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Cooperative games with size-truncated information

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Cooperative games with size-truncated information

Abstract: We study the marginal worth vectors and their convex hull, the so-called Weber set, from the original coalitional game and the transformed one, which is called the Weber set of level k. We prove that the core of the original game is included in each of the Weber set of level k, for any k, and that the Weber sets of consecutive levels form a chain if and only if the original game is 0-monotone. Even if the game is not 0-monotone, the intersection of the Weber sets for consecutive levels is always not empty, what is not the case for non-consecutive ones. Spanish education system.

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1. Introduction

Cost allocation appears in different settings such as pricing policies, use of facilities or the cost-benefit analysis of projects. A tool to analyze cost allocation is its modelling as a cooperative game. For example the analysis of the 1930's Tennessee Valley Authority project (Ransmeier, 1942) takes into account the possible usages of the dams (see, e.g. Young, 1985).

In cooperative game theory, the core of a game consists of all efficient payoff vectors from which no coalition has an incentive to deviate. One well-known core catcher is the Weber set (Weber, 1978, 1988) defined as the convex hull of the marginal worth vectors, and it coincides with the core for convex games (Shapley, 1971). Notice that the Weber set is always non-empty, but the core can be empty.

In an applied context, and assessing a cooperative situation, it is sometimes expensive or complex the evaluation of the worth of the coalitions, and therefore some of the widely used methods take only into account the total amount (worth of the grand coalition), the stand-alone cost (individual worth), or the marginal cost of each agent to the grand coalition (separable cost).

In this paper, we will 'forget' or change the worth of certain coalitions and use solely the worth of some coalitions, either because we do not want to compute the remaining worths or because we just have limited information on the game. Informational costs to assess the worth of the coalitions are also to be considered.

The paper is organized as follows. In Section 2, we introduce the notation to be used. In Section 3, we define the game of level k and prove that the core of the original game is included in the Weber set of any level. We give the necessary and sufficient condition to ensure that the Weber sets of consecutive levels are ordered by inclusion. It turns out that it is the $0-$monotonicity property. Finally we prove that the intersection of the Weber sets for consecutive levels is always non-empty.

2. Preliminaries

Let $N = \{1, 2, ..., n\}$ be a finite set, *the player set.* A cooperative game of transferable utility (a T.U.-game or a game) of n players is given by a pair (N, v), where $v : 2^N \longrightarrow \mathbb{R}$ is a real function over the set 2^N of all subsets of N (*coalitions*) satisfying $v(\emptyset) = 0$. The set of all T.U. games over the set N is denoted by G^N.

As usual, the set $I^*(v)$ is the set of preimputations, and $I(v)$ is the set of imputations:

$$I^*(v) :=\{x \in \mathbb{R}^n \mid \ x_1 + x_2 + ... + x_n = x(N) = v(N)\} \quad \text{and}$$
$$I(v) :=\{x \in I^*(v) \mid \ x_i \geq v(\{i\}),\ i = 1, 2, \ldots, n\}.$$

A game (N, v) with $I(v) \neq \emptyset$ is called *essential.*

Given a vector $x \in \mathbb{R}^n$ and a coalition $S, x(S)$ is defined as $x(S) := \sum_{i \in S} x_i$ if $S \neq \emptyset$ and $x(\emptyset) = 0$. For any game (N, v), the *subgame associated to a coalition S.* $\left(S, v_{|S}\right)$, is defined as $v_{|S}(T) := v(T)$ for all $T \subseteq S$.

A game (N, v) is called *modular* if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. A game (N, v) is called *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

In order to define the Weber set (see Weber (1978)) of a game $v \in G^N$ we need to define first, for any permutation $\theta \in \Pi_N$, the *marginal worth vector*,

$m^\theta(v) \in \mathbb{R}^n$ as

$$m_i^\theta(v) = v(P_{\theta,i} \cup \{i\}) - v(P_{\theta,i}), \quad i = 1, 2, \ldots, n,$$

where $P_{\theta,i} = \{j \in N \mid \theta(j) < \theta(i)\}$ is the set of predecessors of i in the permutation θ.

