INFINITY TOPOI AS LOCALIZATIONS

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Abstract. These notes introduce the notion of infinity topos as an accessible left exact localization of the ∞-category of presheaves over a small ∞-category. To that end, we review the basics of higher categories in the model of quasicategories.

1. Higher categories via quasicategories

In higher category theory, the main objects of study are ∞-categories, which are composed not only of objects and morphisms between objects, but also of n-morphisms between $(n-1)$-morphisms for all $n \geq 1$. Additionally, the composition and identity are weak up to a higher morphism. In particular, we are interested in studying $(\infty, 1)$-categories, from now on denoted simply as ∞-categories, which also have weakly invertible n-morphisms for all $n \geq 2$. If a ∞-categories also has weakly invertible 1-morphisms, it will be called an ∞-groupoid.

Historically, there have been many definitions of ∞-categories [1], and each one is called a *model*. In this section, we present the model of infinity categories as quasicategories in the category of simplicial sets **sSet**, as introduced by Joyal [3] and Lurie [4]. The theory presented follows the notes from Rezk [7] and the wiki of Lurie [5]. A short introduction to the category of simplicial sets is included in Appendix A, and all the constructions of this section have been sum up in a table format at Appendix B.

Let X be a simplicial set, $k \in \mathbb{N}$, and $i : \Lambda^n_k \hookrightarrow \Delta^n$ be the canonical horn inclusion. We say that X has the *k-th horn extension property* if for every $n \in \mathbb{N}$ and every map $f : \Lambda^n_k \to X$, there exists a map $\tilde{f} : \Lambda^n_k \to X$ making the following diagram commute:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \nearrow_{\tilde{f}} & \\ \Delta^n & & \end{array}$$

Definition 1.1. A simplicial set is:

(i) A *quasicategory* (or *weak Kan complex*) if it has the k-th horn extension property for all $0 < k < n$ (only the inner horns). Denote its subcategory as $\mathbf{qCat} \subset \mathbf{sSet}$.
(ii) A *Kan complex* if it has the k-th horn extension property for all $0 \leq k \leq n$ (all the horns). Denote its subcategory as $\mathbf{Kan} \subset \mathbf{sSet}$.

Observe that by definition any Kan complex is a quasicategory. In this model of higher categories, the quasicategories model ∞-categories, and the Kan complexes model ∞-groupoids.

Example 1.2. For any topological space X, the singular complex $\mathrm{Sing}(X)$ is a Kan complex. On the other hand, the nerve $N : \mathbf{Cat} \to \mathbf{sSet}$ sends each category to a quasicategory.

For the rest of this section, let $\mathcal{C}, \mathcal{D}$ be quasicategories, and $x, y, z, t \in \mathcal{C}_0$. Define the collection of *objects* of $\mathcal{C}$ as $\mathrm{Obj}(\mathcal{C}) := \mathcal{C}_0$, and the collection of *morphisms* (or *1-morphisms*) of $\mathcal{C}$ as $\mathrm{Mor}(\mathcal{C}) := \mathcal{C}_1$. In particular, the collection of morphisms between two fixed objects x and y is defined as $\mathrm{Hom}_{\mathcal{C}}(x, y) := \{f \in \mathcal{C}_1 \, \mathrm{id}\, x = d_0(f) \text{ and } y = d_1(f)\}$, with identities $\mathrm{id}_x := s_0(x) \in \mathrm{Hom}_{\mathcal{C}}(x, x)$. Alternatively, $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ can also be denoted by $f : x \to y$. Similarly, the n-simplices of $\mathcal{C}$ will correspond to n-morphisms of $\mathcal{C}$ as viewed as ∞-category.

