Hilbert function of general fat points in $\mathbb{P}^1 \times \mathbb{P}^1$

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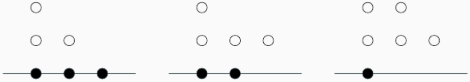

[1] DISMA - Politecnico di Torino, Italy [2] DIME - Università degli Studi di Genova, Italy

Introduction

Polynomial interpolation: simple points

Polynomial Interpolation Problem

Given a set of points $\mathbb{X} = \{P_1, \ldots, P_s\}$ in complex projective space $\mathbb{P}^n$,
how many hypersurfaces of degree d pass through $\mathbb{X}$?

e.g., in the projective plane:

through 2 distinct points there is a unique line

through 5 general points there is a unique conic

Polynomial interpolation: simple points

Let $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$, standard graded polynomial ring.

S_d := $\mathbb{C}$-vector space of homogeneous polynomials of degree d

Hilbert function

Let $I = \bigoplus_{d \geq 0} I_d$ be a homogeneous ideal. The **Hilbert function** of S/I in degree d is

$$\mathrm{HF}_{S/I}(d) := \dim_{\mathbb{C}} S_d / I_d = \dim_{\mathbb{C}} S_d - \dim_{\mathbb{C}} I_d.$$

Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$, then $I(\mathbb{X}) = \wp_1 \cap \ldots \cap \wp_s = \bigoplus_{d \geq 0} I(\mathbb{X})_d \subset S$.

$P = (p_0 : p_1 : p_2) \in \mathbb{P}^2 \rightsquigarrow \wp = (p_1 x_0 - p_0 x_1, p_2 x_0 - p_0 x_2)$

The **Hilbert function** of $\mathbb{X}$ is the Hilbert function of $S/I(\mathbb{X})$.

Polynomial interpolation: simple points

Polynomial Interpolation Problem

Given a set of points $\mathbb{X} = \{P_1, \ldots, P_s\}$ in complex projective space $\mathbb{P}^n$,
what is the Hilbert function of $\mathbb{X}$ in degree d?

Obviously, the answer depends on the position of the points.

[Geramita-Orecchia, 1981] If the points are in **general position**,

$$\mathrm{HF}_{\mathbb{X}}(d) = \min\left\{\binom{n+d}{d}, s\right\}.$$

Proof. If $\{m_1, \ldots, m_N\}$ is the standard monomial basis for S_d, then $\mathrm{HF}_{\mathbb{X}}(d) = \mathrm{rk}\,(m_i(P_j))_{ij}$.

Polynomial interpolation: fat points

Fat points

A **fat point** of **multiplicity** m and **support** at P is the 0-dim scheme given by $\wp^m$.
We denote it by mP.

A **scheme of fat points** is a union of fat points, i.e., $\mathbb{X} = m_1P_1 + \ldots + m_sP_s$ defined by $I(\mathbb{X}) = \wp_1^{m_1} \cap \ldots \cap \wp_s^{m_s}$.

If $m_1 = \ldots = m_s = m$, then
$I(\mathbb{X})$ is the m-th symbolic power of $\wp_1 \cap \ldots \cap \wp_s$.

Remark. $f \in \wp^m$ if and only if $D(f)|_P = 0$, for any $D \in \mathbb{C}[\partial_0, \ldots, \partial_n]_{\leq m-1}$.

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Polynomial interpolation: fat points

Polynomial Interpolation Problem

Let $\mathbb{X} = m_1P_1 + \ldots + m_sP_s$ be a scheme of fat points in $\mathbb{P}^n$,

what is the Hilbert function of $\mathbb{X}$ in degree d?

this is equivalent to asking

Polynomial interpolation: fat points

Polynomial Interpolation Problem

Let $\mathbb{X} = m_1 P_1 + \ldots + m_s P_s$ be a scheme of fat points in $\mathbb{P}^n$,

what is the Hilbert function of $\mathbb{X}$ in degree d?

this is equivalent to asking

Given a set of points $\{P_1, \ldots, P_s\}$ and positive integers $m_1, \ldots, m_s$,

how many hypersurfaces of degree d are singular at P_i of order m_i, for $i = 1, \ldots, s$?

Polynomial interpolation: fat points

Remark. $f \in \wp^m$ if and only if $D(f)|_P = 0$, for any $D \in \mathbb{C}[\partial_0, \dots, \partial_n]_{\leq m-1}$.

Therefore, a fat point of multiplicity m in $\mathbb{P}^n$ imposes $\binom{n+m-1}{n}$ linear equations.

If we assume the points to have general support, the **expected Hilbert function** is

$$exp.\mathrm{HF}_{\mathbb{X}}(d) = \min \left\{ \binom{n+d}{n}, \sum_{i=1}^{s} \binom{n+m_i-1}{n} \right\}.$$

Polynomial interpolation: fat points

Example 1. Let $\mathbb{X} = 2P_1 + \ldots + 2P_5 \subset \mathbb{P}^2$, with general support. We expect to have no quartics through $\mathbb{X}$.

$$exp.\dim I(\mathbb{X})_4 = \binom{4+2}{2} - 5 \cdot 3 = 0.$$

However, there is a unique conic C passing through the point $P_1, \ldots, P_5$.

Hence, $2C \in I(\mathbb{X})_4$. By Bézout's Theorem,

$$\dim I(\mathbb{X})_4 = 1 > 0 = exp.\dim I(\mathbb{X})_4.$$

Polynomial interpolation: fat points

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Example 2. Let $\mathbb{X} = 2P_1 + \ldots + 2P_7 \subset \mathbb{P}^4$, with general support. We expect to have no cubics through $\mathbb{X}$.

$$exp.\dim I(\mathbb{X})_3 = \binom{3+4}{3} - 7 \cdot 5 = 0.$$

However, there is a unique rational normal curve C passing through the P_i's.

The 2-nd secant variety is a cubic surface singular along C, i.e., $\sigma_2(C) \in I(\mathbb{X})_3$, and

$$\dim I(\mathbb{X})_3 = 1 > 0 = exp.\dim I(\mathbb{X})_3.$$

Polynomial interpolation: fat points

[Alexander-Hirschowitz, 1994] Let $\mathbb{X} = 2P_1 + \ldots + 2P_s \subset \mathbb{P}^n$ with general support. Then, the Hilbert function of $\mathbb{X}$ in degree d is as expected, i.e.,

$$\mathrm{HF}_{\mathbb{X}}(d) = \min\left\{\binom{n+d}{n}, 3s\right\},$$

except for:

1. quadrics: $d = 2,\ 2 \leq s \leq n$
2. in $\mathbb{P}^2$: $d = 4,\ s = 5$ [Example 1]
3. in $\mathbb{P}^3$: $d = 4,\ s = 9$;
4. in $\mathbb{P}^4$: $d = 3,\ s = 7$ [Example 2] and $d = 4,\ s = 14$.

