Quasicategories

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1.1 Simplicial sets

We denote by Δ the category whose objects are the sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are order-preserving functions $[n] \to [m]$.

A *simplicial set* is a functor $X \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, where $\mathbf{Set}$ denotes the category of sets. A *simplicial map* $f \colon X \to Y$ between simplicial sets is a natural transformation. The category of simplicial sets with simplicial maps is denoted by $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ or, more concisely, as $\mathbf{sSet}$.

For a simplicial set X, we normally write X_n instead of $X[n]$, and call it the set of *n-simplices* of X. There are injections $\delta_i^n \colon [n-1] \to [n]$ forgetting i and surjections $\sigma_i^n \colon [n+1] \to [n]$ repeating i for $0 \leq i \leq n$ that give rise to functions

$$d_i^n \colon X_n \longrightarrow X_{n-1}, \qquad s_i^n \colon X_{n+1} \longrightarrow X_n,$$

called *faces* and *degeneracies* respectively. Since every order-preserving function $[n] \to [m]$ is a composite of a surjection followed by an injection, the sets $\{X_n\}_{n \geq 0}$ together with the faces d_i^k and degeneracies s_j^ℓ determine uniquely a simplicial set X. Faces and degeneracies satisfy the *simplicial identities*:

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n \text{ if } i < j;$$

$$d_i^{n+1} \circ s_j^n = \begin{cases} s_{j-1}^{n-1} \circ d_i^n \text{ if } i < j; \\ \mathrm{id}_{X_n} \text{ if } i = j \text{ or } i = j+1; \\ s_j^{n-1} \circ d_{i-1}^n \text{ if } i > j+1; \end{cases}$$

$$s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n \text{ if } i \leq j.$$

For $n \geq 0$, the *standard n-simplex* is the simplicial set $\Delta[n] = \Delta(-, [n])$, that is,

$$\Delta[n]_m = \Delta([m], [n])$$

for all $m \geq 0$. Then the Yoneda Lemma implies that

$$X_n = X[n] \cong \mathbf{sSet}(\Delta(-, [n]), X) = \mathbf{sSet}(\Delta[n], X)$$

for each simplicial set X and $n \geq 0$. Consequently, for all simplicial sets X we have

$$X \cong \operatorname*{colim}_{(\Delta \downarrow X)} \Delta[n]$$

as a special case of the Density Theorem. The comma category $(\Delta \downarrow X)$ is called the *category of simplicies* of X.

For $k \geq 0$, the *k-skeleton* of a simplicial set X is the smallest sub-simplicial-set of X containing $X_0, \ldots, X_k$.

The *boundary* of $\Delta[n]$, denoted $\partial\Delta[n]$, is the $(n-1)$-skeleton of $\Delta[n]$. The *kth horn* $\Lambda^k[n]$ is the sub-simplicial-set of $\partial\Delta[n]$ resulting from removing the kth face. The horns with $0 < k < n$ are called *inner horns* and those with $k = 0$ and $k = n$ are *outer horns*.

The *geometric realization* of a simplicial set X is defined as

$$|X| = \operatorname*{colim}_{(\Delta \downarrow X)} \Delta^n$$

where $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1,\ \Sigma t_i = 1\}$ with the Euclidean topology. In other words, $|-|$ is the left Kan extension of the functor from Δ to the category of topological spaces sending $[n]$ to Δ^n.

A simplicial map $f\colon X \to Y$ between simplicial sets is a *weak equivalence* if the induced map $|X| \to |Y|$ is a weak homotopy equivalence of topological spaces, that is, if it induces a bijection of connected components and group isomorphisms

$$\pi_n(X, x) \cong \pi_n(Y, f(x))$$

for $n \ge 1$ and all $x \in X_0$. Two simplicial sets X and Y are called *weakly equivalent* (denoted $X \simeq Y$) if there is a zig-zag of weak equivalences between them:

$$X = W_0 \to W_1 \leftarrow W_2 \to \cdots \to W_{n-1} \leftarrow W_n = Y.$$

1.2 Nerve of a category

The *nerve* of a small category $\mathcal{C}$ is the simplicial set $N\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ defined as

