2-CATEGORY THEORY AND THE HOMOTOPY 2-CATEGORY

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These notes follow appendices B.1, B.2 and section 1.4 in [2].

1. 2-categories

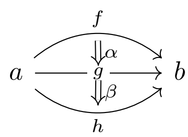
Definition 1. A **2-category** is a category enriched in the category of categories **Cat**.

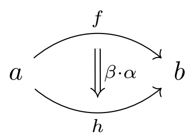
More explicitly, this means that a 2-category $\mathcal{C}$ consists of:

(1) A collection of objects $\mathcal{C}$.
(2) For every pair of objects $a, b \in \mathcal{C}$, a category $\mathcal{C}(a, b)$.
(3) For every object $a \in \mathcal{C}$, a morphism of categories (functor) $\mathrm{id}_a : 1 \to \mathcal{C}(a, a)$.
(4) For every triple of objects $a, b, c \in \mathcal{C}$, a morphism of categories (functor)
$$\circ : \mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$$
satisfying strict associativity and unit.

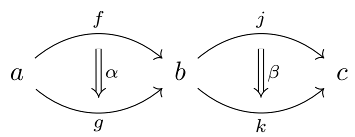
Remark 2. Forgetting the category structure of $\mathcal{C}(x, y)$ one obtains a 1-category, called the underlying category of the 2-category $\mathcal{C}$.

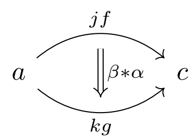
Given three 1-morphisms $f, g, h : a \to b$ with the same source and target, and two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, since α and β are morphisms in the category $\mathcal{C}(a, b)$ we obtain (via composition in the category $\mathcal{C}(a, b)$) a 2-morphism $\beta \cdot \alpha$. This new 2-morphism is called the **vertical composition** of β with α.





Now suppose given two objects $a, b, c \in \mathcal{C}$, two 1-morphisms $f, g : a \to b$, two 1-morphisms $j, k : b \to c$, and two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$. Composition $\mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$ gives a morphism in the category $\mathcal{C}(a, c)$ (a 2-morphism) from $jf := \circ(j, f)$ to $kg := \circ(k, g)$. We denote it by $\beta * \alpha$ and call it the **horizontal composition** of α and β.





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The categorical notion of isomorphism becomes the 2-categorical notion of equivalence:

Definition 3. An **equivalence** between two objects $x, y \in \mathcal{C}$ consists of:

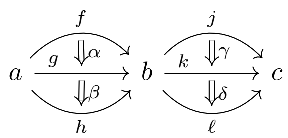
(1) A pair of 1-morphisms $f : x \to y$, $g : y \to x$.
(2) A pair of invertible 2-morphisms $\alpha : \mathrm{id}_x \Rightarrow gf$, $\beta : \mathrm{id}_y \Rightarrow fg$.

If $x, y \in \mathcal{C}$ are such that there exists an equivalence between them, we say that they are **equivalent**.

Definition 4. If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, a **2-functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

(1) A mapping on objects $c \in \mathcal{C} \mapsto Fc \in \mathcal{D}$.
(2) A mapping on 1-morphisms $(f : x \to y) \in \mathcal{C} \mapsto (Ff : Fx \to Fy) \in \mathcal{D}$.
(3) A mapping on 2-morphisms $(\alpha : f \Rightarrow g) \in \mathcal{C}(x, y) \mapsto (F\alpha : Ff \Rightarrow Fg) \in \mathcal{D}(Fx, Fy)$ that respects horizontal and vertical composition, and also horizontal and vertical identites.

If we have a diagram



and we want to obtain a 2-morphism $jf \Rightarrow \ell h$ we can do it in two ways: first applying horizontal composition and then vertical composition, or the other way around. It turns out that the resulting 2-morphism is the same in both cases:

Lemma 5 (Middle-four interchange)**.** *The relation* $(\delta * \beta) \cdot (\gamma * \alpha) = (\delta \cdot \gamma) * (\beta \cdot \alpha)$ *holds.*

Proof. We know that

$$(\delta * \beta) \cdot (\gamma * \alpha) = \circ(\delta, \beta) \cdot \circ(\gamma, \alpha),$$

where $\circ : \mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$.

