Goodwillie calculus I

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1 Introduction

The purpose of Goodwillie calculus is to give a setting in which one can approximate a given functor $F : \mathcal{C} \to \mathcal{D}$ between two ∞-categories (usually Top_* of Sp) by **polynomial functors** $P_n F$. This is done in analogy to ordinary calculus, where one approximates a real valued function by polynomials. The analogy is strong and much of the vocabulary in Goodwillie calculus is made to resemble ordinary calculus. For instance, there are the notions of **polynomial functors**, **derivatives** and **Taylor tower** of a functor.

$$F(X) \to \cdots \to P_n F(X) \to P_{n-1} F(X) \to \cdots \to P_1 F(X)$$

Taylor tower of F applied to an object $X \in \mathcal{C}$.

The analogy, however, is not perfect and breaks down for instance when one looks at the identity functor $I : Top_* \to Top_*$. The Taylor tower of this functor is a highly non-trivial object, in opposition to the identity map in ordinary calculus. Although this might look like bad news, the non-triviality of the Taylor tower of I has very interesting consequences in the study of homotopy types. For example, applying the homotopy functor to the Taylor tower of I (applied to an object X), one obtains a sequence of homotopy groups interpolating between the **homotopy groups** of X and the **stable homotopy groups** of X!

$$\pi_*(X) \to \cdots \to \pi_*(P_n I(X)) \to \pi_*(P_{n-1}(X)) \to \cdots \to \pi_*(P_1 I(X)) \cong \pi_*^s(X)$$

In these notes, we aim to give a very brief introduction to the subject of Goodwillie calculus. We will define all the objects previously mentioned in the general context of ∞-categories, but present examples and results focusing on the ∞-category of pointed topological spaces Top_*, and mostly on the identity functor on this category. The theory was introduced and developed by Tom Goodwillie ([Goo90],[Goo92],[Goo03]) and has since seen further developments with different flavours and several applications (a few instances are [AM99] [BCR07] [KR02] [Heu15] [Heu21] [Wei99] [GW99] [Bök+96]). We follow mainly the survey on chapter 1 of [Mil19].

2 Polynomial functors

Let $\mathcal{C}$ be an ∞-category that admits pushouts.

Definition 1. An n**-cube** in $\mathcal{C}$ is a functor $X : \mathcal{P}(I) \to \mathcal{C}$, where $\mathcal{P}(I)$ is the poset of subsets of some finite set I with cardinality n. An n-cube is **cartesian** if the canonical map

$$X(\varnothing) \to \mathrm{holim}_{\varnothing \neq S \subset I} X(S)$$

is an equivalence and **cocartesian** if

$$\mathrm{hocolim}_{S \subsetneq I} X(S) \to X(I)$$

is an equivalence. Moreover, an n-cube is said to be **strongly cocartesian** if every 2-dimensional face is a pushout.

Remark 2.

- A cartesian (cocartesian) 0-cube is a terminal (initial) object; a cartesian (or cocartesian) 1-cube is an equivalence; a cartesian (cocartesian) 2-cube is a pullback (pushout):

$$\begin{array}{ccccc} X(\varnothing) & \xrightarrow{\sim} & \mathrm{holim}_{\varnothing \neq S \subset I} X(S) & \longrightarrow & X(\{0\}) \\ & & \downarrow & & \downarrow \\ & & X(\{1\}) & \longrightarrow & X(I) \end{array}$$

- For $n \geq 2$, a strongly cocartesian n-cube is cocartesian. For $n = 0, 1$, every n-cube is strongly cocartesian.

Definition 3. (Polynomial functor) A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be n**-excisive** if it takes every strongly cocartesian $(n+1)$-cube in $\mathcal{C}$ to a cartesian $(n+1)$-cube in $\mathcal{D}$. We say that F is **polynomial** if it is n-excisive for some integer n.

Denote by $\mathrm{Exc}_n(\mathcal{C}, \mathcal{D}) \subset \mathrm{Map}(\mathcal{C}, \mathcal{D})$ the ∞-category of n-excisive functors.

