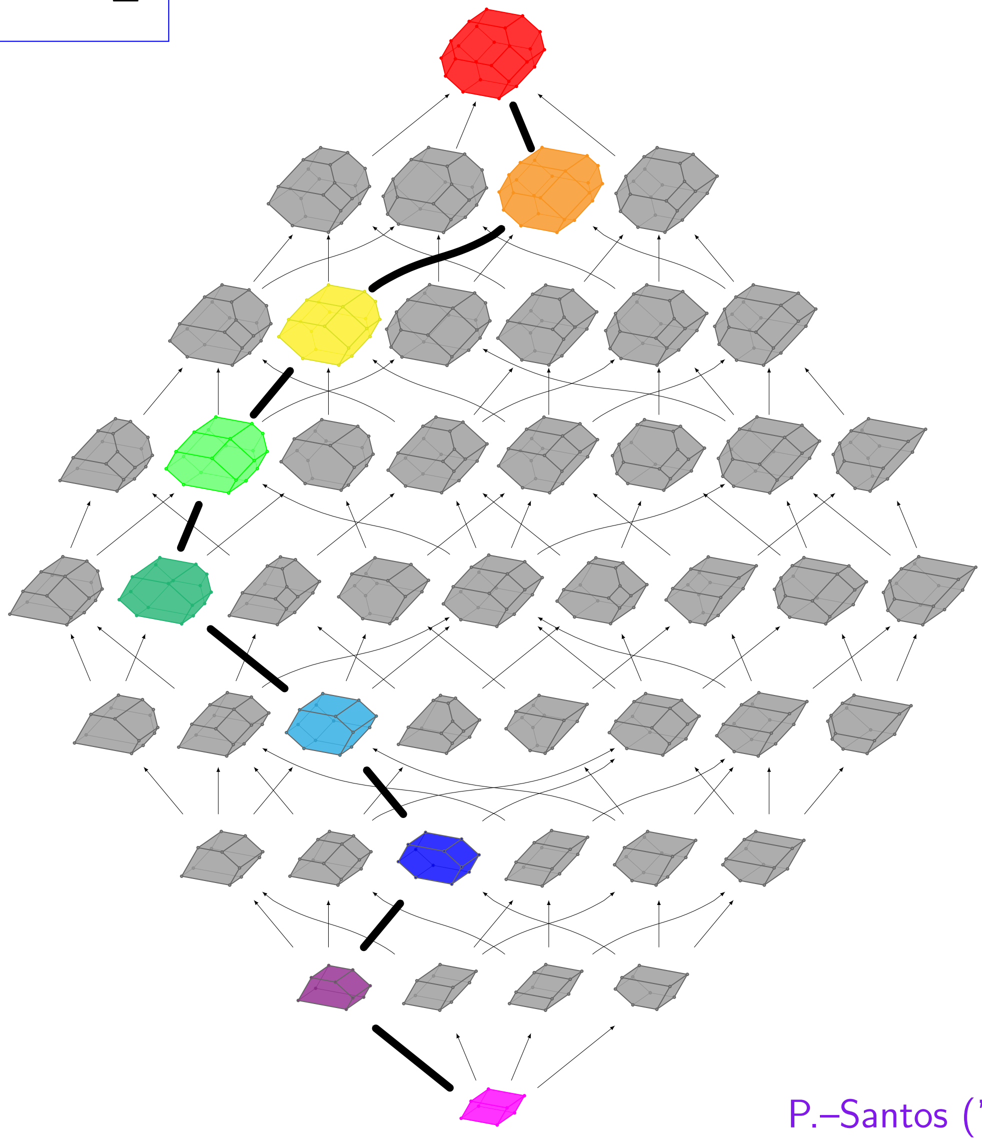
P.–Santos ('19)

Padrol–P.–Ritter ('20$^{+}$)

quotientope = polytope whose normal fan is $\mathcal{F}_{\equiv}$

# QUOTIENTOPES

quotientope = polytope whose normal fan is $\mathcal{F}_{\equiv}$



P.–Santos ('19)

# PART II. BEYOND THE WEAK ORDER

Objective: Extend the weak order beyond the vertices of the permutahedron

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## Chap 5. Facial weak order

Krob–Latapy–Novelli–Phan–Schwer ('01) Palacios–Ronco ('06)

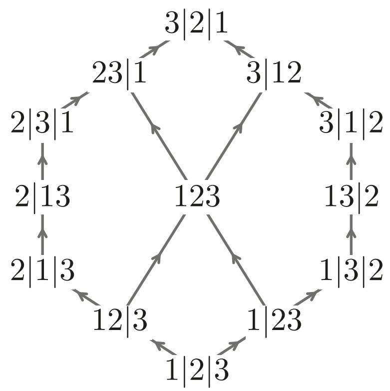
facial weak order = lattice on all faces of $\mathbb{P}\mathrm{erm}(W)$

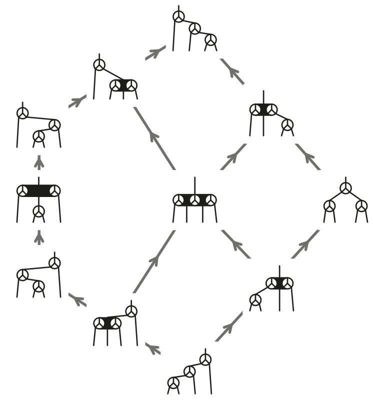
$F \leq G \iff \min F \leq \min G$ and $\max F \leq \max G$

facial lattice congruence = congruence on faces

$F \equiv G \iff \min F \equiv \min G$ and $\max F \equiv \max G$

Dermenjian–Hohlweg–McConville–P. ('18, '19$^+$)



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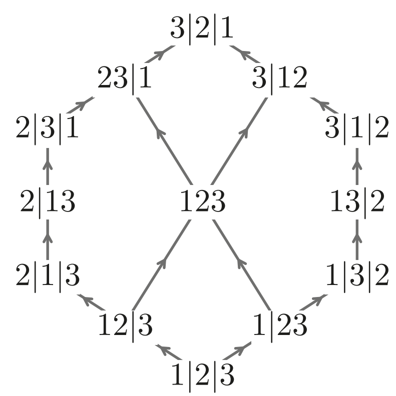
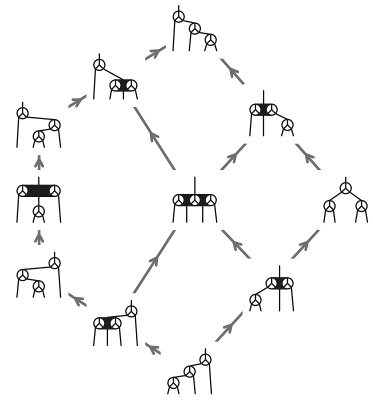
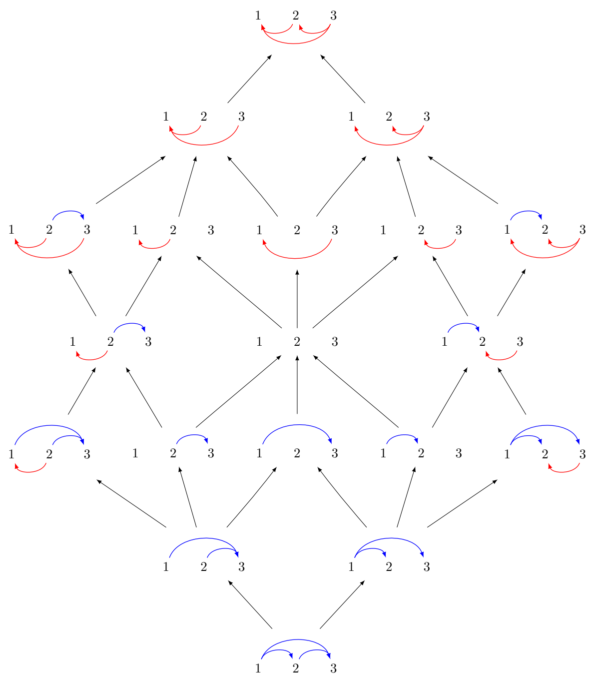




## Chap 6. Weak order on integer posets

integer poset = poset on $[n]$

weak order on integer posets = Chatel–P.–Pons ('19)

$\lhd \le \blacktriangleleft \iff \lhd^{-} \subseteq \blacktriangleleft^{-}$ and $\lhd^{+} \supseteq \blacktriangleleft^{+}$

Hopf algebra on integer posets P.–Pons ('20)

weak order on $\Phi$-posets Gay–P. ('19)

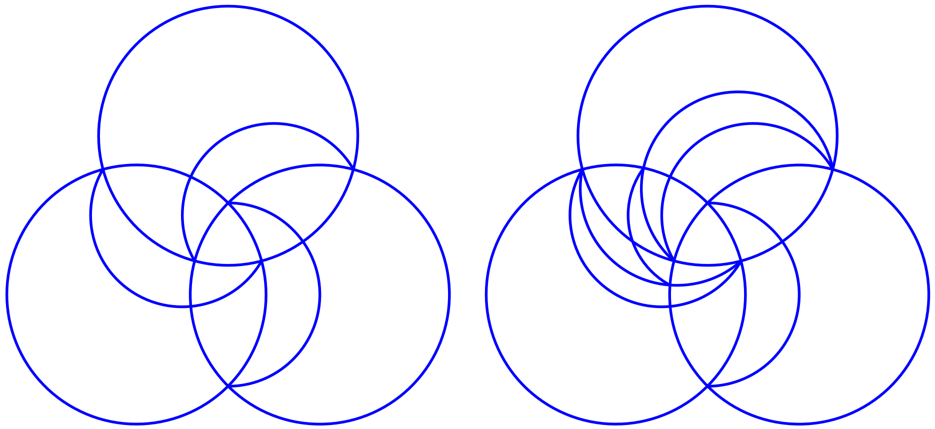
# PART III. CLUSTER ALGEBRAS AND GENERALIZED ASSOCIAHEDRA

cluster complex = simplicial complex constructed from an iterative process of mutations

finite type classification by Weyl groups

$\boldsymbol{g}$-vector fan = fan associated to an initial cluster seed, realizing the cluster complex

Fomin–Zelevinsky ('02, '03, '05, '07)



Objective: Construct polytopal realizations of $\boldsymbol{g}$-vector fans of finite type cluster alg.

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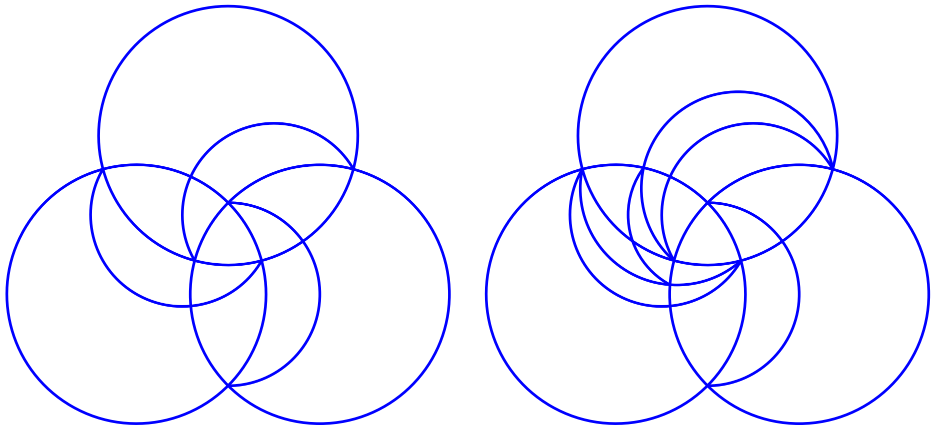


Objective: Construct polytopal realizations of $\boldsymbol{g}$-vector fans of finite type cluster alg.

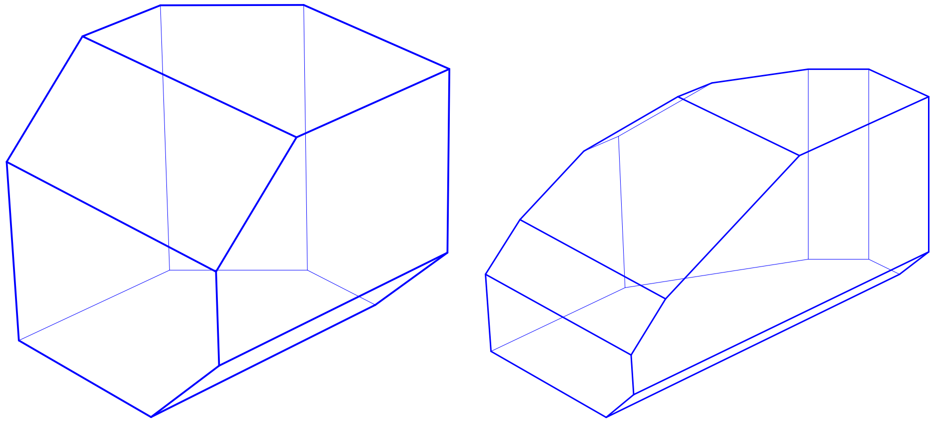
## Chap 7. Polytopal realizations of finite type $\boldsymbol{g}$-vector fans

universal associahedron = polytope whose normal fan contains a copy of each $\boldsymbol{g}$-vector fan

Hohlweg–P.–Stella ('18)

type cone = space of all polytopal realizations

Padrol–Palu–P.–Plamondon ('19$^{+}$)



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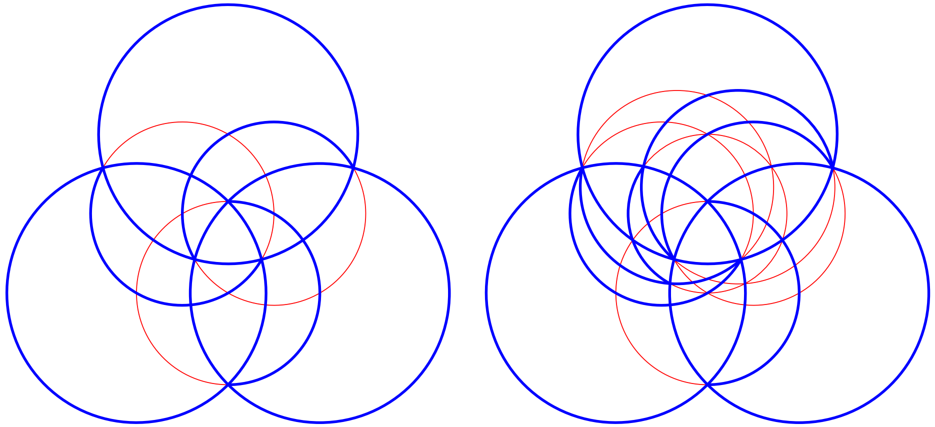

Objective: Construct polytopal realizations of $g$-vector fans of finite type cluster alg.

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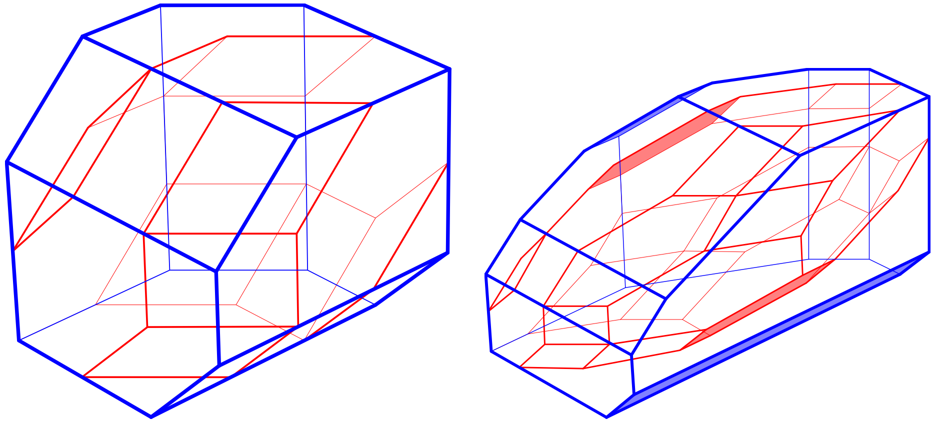

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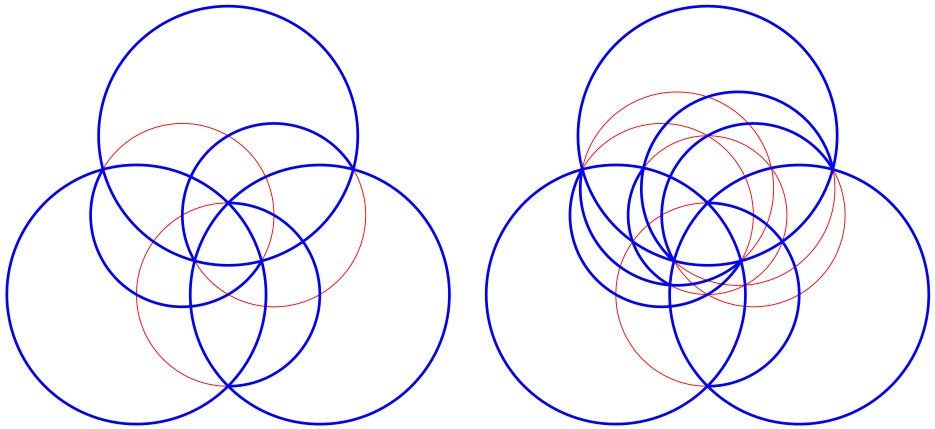

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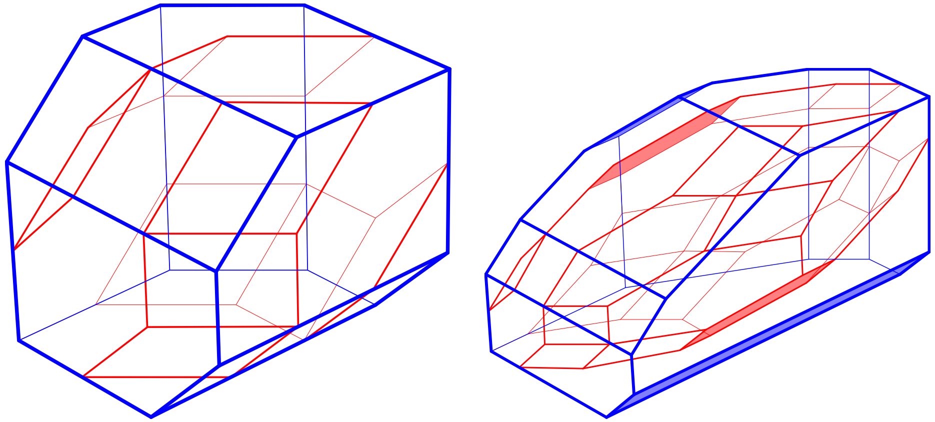


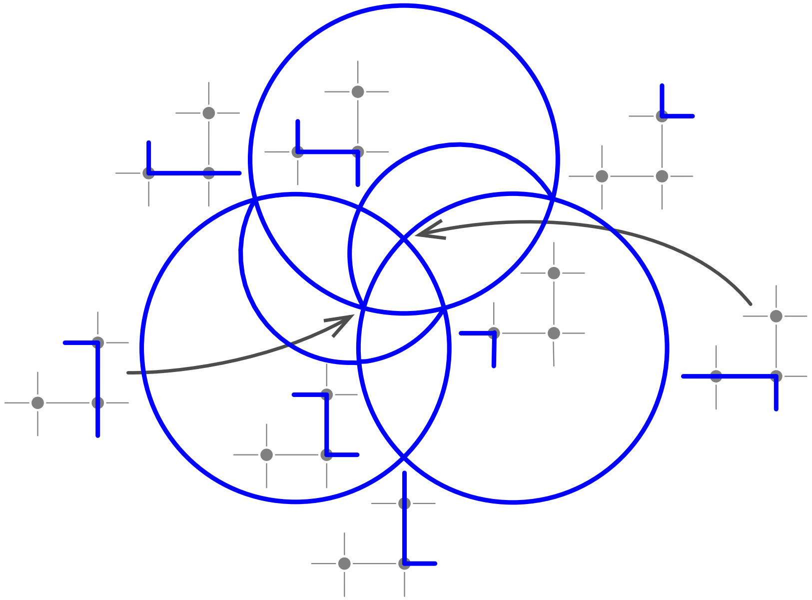
## Chap 8. Brick polytopes of subword complexes

subword complex = generalization of pipe dreams and sorting networks to Coxeter groups

brick polytope = polytope realizing only acyclic facets

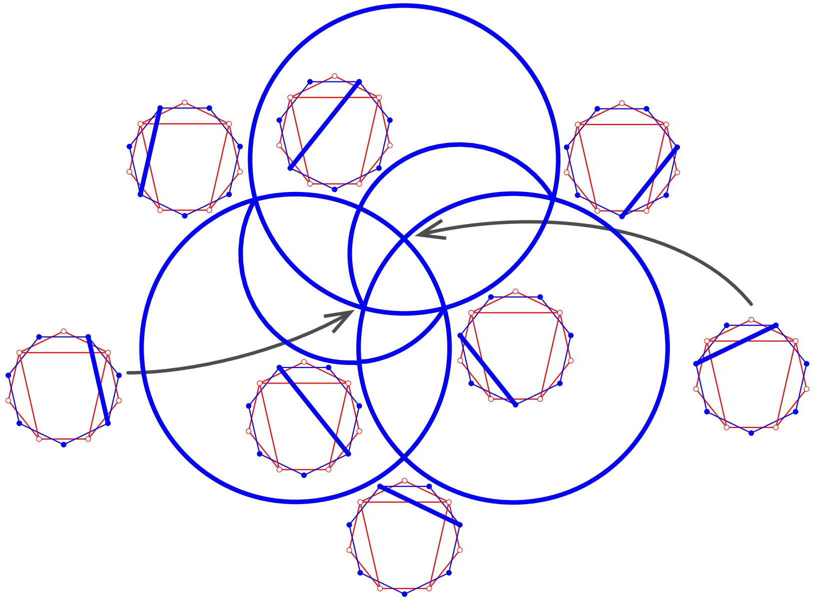
P.–Stump ('15a, '15b)

Two recent generalizations of the associahedron:



Non-kissing complex
of paths on a grid

McConville ('17) Garver–McConville ('17$^+$)



Non-crossing complex
on accordions of a dissection

Baryshnikov ('01) Chapoton ('16)
Garver–McConville ('18)

Objective:
- Explain the connections between non-kissing and non-crossing
- Develop combinatorial and geometric properties of these complexes

Palu–P.–Plamondon ('19)

Objective: • Explain the connections between non-kissing and non-crossing
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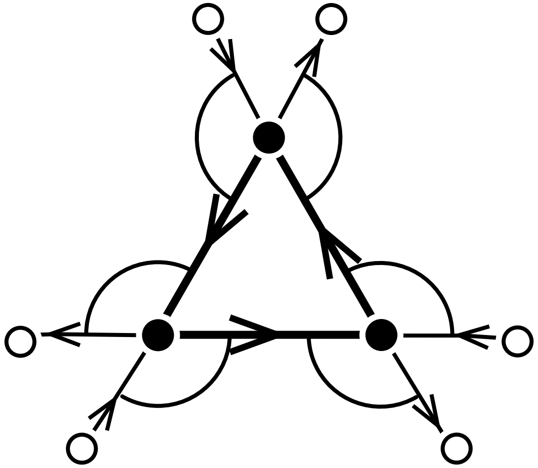
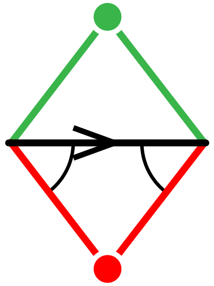
## Chap 9. Non-kissing versus non-crossing

Palu–P.–Plamondon ('19)

locally gentle quiver ⟷ orientable surface with boundary endowed with a pair of dual cellular dissections





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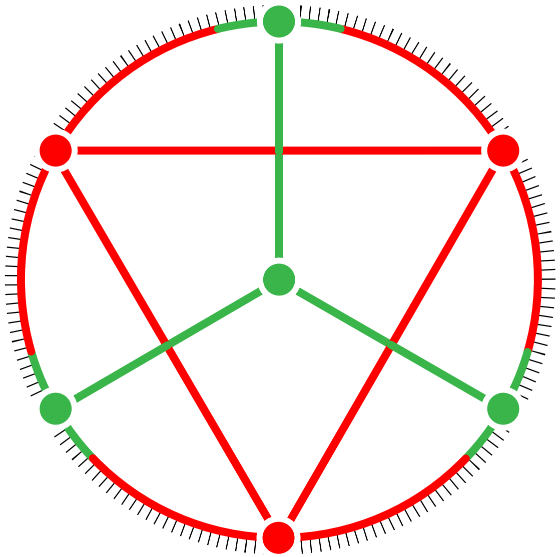

# PART IV. NON-KISSING AND NON-CROSSING COMPLEXES

Objective:
- Explain the connections between non-kissing and non-crossing
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## Chap 9. Non-kissing versus non-crossing