...for higher multiplicity very little is known

Polynomial interpolation: fat points

Beniamino Segre

ALCUNE QUESTIONI SU INSIEMI FINITI DI PUNTI IN GEOMETRIA ALGEBRICA

È con viva commozione che mi accingo a parlare nella mia città natìa, in questa gloriosa Università che mi accolse sedicenne e dove trascorsi il periodo più formativo e determinante dei miei studi. Sono lieto di rivedere qui presenti, e vegeti quasi che da allora non fossero trascorsi ben otto lustri, i miei professori di quegli anni Boggio e Togliatti, ed il professore Terracini del quale fui poi per qualche tempo assistente.

Il mio pensiero si volge con gratitudine ad essi ed agli altri miei Maestri: Corrado Segre con cui mi addottorai, Somigliana del quale pure fui assistente, ed inoltre Peano e Fano, tutti purtroppo scomparsi, ma la cui voce ed i cui insegnamenti ancora mi riecheggiano nel cuore.

Osserviamo anzitutto che, essendo $\delta \geqslant -1$, *risulta sempre* $\sigma > 0$ *quando si supponga* $d < -1$. Altri esempi di sistemi lineari Σ sovrabbondanti, aventi cioè appunto $\sigma > 0$, vengono offerti da

$$\{10 \mid 4^2, 5^3\}$$

e da

$$\{n \mid k_1, k_2\} \qquad \text{con } k_1, k_2 \leq n,\ k_1 + k_2 \geq n+2,$$

per i quali rispettivamente si ha $d = 0$, $\sigma = 2$ e

$$\sigma = (k_1 + k_2 - n)(k_1 + k_2 - n - 1)/2;$$

e va rilevato al riguardo come ciascuno di questi sistemi risulti dotato di componente fissa multipla (rispettivamente: la conica per i cinque punti base contata 4 volte e la retta per i due punti base contata $k_1 + k_2 - n$ volte). Questi ed altri esempi consimili portano a fare ritenere probabile che

Affinchè un sistema lineare completo Σ di curve piane, dotato di un numero finito di punti base assegnati in posizione generica ed avente dimensione virtuale $d \geqslant -1$, sia sovrabbondante (e quindi effettivo, cioè di dimensione $\delta \geqslant 0$) è necessario (ma, come risulta da esempi, non sufficiente) ch'esso possegga qualche componente fissa multipla.

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Polynomial interpolation: fat points

B. Segre 1961

Alcune questioni su insiemi finiti di punti in geometria algebrica (page 72)

"(...) in order that a complete linear system Σ of plane curves, passing through multiple base points in general position with virtual dimension $d \geq -1$ is superabundant (and then effective, i.e., of dimension $\delta \geq 0$), it is necessary (but, from examples, not sufficient) that it has some multiple fixed component."

SHGH Conjecture

A degree d curve C is *negative* for $P_1, \ldots, P_d \in \mathbb{P}^2$ if $(\mathrm{mult}_{P_1}(C))^2 + \ldots + (\mathrm{mult}_{P_d}(C))^2 > d^2$.

[SHGH Conjecture - B. Segre, '61; Harbourne '85; Gimigliano '87; Hirschowitz '89]
If with general support and $f = f_1^{b_1} \cdots f_t^{b_t}$ is the greatest common divisor of $I(\mathbb{X})_j$. Let $\mathcal{N}$ be the set of negative curves for $P_1, \ldots, P_d$. Then,

$$\mathrm{HF}_{\mathbb{X}}(j) = \min\left\{ \binom{j+2}{2}, \sum_{i=1}^{s} \binom{m_i+1}{2} - \sum_{i\,:\,f_i \in \mathcal{N}} \binom{b_i}{2} \right\}.$$

[Castelnuovo, 1891] $d \leq 9$; [Yang, 2007] $m_i \leq 7$;

[Ciliberto-Miranda, 1998] $m_i = m \leq 12$; ...and some other special cases

Multigraded Interpolation

Multigraded Interpolation Problem

We consider the case of $\mathbb{P}^1 \times \mathbb{P}^1$ for simplicity of notation.

Let $S = \mathbb{C}[x_0, x_1; y_0, y_1] = \bigoplus_{(a,b)\in\mathbb{N}^2} S_{a,b}$ be a bi-graded polynomial ring

$S_{a,b}$:= *$\mathbb{C}$-vector space of bi-homogeneous polynomials of bi-degree* (a, b)

Fat points in multiprojective space

A point $P = (Q_1, Q_2) = ((q_{1,0} : q_{1,1}); (q_{2,0} : q_{2,1})) \in \mathbb{P}^1 \times \mathbb{P}^1$ is defined by

$$\wp = I(Q_1) + I(Q_2) = (q_{1,1}x_0 - q_{1,0}x_1, q_{2,1}y_0 - q_{2,0}y_1) \subset S.$$

The **fat point** mP is the 0-dimensional scheme defined by $\wp^m$.
The **scheme of fat points** $\mathbb{X} = m_1P_1 + \ldots + m_sP_s$ is defined by $I(\mathbb{X}) = \wp_1^{m_1} \cap \ldots \cap \wp_s^{m_s}$.

Multigraded Interpolation Problem

Multigraded Hilbert function

Let $I = \bigoplus_{(a,b)\in\mathbb{N}^2} I_{a,b}$ be a bi-homogeneous ideal.

The **Hilbert function** of S/I in bi-degree (a,b) is

$$\mathrm{HF}_{S/I}(a,b) = \dim_{\mathbb{C}} S_{a,b}/I_{a,b} = \dim_{\mathbb{C}} S_{a,b} - \dim_{\mathbb{C}} I_{a,b}.$$

The Hilbert function of a scheme of fat points $\mathbb{X}$ is the Hilbert function of $S/I(\mathbb{X})$.

Multigraded Interpolation Problem

Let $\mathbb{X}$ be a scheme of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$,

what is the Hilbert function of $\mathbb{X}$ in bi-degree (a,b)?

