Modalities

24 October 2022

1 Modal logic

The term *modality* informally refers to ways in which a language can express various relationships to reality or truth. For example, “can”,“could”, “should”, “must” are modal verbs, while “possibly” or “necessarily” are modal adverbs.

Modal logic is a collection of formal systems developed to represent statements about necessity and possibility. The *modal operators* are denoted as $\Diamond$ and $\Box$. For a proposition p, we write $\Diamond\varphi$ to denote “possibly p” and $\Box\varphi$ to denote “necessarily p”. Here one should distinguish between *logical necessity* and *deontic necessity* (that is, moral or legal obligation).

The syntax rules of the modal operators include the following:

- $\Box p \leftrightarrow \neg\Diamond\neg p$.
- $\Diamond p \leftrightarrow \neg\Box\neg p$.
- K: $\Box(p \to q) \to (\Box p \to \Box q)$.
- T: $\Box p \to p$.
- 4: $\Box p \to \Box\Box p$ (idempotence).
- B: $p \to \Box\Diamond p$ (the naturalistic fallacy in deontic logic).
- D: $\Box p \to \Diamond p$.
- 5: $\Diamond p \to \Box\Diamond p$.

Here $K + T + 4$ and $K + T + 5$ are models of *interior algebra*, that is, Boolean algebra with an interior operator and a closure operator (Tarski–Jónsson, 1951).

2 Modalities in homotopy type theory

In type theory, modalities are unary operations on types, and on propositions they reduce to modal logic. Type theory is equipped with monads and comonads on types. Monads in computer science embody a notion of *computation:* if T is the modality of possibility and a is an object of type A, then Ta models computation of A. Thus, computation of type A represents the possibility of a value of type A.

When the underlying type theory is homotopy type theory, modalities are a generalization of traditional modalities in the sense of higher category theory. Hence they have semantics in ∞-categories given by ∞-monads. Details are given in [4].

Homotopy type theory is an internal language for ∞-topoi. Modalities are introduced in homotopy type theory using a localization higher inductive type, and left-exact modalities correspond semantically to subtopoi. Only idempotent monadic modalities are considered. Each idempotent monadic modality can be represented as an operator $\bigcirc\colon U \to U$ on a type universe U equipped with a family of functions $\prod_{(A:U)} A \to \bigcirc A$ yielding a *reflective subuniverse* $U_\bigcirc \equiv \Sigma_{(X:U)} \mathrm{isModal}(X)$ where X is *modal* if the reflection map $X \to \bigcirc X$ is an equivalence.

One of the most important modalities in homotopy type theory is *n-truncation*. More generally, many modalities are instances of *nullifications* with respect to small families, defined as higher inductive types. Given a family $F\colon \prod_{(a:A)} B(a) \to C(a)$, a type X is *F-local* if the induced map $(C(a) \to X) \to (B(a) \to X)$ is an equivalence for all $a : A$. An F-localization of X is a universal F-local type $L_F X$ admitting a map $X \to L_F X$. A modality is *accessible* if it is associated with an F-localization. An accessible modality is left exact if the reflector preserves finite limits.

In the internal logic of topoi, subtopoi are represented by Lawvere–Tierney operators on the subobject classifier, which generate a subtopos by internal sheafification. In dependent type theory, a subtopos is an operation on a type universe and any Lawvere–Tierney operator on the universe of propositions gives rise to a left-exact modality. While in topos theory every left-exact modality arises from a Lawvere–Tierney operator, in ∞-topos theory this is no longer true. The subtopoi that are determined by their behavior on propositions are called *topological* in [3].

3 Modalities in ∞-topoi

3.1 Factorization systems

Let $\mathcal{E}$ be an ∞-category. A *factorization system* in $\mathcal{E}$ consists of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ closed under isomorphisms and composition, such that for every $u\colon A \to B$ in $\mathcal{L}$ and every $v\colon P \to Q$ in $\mathcal{R}$ and for every square $u \to v$ the space of fillers $B \to P$ is contractible (then one says that $\mathcal{L}$ is *left orthogonal* to $\mathcal{R}$ and that $\mathcal{R}$ is *right orthogonal* to $\mathcal{L}$), and every morphism $f\colon X \to Y$ in $\mathcal{E}$ admits a factorization $X \to E \to Y$ with $X \to E$ in $\mathcal{L}$ and $E \to Y$ in $\mathcal{R}$. For such a factorization one denotes $||f|| = E$.