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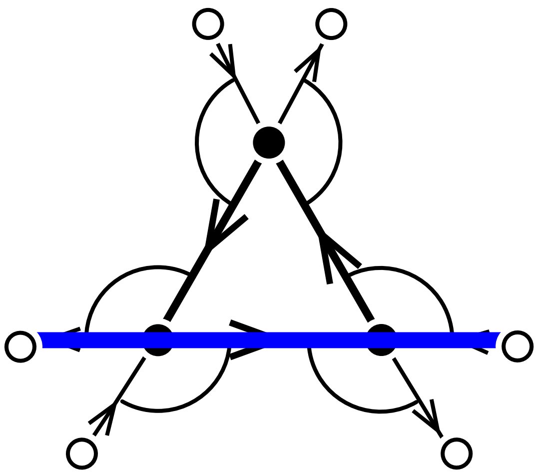

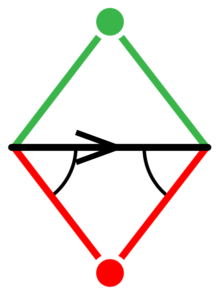

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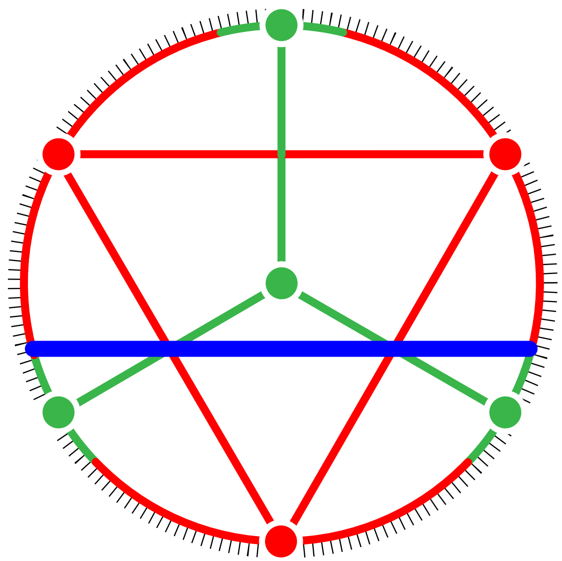

walks ⟷ accordeon or slalom

non-kissing complex ⟷ non-crossing complex

# PART IV. NON-KISSING AND NON-CROSSING COMPLEXES

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## Chap 9. Non-kissing versus non-crossing

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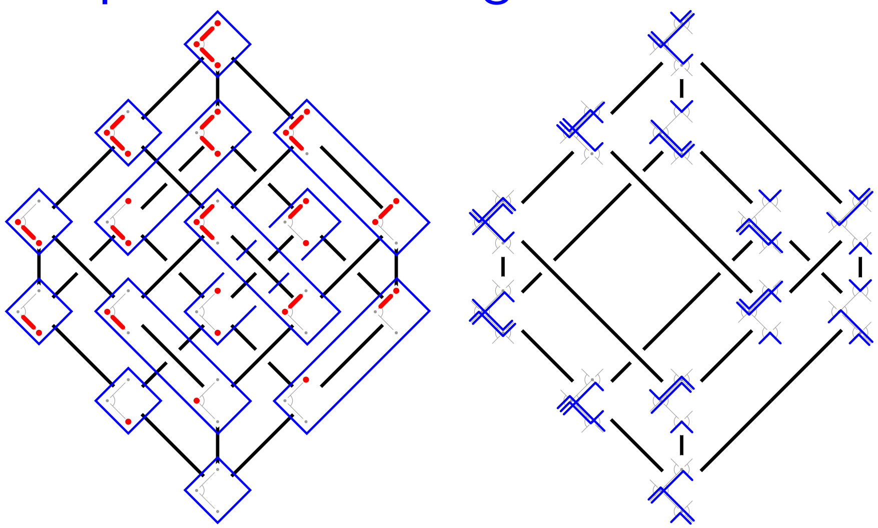
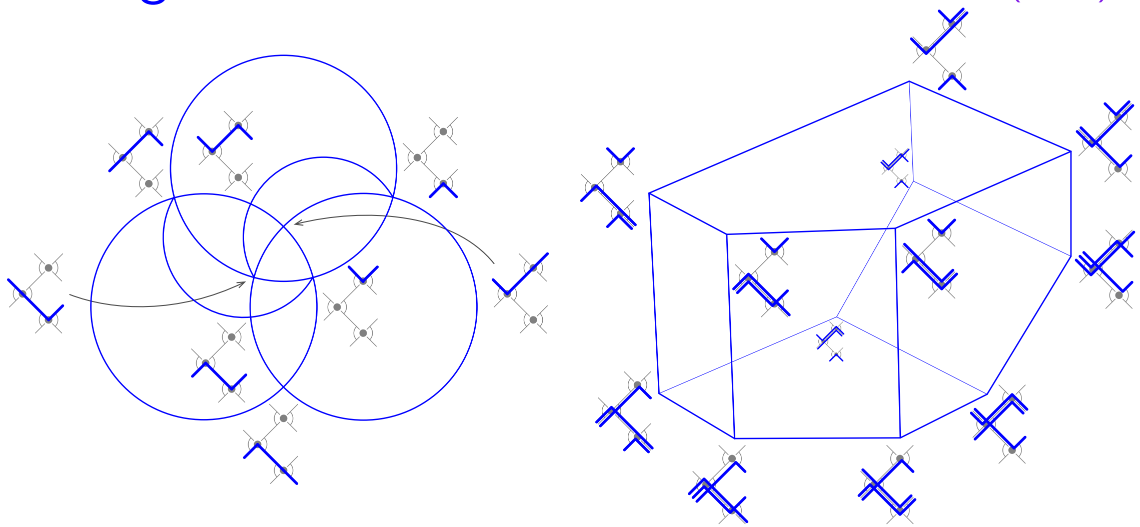
locally gentle quiver $\longleftrightarrow$ orientable surface with boundary endowed with a pair of dual cellular dissections

non-kissing complex $\longleftrightarrow$ non-crossing complex

## Chap 10. Non-kissing lattices and non-kissing associahedra

Palu–P.–Plamondon ('17$^+$)





non-kissing lattice
= quotient of a lattice of biclosed sets

non-kissing associahedron
= polytopal realization of the $\boldsymbol{g}$-vector fan

# FOCUS ON A RANDOM PAGE

Today we will focus on page...

# FOCUS ON A RANDOM PAGE

Today we will focus on page...

# FOCUS ON A RANDOM PAGE

## Today we will focus on page...



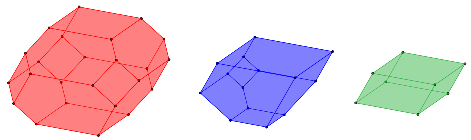


Figure 4.4: Permutahedron (left), associahedron (middle) and cube (right) as quotientopes.

**Theorem 4.8.** *For any lattice congruence $\equiv$ of the weak order on $\mathfrak{S}_n$, and any forcing dominant function $f : \mathcal{A}_n \to \mathbb{R}_{>0}$, the quotient fan $\mathcal{F}(\equiv)$ is the normal fan of the polytope*

$$\mathbb{QT}^f(\equiv) := \big\{ \boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{r}(R) \,|\, \boldsymbol{x} \rangle \le h^f_{\equiv}(R) \text{ for all } \varnothing \neq R \subsetneq [n] \big\}.$$

*In particular, when oriented in the direction $\boldsymbol{\omega} := (n, \dots, 1) - (1, \dots, n) = \sum_{i \in [n]} (n + 1 - 2i)\, \boldsymbol{e}_i$, the graph of $\mathbb{QT}^f(\equiv)$ is the Hasse diagram of the quotient of the weak order by $\equiv$.*

**Remark 4.9.** Note that the definition of the height function ensures that $h^f_{\equiv}(R) \le h^f_{\equiv'}(R)$ and thus $\mathbb{QT}^f(\equiv) \subseteq \mathbb{QT}^f(\equiv')$ when $\equiv$ coarsens $\equiv'$. See Figure 4.5.

### 4.2.3 Minkowski sums of associahedra or shard polytopes

We conclude with an alternative approach to quotientopes recently developed in [PPR20] to study the polytopality of quotient fans beyond the braid arrangement (see also Section A.3).

**Lemma 4.10.** *For any lattice congruence $\equiv$ of the weak order, the quotient fan $\mathcal{F}(\equiv)$ is the common refinement of the quotient fans $\mathcal{F}(\equiv_1), \dots, \mathcal{F}(\equiv_p)$ of the lattice congruences whose arc ideals $\mathcal{I}_{\equiv_1}, \dots, \mathcal{I}_{\equiv_p}$ are the principal upper ideals of the forcing order generated by the minimal elements of the arc ideal $\mathcal{I}_{\equiv}$ of $\equiv$.*

**Lemma 4.11.** *An arc ideal is principal if and only if it corresponds to a Cambrian congruence (possibly of low dimension).*

**Corollary 4.12.** *For any lattice congruence $\equiv$ of the weak order, the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of a Minkowski sum of associahedra.*

In fact, this idea can even been pushed further to obtain realizations of all quotientopes (including associahedra) as Minkowski sums of elementary summands, defined as follows.

**Definition 4.13.** *For an arc $\alpha = (a, b, n, S)$, we define*

- *an $\alpha$-alternating matching as a (possibly empty) sequence $M = \{a_1, b_1, \dots, a_k, b_k\}$ where $a \le a_1 < b_1 < \dots < a_k < b_k \le b$ and $a_i \in S \cup \{a\}$ while $b_i \notin S$ for all $i \in [k]$.*
- *the characteristic vector of this $\alpha$-alternating matching as $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$,*
- *the shard polytope $\mathbb{SP}(\alpha)$ as the convex hull of the characteristic vectors of all $\alpha$-alternating matchings.*

**Proposition 4.14.** *For any arc $\alpha$, the union of the walls of the normal fan of the shard polytope $\mathbb{SP}(\alpha)$ contains the shard $\Sigma(\alpha)$ and is contained in the union of the shards $\Sigma(\beta)$ for the arcs $\beta$ forced by $\alpha$.*

**Corollary 4.15.** *For any lattice congruence $\equiv$ of the weak order, the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of the Minkowski sum of the shard polytopes $\mathbb{SP}(\alpha)$ over all $\alpha \in \mathcal{I}_{\equiv}$.*

**Example 4.16.** For the arc $\alpha = (a, b, n, ]a, b[)$, the $\alpha$-alternating matchings are given by $\varnothing$ and $\{i, b\}$ for $a \le i < b$, so that the corresponding shard polytope $\mathbb{SP}(\alpha)$ is the translation of the standard simplex $\triangle_{[a,b]}$ by the vector $-\boldsymbol{e}_b$. We obtain thus the classical realization of Loday's associahedron as the Minkowski sum of all faces of the standard simplex corresponding to the intervals of $[n]$.

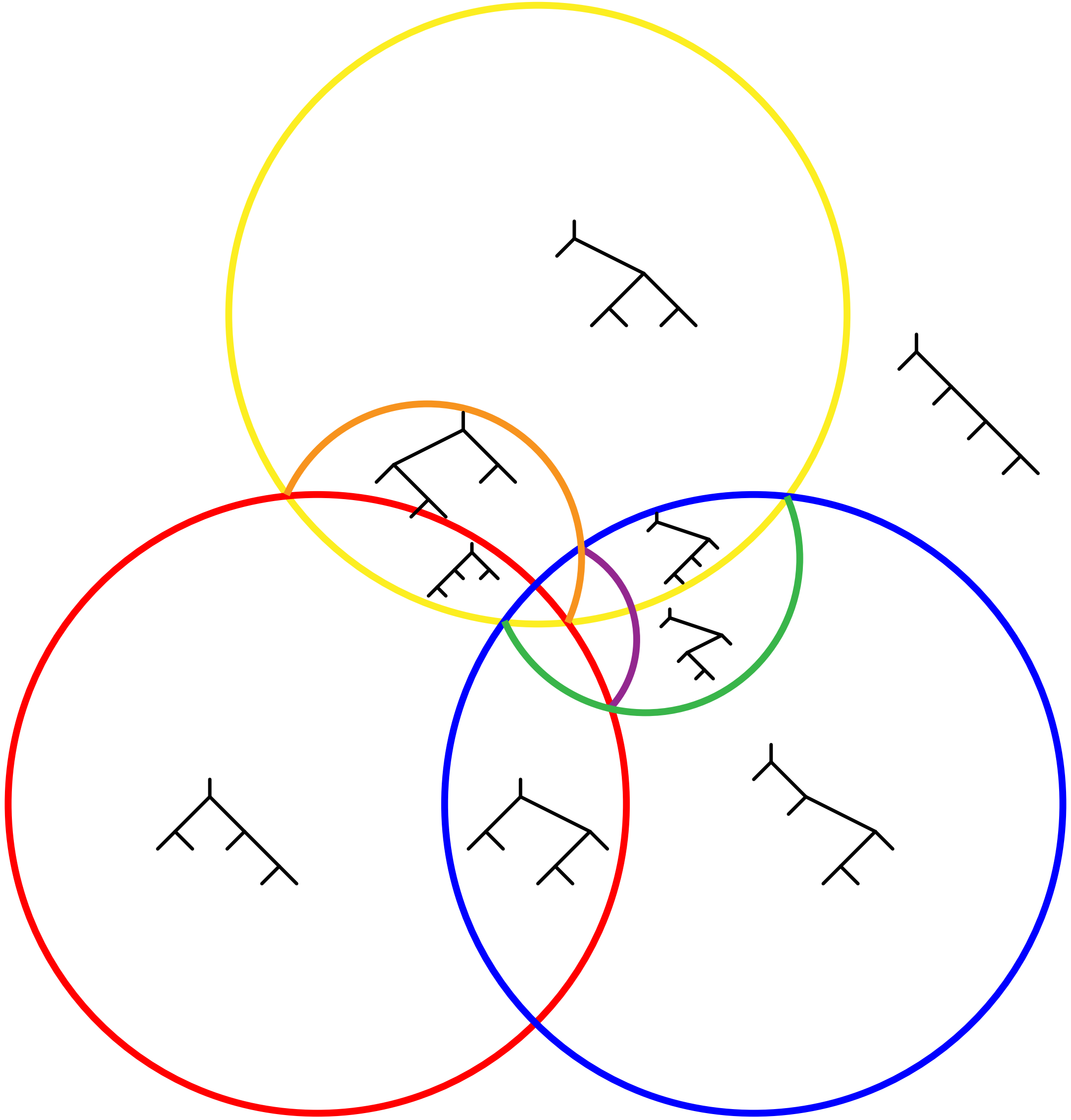
# SHARD POLYTOPES AND QUOTIENTOPES

# QUOTIENT FAN

lattice congruence = equivalence relation on $L$ compatible with meets and joins:

$$x \equiv x' \text{ and } y \equiv y' \text{ implies } x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

quotient fan $\mathcal{F}_\equiv$ = chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations $\sigma$ in the same congruence class of $\equiv$ Reading ('05)



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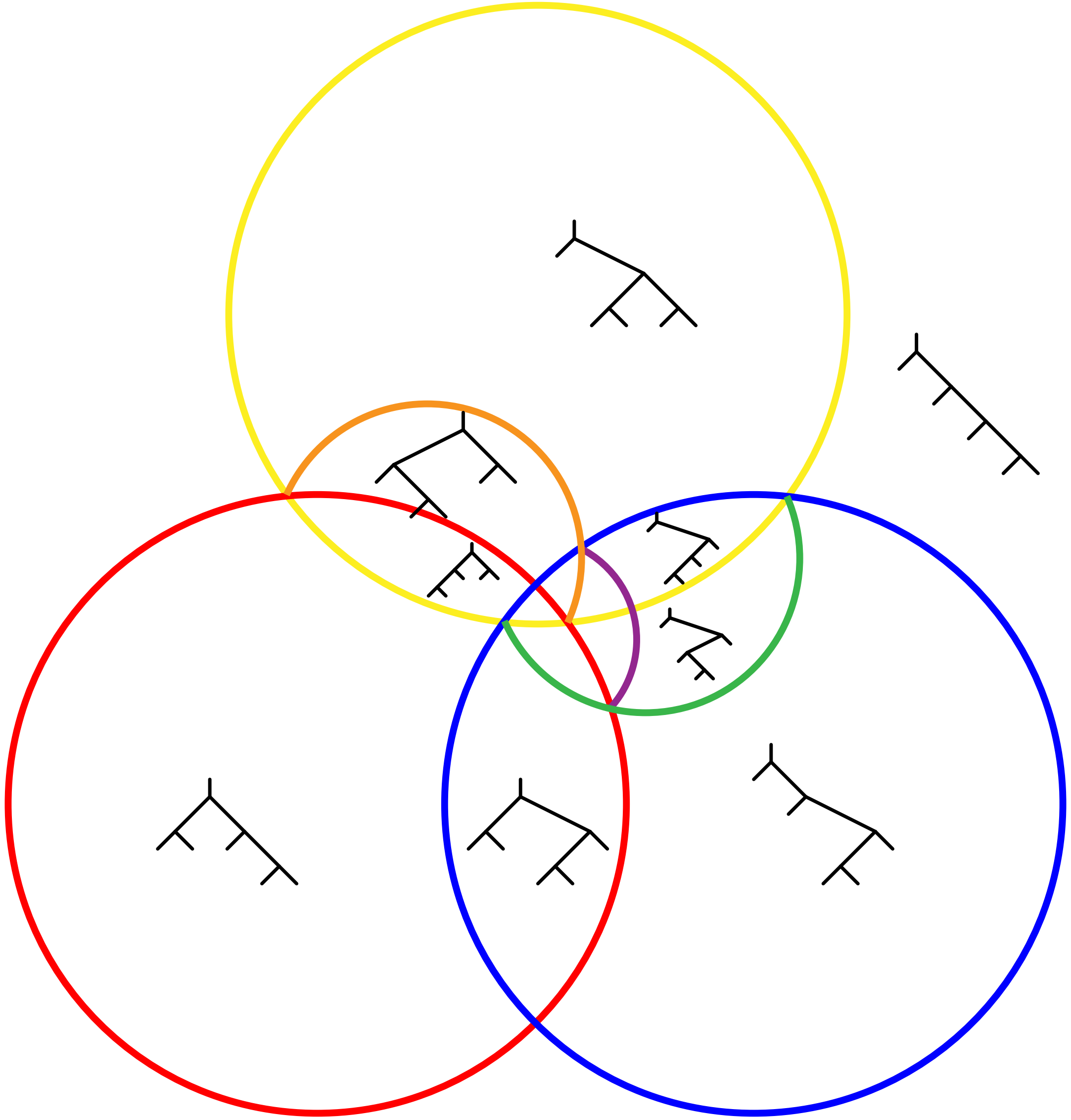
$\boldsymbol{W}_{\equiv}$ = walls of the quotient fan $\mathcal{F}_{\equiv}$

Describe the possible sets of walls $\boldsymbol{W}_{\equiv}$



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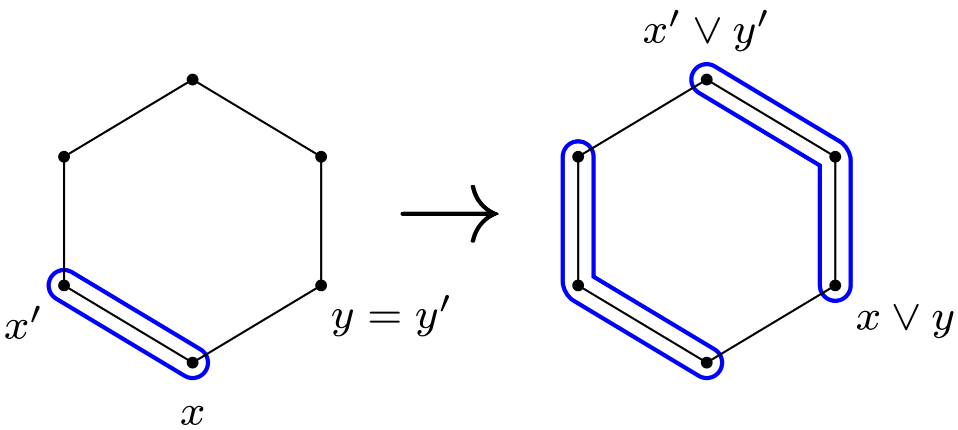
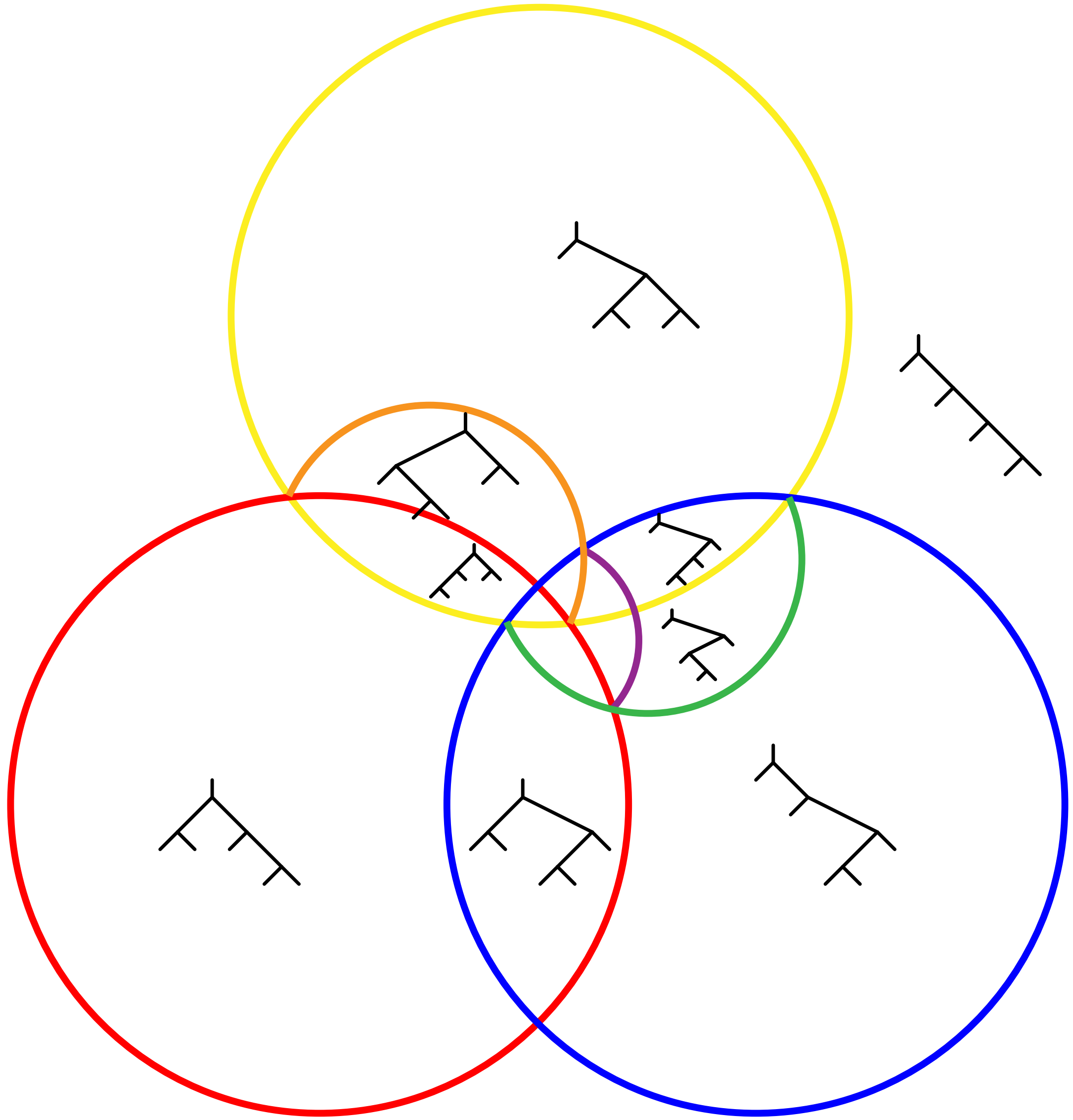
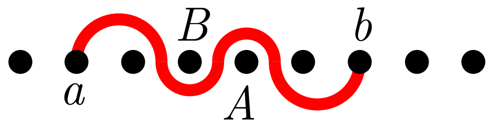