Multigraded Interpolation Problem

If we assume the points to have general support, the **expected Hilbert function** is

$$exp.\mathrm{HF}_{\mathbb{X}}(a,b) = \min\left\{(a+1)(b+1),\ \sum_{i=1}^{s}\binom{m_i+1}{2}\right\}.$$

[Catalisano-Geramita-Gimigliano, 2005] Let $\mathbb{X} = 2P_1 + \ldots + 2P_s \subset \mathbb{P}^1 \times \mathbb{P}^1$, with general support. Then, the Hilbert function of $\mathbb{X}$ in bi-degree (a,b), with $a \geq b$, is as expected, except for $(a,b) = (2k,2),\ s = 2k+1,\ k \geq 1$, where the defect is 1.

They complete the case of double points for any $\underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{t\ times}$. *In particular, it is as expected for* $t \geq 5$.

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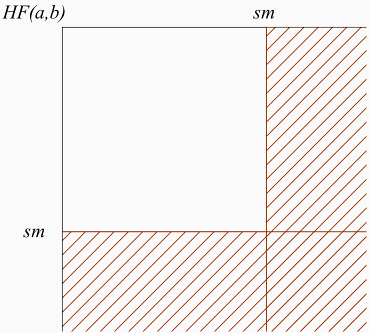
Multigraded Interpolation Problem

[Guardo-Van Tuyl, 2005]

Let $\mathbb{X} = mP_1 + \ldots + mP_s \subset \mathbb{P}^1 \times \mathbb{P}^1$, with general support. The Hilbert function of $\mathbb{X}$ is constant in the a-th row (resp., in the b-th column) for $b \geq sm$ (resp., $a \geq sm$).



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Multigraded Interpolation Problem

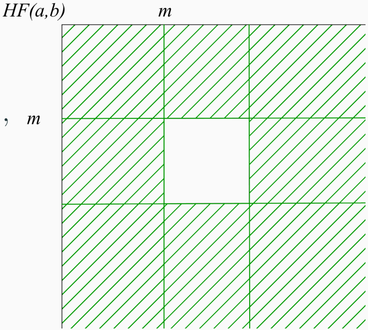
[Carlini-Catalisano-O., 2017]

Let $\mathbb{X} = mP_1 + \ldots + mP_s \subset \mathbb{P}^1 \times \mathbb{P}^1$, with general support. Let $a \geq b$ and $m \geq b$. Then,

$$\mathrm{HF}_{\mathbb{X}}(a,b) = \min\left\{(a+1)(b+1), s\binom{m+1}{2} - s\binom{m-b}{2}\right\},$$

except for $s = 2k+1$ and $a = bk + c + s(m-b)$, with $c = 0, \ldots, b-2$, where

$$\mathrm{HF}_{\mathbb{X}}(a,b) = (a+1)(b+1) - \binom{c+2}{2}.$$



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Multigraded Interpolation Problem

[Carlini-Catalisano-O., 2017]

Let $\mathbb{X} = mP_1 + \ldots + mP_s \subset \mathbb{P}^1 \times \mathbb{P}^1$, with general support. Let $a \geq b$ and $m \geq b$. Then,

$$\mathrm{HF}_{\mathbb{X}}(a,b) = \min\left\{(a+1)(b+1), s\binom{m+1}{2} - s\binom{m-b}{2}\right\}$$

except for $s = 2k+1$ and $a = bk + c + s(m-b)$, with $c = 0, \ldots, b-2$, where

$$\mathrm{HF}_{\mathbb{X}}(a,b) = (a+1)(b+1) - \binom{c+2}{2}.$$

Example. $\mathbb{X} = 5P_1 + \ldots + 5P_5$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	25
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	45	45	45	45
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	59	60	60	60	60	60	60
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	67	69	70	70	70	70	70	70	70	70
5	10	15	20	25	30	35	40	45	50	55	60	65	69	72	74	75	75	75	75	75	75	75	75	75	75
6	12	18	24	30	36	42	48	54	60	65	69	72	74	75	75	75	75	75	75	75	75	75	75	75	75
7	14	21	28	35	42	49	56	63	69	72	74	75	75	75	75	75	75	75	75	75	75	75	75	75	75
8	16	24	32	40	48	56	64	71	74	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
9	18	27	36	45	54	63	71	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
10	20	30	40	50	60	69	74	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
11	22	33	44	55	65	72	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
12	24	36	48	60	69	74	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
13	26	39	52	65	72	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
14	28	42	56	69	74	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
15	30	45	60	72	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
16	32	48	64	74	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
17	34	51	67	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
18	36	54	69	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
19	38	57	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
20	40	59	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
21	42	60	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
22	44	60	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
23	45	60	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
24	45	60	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
25	45	60	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
25	45	60	70	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75

Multigraded Interpolation Problem

[Carlini-Catalisano-O., 2017]

Let $\mathbb{X} = 3P_1 + \ldots + 3P_s \subset \mathbb{P}^1 \times \mathbb{P}^1$, with general support.

Then,

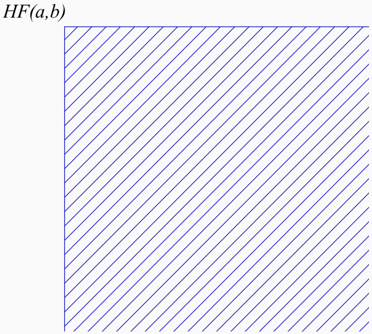
$$\mathrm{HF}_{\mathbb{X}}(a,b) = \min\left\{(a+1)(b+1), 6s - s\binom{3-b}{2}\right\},$$

except for s odd, say $s = 2k+1$, and:

i. $(a,b) = (4k+1, 2)$, where $\mathrm{HF}_{\mathbb{X}}(a,b) = (a+1)(b+1) - 1$;

ii. $(a,b) = (3k, 3)$, where $\mathrm{HF}_{\mathbb{X}}(a,b) = (a+1)(b+1) - 1$;

iii. $(a,b) = (3k+1, 3)$, where $\mathrm{HF}_{\mathbb{X}}(a,b) = 6s - 1$.



