Transformation group theory: Study symmetries of top spaces.

Q. Which spaces admit a free/almost-free/effective action of a certain group?

Group actions in Top: $G \times X \longrightarrow X$ continuous st $\begin{cases} e \cdot x = x \\ g\cdot(h\cdot x) = (gh)\cdot x \end{cases}$

$x \in X$, $G_x := \{ g \in G ;\ gx = x \}$ isotropy subgp at x

$$\begin{cases} \text{Free} : G_x = \{e\} \ \forall x \\ \text{Almost-free} : G_x \text{ finite } \forall x \\ \text{Effective} : \bigcap G_x = \{e\} \end{cases}$$

TALK 1. General considerations of G-actions when $G = \mathbb{T}^r := S^1 \times \overset{(r)}{\cdots} \times S^1$ or $G = (\mathbb{Z}/p\mathbb{Z})^r =: \mathbb{T}_p^r$

Borel Construction $X \longrightarrow X_G = EG \times_G X \longrightarrow BG$

EG: contractible free G-space (always exists!)

$BG = EG/G$ classifying space

$\Longrightarrow$ universal G-bundle $G \longrightarrow EG \longrightarrow BG$

(ex. $\mathbb{Z}^n \to \mathbb{R}^n \to \mathbb{T}^n$)

Consider diag. action:

$$\left.\begin{array}{l} G \times (EG \times X) \longrightarrow EG \times X \\ g\ ,\ (h, x) \longmapsto (hg^{-1}, gx) \\ \text{then } X_G := EG \times X/G \longrightarrow BG \\ \quad [h, x] \longmapsto [h] \end{array}\right\} \text{associated bundle}$$

with $G \nearrow X_G$

Equivariant Cohomology $H_G^*(X) := H^*(X_G)$

- $G = \{e\}$ $\quad H_G^*(X) \cong H(X)$
- $X \simeq *$ $\quad H_G(X) \cong H(BG)$
- Free action $\Rightarrow$ $X_G \xrightarrow{\pi} X/G$ htp equiv & so $H_G(X) \cong H(X/G)$

  $[h, x] \longmapsto [x]$
- Almost-free $\Rightarrow$ π is a $\mathbb{Q}$-htp equivalence

$\longrightarrow$ from the viewpoint of $\mathbb{Q}$-htp theory, free $\equiv$ almost-free

TALK 2. Borel fibration, equiv cohom, Mann-Su Thm

Rmk $H_G^*(X)$ computable by SS // Also $\exists$ $\mathbb{Q}$-htp models!

$\mathbb{Q}$-Homotopy Theory

- $\mathbb{K}$ field. $C^*(X,\mathbb{K})$ is a dga inducing cdga in H^*

commutative cochains problem: find a cdga over $\mathbb{K}$ st
$A(X) \xrightarrow{\sim} C(X,\mathbb{K})$ & $\forall\, X \hookrightarrow Y$ $A(Y) \twoheadrightarrow A(X)$.

- Ex. $A_{dR}(M)$ is a solution when $\mathbb{K}=\mathbb{R}$ & M mfld
- RHT. Solution over $\mathbb{Q}$ given by $A_{pl}(X)$.

  Also: $(\Lambda(V),d) \xrightarrow{\sim} A_{pl}(X)$ minimal model

  If $H^1(X,\mathbb{Q})=0$ then $\pi_*(X)\otimes\mathbb{Q} \cong V^*$
- ⚠ $\mathbb{K}=\mathbb{F}_p$. No solution in gral! Steenrod operations

Example: Model for Borel fibration $X \longrightarrow X_{\mathbb{T}^r} \longrightarrow B\mathbb{T}^r$

$(\Lambda(y_1,\dots,y_r),0) \xrightarrow{\sim} A_{pl}(\mathbb{T}^r)$; $|y_i|=1$

$(\Lambda(x_1,\dots,x_r),0) \xrightarrow{\sim} A_{pl}(B\mathbb{T}^r)$; $|x_i|=2$

$(\Lambda(x_i),0) \longrightarrow (\Lambda(x_i)\otimes\Lambda(V),D) \longrightarrow (\Lambda V,d)$ (min model of X)

allows to realize torus actions!

TALK 3: Intro to RHT; Model of Borel fibration

Free toral ranks $A_{pl}(X_{\mathbb{T}^r})$ gives upper bounds of free toral rks in terms of χ_π

Thm (Halperin) $\mathbb{T}^r \overset{AF}{\circlearrowright} X$ (1-connected, CW-cx w/ $\dim H(X,\mathbb{Q})<\infty$, $\dim \pi(X)\otimes\mathbb{Q}<\infty$)

then $\chi_\pi := \sum(-1)^k \dim \pi_k(X)\otimes\mathbb{Q} \leq -r$

Corollary $\mathbb{T}^r \overset{F}{\circlearrowright} X = S^{n_1}\times\dots\times S^{n_k}$ then $r \leq k_0 = \#$ odd dim'l spheres

↳ idea: if $r>k_0$ then $H^i(X_{\mathbb{T}^r})\neq 0$ for $i\gg$

But free $\Longrightarrow X_{\mathbb{T}^r} \simeq X/\mathbb{T}^r$ finite dim'l !!

TRC $\mathbb{T}^r \overset{AF}{\circlearrowright} X$ compact, Haus, 1-con $\Longrightarrow \dim H(X,\mathbb{Q}) \geq 2^r$.

