Bounds for Waldschmidt constants

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May 10th, 2019

Nagata Conjecture

Conjecture (M. Nagata, 1959)

Let $r \geqslant 9$ sufficiently general points in projective plain be given. If a curve passes through all these points with multiplicity at least m, then the degree of the curve is at least

$$\sqrt{r} \cdot m$$

Theorem (M. Nagata, 1959)

This is true for $r = k^2$ points, where k is an integer.

Symbolic power

We work in projective space $\mathbb{P}^N$, let $R = \mathbb{C}[x_0, \dots, x_N]$ be its coordinate ring.

Definition

For a homogeneous ideal $I \subset R$ we define its m-th symbolic power

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} I_P^m \cap R.$$

If I is radical then

$$I^{(m)} = \{f \in R : f \text{ vanishes at } V(I) \text{ to order } m\}.$$

Initial degree

Definition

For a homogeneous ideal $I \subset R$ we define the initial degree $\alpha(I)$ to be the lowest degree of a non-zero form in I.

A natural geometric question is: Let I be an ideal of a given set (of points, or any closed set (or scheme) in $\mathbb{P}^N$). What is $\alpha(I^{(m)})$?

Waldschmidt constant

Definition

For a homogeneous ideal $I \subset R$ we define the Waldschmidt constant

$$\widehat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

It exists and is equal to the infimum over the sequence above.

Conjecture (Restatement of the Nagata Conjecture)

For an ideal I of $r \geqslant 9$ sufficiently general points

$$\widehat{\alpha}(I) \geqslant \sqrt{r}.$$

Results and methods

In general, only sporadic values of Waldschmidt constant were computed. The most sharp results are for general points in projective plane — this is long and very interesting story about Nagata Conjecture, Segre-Harbourne-Gimigliano-Hirschowitz Conjecture and computing Seshadri constants. Beautiful methods and theories were developped to deal with these conjectures.

Results and methods

For general points in $\mathbb{P}^N$ we have (almost trivial) bound

$$\widehat{\alpha} \geqslant \lfloor \sqrt[N]{r} \rfloor,$$

(r will always denote the number of structures , eg. points, lines).

This bound can be strengthened by an algorithm due to Szemberg, Szpond and myself, leading to

$$\widehat{\alpha} \geqslant \sqrt[N]{r} - \varepsilon,$$

where, in most cases, $\varepsilon < 0.3$.

Results and methods

Theorem

Let $r \leqslant k^N$, let I be the ideal of r sufficiently general points in $\mathbb{P}^N$. Then

$$\widehat{\alpha}(I) \geqslant \frac{r}{k^{N-1}}.$$

Results and methods

Theorem (FGHL-BMSz)

All sets Z of arbitrary points in $\mathbb{P}^2$ with $\widehat{\alpha} < 5/2$ are described. All possible values for $\widehat{\alpha} < 5/2$ are

$$1, \quad 2 - \frac{1}{r-1}, \quad 2, \quad \frac{9}{4}, \quad \frac{16}{7}, \quad \frac{7}{3}, \quad \frac{12}{5}, \quad \frac{17}{7}.$$

As a by-product, we know that a set Z (given by ideal I) having five consecutive jumps by two in a sequence

$$\alpha(I^{(m)}), \alpha(I^{(m+1)}), \alpha(I^{(m+2)}), \alpha(I^{(m+3)}), \ldots$$

lies on a conic.

Results and methods

Theorem

For general lines in $\mathbb{P}^3$ we have the bound

$$\widehat{\alpha} \geqslant \lfloor \sqrt{2.5r} \rfloor$$

and an algorithm giving much sharper bounds.

The ε in this case is usually less than 0.5.

We also know some sporadic values for interesting not-quite-general cases of lines, planes etc. in projective spaces.

Upper bounds

The very well-known upper bound for a Waldschmidt constant for r general points in $\mathbb{P}^N$, equal to $\sqrt[N]{r}$, can be found by looking at number of conditions imposed by "fat points".

This result was carried over to the case of higher dimensional linear subspaces. For example, the Waldschmidt constant for an ideal of r general lines in $\mathbb{P}^3$ is bounded by the largest real root of the polynomial

$$t^3 - 3rt + 2r.$$

Asymptotic Hilbert Polynomial

Definition

Let I be a homogeneous ideal, by $\mathrm{HP}_I(t)$ we denote the Hilbert Polynomial of this ideal,

$$\mathrm{HP}_I(t) = \dim(R/I)_t, \quad t \gg 0.$$

Define asymptotic Hilbert Polynomial

$$\widetilde{\mathrm{HP}}_I(t) = \lim_{m\to\infty} \frac{\mathrm{HP}_{I^{(m)}}(mt)}{m^N}.$$

Under some reasonable assumptions it exists. In some interesting cases it can be computed!

Asymptotic Hilbert Polynomial and a bound

Theorem

Let I be radical homegeneous. Assume that in the sequence $\{\mathrm{depth}(I^{(m)})\}$ there exists a constant subsequence of value $N-c$. Then

$$\left(\frac{\partial}{\partial t}\right)^{[c]}\left(\frac{t^N}{N!}-\widetilde{\mathrm{HP}}_I(t)\right)(\widehat{\alpha}(I))\leqslant 0.$$