Multiprojective-Affine-Projective Method

Multiprojective-Affine-Projective Method

Consider the birational map

$$\varphi : \begin{array}{ccccc} \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{A}^2 & \to & \mathbb{P}^2 \\ ((s_0 : s_1); (t_0 : t_1)) & \mapsto & \left(\frac{s_1}{s_0}, \frac{t_1}{t_0}\right) & \mapsto & \left(1 : \frac{s_1}{s_0} : \frac{t_1}{t_0}\right) = (s_0 t_0 : s_1 t_0 : s_1 t_0). \end{array}$$

[Catalisano-Geramita-Gimigliano, 2002]

Let $\mathbb{X}$ be a set of fat points with general support in $\mathbb{P}^1 \times \mathbb{P}^1$. Then,

$$\mathrm{HF}_{I(\mathbb{X})}(a, b) = \mathrm{HF}_{I(X)}(a + b),$$

where $X = \varphi(\mathbb{X}) + aQ_1 + bQ_2$, with $Q_1 = (0 : 1 : 0)$ and $Q_2 = (0 : 0 : 1)$.

Multiprojective-Affine-Projective Method

Example. We can define it for higher multi-projective spaces.

$$\begin{array}{cccc} \varphi : & \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} & \dashrightarrow & \mathbb{P}^{n_1+n_2+n_3} \\ & ((1 : \ldots : s_{n_1}); (1 : \ldots : t_{n_2}); (1 : \ldots : u_{n_3}) & \mapsto & (1 : \ldots : s_{n_1} : t_1 : \ldots : t_{n_2} : u_1 : \ldots : u_{n_3}) \end{array}$$

[Catalisano-Geramita-Gimigliano, 2002]

Let $\mathbb{X}$ be a set of fat points with general support in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. Then,

$$\mathrm{HF}_{I(\mathbb{X})}(a_1, a_2, a_3) = \mathrm{HF}_{I(X)}(a_1 + a_2 + a_3),$$

where $X = \varphi(\mathbb{X}) + (a_2 + a_3)\Pi_1 + (a_1 + a_3)\Pi_2 + (a_1 + a_2)\Pi_3$, with

$$\Pi_1 = \{(0 : s_1 \ldots : s_{n_1} : 0 : \ldots : 0)\} \simeq \mathbb{P}^{n_1 - 1} \qquad \Pi_3 = \{(0 : \ldots : 0 : u_1 : \ldots : u_{n_3})\} \simeq \mathbb{P}^{n_3 - 1}$$

$$\Pi_2 = \{(0 : \ldots : 0 : t_1 : \ldots : t_{n_2} : 0 : \ldots : 0)\} \simeq \mathbb{P}^{n_2 - 1}$$

Multiprojective-Affine-Projective Method

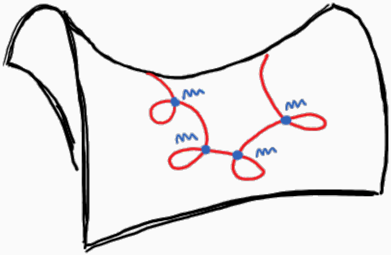
Let $\mathbb{X} = mP_1 + \ldots + mP_s \subset \mathbb{P}^1 \times \mathbb{P}^1$, with general support.

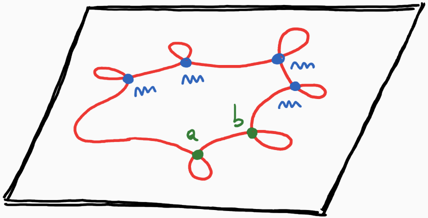
$\iff$

Let $X = aQ_1 + bQ_2 + mP_1 + \ldots + mP_s \subset \mathbb{P}^2$, with general support.

What is $\mathrm{HF}_{I(\mathbb{X})}(a, b)$?

What is $\mathrm{HF}_{I(X)}(a + b)$?





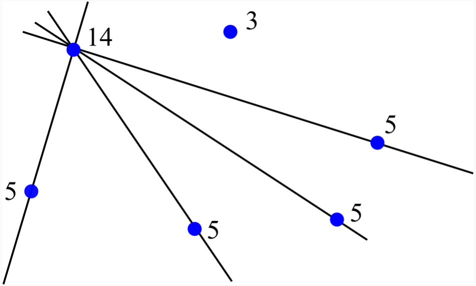
...so, we are back in the standard graded (planar) case!

Example: reduction to $m = b$ case

Example. Let $a = 14$, $b = 3$, $m = 5$ and $s = 4$.

$$exp. \dim I(\mathbb{X})_{14,3} = \max\{0,\ (14+1)(3+1) - 4\cdot 15\} = 0.$$

We consider $X = 14Q_1 + 3Q_2 + 5P_1 + \ldots + 5P_4 \subset \mathbb{P}^2$ and we compute $\dim I(X)_{17}$.



By Bézout's Theorem,

all the lines $\overline{Q_1P_i}$ are fixed components, twice.

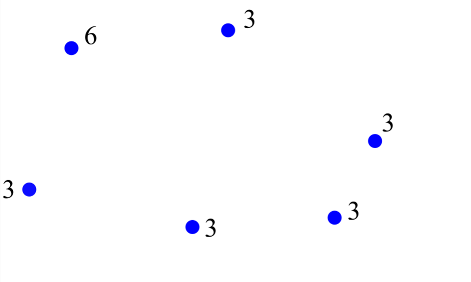
So, they can be removed.

Example: reduction to $m = b$ case

Example. Let $a = 14$, $b = 3$, $m = 5$ and $s = 4$.

$$exp. \dim I(\mathbb{X})_{14,3} = \max\{0,\ (14+1)(3+1) - 4 \cdot 15\} = 0.$$

We consider $X = 14Q_1 + 3Q_2 + 5P_1 + \ldots + 5P_4 \subset \mathbb{P}^2$ and we compute $\dim I(X)_{17}$.



Let $X' = 6Q_1 + 3Q_2 + 3P_1 + \ldots + 3P_4 \subset \mathbb{P}^2$.
Then, $\dim I(X)_{17} = \dim I(X')_9$. Now,

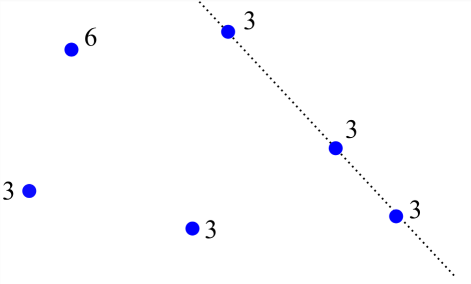
$$exp. \dim I(X')_9 = \max\{0,\ (6+1)(3+1) - 4 \cdot 6\} = 4.$$

Do you remember Beniamo Segre's remark?

