# Toric Varieties for Discrete Geometers

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Sources:

- B. Sturmfels, "Gröbner bases and convex polytopes", Lect. Univ. Series, Amer. Math. Soc., 1995

- D. Cox, "What is a toric variety?" in "Topics in algebraic geometry and geometric modeling" Contemp. Math 334, Amer. Math. Soc., 2003, pp 203-223

(Plus: Fulton, "Introduction to toric varieties", Princeton U. P., 1993.
Cox, Little, Schenk, "Toric varieties", A.M.S, 2011.

# The five ways to a toric variety

(I) The toric variety $V_A$ of a finite set $A \subseteq \mathbb{Z}^d$

(II) The toric variety $V_S$ of an affine semigroup $S \subseteq \mathbb{Z}^d$

(III) The toric variety $\tilde{V}_A$ of a homogeneous pt. set $A \subseteq H \subseteq \mathbb{R}^d$

(IV) The toric variety $X_P$ of a rational polytope $P \subseteq \mathbb{R}^d$

(V) The toric variety $X_\Sigma$ of a rat. polyhedral fan $\Sigma$ in $\mathbb{R}^d$

same

closure in $\mathbb{P}^m$

quotient/restriction

special case of

| | affine | projective | abstract |
|---|---|---|---|
| normal | (*) | (IV) | (V) |
| perhaps not-normal | (I) (II) | (III) | |

(*) (I) is normal $\Longleftrightarrow$ $A$ generates $\mathrm{pos}(A) \cap \mathbb{Z}^d$

(II) is normal $\Longleftrightarrow$ $S = \sigma \cap \mathbb{Z}^d$ for a cone $\sigma$

(0) "Toric varieties for algebraic geometers"

> Def: " A toric variety is a complex algebraic variety $X$ that contains a torus $T \cong (\mathbb{C}^*)^d$ as an open dense subset, together with an action $T \cdot X \longrightarrow X$ of $T$ on $X$ that extends the standard action (multiplication) $T \cdot T \longrightarrow T$

Glossary:

Variety
- affine: the zero set of a system of finitely many polynomials $\subseteq \mathbb{C}^n$
- projective: same, but with homogeneous polynomials $\subseteq \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$
- abstract: (for this talk) an object constructed glueing affine varieties via transition maps.

Normal: "soft version" of smooth. A local preperty, that limits the type of singularitíes that you allow.

Algebraic torus: $T^n = (\mathbb{C}^*)^n$, with coordinate-wise multiplication

Etimology: $\underbrace{(\mathbb{C}^*)^n \cong \underbrace{(S^1)^n}_{\text{topologists' torus}} \times \underbrace{(0,\infty)^n}_{\text{topologically trivial}}}_{\text{as groups, with coordinate-wise multiplication, and } S^1 = \{z \in \mathbb{C} \mid |z| = 1\}}$

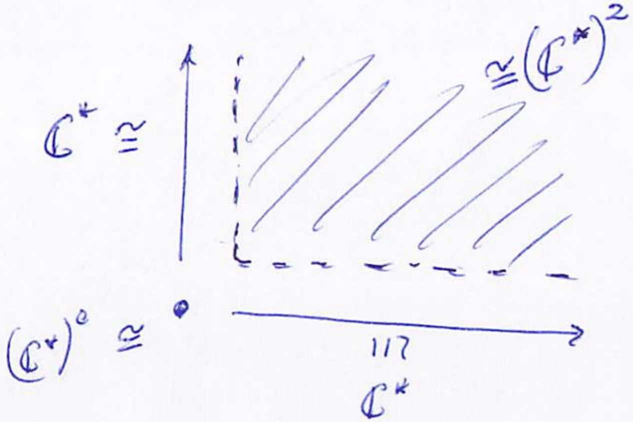
## (0') "Toric varieties for combinatorial topologists":

"Cell complexes" made of tori, with a single top-dimensional torus plus lower tori "glued to it".

