On the type of some semigroups

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Basic definitions

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$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda a_n \mid n, \lambda_1, \ldots, \lambda_n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

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Given a field $\Bbbk$, the *numerical semigroup ring* associated with S and $\Bbbk$ is the one-dimensional domain $\Bbbk[S] = \Bbbk[t^s \mid s \in S]$, where t is an indeterminate. This ring is strictly related to $\Bbbk[[S]] = \Bbbk[[t^s \mid s \in S]]$.

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Consider the following numerical semigroup:

$$S = \langle 5, 6, 14 \rangle = \{0, 5, 6, 10, 11, 12, 14 \rightarrow\}$$

In this case $\nu(S) = 3$, $\mathrm{F}(S) = 13$, and $m(S) = 5$.

The multiplicity and embedding dimension of S are the multiplicity and embedding dimension of $\Bbbk[[S]]$.

Symmetric numerical semigroups

By definition $\mathrm{F}(S) \notin S$. Moreover, if $s \in S$, then $\mathrm{F}(S) - s \notin S$ otherwise $\mathrm{F}(S) = s + (\mathrm{F}(S) - s) \in S$.

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The previous example $S = \langle 5, 6, 14 \rangle = \{0, 5, 6, 10, 11, 12, 14 \rightarrow\}$ is not symmetric because 4 and $13 - 4 = 9$ are not in S.

While $\langle 5, 6 \rangle = \{0, 5, 6, 10, 11, 12, 15, 16, 17, 18, 20 \rightarrow\}$ is symmetric.

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Theorem (Kunz, 1970) *S is symmetric if and only if $\Bbbk[S]$ is Gorenstein.*

This result holds not only for numerical semigroup rings, but also for analytically irreducible rings.

Symmetric numerical semigroups

The set $K(S) = \{z \in \mathbb{Z} \mid \mathrm{F}(S) - z \notin S\}$ is called *canonical ideal* of S. It is not difficult to see that $S \subseteq K(S)$ and that the equality holds if and only if S is symmetric.

The fractional ideal $(t^z \mid z \in K(S))$ is a canonical module of $\mathbb{k}[[S]]$.

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Definition The set of *pseudo-Frobenius numbers* of S is

$$\mathrm{PF}(S) = \{f \in \mathbb{Z} \setminus S \mid f + s \in S \text{ for every } s \in S \setminus \{0\}\}.$$

The *type* of S is $t(S) = |\,\mathrm{PF}(S)| \geq 1$.

The type $t(S)$ is equal to the Cohen-Macaulay type of $\Bbbk[S]$. In particular, S is symmetric if and only if $t(S) = 1$.

Apéry set

How can we find pseudo-Frobenius numbers?

Definition The *Apéry set* of S with respect to $n \in S \setminus \{0\}$ is

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If $x, y \in \mathrm{Ap}(S, n)$, set $x \leq_S y$ if $y = x + s$ for some $s \in S$.

Proposition

- *The minimal elements of* $\mathrm{Ap}(S, n) \setminus \{0\}$ *with respect to* $\leq_S$ *are the minimal generators of* S.
- *The maximal elements of* $\mathrm{Ap}(S, n)$ *with respect to* $\leq_S$ *minus* n *are the pseudo-Frobenius numbers of* S.

Pseudo-symmetric numerical semigroups

If $\mathrm{F}(S)$ is even, it follows that $\mathrm{F}(S)/2 \notin S$. Therefore, since also $\mathrm{F}(S) - \frac{\mathrm{F}(S)}{2} = \frac{\mathrm{F}(S)}{2} \notin S$, the semigroup S is never symmetric.

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Definition A numerical semigroup S is said to be *pseudo-symmetric* if $\mathrm{F}(S)$ is even and $s \in S$ if and only if $\mathrm{F}(S) - s \notin S$ for every $s \in \mathbb{Z} \setminus \{\frac{\mathrm{F}(S)}{2}\}$.