$$N\mathcal{C} = \mathrm{Cat}(-, \mathcal{C});$$

that is, the set $(N\mathcal{C})_n$ of n-simplices of $N\mathcal{C}$ is the set of functors $[n] \to \mathcal{C}$ where $[n]$ is viewed as a category by means of its order $0 \to 1 \to 2 \to \cdots \to n$. Faces and degeneracies of $N\mathcal{C}$ come from the injections $\delta_i^n\colon [n-1] \to [n]$ and surjections $\sigma_i^n\colon [n+1] \to [n]$. In other words, the n-simplices of $N\mathcal{C}$ are sequences of n composable morphisms:

$$(N\mathcal{C})_n = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n\},$$

where the ith face composes $f_{i+1} \circ f_i$ and the ith degeneracy inserts $\mathrm{id}\colon X_i \to X_i$.

Hence we may view $(N\mathcal{C})_0$ as the set of objects of $\mathcal{C}$ and $(N\mathcal{C})_1$ as the set of morphisms of $\mathcal{C}$. For example, the nerve of $[n]$ is the standard n-simplex $\Delta[n]$.

The Yoneda Lemma tells us that the functor $N\colon \mathrm{Cat} \to \mathbf{sSet}$ is fully faithful:

$$\mathbf{sSet}(N\mathcal{C}, N\mathcal{D}) \cong \mathrm{Cat}(\mathcal{C}, \mathcal{D}),$$

that is, the functors $\mathcal{C} \to \mathcal{D}$ are in one-to-one correspondence with the simplicial maps $N\mathcal{C} \to N\mathcal{D}$. Hence we may think of the category Cat of small categories as a subcategory of simplicial sets, by identifying a category with its nerve.

However, $N\colon \mathrm{Cat} \to \mathbf{sSet}$ is very far from being an equivalence of categories, since not every simplicial set is isomorphic to the nerve of a category. For example, for a simplicial set to be a nerve it is necessary that for every pair of consecutive edges $e_1\colon v_1 \to v_2$ and $e_2\colon v_2 \to v_3$ there is a (uniquely determined) edge $e_2 \circ e_1\colon v_1 \to v_3$.

1.3 The Kan condition

A simplicial set X satisfies the *Kan condition* if every map $\Lambda^k[n] \to X$ admits an extension $\Delta[n] \to X$ (called a *filler*). Then we also say that X is a *Kan complex* or a *fibrant* simplicial set. In other words, X is a Kan complex if and only if the map

$$\mathbf{sSet}(\Delta[n], X) \longrightarrow \mathbf{sSet}(\Lambda^k[n], X) \tag{1.1}$$

is surjective for $0 \leq k \leq n$ and for all n.

Example 1.1. If X is a topological space, then the *singular* simplicial set $\operatorname{Sing} X$ has as n-simplices the continuous maps $\Delta^n \to X$, with faces and degeneracies coming from those of $\Delta[n]$. This is an example of a Kan complex, since the geometric realization of every horn $\Lambda^k[n]$ is a strong deformation retract of Δ^n.

The functor $\operatorname{Sing}\colon \mathrm{Top} \to \mathbf{sSet}$ is right adjoint to the geometric realization functor, and moreover the counit map $|\operatorname{Sing} X| \to X$ is a weak homotopy equivalence of topological spaces for every space X, while the unit map $K \to \operatorname{Sing}|K|$ is a weak equivalence of simplicial sets for every simplicial set K.

Example 1.2. A group G can be viewed as a small category with a single object and a morphism for each element of G, with composition corresponding to the multiplication in G. Then the topological space $BG = |NG|$ is called the *classifying space* of the group G and has the property that its fundamental group is isomorphic to G while all its higher homotopy grops are zero. The nerve NG is a Kan complex.

Proposition 1.3. *A simplicial set X satisfies $X \cong N\mathcal{C}$ for some category $\mathcal{C}$ if and only if* (1.1) *is bijective for $0 < k < n$ and $n \geq 2$.*

Proof. The condition given in the statement ensures that all composable arrows have a unique composite. □

Definition 1.4. A simplicial set X is called a *quasicategory* (or a *weak Kan complex*, or an ∞-*category*) if (1.1) is surjective for $0 < k < n$ and $n \geq 2$.