A *functor* between two quasicategories is a map of simplicial sets between them. Because **sSet** is a cartesian closed category, there is an exponential simplicial set

$$\mathrm{Map}(\mathcal{C}, \mathcal{D}) := \mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C} \times \Delta^{\bullet}, \mathcal{D}).$$

If $\mathcal{D}$ is a quasicategory, then $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ is also a quasicategory, modeling the ∞-category of ∞-functors between $\mathcal{C}$ and $\mathcal{D}$. Then, the collection of functors between $\mathcal{C}$ and $\mathcal{D}$ is $\mathrm{Map}_0(\mathcal{C}, \mathcal{D})$, and the collection of *natural transformations* is $\mathrm{Map}_1(\mathcal{C}, \mathcal{D})$.

The composition in quasicategories is not a function like in ordinary category theory, instead it is a relation between three morphisms, and it will only be associative and unital up to a 2-morphism:

Definition 1.3. Let $f : x \to y$, $g : y \to z$, and $h : x \to z$. Then h is a *witness of the composition* (or *composite*) of f and g if there exist a 2-morphism $\sigma \in \mathcal{C}_2$ such that

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, Rightarrow, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

By the definition of quasicategory, for every composable pair (f, g) there exists some witness of the composition h. The weak associativity and weak identity properties can be studied in terms of an internal concept of "homotopy" in a quasicategory:

Definition 1.4. Let $f : x \to y$ and $g : x \to y$. We say that there is a *homotopy* between f and g (or that f and g are *homotopic*), denoted $f \sim g$, if there exists $\sigma \in \mathcal{C}_2$ such that

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] \arrow[d, Rightarrow, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & {} & y \end{tikzcd}$$

In fact, homotopy is an equivalence relation thanks to $\mathcal{C}$ being a quasicategory. Because composition and homotopy are compatible, it is well-defined to denote the homotopy class of composites as $[g] \circ [f]$. Then, for all $f : x \to y$, $g : y \to z$, $h : z \to t$, the following properties follow directly:

- Weak identity: $[f] \circ [\mathrm{id}_x] = [f] = [\mathrm{id}_y] \circ [f]$.
- Weak associativity: $([f] \circ [g]) \circ [h] = [f] \circ ([g] \circ [h])$.

Hence, to any quasicategory $\mathcal{C}$ there is an associated *homotopy category* $h\mathcal{C}$ with objects $\mathcal{C}_0$ and morphisms between any two objects $\mathrm{Hom}_{h\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)/_{\sim}$.

Definition 1.5. A morphism $f : x \to y$ is an *isomorphism* if its image in the homotopy category $[f]$ is an isomorphism in the usual sense of category theory.

As expected, between two objects of a quasicategory there is more structure than just the set of morphisms. In fact, there is an ∞-groupoid between each pair of objects, that is, a Kan complex, which can be realized by the following construction:

Definition 1.6. The *mapping space* $\mathcal{C}(x, y)$ between two objects x and y is the simplicial set defined by the pullback

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^1, \mathcal{C}) \arrow[d, "\pi"] \\ \{(x,y)\} \arrow[r, hook] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

where π is the composition of the restriction $\mathrm{Map}(\Delta^1, \mathcal{C}) \to \mathrm{Map}(\partial\Delta^1, \mathcal{C})$ with the isomorphism $\mathrm{Map}(\partial\Delta^1, \mathcal{C}) \cong \mathcal{C} \times \mathcal{C}$. It can be shown that $\mathcal{C}(x, y)$ is always a Kan complex.

On the other hand, adjunctions can be generalized to the setting of quasicategories using the characterization of by the unit and the counit:

Definition 1.7. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is a *left adjoint* of G (or that G is a *right adjoint* of F), if there exists natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that the following diagrams commute up to 2-morphism:

$$F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{(\mathrm{id}_F,\eta)} F \circ G \circ F \xrightarrow{(\epsilon,\mathrm{id}_F)} \mathrm{id}_{\mathcal{D}} \circ F = F, \qquad F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{\mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F = F$$

$$G = \mathrm{id}_{\mathcal{C}} \circ G \xrightarrow{(\eta,\mathrm{id}_G)} G \circ F \circ G \xrightarrow{(\mathrm{id}_G,\epsilon)} G \circ \mathrm{id}_{\mathcal{D}} = G, \qquad G = \mathrm{id}_{\mathcal{C}} \circ G \xrightarrow{\mathrm{id}_G} G \circ \mathrm{id}_{\mathcal{D}} = G$$

