BOGOMOLOV-TIAN-TODOROV THEOREM FOR CALABI-YAU MANIFOLDS

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1. Introduction

Let M be a complex manifold and B a small neighbourhood of 0 in $\mathbb{C}$. In previous sessions we have seen that giving an almost complex structure on $M \times B$ is equivalent to giving a smooth section $\xi(t) \in A^{01}(M, T_{10}M)$, which corresponds to an integrable almost complex structure over $M \times B$ if and only if it satisfies the Maurer-Cartan equation $\bar{\partial}\xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0$ and $\xi(t)$ depends homomorphically on t.

If we assume that $\xi(t) = \sum_{i \geq 1} \xi_i t^i$ with $\xi_i \in A^{01}(M, T_{10})$, we obtain a system of equations:

$$\bar{\partial}\xi_1 = 0$$

$$\bar{\partial}\xi_2 + \frac{1}{2}[\xi_1, \xi_1] = 0$$

$$\bar{\partial}\xi_3 + \frac{1}{2}[\xi_1, \xi_2] = 0$$

$$\vdots$$

Since ξ_1 is $\bar{\partial}$-closed it represents a cohomology class $[\xi_1] \in H^{01}(M, T_{10})$, called the Kodaira-Spence class of $\xi(t)$. In consequence, it is natural to ask if given $\alpha \in H^{01}(M, T_{10})$, there exists a solution $\xi(t)$ such that $[\xi_1] = \alpha$.

In general, the solution does not exists. For example, A. Douady showed that $M \times \mathbb{C}P^1$, where M is the Iwasawa manifold, is obstructed ([1]), E. Ghys proved that $\mathrm{Sl}(2, \mathbb{C})/\Gamma$ is obstructed for some cocompact lattices Γ ([2]) and S. Rollenske showed that there are obstructions for some complex nilmanifolds ([6]).

However if M has extra properties then it may be possible to always solve the Maurer-Cartan equation for any $\alpha \in H^{01}(M, T_{10})$. The aim of this talk is to prove the following theorem:

Theorem 1.1. *(Bogomolov-Tian-Todorov) Let M be a Calabi-Yau manifold, then for every $v \in H^1(M, T_{10})$ there exists a solution $\xi(t)$ of the Maurer-Cartan equation such that $[\xi_1] = v$.*

2. Calabi-Yau manifolds

2.1. **Contraction on exterior algebras.** Let E be a vector space over $\mathbb{K}$ of dimension n, let E^* be its dual and consider the linear map $E \times E^* \longrightarrow \mathbb{K}$ such that $(v, f) \mapsto f(v)$. Then for any $v \in E$ we define a linear map $i_v : \bigwedge^b E^* \longrightarrow \bigwedge^{b-1} E^*$ such that

$$i_v(f_1 \wedge \cdots \wedge f_b) = \sum_{i=1}^{b} (-1)^{i-1} f_i(v) f_1 \wedge \cdots \wedge \hat{f_i} \wedge \cdots \wedge f_b.$$

For any $v_1 \wedge \cdots \wedge v_a \in \bigwedge^a E$ we can generalize the above linear map to a linear map $i_{v_1\wedge\cdots\wedge v_a} : \bigwedge^b E^* \longrightarrow \bigwedge^{b-a} E^*$ by defining $i_{v_1\wedge\cdots\wedge v_a}(f_1 \wedge \cdots \wedge f_b) = i_{v_1}(i_{v_2}(\ldots i_{v_a}(f_1 \wedge \cdots \wedge f_b)\ldots))$. Then there is an induced bilinear map

$$\bigwedge^a E \times \bigwedge^b E^* \longrightarrow \bigwedge^{b-a} E^*.$$

In particular, for any $\Omega \in \bigwedge^n E^*$ we can define an isomorphism $\eta : \bigwedge^a E \longrightarrow \bigwedge^{n-a} E^*$ such that $\eta(v_1 \wedge \cdots \wedge v_a) = i_{v_1\wedge\cdots\wedge v_a}(\Omega)$.

Recall that we have a similar construction if we have an inner product $\langle\cdot,\cdot\rangle : E \times E \longrightarrow \mathbb{K}$. This inner product induces an inner product on $\bigwedge^k E$ for each k. If we fix an orientation on E, there is a unique $\omega \in \bigwedge^n E$ such that $\langle\omega,\omega\rangle = 1$. Thus we have a liner map

$$* : \bigwedge^a E \longrightarrow \bigwedge^{n-a} E,$$

called the Hodge star operator, which is completely determined by the property that for any $v, w \in \bigwedge^a E$ we have that $v \wedge *w = \langle v, w\rangle\omega$.