The numerical semigroup $\langle 5, 6, 13 \rangle = \{0, 5, 6, 10, 11, 12, 13, 15 \rightarrow\}$ is pseudo-symmetric.

We note that a pseudo-symmetric numerical semigroup has always type 2, but the converse is not true.

Almost symmetric numerical semigroups

We have already noted that if $s \in S$, then $\mathrm{F}(S) - s \notin S$.

Also, we have seen examples in which both s and $\mathrm{F}(S) - s$ are not in S. In this case we say that s and $\mathrm{F}(S) - s$ are *gaps of the second type* for S and we denote the set of these gaps by $L(S)$.

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Definition (Barucci and Fröberg, 1997) A numerical semigroup S is *almost symmetric* if $L(S) \subseteq \mathrm{PF}(S)$, or equivalently $\mathrm{F}(S) - f \in \mathrm{PF}(S)$ for every $f \in \mathrm{PF}(S) \setminus \{\mathrm{F}(S)\}$.

Let $S = \langle 4, 6, 9, 11 \rangle = \{0, 4, 6, 8 \rightarrow\}$. In this case $L(S) = \{2, 5\}$ and $\mathrm{PF}(S) = \{2, 5, 7\}$, then S is almost symmetric.

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The symmetric and pseudo-symmetric semigroups are exactly the almost symmetric semigroups with type 1 and 2 respectively.

Nearly Gorenstein rings

Let $\mathbb{k}$ be a field and let R be a Cohen-Macaulay positively graded $\mathbb{k}$-algebra with graded maximal ideal $\mathfrak{m}$ and canonical module ω_R. The *trace ideal* of ω_R is the ideal

$$\operatorname{tr}(\omega_R) = \sum_{\varphi \in \operatorname{Hom}_R(\omega_R, R)} \varphi(\omega_R).$$

It describes the non-Gorenstein locus of R: if $\mathfrak{p} \in \operatorname{Spec}(R)$, the ring $R_{\mathfrak{p}}$ is not Gorenstein if and only if $\operatorname{tr}(\omega_R) \subseteq \mathfrak{p}$.

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These rings also appear in other previous works by Ding (1993), Huneke-Vraciu (2006), Striuli-Vraciu (2011)...

Computing the trace

With some assumptions on R, it holds that $\operatorname{tr}(\omega_R) = \omega_R\omega_R^{-1}$, where $\omega_R^{-1} = (R :_{Q(R)} \omega_R) = \{x \in Q(R) \mid x\omega_R \subseteq R\}$.

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$$\mathrm{tr}(K(S)) = K(S) + K'(S),$$

where $K'(S) = S - K(S) = \{x \in \mathbb{Z} \mid x + K(S) \subseteq S\}$ and S is a *nearly Gorenstein semigroup* exactly when $S \setminus \{0\} \subseteq \mathrm{tr}(K(S))$.

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Example

Consider the semigroup $S = \langle 5, 6, 9, 13 \rangle = \{0, 5, 6, 9 \rightarrow\}$. Then, $K(S) = \{0, 1, 4, 5, 6, 7, 9 \rightarrow\}$ and $K'(S) = \{5, 9 \rightarrow\}$; it follows that $\mathrm{tr}(K(S)) = \{5, 6, 9 \rightarrow\} = S \setminus \{0\}$. Therefore, S is nearly Gorenstein.

Type of semigroups with few generators

We want to study the type of a numerical semigroup S with respect to its embedding dimension $\nu(S)$, the number of minimal generators.

- If $\nu(S) = 2$, then $t(S) = 1$.
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- If $\nu(S) = 3$, then $t(S) \leq 2$.
- If $\nu(S) = 4$, then there is no bound for $t(S)$.

Example (Backelin, 1987)

Given $n \geq 2$ and $r \geq 3n + 2$, let $s = r(3n + 2) + 3$ and

$$S = \langle s, s + 3, s + 3n + 1, s + 3n + 2 \rangle.$$

Then, $\nu(S) = 4$ and $t(S) \geq 2n + 2$.

