BOREL CONSTRUCTION, EQUIVARIANT COHOMOLOGY AND THE MANN-SU THEOREM

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The aim of this talk is to introduce the concept of equivariant cohomology and explain a theorem of L.N. Mann and J.C. Su as one of its applications. Given a topological space X with a group G acting on it, the goal of equivariant cohomology is to provide a cohomology theory which apart from telling us properties of topological space X, it also gives us information on the group action of G. In order to define the equivariant cohomology of a G-space we will need to review some facts on principal bundles and to introduce the Milnor construction of a Lie group. There is a lot of literature on this topic, we will mainly use [1, 2, 3].

The Mann-Su theorem finds a relationship between the cohomology of a compact manifold M and how "big" certain groups acting on M can be. We will follow their original paper ([4]) and references therein.

1. Equivariant cohomology

Notation 1.1. *In this notes X will always be a CW-complex, M a connected topological manifold and G a Lie group. If there is a group action of G on X then we will say that X is a G-space. We will also assume that the group actions are effective.*

In order to define the Borel construction and equivariant cohomology we first need to introduce free group actions and their relationship with principal G-bundles.

Definition 1.2. *A group action of G on X is free if $G_x = \{e\}$ for all $x \in X$, where e denotes the identity element of G.*

Recall that if $p : E \longrightarrow B$ is a principal G-bundle then there exists a free action of G on E such that p induces a homeomorphism $B \cong E/G$. Thus, every principal G-bundle induces a free action of G. Conversely, if we have a group action of G on X, then the orbit map $\pi : X \longrightarrow X/G$ is a principal G-bundle.

Let X be a G-space and $p : E \longrightarrow B$ a principal G-bundle. The associated bundle construction is a way to construct fiber bundles with fiber X by attaching X to each fiber of a principal G-bundle using the group action of G on X. More precisely, consider the diagonal G-action on $E \times X$ such that $(a, x)g = (ag, g^{-1}x)$ for all $g \in G$ and $(a, x) \in E \times X$. Note that this action is free. The quotient space space $(E \times X)/G$ is denoted by $E \times_G X$ and its elements by $[a, x]$. Then, we have a continuous map $q : E \times_G F \longrightarrow B$ such that $q[e, f] = p(e)$, which is a fiber bundle over B with fiber X.

Theorem 1.3. *(Milnor, 1956) For any Lie group G there exists a contractible space EG such that G acts freely on EG.*

Remark 1.4. 1. *EG is well-defined up to G-homotopy.*

2. *The quotient $EG/G = BG$ is called the classifying space of G. This name is a consequence of the fact that there is a bijection between the set of principal G-bundles over X, $Prin_G(X)$, and $[X, BG]$.*

3. *The principal bundle $EG \longrightarrow BG$ is called the universal principal G-bundle.*

Before sketching the main ideas of the proof, we will focus on an important example:

Example 1.5. *Let $p \in \mathbb{N}$. We will construct EG when $G = \mathbb{Z}/p$. Firstly, we consider the sphere*

$$S^{2n-1} = \{(z_1, ..., z_n) \in C^n : |z_1|^2 + ... + |z_n|^2 = 1\} \subset \mathbb{C}^n.$$

There is a group action of $\mathbb{Z}/p$ on S^{2n-1} such that $a(z_1, ..., z_n) = (e^{\frac{2\pi i a}{p}} z_1, ..., e^{\frac{2\pi i a}{p}} z_n)$ for any $a \in \mathbb{Z}/p$ and $(z_1, ..., z_n) \in \mathbb{C}^n$. It is straightforward to check that this action is free.

Moreover, the inclusion maps $i_{n,m} : S^{2m-1} \longrightarrow S^{2n-1}$ such that $i_{m,n}(z_1, ..., z_n) = (z_1, ..., z_n, 0, ..., 0)$ are $\mathbb{Z}/p$-equivariant, which means that $i_{m,n}(a(z_1, ..., z_n)) = ai_{m,n}((z_1, ..., z_n))$ for all $a \in \mathbb{Z}/p$ and $(z_1, ..., z_n) \in \mathbb{C}^n$.