Hence a simplicial set X is a quasicategory if and only if every map $\Lambda^k[n] \to X$ from an inner horn has a (not necessarily unique) filler $\Delta[n] \to X$, for $n \geq 2$. If α is a 2-simplex of X filling

$$v_0 \xrightarrow{\varphi} v_1 \xrightarrow{\psi} v_2$$

and $\phi\colon v_0 \to v_2$ is the face $d_1^2\alpha$, then we say that ϕ is a *composite* of φ and ψ *witnessed* by α. Thus in a quasicategory composites of arrows exist, although they need not be unique.

Categories and Kan complexes are special cases of quasicategories.

- A category is a quasicategory where fillers for inner horns are unique.
- A Kan complex is a quasicategory where fillers for all horns exist.

A category where all morphisms have inverses is called a *groupoid*.

Proposition 1.5. *A category $\mathcal{C}$ is a groupoid if and only if $N\mathcal{C}$ is a Kan complex.*

Proof. In $N\mathcal{C}$ is a Kan complex, then fillers for maps from the outer horn $\Lambda^0[2]$ yield inverses, which are unique since $\mathcal{C}$ is a category. □

A quasicategory which is a Kan complex is called an ∞-*groupoid*.

Example 1.6. For a topological space X, the *fundamental* ∞-*groupoid* $\pi_\infty X$ is the quasicategory whose 0-simplices are the points of X, whose 1-simplices are the paths in X, and whose n-simplices for $n \geq 2$ are the homotopies $H\colon \sigma \to \tau$ where σ and τ are $(n-1)$-simplices.

1.4 Homotopy category of a quasicategory

If X is a quasicategory, we call *objects* of X its 0-simplices and *morphisms* of X the 1-simplices. The face $d^1_1\colon X_1 \to X_0$ is called *source map* and $d^1_0\colon X_1 \to X_0$ is the *target map*. We denote by $f\colon x \to y$ a morphism with source x and target y.

Every object $x \in X_0$ has an identity morphism, namely $\mathrm{id}_x = s^0_0 x$. Thus id_x is a morphism from x to x since $d^1_0 \circ s^0_0 = d^1_1 \circ s^0_0 = \mathrm{id}_{X_0}$.

If $f, g\colon x \to y$ are morphisms in a quasicategory X, we say that f is *homotopic* to g, and write $f \simeq g$, if there exists a 2-simplex $\sigma \in X_2$ (called a *homotopy* from x to y) with $d^2_0\sigma = g$, $d^2_1\sigma = f$ and $d^2_2\sigma = \mathrm{id}_x$. The homotopy relation is an equivalence relation. The homotopy class of f is denoted by $[f]$.

The *homotopy category* $\mathrm{Ho}(X)$ has X_0 as its set of objects and

$$\mathrm{Ho}(X)(x,y) = \{[f] \mid f\colon x \to y\}.$$

This is indeed a category since composition $[g] \circ [f]$ is well defined and unique, namely $[g] \circ [f] = [h]$ where h is any filler for

$$x \xrightarrow{f} y \xrightarrow{g} z.$$

The extension property for $\Lambda^1[3] \hookrightarrow \Delta[3]$ yields uniqueness of h up to homotopy. The identity morphism in $\mathrm{Ho}(X)$ of an object $x \in X_0$ is the homotopy class $[s^0_0 x]$.

A morphism $f\colon x \to y$ in X is called an *equivalence* if $[f]$ is invertible in $\mathrm{Ho}(X)$, that is, if there exists a morphism $g\colon y \to x$ with $g \circ f \simeq \mathrm{id}_x$ and $f \circ g \simeq \mathrm{id}_y$.

Proposition 1.7 (Joyal)**.** *The homotopy category* $\mathrm{Ho}(X)$ *is a groupoid if and only if* X *is a Kan complex.*

Example 1.8. The homotopy category $\mathrm{Ho}(\pi_\infty X)$ of the fundamental ∞-groupoid of a topological space X is the fundamental groupoid (or Poincaré groupoid) $\pi_1(X)$.