Definition 1.8. A functor $F : \mathcal{C} \to \mathcal{D}$ is:

(i) *Fully faithful* if, for all objects $x, y \in \mathcal{C}$, F induces a functorial weak equivalence of Kan complexes

$$\alpha_{x,y} : \mathcal{C}(x, y) \xrightarrow{\simeq} \mathcal{D}(F(x), F(y)).$$

(ii) *Essentially surjective* if the induced functor of the homotopy categories $hF : h\mathcal{C} \to h\mathcal{D}$ is essentially surjective.

(iii) A *categorical equivalence* if it is both fully faithful and essentially surjective.

(iv) A *localization* of $\mathcal{C}$ if it has a fully faithful right adjoint functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$.

Definition 1.9. Let K be any simplicial set, $y \in \mathcal{C}_0$ an object, $F : K \to \mathcal{C}$ any functor and $\underline{y} : K \to \mathcal{C}$ a constant functor which sends all K to y. Then:

(i) A natural transformation $\alpha : \underline{y} \Rightarrow F$ *exhibits* y *as a limit of* F if α induces a homotopy equivalence of Kan complexes

$$\mathcal{C}(x, y) \longrightarrow \mathrm{Map}(K, \mathcal{C})(\underline{x}, F).$$

(ii) A natural transformation $\beta : F \Rightarrow \underline{y}$ *exhibits* y *as a colimit of* F if β induces a homotopy equivalence of Kan complexes

$$\mathcal{C}(y, z) \longrightarrow \mathrm{Map}(K, \mathcal{C})(F, \underline{z}).$$

The usual definition for preservation of limits and colimits follow from the previous construction. In addition, a limit or colimit is called finite if the diagram simplicial set K is finite. Hence, a functor $G : \mathcal{C} \to \mathcal{D}$ is *left exact* if it preserves finite limits.

2. Infinity topoi as localizations

The higher categorical version of a topos was developed by Toën and Vezzosi [9] (as *Segal topos*), Rezk [8] (as *model topos*) and Lurie [4] (as ∞*-topos*), all three presenting equivalent ∞-categories. The goal of this section is defining ∞-topoi as a particular class of presentable ∞-categories which also carry extra structure. Recall that any model category has an underlying ∞-category (as seen in [4]). Recently, Pavlov [6] has proven that the ∞-category of combinatorial model categories is equivalent to the ∞-category of presentable ∞-categories. All these relations can be summed up in the following diagram:

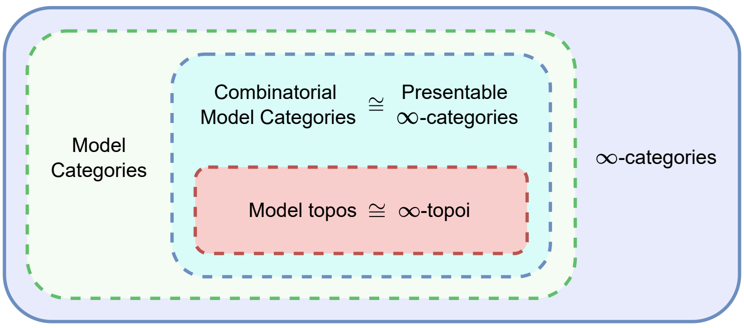

Figure 1. Relation between model categories, ∞-categories and ∞-topoi.

