Additive number theory & monomial curves

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Algebraic combinatorics and its connections to geometry

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My lines in the project

Subline 3.4 Additive combinatorics and commutative algebra

P. Gimenez and M.G.S. *Castelnuovo-Mumford regularity of projective monomial curves via sumsets*. Mediterr. J. Math. **20**, 287 (2023)

Subline 3.3 Semigroup rings

I. García-Marco, P. Gimenez and M.G.S. *Projective Cohen-Macaulay monomial curves and their affine projections*. Soon on arxiv

Additive Numer Theory (ANT)

Let $A = \{a_0 < a_1 < \cdots < a_{n-1}\} \subset \mathbb{Z}$ be a finite set of integers. We will assume that $n = |A| \geq 4$.

- Sumsets of A: $0A := \{0\}$ and for all $s \geq 1$,

$$sA = \{a_{i_1} + \cdots + a_{i_s} \mid 0 \leq i_1 \leq \cdots \leq i_s \leq n-1\}.$$

- Main problem: study the sumsets of A and their cardinality.
- A is in *normal form*, $A = \{a_0 = 0 < a_1 < \cdots < a_{n-1} =: d\}$ with $\gcd(a_1, \ldots, a_{n-1}) = 1$.

Example:

$$A = \{8, 29, 71, 92\} \xrightarrow{-8} \{0, 21, 63, 84\} \xrightarrow{\div 21} A^{(N)} = \{0, 1, 3, 4\}$$

$$sA^{(N)} = [0, 4s],\ \forall s \geq 2 \Rightarrow sA = \{8s + 21i : 0 \leq i \leq 4s\},\ \forall s \geq 2$$

Another example

Example: $A = \{0, 2, 3, 7\}$

- $2A = \{0, 2, 3, 4, 5, 6, 7, 9, 10, 14\} = \{0\} \sqcup [2, 7] \sqcup \{9, 10, 14\}$.
- $3A = \{0\} \sqcup [2, 14] \sqcup \{16, 17, 21\}$.
- $4A = \{0\} \sqcup [2, 21] \sqcup \{23, 24, 28\}$.
- $\ldots$

$$\begin{aligned} \forall s \geq 2,\ sA &= \{0\} \sqcup [2, 7s-7] \sqcup \{7s-5, 7s-4, 7s\} \\ &= \{0\} \sqcup [2, 7s-7] \sqcup (7s - \{0, 4, 5\}) \end{aligned}$$

The Structure Theorem

Theorem [Nathanson; 1972]

Let $A = \{a_0 = 0 < a_1 < \cdots < a_{n-1} = d\} \subset \mathbb{Z}$ be a set in normal form. There exist numbers $c_1, c_2 \in \mathbb{N}$ and sets $C_i \subset [0, c_i - 2]$, for $i = 1, 2$, such that

$$sA = C_1 \sqcup [c_1, sd - c_2] \sqcup (sd - C_2)$$

for all $s \geq \max\{1, s_0^{\mathrm{N}}\}$, with $s_0^{\mathrm{N}} := (n-2)(d-1)d$.

- $s_0^{\mathrm{WCC}} = \left(\sum_{i=2}^{n-2} a_i\right) + d - n + 1$ [Wu, Chen, Chen; 2011]
- $s_0^{\mathrm{GS}} = 2\lfloor \frac{d}{2} \rfloor$ [Granville, Shakan; 2020]
- $s_0^{\mathrm{GW}} = d - n + 2$ [Granville, Walker; 2021]

The Structure Theorem

Example: $A = \{0, 2, 3, 7\}$

- $sA = \{0\} \sqcup [2, 7s - 7] \sqcup \{7s - 5, 7s - 4, 7s\}$, for all $s \geq 2$
- $c_1 = 2$, $c_2 = 7$
- $C_1 = \{0\}$, $C_2 = \{0, 4, 5\}$
- $s_0^{\mathrm{N}} = 84$, $s_0^{\mathrm{WCC}} = 7$, $s_0^{\mathrm{GS}} = 6$, $s_0^{\mathrm{GW}} = 5$

We denote $\sigma = \sigma(A)$ the least nonnegative integer such that the decomposition in the Structure Theorem holds for every $s \geq \sigma(A)$. We call this number the sumsets regularity of A.

