SEMANTICS OF HOMOTOPY TYPE THEORY

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Abstract. These notes study the relation between ∞-topoi and homotopy type theory. This relation can be seen as an instance of the syntax and semantics duality: homotopy type theory behaves like the internal logic of ∞-topoi, and any ∞-topos is a higher categorical model of homotopy type theory. As a consequence, any result proven inside homotopy type theory yields a theorem in any ∞-topos, and in particular in the ∞-category of homotopy types.

1. Motivation and history

The origins of type theory date back to the work of Russel, Church, Lawvere and others. Intuitionistic type theory (ITT) was introduced in the 70s by Per Martin Löf, based on dependent types, constructive logic and identity types. Originally, ITT was studied for 0-dimensional models, like the category of sets, i.e., regarding each type as a set, dependent types as families of sets, and identity types as equalities of elements. But in 1998, Hofmann and Streicher showed that ITT has a model in the category of groupoids, considering identity types as equivalence relations.

The model of ITT in groupoids inspired the search for a higher-dimensional model of ITT, where each type is regarded as an ∞-groupoid, i.e., a homotopy type. The first higher-dimensional models of ITT were found in 2006 independently by Awodey and Warren [4], and Voevodsky [16]. Around the same time, Voevodsky discovered the univalence axiom (UA), which was shown to be admissible in some higher-dimensional models, but not in the 0-dimensional ones. In 2011, Bauer, Lumsdaine, and Shulman introduced higher inductive types (HIT); and shortly after, homotopy type theory (HoTT) was defined as an intuitionistic type theory with the addition of the univalence axiom and higher inductive types:

$$\mathrm{HoTT} = \mathrm{ITT} + \mathrm{UA} + \mathrm{HIT}\,.$$

The people working in the field of HoTT can be divided by their motivation in two groups with non-empty intersection:

- **Synthetic homotopy theory:** Recently, as a result of works of many authors [13, 11, 10, 14, 7, 8], it was shown that HoTT has models in all ∞-topoi. Hence, any result proven inside HoTT is true in any ∞-topos, and in particular in the infinity category of homotopy types.
- **Computer formalization:** Any result proven inside HoTT has an inherent computer formalization. Furthermore, there exists variants to HoTT (like Cubical Type Theory) which provide better computational properties by replacing the univalence axiom with a similar construction.

The rest of these notes focus on a brief introduction of the first point: how results inside homotopy type theory relate to constructions on any ∞-topos.

2. Review of categorical semantics

As in previous notes of this seminar, we denote ∞-category, ∞-groupoid and ∞-topoi to mean the $(\infty, 1)$ categorical constructions in any suitable model. In addition, an ∞-topos denotes a Grothendieck ∞-topos.

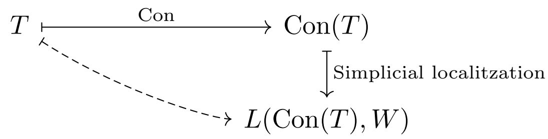
In general, *syntax* is the formal specification of a theory of formal logic, as opposed to *semantics*, which is the interpretation of the syntax of a theory in a model. Some examples of structures with a duality between theories and their models are operads, monads, Lawvere theories and sketches. Each of these may be understood as characterizing a theory, and its models then are the corresponding algebras. Given an abstract theory $\mathbb{T}$, there exists categorical semantics in the form of an adjunction

$$\mathrm{Con} : \mathrm{Theories}(\mathbb{T}) \rightleftarrows \mathrm{Models}(\mathbb{T}) : \mathrm{Lan}$$

where at the left we have the category of presentations of $\mathbb{T}$, and at the right the category of models over $\mathbb{T}$. Here, we understand a *presentation* T of an abstract theory $\mathbb{T}$ as a concrete logical framework (axioms, operations, relations, etc.) which has $\mathrm{Models}(\mathbb{T})$ as the category of models of T. For example, there are several presentations of the theory of groups. The functor Con forms the *category of contexts* (or *syntactic category*) of a theory, and the functor Lan sends any model to its *internal logic*.

The category of contexts is defined as the initial object in the category of models of the given theory. Given a type theory, then its category of contexts is a model which consists of exactly those types and terms which may be constructed (via the relevant natural deduction operations) in the type theory. For example, extensional type theory (an intuitionistic type theory without any higher identities) is precisely the formal language for which locally cartesian closed categories are the semantics.

