INTRODUCTION TO HIGHER TOPOS THEORY

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Abstract. In these notes, we present a selection of properties and examples of ∞-topoi. In the first section, we review the definition of ∞-topos and some basic theory. The second section studies higher sheaves and Grothendieck topologies. The third one is dedicated to the characterizations of ∞-topoi, using Giraud axioms or descent.

These notes assume familiarity with the model of higher categories based on quasicategories. For an introduction into this subject see the notes from the previous day of this seminar, the course notes of Rezk [4], or the wiki of Lurie [2]. The word ∞-category is used as a replacement of quasicategory, and ∞-groupoid as a replacement of Kan complex. Also, denote by $\mathcal{S}$ the ∞-category of all ∞-grupoids, and by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the ∞-category of functors between $\mathcal{C}$ and $\mathcal{D}$, which is modeled by the quasicategory $\mathrm{Map}(\mathcal{C}, \mathcal{D})$. The contents of this document are based on a lecture notes of Rezk [5], the book of Lurie [1], and the notes [3] of a talk of Rezk in a conference called "Toposes online".

1. Basic properties of infinity topoi

Recall from the first lecture of this seminar the definition of a Grothendieck topos as a localization:

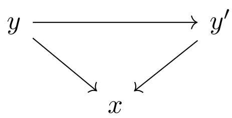
Definition 1.1. A *(Grothendieck) topos* $\mathcal{E}$ is a category together with a left exact localization of the category of presheaves $\mathrm{PSh}(K) := \mathrm{Fun}(K^{\mathrm{op}}, \mathbf{Set})$ on a small category K.

As we saw at the end of the previous lecture, the definition of a higher topos is very similar: it is only necessary to change all the 1-categorical concepts with its ∞-categorical counterparts, replace the category of sets with the ∞-category of ∞-grupoids $\mathcal{S}$, and add the requirement that the localization must be accessible. Observe that in the 1-categorical construction, the localization is also accessible, but this is always true for any localization of presheaves on a small category, and therefore is not an extra condition to a topos.

Definition 1.2. An *∞-topos* $\mathcal{X}$ is a ∞-category together with an accessible left exact localization of the ∞-category of presheaves $\mathrm{PSh}(\mathcal{K}) := \mathrm{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{S})$ on a small ∞-category $\mathcal{K}$.

Given an ∞-category $\mathcal{C}$ and object $x \in \mathcal{C}$, the *slice category* $\mathcal{C}_{/x}$ is an ∞-category with:

- The objects are morphisms $f : y \to x$ for any $y \in \mathcal{C}$.
- The morphisms are commutative triangles:



- The n-morphisms are

$$(\mathcal{C}_{/x})_n = \{\sigma : \Delta^{n+1} \to \mathcal{C} \mid \sigma \text{ sends the final vertex of } \Delta^{n+1} \text{ to } x\}.$$

Some of the most well-known properties and examples of higher topos are:

Proposition 1.3. *For every ∞-topos $\mathcal{X}$:*

(i) *$\mathcal{X}$ is a cartesian closed ∞-category.*
(ii) *For any object $x \in \mathcal{X}$, the slice-category $\mathcal{X}_{/x}$ is itself an ∞-topos.*

Example 1.4. Some examples of ∞-topoi:

(i) Any presheaf $\mathrm{PSh}(\mathcal{K})$ on a small ∞-category $\mathcal{K}$ is trivially an ∞-topos.
(ii) The ∞-category of ∞-grupoids $\mathcal{S}$ is an ∞-topos, because it is equal to the presheaf $\mathrm{PSh}(\mathbf{1}) = \mathrm{Fun}(\mathbf{1}^{\mathrm{op}}, \mathcal{S}) = \mathcal{S}$ where $\mathbf{1}$ is the terminal ∞-category with only one object.

As in the case of ordinary topos, the morphisms between ∞-topos that respect the internal structure are left exact adjunctions:

Definition 1.5. A *geometric morphism* $f : \mathcal{X} \to \mathcal{Y}$ is a functor $f^* : \mathcal{Y} \to \mathcal{X}$ which has a right adjoint $f_* : \mathcal{X} \to \mathcal{Y}$ and f^* is left exact. The ∞-topoi with geometric morphisms form the ∞-category of ∞-topoi, denoted ∞-**Topoi**.

Example 1.6 (Terminal ∞-topos)**.** Because $\mathcal{S} = \mathrm{PSh}(\mathbf{1})$, for any ∞-topos $\mathcal{X}$, a geometric morphism $f : \mathcal{X} \to \mathcal{S}$ is determined by a functor $\varphi : \mathbf{1} \to \mathcal{X}$ such that $\tilde{\varphi} : \mathrm{PSh}(\mathbf{1}) \to \mathcal{X}$ preserves the terminal object and pullbacks of representables. Since the only representable is the terminal object, then there is a unique geometric morphism $\pi : \mathcal{X} \to \mathcal{S}$, and therefore, $\mathcal{S}$ is the terminal ∞-topos.