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Numata conjectured that if $\nu(S) = 4$ and S is almost symmetric, then $t(S) \leq 3$. ***This was proved by Moscariello (2016)***.

Row factorization matrices

Let $S = \langle n_1, \ldots, n_\nu \rangle$, where $n_1, n_2, \ldots, n_\nu$ are minimal generators. For every $f \in \mathrm{PF}(S)$ and every $i = 1, \ldots, \nu$ we have

$$f + n_i = \sum_{j=1}^{\nu} a_{ij} n_j$$

with $a_{ij} \geq 0$ and $a_{ii} = 0$.

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Definition (Moscariello, 2016) A square matrix $A = (a_{ij})$ of order ν is a *row factorization matrix* for f (briefly RF^+ *matrix*), if $a_{ii} = -1$, $a_{ij} \in \mathbb{N}$ when $i \neq j$ and $f = \sum_{j=1}^{\nu} a_{ij} n_j$ for all i.

A pseudo-Frobenius number can have more RF^+ matrices.

Moscariello's idea

Let $S = \langle 4, 7, 10, 13 \rangle$. In this case $\mathrm{PF}(S) = \{3, 6, 9\}$. For example $3 + 4 = 1 \cdot 7, 3 + 7 = 1 \cdot 10, 3 + 10 = 1 \cdot 13, 3 + 13 = 4 \cdot 4$.

$$\mathrm{RF}^{+}(3) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 4 & 0 & 0 & -1 \end{pmatrix},$$

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Proposition (Moscariello, 2016) *Assume that* $f, \mathrm{F}(S) - f \in \mathrm{PF}(S)$. *Let* (a_{ij}) *and* (b_{ij}) *be two* RF^+ *matrices for* f *and* $\mathrm{F}(S) - f$ *resp. Then,* $a_{ij} b_{ji} = 0$ *for every* $i \neq j$.

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If $\nu = 4$, in (a_{ij}) and (b_{ij}) there are at least 12 zeroes and 4 rows with two 0.

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IDEA: *$an_i - n_j$ and $bn_i - n_j$ cannot be both in $\mathrm{PF}(S)$. There are 12 writings $\lambda n_i - n_j$ and these correspond to the rows with two 0.*

Nearly Gorenstein vectors

Let $G(S)$ be the generators of S. Recall that S is almost symmetric if $\mathrm{F}(S) - f \in \mathrm{PF}(S)$ for all $f \in \mathrm{PF}(S)$; equivalently $n + \mathrm{F}(S) - f \in S$ for all $f \in \mathrm{PF}(S)$ and every $n \in \mathrm{G}(S)$.

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Proposition (Moscariello and S., 2021) *A semigroup S is nearly Gorenstein if and only if for every $n_i \in \mathrm{G}(S)$ there exists $f_i \in \mathrm{PF}(S)$ such that $n_i + f_i - f \in S$ for all $f \in \mathrm{PF}(S)$.*

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In particular, an almost symmetric semigroup is nearly Gorenstein.

Definition Let $S = \langle n_1, \ldots, n_\nu \rangle$, where $n_1 < \cdots < n_\nu$ are minimal generators. We call $(f_1, \ldots, f_\nu) \in \mathrm{PF}(S)^\nu$ *nearly Gorenstein vector* for S, briefly NG-*vector*, if $n_i + f_i - f \in S$ for all $f \in \mathrm{PF}(S)$ and $i = 1, \ldots, \nu$.

Hence, S is nearly Gorenstein if and only if it admits a NG-vector.

An example

Let $S = \langle 10, 12, 37, 75 \rangle$. We have $\mathrm{PF}(S) = \{38, 63, 65\}$.

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37 + 38 - 38 = 37 \in S & 37 + 38 - 63 = 12 \in S & 37 + 38 - 65 = 10 \in S \\
75 + 38 - 38 = 75 \in S & 75 + 38 - 63 = 50 \in S & 75 + 38 - 65 = 48 \in S
\end{array}
$$

So, $(65, 63, 38, 38)$ is an NG-vector, and hence S is nearly Gorenstein.

