

# Adjunctions and equivalences.

Recall that if  $K$  is an  $\infty$ -cosmos, the homotopy category  $hK$  is the 2-category defined as follows:

- Objects  $hK$  = Objects of  $K$  = " $\infty$ -categories"  $A, B, C, \dots$
- 1-Morphisms = 0-simplices of  $\text{Fun}(A, B) \in \mathcal{C}at$   $A \xrightarrow{f} B$
- 2-Morphisms = 1-simplices of  $\text{Fun}(A, B)$  Homotopy classes of 1-simplices on  $\text{Fun}(A, B)$  with source  $f$  and target  $g$ 

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \\ \xrightarrow{g} \end{array} B$$

In any 2-category, we have a notion of equivalence between objects:  $A \simeq B$  if there exists a pair of 1-morphisms  $f: A \rightarrow B$  and  $g: B \rightarrow A$  ~~such that~~ and invertible 2-morphisms

$$A \begin{array}{c} \xrightarrow{\cong} \\ \Downarrow \\ \xrightarrow{g \circ f} \end{array} A$$

$$\alpha: id_A \Rightarrow g \circ f$$

and

$$B \begin{array}{c} \xrightarrow{\cong} \\ \Downarrow \\ \xrightarrow{f \circ g} \end{array} B$$

$$\beta: f \circ g \Rightarrow id_B$$

~~We saw last week~~ We saw last week that this notion is the right one for an equivalence of  $\infty$ -cosmos. | Let  $f: A \rightarrow B$  be an  $\infty$ -functor.

$f: A \rightarrow B$  is an equivalence of  $\infty$ -categories in  $K$  ( $\infty$ -cosmos)

~~$\Leftrightarrow$~~   $f$  is an equivalence in  $hK$

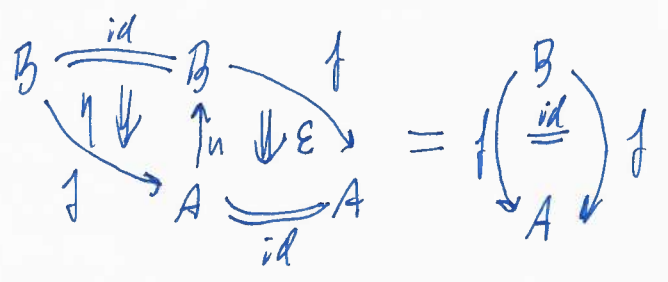
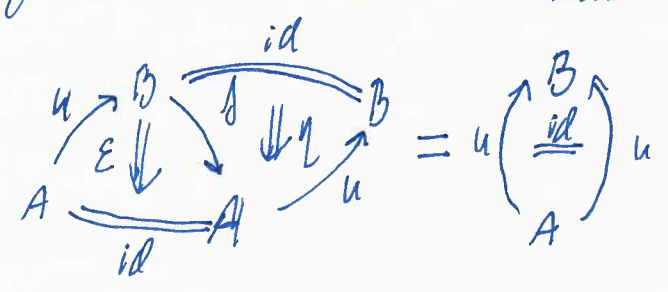
(i)  $\text{Fun}(X, A) \xrightarrow{\sim} \text{Fun}(X, B)$  is an equiv. of  $\mathcal{C}at$   $\forall X \in K$ .

In all the examples we saw of  $\mathcal{A}$ -cosmos, the equations are the equations as  $\mathcal{A}$ -cosmos.

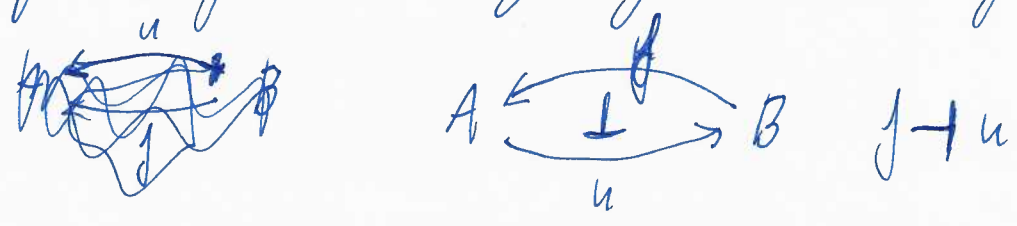
We can do similarly for other notions, like adjunctions.

Definition (Adjunction in an  $\mathcal{A}$ -cosmos). Let  $K$  be an  $\mathcal{A}$ -cosmos and  $A, B \in K$ . An adjunction between  $A$  and  $B$  consists of:

- A pair of  $\mathcal{A}$ -functors  $u: A \rightarrow B$ ,  $f: B \rightarrow A$   $u, f$  are 1-morphisms in  $kK$
- A pair of  $\mathcal{A}$ -natural transformations  $\eta: Id_B \Rightarrow uf$  and  $\epsilon: fu \Rightarrow Id_A$  called unit and counit such that  $\eta, \epsilon$  are 2-morphisms in  $kK$ .



