Homogeneous numerical semigroups, shiftings, and monomial curves of homogeneous type

Santiago Zarzuela

Universitat de Barcelona

Seminari de Geometria Algebraica de Barcelona

29 de juny de 2018

Based on joint work with

Reheleh Jafari

IPM and MIM (Kharazmi University), Tehran

- Motivation: a conjecture of Herzog-Srinivasan.
- Homogeneous semigroups and semigroups of homogeneous type.
- Small embedding dimensions and gluing.
- Asymptotic behavior under shifting.

Motivation: a conjecture of Herzog-Srinivasan

- Let $\underline{\mathbf{n}} := 0 < n_1 < \cdots < n_d$ be a family of positive integers.
- Let $S = \langle n_1, \ldots, n_d \rangle \subseteq \mathbb{N}$ be the semigroup the generated by the family $\underline{\mathbf{n}}$.
- Let K be a field and $K[S] = K[t^{n_1}, \ldots, t^{n_d}] \subseteq K[t]$ be the semigroup ring defined by $\underline{\mathbf{n}}$.

Consider the presentation:

$$0 \longrightarrow I(S) \longrightarrow K[x_1, \ldots, x_d] \xrightarrow{\varphi} K[S] \longrightarrow 0$$

given by $\varphi(x_i) = t^{n_i}$.

• Set $R := K[x_1, \ldots, x_d]$.

For any $i \geq 0$ consider the i-th (total) Betti number of $I(S)$:

$$\beta_i(I(S)) = \dim_K \mathrm{Tor}_i^R(I(S), K)$$

- We call the Betti numbers of $I(S)$ as the Betti numbers of S.

- For any $j \geq 0$ we consider the shifted family

$$\underline{\mathbf{n}} + j := 0 < n_1 + j < \cdots < n_d + j$$

and the semigroup

$$S + j := \langle n_1 + j, \ldots, n_d + j \rangle$$

that we call the j-th shifting of S.

Conjecture (by J. Herzog and H. Srinivasan):

The Betti numbers of $S + j$ are eventually periodic on j with period $n_d - n_1$.

Remarks:

- If we start with S a numerical semigroup, that is

$$\mathrm{g.c.d}(n_1, \ldots, n_d) = 1$$

it may happen that $S + j$ is not anymore a numerical semigroup.

For instance, let $S = \langle 3, 5 \rangle$: then $S + 1 = \langle 4, 6 \rangle$.

- Also, we may start with a family which is a minimal system of generators of S but the shifted family is not anymore a minimal system of generators of $S + j$.

For instance, $S = \langle 3, 5, 7 \rangle$: then $S + 1 = \langle 4, 6, 8 \rangle = \langle 4, 6 \rangle$.

- But if S is minimally generated by $n_1, \ldots, n_d$ then $S + j$ is minimally generated by $n_1 + j, \ldots, n_d + j$ for any $j > n_d - 2n_1$.

The conjecture has been proven to be true for:

- $d = 3$ (A. V. Jayanthan and H. Srinivasan, 2013).
- $d = 4$ (particular cases) (A. Marzullo, 2013).
- Arithmetic sequences (P. Gimenez, I. Senegupta, and H. Srinivasan, 2013).

- In general (Thran Vu, 2014).

Namely, there exists a positive value N such that for any $j > N$ the Betti numbers of $S + j$ are periodic with period $n_d - n_1$.

Remark:

The bound N depends on the Castelnuovo-Mumford regularity of $J(S)$, the ideal generated by the homogeneous elements in $I(S)$.

The proof of Vu is based on a careful study of the simplicial complex defined for the case of numerical semigroups by A. Campillo and C. Marijuan, 1991 (later extended by J. Herzog and W. Bruns, 1997) whose homology provides the Betti numbers of the defining ideal of a monomial curve.

The other main ingredient of the proof by Vu is the following technical result:

Theorem

There exists an integer N such that for all $j > N$, any minimal binomial inhomogeneous generator of $I(S)$ is of the form

$$x_1^{\alpha} u - v x_d^{\beta}$$

where $\alpha, \beta > 0$, and where u and v are monomials in the variables $x_2, \ldots, x_{d-1}$ with

$$\deg x_1^{\alpha} u > \deg v x_d^{\beta}$$

• Assume that S is a numerical semigroup.

• Let $I^*(S)$ be the initial ideal of $I(S)$, that is, the ideal generated by the initial forms of the elements of $I(S)$.