Example: $m = b$ case

Example. Let $X' = 6Q_1 + 3Q_2 + 3P_1 + \ldots + 3P_4 \subset \mathbb{P}^2$, with $exp. \dim I(X')_9 = 4$. By semicontinuity, if $\widetilde{X}'$ is a specialization of X', we have

$$4 = exp. \dim I(X')_9 \leq \dim I(X')_9 \leq \dim I(\widetilde{X}')_9.$$

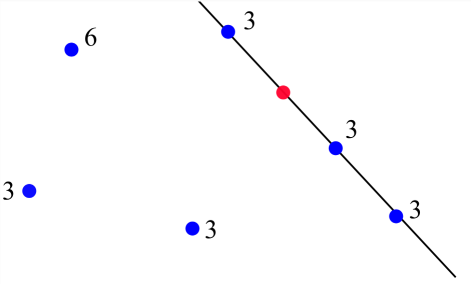


Assume that Q_2, P_3 and P_4 are collinear.

Example: $m = b$ case

Example. Let $\widetilde{X}' = 6Q_1 + 3Q_2 + 3P_1 + \ldots + 3P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. For any point $A \in \mathbb{P}^2$,

$$4 = exp. \dim I(X')_9 \leq \dim I(X')_9 \leq \dim I(\widetilde{X}')_9 \leq \dim I(\widetilde{X}' + A)_9 + 1.$$



Assume that A is collinear with Q_2, P_3, P_4. By Bézout's Theorem, the line is a fixed component and can be removed.

Example: $m = b$ case

Example. Let $\widetilde{X}' = 6Q_1 + 3Q_2 + 3P_1 + \ldots + 3P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. For any point $A \in \mathbb{P}^2$,

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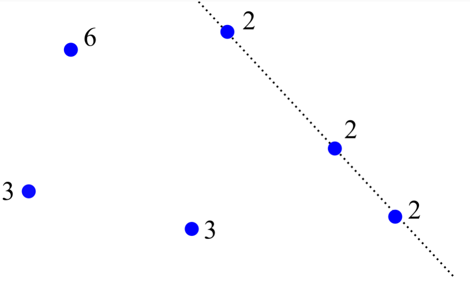


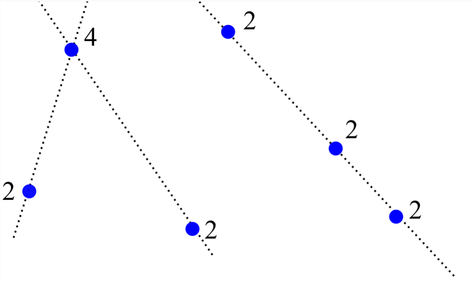
Let $W = 6Q_1 + 2Q_2 + 3P_1 + 3P_2 + 2P_3 + 2P_4$. Then,

$$\dim I(\widetilde{X}' + A)_9 = \dim I(W)_8.$$

Example: $m = b$ case

Example. Let $W = 6Q_1 + 3Q_2 + 3P_1 + 3P_2 + 2P_3 + 2P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. Then,

$$4 = exp.\dim I(X')_9 \leq \dim I(X')_9 \leq \ldots = \dim I(W)_8 + 1.$$



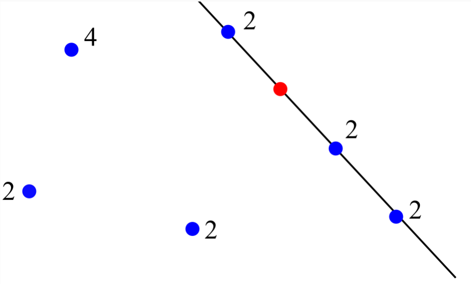
The lines $\overline{Q_1P_1}$, $\overline{Q_1P_2}$ are fixed components.
Let $W' = 4Q_1 + 2Q_2 + 2P_1 + 2P_2 + 2P_3 + 2P_4$,

$$\dim I(W)_8 = \dim I(W')_6.$$

Example: $m = b$ case

Example. Let $W' = 4Q_1 + 2Q_2 + 2P_1 + 2P_2 + 2P_3 + 2P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. For any point $A' \in \mathbb{P}^2$,

$$4 = exp. \dim I(X')_9 \leq \dim I(X')_9 \leq \ldots \leq \dim I(W' + A')_6 + 2.$$

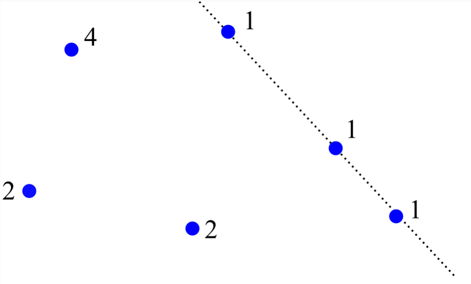


Assume that A' is collinear with Q_2, P_3, P_4. By Bézout's Theorem, the line is a fixed component and can be removed.

Example: $m = b$ case

Example. Let $W' = 4Q_1 + 2Q_2 + 2P_1 + 2P_2 + 2P_3 + 2P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. For any point $A' \in \mathbb{P}^2$,

$$4 = exp.\dim I(X')_9 \leq \dim I(X')_9 \leq \ldots \leq \dim I(W' + A')_6 + 2.$$



Let $W'' = 6Q_1 + Q_2 + 2P_1 + 2P_2 + P_3 + P_4$.
Then,

$$\dim I(W' + A')_6 = \dim I(W'')_5.$$

Example: $m = b$ case

Example. Let $W'' = 4Q_1 + Q_2 + 2P_1 + 2P_2 + P_3 + P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. Then,

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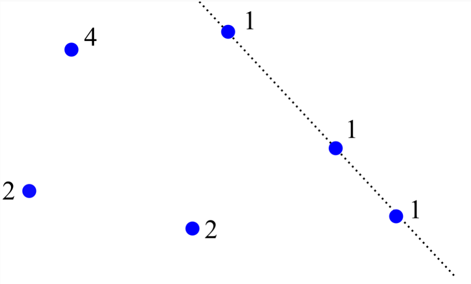


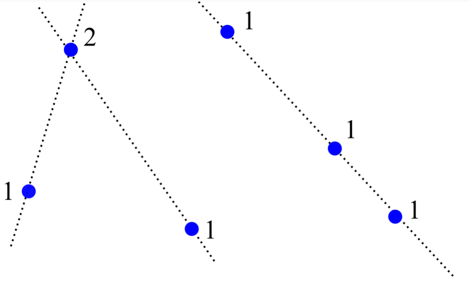
...we continue in a similar way...