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12 + 63 - 38 = 37 \in S & 12 + 63 - 63 = 12 \in S & 12 + 63 - 65 = 10 \in S \\
37 + 38 - 38 = 37 \in S & 37 + 38 - 63 = 12 \in S & 37 + 38 - 65 = 10 \in S \\
75 + 38 - 38 = 75 \in S & 75 + 38 - 63 = 50 \in S & 75 + 38 - 65 = 48 \in S
\end{array}
$$

So, $(65, 63, 38, 38)$ is an NG-vector, and hence S is nearly Gorenstein. Note also that

$$
\begin{array}{lll}
75 + 63 - 38 = 100 \in S & 75 + 63 - 63 = 75 \in S & 75 + 63 - 65 = 73 \in S \\
75 + 65 - 38 = 102 \in S & 75 + 65 - 63 = 77 \in S & 75 + 65 - 65 = 75 \in S
\end{array}
$$

An example

Let $S = \langle 10, 12, 37, 75 \rangle$. We have $\mathrm{PF}(S) = \{38, 63, 65\}$. Note that

$$\begin{array}{lll} 10 + 65 - 38 = 37 \in S & 10 + 65 - 63 = 12 \in S & 10 + 65 - 65 = 10 \in S \\ 12 + 63 - 38 = 37 \in S & 12 + 63 - 63 = 12 \in S & 12 + 63 - 65 = 10 \in S \\ 37 + 38 - 38 = 37 \in S & 37 + 38 - 63 = 12 \in S & 37 + 38 - 65 = 10 \in S \\ 75 + 38 - 38 = 75 \in S & 75 + 38 - 63 = 50 \in S & 75 + 38 - 65 = 48 \in S \end{array}$$

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Therefore, also $(65, 63, 38, 63)$ and $(65, 63, 38, 65)$ are NG-vectors. It is easy to see that these three are the only ones.

RF^- matrices

Let $S = \langle n_1, \ldots, n_\nu \rangle$ be nearly Gorenstein, where $n_1 < \cdots < n_\nu$ are minimal generators. Fix an NG-vector $(f_1, \ldots, f_\nu)$.

For every $f \in \mathrm{PF}(S)$ and for every i such that $f \neq f_i$ we have

$$n_i + f_i - f = \sum_{j=1}^{\nu} b_{ij} n_j$$

with $b_{ij} \geq 0$ and $b_{ii} = 0$.

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with $b_{ij} \geq 0$ and $b_{ii} = 0$. Thus, we can define another matrix:

Definition A square matrix $B = (b_{ij})$ of order ν is an RF^- *matrix* for f if B satisfies the following properties:

- if $f = f_i$, in the i-th row of B there are only zeroes;
- otherwise $b_{ii} = -1$ and $f_i - f = \sum_{j=1}^{\nu} b_{ij} n_j$.

RF^{-} matrices

$S = \langle 10, 12, 37, 75 \rangle$ is nearly Gorenstein since $(65, 63, 38, 63)$ is an NG-vector for S. Then

$$\mathrm{RF}^{+}(38) = \begin{pmatrix} -1 & 4 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 4 & 3 & 1 & -1 \end{pmatrix}, \ \mathrm{RF}^{-}(38) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & -1 \end{pmatrix}$$

$$10 + f_1 - 38 = 1 \cdot 37 \qquad 12 + f_2 - 38 = 1 \cdot 37$$
$$f_3 = 38 \qquad 75 + f_4 - 38 = 4 \cdot 10 + 5 \cdot 12$$

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Lemma *Let* (a_{ij}) *and* (b_{ij}) *be an* RF^+ *and an* RF^- *matrix for* $f \in \mathrm{PF}(S)$ *respectively. Then,* $a_{jk}b_{kj} = 0$ *for every* $j \neq k$.