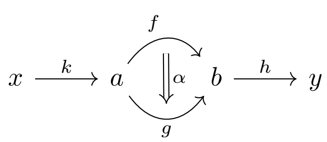
Since $\circ$ is a functor it commutes with the internal compositions in $\mathcal{C}(a, c)$ and $\mathcal{C}(b, c) \times \mathcal{C}(a, b)$. Vertical composition is precisely defined as internal composition, so

$$\circ(\delta, \beta) \cdot \circ(\gamma, \alpha) = \circ(\delta \cdot \gamma, \beta \cdot \alpha).$$

By definition this is precisely $(\delta \cdot \gamma) * (\beta \cdot \alpha)$. This completes the proof. □

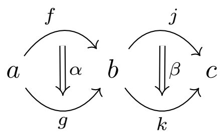
2. Whiskering

Suppose we are in the following situation:



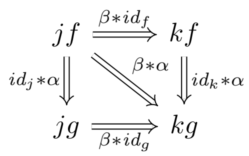
Definition 6. The **whiskered composite** of the 2-morphism α with the 1-morphisms k and h, which we denote as $h\alpha k$, is defined as the horizontal composition $\mathrm{id}_h * \alpha * \mathrm{id}_k$.

Take the diagram

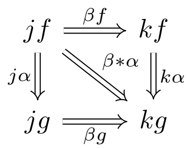


A 2-morphism $jf \Rightarrow kg$ can be obtained by means of the horizontal composition: $\beta * \alpha : jf \Rightarrow kg$. However we can also do it in a different way. The horizontal composition $\beta * \mathrm{id}_f$ is a 2-morphism $jf \Rightarrow kf$, and the horizontal composition $\mathrm{id}_k * \alpha$ gives a 2-morphism $kf \Rightarrow kg$. We can now apply vertical composition and obtain another 2-morphism $jf \Rightarrow kg$. It turns out that these two procedures for obtaining the required 2-morphism yield the same result:

Lemma 7 (Naturality of whiskering)**.** *The following diagram is commutative:*



One can rewrite this commutative diagram in terms of whiskering:

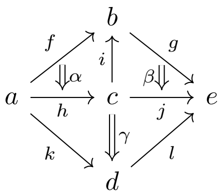


A direct consequence is

Corollary 8. *In the previous context, if three of βf, $k\alpha$, βg, $j\alpha$ are invertible, so is the fourth.*

3. Pasting diagrams

Consider the following diagram in a 2-category $\mathcal{C}$:



The theorem proven in [1] asserts that, whenever we have a diagram of this kind that is *well-formed*, no matter how we compose the 2-morphisms we will always obtain the same resulting 2-morphism from the source 1-morphism (the composite gf) to the target 1-morphism (the composite lk).

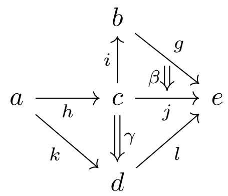
We give a procedure that can be applied to every *well-formed* pasting diagram. First of all, we define:

- The **source of the diagram** is the unique object that never appears as the target of a 1-morphism. In our example it is a.
- The **target of the diagram** is the unique object that never appears as the source of a 1-morphism. In our example it is e.
- The **source 1-cell of the pasting diagram** is the unique composite of 1-morphism such that none of its parts appear as the target of a 2-morphism. In our example it is gf.
- The **target 1-cell of the pasting diagram** is the unique composite of 1-morphisms such that none of its parts appear as the target of a 2-morphism. In our example it is lk.

We proceed as follows:

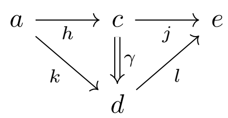
(1) Take a 2-cell whose source 1-morphism appears in the source 1-cell of the pasting diagram. In our example we can take α. Using the operation of whiskering we can construct a 2-cell with source gf (the source 1-cell): $\alpha g : gf \Rightarrow gih$.

(2) Remove the chosen 2-cell, α, from the pasting diagram. We obtain the following pasting diagram, whose source 1-cell is gih and whose target 1-cell remains unchanged:



(3) As before, take a 2-cell whose source 1-morphism appears in the souce 1-cell (gih). We can take β. With the whiskering operation we construct the 2-morphism $h\beta : gih \Rightarrow jh$.

(4) Remove the chosen 2-cell. The obtained pasting diagram has source 1-cell jh:



(5) This diagram already gives a 2-morphism $\gamma : jh \Rightarrow lk$.

(6) We can vertically compose the obtained 2-morphisms,

$$gf \overset{\alpha g}{\Longrightarrow} gih \overset{h\beta}{\Longrightarrow} jh \overset{\gamma}{\Longrightarrow} lk,$$

and finally produce the desired 2-morphism $\gamma \cdot h\beta \cdot \alpha g : gf \Rightarrow lk$.