} Borel model ↓

Algebraic TRC If $\dim H(\Lambda(x_i)\otimes\Lambda(V),D)<\infty$ then $\dim H(\Lambda V,d)\geq 2^r$.

TALK 4: Finite dim'l RHT

p-Torus Actions

$\mathbb{T}_p^r := (\mathbb{Z}/p\mathbb{Z})^r$ elementary abelian p-group of rank r.

$(\mathbb{Z}/p\mathbb{Z})^r \cong \langle e_1, .., e_r \,;\, e_i e_j = e_j e_i,\ e_i^p = 1 \rangle$

p-TRC $\mathbb{T}_p^r \underset{F}{\circlearrowright} X$ then $\dim H^*(X, \mathbb{F}_p) \geq 2^r$

A precursor of p-TRC is the following:

Conjecture $\mathbb{T}_p^r \underset{F}{\circlearrowright} S^{n_1} \times .. \times S^{n_K}$ then $r \leq K$

($r \leq K_\theta$ sharp version)

(**Example** true for $K = 1$)

Thm (HANKE) If $p > 3 \cdot \dim X$ & $\mathbb{T}_p^r \underset{F}{\circlearrowright} S^{n_1} \times .. \times S^{n_K} \Rightarrow r \leq K_\theta$.

Rmks
- Hanke's Thm $\Rightarrow$ Halpern's Coro but NOT conversely: $\exists$ free actions of $\mathbb{T}_p^r$ on $(S^m)^K$ that can't be extended to $\mathbb{T}^r$-actions (exotic)
- $H^*(B\mathbb{T}_p^r, \mathbb{Q}) = 0$ so $\mathbb{Q}$-htp theory seems useless here

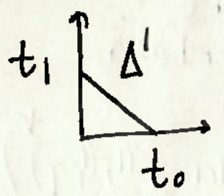
Tame Homotopy Theory

Recall the construction of A_{pl}:

$\Omega_n^* := \Lambda(t_0, .., t_n, dt_0, .., dt_n) / \sum t_i = 1, \sum dt_i = 0$

└ algebra of polynomial forms on

$\Delta^n = \{(t_0, -, t_n) \in \mathbb{R}^{n+1} \,;\, 0 \leq t_i \leq 1 \;\&\; \sum t_i = 1\}$



$\Delta^{n-1} \hookrightarrow \Delta^n \quad (t_0, .., t_{n-1}) \longmapsto (t_0, .., \underset{(i)}{0}, .., t_{n-1})$

$\Delta^{n+1} \longrightarrow \Delta^n \quad (t_0, -, t_{n+1}) \longmapsto (t_0, .., \underset{(i)}{t_i + t_{i+1}}, .., t_{n+1})$

$\Rightarrow$ Get a simplicial cdga $\Omega_\bullet^*$

$A_{pl}(X) := \mathrm{Hom}_{sSet}(\mathrm{Sing}(X), \Omega_\bullet^*)$

$\mathrm{Sing}(X)_n := \{\Delta^n \to X \text{ cont}\}$

$\exists$ quasi-isomorphism $A_{pl}(X) \xrightarrow{\int} C^*(X, \mathbb{Q})$ given by integration of forms.

- $\int$ creates denominators : $\int_{[0,1]} t^{k-1}\,dt = \frac{1}{k}$
- Not much hope for polyn forms to describe $C^*(X,\mathbb{Z})$

Cenkl-Porter complex : Understand what denominators are created!

- $\text{Weight}\left(t_0^{\alpha_0} dt_0^{\varepsilon_0} \cdots t_n^{\alpha_n} dt_n^{\varepsilon_n}\right) := \max_i \{\alpha_i + \varepsilon_i\}$
- If $\text{Weight}(\omega) \leq q$ then $\int \omega \in \mathbb{Q}_q := \mathbb{Z}\left[\frac{1}{p} ; p \leq q\right]$

smallest subring of $\mathbb{Q}$ where all primes $p \leq q$ are invertible ($\mathbb{Q}_0 = \mathbb{Q}_1 = \mathbb{Z}$)

filtered cdga's

$\Omega^*_\circ \rightsquigarrow T_\circ^{*,q}$ (weight) technical issue : use cubical decomp of Δ^n

$T^{*q}(X) := \text{Hom}(\text{Sing}(X), T_\circ^{*,q})$

$\int \Big\downarrow S$ ← multiplicative $T^{*q} \times T^{*q'} \to T^{*q+q'}$

$C^*(X, \mathbb{Q}_q)$

- $H^*(T^{*,0}(X)) \cong H^*(X,\mathbb{Z})$
- $H^*(T^{*q}(X) \otimes \mathbb{F}_p) \cong H^*(X,\mathbb{F}_p)\ \forall\ p > q$.

TALK 5: pf of main thm

TALK 6: Cenkl-Porter complex

T7: Tame Massey, Sq^* & Tame spaces → [Dwyer] X $(r-1)$-connected is tame if $\pi_{r+q}(X)$ is a module over $\mathbb{Q}_q\ \forall\ q \geq 0$

T8: Tame Hirsh Lemma → Principal fibrations are controlled up to a certain degree $H(K(\mathbb{Z},k),\mathbb{Q}_q) \cong \Lambda_{\mathbb{Q}_q}(\text{Hom}(\pi,\mathbb{Q}_q)^k)$ up to degree $k+2q-3$

T9: Tame models for Borel spaces

T10: Proof of Hirsh Lemma