The functor  $f$  is called left adjoint and  $u$  right adjoint



Adjunctions in an  $\mathcal{A}$ -cosmos  $\equiv$  2-categorical notion of adjunction in  $kK$ .

The notion of adjunction for quasi-categories in Lurie is quite different. It is based on viewing adjunctions in terms of correspondences (profunctors).

A correspondence is a functor  $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{sets} \cong M \rightarrow (0, 1)$  <sup>some skipping</sup>  
 $F: \mathcal{C} \rightarrow \mathcal{D}$   $G: \mathcal{D} \rightarrow \mathcal{C}$  corresponds  $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{sets}$   
 $x, y \rightsquigarrow \begin{cases} \mathcal{C}(x, Gy) \\ \mathcal{D}(Fx, y) \end{cases} \rightsquigarrow \text{iff } F, G \text{ adjoint.}$

Definition (Lurie). Let  $A, B$  be quasi-categories. An adjunction between  $A$  and  $B$  is a map  $M \xrightarrow{q} \Delta[1]$  which is both a cartesian fibration and a cocartesian fibration together with an equivalences  $A \cong M_0$   $B \cong M_1$   $m_i = q^{-1}(i)$ .

Theorem: Let  $K$  be the co-limit  $\mathcal{Q} \text{cat}$ , ~~and  $A, B$~~   
~~be quasi-categories. The following are equivalent:~~  
 and let  $M \rightarrow \Delta[1]$  both a cartesian fibration and cocart. fib.  
 then  $B \cong M_0$  and  $A \cong M_1$  and the functors  $f: B \rightarrow A$   
 and  $u: A \rightarrow B$  defined by the (co)cartesian lifts ~~is~~  
 an adjunction in  $\mathcal{K} \text{cat}$ . The converse is also true.  
 (co-limit  $\mathcal{Q} \text{cat}$ )

"To describe properties of adjunction in an  $\infty$ -category, we can work 2-categorically in  $hK$ ."

• Lemma: An adjunction in a 2-category is preserved by any 2-functor.

Examples: Let  $A \begin{matrix} \xleftarrow{\eta} \\ \perp \\ \xrightarrow{\epsilon} \end{matrix} B$  be an adjunction in an  $\infty$ -category.

(i) For any  $\infty$ -category  $X$ , we have adjunctions

$$\text{Func}(X, A) \begin{matrix} \xleftarrow{\eta_*} \\ \perp \\ \xrightarrow{\epsilon_*} \end{matrix} \text{Func}(X, B) \quad \text{hFunc}(X, A) \begin{matrix} \xleftarrow{\eta_*} \\ \perp \\ \xrightarrow{\epsilon_*} \end{matrix} \text{hFunc}(X, B)$$

( of  $A$ -objs in Cat. )

$$\text{Func}(X, -): hK \rightarrow h\text{Cat}$$

(ii) For any simplicial set  $U$ , we have an adjunction

$$A^U \begin{matrix} \xleftarrow{\eta^U} \\ \perp \\ \xrightarrow{\epsilon^U} \end{matrix} B^U$$

+ contravariant versions.

Mention other examples:  $N$  (adjunction) or taking the underlying quiver of a Quillen adjunction.

(iii)  $N: \text{Cat} \rightarrow h\text{Cat}$  send adjunctions between  $\text{Cat}$  to adj. between  $h\text{Cat}$ .

(iv)  $F: \mathcal{E} \rightleftharpoons \mathcal{D}: U$  Quillen pair of simplicial model cat.  $\rightsquigarrow$   $\rightsquigarrow$  underlying quiver-cat.

Let us see some results about adjunctions of  $n$ -categories, that can be deduced formally from the theory of 2-categories.

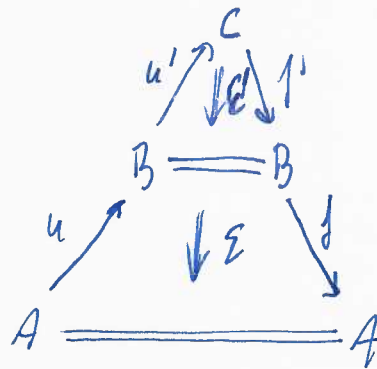
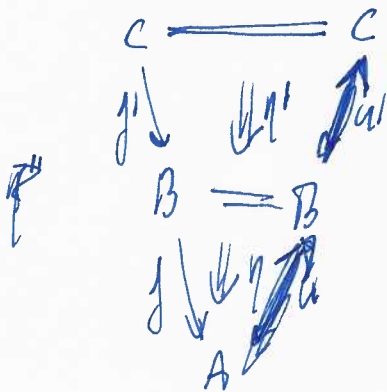
Proposition: If  $C \begin{matrix} \xrightarrow{f'} \\ \perp \\ \xleftarrow{u'} \end{matrix} B$  and  $B \begin{matrix} \xrightarrow{1} \\ \perp \\ \xleftarrow{u} \end{matrix} A$  then

$$C \begin{matrix} \xrightarrow{ff'} \\ \perp \\ \xleftarrow{u'u} \end{matrix} A.$$