UPC BGSMath EXCELENCIA MARÍA DE MAEZTU

Example: $m = b$ case

Example. Let $W''' = 2Q_1 + Q_2 + P_1 + P_2 + P_3 + P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. Then,

$$4 = exp. \dim I(X')_9 \leq \dim I(X')_9 \leq \ldots = \dim I(W''')_3 + 2.$$



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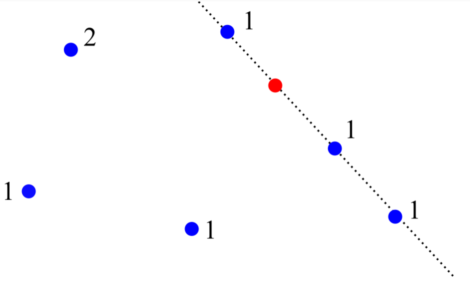
Example: $m = b$ case

Example. Let $W''' = 2Q_1 + Q_2 + P_1 + P_2 + P_3 + P_4 \subset \mathbb{P}^2$, with Q_2, P_3 and P_4 collinear. For any $A'' \in \mathbb{P}^2$,

$$4 = exp. \dim I(X')_9 \leq \dim I(X')_9 \leq \ldots \leq \dim I(W''' + A'')_3 + 3.$$

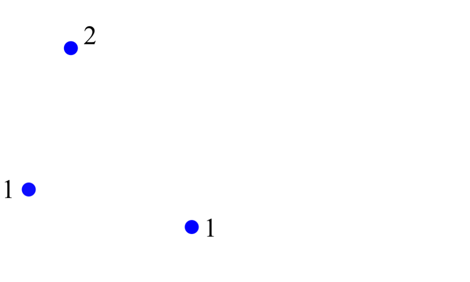


...we continue in a similar way...

Example: $m = b$ case

Example. Let $W'''' = 2Q_1 + P_1 + P_2 \subset \mathbb{P}^2$.
Then,

$$4 = exp. \dim I(X')_9 \leq \dim I(X')_9 \leq \ldots = \dim I(W'''')_2 + 3.$$

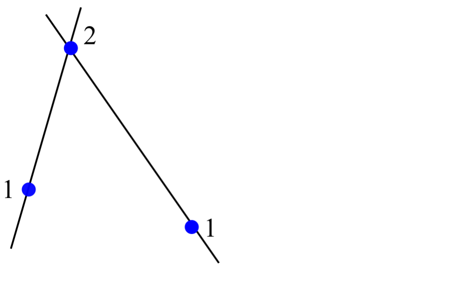


...we continue in a similar way...

Example: $m = b$ case

Example. Let $W'''' = 2Q_1 + P_1 + P_2 \subset \mathbb{P}^2$. Then,

$$4 = exp.\dim I(X')_9 \leq \dim I(X')_9 \leq \ldots = \dim I(W'''')_2 + 3 = 1 + 3 = 4$$



Example: subabundance cases

- The Hilbert function of $\mathbb{X} = 5P_1 + \ldots + 5P_4$ is defective in bi-degree $(14, 3)$, i.e.,
$$\dim I(\mathbb{X})_{14,3} = 4 > 0 = exp. \dim I(\mathbb{X})_{14,3} = \max\{0,\ 15 \cdot 4 - 4 \cdot 15\}.$$

Defectiveness given by four double lines in the base locus

- The Hilbert function of $\mathbb{X} = 3P_1 + \ldots + 3P_4$ is non-defective in bi-degree $(6, 3)$, i.e.,
$$\dim I(\mathbb{X})_{6,3} = 4 = exp. \dim I(\mathbb{X})_{6,3} = \max\{0,\ 7 \cdot 4 - 4 \cdot 6\}.$$

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Therefore,

The Hilbert function of $\mathbb{X} = 3P_1 + \ldots + 3P_s$ is non-defective in bi-degree $(6, 3)$ if $s \leq 4$.

Fixed (a, b), let $s_1 := \left\lfloor \frac{(a+1)(b+1)}{\binom{m+1}{2}} \right\rfloor$. If $I(\mathbb{X})_{a,b}$ has expected dimension for s_1, then it holds also for $s \leq s_1$.

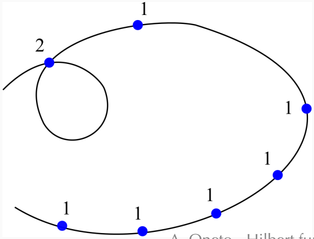
Example: superabundance cases

Example. Let $\mathbb{X} = 3P_1 + \ldots + 3P_5 \subset \mathbb{P}^1 \times \mathbb{P}^1$. Then,

$$exp. \dim I(\mathbb{X})_{6,3} = \max\{0,\ (6+1)(3+1) - 5\cdot 6\} = 0.$$

Fixed (a,b), we consider $s_2 = \left\lceil \frac{(a+1)(b+1)}{\binom{m+1}{2}} \right\rceil$, if $I(\mathbb{X})_{a,b}$ is empty for s_2, then it is empty for $s \geq s_2$.

Now, we have: $\dim I(\mathbb{X})_{6,3} = \dim I(X)_9$, for $X = 6Q_1 + 3Q_2 + 3P_1 + \ldots + 3P_5 \subset \mathbb{P}^2$.



If $Y = 2Q_1 + Q_2 + P_1 + \ldots + P_5 \subset \mathbb{P}^2$, then

$$\dim I(Y)_3 = \binom{3+2}{2} - 3 - 6 = 1.$$

There is a (unique) plane cubic $C \in I(Y)_3$.

Example: superabundance cases

Example. Let $\mathbb{X} = 3P_1 + \ldots + 3P_5 \subset \mathbb{P}^1 \times \mathbb{P}^1$. Then,

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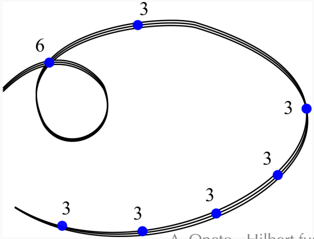


Therefore, $3C \in I(X)_9$!

By Bézout's Theorem, $\dim I(X)_9 = 1$.

Defective case!

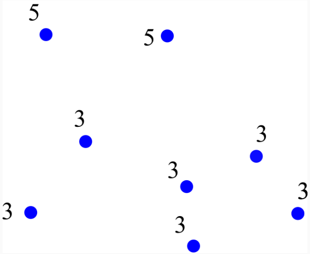
It follows also that $I(X)_9$ is empty for $s > 5$.

Example: case $b > m = 3$.