The projective monomial curve $\mathfrak{C}_A$

- Let k be an infinite field and $A \subset \mathbb{N}$ a set in normal form
- $\mathfrak{C}_A$ the *projective monomial curve parametrized by A*,
  $\left\{(v^d : u^{a_1}v^{d-a_1} : \cdots : u^{a_{n-2}}v^{d-a_{n-2}} : u^d) \mid (u:v) \in \mathbb{P}^1_k\right\} \subset \mathbb{P}^{n-1}_k$,
  $\mathfrak{C}_A = V(\ker\varphi)$, where $\varphi : k[x_0, \ldots, x_{n-1}] \to k[u,v]$ is the k-algebra morphism defined by $\varphi(x_i) = u^{a_i}v^{d-a_i}$
- $\ker\varphi$ is homogeneous, prime and binomial, i.e., it is a toric ideal
- $k[\mathfrak{C}_A] = k[x_0, \ldots, x_{n-1}]/\ker\varphi$: coordinate ring of $\mathfrak{C}_A$

Betti diagram and Castelnuovo-Mumford regularity

Consider $R = k[x_0, \dots, x_{n-1}]$ with the standard graduation.
Let $0 \to F_p \to \cdots \to F_0 \to k[\mathfrak{C}_A] \to 0$ be a **minimal graded free resolution** of $k[\mathfrak{C}_A]$, where $F_i = \oplus_j R(-j)^{\beta_{i,j}}$ for all i.
$\{\beta_{i,j}\}_{i,j}$ are the *graded Betti numbers of* $k[\mathfrak{C}_A]$. We can arrange them in a table, called the *Betti diagram* of $k[\mathfrak{C}_A]$:

	i
j	$\beta_{i,i+j}$

- Projective dimension: $\mathrm{pd}(k[\mathfrak{C}_A]) = \max\{i : \beta_{i,j} \neq 0 \text{ for some } j\}$
  Auslander-Buchsbaum formula: $\mathrm{pd}(k[\mathfrak{C}_A]) = n - \mathrm{depth}(k[\mathfrak{C}_A])$
- Castelnuovo-Mumford reg.: $\mathrm{reg}(k[\mathfrak{C}_A]) = \max\{j - i : \beta_{i,j} \neq 0\}$

Hilbert function

The *Hilbert function of* $k[\mathfrak{C}_A]$ is the function $\mathrm{HF}_A : \mathbb{N} \to \mathbb{N}$ given by

$$\mathrm{HF}_A(s) = \dim_k k[\mathfrak{C}_A]_s$$

$$\mathrm{HF}_A(s) = \sum_{i,j \mid \beta_{i,j} \neq 0} (-1)^i \beta_{i,j} \binom{s-j+n-1}{n-1}$$

Hilbert polynomial

There exists a polynomial $\mathrm{HP}_A(T) \in k[T]$, the Hilbert polynomial of $k[\mathfrak{C}_A]$, such that $\mathrm{HF}_A(s) = \mathrm{HP}_A(s)$ for all $s \gg 0$.

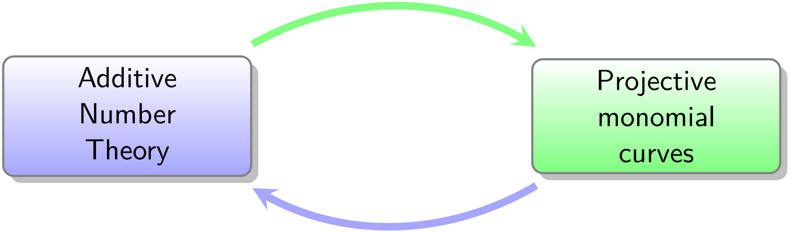
- regularity of the Hilbert function:

$$\begin{aligned} \mathrm{r}(k[\mathfrak{C}_A]) &= \min\{s \in \mathbb{N} : \mathrm{HF}_A(s') = \mathrm{HP}_A(s'),\ \forall s' \geq s\} \\ &= \min\{s \in \mathbb{N} : \mathrm{HF}_A(s'+1) - \mathrm{HF}_A(s') = d,\ \forall s' \geq s\} \end{aligned}$$

The association $A \to \mathfrak{C}_A$

J. Elias. *Sumsets and projective curves*. Mediterr. J. Math. **19**, 177 (2022)

Theorem.
$|sA| = \mathrm{HF}_A(s)$, for all $s \in \mathbb{N}$.