2. Higher sheaves

One of the main characterizations of the topoi is the equivalence between the Grothendieck topologies and left exact localizations of presheaves categories. In particular, any topos is a category of sheaves over a category with some Grothendieck topology. As a result, there is the following diagram of equivalences:

$$\{\text{Topoi}\} \iff \{\text{Categories of sheaves on } C\} \iff \{\text{Grothendieck topologies on } C\}$$

When taking into account higher topos, this characterization does not hold in general. But a similar construction can be done, defining the higher categorical version of a sheaf.

A *sieve on an object* $x \in \mathcal{C}$ is a full sub-category $\mathcal{D}_x \subset \mathcal{C}_{/x}$ which is closed under precomposition with morphisms in $\mathcal{C}_{/x}$. For S a sieve on $x \in \mathcal{C}$ and $f : y \to x$ a morphism into x, the *pullback sieve* f^*S on y is the one spanned by all those morphisms into y that become equivalent to a morphism in S after postcomposition with f.

Definition 2.1. A *Grothendieck topology* on an ∞-category $\mathcal{C}$ is an assignment to each object $x \in \mathcal{C}$ of a collection of sieves on x, called *covering sieves*, such that:

(i) (Trivial sieve covers) For each $x \in \mathcal{C}$, the trivial sieve $\mathcal{C}_{/x} \subseteq \mathcal{C}_{/x}$ on x is a covering sieve.
(ii) (Pullback of a sieve covers) If S is a covering sieve on x and $f : y \to x$ a morphism, then the pullback sieve f^*S is a covering sieve on y.
(iii) (Sieve covers if its pullbacks cover) For S a covering sieve on x and T any sieve on x, if the pullback sieve f^*T for every $f \in S$ is covering, then T itself is covering.

For the rest of this section, let $\mathcal{K}$ be a small ∞-category. A map $f : x \to y$ of $\mathcal{C}$ is called *monomorphism* if, for each $z \in \mathcal{C}$, the induced map $\mathcal{C}(z, x) \to \mathcal{C}(z, y)$ has all homotopy fibers empty or contractible. Given a localization $i : \mathcal{C} \leftrightarrows \mathrm{PSh}(\mathcal{K}) : L$, let

$$\overline{S} = \{f \in \mathrm{PSh}(\mathcal{K}) \mid L(f) \text{ is an iso}\}$$

be the class of maps inverted by L. Then, L is a *topological localization* if $\overline{S}$ is stable under pullback and the smallest strongly saturated class of morphisms containing a subclass of monomorphisms $S \subset \overline{S}$. For more details on strongly saturated classes see [1]. Then, the definition of higher sheaves follows:

Definition 2.2. An ∞-category of ∞*-sheaves* $\mathrm{Sh}(\mathcal{K}, \mathcal{T})$ is a topological localization of $\mathrm{PSh}(\mathcal{K})$.

It can be proven that any sieve on x is an equivalence class of monomorphisms $U \to j(x)$ in $\mathrm{PSh}(\mathcal{K})$, with $j : \mathcal{K} \to \mathrm{PSh}(\mathcal{K})$ the ∞-Yoneda embedding. Those equivalence classes of monomorphisms are usually called *covering monomorphisms*. Hence, the relation between ∞-sheaves and Grothendieck topologies is not immediate, but can be realized in the following theorem:

Theorem 2.3. $\mathrm{Sh}(\mathcal{K},\mathcal{T})$ *is a topological localization of* $\mathrm{PSh}(\mathcal{K})$ *if, and only if, there exists a Grothendieck topology* $\mathcal{T}$ *on* $\mathcal{K}$ *such that:*

- $\mathrm{Sh}(\mathcal{K},\mathcal{T})$ *is the full subcategory of* $\mathrm{PSh}(\mathcal{K})$ *spanned by objects* $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{S}$ *such that*

$$F(x) = \mathrm{PSh}(\mathcal{K})(j(x), F) \to \mathrm{PSh}(\mathcal{K})(U, F)$$

  is an equivalence for all covering monomorphisms $p : U \to j(x)$ *of* $\mathcal{T}$.
- *The inclusion functor* $\mathrm{Sh}(\mathcal{K},\mathcal{T}) \hookrightarrow \mathrm{PSh}(\mathcal{K})$ *admits a left adjoint* $a : \mathrm{PSh}(\mathcal{K}) \to \mathrm{Sh}(\mathcal{K},\mathcal{T})$ *which is left exact.*

It can be seen that any topological localization is a left exact localization, and therefore any ∞-category of ∞-sheaves is an ∞-topos. But the converse has not been proved yet. By this fact and the previous theorem, the diagram of equivalences in the higher categorical setting is:

$$\{\infty\text{-Topoi}\} \Longleftarrow \{\infty\text{-categories of } \infty\text{-sheaves on } \mathcal{K}\} \iff \{\text{Grothendieck topologies on } \mathcal{K}\}$$

Example 2.4. Let X be a paracompact topological space, and $\mathcal{O}_X$ be the poset of open set on X, with morphisms being the inclusions. In particular, $\mathcal{O}_X$ has a Grothendieck topology where the covering sieves correspond to families $\{U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$.

