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An introduction to Grothendieck topoi

0. A bit of history

- Grothendieck (1960's): Weil cohomology in algebraic geometry
  "Topoi are generalized topological spaces"

- Lawvere - Tierney (early 1970's): Elementary axiomatization of the category of sets (without "$\in$")
  "Topos are generalized categories of sets"

1. Motivating example: Sheaves on a space.

Let X be a top. space and $\mathcal{O}(X)$ the poset of its open subsets. The category of presheaves on X is $\mathrm{Psh}(X) = \mathrm{Fun}(\mathcal{O}(X)^{op}, \mathrm{Sets})$.
For each open $U \subset X$, we have a set $F(U)$ called the set of sections over U. For each $V \subset U$, we have a map $F(U) \longrightarrow F(V)$ restriction from U to V

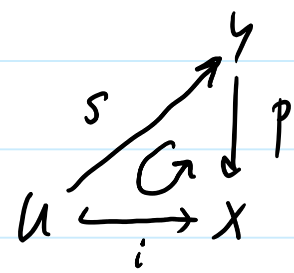
$$s \longmapsto s|_V$$

• Example: $F_c(U) = \{$continuous functions $U \longrightarrow \mathbb{R}\}$
($\underbrace{\text{bounded}}$): F_b

We can ask the following: If $\{U_i\}_{i\in I}$ is a family of open subsets and $s_i \in F(U_i)$, is there $s \in F(\bigcup_{i\in I} U_i)$ such that $s|_{U_i} = s_i$ $\forall i \in I$? For this we need:
Matching condition: $s_i|_{U_i\cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i,j$.

• Definition: A sheaf F is a presheaf such that for every family $\{U_i\}_{i\in I}$ of open subsets and every matching family $\{s_i\}_{i\in I}$ there is a unique section $s \in F(\bigcup_{i\in I} U_i)$ such that $s|_{U_i} = s_i$ $\forall i$.
Denote by $Sh(X)$ the category of sheaves

Example: F_c is a sheaf. F_b is not a sheaf (boundness is not a local property)
Example: Let $Y \xrightarrow{p} X$ be a map in Top (bundle over X)
We can define the sheaf of sections $F: \mathcal{O}(X)^{op} \longrightarrow Sets$ by $F(U) = \{s : U \longrightarrow Y \mid p \circ s = i\}$



We can also describe the category of sheaves in a more formal way as follows:
Let Top/X denote the slice category and consider the functor

$$\begin{aligned} I : \mathcal{O}(X) &\longrightarrow Top/X \\ U &\longmapsto (U \hookrightarrow X). \end{aligned}$$

We have the following diagram:

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, "I"] \arrow[d, "y"'] & Top/X \arrow[dl, dashed, shift left] \\ Psh(X) \arrow[ur, dashed, shift left] & \end{tikzcd}$$

$$\rotatebox{90}{$=$}$$

$$Fun(\mathcal{O}(X)^{op}, Sets)$$

This gives an adjunction $- \otimes I : Psh(X) \rightleftarrows Top/X : Hom(I, -)$
The right adjoint is given by

$$Hom(I, Y \to X)(U) = Top/X(U \hookrightarrow X, Y \to X)$$

i.e. $Hom(I, Y \xrightarrow{p} X)$ is the sheaf of sections of p.

Every adjunction restricts to an equivalence between full subcategories (unit iso, counit iso). In this case

$$Sh(X) \underset{\cong}{\rightleftarrows} \acute{E}t(X)$$

étale bundles $p : Y \to X$ local homeomorphism.

Every sheaf is isomorphic to the sheaf of sections of a local homeomorphism.
($Y \xrightarrow{p} X$ is a local homeo. if $\forall y \in Y$ $\exists$ open $y \in U$ s.t. $p|_U$ is a homeomorphism).

If X is "good" (e.g. T_2) we can recover X from its category of sheaves (lattice of subobjects of the terminal object in $Sh(X)$ is the lattice of open subsets)

2. Grothendieck topos

Is an "abstraction" of $Sh(X)$.

- **Definition**: Let $\mathcal{C}$ be a category with finite limits. A Grothendieck topology J on $\mathcal{C}$ is a family of morphisms $\{f_i : U_i \longrightarrow X\}_{i \in I}$ called coverings s.t.:

(i) If $\{f_i : U_i \to X\}$ is a covering and $g : Y \longrightarrow X$ then $\{U_i \times_X Y \longrightarrow Y\}$ is a covering.

(ii) If $\{f_i : U_i \to X\}$ is a covering and $\{g_j : V_j \to X\}$ is a collection such that $\{V_j \times_X U_i \longrightarrow U_i\}$ is a covering, then $\{g_j : V_j \to X\}$ is a covering.