The convex hull of the set of marginal worth vectors is the Weber set,

$$Web(v) := convex\{m^\theta(v)\}_{\theta \in \Pi_N}.$$

Any marginal worth vector is a preimputation, i.e. efficient, and moreover, the Weber set is always non-empty, convex and compact.

It is known that for convex games, the core and the Weber set coincide, they are non-empty and included in the imputation set (see Shapley (1971) and Weber (1988)). In general, as it is well known (see Derks (1992) and Weber (1988)), the Weber set contains the core of a game, which is defined by:

$$C(v) := \{x \in I(v) \ \mid \ x(S) \geq v(S), \forall S \in 2^N\}.$$

A game (N, v) is called $0-$*monotone* if $v(S) + \sum\limits_{i \in T \setminus S} v(\{i\}) \leq v(T)$, for all $S \subseteq T \subseteq N$. This is equivalent to the following condition: $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ for all $S \subseteq N$ and for all $i \notin S$. Notice that this implies $Web(v) \subseteq I(v)$. If the game is $0-$monotone, it is totally essential, i.e. $v(S) \geq \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$.

3. Main results

We define now the game of level k, where the worth of all coalitions with size equal or below k has been substituted by the modular game arising from the individual worths. This is related to the filtering of the game by an hypergraph or conference structure (see Myerson (1980) or van den Nouweland et al. (1992)).

Given a game (N, v) we define *the game of level* k, (N, v_k), for $k = 1, \ldots, n-1$, as

$$v_k(S) := \begin{cases} \sum_{j \in S} v(\{j\}) & \text{if } |S| \leq k, \\ v(S) & \text{if } |S| > k. \end{cases}$$

Notice that $v_1 = v$, and $I(v) = I(v_k)$, for all $k \in \{1, \ldots, n-1\}$, and it is easy to see that if the game v is 0-monotone, v_k is 0-monotone. Moreover, if $k, k' \in \{1, \ldots, n-1\}$, we have $(v_k)_{k'} = (v_{k'})_k = v_{\max\{k,k'\}}$.

For any cooperative game (N, v), the *Weber set of level* k *of the game* v is the Weber set of the game v_k, that is $Web(v_k) = convex\left\{m^\theta(v_k)\right\}_{\theta \in \Pi_N}$. Then, if the game v is essential, we have $I(v) = Web(v_{n-1})$. If the game is 0-monotone, $Web(v_k) \subseteq Web(v_{n-1}) = I(v)$, for any $k = 1, \ldots, n-1$.

Since $C(v) \subseteq C(v_k)$, the core of the game is always included in the Weber set of any level, that is

$$C(v) \subseteq Web(v_k), \qquad \text{for } k \in \{1, \ldots, n-1\}.$$

It could seem that also $Web(v) \subseteq Web(v_k)$ for $k \in \{1, \ldots, n-1\}$, but the following example shows that this is not true in general. Consider the following game (N, v) of three players: $N = \{1, 2, 3\}$ and $v(S) = 0$, for $|S| = 1, v(S) = -1$, for $|S| = 2$, and $v(N) = 0$. In this case $Web(v) =$ $convex\left\{\overline{(0, -1, 1)}\right\}$ where the overline means all the possible permutations of these digits among the coordinates. Notice that $Web(v_2) = \{(0, 0, 0)\}$, and we obtain the strict inclusion $Web(v) \supset Web(v_2)$.

In the next theorem we show that the $0-$monotonicity of the game is the characterization of the chain inclusion of the Weber sets of different levels.

Theorem 3.1. *Let (N, v) be a game. Then the following statements are equivalent:*

1. (N, v) *is* 0*-monotone,*
2. $$Web(v) = Web(v_1) \subseteq Web(v_2) \subseteq \ldots \subseteq Web(v_{n-1}) = I(v).$$

Proof. $1. \to 2.$

We will prove that $Web(v_{k-1}) \subseteq Web(v_k)$ for all $k \in \{2, \ldots, n-1\}$ by induction on the cardinality of N. If $|N| = 3$, it is the inclusion $Web(v) \subseteq I(v)$, satisfied for $0-$monotone games.