When considering a type theory with higher-dimensional models, like homotopy type theory, we cannot make a direct interpretation into higher dimensional categories, because the type theory does not encode the higher terms inside types. Instead, we can consider a modification of the 1-categorical semantics in a Quillen model category, and compose it with a simplicial localization to reach higher categories. If T denotes a type theory with higher-dimensional models, then the higher semantics are:



An ∞-category $\mathcal{C}$ is *locally cartesian closed* if, for each object $x \in \mathcal{C}$, the slice $\mathcal{C}/x$ is a cartesian closed ∞-category, i.e., for any $f : x \to y$ in $\mathcal{C}$, the pullback functor $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ has a right adjoint $\prod_f$. In particular, every ∞-topos is locally cartesian closed.

The semantics of homotopy type theory in the most general case continue to be an open problem. When HoTT was discovered, the main figures in the field decided on some set of conjectures on what the semantics should be:

Conjecture 2.1 (Joyal [9], Awodey [3])**.** *Assume that there exists a suitable definition of elementary ∞-topoi, generalizing elementary topoi to a higher-dimensional setting. Then:*

(i) *Intuitionistic type theory is the internal logic to locally cartesian closed ∞-categories.*
(ii) *Homotopy type theory is the internal logic to elementary ∞-topoi.*

In the recent years, a great community effort has achieved several partial results for both conjectures, and they enable the interpretation of results in the majority of cases:

Theorem 2.2 (Cisinski, Shulman [14])**.** *Every presentable and locally cartesian closed ∞-category has a presentation by an intuitionistic type theory.*

Theorem 2.3 (Shulman [13])**.** *Homotopy type theory with universes weakly à la Tarski can be interpreted in any ∞-topos.*

3. Higher semantics of homotopy type theory

This section follows the same direction as the talk "Reasoning in an ∞-topos with homotopy type theory" [5] made by Dan Christensen at the Topos Institute Colloquium on 2021.

In type theory, the basic objects of study are called *types*. Each type is like a collection of elements: the notation $a : A$ means that a is an element of type A. In non-homotopical type theories, types are usually thought as sets, but in HoTT we think of types as objects in an ∞-topos. Then, an element $a : A$ corresponds to a generalized element $a : 1 \to A$ in an ∞-topos. In addition to types, a type theory is also composed of a set of predefined *rules*, governing how we can construct types, or introduce, use and compute elements.

For example, for each two types A and B, there is a *function type* denoted $A \to B$, which can be seen to correspond to the internal exponential object B^A in any ∞-topos. This type is defined by the following three rules:

(i) If $f(x)$ is an expression of type B whenever x is of type A, then there exists a function

$$x \mapsto f(x) : A \to B.$$

(ii) If $f : A \to B$ is a function, and $a : A$ an element, then the application is $f(a) : B$.

(iii) For any function $x \mapsto f(x) : A \to B$ constructed by (i), and any element $a : A$, then the application by (ii) recovers $f(a) : B$ as expected.

Using the function type and a well-defined inductive generation (see [15, Chapter 5] for more details), most basic constructions in type theory follow as expected:

- The *empty type* $\emptyset$ is the free type with no generators, and the initial object of an ∞-topos.
- The *one-point type* $\mathsf{1}$ is the free on one generator $* : \mathsf{1}$, and semantically corresponds to the terminal object.
- The *product type* $A \times B$ is generated by the projections $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$, and it corresponds to the categorical products.
- The *coproduct type* $A + B$ is generated by the inclusions $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$, and it corresponds to the categorical coproducts.
- The type of *natural numbers* $\mathbb{N}$ is generated by an element $0 : \mathbb{N}$ and a function $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$, and it corresponds to a homotopy natural numbers object.

In addition to the previous constructions, HoTT assumes the existence of a *universe type* $\mathcal{U}$ closed under type-forming operations (i.e., a function between types of the universe is in the universe). Then, saying that A is a type is equivalent to $A : \mathcal{U}$. In an ∞-topos, the existence of $\mathcal{U}$ is equivalent to the property of descend (which is also equivalent to the existence of an object classifier). The universe is a handy tool to understanding type dependency:

- A *dependent type* $P : A \to \mathcal{U}$ is a type family depending on A.
- The *dependent product* $\prod_{x:A} P(x)$ is defined with the same rules as the type of functions, but with each dependent product $f : \prod_{x:A} P(x)$ sending a to $f(a) : P(a)$, i.e., the codomain varies for each element in the domain.
- The *dependent sum* $\sum_{x:A} P(x)$ is freely generated by projections like the ordinary product, but in this case it's elements are pairs (a, b) where $a : A$ and $b : P(a)$.