2.2. **Review on Kähler manifolds.** In order to find a solution $\xi(t) = \sum_{i\geq 0} \xi_i t^i$ we will need to choose ξ_i in a clever way by using the Hodge decomposition and the $\partial\bar{\partial}$ lemma.

Assume that M is a compact complex manifold with and hermitian metric h. Then TM is an orientable vector bundle of real dimension $2n$. We can extend the Hodge star operator to vector bundles, $* : \bigwedge^a TM \longrightarrow \bigwedge^{2n-a} TM$. Moreover, one can see that the Hodge star operator induces a map $* : A^{p,q}(M) \longrightarrow A^{n-p,n-q}(M)$. Thus we can define new operators $\partial^* = -*\partial* : A^{p,q}(M) \longrightarrow A^{p-1,q}(M)$ and $\bar{\partial}^* = -*\bar{\partial}* : A^{p,q}(M) \longrightarrow A^{p,q-1}(M)$.

Finally, there are operators $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and spaces $\mathcal{H}^{p,q}_{\partial} = \{\alpha \in A^{p,q}(M) : \Delta_\partial\alpha = 0\}$ and $\mathcal{H}^{p,q}_{\bar{\partial}} = \{\alpha \in A^{p,q}(M) : \Delta_{\bar{\partial}}\alpha = 0\}$ (a form in any of these spaces is called harmonic). Then we have a way to decompose $A^{p,q}(M)$ by using these subspaces:

Theorem 2.1. *(Hodge decomposition theorem) Let (M, h) be a compact Kähler manifold. Then there exists two natural decompositions*

$$A^{p,q}(M) = \partial A^{p-1,q}(M) \oplus \mathcal{H}^{p,q}_{\partial}(M) \oplus \partial^* A^{p+1,q}(M)$$

and

$$A^{p,q}(M) = \bar{\partial} A^{p,q-1}(M) \oplus \mathcal{H}^{p,q}_{\bar{\partial}}(M) \oplus \bar{\partial}^* A^{p,q+1}(M).$$

Moreover, we have that $\mathcal{H}^{p,q}_{\bar{\partial}}(M) = \mathcal{H}^{p,q}_{\partial}(M) \cong H^{p,q}(M)$.

We will also use the following relations:

Theorem 2.2. *($\partial\bar{\partial}$ lemma) Let M be a compact Kähler manifold. If α is a d-closed form of type (p, q) then the following are equivalent:*

(1) *The form α is d-exact.*

(2) *The form α is ∂-exact.*

(3) *The form α is $\bar{\partial}$-exact.*

(4) *The form α is $\partial\bar{\partial}$-exact.*

2.3. **Calabi-Yau manifolds.**

Definition 2.3. *A compact Kähler manifold of dimension n is Calabi-Yau if the canonical line bundle associated to the holomorphic tangent bundle $T_{10}M$, denoted by $K_M = \bigwedge^n T_{10}M$, is trivial.*

Remark 2.4. *One can find slightly inequivalent definitions of Calabi-Yau manifolds in the literature. For example, the next three conditions below have been used to define a Calabi-Yau manifold. Let M be a compact Kähler manifold, then:*

(1) *M is Calabi-Yau if the holonomy is* $\mathrm{SU}(n)$.

(2) *M is Calabi-Yau the canonical bundle is trivial.*

(3) *M is Calabi-Yau if the first Chern class $c_1(M)$ vanishes.*

Each condition is weaker than the one above it. For example, condition (2) *implies that the holonomy of M is contained in* $\mathrm{SU}(n)$ *(*[4, Corollary 6.2.5]*). Hence, a complex tori is Calabi-Yau in the sense of* (2) *(the definition that we take) but not in the sense of* (1).

