Adjunctions and equivalences.

Recall that if $\mathcal{K}$ is an ∞-cosmos, the homotopy category $h\mathcal{K}$ is the 2-category defined as follows:

- Objects $h\mathcal{K}$ = Objects of $\mathcal{K}$ = "∞-categories" $A, B, C, \dots$
- 1-Morphisms = 0-simplices of $\mathrm{Fun}(A,B) \in \mathcal{Q}Cat$ $\quad A \xrightarrow{f} B$
- 2-Morphisms = ~~1-simplices of $F(A,B)$~~ Homotopy classes of 1-simplices in $\mathrm{Fun}(A,B)$ with source f and target g $\quad$ $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$

In any 2-category, we have a notion of equivalence between objects: ~~A, B~~ : $A \simeq B$ if there exists a pair of 1-morph $f : A \to B$ and $g : B \to A$ ~~such that~~ and invertible 2-morph

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, phantom, "\cong\Downarrow\alpha"] \arrow[r, bend right, "g \circ f"'] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, bend left] \arrow[r, phantom, "\cong\Uparrow\beta"] \arrow[r, bend right, "f \circ g"'] & B \end{tikzcd}$$

$\alpha : id_A \Rightarrow g \circ f$ $\qquad\qquad$ $\beta : f \circ g \Rightarrow id_B$

We saw last week that this notion is the right one for an equivalence of in an ∞-cosmos. | Let $f : A \to B$ be an ∞-functor.

$f : A \to B$ is an equivalence of ∞-categories in $\mathcal{K}$ (∞-cosmos)

(ii) f is an equivalence in $h\mathcal{K}$

(i) $\mathrm{Fun}(X,A) \xrightarrow{\simeq} \mathrm{Fun}(X,B) : f_*$ is an equiv. of ~~qcat~~ $\forall X \in \mathcal{K}$.

In all the examples we saw of ∞-cosmos, the equivalences are the equivalences as ∞-cosmoi.

We can do similarly for other notions, like adjunctions.

- **Definition** (Adjunction in an ∞-cosmos). Let K be an ∞-cosmos and $A, B \in K$ (∞-cats). An adjunction between A and B consists of:
  - A pair of ∞-functors $u : A \to B$, $f : B \to A$ — u, f are 1-morph in hK
  - A pair of ∞-natural transformations $\eta : Id_B \Rightarrow uf$ and $\varepsilon : fu \Rightarrow Id_A$ called unit and counit such that — η, ε are 2-morph of hK.

$$
\begin{tikzcd}
 & B \arrow[rr, bend left, "id"] \arrow[dr, "f"] \arrow[d, phantom, "\varepsilon\Downarrow"] & \arrow[d, phantom, "\Downarrow\eta"] & B \\
A \arrow[ur, "u"] \arrow[rr, bend right, "id"'] & {} & A \arrow[ur, "u"'] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow id"] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, bend left, "id"] \arrow[dr, "f"'] \arrow[r, phantom, "\eta\Downarrow"] & B \arrow[dr, bend left, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
 & A \arrow[u, "u"'] \arrow[r, bend right, "id"'] & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow id"] & A
\end{tikzcd}
$$

The functor f is called left adjoint and u right adjoint

$$
\begin{tikzcd}
A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"']
\end{tikzcd}
\qquad f \dashv u
$$

Adjunctions in an ∞-cosmos $\equiv$ 2-categorical notion of adjunction in hK.

The notion of adjunction for quasicategories in Lurie is quite different. It is based on viewing adjunction ~~as~~ in terms of correspondences (profunctors).

A correspondence is a functor $\mathcal{C}^{op}\times\mathcal{D}\to$ sets $\equiv \underset{\text{some category}}{M}\to(0\to1)$

$F:\mathcal{C}\to\mathcal{D}$ $\quad G:\mathcal{D}\to\mathcal{C}$ $\quad$ correspondences $\mathcal{C}^{op}\times\mathcal{D}\to$ sets

$X, Y \rightsquigarrow \begin{cases}\mathcal{C}(X,GY)\\ \mathcal{D}(FX,Y)\end{cases}$ $\cong$ iff F, G adjoint.

- **Definition** (Lurie). Let A, B be quasicategories. An adjunction between A and B is a map $M\xrightarrow{q}\Delta[1]$ which is both a cartesian fib and a cocartesian fibration together with ~~and~~ an equivalences $A\simeq M_0$ $\quad B\simeq M_1$ $\quad M_i=q^{-1}(i)$.

- **Theorem**: Let K be the ∞-cosmos QCat, ~~and A, B be quasicategories. The following are equivalent.~~ and let $M\twoheadrightarrow\Delta[1]$ both a cartesian fib and cocart. fib. then $B\simeq M_0$ and $A\simeq M_1$ and the functor $f:B\to A$ and $u:A\to B$ defined by the (co)cartesian lift ~~give an~~ is an adjunction in hQCat. The converse is also true.
 (∞-cosmos QCat)

"To describe properties of adjunctions in an ∞-cosmos, we can work 2-categorically in $h\mathcal{K}$."