We have obtained a family of groups actions of $\mathbb{Z}/p$ on spaces that are not contractible. We will use them to construct a $\mathbb{Z}/p$-action on a contractible space. Note that we have a chain of $\mathbb{Z}/p$-equivariant inclusions $S^1 \subseteq S^3 \subseteq S^5 \subseteq \cdots$. Then we can define the space

$$S^\infty = \bigcup_{n \in \mathbb{N}} S^{2n-1}$$

with the topology induced by the inclusions. That is, $U \subset S^\infty$ is open if and only if $S^{2n-1} \cap U$ is open in S^{2n-1} for all $n \in \mathbb{N}$. A point in S^∞ is a sequence $(z_1, z_2, \dots)$ where $z_i \in \mathbb{C}$, only finitely many of them are non-zero and $\sum |z_i|^2 = 1$. The free group actions of $\mathbb{Z}/p$ on S^{2n-1} for $n \geq 1$ induce a free group action of $\mathbb{Z}/p$ on S^∞. In addition, we have the well-known result:

Lemma 1.6. *S^∞ is contractible.*

Proof. Let $\mathbf{1} : S^\infty \longrightarrow S^\infty$ be the constant map $\mathbf{1}(z_1, z_2, \dots) = (1, 0, \dots)$ and let $\sigma : S^\infty \longrightarrow S^\infty$ be the continuos shift map such that $\sigma(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$. Then, one can see that the map $H : S^\infty \times I \longrightarrow S^\infty$ such that

$$H(z, t) = \frac{(1-t)z + t\mathbf{1}(z) + (t - t^2)\sigma(z)}{|(1-t)z + t\mathbf{1}(z) + (t - t^2)\sigma(z)|}$$

is the desired homotopy which makes S^∞ contractible.

□

In consequence, we have that $E\mathbb{Z}/p \cong S^\infty$. The quotients $S^{2n-1}/(\mathbb{Z}/p) \cong L^n_p$ are known as lens spaces. Thus, $B\mathbb{Z}/p$ is usually denoted by L^∞_p.

Note that the action of $\mathbb{Z}/p$ on S^{2n-1} is the restriction of the action of $S^1 \subset \mathbb{C}^*$ on S^{2n-1}. We can use the same reasoning to show that $ES^1 \cong S^\infty$ and $BS^1 \cong \mathbb{C}P^\infty$.

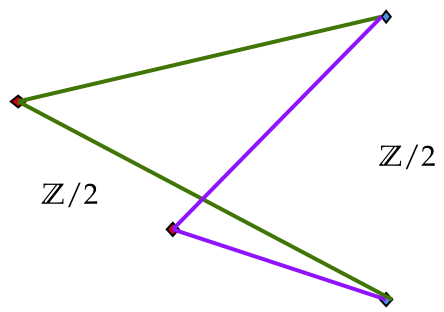
Recall that given two topological spaces X and Y, we define the join $X * Y$ to be $X \times Y \times I/ \sim$, where we have the equivalence relation $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$

and $y, y' \in Y$. Moreover, if X and Y are G-spaces then $X * Y$ is is a G-space where the action fulfils that $g[x, y, t] = [gx, gy, t]$ for all $[x, y, t] \in X * Y$ and $g \in G$.

We are now ready to explain the Milnor construction of EG for an arbitrary Lie group G. Firstly, note that G (as a group) acts freely on itself (as a topological space) by right multiplication. Explicitly, given $g, h \in G$ we have that $g(h) = gh$. We set $EG(0) = G$ and $EG(n) = EG(n-1) * G$. The free action of G on G induces a free action of G on each $EG(n)$. On the other hand, $EG(n)$ is $n-1$-connected for $n \geq 1$.

We have a sequence of G-equivariant inclusions $EG(0) \subseteq EG(1) \subseteq EG(2) \subseteq \cdots$. Therefore, like in the above example, we can define EG to be the colimit of these inclusions. The space EG can be constructed as the infinite join $G * G * G * \cdots$. This space is contractible and has a free action of G.