(iii) If $\{f_i : U_i \to X\}$ is a collection such that some f_i admits a section (i.e. $\exists s : X \to U_i$ s.t. $f_i \circ s = id_X$) then $\{f_i : U_i \to X\}$ is a covering.

The pair $(\mathcal{C}, \mathcal{T})$ is called a site

Example: For any category $\mathcal{C}$, the trivial topology is $\mathcal{T}$ = all morphisms

Example: Let $\mathcal{C} = \mathcal{O}(X)$ and $\mathcal{T} = \{\{U_i \to U\}_{i\in I}\}$ such that $U = \bigcup_{i\in I} U_i$. covering topology

Example: (Alg. Geometry). Let X be a scheme and $\mathcal{C} = \mathrm{Schemes}/_X$ ($U \to X$ étale).

$\mathcal{T} = \{\{U_i \to U\}_{i\in I}\}$ such that $\coprod_i U_i \to U$ is surjective $\mathcal{T}$ = small étale site of X.

If $(\mathcal{C}, \mathcal{T})$ is a site, a presheaf $F : \mathcal{C}^{op} \to \mathrm{Sets}$ is a sheaf if for every covering $\{U_i \to X\}_{i\in I}$ the diagram

$$F(X) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_X U_j)$$

is an equalizer.

• Definition: $Sh(\mathcal{C}, T)$ is the full subcategory of $Psh(\mathcal{C})$ spanned by the sheaves

• Definition: A Grothendieck topos $\mathcal{D}$ is a category equivalent to $Sh(\mathcal{C}, T)$ for some site $(\mathcal{C}, T)$.

Example: $Sh(X)$ is $Sh(\mathcal{O}(X), T)$ where T is the covering topology. $Psh(\mathcal{C})$ is $Sh(\mathcal{C}, T)$ where T = trivial topology.

Remark: There can be different sites with the same category of sheaves

• Theorem: The inclusion $Sh(\mathcal{C}, T) \hookrightarrow Psh(\mathcal{C})$ admits a left adjoint $L : Psh(\mathcal{C}) \to Sh(\mathcal{C}, T)$ called sheafification functor, that preserves finite limits.

• **Theorem**: Let $\mathcal{D}$ be a category. TFAE:

(i) $\mathcal{D}$ is a Grothendieck topos, i.e. $\mathcal{D} \simeq Sh(\mathcal{C}, T)$

(ii) $\mathcal{D}$ is a left exact localization of $Psh(\mathcal{C})$
(i.e. $\exists$ fully faithful embedding $\mathcal{D} \hookrightarrow Psh(\mathcal{C})$ which admits a left adjoint preserving finite limits)

(iii) $\mathcal{D}$ satisfies Giraud's axioms

1. $\mathcal{D}$ is locally presentable.
2. Colimits in $\mathcal{D}$ are universal
3. Coproducts are disjoint
4. Equivalence relations in $\mathcal{D}$ are effective.
(kernel pairs of their coequalizer)

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② $f: S \longrightarrow T$ $\quad f^*: \mathcal{E}/T \longrightarrow \mathcal{E}/S$ $\quad f^*$ preserves colimits

$X \longrightarrow T$

$$\begin{tikzcd} X \times_T S \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "f"] \\ X \arrow[r] & T \end{tikzcd}$$

③

$$\begin{tikzcd} 0 \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y \arrow[r] & Y \sqcup X \end{tikzcd}$$

initial object.

• **Remark**: Condition (ii) without the finite limit condition characterizes locally presentable cats.

Definition: Let $\mathcal{D}$ and $\mathcal{E}$ be topoi. A geometric morphism $f : \mathcal{D} \longrightarrow \mathcal{E}$ is an adjunction

$$f^* : \mathcal{E} \rightleftarrows \mathcal{D} : f_*$$

such that the left adjoint f^* preserves finite limits.

Example: For any space X, the sheafification functor $\mathrm{Psh}(X) \longrightarrow \mathrm{Sh}(X)$ preserves finite limits.

Example: For every functor $f : \mathcal{C} \longrightarrow \mathcal{D}$, the functor $f^* : \mathrm{Psh}(\mathcal{D}) \longrightarrow \mathrm{Psh}(\mathcal{C})$ has both a left adjoint $f_!$ and a right adjoint f_*. Hence (f^*, f_*) is a geometric morphism $\mathrm{Psh}(\mathcal{C}) \longrightarrow \mathrm{Psh}(\mathcal{D})$.

- References...

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