Contact structures with singularities

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Overview

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   - b-Contact geometry
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   - Geometry of Critical Set
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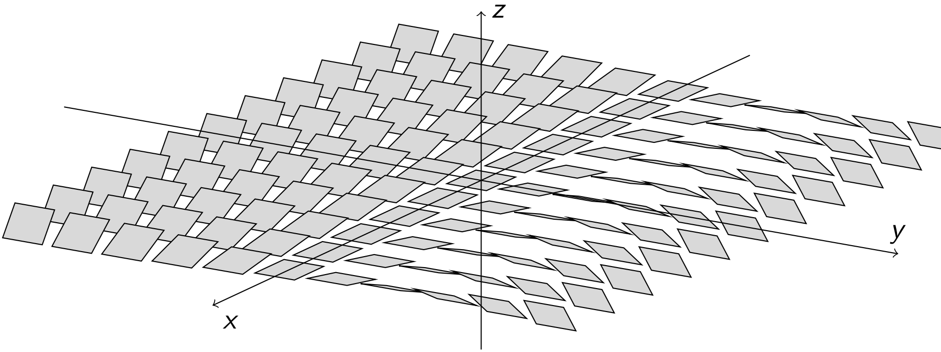
   - Existence of b^m-contact structures

Introduction

Contact

Definition

$(M^{2n+1}, \ker \alpha)$ where $\alpha \in \Omega^1(M)$ satisfies $\alpha \wedge (d\alpha)^n \neq 0$ is a contact manifold.



$$(\mathbb{R}^3, \ker(dz + xdy))$$

Definition

The Reeb vector field R_α defined by the equations

$$\begin{cases} i_{R_\alpha}\alpha = 1 \\ i_{R_\alpha}d\alpha = 0. \end{cases}$$

Theorem (Gray stability)

Let $\ker\alpha_t$, $t \in [0,1]$, *be a smooth family of contact structures on* M *compact. Then there exists a isotopy* ψ_t *such that* $\psi_t^*\alpha_t = \lambda_t\alpha_0$ *for* $\lambda_t : M \to \mathbb{R}^+$.

Theorem (Darboux theorem for contact manifolds)

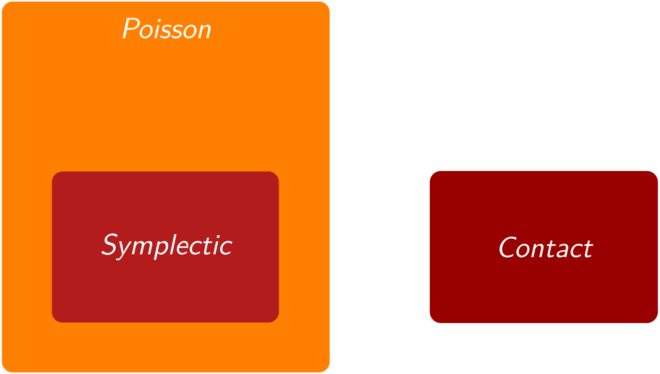
Let $(M^{2n+1}, \ker\alpha)$ *be a contact manifold and let* $p \in M$. *Then there exists an open neighbourhood* $\mathcal{U} \ni p$ *with coordinates* $x_1, \dots, x_n, y_1, \dots, y_n, z$ *such that* $\alpha|_{\mathcal{U}} = dz + \sum_{i=1}^n x_i dy_i$.

Symplectic

Contact

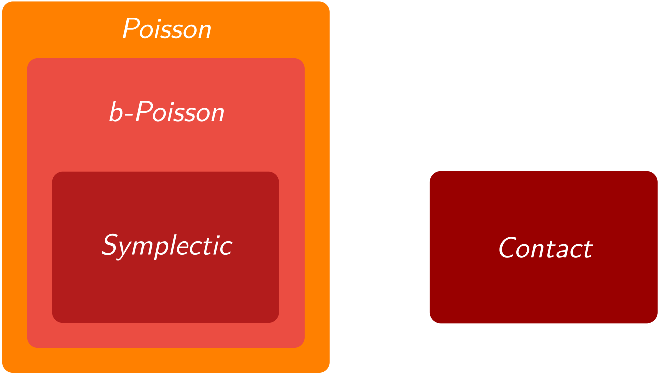
Definition

A symplectic manifold (M, ω) is a manifold equipped with a non-degenerate, closed 2-form.