4. The homotopy 2-category of an ∞-cosmos

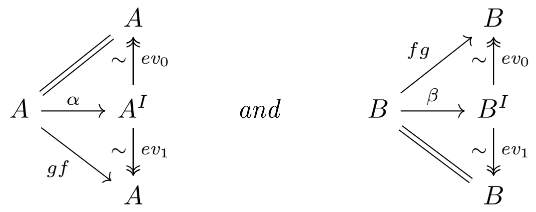
Let $\mathcal{K}$ be an ∞-cosmos. Recall that this means that $\mathcal{K}$ is a category enriched in quasicategories that satisfies a bunch of axioms: the completeness axioms, and the isofibration axioms. We think of elements in an ∞-cosmos as ∞-categories of a particular kind, and for every pair of ∞-categories $A, B \in \mathcal{K}$ there exists a quasi-category $\mathrm{Fun}(A, B)$, called the functor space of A and B.

Definition 9. Let $\mathcal{K}$ be an ∞-cosmos. Its **homotopy category** is the 2-category $h\mathcal{K}$ defined as follows:

- Objects: It has the same objects as $\mathcal{K}$ (i.e. ∞-categories).
- 1-morphisms: For every pair of objects, the 1-morphisms between them are the elements in $\mathrm{Fun}(A, B)_0$.
- 2-morphisms (∞-**natural transformations**): For every pair of 1-morphisms $f, g \in \mathrm{Fun}(A, B)_0$, the 2-morphisms between them are the homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$ with source f and target g.

Theorem 10 (Equivalences are equivalences)**.** *Let $\mathcal{K}$ be an ∞-cosmos, and let $f : A \to B$ be an ∞-functor. The following are equivalent:*

(1) *For every object $X \in \mathcal{K}$ the post-composition map $f_* : Fun(X, A) \to Fun(X, B)$ defines an equivalence of quasi-categories.*
(2) *The ∞-functor f is part of the data of an equivalence in the homotopy 2-category $h\mathcal{K}$.*[1]
(3) *There exists an ∞-functor $g : B \to A$ and maps (in the ∞-cosmos $\mathcal{K}$)*



making the above diagrams commute.

Proof.

(1) $\implies$ (2): The functor f_* induces an equivalence hf_* in the homotopy categories of the quasi-categories.

- If one fixes $X = B$, the equivalence $hf_* : h\mathrm{Fun}(B, A) \to h\mathrm{Fun}(B, B)$ allows to find some $g \in h\mathrm{Fun}(B, A)$ and an ∞-natural isomorphism β such that $\beta : hf_*(g) \cong \mathrm{id}_B \in h\mathrm{Fun}(B, B)$. Because $hf_*(g) = fg$ one obtains $\beta : fg \Rightarrow \mathrm{id}_B$.
- Analogously, if one fixes $X = A$ one obtains an ∞-natural isomorphism $\alpha : gf \Rightarrow \mathrm{id}_A$.

The data above implies (2).

(2) $\implies$ (3): Specifying ∞-natural isomorphisms $\alpha : gf \Rightarrow \mathrm{id}_A$ and $\beta : fg \Rightarrow \mathrm{id}_B$ gives the data of the diagrams in (3).

(3) $\implies$ (1): The functor $\mathrm{Fun}(X, -)$ sends the data of (3) to the data of (1). □

[1] This means that, in the homotopy category $h\mathcal{K}$, there exists an ∞-functor $g : B \to A$, and two invertible ∞-natural isomorphisms $\alpha : \mathrm{id}_A \Rightarrow gf$ and $\beta : \mathrm{id}_y \Rightarrow fg$.

Using this characterization of equivalence of ∞-categories one can easily show the following, which would be much harder if we didn't have (2):

Corollary 11. *Equivalences of ∞-categories $A' \xrightarrow{\sim} A$ and $B \xrightarrow{\sim} B'$ induce an equivalence of quasicategories $Fun(A, B) \xrightarrow{\sim} Fun(A', B')$.*

Proof. The functors $\mathrm{Fun}(A, -) : \mathcal{K} \to \mathbf{qCat}$ and $\mathrm{Fun}(-, B') : \mathcal{K}^{\mathrm{op}} \to \mathbf{qCat}$ induce 2-functors $h\mathrm{Fun}(A, -) : h\mathcal{K} \to h\mathbf{qCat}$ and $h\mathrm{Fun}(-, B') : h\mathcal{K}^{\mathrm{op}} \to h\mathbf{qCat}$. These 2-functors preserve equivalences, so we immediately deduce the corollary. □

Definition 12. Let $\mathcal{K}$ be an ∞-cosmos, and $A \in \mathcal{K}$ an ∞-category. Its **homotopy category** is the 1-category defined as

$$hA := h\mathrm{Fun}(1, A),$$

where 1 denotes the terminal ∞-category in $\mathcal{K}$ (its existence is guaranteed by the completeness axioms), and $h\mathrm{Fun}(1, A)$ denotes the homotopy category of the quasicategory $\mathrm{Fun}(1, A)$.