$\boxed{\text{Example: } \mathbb{C}^2}$

$$\mathbb{C}^2 = (\mathbb{C}^*)^2 \cup \begin{matrix} \mathbb{C}^* \times \{0\} \\ \{0\} \times \mathbb{C}^* \end{matrix} \cup \{(0,0)\}$$

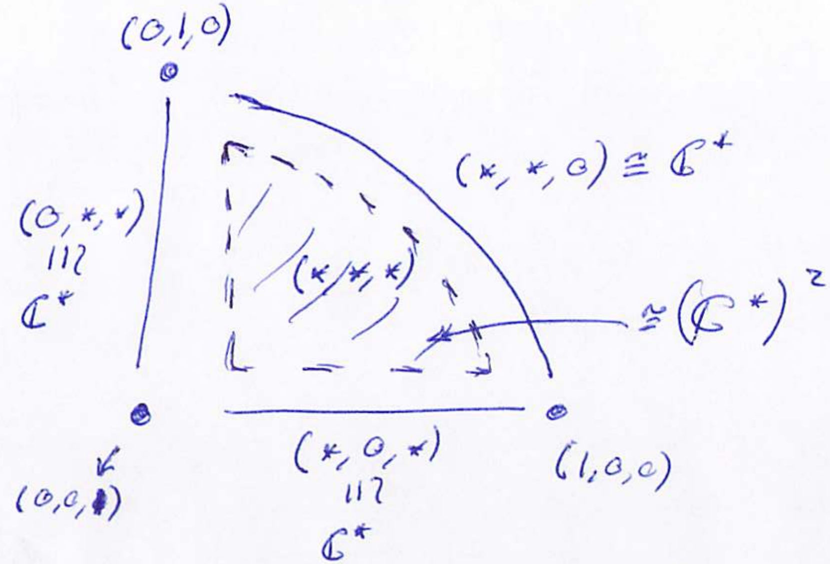
dim 2    2× dim 1    dim 0




$\mathbb{C}^2$ = "gluing of 4 tori, with f-vector (1, 2, 1)"

$\boxed{\text{Example: } \mathbb{P}^2}$

$$\mathbb{P}^2 = \{(x,y,z) \in \mathbb{C}^3 \setminus \{(000)\}\} \big/ \text{dilation}$$




$\mathbb{P}^2$ = "glueing of 7 tori, with f-vector (3,3,1)"

## (I) $\boxed{\text{The toric variety of } A = \{a_1, \dots, a_m\} \subseteq \mathbb{Z}^d}$

Embed $\phi : (\mathbb{C}^*)^d \hookrightarrow \mathbb{C}^m$ via $A$:

$$t = (t_1, \dots, t_d) \longmapsto (t^{a_1}, \dots, t^{a_m})$$

remark: negative exponents are ok since $t_i \neq 0$

Let $\boxed{V_A := \text{closure}\left(\text{Im}\,(\mathbb{C}^*)^d\right)}$

<u>Remarks</u>:

- linear relations in $A$ $\longrightarrow$ binomials vanishing on $V_A$
- In fact: $\boxed{I_A := I_{V(A)} \text{ is generated by those binomials}}$ (Sturmfels Prop. 4.11)
- $0 \in V_A \iff \text{pos}(A)$ is "pointed" or "strongly convex"
- More generally:

THE "STRUCTURE THEOREM" OF TORIC VARIETIES (my naming, see appendix A1) $\Longrightarrow$

| | |
|---|---|
| "Nonnegative covectors of $A$" | $\longleftrightarrow$ 0/1 points in $V_A$ (limits as $t \to 0$ of 1 parameter subgroups) |
| $\updownarrow$ | $\updownarrow$ |
| Faces of $\text{pos}(A)$ | torus orbits in $V_A$ ("complex of tori") |

## (II) The toric variety of an affine semigroup

**Semigroup of $A = \{a_1, \dots, a_m\}$** : $S_A := \{ n_1 a_1 + \dots + n_m a_m \mid n_i \in \mathbb{Z}_{\geq 0} \}$

**Affine semigroup** := $S_A$ for some $A$

**Remark:** if $A \subseteq A' \subseteq S_A$, then $\boxed{V_A \cong V_{A'}}$

$V_A \subseteq \mathbb{C}^m$, $V_{A'} \subseteq \mathbb{C}^{m'}$, $m' = |A'| > m$

new coords. depend on old ones polynomially

forget coords

(adding positive combinations of existing points in $A$ does not change the toric variety)

In particular, we can call $V_A$ the "toric variety of $S_A$".