Proof: Consider the units and counits

$$\eta: id_B \Rightarrow u'f' \quad \epsilon: fu \Rightarrow id_A, \quad \eta': id_C \Rightarrow u'f' \quad \epsilon': f'u' \Rightarrow id_B$$

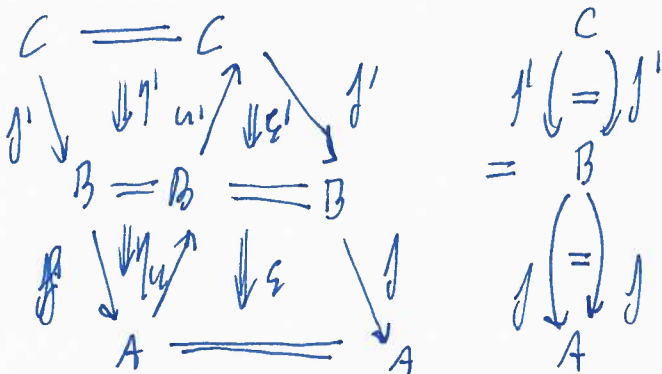
We define the unit and counit of  $C \begin{matrix} \xrightarrow{ff'} \\ \perp \\ \xleftarrow{u'u} \end{matrix} A$  as follows



$$\eta'': id_C \Rightarrow u'u'ff'$$

$$\epsilon'': ff'u'u \Rightarrow id_A$$

The triangle inequalities hold



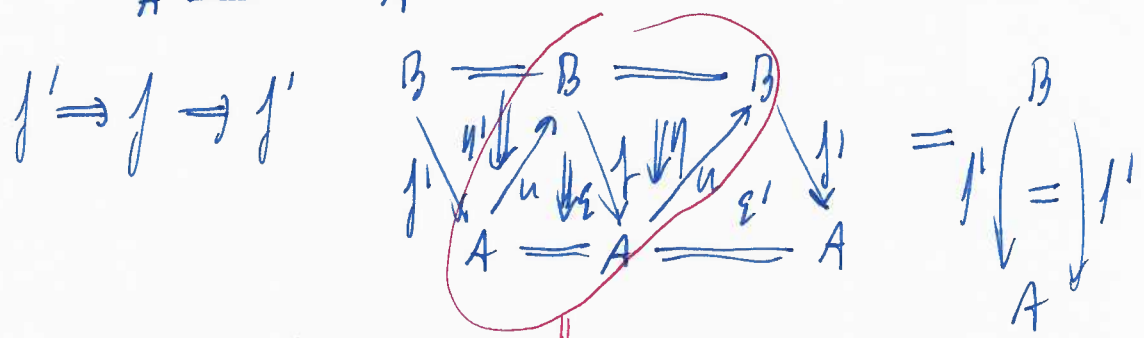
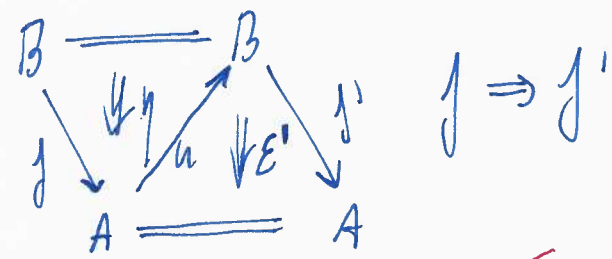
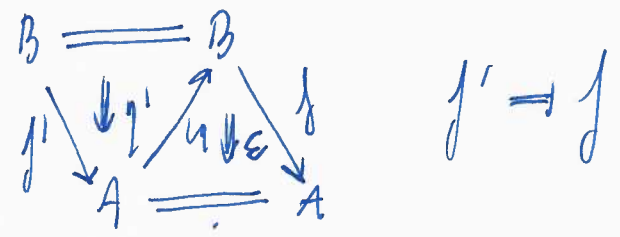
The other one is similar.

Proposition: If  $f \dashv u$  and  $f' \dashv u$ , then  $f \cong f'$ .  
 Conversely, if  $f \dashv u$  and  $f \cong f'$ , then  $f' \dashv u$ .

Proof:  $A \begin{matrix} \xleftarrow{1} \\ \xrightarrow{f} \\ \xrightarrow{f'} \end{matrix} B$

$f \dashv u$   $\eta: id_B \Rightarrow uf$   $\epsilon: fu \Rightarrow id_A$   $f' \dashv u$   $\eta': id_B \Rightarrow uf'$   $\epsilon': f'u \Rightarrow id_A$

We need to prove that  $f \cong f'$ . We define 2-morphisms  $f \Rightarrow f'$  &  $f' \Rightarrow f$ .



Similarly  $f \Rightarrow f' \Rightarrow f$ .