Another example of Calabi-Yau manifold is a nonsingular hypersurfaces of degree $n+1$ in $\mathbb{C}P^n$ with $n \geq 3$. Recall that a hypersurface of degree d in $\mathbb{C}P^n$ is of the form $X = \{[z_0, ..., z_n] : f(z_0, ..., z_n) = 0\}$, where f is a non-zero homogeneous polynomial of degree d. If X is nonsingular, then X is a compact Kähler manifold of dimension $n-1$ such that $c_1(X) = 0$ if and only if $d = n+1$. However, one can see that X has holonomy $\mathrm{SU}(n-1)$ *for $n \geq 3$ and therefore it is Calabi-Yau in the sense of* (1) *(see* [4, 6.7] *for the details and generalizations of this construction).*

Since the canonical line bundle is trivial, we can fix a trivializing section $\Omega \in H^0(M, K_M)$, which we regard as a holomorphic volume form. Then by extending the concept of contractions to vector bundles, we define an isomorphism

$$\eta : \bigwedge^{a} T_{10}M \longrightarrow \bigwedge^{n-a} T_{10}M^*.$$

More precisely, if in local coordinates we have that $\Omega = f dz_1 \wedge \cdots \wedge dz_n$, then

$$\eta(\tfrac{\partial}{\partial z_{i_1}} \wedge \cdots \wedge \tfrac{\partial}{\partial z_{i_r}}) = (-1)^{(\Sigma i_j)-r} f dz_1 \wedge \cdots \wedge d\hat{z}_{i_1} \wedge \cdots \wedge d\hat{z}_{i_r} \wedge \cdots \wedge dz_n$$

for $i_1 < \cdots < i_r$.

This isomorphism induces a canonical isomorphism $\eta : A^{0,q}(M, \bigwedge^p T_{10}) \longrightarrow A^{n-p,q}(M)$.

3. Tian-Todorov lemma and the proof of BTT theorem

We define a new operator in a similar way we defined ∂^*.

Definition 3.1. *The operator $\triangle : A^{0,q}(M, \bigwedge^p T_{10}) \longrightarrow A^{0,q}(M, \bigwedge^{p-1} T_{10})$ is defined as $\triangle = \eta^{-1} \circ \partial \circ \eta$.*

Remark 3.2. *The operator ∂ is not well defined in $A^{0,q}(M, \bigwedge^p T_{10})$. Thus, we can think the operator $\triangle$ as an alternative to ∂.*

Warning!: The operator $\triangle$ has no relation with the Laplacians Δ, $\Delta_{\bar\partial}$ and Δ_∂. Indeed, we have not used any metric yet, so strictly speaking, there is no Laplacian.

Lemma 3.3. *The operator $\bar\partial$ commutes with η, $\bar\partial \circ \eta = \eta \circ \bar\partial$. Moreover, we have that $\triangle \circ \bar\partial = -\bar\partial \circ \triangle$.*

Proof. The first claim is proved locally (see [3, Lemma 6.1.8]). For the second claim, we have that $\bar\partial\triangle(\alpha) = \eta^{-1}\partial\bar\partial\eta(\alpha) = -\eta^{-1}\bar\partial\partial\eta\alpha = -\bar\partial\triangle(\alpha)$ for any $\alpha \in A^{0,q}(M, \bigwedge^p T_{10})$. □

Note that $\triangle^2 = (\eta^{-1}\partial\eta)(\eta^{-1}\partial\eta) = 0$. Therefore, it could be used as a differential, but we do not have the Leibniz rule, as the following lemma shows:

Lemma 3.4. *(Tian-Todorov lemma) Let $\alpha \in A^{0,p}(M, T_{10})$ and $\beta \in A^{0,q}(M, T_{10})$, then*

$$\triangle(\alpha \wedge \beta) = \triangle(\alpha) \wedge \beta + (-1)^p \alpha \wedge \triangle(\beta) + (-1)^{p+1}[\alpha, \beta].$$

The proof of the equality is done locally, by setting $\alpha = ad\bar z_I \otimes \frac{\partial}{\partial z_i}$ and $\beta = bd\bar z_J \otimes \frac{\partial}{\partial z_j}$. The main idea of the proof is that we can reduce the computation to the case where $p = q = 0$ (for the details see [3, Lemma 6.1.9]).

Corollary 3.5. *Let $\alpha \in A^{0,p}(M, T_{10})$ and $\beta \in A^{0,q}(M, T_{10})$, then:*

(1) *If α and β are $\bar\partial$-closed, then $[\alpha, \beta]$ is also $\bar\partial$-closed.*

(2) *If $\eta(\alpha)$ and $\eta(\beta)$ are ∂-closed, then $\eta[\alpha, \beta]$ is ∂-exact.*

Proof. (1) It is a direct consequence of lemma 3.3 ($\triangle \circ \bar\partial = -\bar\partial \circ \triangle$).