Examples:

- Let $\mathcal{L}$ be all morphisms in $\mathcal{E}$ and $\mathcal{R}$ be the class of isomorphisms. Then $(\mathcal{L}, \mathcal{R})$ is a factorization system. The same is true with these two classes reversed.

- In the ∞-category $\mathcal{S}$ of spaces, a *monomorphism* is a map that is isomorphic to the inclusion of a union of connected components, and a *surjection* is a map u such that $\pi_0(u)$ is surjective. Then $(\mathrm{Surj}, \mathrm{Mono})$ is a factorization system. This example generalizes to any ∞-topos; see [1]. A morphism $f\colon X \to Y$ is a monomorphism if the square $\mathrm{id}_X \to f$ is a pullback. A morphism is *surjective* if it is left orthogonal to the class of monomorphisms. Then $f\colon X \to Y$ is surjective if and only if the base change functor $f^*\colon \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$ reflects isomorphisms (*base change* means pullback along f).

- If $\mathcal{E}$ is an ∞-topos with finite limits, the *diagonal* $\Delta(X)$ of an object X is the canonical morphism $X \to X \times X = X^{S^0}$, and the diagonal of a morphism $u\colon A \to B$ is the canonical morphism $\Delta(u)$ from A to the pullback of u with itself. Thus $\Delta(X) = \Delta(X \to 1)$ where 1 is the terminal object. We can iterate to $\Delta^n(X)\colon X \to X^{S^{n-1}}$ for $n \geq 0$, where $S^{-1} = \emptyset$.

  For any $-1 \leq n < \infty$, a morphism f in $\mathcal{E}$ is *n-truncated* if the iterated diagonal morphism $\Delta^{n+2}(f)$ is invertible. A morphism f is *n-connected* if the iterated diagonal morphisms $\Delta^{k+1}(X)\colon X \to X^{S^k}$ are surjective for $-1 \leq k \leq n$. Then (n-connected, n-truncated) is a factorization system in $\mathcal{E}$. An object A is n-truncated if the diagonal map $A \to A^{S^{n+1}}$ is invertible. The full subcategory of n-truncated objects is reflective and the reflector τ_n takes an object X to its n-truncation $\tau_n(X)$. Then an object X is n-connected if $\tau_n(X) = 1$. A morphism $f\colon A \to B$ is n-truncated if and only if the object (A, f) of $\mathcal{E}_{/B}$ is n-truncated, and similarly for n-connected.

If $(\mathcal{L}, \mathcal{R})$ is a factorization system in $\mathcal{E}$, then $\mathcal{R}$ is closed under limits and base change (this means that if $v\colon P \to Q$ is in $\mathcal{R}$ and $F \to P$ is a pullback along a map $G \to Q$ then $F \to G$ is in $\mathcal{R}$), and $\mathcal{L}$ is closed under colimits and cobase change. Moreover, $\mathcal{L} \cap \mathcal{S}$ is the class of isomorphisms.

A class of morphisms $\mathcal{M}$ is *saturated* if it contains the isomorphisms and it is closed under composition and colimits. Every class of morphisms is contained in a smallest saturated class. If $(\mathcal{L}, \mathcal{R})$ is a factorization system then $\mathcal{L}$ is saturated.

3.2 Reflective subcategories

An object X in an ∞-category $\mathcal{E}$ is *local* with respect to a morphism $u\colon A \to B$ if the map of spaces $\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$ is invertible in $\mathcal{S}$. For a class of morphisms Σ, we denote by $\mathrm{Loc}(\mathcal{E}, \Sigma)$ the full subcategory of $\mathcal{E}$ whose objects are the Σ-local objects.

A reflector $\phi\colon \mathcal{E} \to \mathcal{F}$ is *accessible* if $\mathcal{F} = \mathrm{Loc}(\mathcal{E}, \Sigma)$ for some **set** Σ of morphisms. An ∞-category is *presentable* if it is an accessible localization of a presheaf category $\mathrm{Psh}(\mathcal{K})$ for a small ∞-category $\mathcal{K}$. (A subcategory $\mathcal{C}$ is *full* if the inclusion functor $\mathcal{C} \to \mathcal{E}$ is fully faithful, i.e., the morphism $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{E}}(X, Y)$ is invertible for all X and Y in $\mathcal{C}$.)

Thus an ∞-category $\mathcal{E}$ is an ∞-topos if it is an accessible left-exact localization of $\mathrm{PSh}(\mathcal{K})$ for some $\mathcal{K}$ small.