1.5 Simplicial enrichment

Denote by $\Delta^{\{i_0,\ldots,i_k\}} \subseteq \Delta[n]$ the k-simplex spanned by the vertices $i_0, \ldots, i_k$.

A homotopy from a morphism $f\colon x \to y$ to a morphism $g\colon x \to y$ can be viewed as a simplicial map $\sigma\colon \Delta[2] \to X$ with

$$\sigma|_{\Delta^{\{0,1\}}} = \mathrm{id}_x, \qquad \sigma|_{\Delta^{\{2\}}} = y,$$

that is, $d^2_2\sigma = s^0_0 x$ and $d^1_0 d^2_1 \sigma = y$.

Definition 1.9. If x and y are objects of a quasicategory X, an *n-morphism* from x to y is a simplicial map $\sigma\colon \Delta[n] \to X$ with

$$\sigma|_{\Delta\{0,\dots,n-1\}} = \mathrm{id}_x, \qquad \sigma|_{\Delta\{n\}} = y,$$

that is, $d_n^n\sigma = s_0^{n-2} \cdots s_0^0 x$ and $d_0^1 \cdots d_{n-1}^n \sigma = y$.

Then there is a simplicial set $\mathrm{map}(x,y)$ whose set of n-simplices $\mathrm{map}(x,y)_n$ is the set of $(n+1)$-morphisms from x to y for all $n \geq 0$. We call it the *space of morphisms from x to y*.

Proposition 1.10. *If x and y are any two objects of a quasicategory X, then* $\mathrm{map}(x,y)$ *is a Kan complex.*

Hence $\mathrm{map}(x,y)$ is also a quasicategory. It follows from the definitions that

$$\mathrm{Ho}(X)(x,y) = \pi_0 \,\mathrm{map}(x,y),$$

where π_0 denotes the set of connected components of a simplicial set.

1.6 Equivalences of quasicategories

If X and Y are simplicial sets, then there is a simplicial set $\mathrm{Map}(X,Y)$ with

$$\mathrm{Map}(X,Y)_n = \mathbf{sSet}(\Delta[n] \times X, Y)$$

for all n. Then, for all objects x and y of a quasicategory X, the space of maps $\mathrm{map}(x,y)$ is the pull-back of the source-and-target map

$$(s,t)\colon\ \mathrm{Map}(\Delta[1],X) \longrightarrow \mathrm{Map}(\partial\Delta[1],X) = X \times X$$

along

$$(x,y)\colon \Delta[0] \longrightarrow X \times X.$$

This is analogous to the fact that, for a category $\mathcal{C}$ and two objects x and y of $\mathcal{C}$, the set $\mathcal{C}(x,y)$ is the pull-back of the source-and-target map $\mathrm{Mor}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ along the constant map (x,y).

Proposition 1.11. *Let X and Y be simplicial sets.*

(i) *If Y is a Kan complex, then* $\mathrm{Map}(X,Y)$ *is a Kan complex.*

(ii) *If Y is a quasicategory, then* $\mathrm{Map}(X,Y)$ *is a quasicategory.*

A *functor* of quasicategories $u\colon X \to Y$ is a map of simplicial sets from X to Y. Given two functors $u, v\colon X \to Y$, a *natural transformation* $f\colon u \to v$ is a 1-simplex $\eta \in \mathrm{Map}(X,Y)_1$ with $d_0^1\eta = u$ and $d_1^1\eta = v$.

If X and Y are quasicategories, then $\mathrm{Map}(X,Y)$ is also denoted by $\mathrm{Fun}(X,Y)$ and called the *quasicategory of functors* from X to Y. Hence $\mathrm{Fun}(X,Y)$ is a simplicial set whose vertices are the functors $u\colon X \to Y$ and whose 1-simplices are the natural transformations $\eta\colon u \to v$.

Recall that an *equivalence of categories* is a fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ which is essentially surjective, that is, for every object $Y \in \mathcal{D}$ there is an object $X \in \mathcal{C}$ with $FX \cong Y$. Next we define the analogous concept for quasicategories.