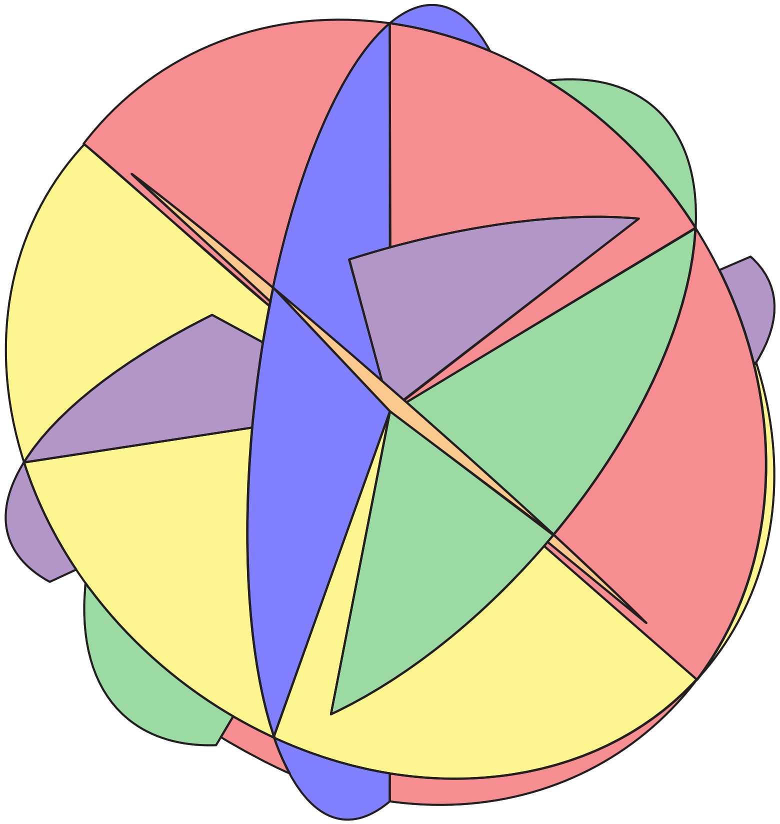



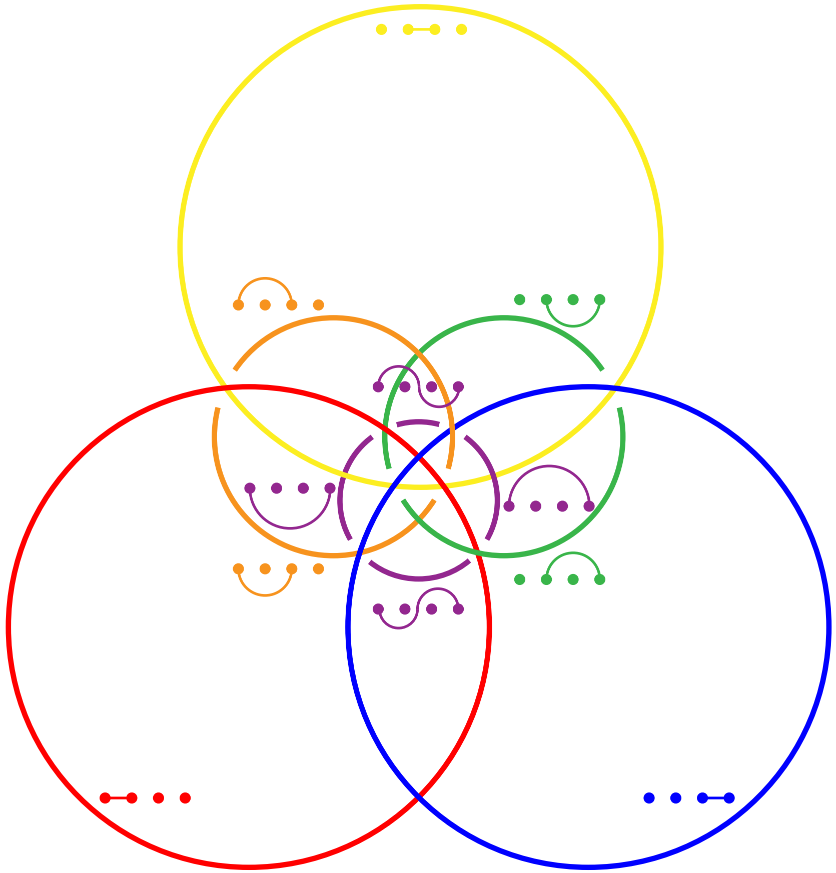
# ARCS AND SHARDS

<u>arc</u> $(a, b, A, B)$ with $1 \leq a < b \leq n$ and $A \sqcup B = \,]a, b[$




<u>shard</u> $\Sigma(a, b, A, B) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{a'} \leq x_a = x_b \leq x_{b'} \text{ for all } a' \in A \text{ and } b' \in B \right\}$





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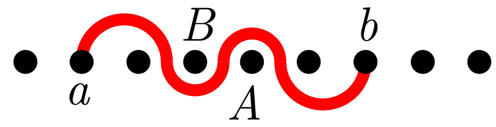


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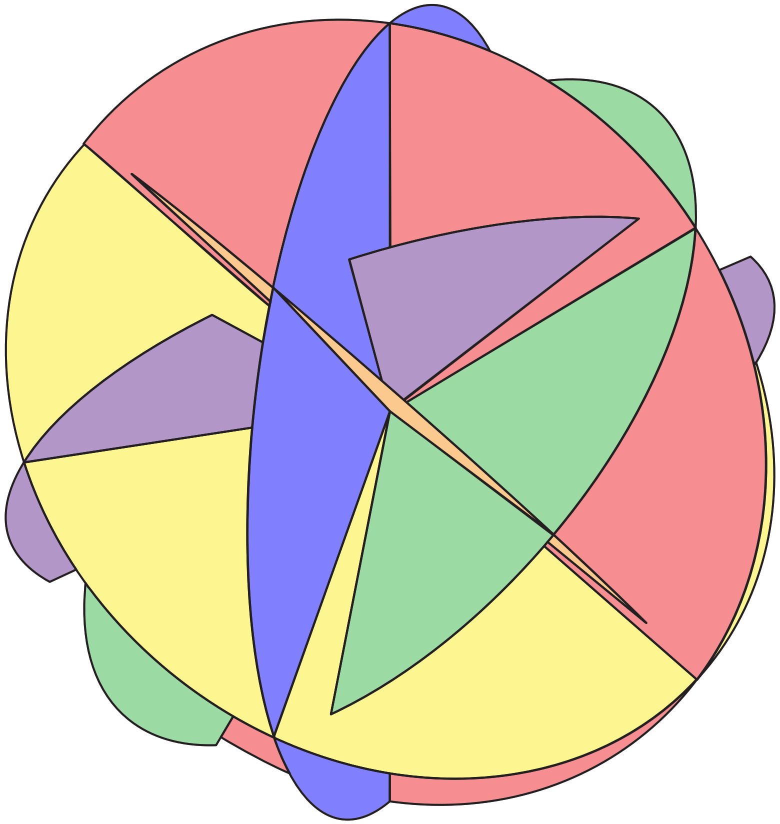

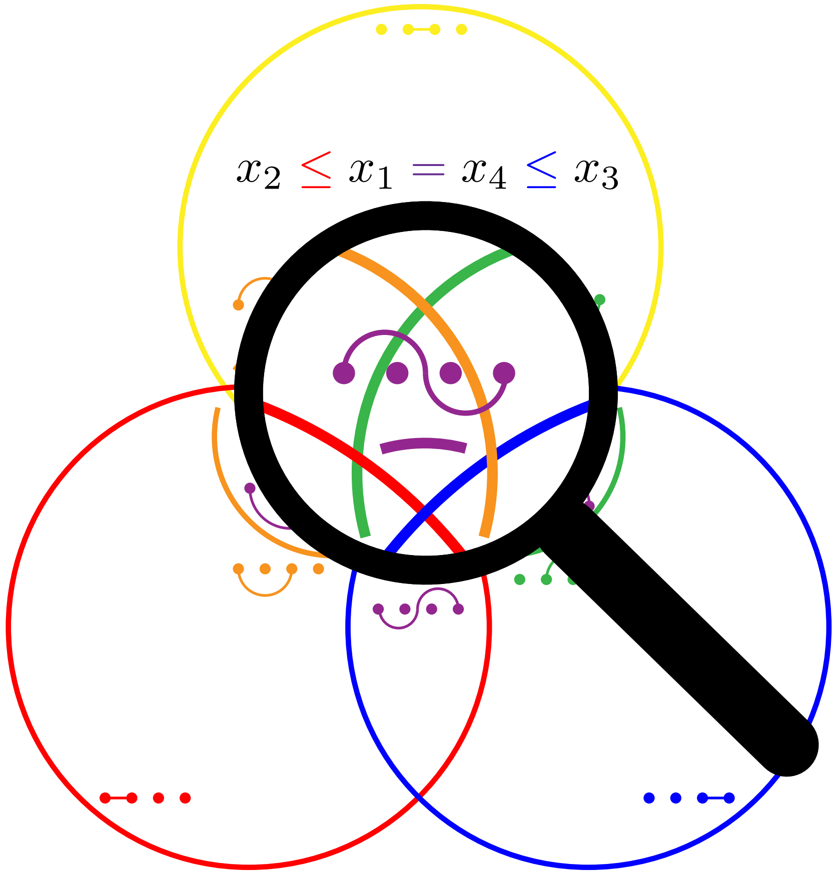

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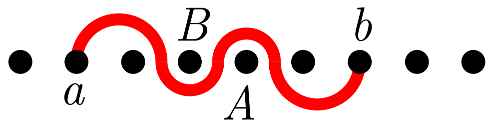
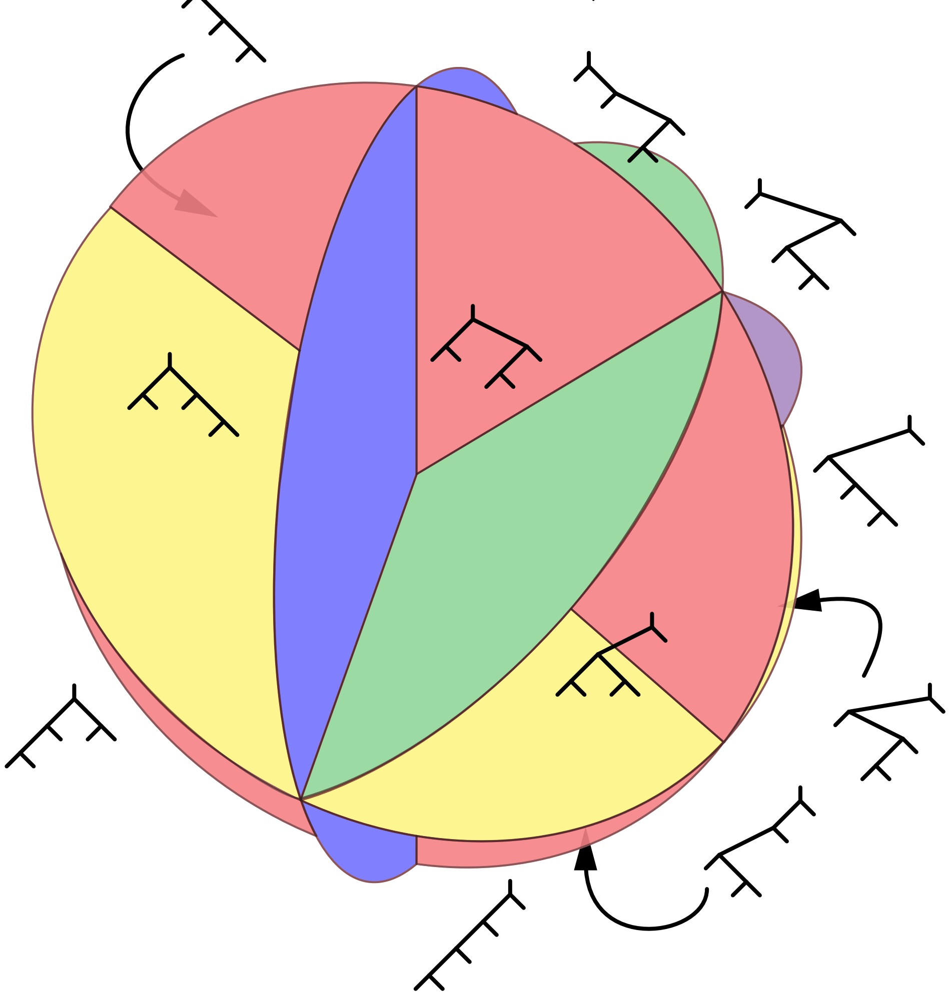
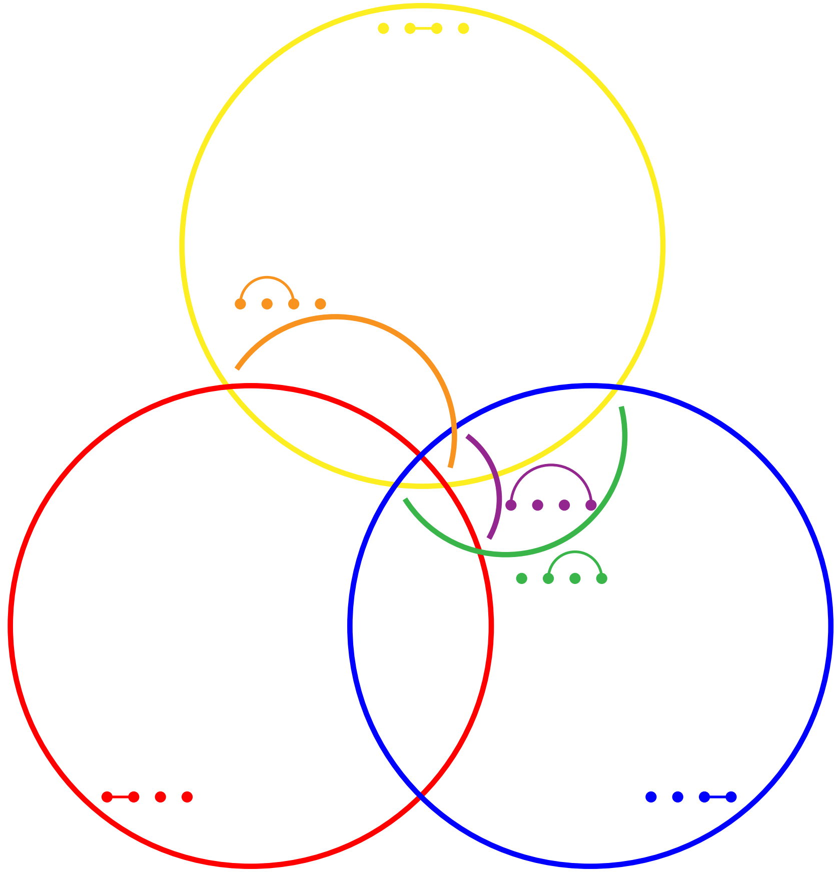


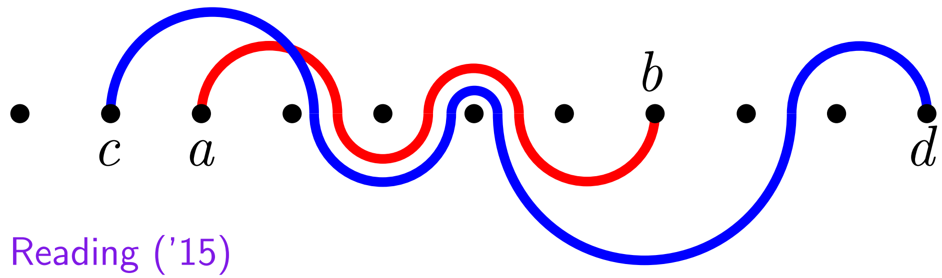

shard $\Sigma(a, b, A, B) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_{a'} \leq x_a = x_b \leq x_{b'} \text{ for all } a' \in A \text{ and } b' \in B \}$



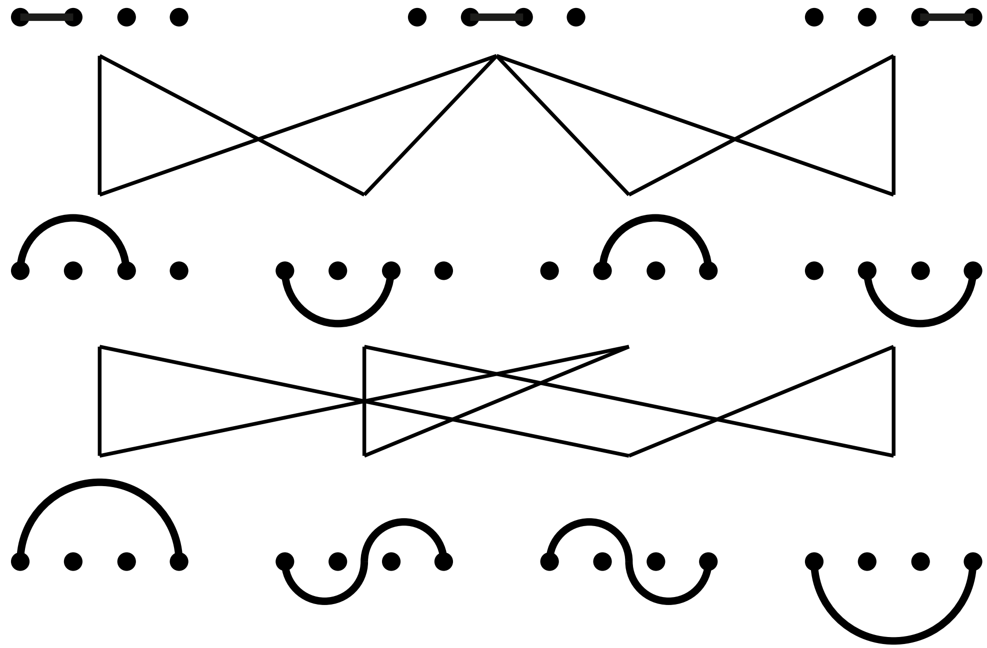


The set of walls $\boldsymbol{W}_{\equiv}$ of the quotient fan $\mathcal{F}_{\equiv}$ is a union of shards $\boldsymbol{\Sigma}_{\equiv}$ Reading ('05)

$\Sigma(a, b, A, B)$ forces $\Sigma(c, d, C, D) =$
$c \leq a < b \leq d$ and $A \subseteq C$ and $B \subseteq D$




Reading ('15)



# FORCING

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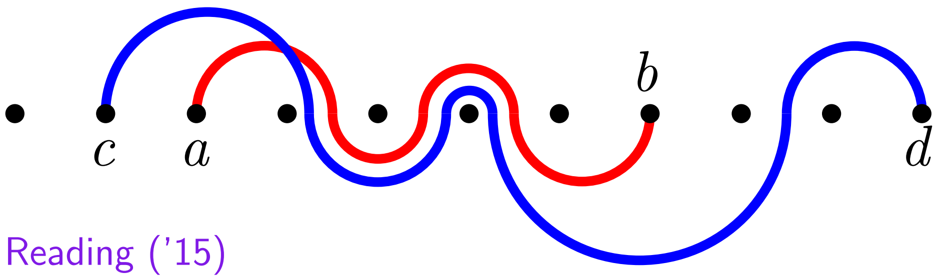


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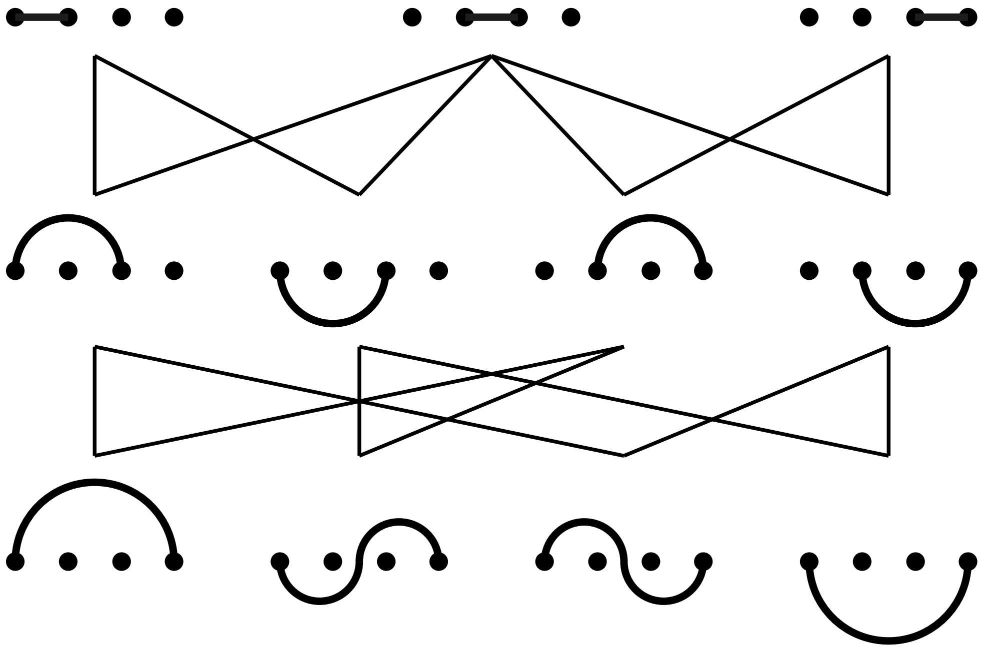
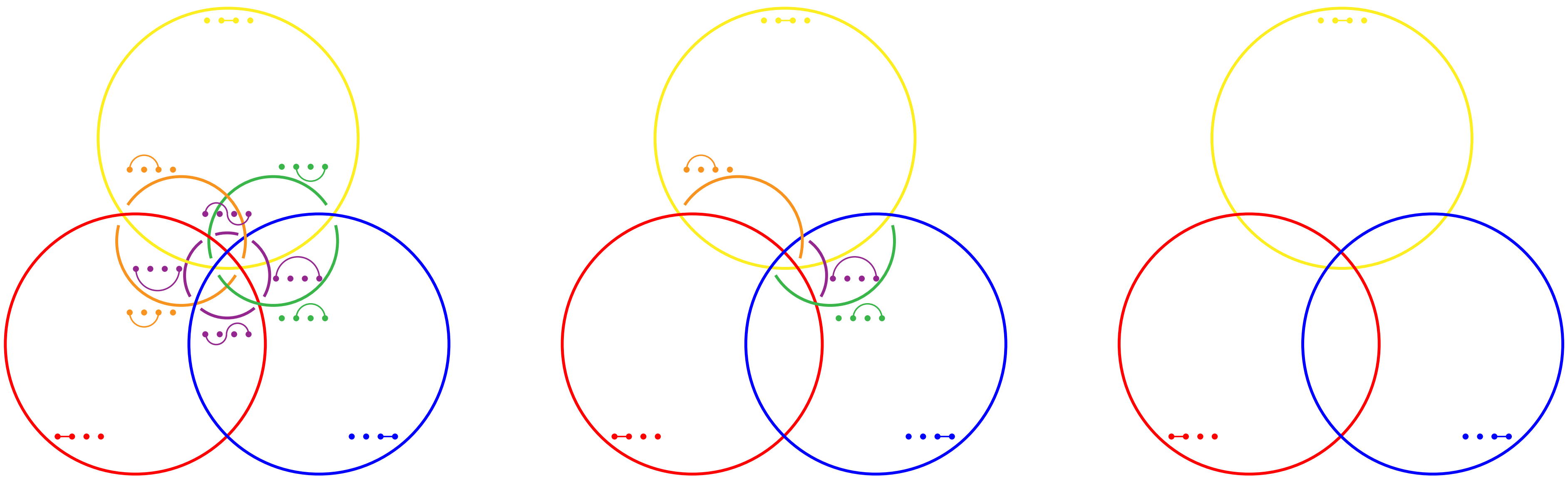


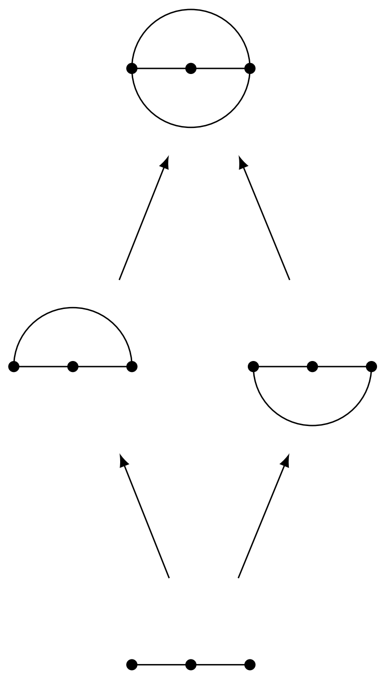
TFAE for a set of shards $\boldsymbol{\Sigma}$:

- there is a congruence $\equiv$ with $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\equiv}$
- $\boldsymbol{\Sigma}$ is an upper ideal in forcing order