As proven by Bergner [2], there is a model of higher categories in the category of simplicially enriched categories, denoted **sCat**, where the ∞-categories correspond to categories enriched in Kan complexes. Consider the subcategory of all Kan complexes **Kan**, it is in particular enriched in Kan complexes if we consider the mapping spaces Map, hence $\mathbf{Kan} \in \mathbf{sCat}$. In addition, the homotopy coherent nerve functor $N^{hc} : \mathbf{sCat} \to \mathbf{sSet}$ realizes a Quillen equivalence, and sends any category enriched in Kan complexes to a quasicategory. Therefore, there exists a quasicategory $\mathcal{S} := N^{hc}(\mathbf{Kan})$ which corresponds to the quasicategory of all ∞-grupoids. Then, the quasicategory of ∞-presheaves on $\mathcal{C}$ is by definition

$$\mathrm{PSh}_\infty(S) := \mathrm{Map}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

Let κ denote a regular cardinal. To formalize the theory of presentable ∞-categories first we need to introduce several technical concepts:

- A κ-small simplicial set is a simplicial set with less than κ non-degenerate simplices.
- $\mathcal{C}$ admits *κ-filtered colimits* if every diagram $F : K \to \mathcal{C}$ indexed by a κ-small simplicial set K admits a point $y \in \mathcal{C}_0$ and a natural transformation $\beta : F \Rightarrow \underline{y}$. Furthermore, a functor $G : \mathcal{C} \to \mathcal{D}$ *preserves κ-filtered colimits* if sends the κ-filtered colimits of $\mathcal{C}$ to colimits in $\mathcal{D}$.
- $\mathcal{C}$ is *essentially small* if there is a regular cardinal κ such that there exists a κ-small quasicategory $\mathcal{C}'$ and a categorical equivalence $\mathcal{C}' \to \mathcal{C}$.
- $\mathcal{C}$ is *locally small* if for all objects $x, y \in \mathcal{C}_0$ the mapping space $\mathcal{C}(x, y)$ is essentially small.
- If $\mathcal{C}$ is a quasicategory with κ-filtered colimits, then an object $x \in \mathcal{C}$ is called *κ-compact* if the mapping space functor $\mathcal{C}(x, -) : \mathcal{C} \longrightarrow \mathcal{S}$ preserves κ-filtered colimits.
- A quasicategory $\mathcal{C}$ is *accessible* if there is a regular cardinal κ such that:
  - $\mathcal{C}$ is locally small.
  - $\mathcal{C}$ admits κ-small filtered colimits
  - The full subcategory $\mathcal{C}_\kappa \subset \mathcal{C}$ of κ-compact objects is essentially small.
  - $\mathcal{C}_\kappa$ generates $\mathcal{C}$ under small, κ-filtered colimits.

  Furthermore, a functor $F : \mathcal{C} \to \mathcal{D}$ is *accessible* if $\mathcal{C}$ is an accessible quasicategory and there is a regular cardinal κ such that F preserves κ-small filtered colimits.

Definition 2.1. A quasicategory $\mathcal{C}$ is *(locally) presentable* if $\mathcal{C}$ is accessible and has all small colimits.

Theorem 2.2. *A quasicategory $\mathcal{C}$ is presentable if, and only if, there exists an accessible localization of the category of presheaves on a small quasicategory $\mathcal{K}$, i.e., there are two adjoint functors*

$$\mathcal{C} \underset{i}{\overset{L}{\underset{\hookrightarrow}{\overset{\leftarrow}{\bot}}}} \mathrm{PSh}_\infty(\mathcal{K})$$

where i is fully faithful and accessible.

Finally, the following definition generalizes the usual definition of 1-topos replacing **Set** with the quasicategory of ∞-groupoids $\mathcal{S}$:

Definition 2.3. An *∞-topos* $\mathbf{H}$ is a quasicategory together with an accessible left exact localization of the category of presheaves on a small quasicategory $\mathcal{K}$, i.e., there are two adjoint functors

$$\mathbf{H} \underset{i}{\overset{L}{\underset{\hookrightarrow}{\overset{\leftarrow}{\bot}}}} \mathrm{PSh}_\infty(\mathcal{K})$$

where i is fully faithful and accessible, and L is left exact.