Example 13. Take the ∞-cosmos of quasi-categories $\mathcal{K} = \mathbf{qCat}$. This notion of homotopy category of a quasicategory $A \in \mathbf{qCat}$ coincides with the previously defined notion. Since the terminal object of $\mathbf{qCat}$ is $\Delta[0]$,

$$hA = h\mathrm{Fun}(1, A) = h\mathrm{Map}(\Delta[0], A) = h\{\mathrm{Hom}_{\mathbf{Set}}(\Delta[0] \times \Delta[n], A)\}_n \cong$$

$$\cong h\{\mathrm{Hom}_{\mathbf{Set}}(\Delta[n], A)\}_n \cong hA,$$

where the first occurrence of hA denotes the homotopy category of the ∞-category A in the ∞-cosmos $\mathbf{qCat}$, and latter one denotes the homotopy category of the quasi-category A.

5. An example

The category of categories **Cat** can be seen as a 2-category as follows:

(1) Objects: All categories.
(2) 1-morphisms: A 1-morphism between two objects (categories) A and B is a functor $F : A \to B$.
(3) 2-morphisms: If $A, B \in \mathbf{Cat}$ are two objects (categories) and $F, G : A \to B$ are two 1-morphisms (functors), a 2-morphism from F to G is a natural transformation $\eta : F \Rightarrow G$.

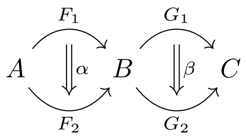
This data does not define a 2-category. We still need to define the identity functor $\mathrm{id}_A : \mathbb{1} \to \mathbf{Cat}(A, A)$ and the composition rule $\circ : \mathbf{Cat}(B, C) \times \mathbf{Cat}(A, B) \to \mathbf{Cat}(A, C)$.

For every object (category) $A \in \mathbf{Cat}$ the functor id_a is defined as

- The unique object in $\mathbb{1}$ is sent to the identity morphism from the category A to itself.
- The unique arrow in $\mathbb{1}$ is sent to the identity natural transformation from the identity functor of A to the identity functor of A.

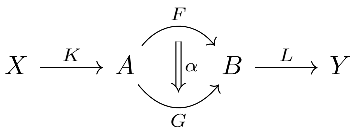
For every three objects (categories) $A, B, C \in \mathbf{Cat}$ the composition $\circ$ is defined as

- If $G \in \mathbf{Cat}(B, C)$ and $F \in \mathbf{Cat}(A, B)$ are 1-morphisms (functors), $\circ(G, F)$ is defined as the composite functor.
- If $\alpha : G_1 \Rightarrow G_2$ is a 2-morphism (natural transformation) between the 1-morphisms $G_1, G_2 \in \mathbf{Cat}(B, C)$, and $\beta : F_1 \Rightarrow F_2$ is a 2-morphism (natural transformation) between the 1-morphisms $F_1, F_2 \in \mathbf{Cat}(A, C)$, the composition $\circ(\alpha, \beta)$ is the 2-morphism from $\circ(G_1, F_1)$ to $\circ(G_2, F_2)$ given by the Godement product[2] of the natural transformations α and β.



Strict associativity and unit axioms should be checked. Vertical composition is given by composition of functors, and horizontal composition is given by the Godement product.

Given a diagram



the whiskered composite $L\alpha K$ is the following 2-morphism (natural transformation):

$$x \in X, \quad (L\alpha K)_x = (\mathrm{id}_L * \alpha * \mathrm{id}_K)_x = (\mathrm{id}_L)_{K(x)} = \cdots = L(\alpha_{K(x)}).$$

An equivalence from $A \in \mathbf{Cat}$ to $B \in \mathbf{Cat}$ is a pair of 1-morphisms (functors) $F : A \to B$ and $G : B \to A$ together with invertible 2-morphisms (natural isomorphisms) $\varepsilon : \mathrm{id}_B \Rightarrow FG$ and $\eta : \mathrm{id}_A \Rightarrow GF$. This notion recovers the usual definition of equivalent categories.

References

[1] Power. "A 2-Categorical Pasting Theorem". In: *Journal of Algebra* (1990).
[2] E. Riehl and D. Verity. *Elements of ∞-category theory.*

[2] With the notations above, the Godement product of α and β is defined, on an element $a \in A$ of the category A, as $(\beta * \alpha)_a := \beta_{F_2(\alpha)} \circ G_1(\alpha_a)$