Example. Consider $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$. Then,

$$exp. \dim I(X)_{10} = \max\{0,\ (5+1)(5+1) - 6 \cdot 6\} = 0.$$

We can try similarly as before.



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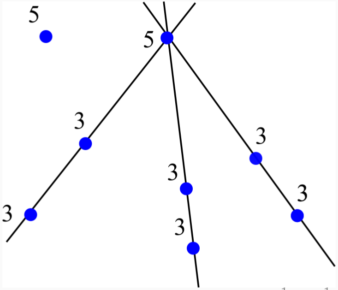

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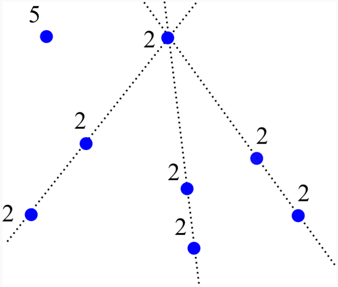
We can try similarly as before. ???



Méthode d'Horace differentiel

Méthode d'Horace

[Castelnuovo's Inequality]

Let X be a 0-dimensional scheme in $\mathbb{P}^n$. Given a hypersurface $H = \{\ell = 0\}$, we have the short exact sequence given by the restriction map

$$0 \to [I(X;\ \mathbb{P}^n) : (\ell)]_{d-1} \longrightarrow I(X;\ \mathbb{P}^n)_d \longrightarrow I(X \cap H;\ H)_d.$$

Residue	$\mathrm{Res}_H(X)$:	the scheme defined by $I(X) : (\ell)$, i.e., $I(X \setminus (X \cap H))$
Trace	$\mathrm{Tr}_H(X)$:	the scheme theoretical intersection $X \cap H \subset H$.

Hence,

$$exp. \dim I(X)_d \leq \dim I(X)_d \leq \dim I(\mathrm{Res}_H(X))_{d-1} + \dim I(\mathrm{Tr}_H(X))_d.$$

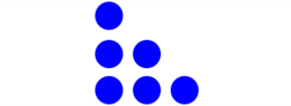
Vertically Graded schemes

Example. Consider the triple point defined by $I = (x_1, x_2)^3$. In $\mathcal{O}_{\mathbb{P}^2,P} \simeq \mathbb{C}[\![x_1, x_2]\!]$,

$$I = I_0 \oplus I_1 \cdot x_2 \oplus I_2 \cdot x_2^2 \oplus (x_2^3),$$

with

$$I_0 = (x_1^3); \quad I_1 = (x_1^3, x_1^2); \quad I_2 = (x_1, x_1^2, x_1^3).$$



Hence, $\mathbb{C}[\![x_1, x_2]\!]/I$ is the 6-dimensional vector space

$$\langle 1, x_1, x_1^2 \rangle \oplus \langle x_2, x_1 x_2 \rangle \oplus \langle x_2^2 \rangle.$$

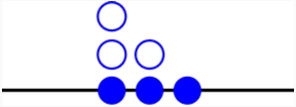
Vertically Graded schemes

Example. Consider the triple point defined by $I = (x_1, x_2)^3$. In $\mathcal{O}_{\mathbb{P}^2, P} \simeq \mathbb{C}[[x_1, x_2]]$,

In the standard specialization,

Residue: $I(\mathrm{Res}_{x_1}(3P)) = I : (x_2) = (x_1^2, x_1x_2, x_2^2) = (x_1, x_2)^2$

$\mathbb{C}[[x_1, x_2]]/I(\mathrm{Res}_{x_1}(3P))$ is the 3-dim vector space $\langle 1, x_1 \rangle \oplus \langle x_2 \rangle$.



Trace: $I(\mathrm{Tr}_{x_2}(3P)) = I \otimes \mathbb{C}[[x_1, x_2]]/(x_2) = (x_1)^3$.

$\mathbb{C}[[x_1]]/I(\mathrm{Tr}_{x_2}(3P))$ is the 3-dim vector space $\langle 1, x_1, x_1^2 \rangle$.

Méthode d'Horace

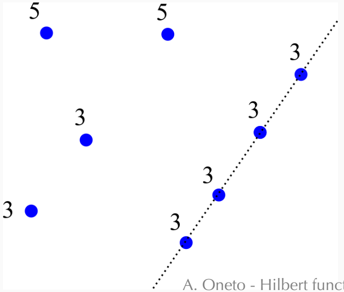
Le Serment des Horaces - J.-L. David (1785)

Méthode d'Horace

Sometimes the arithmetic do not allow to do computations so easily...

Example. Consider $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$ with

$$exp. \dim I(X)_{10} = 0 = (5+1)(5+1) - 6 \cdot 6.$$



Too many conditions on the line.

We need 11 *conditions on the line*

Méthode d'Horace

Sometimes the arithmetic do not allow to do computations so easily...

Example. Consider $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$ with

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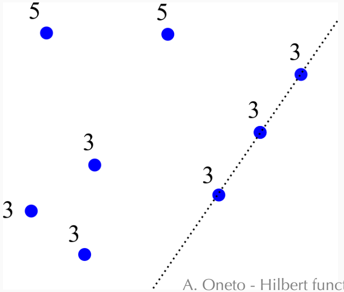


Too few conditions on the line.

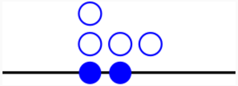
We need 11 conditions on the line

Méthode d'Horace differentiel

Example. Consider the triple point defined by $I = (x_1, x_2)^3$. In $\mathcal{O}_{\mathbb{P}^2, P} \simeq \mathbb{C}[\![x_1, x_2]\!]$, with differential specialization,

2-nd Residue:

$$I(\mathrm{Res}^1_{x_2}(3P)) = I + (I : (x_2)^2) \cdot (x_2) = (x_1^3, x_1 x_2, x_2^2),$$
$$\mathbb{C}[\![x_1, x_2]\!]/I(\mathrm{Res}^2_{x_2}(3P)) = \langle 1, x_1, x_1^2 \rangle \oplus \langle x_2 \rangle.$$

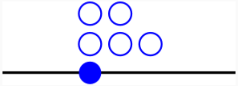


2-nd Trace:

$$I(\mathrm{Tr}^1_{x_2}(3P)) = (I : (x_2)) \otimes \mathbb{C}[\![x_1, x_2]\!]/(x_2) = (x_1)^2,$$
$$\mathbb{C}[\![x_1]\!]/I(\mathrm{Tr}^1_{x_2}(3P)) = \langle 1, x_1 \rangle.$$