Example 1.7. *We can compute the Milnor construction when $G = \mathbb{Z}/2$. Since $\mathbb{Z}/2 \cong S^0$, we can use that $S^m * S^n \cong S^{m+n+1}$ to conclude that $E\mathbb{Z}/2(n) = S^n$ for all $n \geq 0$. It can also be seen that the action of $\mathbb{Z}/2$ induced on $E\mathbb{Z}/2(n)$ is the antipodal action (in consequence $E\mathbb{Z}/2(n)/(\mathbb{Z}/2) \cong \mathbb{R}P^n$). Therefore $E\mathbb{Z}/2 \cong S^\infty$ and $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$.*



$\mathbb{Z}/2 * \mathbb{Z}/2 \cong S^1$

Remark 1.8. *If G_1 and G_2 are two Lie groups, then $E(G_1 \times G_2) \cong EG_1 \times EG_2$ and $B(G_1 \times G_2) \cong BG_1 \times BG_2$.*

Definition 1.9. *Let X be a G-space, the Borel construction of X is the space $X_G = EG \times_G X$. The equivariant cohomology of X is*

$$H^*_G(X) = H^*(X_G).$$

Remark 1.10.

1. *By using the universal principal G-bundle and the associated bundle construction, we can find a fibration*
$$X \longrightarrow X_G \longrightarrow BG.$$
This fibration is known as the Borel fibration.
2. *If the action of G on X is free, we can use the principal G-bundle $\pi : X \longrightarrow X/G$ to construct a fibration*
$$EG \longrightarrow X_G \longrightarrow X/G.$$
Since EG is contractible, then $H^*_G(X) \cong H^*(X/G)$.
3. *If $X = \{pt\}$, then $H^*_G(X) = H^*(BG)$. More in general, the map $X_G \longrightarrow BG$ induces a structure of $H^*(BG)$-module on $H^*_G(X)$.*

4. *If G is a discrete group, then $H^*(BG)$ coincides with the group cohomology $H^*(G)$. For example, if p is a prime then $H^i(B\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$ for all $i \geq 0$.*

2. Bounding elementary p-group actions

In this section p will always denote a prime number. An elementary p-group is a a group of the form $(\mathbb{Z}/p)^r$ (they are also called p-tori). The number r is called the rank of the group. In this section we will explain some results that find a relationship between the cohomology of a space and the rank of the elementary p-groups acting on it.

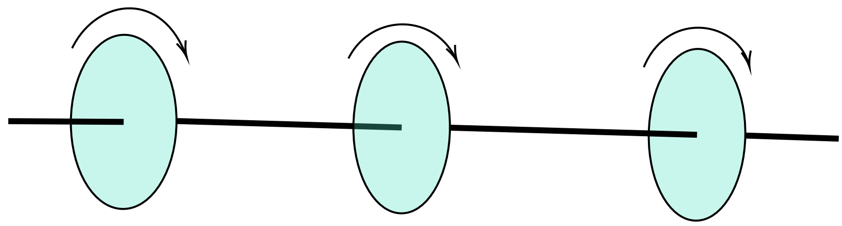
Theorem 2.1. *(Mann-Su, 1962) Let M be a compact manifold of dimension n. Let $B_p = \sum_{i=0}^{n} \dim H^i(M, \mathbb{Z}/p)$. There exist a number C_p only depending on n and B_p such that if $(\mathbb{Z}/p)^r$ acts effectively on M then $r \leq C_p$.*

We will only give a sketch of the proof of this theorem. Assume that we have an effective action of $G = (\mathbb{Z}/p)^r$ on M, then:

1. We can reduce to the study of the case where the action is free. If we only consider topological manifolds and continuous group actions, we need to study actions of $(\mathbb{Z}/p)^r$ with fixed points an use results by A. Borel. If we put the extra assumption that M is a smooth manifold and the action of $(\mathbb{Z}/p)^r$ is smooth then we can construct an invariant Riemannian metric on M and lift the action on M to an free action of $(\mathbb{Z}/p)^r$ on the total space of the orthonormal frame bundle $Fr(M)$ (which is a principal $\mathrm{O}(n)$-bundle). Since $\mathrm{O}(n)$ is a compact Lie group, $Fr(M)$ is still a compact manifold. These reductions will change the value of C_p, but we can control this change using only n and B_p.
2. We can use the Borel fibration $M \longrightarrow M_G \longrightarrow BG$ together with the Serre spectral sequence to obtain a convergent spectral sequence
$$E_2^{s,t} = H^s(BG, \mathcal{H}^t(M, \mathbb{Z}/p)) \implies H_G^{s+t}(M),$$
where we use the calligraphic letter $\mathcal{H}$ to denote the cohomology with local coefficients.
3. Since $H^i(M, \mathbb{Z}/p) = 0$ for all $i > n$, the spectral sequence collapses at the page $n+1$. We have that $E_\infty = E_{n+1}$.
4. Since the action is free, we have that $H_G^i(M) = H^i(M/G) = 0$ for all $i > n$. Using this fact and doing some computations on the spectral sequence, we obtain the inequality
$$\dim E_2^{s+1,0} \leq \sum_{j=0}^{n} \dim E_2^{s-j,j}.$$
5. On the other hand
$$\dim E_2^{s,t} \leq \binom{s-r-1}{r-1} \dim H^t(M, \mathbb{Z}/p),$$
where the first part is precisely $\dim H^s(BG, \mathbb{Z}/p)$, which can be computed using the Künneth formula. Moreover
$$\dim E_2^{n+1,0} = \binom{n-r}{r-1}.$$

6. By using the inequalities of points 4 and 5 and doing some computations we can find that
$$r \leq \frac{\sqrt{n^2 + 4n(n+1)B_p} - n}{2} = C_p.$$

Remark 2.2. 1. *We only need M to be a manifold to be able to reduce the theorem to the study of free actions. If we assume that the action is free we can assume that the G-space is a CW-complex with some finiteness conditions on the cohomology. Nevertheless, in the general case the condition of M being a manifold is necessary. For any $r > 0$ we can construct a contractible 2-complex X_r which admits an action of $(\mathbb{Z}/p)^r$ as shown in the image below. Each component of $(\mathbb{Z}/p)^r$ rotates one of the disks and fixes the line through the origin.*



2. *We cannot find a constant C_p bounding r which only depends on the dimension. For any $r > 0$, it is possible to construct a surface S_r which admit an action of $(\mathbb{Z}/p)^{2r}$.*

3. *We can ask how sharp is this bound. For example, if $M = S^1$ (so $n = 1$ and $B_p = 2$) then $C_p \simeq 1,47$. Therefore, if $(\mathbb{Z}/p)^r$ acts freely on S^1 then $r = 1$, as it was already known. However, it is possible to find much better bounds if we only focus on some specific manifold:*

   Theorem 2.3. *(Smith, 1960) Assume that $(\mathbb{Z}/p)^r$ acts on S^n, then*
   $$r \leq \begin{cases} \frac{n+1}{2} & p \text{ odd} \\ n+1 & p = 2 \end{cases}.$$

   Note that this bound is also for non-free group actions.

Finally, we will state a similar theorem by Carlsson and Baumgarter. Assume that X is a G-space. Then we have a linear action of G on the cohomology $H^*(X,k)$, where k is a field. In particular, $H^*(X,k)$ has a structure of $k[G]$-module. Given a $k[G]$-module A, we define
$$l(A) = \min\{l \in \mathbb{N} : I^l A = 0\},$$
where I denotes the augmentation ideal of $k[G]$. Then:

Theorem 2.4. *(Carlsson, 1983 ($p = 2$), Baumgarter, 1990 (p odd)) Let $G = (\mathbb{Z}/p)^r$ and let X be free G-space such that there exist $n \in \mathbb{N}$ which fulfils that $H^i(X/G, \mathbb{Z}/p) = 0$ for all $i > n$. Then*
$$\sum_{i=0}^{\infty} l(H^i(M, \mathbb{Z}/p)) \geq r + 1.$$

Corollary 2.5. *If the cohomology of X is isomorphic to the cohomology of the product of s spheres of dimension m, $S^m \times \cdots \times S^m$, and the action of G on X induces a trivial action on the cohomology then $s \geq r$.*

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