Suppose now that the inclusion holds up to $n-1$ players. We will prove that any marginal worth vector $m^{\theta}(v_{k-1})$ belongs to $Web(v_k)$.

Let us consider first the case $k < n-1$, and suppose $\theta = (i_1, \ldots, i_{n-1}, i_n) \in \Pi_N$.

The subgame $v_{|N\setminus\{i_n\}}$ satisfies the induction hypothesis and the permutation $\tilde{\theta} = (i_1, \ldots, i_{n-1})$ belongs to $\Pi_{N\setminus\{i_n\}}$. Notice that for $k < |S|, (v_k)_{|S} = \left(v_{|S}\right)_k$. Then,

$$m^{\tilde{\theta}}\left(v_{k-1_{|N\setminus\{i_n\}}}\right) = \sum_{\tilde{\pi}\in\Pi_{N\setminus\{i_n\}}} \lambda_{\tilde{\pi}} m^{\tilde{\pi}}\left(v_{k_{|N\setminus\{i_n\}}}\right)$$

with $\lambda_{\tilde{\pi}} \geq 0$ and $\sum_{\tilde{\pi}\in\Pi_{N\setminus\{i_n\}}} \lambda_{\tilde{\pi}} = 1$. Now, since

$$m_j^{\theta}(v_k) = \begin{cases} m_j^{\tilde{\theta}}\left(v_{k_{|N\setminus\{i_n\}}}\right) & \text{if} \quad j \neq i_n \\ v_k(N) - v_k(N\setminus\{i_n\}) & \text{if} \quad j = i_n, \end{cases}$$

we complete each permutation $\tilde{\pi}$ with the player i_n who enters in the last place and obtain $\pi \in \Pi_N$. Therefore

$$m^{\theta}\left(v_{k-1}\right) = \sum_{\tilde{\pi} \in \Pi_{N \setminus \{i_n\}}} \lambda_{\tilde{\pi}} m^{\pi}\left(v_k\right)$$

with $\lambda_{\tilde{\pi}} \geq 0$ and $\sum_{\tilde{\pi} \in \Pi_{N \setminus \{i_n\}}} \lambda_{\tilde{\pi}} = 1$, showing that $m^{\theta}\left(v_{k-1}\right) \in Web(v_k)$.

For the case $k = n-1$, just recall that for $0-$monotone games $Web(v_k) \subseteq Web(v_{n-1}) = I(v)$ for all k and therefore $Web(v_{n-2}) \subseteq Web(v_{n-1}) = I(v)$.

2. $\rightarrow$ 1.

From $Web(v) \subseteq I(v)$, we obtain that $v(\{i\}) \leq v(S \cup \{i\}) - v(S)$ for all $S \subseteq N$ and for all $i \notin S$. This is a equivalent condition to $0-$monotonicity. □

From the previous theorem, $0-$monotonicity is the characterization of the chain inclusion. The next example shows that we can obtain a reverse chain. Consider $N = \{1, 2, 3, 4\}$ and the game (N, v) defined by

$$v(S) = 0, \text{ for } |S| = 1, \qquad v(S) = -1, \text{ for } |S| = 2,$$
$$v(S) = 1, \text{ for } |S| = 3, \qquad \text{and } \ v(N) = 0.$$

In this case, $Web(v_1) = Web(v) = convex\left\{\overline{(0, -1, 2, -1)}\right\}$ where the overline means all the possible permutations of these digits among the coordinates. Also $Web(v_2) = convex\left\{\overline{(0, 0, 1, -1)}\right\}$ and $Web(v_3) = I(v) = \{(0, 0, 0, 0)\}$.

Therefore we have the following strict inclusions

$$Web(v_1) \supset Web(v_2) \supset Web(v_3) = I(v).$$

Something more surprising is that the intersection of two Weber sets of different levels can be empty, as Example 3.1 in Martínez-de-Albéniz and Rafels (1998) shows. To make this paper self-contained, we state the example below.