Example 4. For the degenerate cases $n = -1, 0$, F is (-1)-excisive if and only if $F(X)$ is the terminal object for every $X \in \mathcal{C}$; F is 0-excisive if it is homotopically constant, *i.e.* F takes every morphism in $\mathcal{C}$ to an equivalence in $\mathcal{D}$.

Example 5. F is 1-excisive if and only if it takes pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. When $\mathcal{C} = \mathcal{D} = Top_*$, the prototypical example of an 1-excisive functor is

$$X \mapsto \Omega^\infty(E \wedge X)$$

where E is some spectrum. These classify the functors which are 1-excisive, reduced (i.e. that preserve the null object) and finitary (*i.e.* that preserve filtered colimits).

Example 6. The identity functor $I : \mathcal{C} \to \mathcal{C}$ is **not** in general n-excisive for any n. In fact, we will see that functors can typically be approximated by polynomial functors and the approximations of $I : Top_* \to Top_*$ provide interesting decompositions of spaces, which would not exist if I were to be polynomial.

Lemma 7. If $F : \mathcal{C} \to \mathcal{D}$ is n-excisive, then it is also $(n+1)$-excisive. Hence, there is a sequence of inclusions

$$\mathrm{Exc}_0(\mathcal{C}, \mathcal{D}) \subset \mathrm{Exc}_1(\mathcal{C}, \mathcal{D}) \subset \cdots \subset \mathrm{Exc}_n(\mathcal{C}, \mathcal{D}) \subset \mathrm{Exc}_{n+1}(\mathcal{C}, \mathcal{D}) \subset \cdots$$

3 Polynomial approximation of functors

In ordinary calculus, one approximates a function around a given point $x \in \mathbb{R}^n$. The Goodwillie analogue of that will be to fix an object $X \in \mathcal{C}$ and objects in "a neighborhood of X" will mean objects that come equipped with a morphism to X, that is objects in the slice ∞-category $\mathcal{C}_{/X}$.

Definition 8. We say that functors $\mathcal{C} \to \mathcal{D}$ admit n-excisive approximations at X if the inclusion

$$\mathrm{Exc}_n(\mathcal{C}_{/X}, \mathcal{D}) \hookrightarrow \mathrm{Map}(\mathcal{C}_{/X}, \mathcal{D})$$

has a left adjoint, which we denote by P_n^X.

Note that functors in $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ restrict to $\mathrm{Map}(\mathcal{C}_{/X}, D)$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we will denote by $P_n^X F : \mathcal{C}_{/X} \to D$ the image by P_n^X of the restriction of F to $\mathrm{Map}(\mathcal{C}_{/X}, D)$.

Theorem 9. *(Goodwillie [Goo03], Lurie [Lur]) Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories such that $\mathcal{C}$ has pushouts and $\mathcal{D}$ has sequential colimits and finite limits, which commute. Then functors $\mathcal{C} \to \mathcal{D}$ admit n-excisive approximations at any object $X \in \mathcal{C}$.*

Let us consider from now on ∞-categories $\mathcal{C}$ and $\mathcal{D}$ satisfying the conditions of the previous theorem (Top_* and Sp are examples of such).

Example 10. The 0-excisive approximation to F at X is equivalent to the constant functor with value $F(X)$.

Let us assume from now on, without loss of generality, that X is the terminal object of $\mathcal{C}$. In that way, the n-excisive approximation to $F : \mathcal{C} \to \mathcal{D}$ is again a functor $\mathcal{C} \to \mathcal{D}$ and will be denoted by $P_n F$.

Example 11. The 1-excisive approximation $P_1 F$ to a functor $F : Top_* \to Top_*$, when applied to a finite CW-complex Y, is of the form

$$P_1 F(Y) \simeq \Omega^\infty(\partial_1 F \wedge Y)$$

where $\partial_1 F$ is a spectrum which we'll call the **first derivative** of F (at the one point space $*$).

Example 12. The 1-excisive approximation to the identity functor $I : Top_* \to Top_*$ is the stable homotopy functor

$$P_1 I(Y) \simeq \Omega^\infty \Sigma^\infty Y,$$

or, equivalently, $\partial_1 I \simeq S^0$, the sphere spectrum.