- $I^*(S) \subset K[x_1, \ldots, x_d]$ is an homogeneous ideal. It is the definition ideal of the tangent cone of S:

$$G(S) = \bigoplus_{k \geq 0} \frac{M^k}{M^{k+1}}$$

where M is the maximal ideal $(t^{n_1}, \ldots, t^{n_d}) \subset K[S]$.

Turning around the above result by Vu, J. Herzog and D. I. Stamate, 2014, have shown that for any $j > N$,

$$\beta_i(I(S+j)) = \beta_i(I^*(S+j)) \text{ for all } i \geq 0$$

In particular, for any $j > N$, $G(S)$ is Cohen-Macaulay.

The condition

$$\beta_i(I(S+j)) = \beta_i(I^*(S+j)) \text{ for all } i \geq 0$$

corresponds to the definition of varieties of homogeneous type.

So what Herzog-Stamate have shown is that for a given monomial curve defined by a numerical semigroup S, all the monomial curves defined by $S+j$ are of homogeneous type for $j \gg 0$.

Our purpose is to understand this fact from the point of view of the Apéry sets.

Also, to provide a bound which only depends on the initial data of the family $\underline{\mathbf{n}}$.

- For that, we will give a condition on the Apéry set of S with respect to its multiplicity, that jointly with the Cohen-Macaulay property of $G(S)$ will be nearby equivalent to the condition by Vu.

- Then, we will show that these conditions eventually hold for $S+j$, with a bound L that we can easily compute in terms $n_1, \ldots, n_d$.

Moreover, this bound will only depend on what may be called the class of the shifted semigroups.

- And finally, we will obtain the results by Herzog-Stamate on the Betti numbers of the tangent cone as a consequence of the previous considerations.

Homogeneous semigroups
and
semigroups of homogeneous type

• Let $\mathbf{a} = (a_1, \ldots, a_d)$ a vector of non-negative integers. Then we define the total order of **a** as $|\mathbf{a}| = \sum_{i=1}^{d} a_i$.

We also set $s(\mathbf{a}) = \sum_{i=1}^{d} a_i n_i \in S$.

• Given an expression of an element $s \in S$: $s = \sum_{i=1}^{d} a_i n_i$ we call the vector $\mathbf{a} = (a_1, \ldots, a_d)$ a factorization of s.

Then, we define the order of s as the maximum total order among the factorizations of s.

• An expression of s is then said to be maximal if the total order of its factorization is the order of s.

A factorization of an element whose total order is maximal is called a maximal factorization.

• A subset $T \subset S$ is said to be homogeneous if all the expressions of elements in T are maximal.

• Remember that given $S = \langle n_1, \ldots, n_d \rangle \subseteq \mathbb{N}$ and $s \in S$, the Apéry set of S with respect to s is defined as

$$AP(S) = \{x \in S \mid x - s \notin S\}$$

It is always a finite set. If S is a numerical semigroup, it has exactly s elements.

Definition

We then say that S is homogeneous if the Apéry set $\mathrm{AP}(S, e)$ is homogeneous, where $e = n_1$ is the multiplicity of S.

- If $d = 2$ then S is homogeneous.

- If $e = d$ (maximal embedding dimension) or $e = d - 1$ (almost maximal embedding dimension) then S is homogeneous.

- Let $b > a > 3$ be coprime integers. Then, the semigroup

$$H_{a,b} = \langle a, b, ab - a - b \rangle$$

is a Frobenius semigroup (it is obtained from $\langle a, b \rangle$ by adding its Frobenius number). Then, $H_{a,b}$ is homogeneous.

(On can see that in this case, the tangent cone $G(H_{a,b})$ is never Cohen-Macaulay.)

• We call a generalized arithmetic sequence a family of integers of the form

$$n_0, n_i = hn_0 + it$$

where t and h are positive integers and $i = 1, ..., d$.

If S is generated by a generalized arithmetic sequence then S is homogeneous.

- For $\mathbf{a} = (a_1, \ldots, a_d)$ we denote by $x^{\mathbf{a}}$ the monomial $x_1^{a_1} \cdots x_d^{a_d}$.

- And remember that the defining ideal $I(S)$ may be generated by binomials of the form $x^{\mathbf{a}} - x^{\mathbf{b}}$.