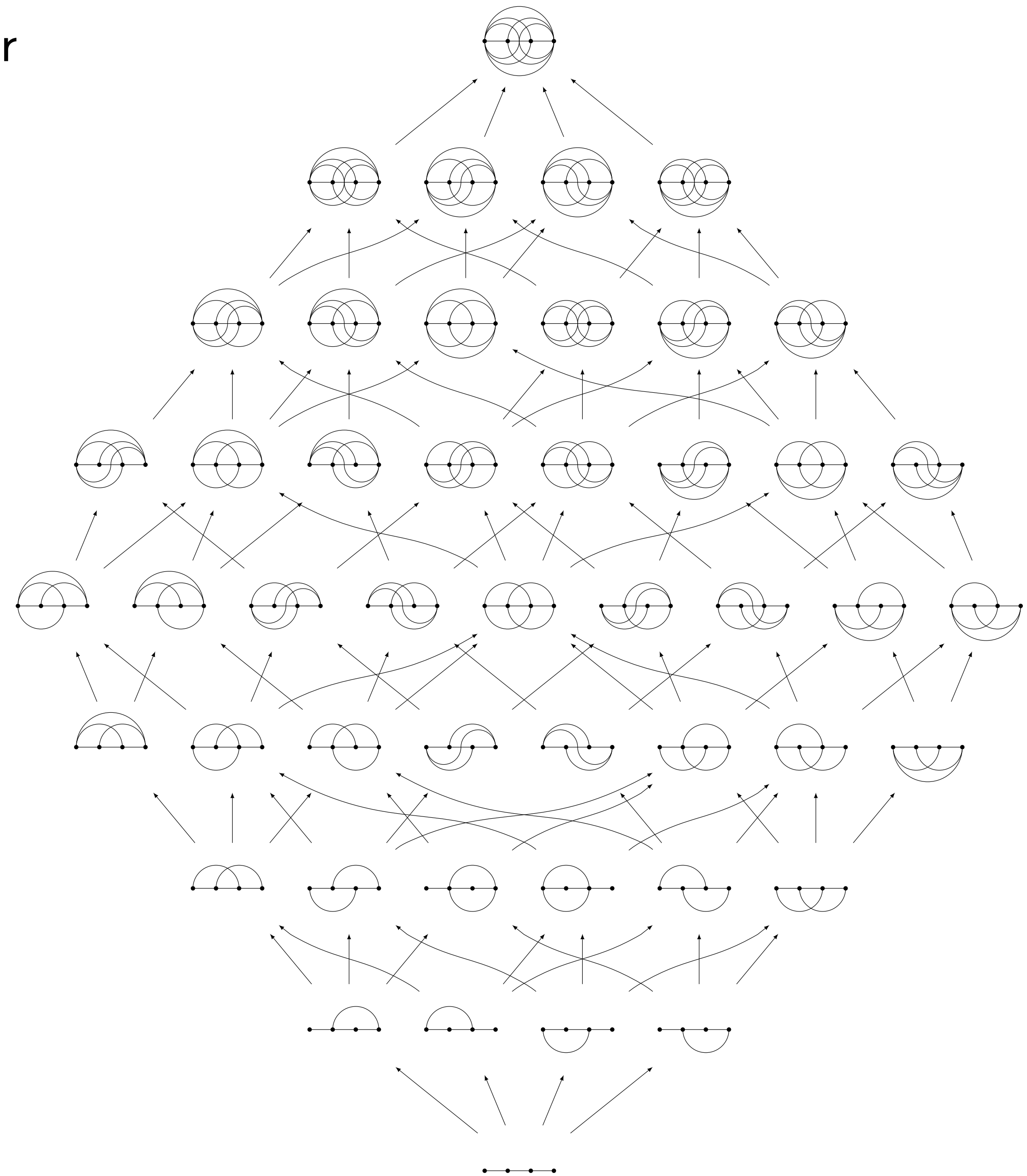
# SHARD IDEALS

shard ideal = upper ideal in forcing order



essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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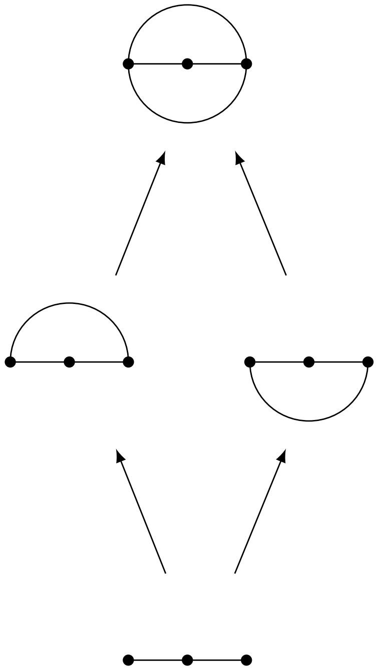

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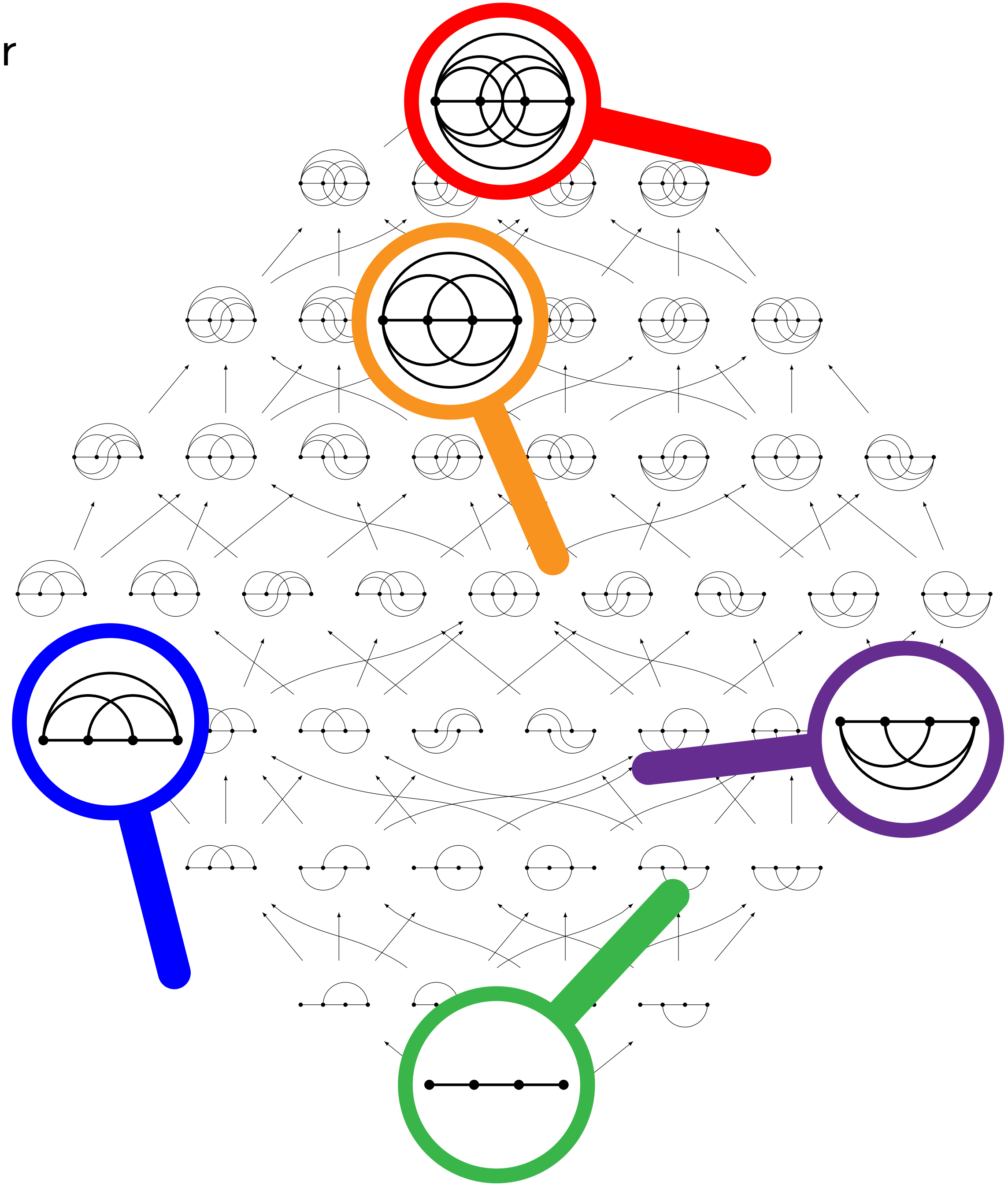
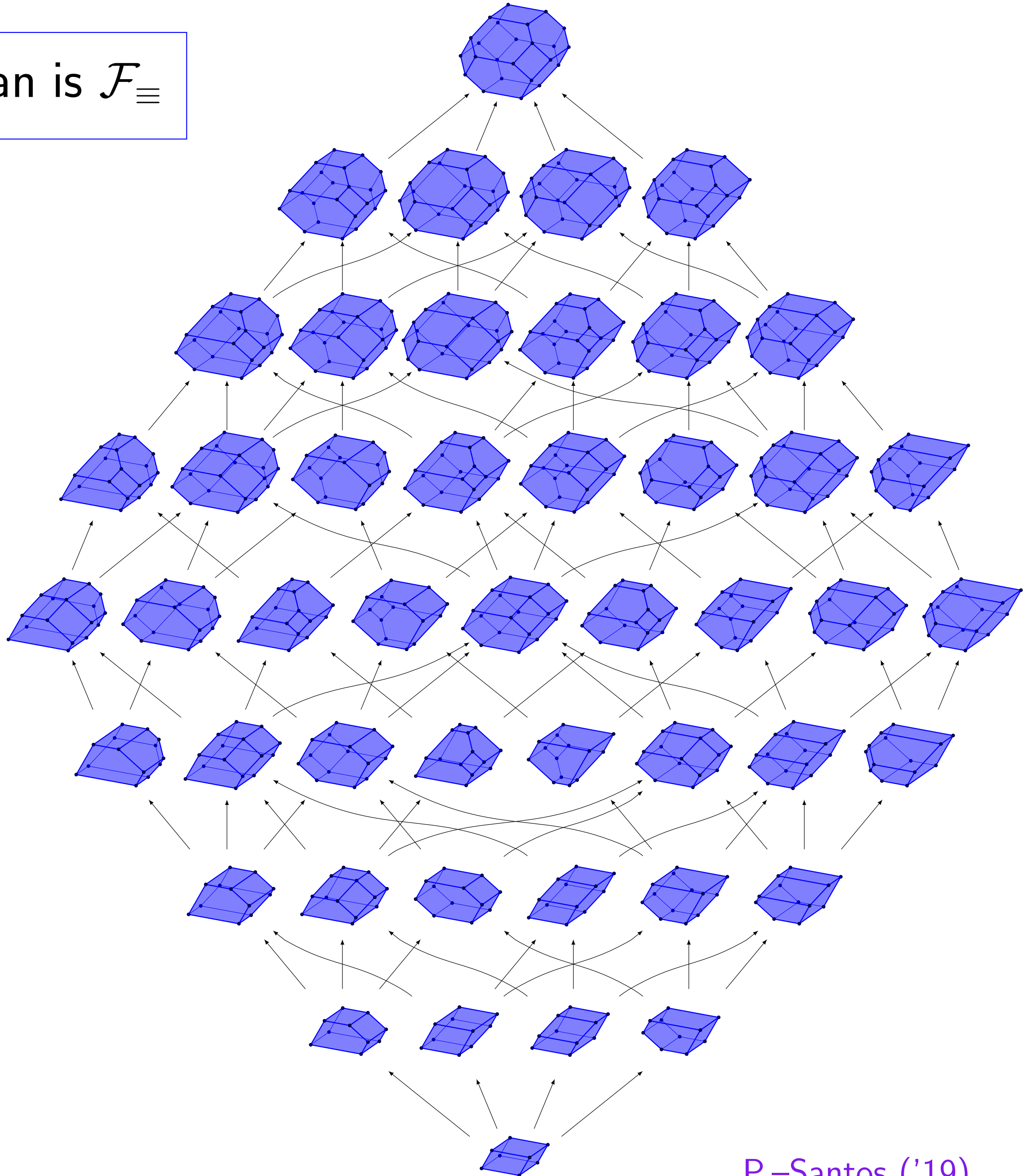


# QUOTIENTOPES

quotientope = polytope whose normal fan is $\mathcal{F}_{\equiv}$



P.–Santos ('19)

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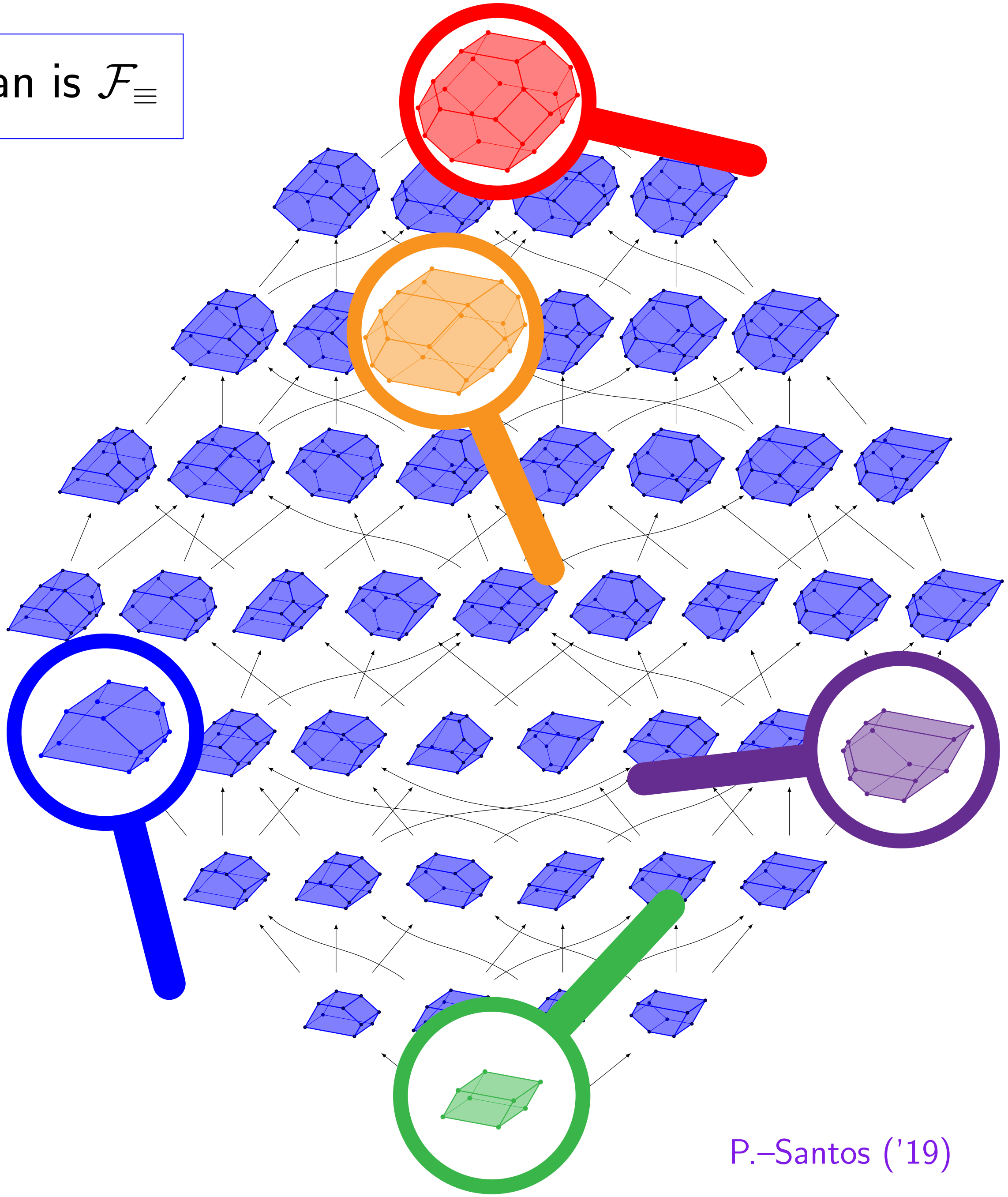
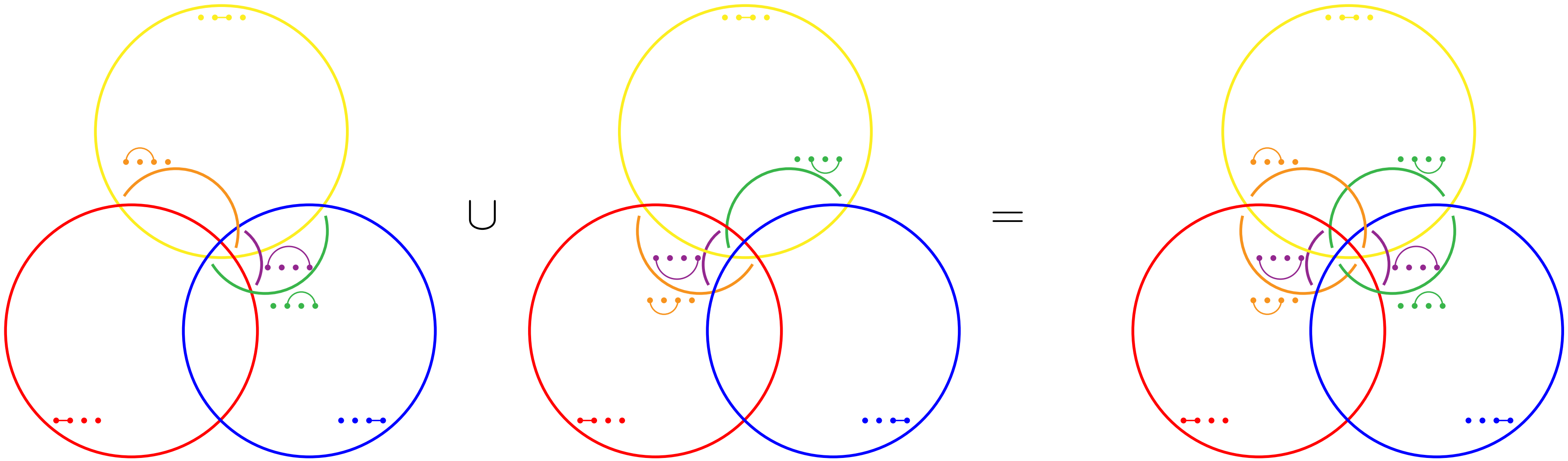


P.–Santos ('19)

# INTERSECTIONS OF CONGRUENCES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
then the quotient fan $\mathcal{F}_{\equiv}$ is the common refinement of the quotient fans $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$



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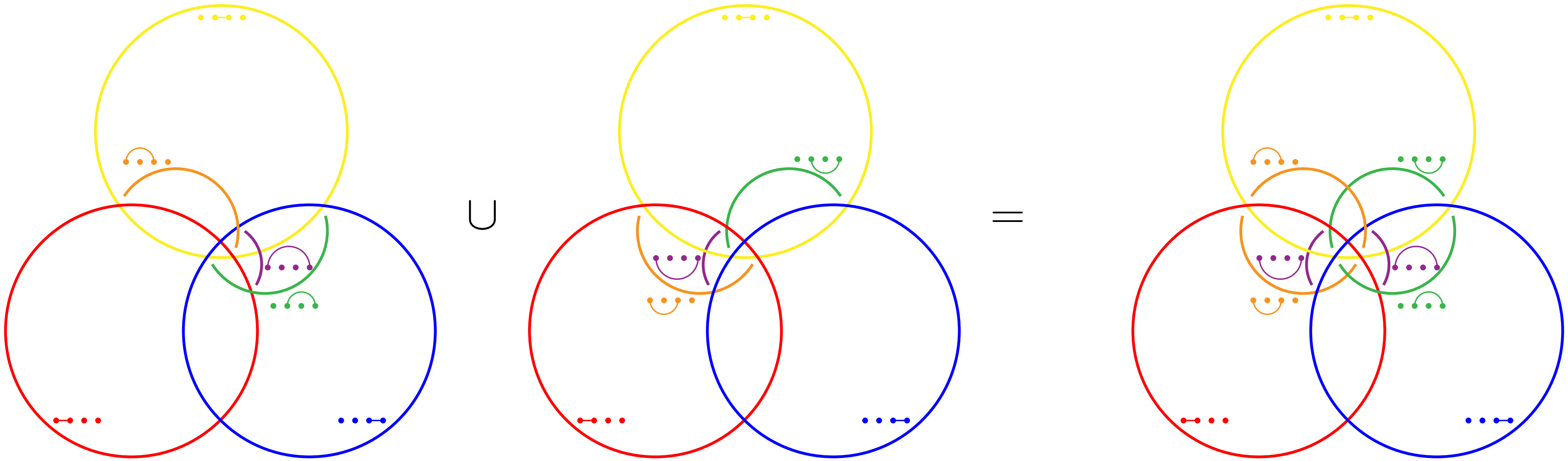
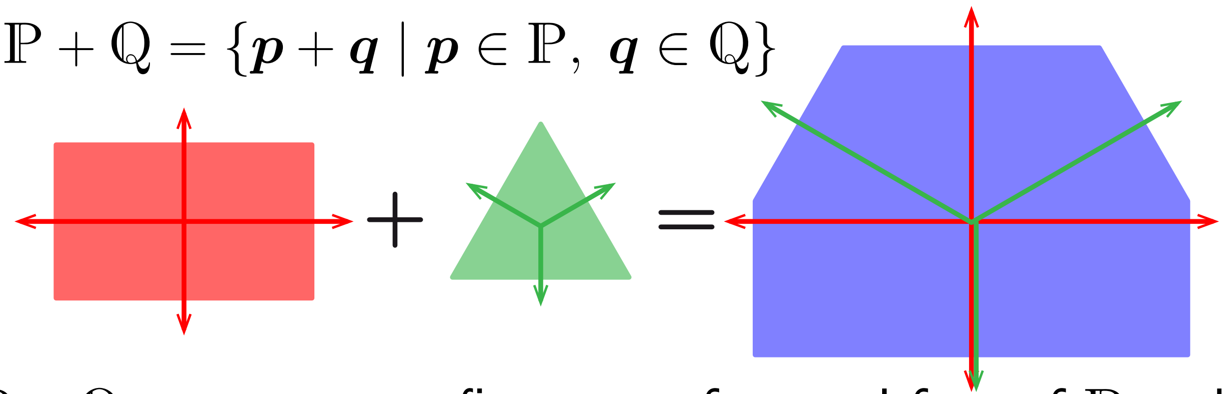


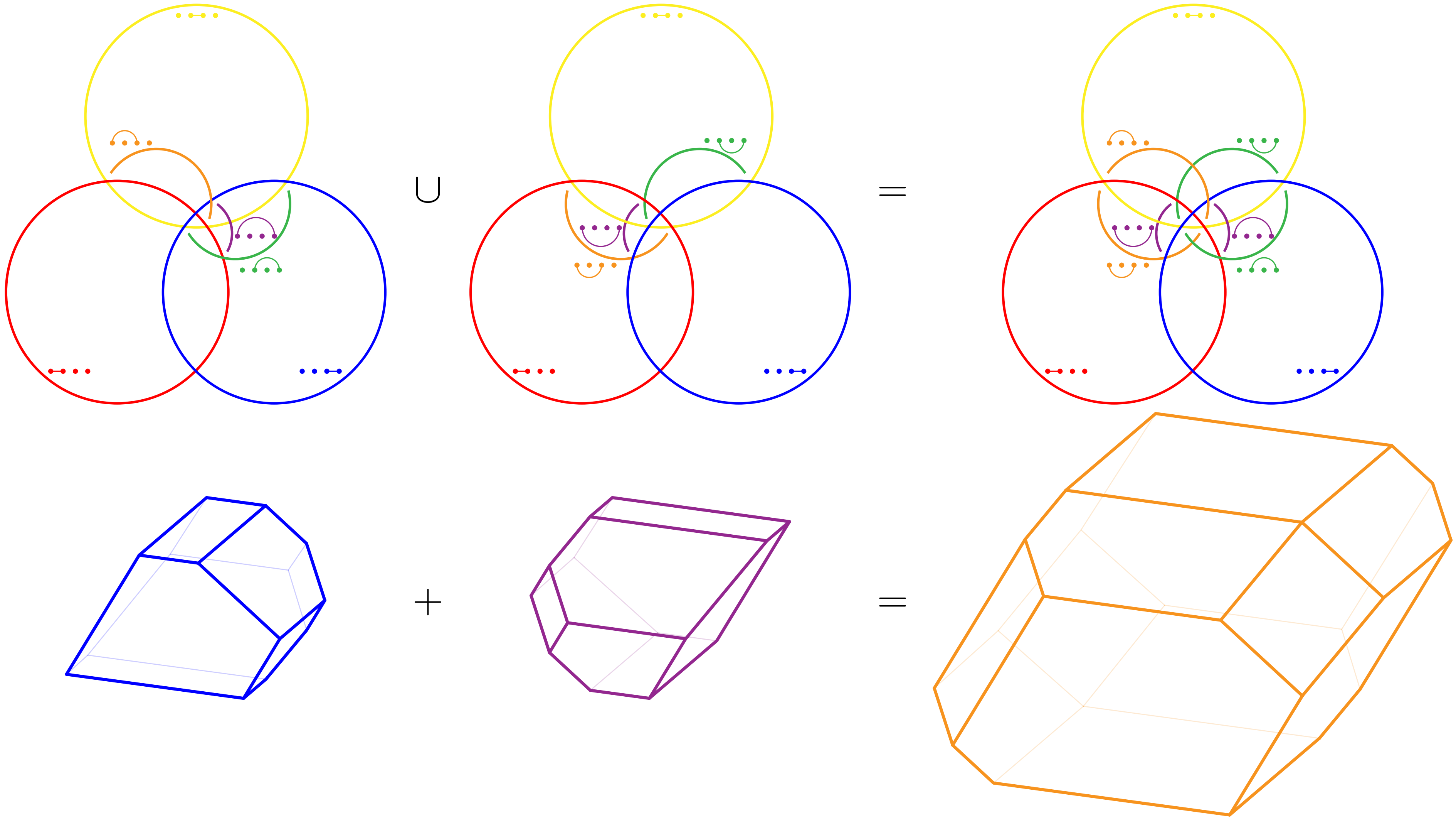
Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$



Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$

# MINKOWSKI SUMS OF QUOTIENTOPES

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \dots, \equiv_k$,
then the quotient fan $\mathcal{F}_\equiv$ is the common refinement of the quotient fans $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$,
and a Minkowski sum of quotientopes for $\mathcal{F}_{\equiv_1}, \dots, \mathcal{F}_{\equiv_k}$ is a quotientope for $\mathcal{F}_\equiv$