**Special (important) case:** $S_\sigma = \sigma \cap \mathbb{Z}^d$ for a rat. polyhedral cone $\sigma$.

$S_\sigma$ is an "affine semigroup"

**Gordon's Lemma:** $S_\sigma$ is finitely generated.

If $\sigma$ is pointed: It has a minimal generating set: the Hilbert basis of $S_\sigma$

In this case $V_\sigma := V_{Hilbert(\sigma)}$ is normal. In fact:

[Thm 13.5, Sturmfels]

normal affine toric varieties (modulo algebraic isomorphism) $\longleftrightarrow$ rational polyhedral cones in $\mathbb{R}^d$ (modulo unimodular integer transformation)

## (III) The *projective* toric variety of a homogeneous $A$.

$A$ is homogeneous $:=$ $\exists$ lin. funct. $f$ with $f(A) = \text{constant} \neq 0$ (that is: $A$ is contained in an affine hyperplane away from $0$)

Sturmfels, Ch 13, calls this "graded"

In this case:

$\{$ linear relations in $A \}$ $=$ $\{$ affine relations in $A \}$ (affine geometry $=$ linear algebra restricted to an affine hyperplane)

(Sturmfels lemma 4.14) $\Longrightarrow$ the binomial ideal $I_A$ is homogeneous

$\Longrightarrow$ the affine variety $V_A \subseteq \mathbb{C}^m$ is a union of lines

$\Longrightarrow$ it gives a projective variety $\tilde{V}_A$ in $\mathbb{P}^{m-1}$

For homogeneous $A$:

$$\left\{ \begin{array}{c} \text{poset of tori} \\ \text{in } \tilde{V}_A \end{array} \right\} = \left\{ \begin{array}{c} \text{poset of} \\ \text{faces of} \\ \text{conv}(A) \end{array} \right\}$$

$\| \qquad\qquad\qquad\qquad \|$

poset of tori of $V_A$, excluding $0$ and reducing dimensions by 1 — poset of faces of pos$(A)$, excluding $0$ and reducing dimensions by 1

## (IV) The projective toric variety of a polytope.

Now let $P$ be a lattice polytope in $\mathbb{R}^d$.

We can build the projective toric variety of $P \cap \mathbb{Z}^d$, but this is not what people call "toric variety of $P$".

Remark: if we dilate $P$ by an integer factor $\nu \in \mathbb{N}$, at some point $V_{\nu P \cap \mathbb{Z}^d}$ becomes normal and after that it does not change (modulo isomorphism).

We define

$$X_P = \widetilde{V_{\nu P \cap \mathbb{Z}^d}}, \quad \text{for } \nu \text{ "large enough"}$$

($\nu = d$ is enough)

See appendix A.2

$$\boxed{\text{Remark: affine charts in } X_P \longleftrightarrow \text{vertex (or face) figures in } P}$$

Remark: If $P$ and $P'$ are normally equivalent → same normal fan then $X_P \cong X_{P'}$ because the construction depends only on the "vertex cones" $\{ \mathrm{pos}(P - v) \mid v \text{ a vertex of } P \}$

# (V) The toric variety of a fan

Let $\mathcal{N}$ be a rat. polyh. fan in $\mathbb{R}^d$ (E.g. the normal fan of a $P$)

① For each cone $\sigma \in \mathcal{N}$ (i.e. face $F$ of $P$) consider the polar cone $\sigma^\vee$ (that is, the face cone $\mathrm{pos}(P-F)$) and build the affine normal toric variety $U_\sigma := V_{\sigma^\vee \cap \mathbb{Z}^d}$ (the affine chart of the $X_P$ in the previous slide).

② Glue these affine charts in the natural way (e.g. they all contain one and the same torus $T \cong (\mathbb{C}^*)^d$)

Remark: • this recipe works for arbitrary fans.
• all normal toric varieties (in the sense of def ⓐ) can be constructed in this way.

| Fan | | Variety |
| --- | --- | --- |
| complete | $\longleftrightarrow$ | complete ($\Leftrightarrow$ compact) |
| polytopal | $\longleftrightarrow$ | projective |
| "regular" | $\longleftrightarrow$ | quasi-projective. |

## (A.1) $\boxed{\text{The structure theorem}}$

Let $A = \{a_1, \dots, a_m\} \subseteq \mathbb{Z}^d$. Let $C \subseteq [m]$.

