Waldschmidt constants

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Schneider-Lang Theorem in one variable

Theorem

Let $f_1, \dots, f_k$ be meromorphic functions in $\mathbb{C}$ with f_1, f_2 algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $j = 1, \dots, k$

$$f_j' \in \mathbb{K}[f_1, \dots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C} \ : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \dots, k\}$$

is finite.

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Corollary (Hermite-Lindemann)

For $\omega \in \mathbb{C}^*$ at least one of the numbers ω, $\exp(\omega)$ is transcendental.

Schwarz Lemma in one variable

Theorem

Let f be an analytic function in a disc $\{|z| \leq R\} \subset \mathbb{C}$ with at least N zeroes in a disc $\{|z| \leq r\}$ with $r < R$. Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^N |f|_R,$$

where

$$|f|_\gamma = \sup_{|z|\leq\gamma} |f(z)|.$$

Schneider-Lang Theorem in several variables

Theorem (Bombieri 1970)

Let $f_1, \dots, f_k$ be meromorphic functions in $\mathbb{C}^n$ with $f_1, \dots, f_{n+1}$ algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $i = 1, \dots, n$, $j = 1, \dots, k$

$$\frac{\partial}{\partial z_i} f_j \in \mathbb{K}[f_1, \dots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C}^n \ : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \dots, k\}$$

is contained in an algebraic hypersurface.

Hörmander version of Schwarz lemma in several variables

Theorem

Let $S \subset \mathbb{C}^n$ be a finite set. Let m be a positive integer. There exists $M(m) > 0$ such that there exists $r > 0$ such that for $R > r$ and a function f analytic in the ball $\{|z| \leq R\} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of S

$$|f|_r \leq \left(\frac{c(n) \cdot r}{R}\right)^{M(m)} |f|_R,$$

where $c(n)$ is a constant depending only on n.

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Make the statement effective. In particular: what is the maximal value of $M(m)$?

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$$|f|_r \leq \left(\frac{\exp(n) \cdot r}{R}\right)^{\alpha(mS)} |f|_R,$$

where $\alpha(mS)$ is the initial degree of $I_S^{(m)}$.

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Remark

The constant $\alpha(mS)$ is optimal.

Symbolic powers

Definition

Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_0, \ldots, x_N]$ be the ring of polynomials. For a homogeneous ideal $0 \neq I \subsetneq R$ its *m-th symbolic power* is

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} \left(I^m R_P \cap R \right).$$

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Theorem (Zariski-Nagata)

Let $X \subset \mathbb{P}^N(\mathbb{K})$ be a projective variety (in particular reduced). Then $I(X)^{(m)}$ is generated by all forms which vanish along X to order at least m.

Symbolic powers of ideals of points

Let $Z = \{P_1, \ldots, P_s\}$ *be a finite set of points in* $\mathbb{P}^N(\mathbb{K})$. *Then*

$$I(Z) = I(P_1) \cap \ldots \cap I(P_s)$$

and

$$I(Z)^{(m)} = I(P_1)^m \cap \ldots \cap I(P_s)^m$$

for all $m \geq 1$.

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Remark

The Waldschmidt constant can be computed by subsequences.

Completion of a list by Chudnovsky

Theorem (Farnik, Gwoździewicz, Hejmej, Lampa-Baczyńska, Malara, Szpond 2017)

There exists a complete classification of all configurations Z of points in $\mathbb{P}^2$ with

$$\widehat{\alpha}(Z) \leq \frac{5}{2}.$$

In particular all Waldschmidt constants for up to 8 points are known.

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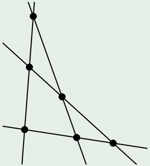
Example



Expected values of Waldschmidt constants for points

Nagata-type Conjecture

Let I be a saturated ideal of $s \gg 0$ very general points in $\mathbb{P}^N(\mathbb{C})$. Then

$$\widehat{\alpha}(I) = \sqrt[N]{s}.$$

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Remark

The Conjecture holds for

$$s = k^N$$

very general points.

Chudnovsky and Demailly Conjectures

Conjecture (Chudnovsky)

Let I be a saturated ideal of points in $\mathbb{P}^N(\mathbb{K})$. Then

$$\widehat{\alpha}(I) \geq \frac{\alpha(I) + N - 1}{N}.$$

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Let I be a saturated ideal of points in $\mathbb{P}^N(\mathbb{K})$. Then

$$\widehat{\alpha}(I) \geq \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}.$$

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$$\widehat{\alpha}(I) \geq \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}.$$

Remark

The Chudnovsky Conjecture is the $m = 1$ case of the Demailly Conjecture.

Containment and a Chudnovsky-type statement

Theorem (Ein, Lazarsfeld, Smith; Hochster, Huneke)

Let I be a saturated ideal in $\mathbb{K}[x_0,\dots,x_N]$. Then for all $m \geq Nr$

$$I^{(m)} \subset I^r.$$

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Corollary (Earlier proof by Skoda with analytic methods)

Let I be a saturated ideal of points in $\mathbb{P}^N(\mathbb{K})$. Then

$$\widehat{\alpha}(I) \geq \frac{\alpha(I)}{N}.$$

Containment needed for the Demailly Conjecture

Conjecture (Bocci, Harbourne, Huneke)

Let I be a radical ideal of a finite set of points in $\mathbb{P}^N$. Let M be the irrelevant ideal. Then there is the containment

$$I^{(rN-(N-1))} \subset M^{(r-1)(N-1)} I^r$$

for all $r \geq 1$.