(2) If $\eta(\alpha)$ and $\eta(\beta)$ are ∂-closed, then $\eta\triangle(\alpha) = \partial\eta(\alpha) = 0$ and $\eta\triangle(\beta) = 0$. Since η is an isomorphism, we have that $\triangle(\alpha \wedge \beta) = (-1)^{p+1}[\alpha, \beta]$ which implies that $\eta[\alpha, \beta] = \partial(-1)^{p+1}\eta(\alpha \wedge \beta)$.

□

With these results we are ready to proof the BTT theorem.

Theorem 3.6. *(Bogomolov-Tian-Todorov theorem) Let M be a Calabi-Yau manifold and let $v \in H^1(M, T_{10})$. Then there exists a formal power series $\sum_{i \geq 1} \xi_i t^i$ with $\xi_i \in A^{0,1}(M, T_{10})$ which satisfies the Maurer-Cartan equations, with $[\xi_1] = v$ and such that $\eta(\xi_i) \in A^{n-1,1}(M)$ is ∂-exact for all $i > 1$.*

Proof. We construct the formal power series recursively, so we need to start by choosing a good candidate for ξ_1. Let $\zeta \in A^{0,1}(M, T_{10})$ be any representative of v. Then ζ is $\bar\partial$-closed, which implies that the form $\eta(\zeta)$ is also $\bar\partial$-closed. We can choose a representative of v whose image by η is also harmonic (and therefore ∂-closed). Indeed, since $\bar\partial\eta(\zeta) = 0$, we have that $\eta(\zeta) = \bar\partial\phi + w \in \bar\partial A^{n-1,0}(M) \oplus \mathcal{H}^{n-1,1}_{\bar\partial}(M)$. Then, we have that $\zeta - \eta^{-1}\bar\partial\phi$ is the desired representative. Hence we choose ξ_1 to be a representative of v such that $\eta(\xi_1)$ is harmonic. Now we want to solve the equation $\bar\partial\xi_2 = -[\xi_1, \xi_1]$.

Since M is compact Kähler we have that $\eta(\xi_1) \in \mathcal{H}^{n-1,1}_{\bar\partial}(M) = \mathcal{H}^{n-1,1}_{\partial}(M)$, which implies that $\eta(\xi_1)$ is ∂-closed. By corollary 3.5, we have that $\eta[\xi_1, \xi_1]$ is $\bar\partial$-closed and ∂-exact. In consequence, $\eta[\xi_1, \xi_1]$

is d-closed and by the $\partial\bar{\partial}$-lemma, there exists $\gamma \in A^{n-2,0}(M)$ such that $\bar{\partial}\partial\gamma = \eta[\xi_1, \xi_1]$. Then we can choose $\xi_2 = -\eta^{-1}(\partial\gamma)$.

Assume that we have found the firsts $\xi_1, \xi_2, ..., \xi_{k-1} \in A^{0,1}(M, T_{10})$ of the formal power series satisfying the conditions of the theorem. Firstly, note that $\eta[\xi_i, \xi_{k-i}]$ is ∂-exact for $0 < i < k$ by corollary 3.5. To repeat the same argument as above we need to see that $\sum_{0<i<k}[\xi_i, \xi_{k-i}]$ is $\bar{\partial}$-closed. We have

$$\bar{\partial}(\sum_{0<i<k}[\xi_i, \xi_{k-i}]) = \sum_{0<i<k}([\bar{\partial}\xi_i, \xi_{k-i}] + [\xi_i, \bar{\partial}\xi_{k-i}]).$$

By induction hypothesis, we have that $\bar{\partial}\xi_i = -\sum_{0<j<i}[\xi_j, \xi_{i-j}]$ for all $0 < i < k$. By using these relations in the equation above an reordering, we obtain

$$\bar{\partial}(\sum_{0<i<k}[\xi_i, \xi_{k-i}]) = -\sum_{0<i<k}\sum_{0<j<i}[[\xi_j, \xi_{i-j}], \xi_{k-i}] + \sum_{0<i<k}\sum_{0<l<i}[\xi_{k-i}, [\xi_l, \xi_{l-i}]].$$

Then, we use that $[\alpha, \beta] = -[\beta, \alpha]$ for $\alpha \in A^{0,2}(M, T_{10})$ to conclude that the last equation is 0 and $\sum_{0<i<k}[\xi_i, \xi_{k-i}]$ is $\bar{\partial}$-closed.