# MINKOWSKI SUMS OF ASSOCIAHEDRA

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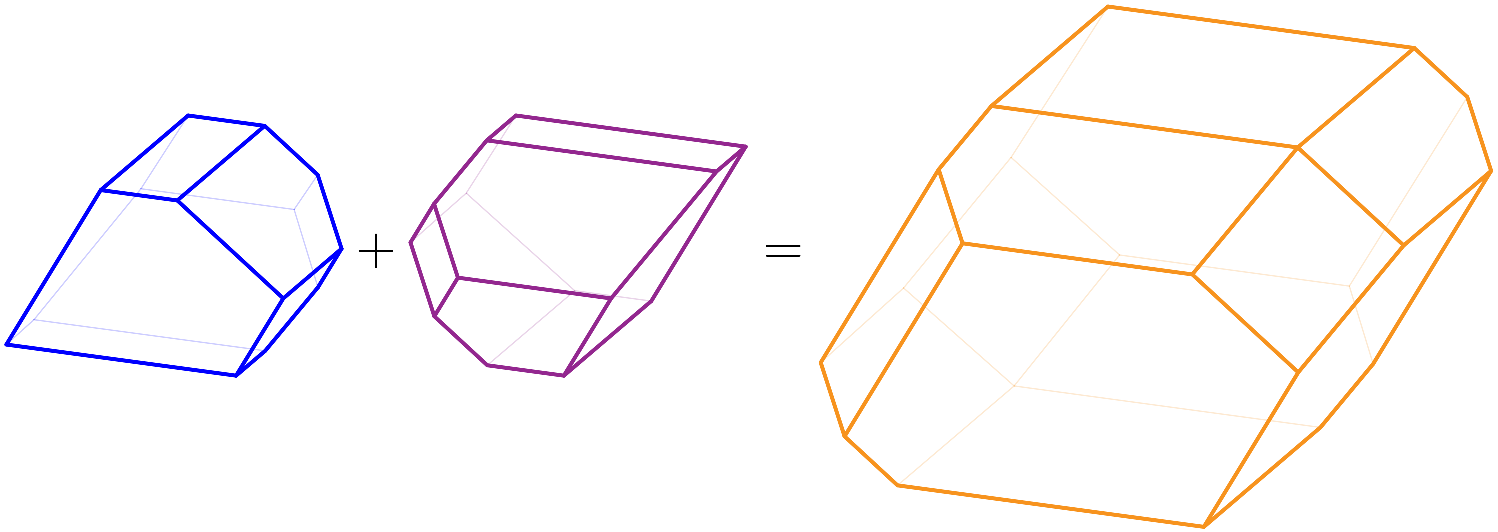
Principal arc ideals are Cambrian congruences

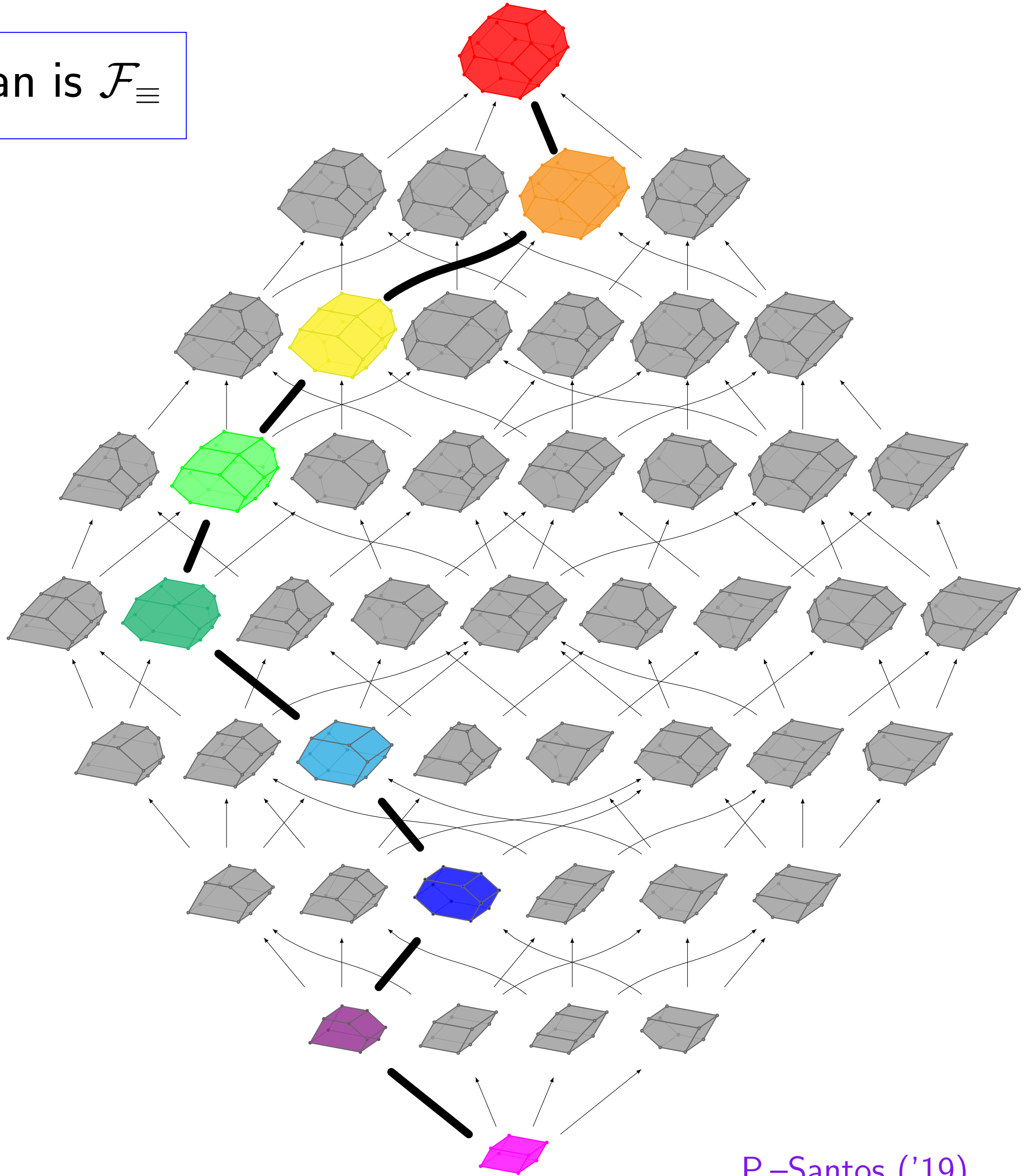
Any quotient fan is realized by a Minkowski sum of (low dim.) associahedra



Padrol-P.-Ritter ('$20^+$)

# MINKOWSKI SUMS OF ASSOCIAHEDRA

quotientope $=$ polytope whose normal fan is $\mathcal{F}_{\equiv}$



POLYWOOD

P.–Santos ('19)

# SHARD POLYTOPES

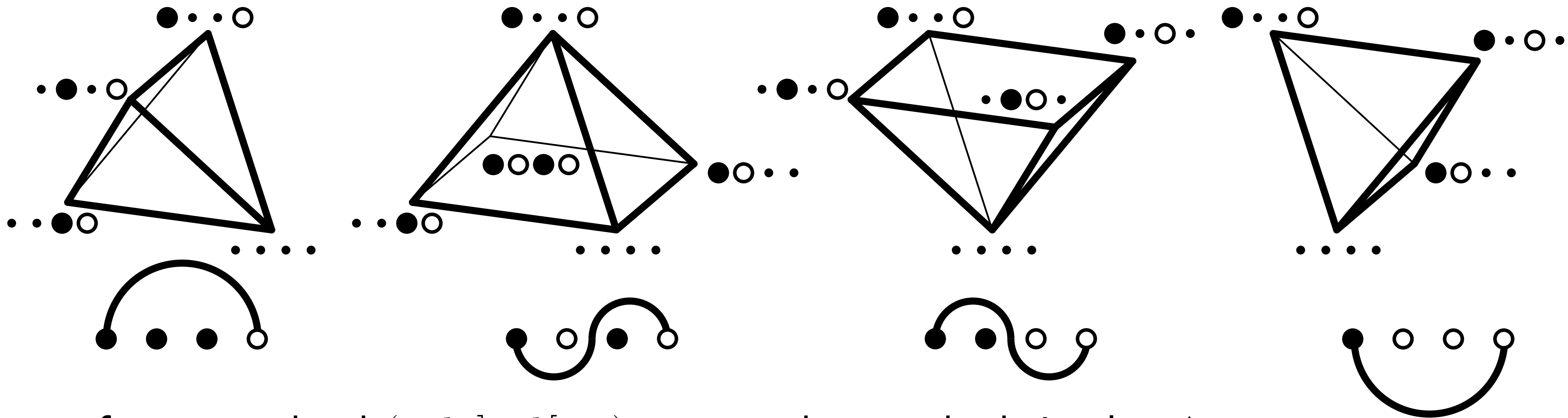
for a shard $\Sigma = \Sigma(a, b, A, B)$, define

- $\Sigma$-matching $=$ sequence $a \le a_1 < b_1 < \cdots < a_k < b_k \le b$ where $\begin{cases} a_i \in \{a\} \cup A \\ b_i \in B \cup \{b\} \end{cases}$
- characteristic vector $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$

shard polytope $\mathbb{SP}(\Sigma)$ $= \operatorname{conv}\left\{\chi(M) \;\middle|\; M\ \Sigma\text{-matching}\right\}$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{cl} x_j = 0 & \text{for all } j \in [n] \smallsetminus [a,b] \\ 0 \le x_{a'} \le 1 & \text{for all } a' \in \{a\} \cup A \\ -1 \le x_{b'} \le 0 & \text{for all } b' \in B \cup \{b\} \\ 0 \le \sum_{i \le j} x_i \le 1 & \text{for all } j \in [n] \end{array} \right\}$$

Padrol-P.-Ritter ($20^+$)



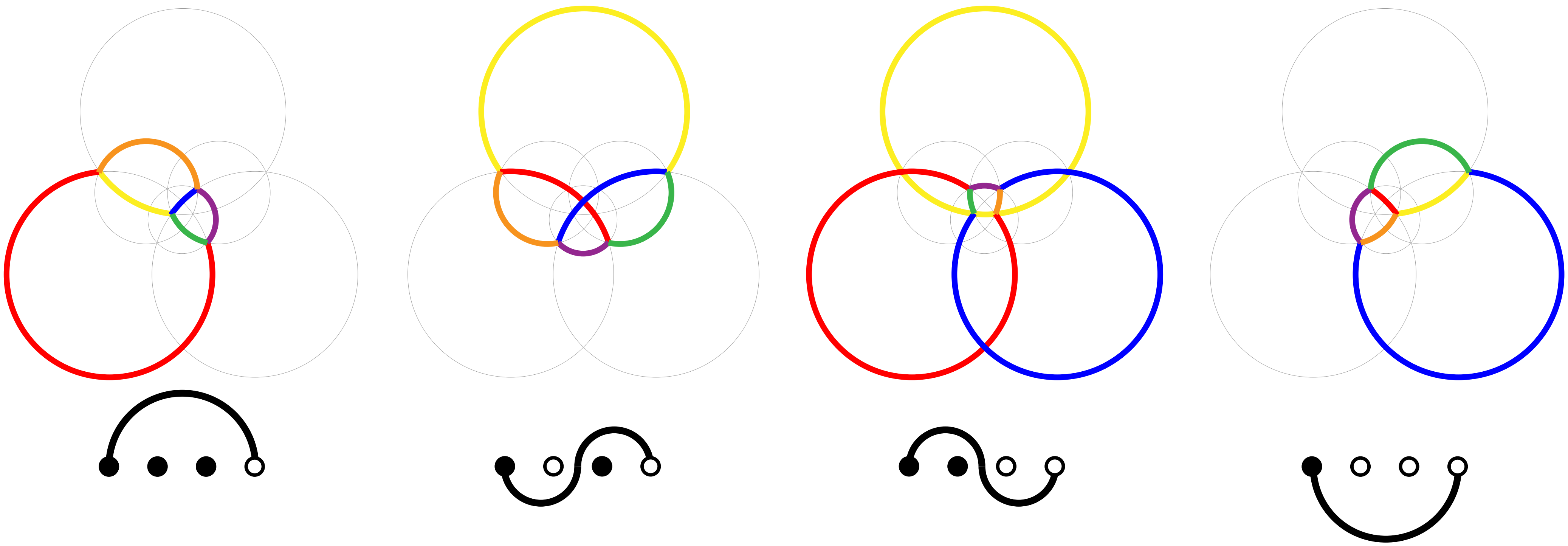
exm: for an up shard $(a, b, ]a, b[, \varnothing)$, we get the standard simplex $\triangle_{[a,b]} - \boldsymbol{e}_b$

# SHARD POLYTOPES

shard polytope $\mathbb{SP}(\Sigma) = \operatorname{conv}\left\{\chi(M) \;\middle|\; M\ \Sigma\text{-matching}\right\}$ Padrol-P.-Ritter (20[+])

The union of the walls of the normal fan of the shard polytope $\mathbb{SP}(\Sigma)$

- contains the shard $\Sigma$,
- is contained in the union of the shards forcing $\Sigma$



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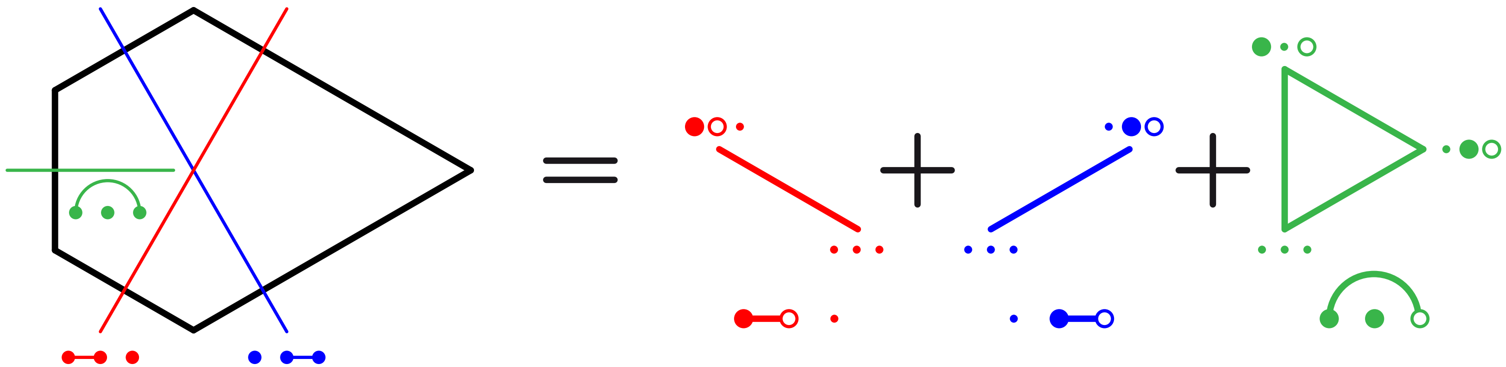

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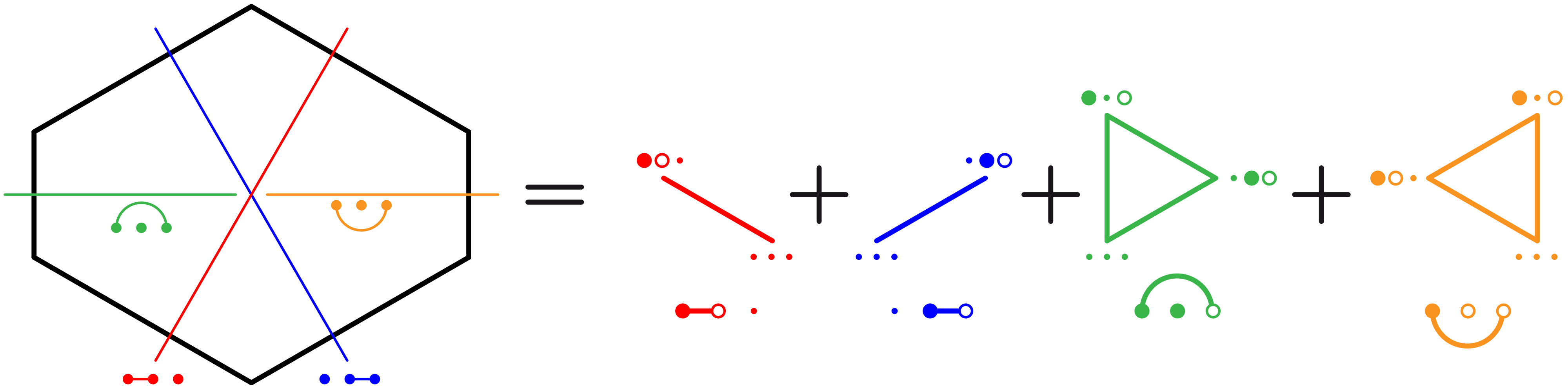


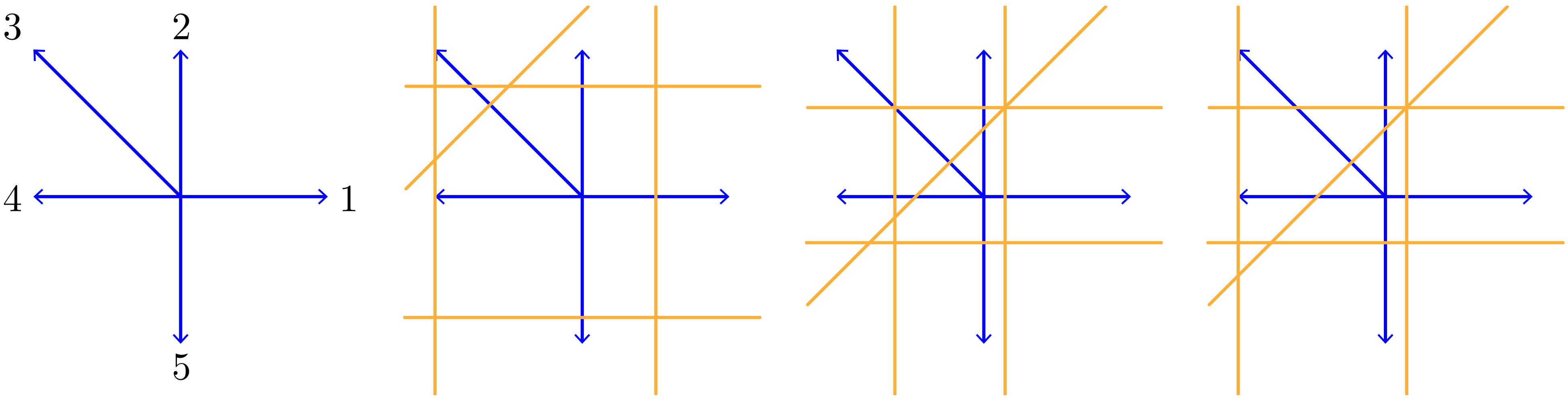
# SHARD POLYTOPES AND TYPE CONES

# CHOOSING RIGHT-HAND-SIDES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with $N$ rays

$\boldsymbol{G} = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\}$




A B C

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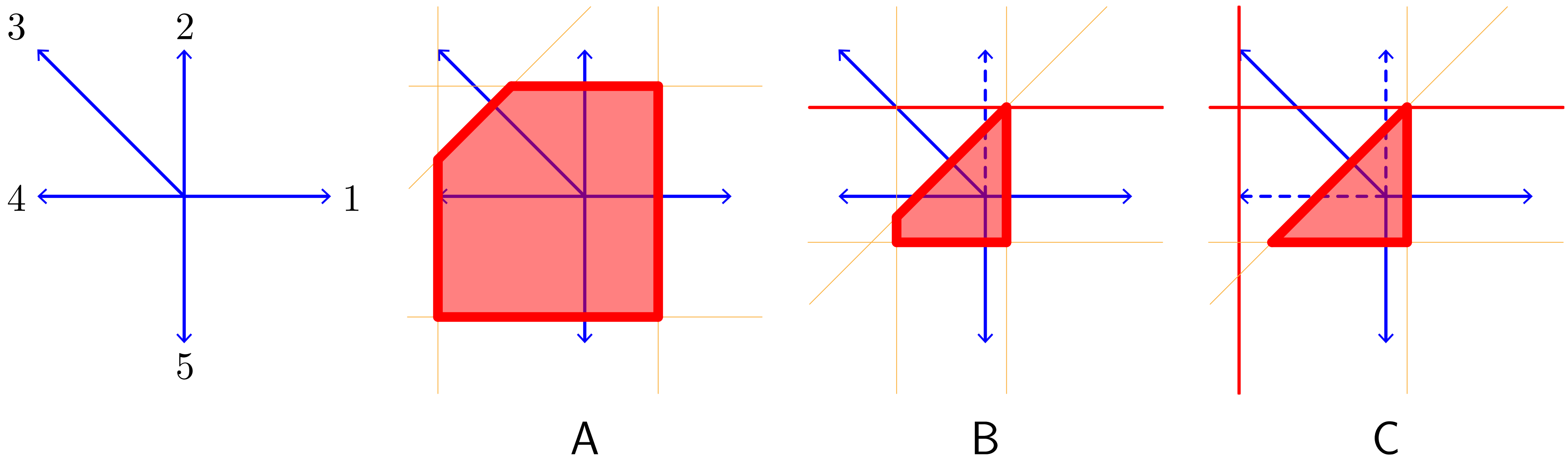

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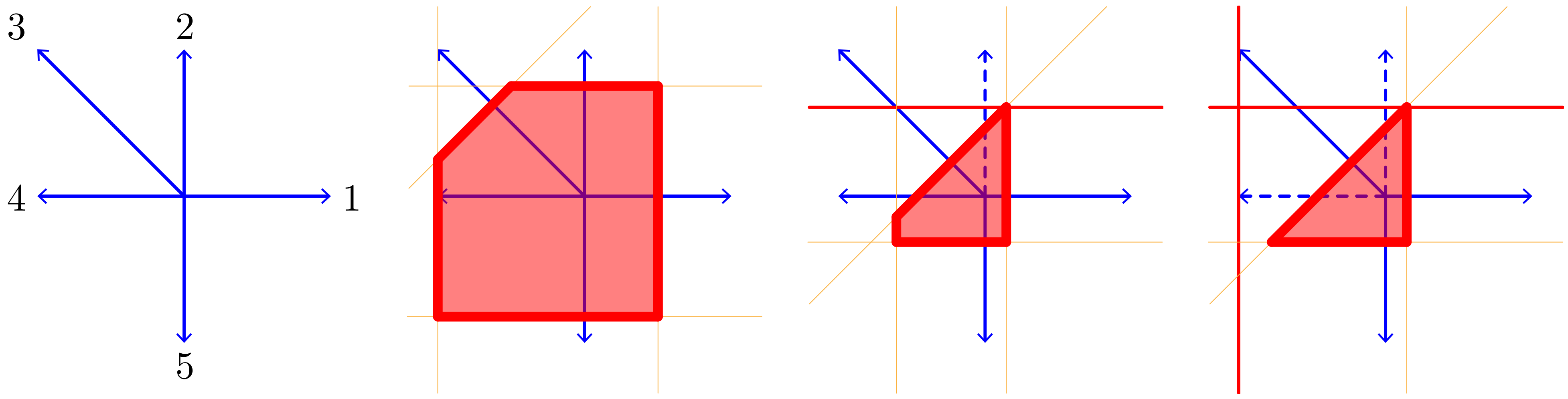


A B C

When is $\mathcal{F}$ the normal fan of $\mathbb{P}_{\boldsymbol{h}}$?