- **Lemma**: An adjunction in a 2-category is preserved by any 2-functor.

Examples: Let
$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$
be an adjunction in an ∞-cosmos.

(i) For any ∞-category X, we have adjunctions

$$\begin{tikzcd} \mathrm{Fun}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(X,B) \arrow[l, bend right, "f_*"'] \end{tikzcd} \qquad \begin{tikzcd} h\mathrm{Fun}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & h\mathrm{Fun}(X,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

of ∞-cats $\qquad\qquad$ in Cat.

$\mathrm{Fun}(X,-): h\mathcal{K} \longrightarrow h\mathcal{Q}Cat$

(ii) For any simplicial set U, we have an adjunction

$$\begin{tikzcd} A^U \arrow[r, bend right, "u^U"'] \arrow[r, phantom, "\perp"] & B^U \arrow[l, bend right, "f^U"'] \end{tikzcd}$$

+ contravariant versions.

Mention other examples: N(adjunction) or taking the underlying quat of a Quillen adjunction.

(iii) $N: Cat \to h\mathcal{Q}Cat$ send adjunct. between 1-cat to adj. between QCat.

(iv) $F: \mathcal{E} \rightleftarrows \mathcal{D}: U$ Quillen pair of simplicial model cats. $\rightsquigarrow$
$\rightsquigarrow$ underlying quasi-cats.

Let us see some results about adjunctions of ∞-categories, that can be deduced formally from the theory of 2-categories.

- **Proposition**: If $C \underset{u'}{\overset{f'}{\underset{\perp}{\rightleftarrows}}} B$ and $B \underset{u}{\overset{f}{\underset{\perp}{\rightleftarrows}}} A$ then

$$C \underset{u'u}{\overset{ff'}{\underset{\perp}{\rightleftarrows}}} A.$$

Proof: Consider the units and counits

$\eta : id_B \Rightarrow uf \quad \varepsilon : fu \Rightarrow id_A, \quad \eta' : id_C \Rightarrow u'f' \quad \varepsilon' : f'u' \Rightarrow id_B$

We define the unit and counit of $C \underset{u'u}{\overset{ff'}{\underset{\perp}{\rightleftarrows}}} A$ as follows

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "f'"'] \arrow[dr, phantom, "\Downarrow\eta'"] & C \\
B \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta"] & B \arrow[u, "u'"'] \\
A \arrow[ur, "u"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & C \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\varepsilon'"] & & \\
 & B \arrow[ur, "u'"] \arrow[rr, equal] & {} \arrow[d, phantom, "\Downarrow\varepsilon"] & B \arrow[dr, "f"] & \\
A \arrow[ur, "u"] \arrow[rrrr, equal] & & {} & & A
\end{tikzcd}$$

$\eta'' : id_C \Rightarrow u'uff' \qquad\qquad \varepsilon'' : ff'u'u \Rightarrow id_A$

The triangle inequalities hold

$$\begin{tikzcd}
C \arrow[rrr, equal] \arrow[dr, "f'"'] & {} \arrow[d, phantom, "\Downarrow\eta'"] & & C \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\varepsilon'"] & \\
& B \arrow[r, equal] \arrow[d, "f"'] & B \arrow[ur, "u'"] \arrow[rr, equal] \arrow[dl, phantom, "\Downarrow\eta"] & {} \arrow[d, phantom, "\Downarrow\varepsilon"] & B \arrow[d, "f"] \\
& A \arrow[ur, "u"'] \arrow[rrr, equal] & & {} & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[d, bend right, "f'"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "="] \\
B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\
A
\end{tikzcd}$$

The other one is similar.

• **Proposition**: If $f \dashv u$ and $f' \dashv u$, then $f \cong f'$.
Conversely, if $f \dashv u$ and $f \cong f'$, then $f' \dashv u$.

Proof:
$$\begin{tikzcd} A \arrow[r, "u"'] & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \end{tikzcd}.$$

$f \dashv u$ $\quad \eta : id_B \to uf \quad \varepsilon : fu \Rightarrow id_A \quad \eta' : id_B \Rightarrow uf' \quad \varepsilon' : f'u \Rightarrow id_A$

$f' \dashv u$

We need to prove that $f \cong f'$. We define 2-morphism $f \Rightarrow f'$ & $f' \Rightarrow f$.

$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f'"'] & \arrow[d, phantom, "\Downarrow\eta'"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
& A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd} \qquad f' \Rightarrow f$$

$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\varepsilon'"] & \\
& A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd} \qquad f \Rightarrow f'$$

$f' \Rightarrow f \Rightarrow f'$
$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f'"'] & \arrow[d, phantom, "\Downarrow\eta'"] & B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f'"] \arrow[d, phantom, "\varepsilon'"] & \\
& A \arrow[ur, "u"] \arrow[rr, equal] & & A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd}
= \begin{tikzcd} B \arrow[d, bend right, "f'"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

$$= \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd}$$

Similarly $f \Rightarrow f' \Rightarrow f$.