Then, comparing the definition of ∞-topos with Theorem 2.2, it is obvious that all ∞-topos are presentable quasicategories, and, in fact, the following characterization holds:

Theorem 2.4. *A quasicategory $\mathbf{H}$ is a ∞-topos if, and only if, it is a presentable quasicategory and the localization from Theorem 2.2 is left exact.*

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Appendix A. Simplicial Sets

Definition A.1. The *simplex category* Δ has as objects the sets $[n] := \{0, 1, \dots, n\}$ for all $n \in \mathbb{N}$, and as morphisms the non-decreasing set functions $[n] \to [m]$ for all $n, m \in \mathbb{N}$.

Definition A.2. A *simplicial set* is a presheaf on Δ, i.e., a functor from Δ^{op} to $\mathbf{Set}$. The simplicial sets together with the natural transformations between them form the category of simplicial sets, denoted by $\mathbf{sSet} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$.

Any simplicial set X has a set for each $[n]$, denoted $X[n]$ or X_n. In the simplex category there are two special types of morphisms: the *injections* $\delta_i^n : [n-1] \to [n]$ and the *surjections* $\sigma_i^n : [n+1] \to [n]$, both defined for every $n \in \mathbb{N}$ and every $0 \le i \le n$ by

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \ge i \end{cases} \qquad \sigma_i^n(j) = \begin{cases} j & \text{if } j \le i \\ j-1 & \text{if } j > i \end{cases}$$

Because any simplicial set X is a functor, these morphisms induce functions, called *faces* $d_i^n := X(\delta_i^n) : X_n \to X_{n-1}$ and *degeneracies* $s_i^n := X(\sigma_i^n) : X_n \to X_{n+1}$. Furthermore, every morphism in the simplex category can be expressed as a composition of surjections and injections.

From the properties of the injections and surjections, we can derive the following *simplicial identities*, that all simplicial set have to satisfy:

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n \qquad \text{if } i < j$$

$$d_i^{n+1} \circ s_j^n = \begin{cases} s_{j-1}^{n-1} \circ d_i^n & \text{if } i < j \\ \mathrm{id}_{X_n} & \text{if } i = j \text{ or } i = j+1 \\ s_j^{n-1} \circ d_{i-1}^n & \text{if } i > j+1 \end{cases}$$

$$s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n \qquad \text{if } i \le j$$

Therefore, a simplicial set is determined by the sets $\{X_n\}_{n \in \mathbb{N}}$ together with the faces and degeneracies maps satisfying the simplicial identities.

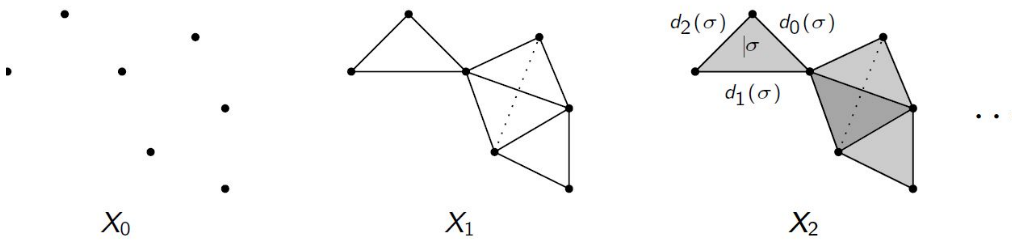


Figure 2. A simplicial set $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$, from nLab wiki.

Example A.3. Some of the most well-known examples are:

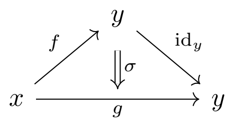
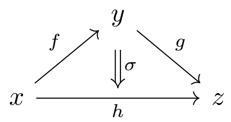
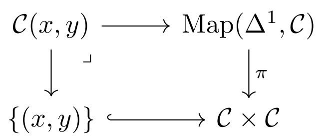
(i) The *standard n-simplex* is the simplicial set defined by $\Delta^n := \mathrm{Hom}_\Delta(-, [n])$. By the Yoneda lemma, for each simplicial set X and each $n \in \mathbb{N}$, we have
$$X_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{Hom}_\Delta(-, [n]), X) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X).$$

(ii) The *boundary* of Δ^n, denoted $\partial\Delta^n$, is defined as the subset of all maps of Δ^n which are not surjective.