For such binomials we have that $s(\mathbf{a}) = s(\mathbf{b})$ and so both $\mathbf{a}$ and $\mathbf{b}$ provide factorizations of the same element $s \in S$.

- $I(S)$ is called generic if it is generated by binomials with full support.

In this case we have that $\mathrm{AP}(S, n_i)$ is homogeneous for any i.

- A family of elements of $I(S)$ such that their initial forms generate $I^*(S)$ is called a standard basis.

Any standard basis is system of generators of $I(S)$ (but not conversely).

And finding minimal systems of generators of $I(S)$ which are also a standard basis is not easy.

Theorem (1)

The following are equivalent:

1. *S is homogeneous and $G(S)$ is Cohen-Macaulay.*
2. *There exists a minimal set of binomial generators E for $I(S)$ such that for all $x^{\boldsymbol{a}} - x^{\boldsymbol{b}} \in E$ with $|\boldsymbol{a}| > |\boldsymbol{b}|$, we have $a_1 \neq 0$.*
3. *There exists a minimal set of binomial generators E for $I(S)$ which is a standard basis and for all $x^{\boldsymbol{a}} - x^{\boldsymbol{b}} \in E$ with $|\boldsymbol{a}| > |\boldsymbol{b}|$, we have $a_1 \neq 0$.*

Not any minimal generating set of $I(S)$ satisfies the properties of the previous result

The proof partly consists in constructing a set of generators satisfying these properties from any minimal set of generators, and then removing superfluous generatros.

Example (2)

Let $S =: \langle 8, 10, 12, 25 \rangle$. We have that

$$\mathrm{AP}(S, 8) = \{25, 10, 35, 12, 37, 22, 47\}$$

It can be seen that it is an homogeneous set and that $G(S)$ is Cohen-Macaulay.

Example (2 cont.)

The set

$$G_1 = \{x_1^3 - x_3^2, x_2^5 - x_4^2, x_1x_3 - x_2^2\}$$

is a minimal generating set of $I(S)$.

We can change $x_2^5 - x_4^2$ by the two binomials $x_1x_2^3x_3 - x_2^5$ and $x_1x_2^3x_3 - x_4^2$. Then, the set

$$G_2 = \{x_1^3 - x_3^2, x_1x_2^3x_3 - x_2^5, x_1x_2^3x_3 - x_4^2, x_1x_3 - x_2^2\}$$

is a generating set that satisfies the properties of the previous proposition. Removing the superfluous generator $x_1x_2^3x_3 - x_2^5$ we get the minimal generating set

$$G_3 = \{x_1^3 - x_3^2, x_1x_2^3x_3 - x_4^2, x_1x_3 - x_2^2\}$$

Remember that:

Definition

We say that S is of homogeneous type if $\beta_i(S) = \beta_i(G(S))$ for all $i \geq 0$.

Inspired by the proof of the main result by Herzog-Stamate we have that:

Proposition (3)

Let S be a homogeneous semigroup such that $G(S)$ is Cohen-Macaulay. Then S is of homogenous type.

- Assume that $G(S)$ is a complete intersection.

Then S is also a complete intersection and both S and $G(S)$ have the same number of minimal generators. So we have that S is of homogeneous type.

The following case is of particular interest:

Corollary (4)

Let S be a numerical semigroup generated by a generalized arithmetic sequence. Then S is of homogeneous type.

(The Cohen-Macaulay property of the tangent cone was proven in this case by L. Sharifan and R. Zaare-Nahandi, 2009.)

Numerical semigroups of homogeneous type are not always homogeneous:

Example (5)

Let $S := \langle 15, 21, 28\rangle$. Then S is of homogeneous type. The defining ideal is generated by a standard basis:

$$I(S) = (x_2^4 - x_3^3, x_1^7 - x_2^5)$$

but it is not homogeneous:

$$3 \times 28 = 4 \times 21 = 84 \in \mathrm{AP}(S, 15)$$

In this case we also have that $G(S)$ is a complete intersection.

Small embedding dimensions and gluing

Now, we study some particular cases. We start with embedding dimension $d = 3$ and the following remarks:

- If S is not symmetric, S is always homogeneous (and so S is of homogeneous type if and only if $G(S)$ is Cohen-Macaulay).

This the case for $S = \langle 3, 5, 7 \rangle$.

- If S is symmetric, S is not necessarily homogeneous neither of homogeneous type.

This is the case for $S = \langle 7, 8, 20 \rangle$.