3-rd Residue:

$$I(\mathrm{Res}^2_{x_2}(3P)) = I + (I : (x_2)^3) \cdot (x_2^2) = (x_1^3, x_1^2 x_2, x_2^2),$$
$$\mathbb{C}[\![x_1, x_2]\!]/I(\mathrm{Res}^2_{x_2}(3P)) = \langle 1, x_1, x_1^2 \rangle \oplus \langle x_2, x_1 x_2 \rangle.$$



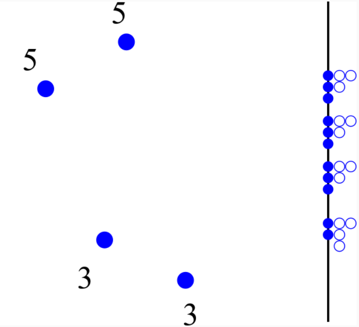
3-rd Trace:

$$I(\mathrm{Tr}^2_{x_2}(3P)) = (I : (x_2^2)) \otimes \mathbb{C}[\![x_1, x_2]\!]/(x_2) = (x_1),$$
$$\mathbb{C}[\![x_1]\!]/I(\mathrm{Tr}^1_{x_2}(3P)) = \langle 1 \rangle.$$

Example: méthode d'Horace differentiel

Example. Let $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$.

$$exp. \dim I(X)_{10} = 0 = (5+1)(5+1) - 6 \cdot 6.$$

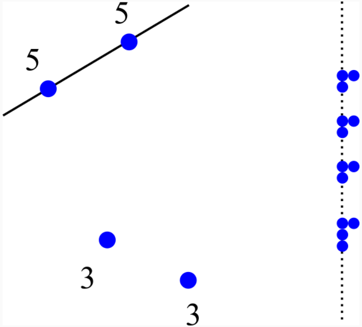


Degree 10

Example: méthode d'Horace differentiel

Example. Let $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$.

$$exp. \dim I(X)_{10} = 0 = (5+1)(5+1) - 6 \cdot 6.$$

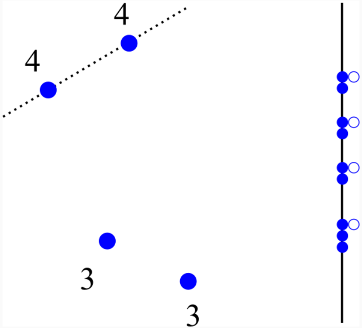


Degree 9

Example: méthode d'Horace differentiel

Example. Let $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$.

$$exp. \dim I(X)_{10} = 0 = (5+1)(5+1) - 6 \cdot 6.$$

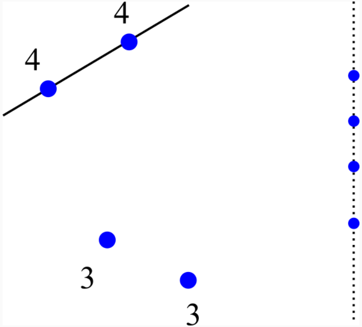


Degree 8

Example: méthode d'Horace differentiel

Example. Let $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$.

$$exp. \dim I(X)_{10} = 0 = (5+1)(5+1) - 6 \cdot 6.$$

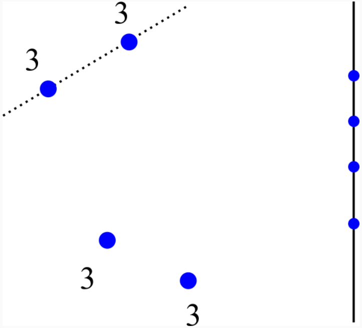


Degree 7

Example: méthode d'Horace differentiel

Example. Let $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6 \subset \mathbb{P}^2$.

$$exp. \dim I(X)_{10} = 0 = (5+1)(5+1) - 6 \cdot 6.$$



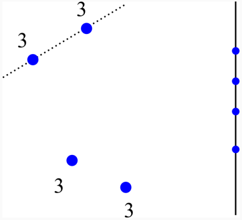
Degree 6

Example: méthode d'Horace differentiel

Example. Let $X = 5Q_1 + 5Q_2 + 3P_1 + \ldots + 3P_6$ and

$W_1 = 3Q_1 + 3Q_2 + 3P_1 + 3P_2$ and $W_2 = P_3 + \ldots + P_6$, with W_2 collinear.

$$\dim I(X)_{10} = \dim I(W_1 + W_2)_6 \overset{(1)}{=} \dim I(W_1)_6 - 4 =$$
$$\overset{(2)}{=} (4+1)(4+1) - 2 \cdot 6 - 4 = 16 - 12 - 4 = 0.$$

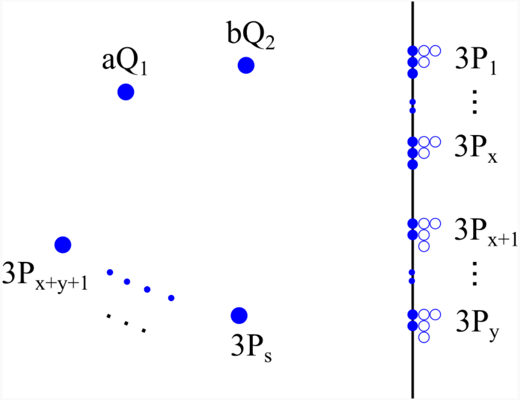


(1) **[Catalisano-Geramita-Gimigliano]**: by a technical lemma, four collinear points give independent conditions

(2) as regards W_1, we are back to the case $m = b = 3$ and we know how to do it!

Conclusion

This procedure works and let us compute the whole Hilbert function of triple points!



¡Gracias!

BIBLIOGRAPHY

E. Carlini, M.V. Catalisano, A. Oneto.
Hilbert functions of general fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. arXiv:1711.06193

J. Alexander, A. Hirschowitz. **La méthode d'Horace elatée: application à l'interpolation en degrée quatre.** *Inventiones mathematicae, 107(1):585–602 (1992).*

A. Gimigliano, M. Idà. **The ideal resolution for generic 3-fat points in $\mathbb{P}^2$.** *Journal of Pure and Applied Algebra, 187(1):99–128 (2004).*

M.V. Catalisano, A.V. Geramita, A. Gimigliano. **Higher secant varieties of Segre-Veronese varieties**. *Proceeding of the Conference "Varieties with Unexpected Properties", Siena, June 8–13, 2004 (2005).*