Let $(\mathcal{L}, \mathcal{R})$ be a factorization system in an ∞-category $\mathcal{E}$ with a terminal object 1. Let $\mathcal{R}[1]$ denote the full subcategory of $\mathcal{E}$ whose objects are those X for which $p_X\colon X \to 1$ is in $\mathcal{R}$. Let us denote $||X|| = ||p_X||$ and let $\eta_X\colon X \to ||X||$ be the corresponding morphism. Then $||X|| \in \mathcal{R}[1]$, and η_X is a reflection of X onto $\mathcal{R}[1]$. Hence $\mathcal{R}[1]$ is reflective.

In fact a morphism $g\colon X \to X'$ is a reflection onto $\mathcal{R}[1]$ if and only if $X' \in \mathcal{R}[1]$ and $g \in \mathcal{L}$. The functor $|| - ||$ inverts all maps in $\mathcal{L}$.

If Σ is a set of morphisms and $\mathcal{E}$ is presentable, then the localization $\mathcal{E} \to \mathrm{Loc}(\mathcal{E}, \Sigma)$ is equivalent to the localization $\mathcal{E} \to \mathcal{R}[1]$ associated with the factorization system $(\mathcal{L}, \mathcal{R})$ generated by Σ, that is, in which $\mathcal{L}$ is the saturation of Σ.

3.3 Modalities

A factorization system $(\mathcal{L}, \mathcal{R})$ in an ∞-category $\mathcal{E}$ with finite limits is a *modality* if $\mathcal{L}$ is closed under base change. A modality $(\mathcal{L}, \mathcal{R})$ is *left-exact* if, in addition, $\mathcal{L}$ is closed under finite limits.

Examples:

- Our previous examples of factorization systems are modalities.
- For every left-exact reflector $\phi\colon \mathcal{E} \to \mathcal{F}$ the class $\mathcal{L}_\phi$ of morphisms rendered invertible by ϕ is the left class of a modality $(\mathcal{L}_\phi, \mathcal{R}_\phi)$.

There is a bijection between left-exact modalities in $\mathcal{E}$ and left-exact localizations in $\mathcal{E}$, sending every reflector ϕ to $(\mathcal{L}_\phi, \mathcal{R}_\phi)$ where $\mathcal{L}_\phi$ is the class of morphisms rendered invertible by ϕ.

If $u\colon A \to B$ is a morphism in an ∞-topos $\mathcal{E}$, then an object X is *u-modal* if it is local with respect to every base change u' of u. If Σ is a class of maps in $\mathcal{E}$, we say that an object X is *Σ-modal* if it is u-modal for every $u \in \Sigma$.

Let $\mathcal{E}$ be an ∞-topos. A class of morphisms $\mathcal{L}$ is *acyclic* if it contains the isomorphisms and it is closed under composition, colimits, and base change, i.e., if it is saturated and closed under base change. Every class of maps is contained in a smallest acyclic class. Then it follows that a factorization system $(\mathcal{L}, \mathcal{R})$ in an ∞-topos $\mathcal{E}$ is a modality if and only if its left class $\mathcal{L}$ is acyclic.

A *covering topology* on an ∞-topos $\mathcal{E}$ is an acyclic class $\mathcal{C}$ containing the class of surjections. The Grothendieck topology associated to a covering topology is the intersection of $\mathcal{C}$ with the class of monomorphisms. Hence the class of surjections is the smallest covering topology and the class of all maps is the largest.

3.4 Modalities for spaces and spectra

For every space W the factorization system generated by $W \to *$ is a modality. Hence the modalities in $\mathcal{S}$ include the *nullifications*.

Modalities in a stable ∞-category correspond to *t-structures*, while a left-exact modality is a triangulated subcategory. A class of morphisms $\mathcal{M}$ is closed under base change if and only if for every map $f\colon X \to Y$ in $\mathcal{M}$ the map $F \to 0$ is in $\mathcal{M}$, where F denotes the fibre of f.

References:

[1] M. Anel, G. Biedermann, E. Finster, A. Joyal, Left-exact localizations of ∞-topoi I: Higher sheaves, arXiv:2101.02791v4 [math.CT] (2022).

[2] M. Anel, G. Biedermann, E. Finster, A. Joyal, Left-exact localizations of ∞-topoi II: Grothendieck topologies, arXiv:2201.01236v1 [math.CT] (2022).

[3] J. Lurie, *Higher Topos Theory*, Annals of Mathematical Studies vol. 170, Princeton University Press, Princeton, 2009.

[4] E. Rijke, M. Shulman, B. Spitters, Modalities in homotopy type theory, *Logical Methods in Computer Science* **16** (2020), 2:1–2:79.