# WALL-CROSSING INEQUALITIES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with $N$ rays

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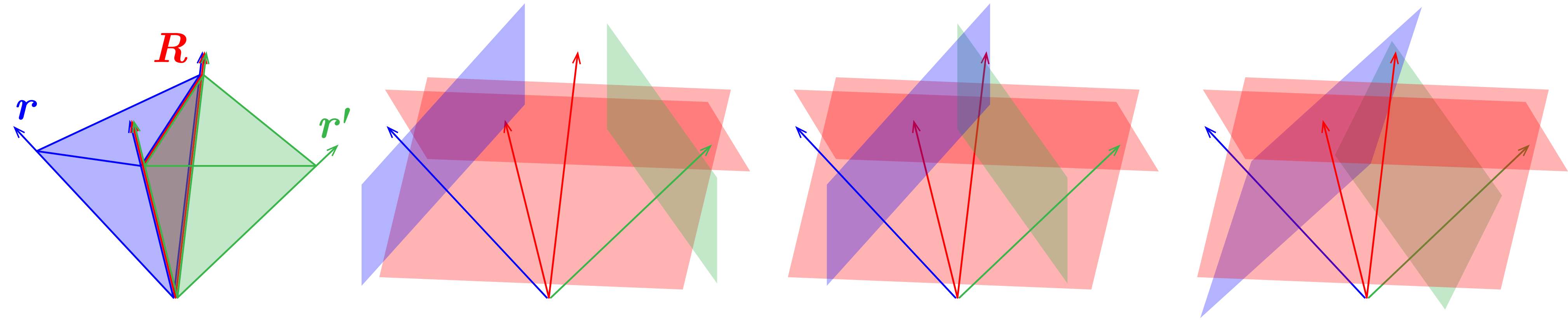

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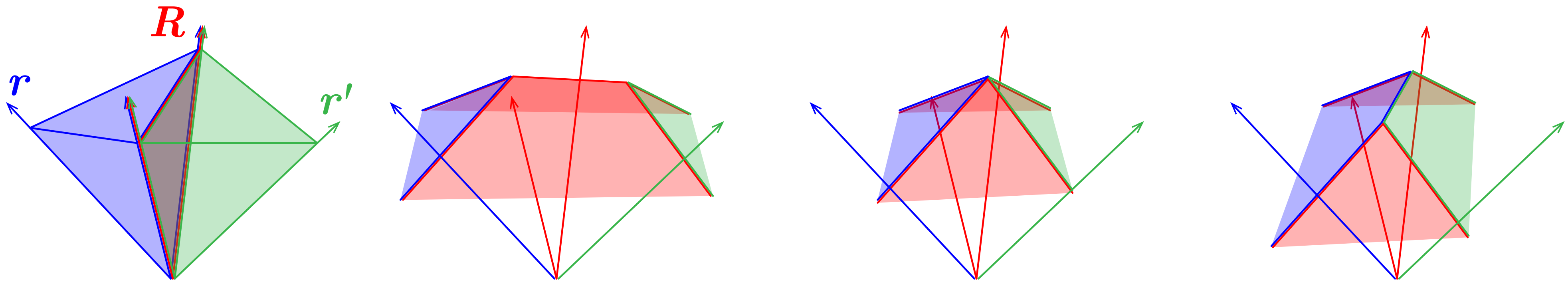
for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$



# WALL-CROSSING INEQUALITIES

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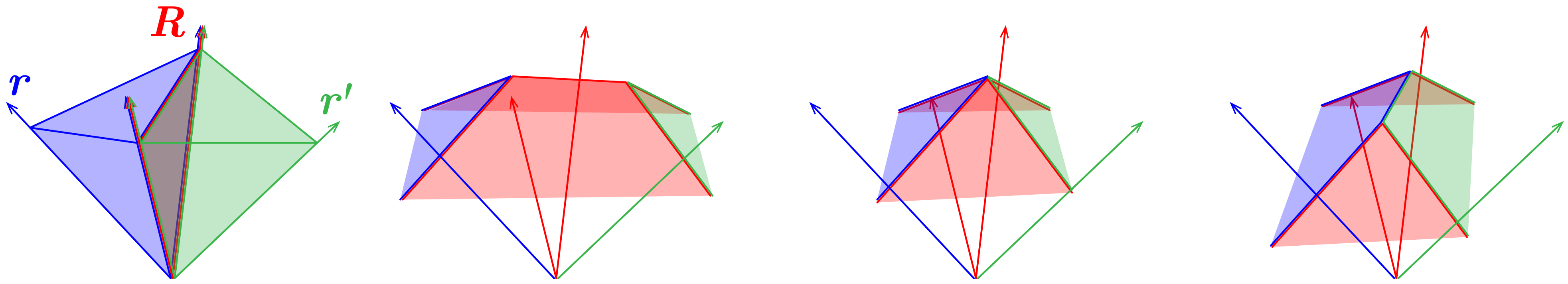


wall-crossing inequality for a wall $\boldsymbol{R}$ = $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r}'\}} \alpha_{\boldsymbol{R},\boldsymbol{s}} \, h_{\boldsymbol{s}} > 0$ where

- $\boldsymbol{r}, \boldsymbol{r}'$ = rays such that $\boldsymbol{R} \cup \{\boldsymbol{r}\}$ and $\boldsymbol{R} \cup \{\boldsymbol{r}'\}$ are chambers of $\mathcal{F}$
- $\alpha_{\boldsymbol{R},\boldsymbol{s}}$ = coeff. of unique linear dependence $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r}'\}} \alpha_{\boldsymbol{R},\boldsymbol{s}} \, \boldsymbol{s} = 0$ with $\alpha_{\boldsymbol{R},\boldsymbol{r}} + \alpha_{\boldsymbol{R},\boldsymbol{r}'} = 2$

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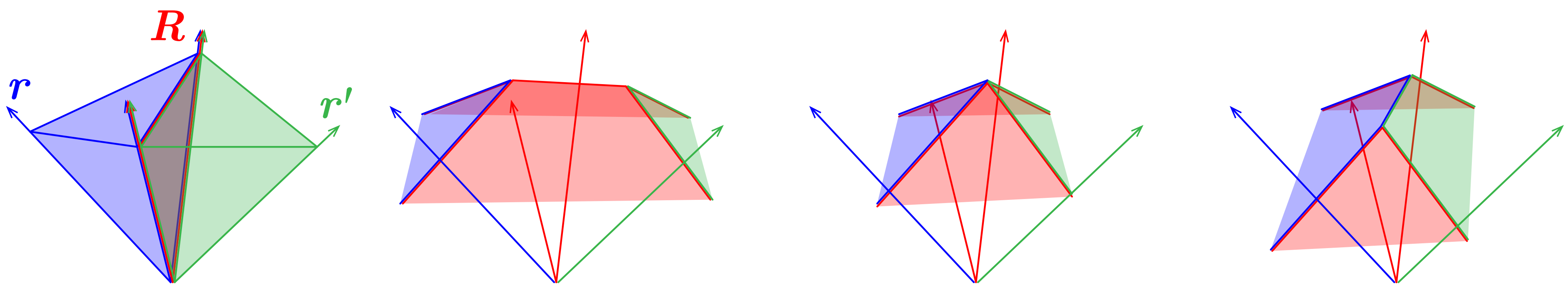


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$\mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}$ $\iff$ $\boldsymbol{h}$ satisfies all wall-crossing inequalities of $\mathcal{F}$

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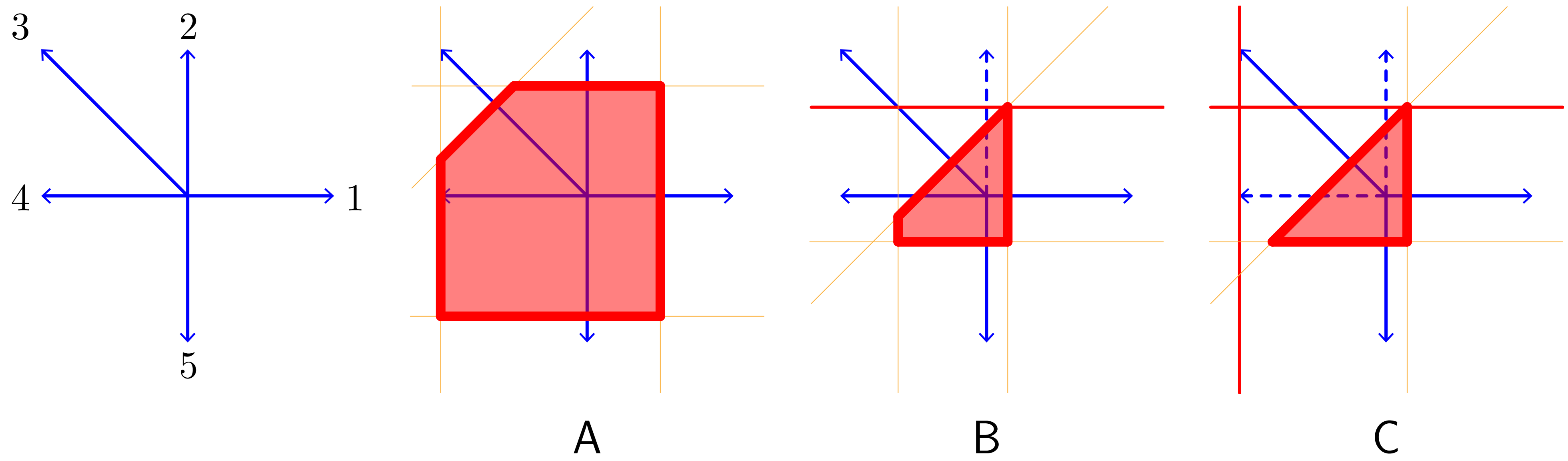



wall-crossing inequalities:

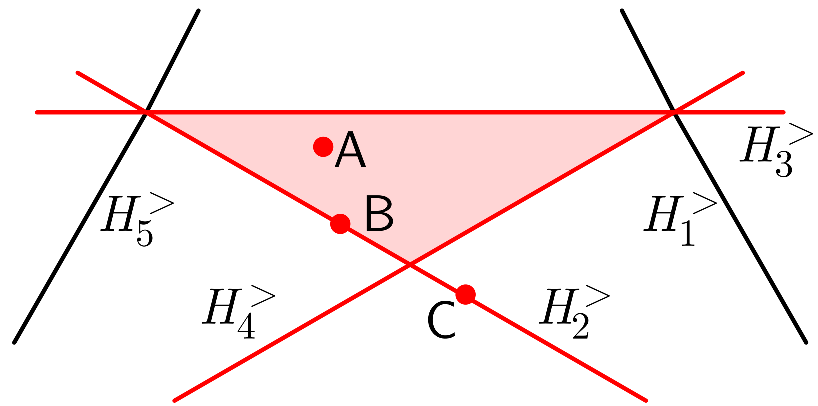
wall 1 : $h_2 + h_5 > 0$

wall 2 : $h_1 + h_3 > h_2$

wall 3 : $h_2 + h_4 > h_3$

wall 4 : $h_3 + h_5 > h_4$

wall 5 : $h_1 + h_4 > 0$



# TYPE CONE

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<u>type cone</u> $\mathbb{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ McMullen ('73)

$= \{\boldsymbol{h} \in \mathbb{R}^N \mid \mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}\}$

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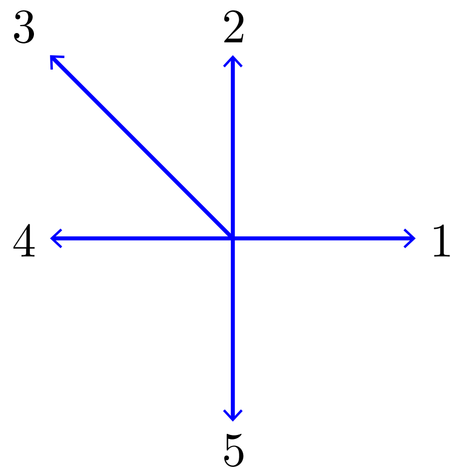


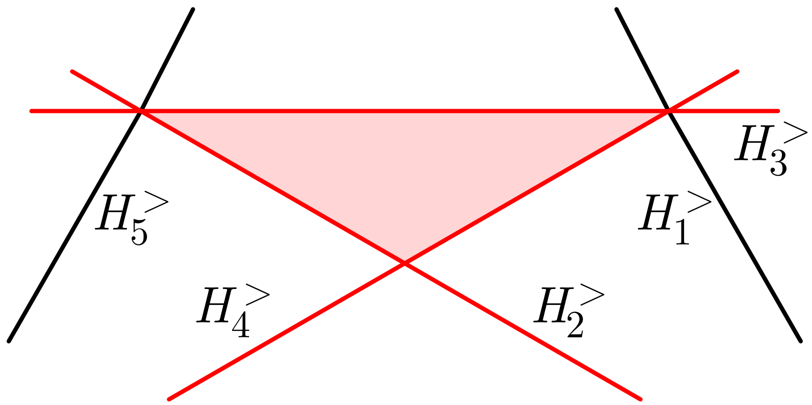

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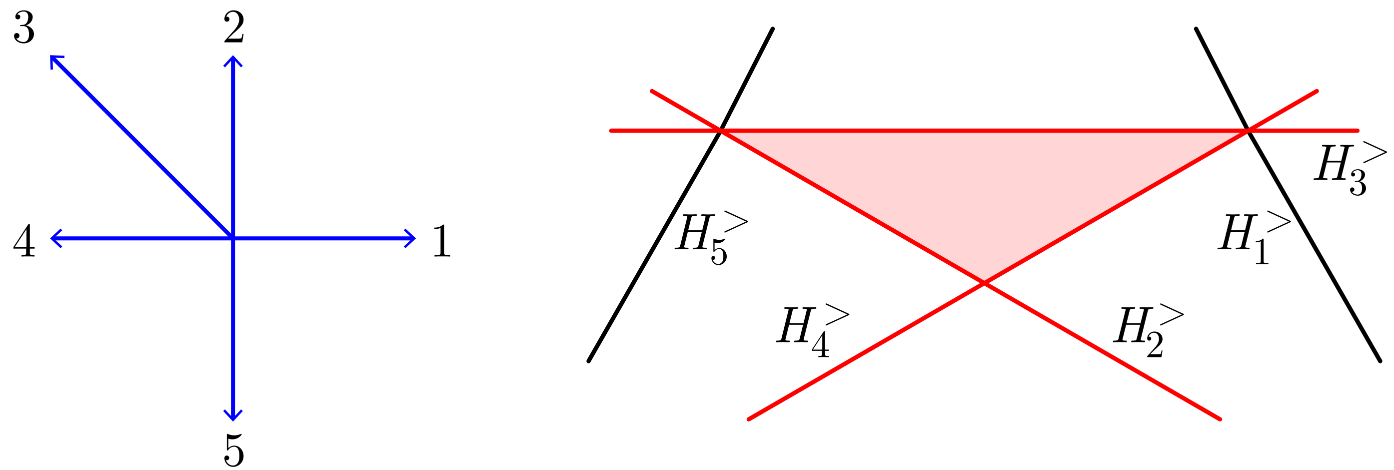


some properties of $\mathbb{TC}(\mathcal{F})$:

- $\mathbb{TC}(\mathcal{F})$ is an open cone (dilations preserve normal fans)
- $\mathbb{TC}(\mathcal{F})$ has lineality space $\boldsymbol{G}\,\mathbb{R}^n$ (translations preserve normal fans)
- dimension of $\mathbb{TC}(\mathcal{F})/\boldsymbol{G}\,\mathbb{R}^n = N - n$

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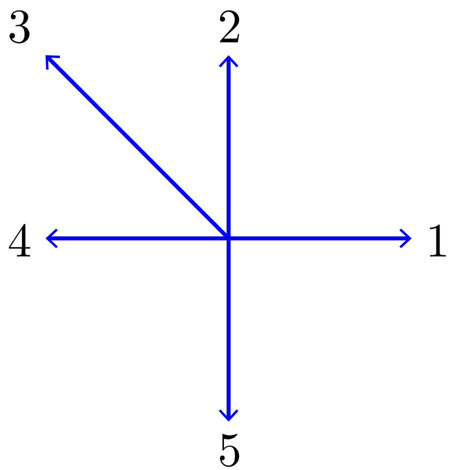
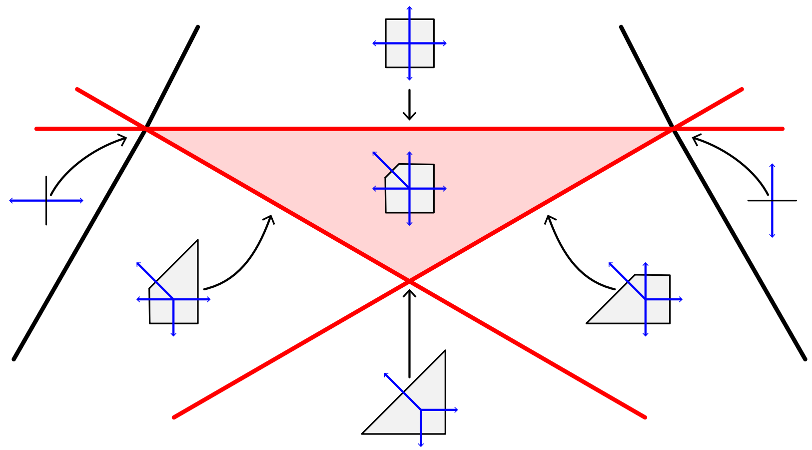


some properties of $\mathbb{TC}(\mathcal{F})$:

- closure of $\mathbb{TC}(\mathcal{F})$ = polytopes whose normal fan coarsens $\mathcal{F}$ = deformation cone
- Minkowski sums $\longleftrightarrow$ positive linear combinations

# SIMPLICIAL TYPE CONE

Assume that the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial

$\boldsymbol{K} = (N-n) \times N$-matrix whose rows are inner normal vectors of the facets of $\mathbb{TC}(\mathcal{F})$

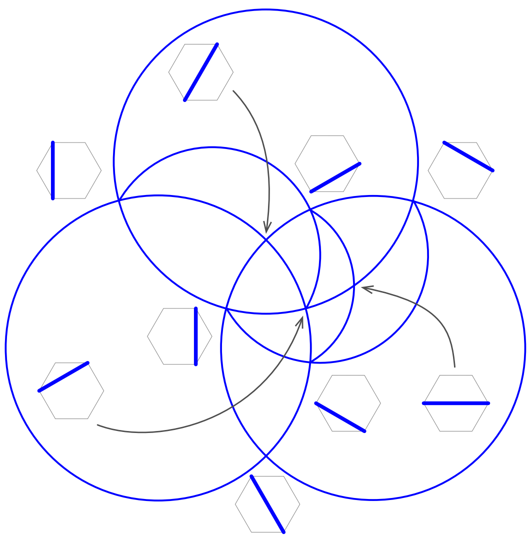
All polytopal realizations of $\mathcal{F}$ are affinely equivalent to

$$\mathbb{R}_{\boldsymbol{\ell}} = \left\{ \boldsymbol{z} \in \mathbb{R}^N \;\middle|\; \boldsymbol{z} \geq 0 \text{ and } \boldsymbol{K}\boldsymbol{z} = \boldsymbol{\ell} \right\}$$

for any positive vector $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$

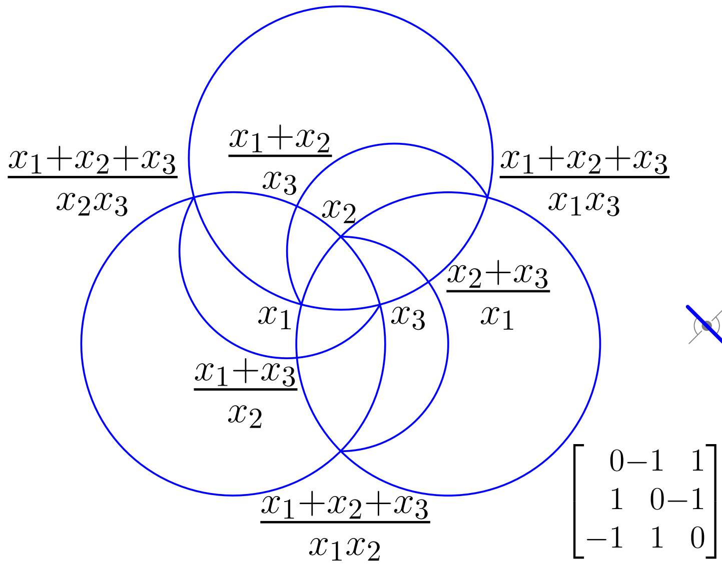
Padrol–Palu–P.–Plamondon ('19$^+$)

Fundamental exms: $\boldsymbol{g}$-vector fans of cluster-like complexes



sylvester fans

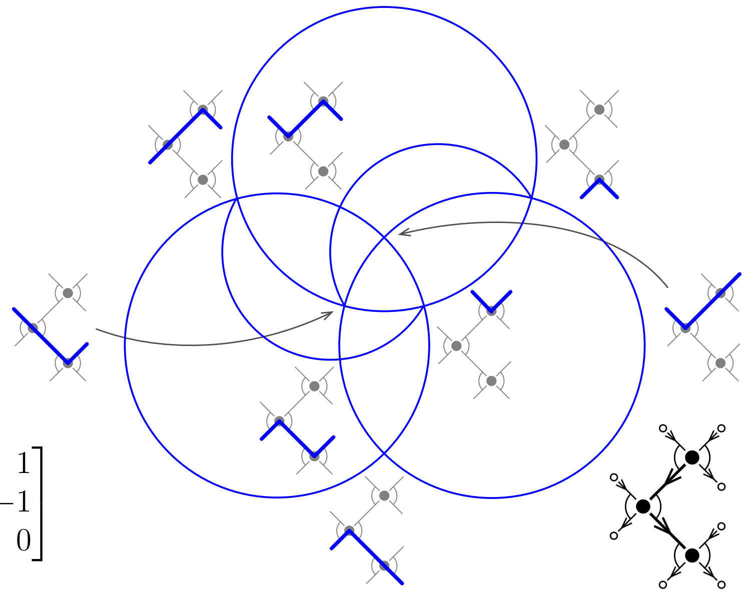
Arkani-Hamed–Bai–He–Yan ('18)




finite type $\boldsymbol{g}$-vector fans

wrt any seed (acyclic or not)

BMDMTY ('18$^+$)

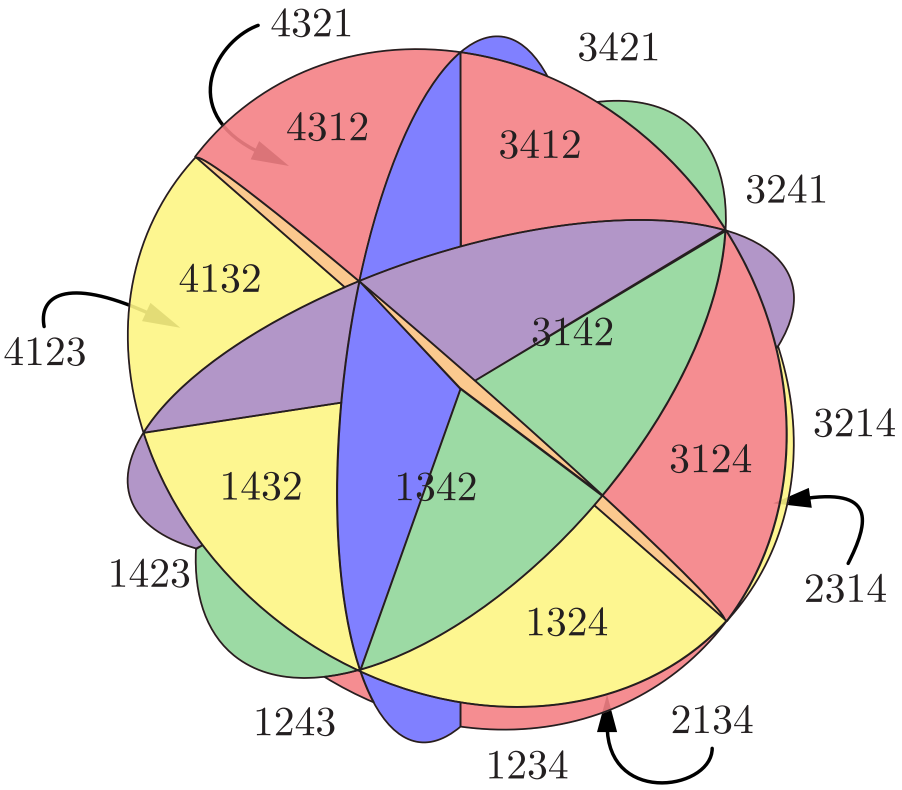


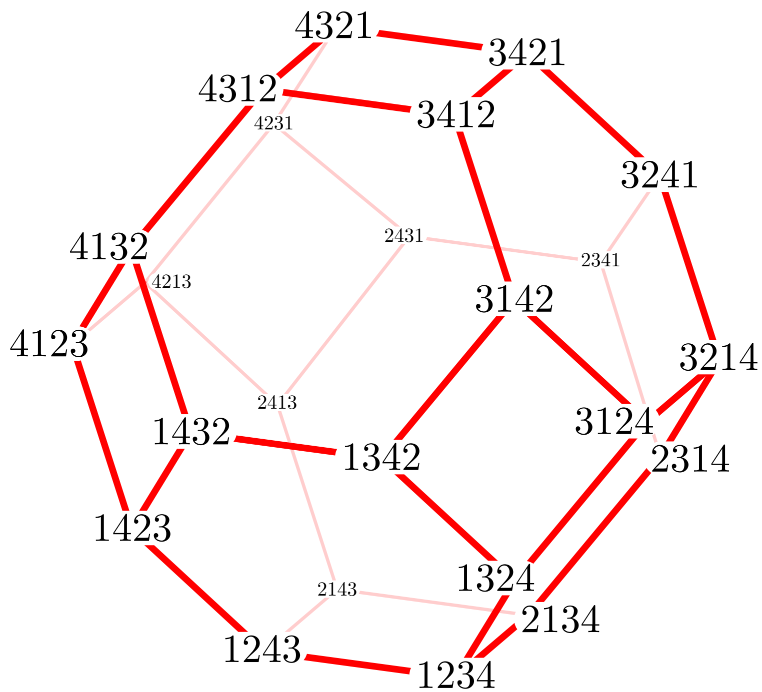
finite gentle fans

for brick and $2$-acyclic quivers

Palu–P.–Plamondon ('18)