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Lemma *Let (a_{ij}) and (b_{ij}) be an RF^+ and an RF^- matrix for $f \in \mathrm{PF}(S)$ respectively. Then, $a_{jk}b_{kj} = 0$ for every $j \neq k$.*

We have two matrices associated with a pseudo-Frobenius number!

Upper bounds for the type

Theorem (Moscariello and S., 2021) *If $S = \langle n_1, n_2, n_3, n_4 \rangle$ is nearly Gorenstein, then $t(S) \leq 3$.*

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There exist nearly Gorenstein numerical semigroups s.t. $\nu(S) < t(S)$.

If $S = \langle 64, 68, 73, 77, 84, 93 \rangle$, then $\nu(S) = 6$ and $t(S) = 9$, since $\mathrm{PF}(S) = \{159, 179, 188, 195, 197, 206, 215, 394, 403\}$.

Affine semigroups

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathbb{N}^d$, where d is a positive integer. The associated *affine semigroup* is the semigroup

$$S = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i \mid \lambda_i \in \mathbb{N} \text{ for } i = 1, \ldots, n \right\}.$$

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We always assume that S is fully embedded in $\mathbb{N}^d$.

Given a field $\Bbbk$, the *affine semigroup ring* associated with S and $\Bbbk$ is the d-dimensional subalgebras of $\Bbbk[x_1, \ldots, x_d]$ given by

$$\Bbbk[S] = \Bbbk[\boldsymbol{x}^{\boldsymbol{a}} \mid \boldsymbol{a} \in S],$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)$ and $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}$.

Simplicial affine semigroups

$$\text{cone}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i \mid \lambda_i \in \mathbb{R} \text{ for } i = 1, \ldots, n \right\}$$

is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^d$, each of whose bounding hyperplanes contains the origin.

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We assume that S is simplicial and $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_d$ are the componentwise smallest non-zero vectors of S of each extremal ray of $\text{cone}(S)$.

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This is a finite set. If $x, y \in \mathrm{Ap}(S, E)$, set $x \leq_S y$ if $y = x + s$ for $s \in S$.

Proposition (Jafari and Yaghmaei, 2021) *If $\Bbbk[S]$ is Cohen-Macaulay, its type $t(S)$ is equal to the number of maximal elements in $\mathrm{Ap}(S, E)$ with respect to $\leq_S$.*

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Let G be the group generated by $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_d$.

Proposition (Rosales and García-Sánchez, 1998) *TFAE:*

- *$\Bbbk[S]$ is Cohen-Macaulay;*
- *For all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \operatorname{Ap}(S, E)$, if $\boldsymbol{w}_1 - \boldsymbol{w}_2 \in G$, then $\boldsymbol{w}_1 = \boldsymbol{w}_2$.*

Trace

Let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_t$ be the maximal elements in $\mathrm{Ap}(S, E)$ wrt $\leq_S$.

Proposition (Jafari, Zarzuela Armengou) $\Bbbk[S]$ *is nearly Gorenstein if and only if for every* $\boldsymbol{a}_i$ *there exists* $\boldsymbol{m}_i$ *s.t.* $\boldsymbol{a}_i + \boldsymbol{m}_i - \boldsymbol{m}_j \in S$ *for all* $j = 1, \ldots, t$.

Therefore, the notion of *NG-vector* can be defined also in this context.

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Therefore, the notion of *NG-vector* can be defined also in this context.

Actually, a more general result holds.

Theorem *The trace of the canonical module of $\Bbbk(S)$ is*

$$\{\boldsymbol{b} \mid \text{there exists } i \text{ s.t. } \boldsymbol{b} + \boldsymbol{m}_i - \boldsymbol{m}_j \in S \text{ for all } j = 1, \ldots, t\}.$$

Type

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Recalling that for a numerical semigroup with embedding dimension 4 the type is at most three, it is natural to ask:

Question *Let S be a nearly Gorenstein affine semigroup with embedding dimension $d+3$. Is $t(S) \leq d+2$?*

THANK YOU!