(iii) The *k-th horn* Λ_k^n is the sub-simplicial-set of Δ^n obtained from removing the k-th face. The horns with $0 < k < n$ are usually called *inner horns*, and the ones with $k = 0$ or $k = n$ are the *outer horns*.

Appendix B. Summary of the model of quasicategories

Let $\mathcal{C}, \mathcal{D}$ be quasicategories, and $x, y, z, t \in \mathcal{C}_0$.

Higher categorical concept	**Definition in quasicategories**
Objects $\mathrm{Obj}(\mathcal{C})$	$\mathcal{C}_0$
1-morphisms $\mathrm{Mor}(\mathcal{C})$	$\mathcal{C}_1$
n-morphisms	$\mathcal{C}_n$
1-morphisms $x \to y$ (or $\mathrm{Hom}_{\mathcal{C}}(x,y)$) between x and y	$\{f \in \mathcal{C}_1 \mid x = d_0(f) \text{ and } y = d_1(f)\}$
∞-functors $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ between $\mathcal{C}$ and $\mathcal{D}$	$\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C} \times \Delta^\bullet, \mathcal{D})$
$f : x \to y$ and $g : x \to y$ are *homotopic* $f \sim g$	$\exists \sigma \in \mathcal{C}_2$ s.t. 
$h : x \to z$ is a *witness of the composition* of $f : x \to y$ and $g : y \to z$	$\exists \sigma \in \mathcal{C}_2$ s.t. 
Homotopy category $h\mathcal{C}$	Objects $\mathcal{C}_0$ and morphisms $\mathrm{Hom}_{h\mathcal{C}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(x,y)/\sim$
$f : x \to y$ is an *isomorphism*	$[f]$ is an isomorphism in $h\mathcal{C}$
Mapping space $\mathcal{C}(x,y)$ between x and y	
$F : \mathcal{C} \to \mathcal{D}$ is a *left adjoint* of $G : \mathcal{D} \to \mathcal{C}$	$\exists\, \eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ s.t. $[\epsilon \circ \mathrm{id}_F] \circ [\mathrm{id}_F \circ \eta] = [\mathrm{id}_F]$ and $[\mathrm{id}_G \circ \epsilon] \circ [\eta \circ \mathrm{id}_G] = [\mathrm{id}_G]$
$F : \mathcal{C} \to \mathcal{D}$ is *fully faithful*	$\forall\, x, y \in \mathcal{C},\ \alpha_{x,y} : \mathcal{C}(x,y) \xrightarrow{\simeq} \mathcal{D}(F(x), F(y))$
$F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective*	$\forall\, y \in \mathcal{D}_0, \exists\, x \in \mathcal{C}_0$ s.t. $F(x) \cong y$
$F : \mathcal{C} \to \mathcal{D}$ is a *categorical equivalence*	Fully faithful and essentially surjective.
$L : \mathcal{C} \to \mathcal{D}$ is a *localization* of $\mathcal{C}$	L has a fully faithful right adjoint functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$
$\alpha : \underline{y} \Rightarrow F$ *exhibits* y *as a limit of* $F : K \to \mathcal{C}$	α induces a weak equivalence $\mathcal{C}(x,y) \to \mathrm{Map}(K, \mathcal{C})(\underline{x}, F)$
$\beta : F \Rightarrow \underline{y}$ *exhibits* y *as a colimit of* $F : K \to \mathcal{C}$	β induces a weak equivalence $\mathcal{C}(y,z) \to \mathrm{Map}(K, \mathcal{C})(F, \underline{z})$