An improvement towards the Chudnovsky Conjecture

Theorem (Esnault – Viehweg 1983)

Let I be a radical ideal of a finite set of points in $\mathbb{P}^N(\mathbb{C})$ with $N \geq 2$. Then

$$\frac{\alpha(I^{(k)})+1}{k+N-1} \leq \frac{\alpha(I^{(m)})}{m},$$

holds for all $k, m \geq 1$.

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Let I be a radical ideal of a finite set of points in $\mathbb{P}^N(\mathbb{C})$ with $N \geq 2$. Then

$$\frac{\alpha(I^{(k)})+1}{k+N-1} \leq \frac{\alpha(I^{(m)})}{m},$$

holds for all $k, m \geq 1$. In particular, for all $k \geq 1$

$$\frac{\alpha(I^{(k)})+1}{k+N-1} \leq \widehat{\alpha}(I).$$

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holds for all $k, m \geq 1$. In particular, for all $k \geq 1$

$$\frac{\alpha(I^{(k)})+1}{k+N-1} \leq \widehat{\alpha}(I).$$

Corollary

The Demailly Conjecture holds in $\mathbb{P}^2(\mathbb{C})$

$$\frac{\alpha(I^{(k)})+1}{k+1} \leq \widehat{\alpha}(I).$$

General points

Theorem (Dumnicki-Tutaj-Gasińska, Fouli-Mantero-Xie 2016)

The Chudnovsky Conjecture holds for general points in $\mathbb{P}^N$.

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Theorem (Malara, Sz., Szpond 2017)

The Demailly Conjecture

$$\widehat{\alpha}(I) \geq \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}$$

holds for $s \geq (m+1)^N$ *very general points in* $\mathbb{P}^N$.

Combinatorial inequality and a naive lower bound on the Waldschmidt constant

Lemma

For all $N \geq 3$, $m \geq 1$ *and* $k \geq m+1$ *there is*

$$\binom{k(m+N-1)+1}{N} \geq \binom{m+N-1}{N}(k+1)^N.$$

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Theorem

Let Z be a set of s very general points in $\mathbb{P}^N$. Then

$$\widehat{\alpha}(Z) \geq \lfloor \sqrt[N]{s} \rfloor.$$

Waldschmidt decomposition

Definition (Waldschmidt decomposition in $\mathbb{P}^N$)

Let $H \cong \mathbb{P}^{N-1}$ be a hyperplane in $\mathbb{P}^N$ and let Z be a subscheme in H. Let D be a divisor of degree d in $\mathbb{P}^N$. The *Waldschmidt decomposition of D with respect to H and Z* is the sum of $\mathbb{R}$-divisors

$$D = D' + \lambda \cdot H$$

such that $\deg(D') = d - \lambda$,

$$\frac{d-\lambda}{\mathrm{mult}_Z D'} \geq \widehat{\alpha}(H; \mathcal{O}_H(1), Z) \tag{1}$$

and λ is the least non-negative real number such that (1) is satisfied.

Lower bound on Waldschmidt constants

Theorem (Dumnicki, Sz., Szpond)

Let $H_1, \ldots, H_s$ be $s \geq 2$ mutually distinct hyperplanes in $\mathbb{P}^N$. Let $a_1 \geq \ldots \geq a_s > 1$ be real numbers such that

$$\begin{cases} a_1 - 1 > 0 \\ a_1 a_2 - a_1 - a_2 > 0 \\ \vdots \qquad \vdots \\ a_1 \ldots a_{s-1} - \sum_{i=1}^{s-1} a_1 \ldots \widehat{a_i} \ldots a_{s-1} > 0 \end{cases}$$

and

$$a_1 \ldots a_s - \sum_{i=1}^{s} a_1 \ldots \widehat{a_i} \ldots a_s \leq 0.$$

Lower bound on Waldschmidt constants

Theorem (Dumnicki, Sz., Szpond)

Let

$$Z_i = \{P_{i,1}, \ldots, P_{i,r_i}\} \in H_i \setminus \bigcup_{j \neq i} H_j$$

be the set of r_i *points such that*

$$\widehat{\alpha}(H_i; Z_i) \geq a_i$$

and let $Z = \bigcup_{i=1}^{s} Z_i$. *Finally, let*

$$q := \frac{a_1 \ldots a_{s-1} - \sum_{i=1}^{s-1} a_1 \ldots \widehat{a_i} \ldots a_{s-1}}{a_1 \ldots a_{s-1}} \cdot a_s + s - 1.$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; Z) \geq q.$$

An application

Proposition

Let s be a positive integer and let k be an integer in the range $1 \leq k \leq s$. Let Z be a set of

$$r \geq r_k = k(s+1)^{N-1} + (s+1-k)s^{N-1}$$

very general points in $\mathbb{P}^N$. Then

$$\widehat{\alpha}(Z) \geq s+1-\frac{s+1-k}{s+1}.$$

Closing