The *presheaves* on X are presheaves $\mathrm{PSh}(\mathcal{O}_X)$ on $\mathcal{O}_X$. A presheaf on X $F : {\mathcal{O}_X}^{\mathrm{op}} \to \mathcal{S}$ is a *sheaf* on X if, for every open cover $\{U_i\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$, the map

$$F(U) \to \lim_{J\in P_f(I)} F\left(\bigcap_{i\in J} U_j\right)$$

is an equivalence, where $P_f(I)$ is the poset of finite non-empty subsets of I.

3. Characterization of infinity topoi

The definition of higher topos depends on finding a suitable small ∞-category, but sometimes, a more intrinsic characterization is needed:

Theorem 3.1. $\mathcal{X}$ *is an* ∞*-topos if and only if*

(i) *it is presentable,*
(ii) *colimits are universal, and*
(iii) *colimits satisfy descent.*

Recall that for any morphism $f : x \to y$, there is an induced pullback functor $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ which sends a morphism $g : z \to y$ to the pullback g^*:

$$\begin{array}{ccc} z \times_y x & \xrightarrow{g^*} & x \\ \downarrow & \lrcorner & \downarrow f \\ z & \xrightarrow[g]{} & y \end{array}$$

The concept of universal colimit is a direct generalization of the one seen for ordinary categories:

Definition 3.2. Let $\mathcal{C}$ be a cocomplete and finitely complete ∞-category. Then, the colimits of $\mathcal{C}$ are *universal* if for all morphisms $f : x \to y$, the induced pullback functor $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ preserves colimits.

As a particular case, we will review the condition of descent applied only to pushouts. Consider the following diagram

$$\begin{array}{ccccccc} y_1 & \longleftarrow & y_0 & \longrightarrow & y_2 & & y \\ \downarrow & \llcorner & \downarrow & \lrcorner & \downarrow & \xRightarrow{\text{ho. po.}} & \downarrow \\ x_1 & \longleftarrow & x_0 & \longrightarrow & x_2 & & x \end{array}$$

where the two squares are homotopy pushouts, and the homotopy colimits of the diagram builds a morphism $y \to x$. Then, if pushouts satisfy descent, for any $i \in \{0,1,2\}$ the diagrams

$$\begin{array}{ccc} y_i & \longrightarrow & y \\ \downarrow & \lrcorner & \downarrow \\ x_i & \longrightarrow & x \end{array}$$

are homotopy pullbacks.

Example 3.3. Descent applied only to homotopy pushouts holds in **Top** or **sSet**, but not in **Set** applied to ordinary pushouts. Take any space X and a continuous function $f : X \to X$, then given the following diagram

$$\begin{array}{ccccc} X & \xleftarrow{(\mathrm{id},\mathrm{id})} & X \amalg X & \xrightarrow{(\mathrm{id},f)} & X \\ \downarrow & \llcorner & \downarrow & \lrcorner & \downarrow \\ * & \longleftarrow & \{1,2\} & \longrightarrow & * \end{array} \quad \xRightarrow{\text{ho. po.}} \quad \begin{array}{c} C_f \\ \downarrow p \\ S^1 \end{array}$$

where $C_f = X \times [0,1]/_{(x,1)\sim(fx,0)}$. Because descend holds, if f is a homeomorphism, p is a fiber bundle.

Now we want to study the general formulation for descent. Let $\mathrm{Cart}(\mathcal{C}) \subseteq \mathrm{Fun}([1],\mathcal{C})$ be the subcategory containing all objects and whose morphisms $f' \to f$ are pullback squares:

$$\begin{array}{ccc} x' & \longrightarrow & x \\ f' \downarrow & & \downarrow f \\ y' & \longrightarrow & y \end{array}$$

Definition 3.4. The colimits of an ∞-category $\mathcal{C}$ *satisfy descent* if $\mathrm{Cart}(\mathcal{C})$ has all small colimits and the inclusion $\mathrm{Cart}(\mathcal{C}) \to \mathrm{Fun}([1],\mathcal{C})$ preserves small colimits.

Another useful characterization, which shares the firsts two points with the previous one, comes as a generalization of Giraud axioms in a Grothendieck topos:

Theorem 3.5 (Töen-Vezzosi [6], Lurie [1]). *$\mathcal{X}$ is an ∞-topos if and only if*

(i) *it is presentable,*
(ii) *colimits are universal,*
(iii) *coproducts are disjoint, and*
(iv) *all groupoid objects in $\mathcal{X}$ are effective.*

Observe that the only condition that has been rewritten from the original topos theoretic definition is the last one. In the original one, it asked for all equivalence relations to be effective, instead of all groupoid objects. For more details about this last condition see [5]. The definition of the coproducts being disjoint is a direct generalization of the same condition for ordinary categories:

Definition 3.6. For any pair x_1 and x_2 of objects with coproduct $x_1 \amalg x_2$, the following commutative square is a pullback:

$$\begin{array}{ccc} 0 & \dashrightarrow & x_2 \\ \vdots & \lrcorner & \downarrow \\ x_1 & \longrightarrow & x_1 \amalg x_2 \end{array}$$

References

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