Definition

A Poisson manifold (M, Π) is a manifold equipped with bi-vector field Π that satisfies $[\Pi, \Pi] = 0$.


Poisson
b-Poisson
Symplectic
Contact

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a *b*-**Poisson structure** on (M, Z).

2002: Radko classified *b*-Poisson surfaces.

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2012: Guillemin–Miranda–Pires: Local normal forms, ...

Theorem (Guillemin–Miranda–Pires)

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \dots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

Symplectic foliation

- On $M \setminus Z$: symplectic leaves
- On Z: codimension 2 symplectic leaves

Away from Z: $\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$.

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$$\Pi = {x_1}^{m} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

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Dual formulation

Assume $Z = f^{-1}(0)$.

$$\{\text{set of vector fields tangent to } Z\} = \langle f\frac{\partial}{\partial f}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\rangle$$

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<u>Serre–Swan</u>: Existence of a bundle having the *b*-vector fields as sections and denote it ${}^bTM$ and its dual ${}^bT^*M$.

$${}^b\Omega^k(M) = \Lambda^k({}^bT^*M)$$

$$\omega = \alpha \wedge \frac{df}{f} + \beta \text{ where } \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M).$$

$$d(\alpha \wedge \frac{df}{f} + \beta) := d\alpha \wedge \frac{df}{f} + d\beta.$$

Definition

(M^{2n}, Z, ω) is *b*-symplectic if it is equipped with $\omega \in {}^b\Omega^2(M)$ that is closed and everywhere of maximal rank as element of $\Lambda^2({}^bT^*M)$.

Examples:

- $(\mathbb{R}^{2n}, \frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i)$
- $(S^2, \frac{dh}{h} \wedge d\theta)$

Theorem (Guillemin–Miranda–Pires)

There is a one to one correspondance between b-symplectic and b-Poisson manifolds.

Definition

(M^{2n}, Z, ω) is b^m-symplectic if it is equipped with $\omega \in {}^{b^m}\Omega^2(M)$ that is closed and everywhere of maximal rank as element of $\Lambda^2({}^{b^m}T^*M)$.

Examples:

- $(\mathbb{R}^{2n}, \frac{dx_1}{x_1{}^m} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i)$
- $(S^2, \frac{dh}{h^m} \wedge d\theta)$

Theorem (Guillemin–Miranda–Pires)

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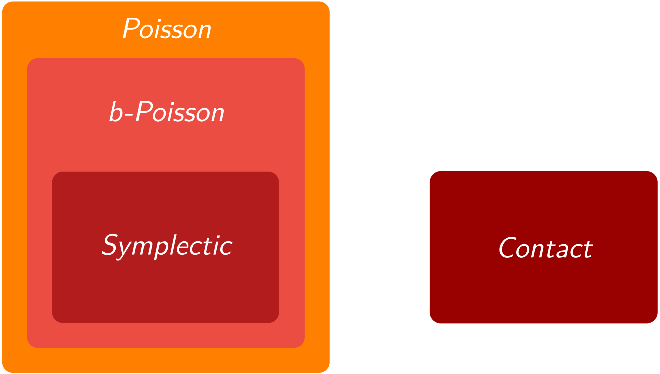
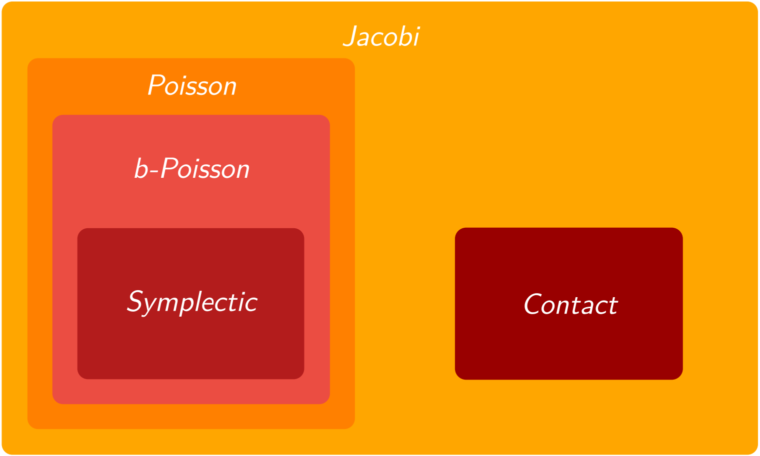
Singular contact manifolds