Finally, by the $\partial\bar{\partial}$-lemma, there exists $\gamma_k \in A^{n-2,0}(M)$ such that $\bar{\partial}\partial\gamma_k = \eta(\sum_{0<i<k}[\xi_i, \xi_{k-i}])$. Therefore $\xi_k = \eta^{-1}\partial\gamma_k$ is the next coefficient of the power formal series. □

Remark 3.7. (1) *There may be other solutions of Maurer-Cartan equation which do not satisfy the extra condition that $\eta(\xi_i)$ is ∂-exact for $i > 1$. In fact, $\eta(x_i)$ do not need to be ∂-closed in general. Even with the extra assumption that $\eta(\xi_1)$ is harmonic the constructed solution is not unique, since in any step we may change $\eta(\xi_k)$ by a $\partial\bar{\partial}$-exact form.*

(2) *There is a procedure to transform any solution to a convergent solution by using analysis. The main idea is that the formal solution converges if ξ_i are $\bar{\partial}^*$-exact for all i.*

(3) *The BTT theorem is surprising in the following sense. Recall that we have seen that the obstructions to construct the formal power series are in $H^2(M, T_{10})$ (in particular the formal power series always exists if $H^2(M, T_{10}) = 0$). If M is Calabi-Yau, then $H^2(M, T_{10}) \cong H^{n-1,2}(M)$, which is usually non-zero. For example if M is a Calabi-Yau 3-fold, then $H^2(M, T_{10})$ is dual to $H^{1,1}(M)$, which is always non-zero since M is Kähler.*

(4) *The condition of M being Kähler is necessary. For example* $\mathrm{Sl}(2, \mathbb{C})/\Gamma$ *from* [2] *or the nilmanifolds from* [6] *have trivial canonical bundle but they are obstructed.*

4. The BTT theorem from the viewpoint of DGLA

Let L be a DGLA and A a local artinian $\mathbb{C}$-algebra (in our case $A = \mathbb{C}[t]/(t^n)$ or $\mathbb{C}[|t|]$). Recall that we have defined functors $\mathrm{MC}_L : \mathbf{Art} \longrightarrow \mathbf{Set}$ such that $\mathrm{MC}_L(A) = \{x \in L^1 \otimes \mathfrak{m}_a : dx + \frac{1}{2}[x, x] = 0\}$ and $Def_L : \mathbf{Art} \longrightarrow \mathbf{Set}$ such that $Def_L(A) = \mathrm{MC}_L(A)/\sim$, where two elements $x, y \in \mathrm{MC}_L(A)$ are equivalent if and only if there exists $a \in L^0 \otimes \mathfrak{m}_A$ such that

$$y - x = \sum_{n=0}^{\infty} \frac{[a, [a, x] - da]^n}{(n+1)!}.$$

We have also seen that if L and L' are two weakly equivalent DGLA then $Def_L \cong Def_{L'}$. This result is helpful when L' has some extra properties that allows us to simplify the deformation functor. For example, if L is abelian (the Lie bracket vanishes) then $Def_L(A) = H^1(L) \otimes \mathfrak{m}_A$.

In our case, the DGLA is $KS_M = (\bigoplus_{i\geq 0} A^{0,i}(M, T_{10}), [\cdot,\cdot], \bar{\partial})$.

Theorem 4.1. *Let M be a Calabi-Yau manifold, then KS_M is quasi-isomorphic to an abelian DGLA.*

We provide a sketch of the proof (see [5, Theorem VII.11] for the details).

Proof. Firstly, we use the map η to induce a DGLA structure on $L^{n-1,*} = \bigoplus_{i\geq 0} A^{n-1,i}(M)$, which is isomorphic to KS_M. Because of corollary 3.5 (2) of the Tian-Todorov lemma, we have that $Q^* = \operatorname{Ker} \partial \cap L^{n-1,*}$ is a DGL subalgebra of $L^{n-1,*}$.

We consider the complex $(R^*, \bar{\partial})$, where $R^i = \frac{Q^i}{\partial L^{n-2,i}}$. If we endow $(R^*, \bar{\partial})$ with the trivial Lie bracket, then the projection $Q^* \longrightarrow R^*$ is a DGLA morphism by the Tian-Todorov lemma.

The last step is to see that the DGLA morphisms

$$L^{n-1,*} \longleftarrow Q^* \longrightarrow R^*$$

are quasi-isomorphisms, but this is a consequence of the $\partial\bar{\partial}$-lemma. □

Corollary 4.2. *Let M be a Calabi-Yau manifold. Then*

$$Def_M(\mathbb{C}[t]/(t^{n+1})) \longrightarrow Def_M(\mathbb{C}[t]/(t^2))$$

is surjective for every $n \geq 2$.

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