Proposition (6)

Assume $d = 3$. Then the following are equivalent:

1. *S is of homogeneous type.*
2. $\beta_1(S) = \beta_1(G(S))$.
3. *$G(S)$ is Cohen-Macaulay, and S is homogeneous or $I(S)_*$ is generated by pure powers of x_2 and x_3.*
4. *Either $G(S)$ is a complete intersection or S is non symmetric homogeneous with Cohen-Macaulay tangent cone.*

For embedding dimension $d = 4$ we start with the following observation:

- S is not necessarily homogeneous neither of homogeneous type.

This is the case for $S = \langle 16, 18, 21, 27 \rangle$ (example taken from D'Anna-Micale-Smartano, 2013).

S is a complete intersection and $G(S)$ is Gorenstein but not a complete intersection.

In fact, we are able to find examples of both, symmetric and pseudo-symmetric numerical semigroups of embedding dimension 4 and arbitrary multiplicity m which are

- not of homogeneous type,

- neither homogeneous.

(Taken from he book of P. A. García Sánchez and J. C. Rosales, 2009).

We also studied several other examples with embedding dimension 4 of homogeneous type with non-complete intersection tangent cone.

- In all cases we had that they are homogeneous.

So we could ask if for $d > 3$ are there numerical semigroup of homogeneous type, but not homogeneous and with non-complete intersection tangent cone.

The answer is positive. In fact, F. Strazzanti provided numerous examples, as the following one:

Example (7)

- $S = \langle 7, 8, 11, 12 \rangle$.

- $\mathrm{AP}(S, 7) = \{0, 8, 11, 12, 16, 20, 24\}$ and $24 = 3 \times 8 = 2 \times 12$.

So S is not homogeneous.

- By looking at the Apéry set one can also check that $G(S)$ is not Gorenstein, so $G(S)$ is not a complete intersection.

- One can compute the defining ideal of S (with GAP) and then both a standard basis and the free resolutions.

Both have the same Betti numbers: 1, 6, 8, 3. So S is of homogeneous type.

Now we study what happens under gluing in some cases.

Remember that given two numerical semigroups:

$$S_1 = \langle m_1, \ldots, m_d \rangle, \ S_2 = \langle n_1, \ldots, n_k \rangle$$

and p, q two co-prime positive integers such that

$$p \notin \{m_1, \ldots, m_d\}, \ q \notin \{n_1, \cdots, n_k\}$$

the numerical semigroup

$$S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, pn_k \rangle$$

is called a gluing of S_1 and S_2. If $S_2 = \mathbb{N}$ we then say that S is an extension of S_1.

First of all we observe that to be homogeneous is not preserved by gluing, even for extensions:

Example (4, revisited)

Let $S := \langle 15, 21, 28 \rangle$. Then S is not homogeneous.

But S is an extension of $S_1 = \langle 5, 7 \rangle$ with $q = 3$ and $p = 28$.

We have the following characterization of homogeneity:

Proposition (8)

Let S be a gluing of S_1 and S_2, $s \in S$ and let

$$n = \min\{n \in \mathbb{N}; np \notin AP(S_1, s)\}$$

Then the following are equivalent:

(1) $AP(S, qs)$ *is homogeneous.*

(2) $AP(S_1, s)$ *and* $AP(S_2, nq)$ *are homogeneous, and if* $n > 1$*, then* $ord_{S_1}(p) = ord_{S_2}(q)$.

Corollary (9)

Let S_1 be homogeneous and $S_2 = \mathbb{N}$. Then S is homogeneous if and only if one of the following conditions hold:

(1) $q = ord_{S_1}(p)$.

(2) $p \notin AP(S_1, m_1)$.

Corollary (10)

Let S_1 be homogeneous with Cohen-Macaulay tangent cone and $S_2 = \mathbb{N}$. For each positive integer q, if $p \in S_1 \setminus AP(S_1, m_1)$ with $ord_{S_1}(p) \geq q$, then S is of homogeneous type.

By gluing we may also construct infinite families which are not homogeneous with complete intersection tangent cone:

Proposition (11)

Let $d \geq 3$ be an integer. Then there exist infinitely many numerical semigroups with complete intersection tangent cones of embedding dimension d, which are not homogeneous.