The Structure Theorem revisited

Structure Theorem: Given a finite set $A \subset \mathbb{Z}$ in normal form, there exist $c_1, c_2 \in \mathbb{N}$ and finite sets $C_i \subset [0, c_i - 2]$, $i = 1, 2$, such that

$$sA = C_1 \sqcup [c_1, sd - c_2] \sqcup (sd - C_2), \text{ for all } s \gg 0.$$

We define:

- $\mathcal{S}_1 = \langle A \rangle \to$ numerical semigroup generated by A
- $\mathcal{S}_2 = \langle d - A \rangle \to$ numerical semigroup generated by $d - A$

Theorem [Elias; 2022]

Using the notations above, for $i = 1, 2$ one has that:

1. c_i is the conductor of $\mathcal{S}_i$, i.e. $c_i = \max(\mathbb{N} \setminus \mathcal{S}_i) + 1$.
2. $C_i = \mathcal{S}_i \cap [0, c_i - 2]$.

The Structure Theorem revisited

Theorem [Gimenez, G.-S.; 2023]

The sumsets regularity of A, $\sigma = \sigma(A)$, can be calculated as follows:

$$\sigma = \max\left\{ \mathrm{r}(k[\mathfrak{C}_A]), \left\lceil \frac{c_1 + c_2}{d} \right\rceil \right\}$$

Corollary: G-W's bound for σ

- $\mathrm{r}(k[\mathfrak{C}_A]) \leq \mathrm{reg}(k[\mathfrak{C}_A])$
- Gruson-Lazarsfeld-Peskine theorem (= Eisenbud-Goto conj.):

$$\mathrm{reg}(k[\mathfrak{C}_A]) \leq d - n + 2$$

- $\lceil \frac{c_1+c_2}{d} \rceil \leq d - n + 1$

$$\Rightarrow \sigma \leq d - n + 2 =: s_0^{GW}$$

Bounds for the regularity in terms of σ

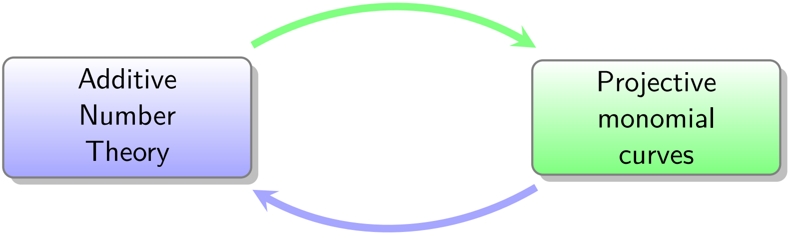
Theorem [Gimenez, G.-S.; 2023]

- If $\sigma = \mathrm{r}(k[\mathfrak{C}_A]) \geq \lceil \frac{c_1+c_2}{d} \rceil$, then $\sigma \leq \mathrm{reg}(k[\mathfrak{C}_A]) \leq \sigma + 1$.
- If $\sigma = \lceil \frac{c_1+c_2}{d} \rceil > \mathrm{r}(k[\mathfrak{C}_A])$, then $\lceil \frac{\sigma}{2} \rceil + 1 \leq \mathrm{reg}(k[\mathfrak{C}_A]) \leq \sigma + 1$.

A	$\mathrm{r}(k[\mathfrak{C}_A])$	$\lceil \frac{c_1+c_2}{d} \rceil$	σ	$\mathrm{reg}(k[\mathfrak{C}_A])$
$\{0,1,3,11,13\}$	5	1	5	5
$\{0,1,3,5,6,12\}$	1	1	1	2
$\{0,4,5,9,16\}$	2	3	3	3
$\{0,5,9,11,20\}$	3	4	4	5

Open conjecture. $\sigma \leq \mathrm{reg}(k[\mathfrak{C}_A]) \leq \sigma + 1$

Summing up so far



$s \mapsto \lvert sA \rvert$	$\longleftrightarrow$	$s \mapsto \mathrm{HF}_A(s)$
Structure Theorem	$\longleftarrow$	Properties of $\mathfrak{C}_A$
Sumsets regularity	$\longleftrightarrow$	Cast.-Mumf. regularity
G-W's bound	$\longleftrightarrow$	G-L-P's Theorem

What next?

Goal: Play the same game with $A \subset \mathbb{N}^m$.