4 Taylor tower

Definition 13. The **Taylor tower** of $F : \mathcal{C} \to \mathcal{D}$ at $X \in \mathcal{C}$ is the sequence of natural transformations:

$$F \to \cdots \to P_{n+1}^X F \to P_n^X F \to \cdots \to P_1^X F \to P_0^X F \simeq F(X)$$

where $P_{n+1}^X F \to P_n^X F$ comes from the universal property of P_{n+1}^X :

$$\mathrm{Nat}_{\mathrm{Exc}_{n+1}(\mathcal{C},\mathcal{D})}(P_{n+1}^X F, P_n^X F) \cong \mathrm{Nat}_{\mathrm{Map}(\mathcal{C},\mathcal{D})}(F, P_n^X F)$$

Like in ordinary calculus, the idea now is to recover the value $F(Y)$ for some $f : Y \to X \in \mathcal{C}_{/X}$ using the Taylor tower of F at X.

Definition 14. The taylor tower of $F : \mathcal{C} \to \mathcal{D}$ at $X \in \mathcal{C}$ converges at $Y \in \mathcal{C}_{/X}$ if the induced map

$$F(Y) \to \mathrm{holim}_n P_n^X F(Y)$$

is an equivalence in $\mathcal{D}$.

Remark 15. The question of convergence is both very difficult and very important, especially for general ∞-categories $\mathcal{C}$ and $\mathcal{D}$. There is, however, a rich theory developed by Goodwillie [Goo92] [Goo03] to study convergence in the setting of topological spaces and spectra. We shall not go into details in this notes, only mention a case where the identity functor $I : Top_* \to Top*$ converges.

Example 16. The Taylor tower of $I : Top_* \to Top_*$ at $*$ converges on simply connected spaces.

Assuming convergence of the taylor tower of a functor $F : \mathcal{C} \to \mathcal{D}$, one then wishes to better understand the structure of the tower, in particular its layers.

Let us take a functor $F : \mathcal{C} \to \mathcal{D}$ and suppose further that $\mathcal{D}$ is a pointed ∞-category.

Definition 17. The **n-th layer** of the Taylor tower of F at X is the functor $D_n^X F : \mathcal{C}_{/X} \to \mathcal{D}$ given by

$$D_n^X F(Y) = \mathrm{hofib}(P_n^X F(Y) \to P_{n-1}^X F(Y)).$$

These functors play the role of homogeneous polynomials in ordinary calculus.

Example 18. For functors $F : Top_* \to Top_*$ and finite CW-complexes X, the n-th layer of F at $*$ is determined by single a spectrum $\partial_n F$, referred to as the **n-th derivative of** F (at $*$), with a symmetric group action:

$$D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$$

Example 19. The n-th derivative of $I : Top_* \to Top_*$ is (non-equivariently) equivalent to a wedge of $(n-1)!$ copies of the $(1-n)$-sphere spectrum [Joh95]. In particular, $\partial_1 I \simeq S^0$ and $\partial_2 I \simeq S^{-1}$, so

$$D_2 I(X) \simeq \Omega^\infty \Sigma^{-1} (\Sigma^\infty X)^{\wedge 2}_{h\Sigma_2}.$$

One can use this to compute the second approximation $P_2 I$ to I as it fits in the fiber sequence $D_2 I \to P_2 I \to P_1 I$.

The derivatives of the identity have nice properties when applied to odd-dimensional spheres.

Theorem 20. *([AM99], Proposition 3.1) Let X be an odd-dimensional sphere. The spectrum*

$$(\partial_n I \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$$

is rationally contractible for $n > 1$.

Corollary 21. (Serre, 1953) If X is an odd-dimensional sphere, the map $X \to \Sigma^\infty \Omega^\infty X$ is a rational homotopy equivalence.

We know then that the homology of $(\partial_n I \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$ is torsion. The next theorem gives us information about how the torsion is distributed among the layers.

Theorem 22. *([AM99], [AD01]) Let X be an odd-dimensional sphere and p a prime. The homology of $(\partial_n I \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$ with mod p coefficients is non-trivial only if n is a power of p.*

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