Example 3.1. *Consider* $N = \{1, 2, 3, 4\}$ *and the following 4-player game defined by:*

$$\begin{aligned}
&v(S) = 0, \text{ for } |S| = 1,\\
&v(S) = -2, \text{ for } |S| = 2, \text{ and } S \neq \{3,4\}, \qquad v(\{3,4\}) = 10,\\
&v(S) = -3, \text{ for } |S| = 3, \text{ and } \{3,4\} \not\subset S,\\
&v(S) = 8, \text{ for } |S| = 3, \text{ and } \{3,4\} \subset S, \text{ and}\\
&v(N) = 5.
\end{aligned}$$

Players 1 and 2 are symmetric (substitutes), and any marginal contribution of these players to any non-empty coalition is negative. Thus, in any of the marginal worth vectors, the payoffs assigned to players 1 and 2 are non-positive, and at least one of them is strictly negative. Any convex combination of the marginal worth vectors will have the same property, i.e., the payoffs assigned to players 1 and 2 are non-positive, and at least one of them is strictly negative. These convex combinations cannot be imputations, because the payoffs of any player must be greater or equal than the worth of the individual coalition, i.e. 0*. Therefore, the Weber set of the previous game does not contain any imputation. Notice that, since the game is essential,* $I(v) = Web(v_3)$ *and* $Web(v) = Web(v_1)$*. Then*

$$Web(v_3) \cap Web(v_1) = I(v) \cap Web(v) = \emptyset.$$

We have seen that the intersection of the Weber sets for non-consecutive levels can be empty, but now we now show that the intersection of two Weber sets of consecutive levels is always non-empty.

Theorem 3.2. *Let* (N,v), *be a game and* $k \in \{2,\ldots,n-1\}$. *Then*

$$Web(v_{k-1}) \cap Web(v_k) \neq \emptyset.$$

Proof. We will prove it by induction on the cardinality of N. If $|N| = 3$, it is stated in the Appendix.

Assume then that $|N| > 3$. If $k < n-1$, consider coalition $S = \{1,2,...,n-1\}$ and apply the induction hypothesis on the game $v_{|S}$. We have

$$Web(v_{k-1|S}) \cap Web(v_{k|S}) \neq \emptyset.$$

Notice that for $k < |S|, (v_k)_{|S} = \left(v_{|S}\right)_k$. Then, for any $\hat{x} \in Web(v_{k-1|S}) \cap Web(v_{k|S})$, define vector $x \in \mathbb{R}^n$ in the following way:

$$x_i = \begin{cases} \hat{x}_i & \text{if} \quad i \in S \\ v\left(N\right) - v\left(N \backslash \{n\}\right) & \text{if} \quad i = n. \end{cases}$$

Each permutation $\hat{\theta} = (i_1, i_2, ..., i_{n-1}) \in \Pi_S$ can be completed to obtain permutation $\theta = (i_1, i_2, ..., i_{n-1}, n) \in \Pi_N$, and we have

$$m_j^{\theta}\left(v_k\right) = \begin{cases} m_j^{\hat{\theta}}\left(v_{k_{|S}}\right) & \text{if} \quad j \neq n, \\ v_k\left(N\right) - v_k\left(N \backslash \{n\}\right) & \text{if} \quad j = n, \end{cases}$$

and also

$$m_j^{\theta}\left(v_{k-1}\right) = \begin{cases} m_j^{\hat{\theta}}\left(v_{k-1_{|S}}\right) & \text{if} \quad j \neq n, \\ v_{k-1}\left(N\right) - v_{k-1}\left(N \backslash \{n\}\right) & \text{if} \quad j = n. \end{cases}$$

Thus, it is easy to see that $x \in Web(v_{k-1}) \cap Web(v_k)$.