Asymptotic behavior under shifting

- Let $m_i := n_d - n_i$, for all $1 \leq i \leq d$.
- Let $g := gcd(m_1, \dots, m_{d-1})$ and $T := \langle \frac{m_1}{g}, \dots, \frac{m_{d-1}}{g} \rangle$.

- Let

$$L := m_1 m_2 \left(\frac{gc + m_1}{m_{d-1}} + d \right) - n_d$$

where c is the conductor of T.

Theorem (12)

Let $j > L$ and $s \in S + j$. If $\mathbf{a}$, $\mathbf{a'}$ are two factorizations of s with $|\mathbf{a}| > |\mathbf{a'}|$, then there exists a factorization $\mathbf{b}$ of s such that $|\mathbf{a}| = |\mathbf{b}|$ and $b_1 \neq 0$.

Corollary (13)

For any $j > L$, the j-th shifted numerical semigroup $S + j$ is homogeneous and $G(S + j)$ is Cohen-Macaulay. In particular, $S + j$ is of homogeneous type.

Proof:

Take E any system of binomials generators of $I(S + j)$. By the previous Theorem 12, for any binomial $x^{\mathbf{a}} - x^{\mathbf{a'}} \in E$ such that $|\mathbf{a}| > |\mathbf{a'}|$, there exists a binomial $x^{\mathbf{a}} - x^{\mathbf{b}}$ such that $|\mathbf{a}| = |\mathbf{b}| > |\mathbf{a'}|$ and $b_1 \neq 0$. Then, substituting $x^{\mathbf{a}} - x^{\mathbf{a'}}$ by $x^{\mathbf{a}} - x^{\mathbf{b}}$ and $x^{\mathbf{b}} - x^{\mathbf{a'}}$ and then refining to a minimal system of generators, we get that $S + j$ fulfills condition (2) in Theorem 1 and so we are done.

Remark:

The bound L is not optimal.

For instance, for a given numerical semigroup:

$$S_k = \langle k, k+a, k+b \rangle$$

D. Stamate, 2015, has found the bound

$$k_{a,b} = \max\{b(\frac{b-a}{g} - 1), b\frac{a}{g}\}$$

such that S_k is of homogeneous type if $k > k_{ab}$. Compared with ours, this is a better bound.

Now, we may consider the differences $s_i = n_d - n_{d-i}$ for all $1 \leq \cdots \leq i \leq \cdots \leq d-1$.

Then, the sequence of integers $\mathbf{n}$ only depends on these differences and n_1.

We call these differences the shifting type of $\mathbf{n}$.

Sequences with the same shifting type are shifted one from the other.

Hence we can find among all the sequences with the same shifting type the one with the smallest n_1 such that the corresponding semigroup is numerical.

Then, all numerical semigroups with such a shifting type are shifted from this numerical semigroup. Fixing the bound L for it and setting $L' = L + n_1$ we get that L' only depends on the shifting type.

Hence, for any numerical semigroup S with this given shifting type and multiplicity $e > L'$, S is homogeneous and $G(S)$ is Cohen-Macaulay.

On the other hand, the width of a numerical semigroup S is defined as the difference $\mathrm{wd}(S) = n_d - n_1$.

It is clear that for a given width, there exist only a finite number of possible shifting types for a numerical semigroup having this width. So we may conclude that:

Proposition (13)

Let $w \geq 2$. Then, there exists a positive integer W such that all numerical semigroups S, with $wd(S) \leq w$ and multiplicity $e \geq W$, are homogeneous and $G(S)$ is Cohen-Macaulay.

SOME REFERENCES

R. Jafari, S. Zarzuela, *Homogeneous numerical semigroups*, Semigroup Forum, published online: 17 April 2018.

A. V. Jayanthan and Hema Srinivasan, *Periodic occurrence of complete intersection monomial curves*, Proc. Amer. Math. Soc. **141** (2013), 4199-4208.

P. Gimenez, I. Sengupta, and Hema Srinivasan, *Minimal graded free resolutions for monomial curves defined by arithmetical sequences*, J. Algebra **388** (2013), 249–310.

Thanh Vu, *Periodicity of Betti numbers of monomial curves*, J. Algebra **418** (2014), 66–90.

J. Herzog and D. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, J. Algebra **418** (2014), 8–28.

D. Stamate, *Asymptotic properties in the shifted family of a numerical semigroup with few generators*, Semigroup Forum **93** (2016), 225–246.

Muchas gracias por vuestra atención

¡Y BUEN VERANO PARA TODOS!