L. Colarte-Gómez, J. Elias & R.M. Miró-Roig. Sumsets and Veronese varieties. *Collect. Math.* **74**, 353–374 (2022)

Starting point: $A \subset \mathbb{N}^2$ such that the semigroup $\langle \tilde{A} \rangle \subset \mathbb{N}^3$ is simplicial

$A \mapsto$ projective monomial surface $\mathcal{S}_A$

Work in progress...

Meanwhile: projective and affine monomial curves

$a_0 = 0 < a_1 < \cdots < a_n = d$, a sequence of relatively prime integers.

Projective mon. curve: $\mathcal{C}$

- $(x_0 : \cdots : x_n) \in \mathbb{P}_k^n$ s.t. $x_i = u^{a_i} v^{d-a_i}$, $i = 0, \ldots, n$, with $(u : v) \in \mathbb{P}_k^1$
- $k[\mathcal{C}] = k[x_0, \ldots, x_n]/I_{\mathcal{A}}$ its coordinate ring

Affine mon. curve: $\mathcal{C}_1$

- $(x_1, \ldots, x_n) \in \mathbb{A}_k^n$ s.t. $x_i = u^{a_i}$, $i = 1, \ldots, n$, with $u \in \mathbb{A}_k^1$
- $k[\mathcal{C}_1] = k[x_1, \ldots, x_n]/I_{\mathcal{A}_1}$ its coordinate ring

Remark. $\beta_i(k[\mathcal{C}]) \geq \beta_i(k[\mathcal{C}_1])$ for all i

Q. When does the equality hold for all i?

$$\beta_i(k[\mathcal{C}]) = \beta_i(k[\mathcal{C}_1]),\ \forall i \Rightarrow k[\mathcal{C}] \text{ is Cohen-Macaulay}$$

Looking for a combinatorial sufficient condition

Apery sets

Apery set Ap_1

- $\mathcal{S}_1 = \langle a_1, \ldots, a_n \rangle \subset \mathbb{N}$ numerical semigroup
- $\mathrm{Ap}_1 := \{y \in \mathcal{S}_1 \mid y - d \notin \mathcal{S}_1\}$.
- $(\mathrm{Ap}_1, \leq_1)$ is a poset, where $y \leq_1 z \Leftrightarrow z - y \in \mathcal{S}_1$.

Apery set $\mathrm{AP}_{\mathcal{S}}$

- $\mathcal{S} = \langle (a_0, d - a_0), (a_1, d - a_1), \ldots, (a_n, d - a_n) \rangle \subset \mathbb{N}^2$.
- $\mathrm{AP}_{\mathcal{S}} := \{\mathbf{y} \in \mathcal{S} \mid \mathbf{y} - (d, 0) \notin \mathcal{S}, \mathbf{y} - (0, d) \notin \mathcal{S}\}$
- $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is a poset, where $\mathbf{y} \leq_{\mathcal{S}} \mathbf{z} \Leftrightarrow \mathbf{z} - \mathbf{y} \in \mathcal{S}$.

Theorem

$(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1) \Rightarrow \beta_i(k[\mathcal{C}]) = \beta_i(k[\mathcal{C}_1])$ for all i.

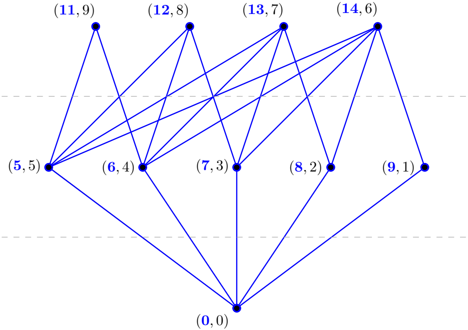
Example: arithmetic sequences

Arithmetic sequence:

$0 < a_1 < a_1 + e < a_1 + 2e < \cdots < a_1 + (n-1)e \qquad \gcd(a_1, e) = 1$

$$(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1) \iff a_1 > n - 2$$

Example: Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ for the sequence $5 < 6 < 7 < 8 < 9 < 10$



References

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- P. Gimenez and M.G.S. *Castelnuovo-Mumford regularity of projective monomial curves via sumsets*. Mediterr. J. Math. **20**, 287 (2023)

Higher dimensions (work in progress):

- L. Colarte-Gómez, J. Elias & R.M. Miró-Roig. Sumsets and Veronese varieties. *Collect. Math.* **74**, 353–374 (2022)

Betti numbers of projective and affine monomial curves:
Soon on arxiv & talk at EACA'24

¡Gracias! *Thank you!*

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