If $k = n-1$ consider the auxiliary game $(N \setminus \{1\}, w)$ defined by

$$w(T) := v_{k-1}\left(T \cup \{1\}\right) - v_{k-1}\left(\{1\}\right), \quad \text{if} \;\; T \subseteq N \setminus \{1\}.$$

Then, we have

$$w(T) = \sum_{j \in T} v(\{j\}), \text{ if } |T| + 1 \leq k - 1 = n - 2, \text{ and}$$

$$w(T) = v(T \cup \{1\}) - v(\{1\}), \text{ if } |T| + 1 > k - 1.$$

Notice that $w(N \setminus \{1\}) = v(N) - v(\{1\})$, and $w_{k-2} = w$. Therefore if $|T| > k - 1$,

$$w_{k-1}(T) = v_{k-1}(T \cup \{1\}) - v_{k-1}(\{1\}) =$$
$$= v_k(T \cup \{1\}) - v_k(\{1\}),$$

and if $|T| \leq k - 1$ we have $w_{k-1}(T) = v_k(T \cup \{1\}) - v_k(\{1\})$.

By induction hypothesis applied to $(N \setminus \{1\}, w)$, we have

$$Web(w_{k-2}) \cap Web(w_{k-1}) \neq \emptyset,$$

and for each $\hat{x} \in Web(w_{k-2}) \cap Web(w_{k-1})$, define vector $x \in \mathbb{R}^n$ by

$$x_i = \begin{cases} v(\{1\}) & \text{if } \quad i = 1, \\ \hat{x}_i & \text{if } \ i \in N \setminus \{1\}. \end{cases}$$

For each permutation $\hat{\theta} = (i_1, i_2, ..., i_{n-1}) \in \Pi_{N \setminus \{1\}}$, consider the permutation $\theta = (1, i_1, i_2, ..., i_{n-1}) \in \Pi_N$. We have

$$m_j^{\theta}(v_k) = \begin{cases} v(\{1\}) & \text{if } \ j = 1, \\ m_j^{\hat{\theta}}(w_{k-1}) & \text{if } \ j \neq 1, \end{cases}$$

and also

$$m_j^{\theta}(v_{k-1}) = \begin{cases} v(\{1\}) & \text{if } \ j = 1, \\ m_j^{\hat{\theta}}(w_{k-2}) & \text{if } \ j \neq 1. \end{cases}$$

Therefore, for each $\hat{x} \in Web(w_{k-2}) \cap Web(w_{k-1})$ we find $x \in Web(v_{k-1}) \cap Web(v_k)$, that is

$$Web(v_{k-1}) \cap Web(v_k) \neq \emptyset.$$

□

4. An example

In this section we present an example from Alegre and Claramunt (1995). This numerical example of 6 players establishes the characteristic function of a game v and computes its Shapley value to allocate the total saving. We use our previous setting and compute the Shapley value for the game v_4, i.e. we use the individual worth and the worth of coalitions of 5 players.

Example 4.1. *In this example there are 6 participants, one of whom is 30 years old, whilst the others are 20. The benefits are prepayable retirement annuities, the 20-year-old participants having an amount of 1,000,000 pesetas, and the 30 year old 5,000,000 pesetas. The effective rate of interest is 6% per year, the maximum admitted likelihood of insolvency is 5% and the mortality table used is that for Spanish males. The analysis of the example gives a total saving to allocate of* $2,086,456.37$ *ptas.*

In Table 1 the characteristic function v *is given. It gives the corresponding savings in the solvency cost. The different coalitions are indicated with the first numbers of the ages of players that form the coalition, i.e. the coalition 223 is formed by two persons 20 year old and one person 30 year old.*

The Shapley value of the game v*, which is an average of the marginal worth vectors (Shapley, 1953), is given, for each player:* $Sh_2(v) = 289\,628.74$ *and* $Sh_3(v) = 638\,312.66$*. It is in the core of the game.*

Table 1: Solvency cost savings for coalitions

Coalition	Characteristic game function
2	0
3	0
22	173 972.07
23	416 528.13
222	415 004.80
223	823 324.36
2222	730 518.84
2223	1 228 945.90
22222	1 059 513.53
22223	1 653 630.45
222223	2 086 456.37

The Shapley value of the game v_4 with size-truncated information is, for each type of players: $Sh_2(v_4) = 327\ 938.83$ *and* $Sh_3(v_4) = 446\ 762.21$. *It is also in the core of the game.*

Appendix

Here we prove that for games with three players, the intersection between the Weber set of level 1 and 2 is always non-empty. Notice that if the game is essential $I(v) = Web_2(v)$, and in this case, the result is stated in Martínez-de-Albéniz and Rafels (1998).