Poisson
b-Poisson
Symplectic
Contact


Jacobi
Poisson
b-Poisson
Symplectic
Contact

Jacobi manifolds

Definition

A Jacobi structure on a manifold M is a Lie algebra on $C^\infty(M)$ that is of local type, i.e. it is a bilinear, bidifferential operator satisfying Jacobi identity.

Jacobi manifolds

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Theorem (Lichnerowicz, Kirillov)

A Jacobi bracket is necessarily of the form

$$\{f, g\} = \Lambda(df, dg) + f(Rg) - g(Rf),$$

where $\Lambda \in \mathfrak{X}^2(M)$ *and* $R \in \mathfrak{X}(M)$ *satisfy*

- $[\Lambda, \Lambda] = 2R \wedge \Lambda,$
- $[\Lambda, R] = \mathcal{L}_R \Lambda = 0.$

Examples

- Poisson manifolds: $R = 0$.
- Contact manifolds $(M, \ker \alpha)$: R Reeb vector field, $\Lambda(df, dg) := d\alpha(X_f, X_g)$.
- Locally conformally symplectic (l.c.s.) manifolds (M, ω, α): $\Lambda(df, dg) := dg(\omega^\sharp df)$ and $R := \omega^\sharp \alpha$.

Remark

If (M, Λ, R) Jacobi then $(M \times \mathbb{R}, e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge R))$ is Poisson.

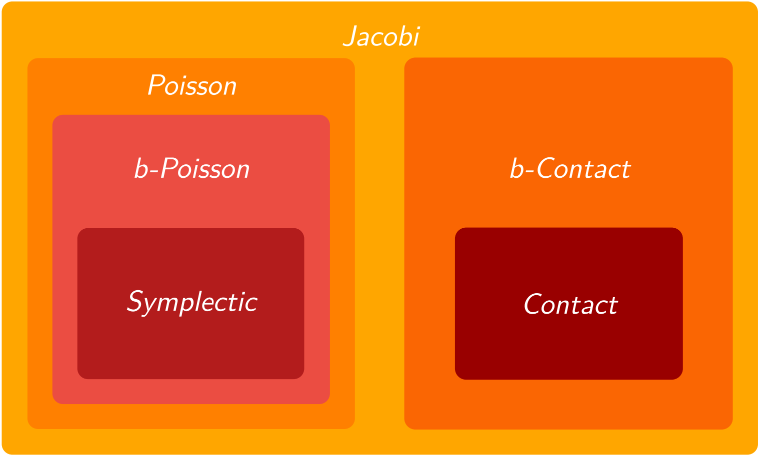
Characteristic leaves

Definition

The Hamiltonian vector fields are defined by $X_f := \Lambda^\sharp(df) + fR$.

$\mathfrak{F}(M) = \{X_f | f \in C^\infty(M)\} = \mathrm{Im}\Lambda^\sharp + \langle R \rangle$ is integrable.

- $R \in \mathrm{Im}\Lambda^\sharp$: even-dimensional leaves: l.c.s.
- $R \notin \mathrm{Im}\Lambda^\sharp$: odd-dimensional leaves: contact.



Definition

(M^{2n+1}, Z) is b-contact if there exists $\alpha \in {}^b\Omega^1(M)$ satisfying $\alpha \wedge (d\alpha)^n \neq 0$.

b-Jacobi

Definition

A Jacobi manifold (M^{2n+1}, Λ, R) is b-Jacobi if $\Lambda^n \wedge R \pitchfork 0$.

Theorem

There is a one to one correspondance between b-Jacobi and b-contact.

Examples

- $(\mathbb{R}^3, \ker(dx + y\frac{dz}{z})),\ R = \frac{\partial}{\partial x}$.
- $(\mathbb{R}^3, \ker(\frac{dz}{z} + xdy)),\ R = z\frac{\partial}{\partial z}$.