# SUBMODULAR FUNCTIONS




closed type cone of braid fan = {deformed permutahedra} = {submodular functions}

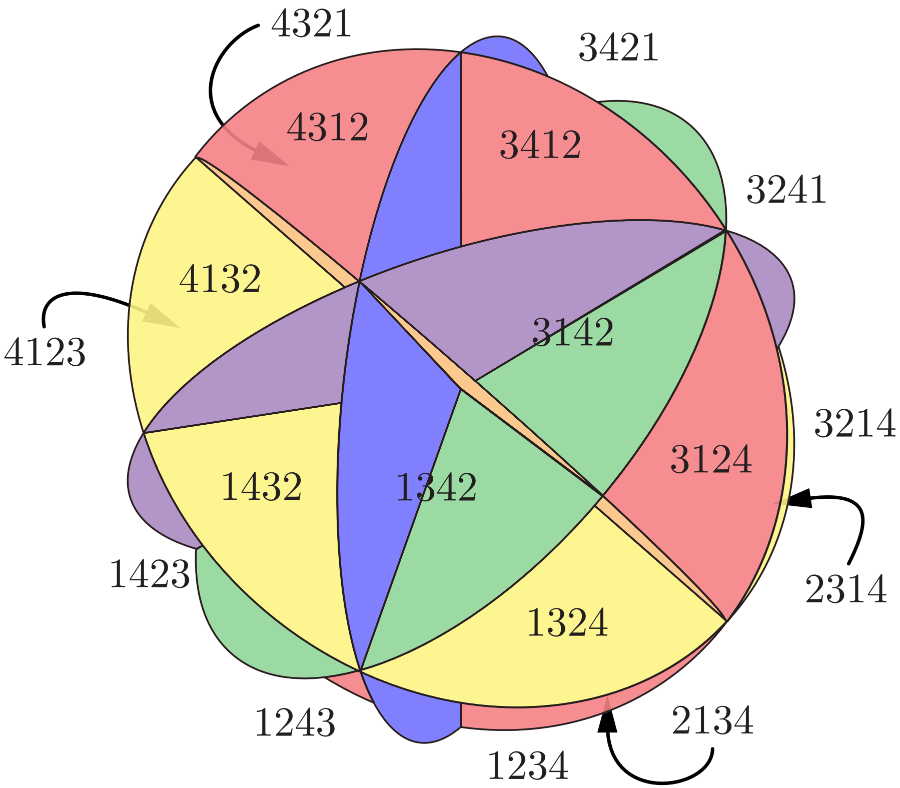
deformed permutahedron = polytope whose normal fan coarsens the braid fan

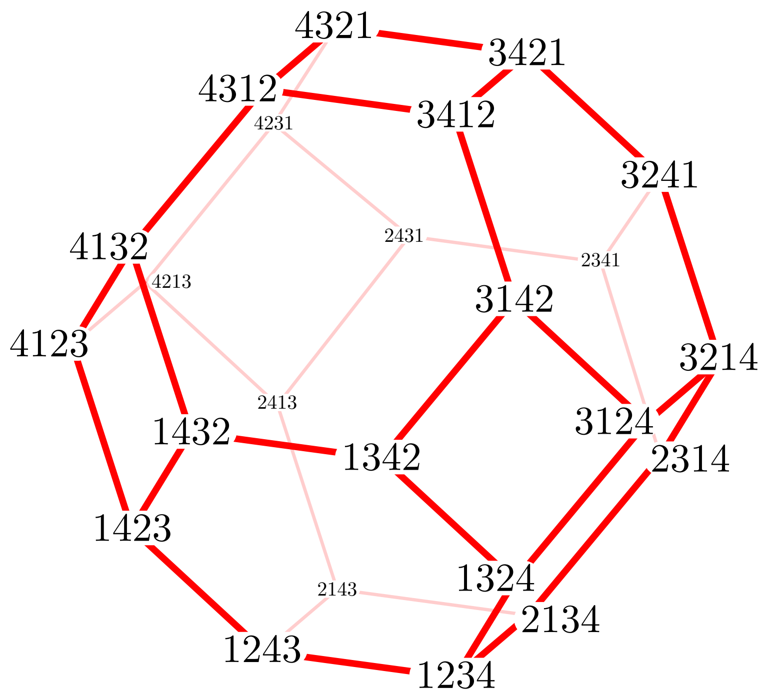
$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \Big\{ \boldsymbol{x} \in \mathbb{R}^n \;\Big|\; \langle\, \mathbb{1} \mid \boldsymbol{x} \,\rangle = z_{[n]} \text{ and } \langle\, \mathbb{1}_R \mid \boldsymbol{x} \,\rangle \ge z_R \text{ for all } R \subseteq [n] \Big\}$$

for some vector $\boldsymbol{z} \in \mathbb{R}^{2^{[n]}}$ such that $z_R + z_S \le z_{R \cup S} + z_{R \cap S}$ and $z_{\varnothing} = 0$

Postnikov ('09) Postnikov–Reiner–Williams ('08)

# SUBMODULAR FUNCTIONS

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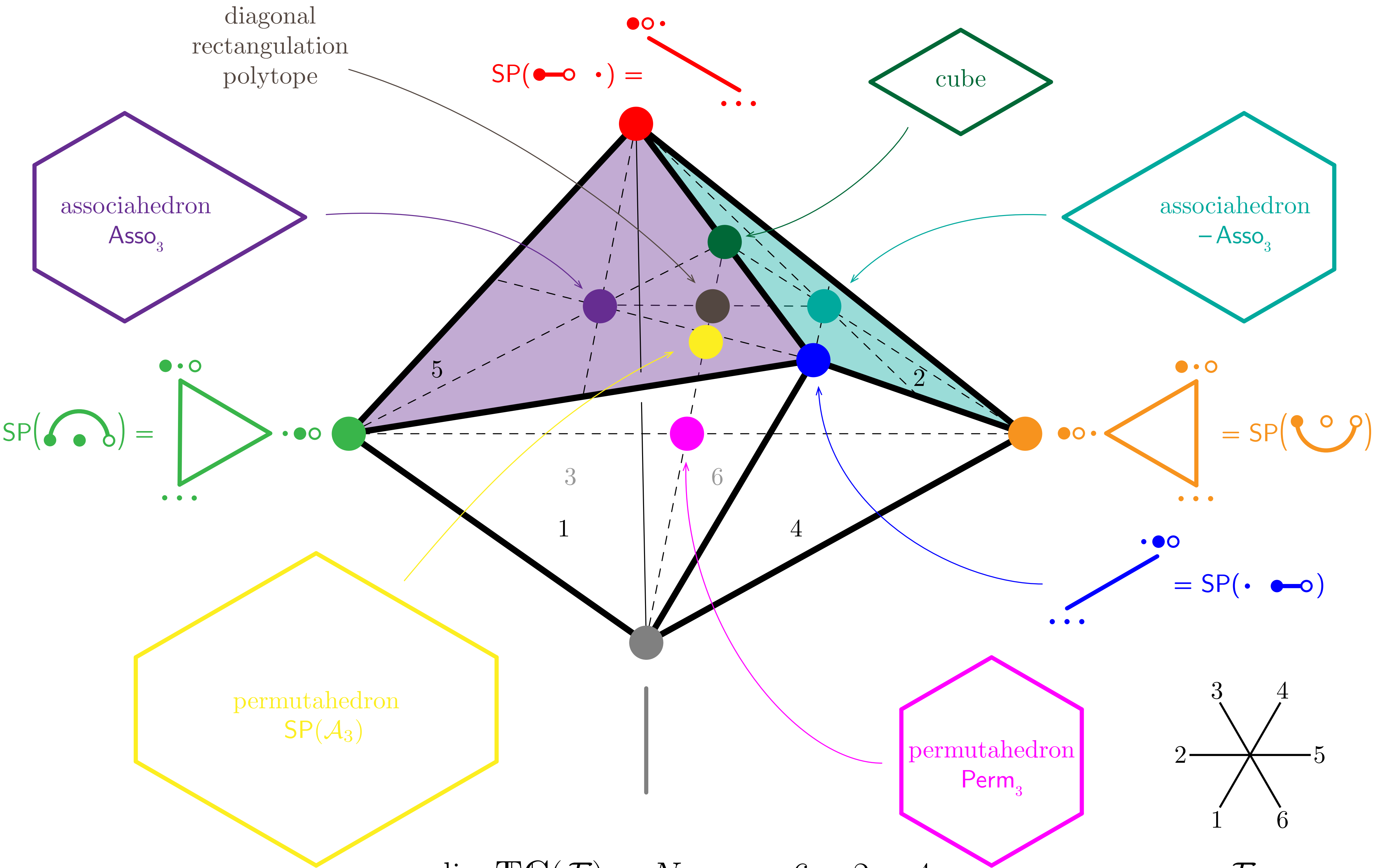
$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n_{\ge 0} \;\middle|\; \langle\, \mathbb{1} \mid \boldsymbol{x} \,\rangle = z_{[n]} \text{ and } \langle\, \mathbb{1}_R \mid \boldsymbol{x} \,\rangle \ge z_R \text{ for all } R \in \mathcal{J} \right\}$$

for some vector $\boldsymbol{z} \in \mathbb{R}^{2^{[n]}}$ such that $z_R + z_S \le z_{R \cup S} + z_{R \cap S}$ and $z_\varnothing = z_{\{i\}} = 0$,
where $\mathcal{J} = \{J \subset [n] \mid |J| \ge 2\}$

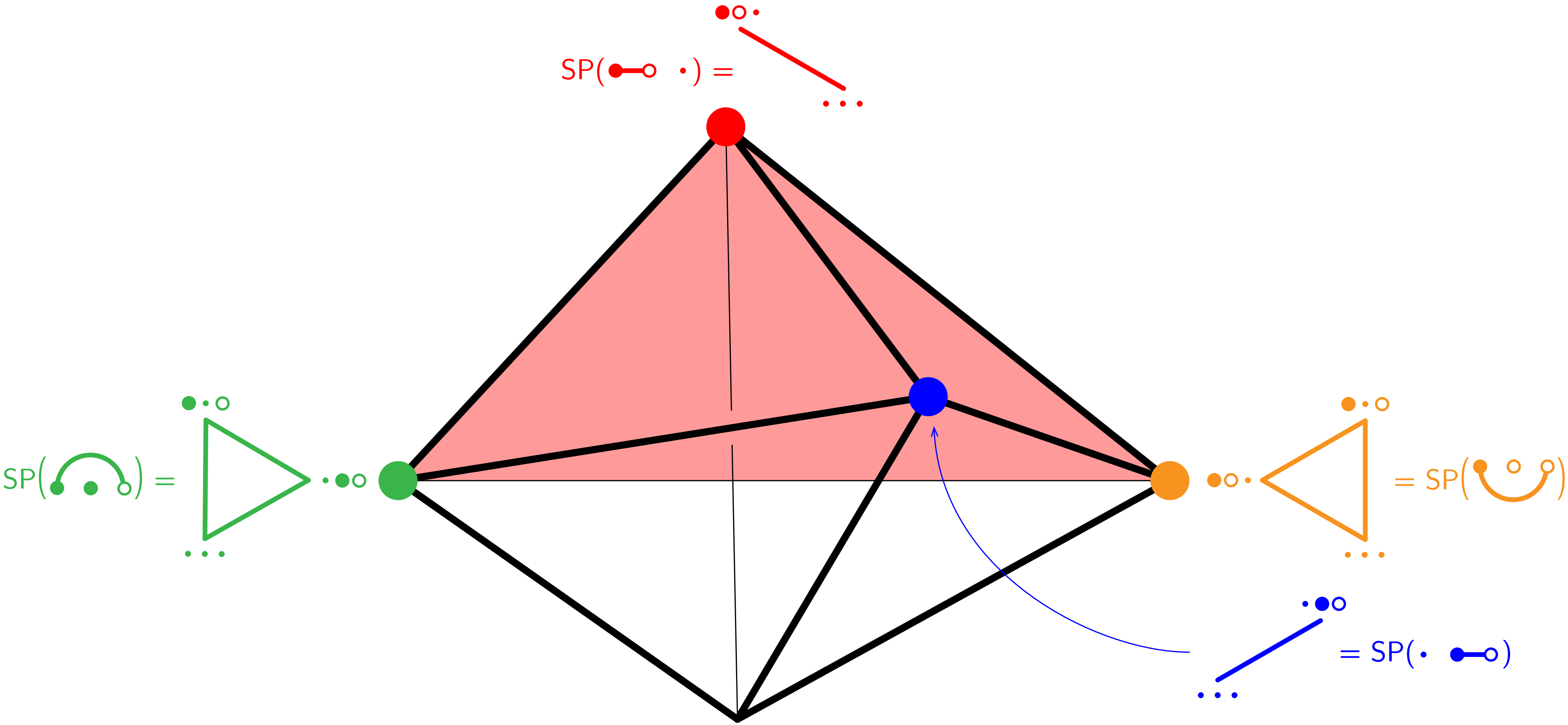
Postnikov ('09) Postnikov–Reiner–Williams ('08)

# SUBMODULAR FUNCTIONS



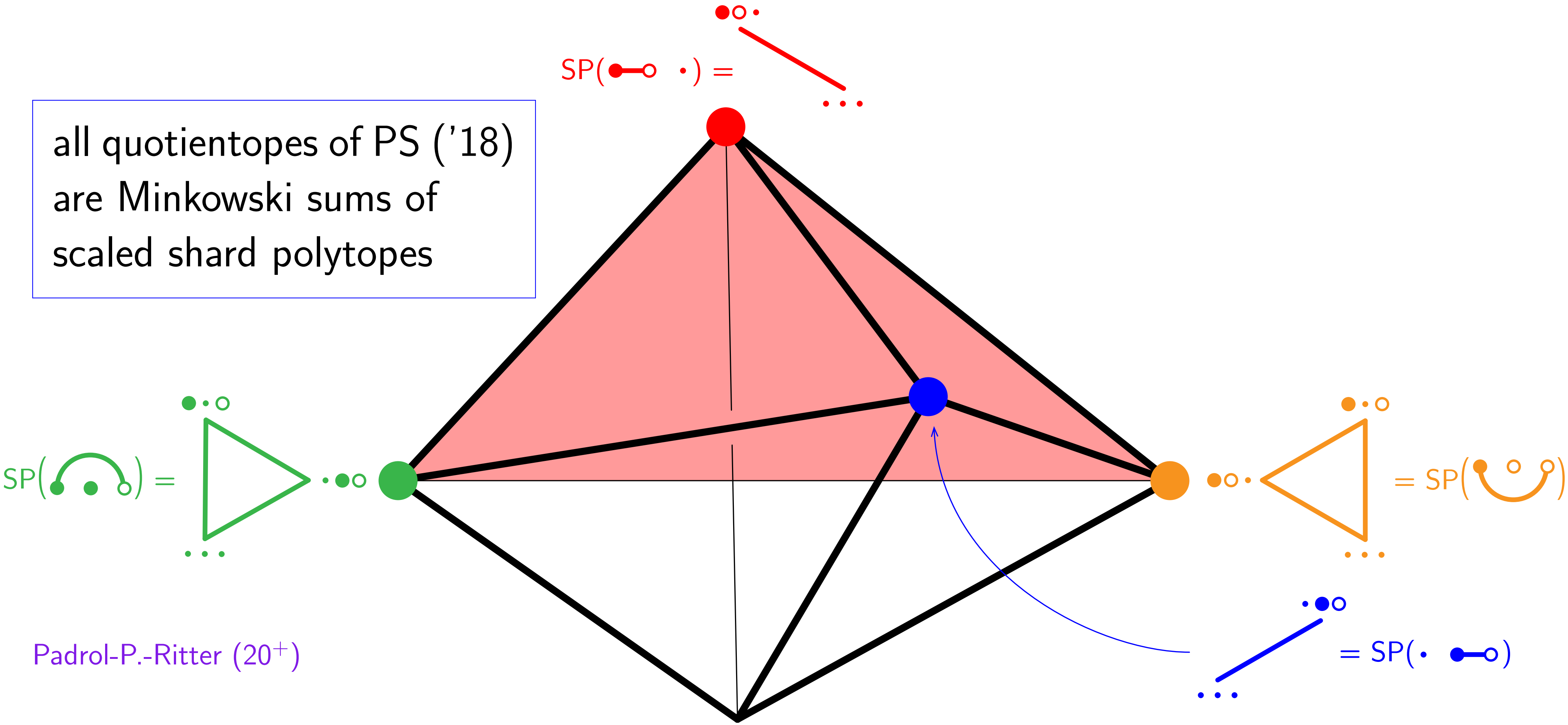

$$\dim \mathbb{TC}(\mathcal{F}) = N - n = 6 - 2 = 4$$

# SUBMODULAR FUNCTIONS



# SUBMODULAR FUNCTIONS

all quotientopes of PS ('18) are Minkowski sums of scaled shard polytopes




Padrol-P.-Ritter ($20^+$)

# SUBMODULAR FUNCTIONS

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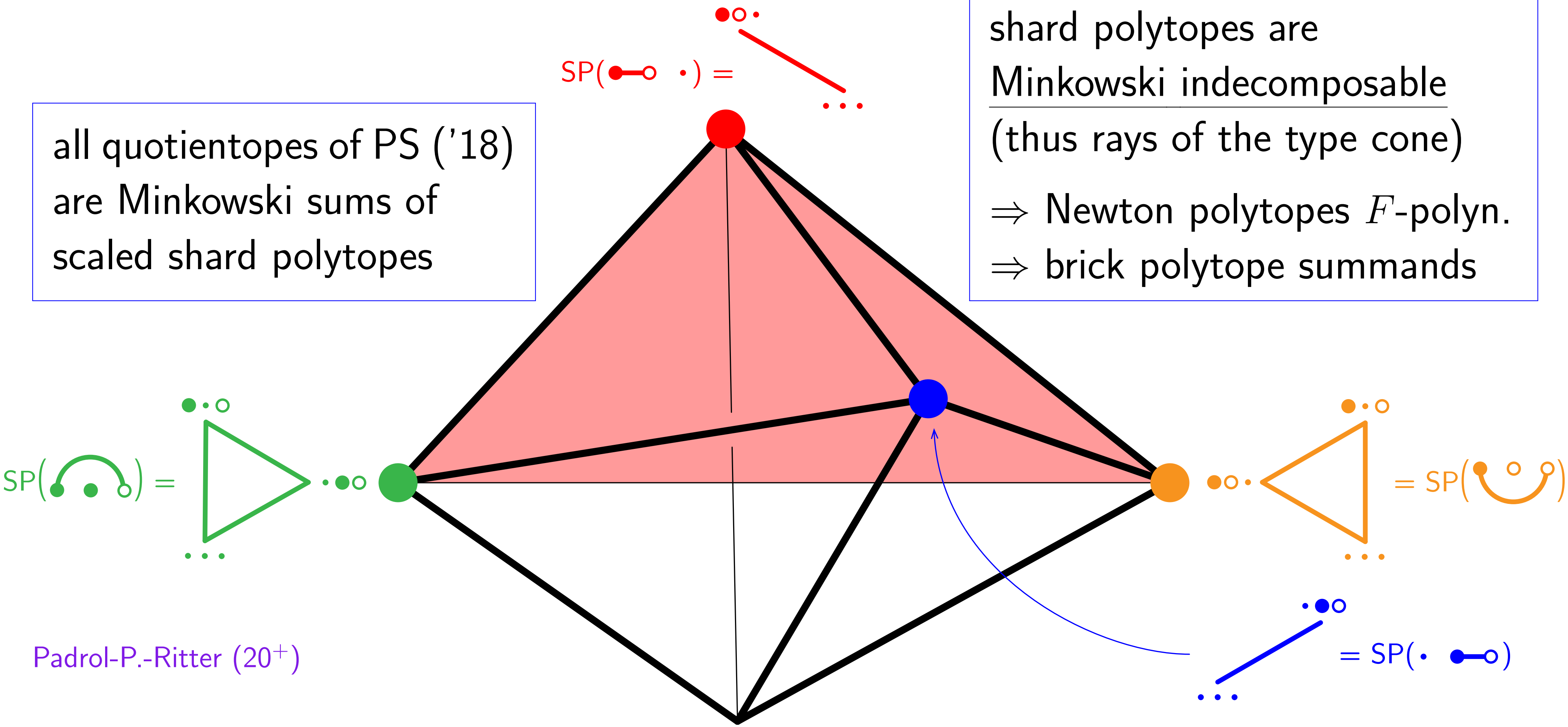
shard polytopes are Minkowski indecomposable (thus rays of the type cone)

$\Rightarrow$ Newton polytopes $F$-polyn.

$\Rightarrow$ brick polytope summands

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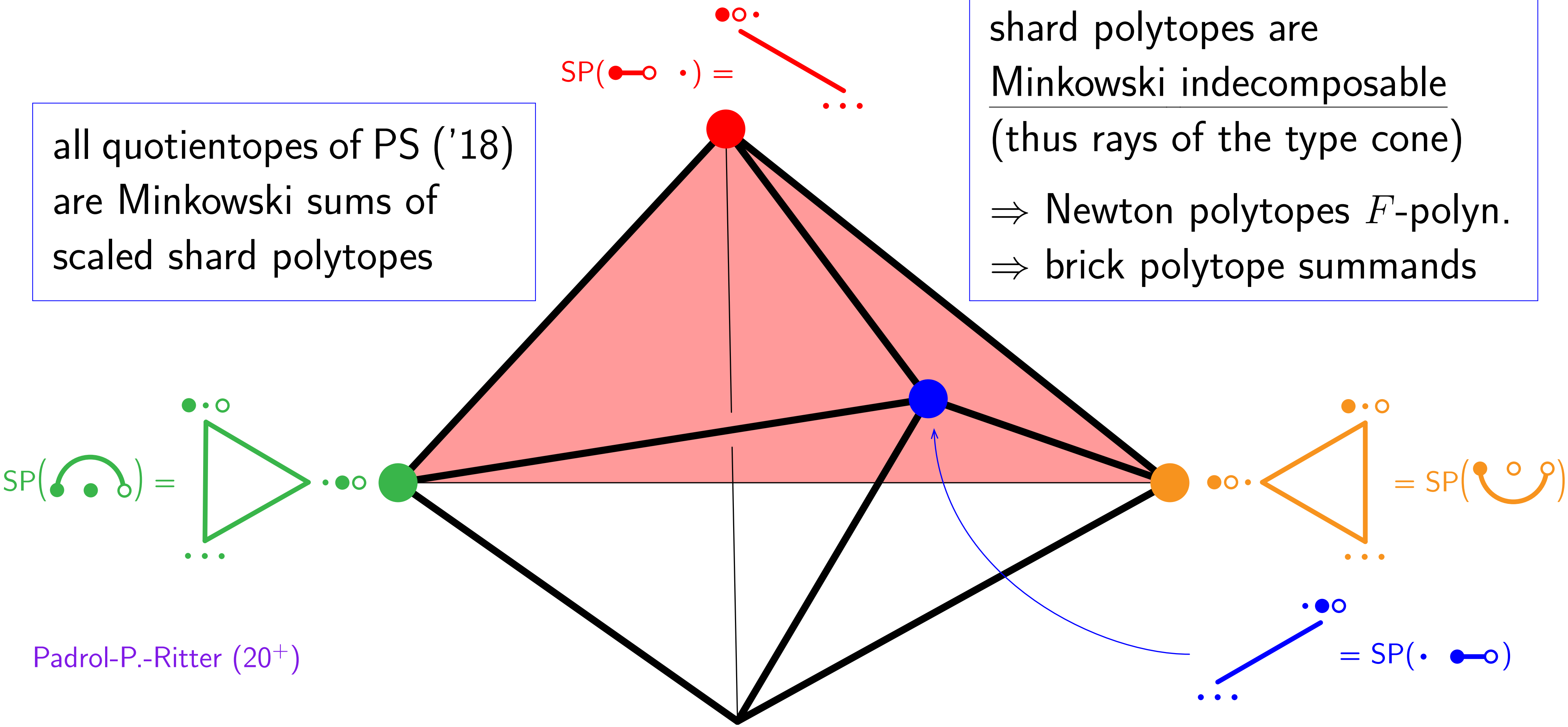



Padrol-P.-Ritter (20[+])

Any deformed permutahedron is a Minkowski sum and difference of shard polytopes

$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \sum_{J \in \mathcal{J}} y_J \triangle_J = \sum_{I \in \mathcal{J}} s_I \, \mathbb{SP}(\Sigma_I)$$

with explicit (combinatorial) exchange matrices between the parameters $\boldsymbol{s}$, $\boldsymbol{y}$ and $\boldsymbol{z}$

# OPEN QUESTIONS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
base region $B$ = distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$
inversion set of a region $C$ = set of hyperplanes of $\mathcal{H}$ that separate $B$ and $C$
poset of regions PR$(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

The poset of regions PR$(\mathcal{H}, B)$ — Björner-Edelman-Ziegler ('90)

- is never a lattice when $B$ is not a simplicial region
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement

If PR$(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a congruence of PR$(\mathcal{H}, B)$, the cones obtained by glueing the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan $\mathcal{F}_\equiv$