Proposition 4.1. *Let (N, v) be a game and $|N| = 3$. Then*

$$Web_1(v) \cap Web_2(v) \neq \emptyset.$$

Proof. Notice that since $Web(v_1) = Web(v)$, it is defined as the convex hull of the following vectors:

$$m^{123}(v) = (v(\{1\}), v(\{1,2\}) - v(\{1\}), v(N) - v(\{1,2\})),$$
$$m^{132}(v) = (v(\{1\}), v(N) - v(\{1,3\}), v(\{1,3\}) - v(\{1\})),$$
$$m^{213}(v) = (v(\{1,2\}) - v(\{2\}), v(\{2\}), v(N) - v(\{1,2\})),$$
$$m^{231}(v) = (v(N) - v(\{2,3\}), v(\{2\}), v(\{2,3\}) - v(\{2\})),$$
$$m^{312}(v) = (v(\{1,3\}) - v(\{3\}), v(N) - v(\{1,3\}), v(\{3\})),$$
$$m^{321}(v) = (v(N) - v(\{2,3\}), v(\{2,3\}) - v(\{3\}), v(\{3\})).$$

The set $Web(v_2)$ is the convex hull of the following vectors:

$$m^{123}(v_2) = m^{213}(v_2) = (v(\{1\}), v(\{2\}), v(N) - v(\{1\}) - v(\{2\})),$$
$$m^{132}(v_2) = m^{312}(v_2) = (v(\{1\}), v(N) - v(\{1\}) - v(\{3\}), v(\{3\})),$$
$$m^{231}(v_2) = m^{321}(v_2) = (v(N) - v(\{2\}) - v(\{3\}), v(\{2\}), v(\{3\})).$$

If the intersection in the statement were empty, since they are non-empty, compact and convex sets, there is an separating hyperplane $\pi(x_1, x_2, x_3) = Ax_1 + Bx_2 + Cx_3 + D = 0$ such that these sets are in different semispaces (see e.g. Theorem 2.4.6 in Webster, 1994). Then,

$$\pi(m^{123}(v)) = Av(\{1\}) + B[v(\{1,2\}) - v(\{1\})] + C[v(N) - v(\{1,2\})] + D < 0,$$
$$\pi(m^{132}(v)) = Av(\{1\}) + B[v(N) - v(\{1,3\})] + C[v(\{1,3\}) - v(\{1\})] + D < 0, \text{ and}$$
$$\pi(m^{123}(v_2)) = Av(\{1\}) + Bv(\{2\}) + C[v(N) - v(\{1\}) - v(\{2\})] + D > 0,$$
$$\pi(m^{132}(v_2)) = Av(\{1\}) + B[v(N) - v(\{1\}) - v(\{3\})] + Cv(\{3\}) + D > 0.$$

Now, substracting any of the last inequalities from the two first ones, we obtain the following inequalities:

$$[B - C][v(\{1,2\}) - v(\{1\}) - v(\{2\})] < 0, \quad (1)$$
$$[B - C][v(N) - v(\{1,3\}) - v(\{2\})] < 0, \quad (2)$$
$$[C - B][v(N) - v(\{1,2\}) - v(\{3\})] < 0, \text{ and} \quad (3)$$
$$[C - B][v(\{1,3\}) - v(\{1\}) - v(\{3\})] < 0. \quad (4)$$

We can find the analogous conditions for marginal worth vectors whose permutations begin with 2 :

$$[A - C][v(\{1,2\}) - v(\{1\}) - v(\{2\})] < 0, \quad (5)$$
$$[A - C][v(N) - v(\{1,2\}) - v(\{3\})] < 0, \quad (6)$$
$$[C - A][v(N) - v(\{2,3\}) - v(\{1\})] < 0, \text{ and} \quad (7)$$
$$[C - A][v(\{2,3\}) - v(\{2\}) - v(\{3\})] < 0. \quad (8)$$

From (1) and (5), we obtain $[B - C][A - C] > 0$ and from (3) and (6) we obtain $[C - B][A - C] > 0$, a contradiction. □

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