Notation:

- $x|_C$ = "$x$ restricted to $C$", for $x \in \mathbb{C}^m$
- $1_C$ = "the 0/1 vector with support $C$"

T.F.A.E:

1. $\exists\, c : \mathbb{R}^d \to \mathbb{R}$ with
$$c(a_i) \begin{cases} = 0 & \text{if } i \notin C \\ > 0 & \text{if } i \in C \end{cases}$$
"positive covectors" = faces of pos(A)

2. $\nexists$ linear dependence $\sum_{i=1}^m \lambda_i a_i = 0$ with $\lambda_i \geq 0\ \forall i$ and $\lambda_i = 0$ if $i \notin C$

positive dependencies

3. $1_{[m] \setminus C} \in V_A$

4. $V_A$ contains an $x$ with $\operatorname{supp}(x) = [m] \setminus C$

proof: $\boxed{1 \Leftrightarrow 2}$ is Farkas Lemma (or, "vector/covector orthogonality")

$\boxed{3 \Rightarrow 4}$ is trivial

$\boxed{1 \Rightarrow 3}$ Let $c$ be as in 1 and notice that:
$$\lim_{\theta \to 0} \underbrace{(\theta^{c a_1}, \dots, \theta^{c a_m})} = 1_{[n] \setminus C}$$
As $\theta$ varies in $\mathbb{C}^*$ this is a "1-parameter subgroup" in $\mathrm{Im}(\phi)$.

$\boxed{4 \Rightarrow 2}$ Suppose 2 fails, and let $\lambda$ be as in 2. Since $\lambda \geq 0$, its binomial is of the form $x^\lambda - 1 = 0$

In particular, $\forall x = (x_1, \dots, x_m) \in V_A$
$$|x_1|^{\lambda_1} \cdot \ldots \cdot |x_m|^{\lambda_m} = 1$$
$\Rightarrow$ some $x_i$ with $i \in C$ has $|x_i|$ nonzero

$\Rightarrow$ No $x \in V_A$ has support $\subseteq [n] \setminus C$ □

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That is: faces of pos(A) = $\underbrace{\text{stratification of } V_A \text{ by support}}$ = $\underbrace{\text{0/1 points in } V_A}$

(Cox, Sect. 9) for "abstract" t.v., via fans

as posets, and as labelled by their $C$

strata are the orbits of the $(\mathbb{C}^*)^d$ action, and they are themselves tori

limit points of 1-parameter subgroups

## (A.2) The local structure (affine charts) of $X_P$

For any homogeneous $A$ we have

$$\begin{array}{ccccc} (\mathbb{C}^*)^d & \xrightarrow{\ \phi\ } & Im(\phi) \subseteq V_A & \longrightarrow & V_{\tilde{A}} \\ t & \longmapsto & (t^{a_1},\dots,t^{a_m}) & & \end{array}$$

quotient $\mathbb{P}^{m-1} = \mathbb{C}^m/\sim$

Each of the $m$ coordinates in $\mathbb{P}^{m-1}$ gives us an affine chart, obtained by setting it $= 1$.

That is, the $i$-th chart is the affine t.v. of $A-\{a_i\}$:

$$\boxed{\begin{array}{ccc} (\mathbb{C}^*)^d & \xrightarrow{\ \phi_i\ } & Im(\phi_i) \subseteq V_{A-\{a_i\}} = \overline{Im(\phi_i)} \\ t & \longmapsto & (t^{a_1-a_i},\dots,1,\dots,t^{a_m-a_i}) \end{array}}$$

That is:

(Sturmfels Lemma 13.10) Affine charts of $V_{\tilde{A}}$ = Affine toric varieties of the "vertex figures" (*) or "contractions" of $A$

polytope language

oriented matroid language

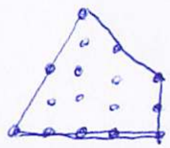
(*) strictly speaking, we have one chart for each lattice point in $P$, not only vertices

In particular:

As we dilate $P$, the affine charts depend on more and more points in the vertex cones.

After we get a Hilbert basis in every cone, the affine charts change no more.

$P =$ 

$2P =$ , etc.

On the other hand, since normality is local, when all affine charts are normal so is $V_{\tilde{A}}$