In an ∞-topos, a dependent type $P : A \to \mathcal{U}$ corresponds to a map $P' : E \to A$ over A, where for each element $a : 1 \to A$, the fiber of P' gives an object which corresponds to the type $P(a)$. In addition, the dependent sum type $\prod_{x:A} P(x)$ is E, the domain of P'. The dependent product is obtained by applying the right adjoint to pullback along $A \to 1$, which by definition only exists in locally cartesian ∞-categories.

Example 3.1. The parity of a natural number can be defined as the dependent type $\mathsf{parity} : \mathbb{N} \to \mathcal{U}$ such that

$$\mathsf{parity}(n) = \begin{cases} \emptyset & \text{if } n \text{ is even,} \\ \mathsf{1} & \text{if } n \text{ is odd.} \end{cases}$$

The dependent sum type $\sum_{n:\mathbb{N}} \mathsf{parity}(n)$ can be shown to be isomorphic to the natural numbers type, and therefore corresponds to a natural numbers object. Finally, there is no dependent products $f : \prod_{n:\mathbb{N}} \mathsf{parity}(n)$, because there is no image for even elements.

Any type can be interpreted as a logical proposition: it is considered to be true if it is inhabited, i.e., if it can be proven to have at least one element. This relation between type theory constructors and logical operators can be seen in Table 1. For more information about this relation, see [15, Section 1.11].

First order logic	**Type theory**
True	1
False	$\emptyset$
Negation of A	$A \to \emptyset$
A and B	$A \times B$
A or B	$A + B$
A implies B	$A \to B$
A if and only if B	$(A \to B) \times (B \to A)$
$\forall\ x \in A,\ B(x)$	$\prod_{x:A} B(x)$
$\exists\ x \in A,\ B(x)$	$\sum_{x:A} B(x)$

TABLE 1. Propositions as types

The *identity type* is a type family $\bullet = \bullet : A \times A \to \mathcal{U}$. In proof theoretic terms, an element of $a = b$ can be thought as a proof that a is identified to b, for all $a, b : A$. As a type, it is inductively generated by the reflexivity elements $\mathrm{id}_a : a = a$ for any $a : A$. In any ∞-topos, this family is represented by the diagonal map $A \to A \times A$, and two elements of a type are equal if the corresponding maps $a : 1 \to A$ and $b : 1 \to B$ are homotopic.

We say that a function $f : A \to B$ is an *equivalence* if the following type is inhabited:

$$\mathsf{isEquiv}(f) := \left(\sum_{g:B\to A} g \circ f = \mathrm{id}_A \right) \times \left(\sum_{h:B\to A} f \circ h = \mathrm{id}_B \right).$$

An equivalence between two types corresponds to an isomorphism in an ∞-topos (i.e., to a weak equivalence of the underlying model category). Then, the type of all equivalences between A and B is defined as

$$A \simeq B := \sum_{f:A\to B} \mathsf{isEquiv}(f).$$

For any types A and B, there exists a function $\mathsf{idToEquiv} : A = B \to A \simeq B$. Then, the *univalence axiom* says that $\mathsf{idToEquiv}$ is an equivalence. This axiom is an assertion about the universe $\mathcal{U}$, and it is the only property in the construction so far which is false in **Set**, but true in any ∞-topos.

Finally, inductive types can be generalized to *higher inductive types*, where we not only generate by elements and functions, but also using identities, identities between identities, and so forth. In 2019, Lumsdaine and Shulman [11] showed that higher inductive types can be modelled in any ∞-topos as "∞-algebras over ∞-monads".

Example 3.2. The circle is a higher inductive type generated by an element $\mathsf{base} : \mathbb{S}^1$ and an identity (thought as a path) $\mathsf{loop} : \mathbb{S}^1 = \mathbb{S}^1$. Semantically, it corresponds to the suspension of $1 \sqcup 1$ as a homotopy pushout in any ∞-topos.

Remark 3.3. When proving things inside HoTT, we can only use purely homotopical arguments. For example, we cannot use the law of excluded middle, the axiom of choice or the Whitehead theorem, because they are not true in every ∞-topos. Therefore, sometimes new proofs need to be found to generalize a result from the homotopy theory of spaces to any infinity topos.

There are many examples where research about results inside HoTT has lead to new results and theories for any ∞-topos. Some examples include: the generalized Blakers-Massey theorem [1], modalities [12, 2], and non-accessible localizations [6].

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