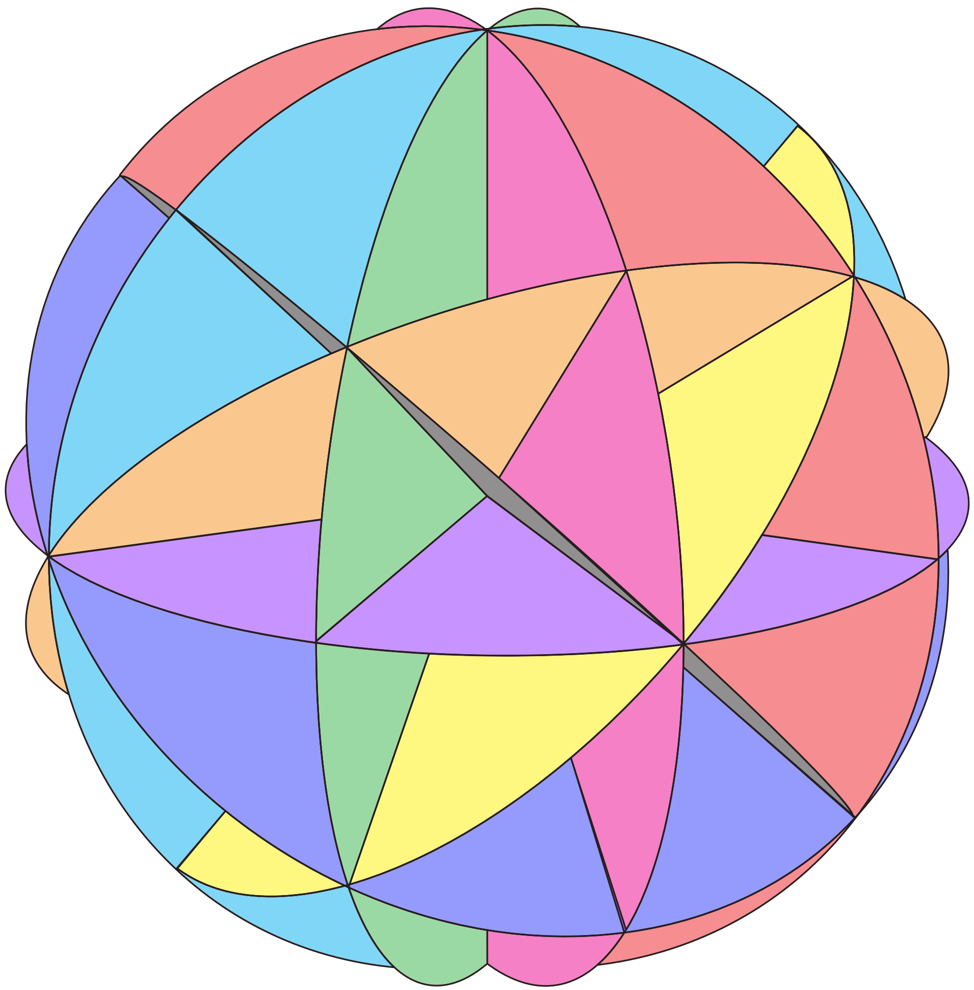
Reading ('05)

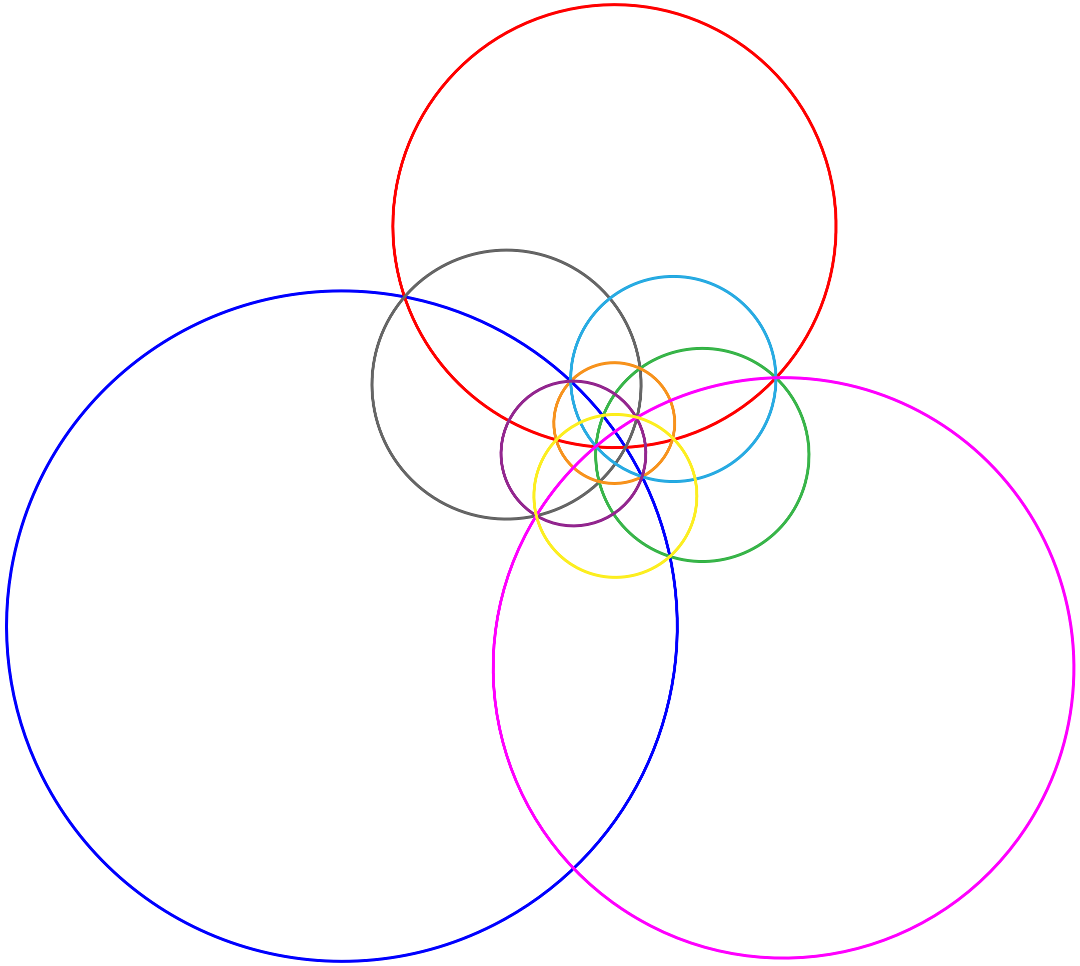
Is the quotient fan $\mathcal{F}_\equiv$ always polytopal?

# OPEN QUESTIONS

shard = piece of hyperplane obtained after cutting all rank 2 subgroups

shard poset = (pre)poset of forcing relations among shards





Reading ('03)

# OPEN QUESTIONS

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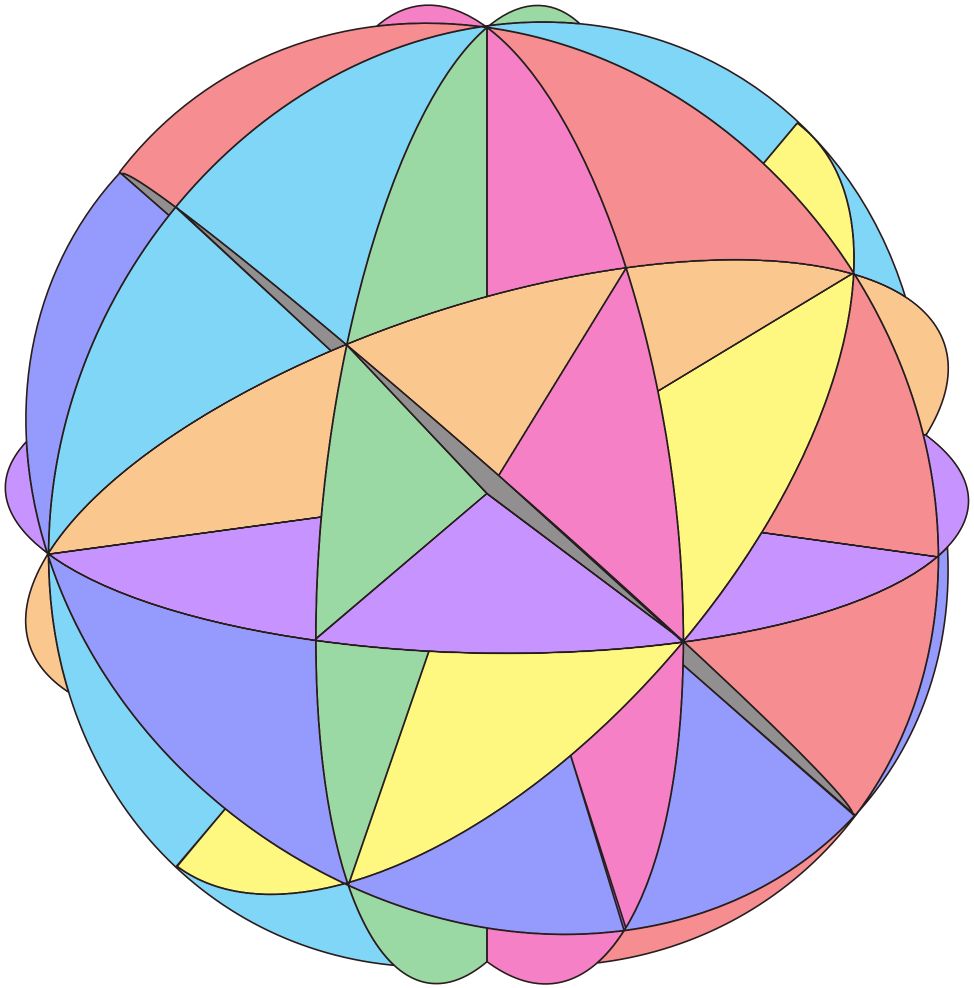

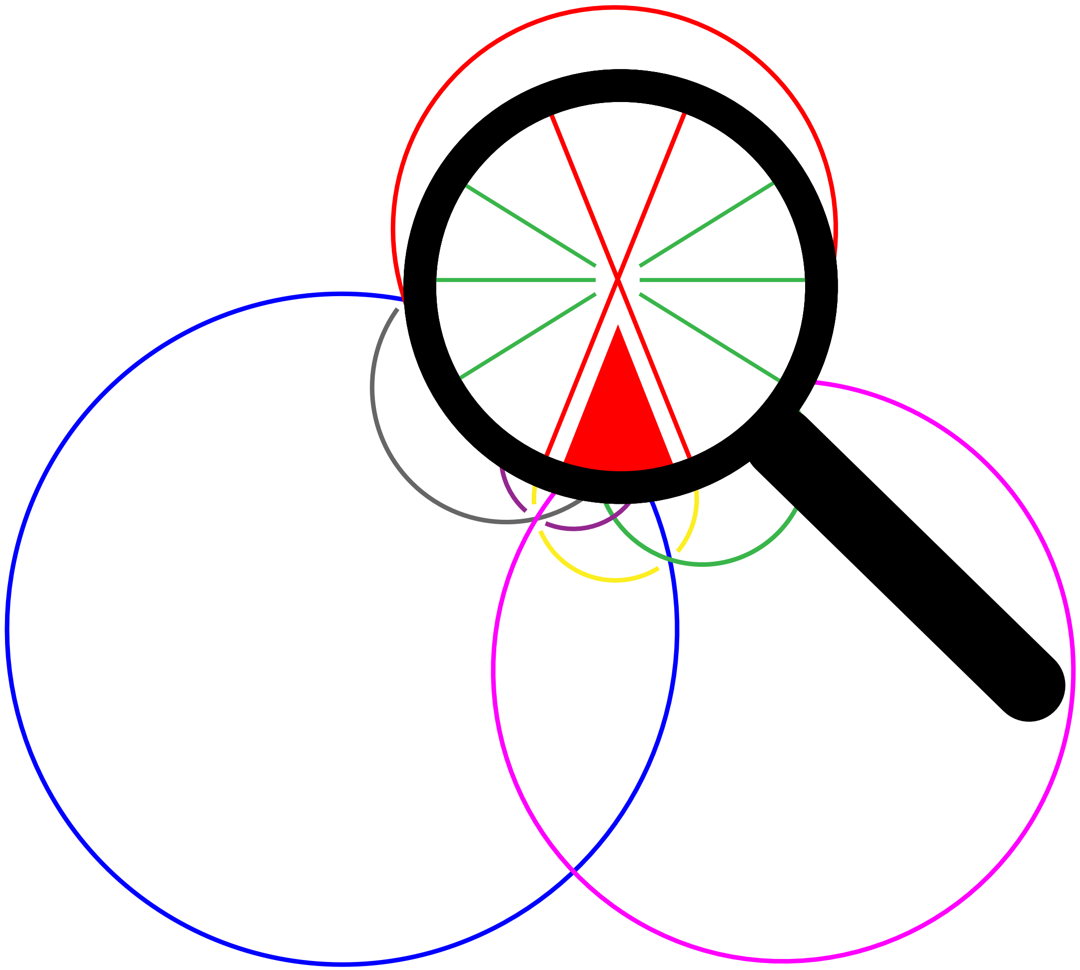

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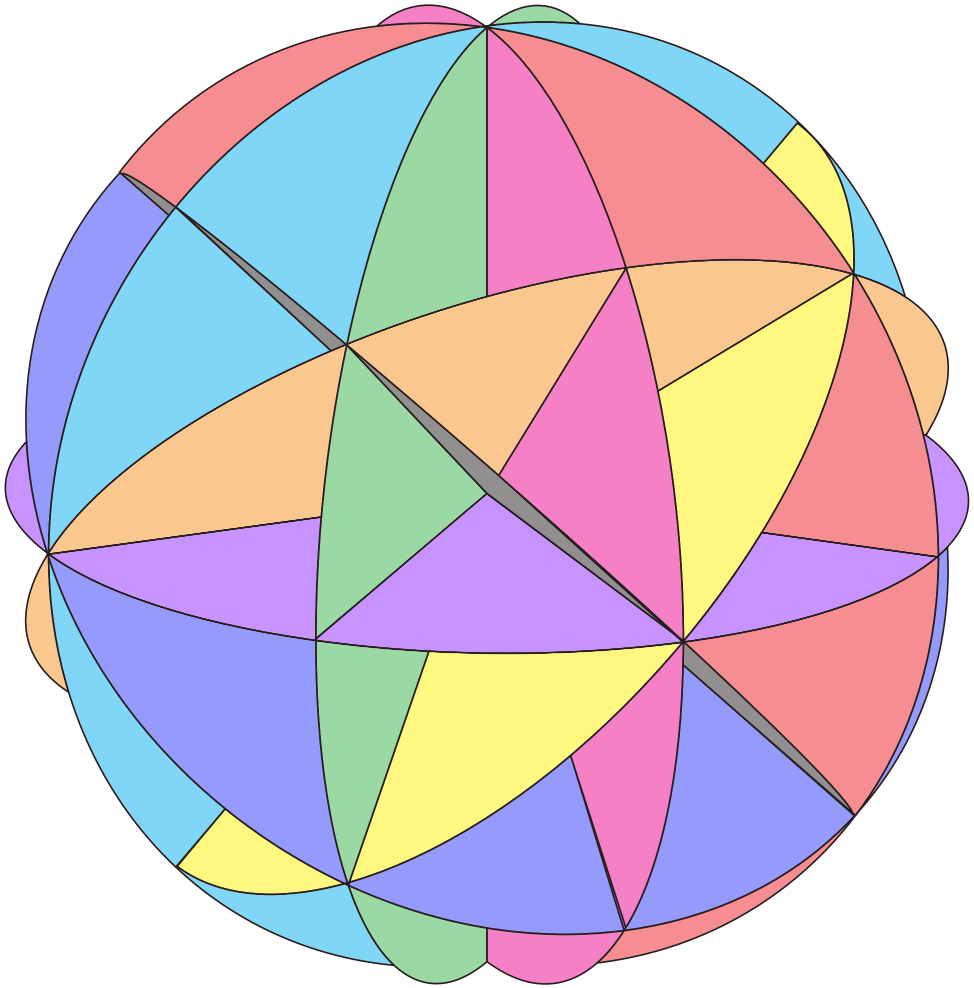

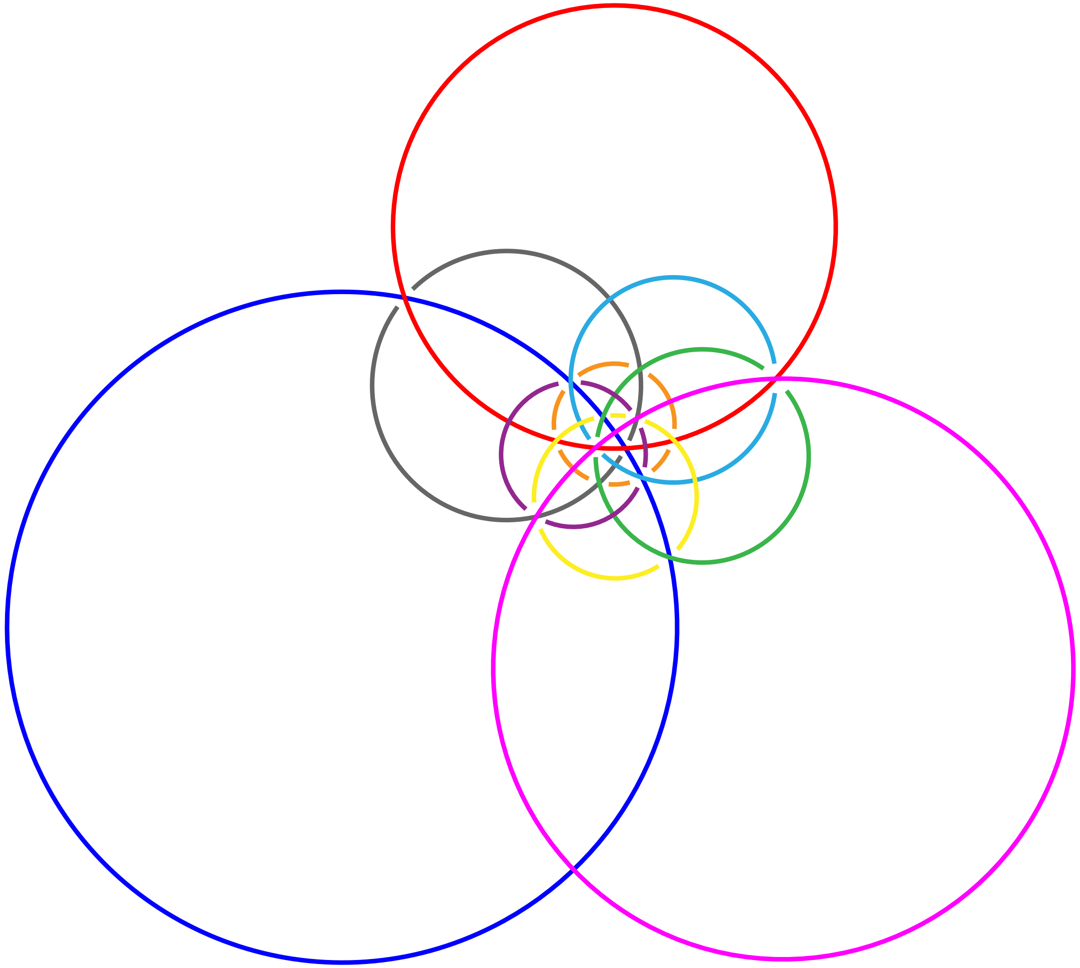

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shard polytope of a shard $\Sigma$ = polytope such that the union of the walls of its normal fan

- contains the shard $\Sigma$,
- is contained in the union of the shards forcing $\Sigma$

Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

If any shard $\Sigma$ admits a shard polytope $\mathbb{SP}(\Sigma)$, then

- for any lattice congruence $\equiv$ of PR$(\mathcal{H}, B)$, the quotient fan $\mathcal{F}_{\equiv}$ is the normal of the Minkowski sum of the shard polytopes $\mathbb{SP}(\Sigma)$ for $\Sigma$ in the shard ideal $\boldsymbol{\Sigma}_{\equiv}$
- if the arrangement $\mathcal{H}$ is simplicial, then the shard polytopes $\mathbb{SP}(\Sigma)$ form a basis for the type cone of the fan defined by $\mathcal{H}$ (up to translation)

Padrol-P.-Ritter ($20^{+}$)

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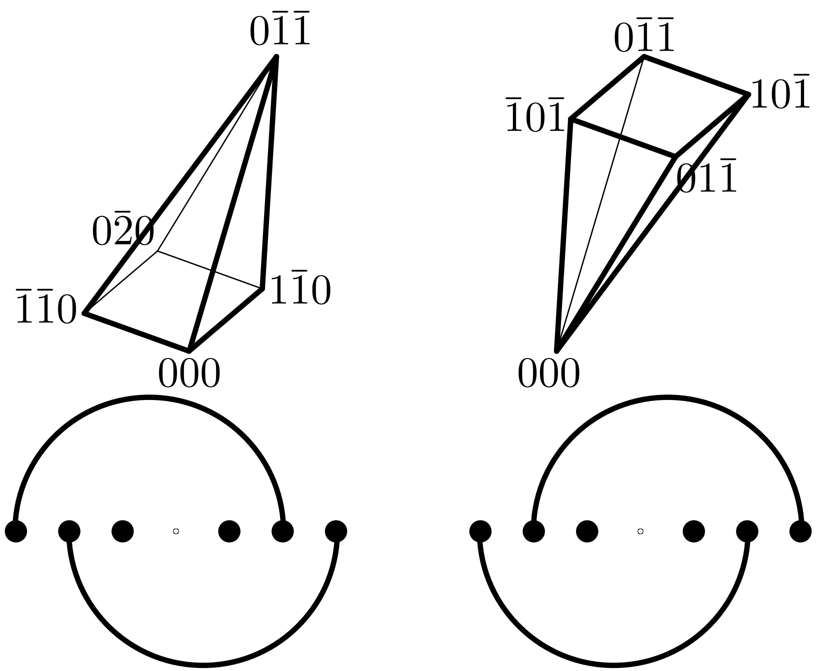
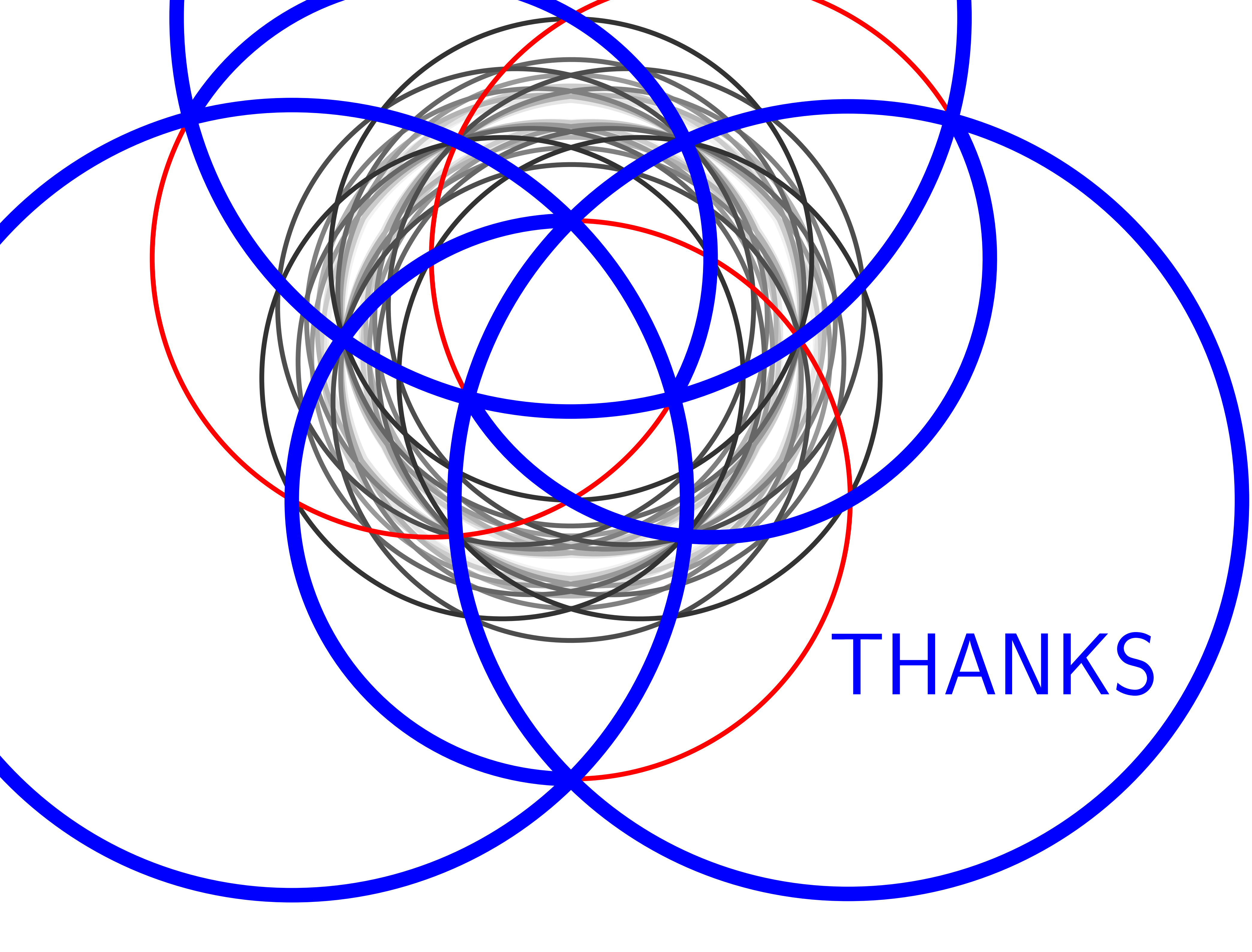
Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

For crystallographic arrangements,
Newton polytopes of $F$-polynomials
all seem to be shard polytopes,
but some shards are missing...




THANKS