A functor $u\colon X \to Y$ of quasicategories is called *fully faithful* if the induced map

$$\mathrm{map}_X(x, y) \longrightarrow \mathrm{map}_Y(u(x), u(y))$$

is a weak equivalence of simplicial sets for all $x, y \in X_0$.

A functor $u\colon X \to Y$ of quasicategories is *essentially surjective* is for every object $y \in Y_0$ there is an object $x \in X_0$ and an equivalence $u(x) \to y$, i.e., if the induced functor $\mathrm{Ho}(X) \to \mathrm{Ho}(Y)$ is essentially surjective.

Definition 1.12. An *equivalence* of quasicategories is an essentially surjective fully faithful functor.

An equivalence of quasicategories is also called a *Dwyer–Kan equivalence*. If $u\colon X \to Y$ is an equivalence of quasicategories, then the induced functor of homotopy categories $\mathrm{Ho}(X) \to \mathrm{Ho}(Y)$ is an equivalence, and, moreover,

$$\pi_n(\mathrm{map}_X(x, y), f) \cong \pi_n(\mathrm{map}_Y(u(x), u(y)), u(f))$$

for $n \geq 1$ and all morphisms $f\colon x \to y$ in X.

Theorem 1.13 (Joyal)**.** *A functor $u\colon X \to Y$ is an equivalence of quasicategories if and only if the induced functor*

$$\mathrm{Ho}(\mathrm{Map}(Y, C)) \longrightarrow \mathrm{Ho}(\mathrm{Map}(X, C))$$

is an equivalence of categories for every quasicategory C.

1.7 Adjunctions

A functor $u\colon X \to Y$ between quasicategories is *left adjoint* to a functor $v\colon Y \to X$ if there are natural transformations

$$\eta\colon \mathrm{id}_X \longrightarrow vu, \qquad \varepsilon\colon uv \longrightarrow \mathrm{id}_Y$$

for which the composites

$$\mathrm{map}_Y(u(x), y) \xrightarrow{v} \mathrm{map}_X(v(u(x)), v(y)) \xrightarrow{(\eta_x)^*} \mathrm{map}_X(x, v(y))$$

and

$$\mathrm{map}_X(x, v(y)) \xrightarrow{u} \mathrm{map}_Y(u(x), u(v(y))) \xrightarrow{(\varepsilon_y)_*} \mathrm{map}_Y(u(x), y)$$

are weak equivalences of simplicial sets for all $x \in X_0$ and $y \in Y_0$.

If $u\colon X \to Y$ and $v\colon Y \to Y$ form an adjoint pair, then the induced functors

$$\mathrm{Ho}(u)\colon \mathrm{Ho}(X) \longrightarrow \mathrm{Ho}(Y), \qquad \mathrm{Ho}(v)\colon \mathrm{Ho}(Y) \longrightarrow \mathrm{Ho}(X)$$

form an adjoint pair as well, since there are natural bijections

$$\mathrm{Ho}(Y)(u(x), y) = \pi_0 \,\mathrm{map}(u(x), y) \cong \pi_0 \,\mathrm{map}(x, v(y)) = \mathrm{Ho}(X)(x, v(y)).$$

Definition 1.14. A functor $u\colon X \to Y$ between quasicategories is called a *reflection* if it has a fully faithful right adjoint.

The universal property of a reflection can be formulated as follows. If $u\colon X \to Y$ is a reflection and we consider the class S of morphisms $f\colon a \to b$ in X such that $u(f)\colon u(a) \to u(b)$ is an equivalence, then, for every quasicategory Z, composing with u defines a fully faithful functor

$$\mathrm{Fun}(Y, Z) \longrightarrow \mathrm{Fun}(X, Z)$$

whose essential image is the collection of functors $v\colon X \to Z$ such that $v(s)$ is an equivalence for all $s \in S$.

The composite $\ell\colon X \to X$ of a reflection $X \to Y$ with its right adjoint $Y \to X$ is called a *localization*.

If $\ell\colon X \to X$ is a localization, then there is a natural transformation $\eta\colon \mathrm{id}_X \to \ell$ such that the morphisms $\ell(\eta_x)$ and $\eta_{\ell(x)}$ from $\ell(x)$ to $\ell(\ell(x))$ are homotopic and each of them is an equivalence.

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