Remark: The rank of $\ker \alpha$ can change!

Theorem (b-Darboux Theorem)

Let (M, Z, α) b-contact, $z \in Z$. There exists a local chart $(\mathcal{U}, z, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $z = 0$ and

1. *if $R_p \neq 0$*
   1. *ξ_p is singular, then in $\mathcal{U}$*

      $$\alpha = dx_1 + y_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

   2. *ξ_p is regular, then in $\mathcal{U}$*

      $$\alpha = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

2. *if $R_p = 0$, then at p*

   $$\alpha = \frac{dz}{z} + \sum_{i=1}^{n} x_i dy_i.$$

Stability of b-contact structures

Theorem

Let (M, Z) compact b-manifold and let (ξ_t), $t \in [0,1]$ be a smooth path of b-contact structures. Then there exists an isotopy ϕ_t preserving the critical set Z such that $(\phi_t)_*\xi_0 = \xi_t$, or equivalently, $\phi_t^*\alpha_t = \lambda_t\alpha_0$ for a non-vanishing function λ_t.

Symplectization and Contactization

- Poissonization of a b-Jacobi manifold gives a b-Poisson manifold
- $(M, Z, \ker\alpha)$ b-contact $\implies$ $(M \times \mathbb{R}, Z \times \mathbb{R}, d(e^t\alpha))$ b-symplectic.
- (M, Z, ω) b-symplectic with Liouville vector field X and H hypersurface $\pitchfork X$. Then $(H, H \cap Z, i_X\omega)$ is b-contact.

Example

Take $(\mathbb{R}^4, \omega = \frac{dz}{z} \wedge dt + dx \wedge dy)$ and the Liouville vector field $X = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$.

$$i_X\omega = t\frac{dz}{z} + xdy$$

- $H_1 = \{(1, y, z, t) | y, z, t \in \mathbb{R}\} \pitchfork X$ and $i_X\omega|_{H_1} = dy + t\frac{dz}{z}$.
- $H_2 = \{(x, y, z, 1) | x, y, z \in \mathbb{R}\} \pitchfork X$ and $i_X\omega|_{H_2} = \frac{dz}{z} + xdy$.

Geometry of Critical Set

First local model:

- Contact leaf where the foliation is singular,
- Two l.c.s. leaves where the foliation is regular.

Second local model:
The induced structure is l.c.s.

Global results

Theorem

Let M^{2n+1} be an almost contact manifold. Then M admits a contact structure.

1. Gromov ('69) for M open
2. Martinet–Lutz ('71, '77) for $\dim M = 3$
3. Geiges, Thomas, Casals, Pancholi, Presas, Etnyre for $\dim M = 5$
4. Borman–Eliashberg–Murphy ('15) for all dimensions

Existence of b^m-contact structures

Question

Given a $(2n+1)$-manifold M with an embedded hypersurface Z does there exist a b^m-contact structure on (M, Z)?

Convex surfaces

Definition

A hypersurface $\Sigma \subset (M, \alpha)$ is convex if there exist a vector field X satisfying $\mathcal{L}_X \alpha = g\alpha$ that is transverse to Σ.

Theorem (Giroux)

In dimension 3*, all surfaces are* C^∞*-close to convex ones.*

Corollary (vertically invariant)

There is a tubular neighbourhood around $\Sigma = \{z = 0\}$ *such that*

$$\alpha = udz + \beta$$

where $\beta \in \Omega^1(Z)$ *and* $u \in C^\infty(Z)$.

Replace z by a function f_ϵ satisfying

- $f_\epsilon(x) = x$ for $x \in \mathbb{R} \setminus [-2\epsilon, 2\epsilon]$,
- $f_\epsilon(x) = -\frac{1}{x^{2m-1}}$ for $x \in [-\epsilon, 0[\cup]0, \epsilon]$,
- $f_\epsilon'(x) > 0$.

Then $\alpha_\epsilon := udf_\epsilon + \beta$ is a b^{2m}-contact form coinciding with α outside of an ϵ-neighbourhood of Z.

Theorem

Let $(M, \ker\alpha)$ *be a* $(2n+1)$*-dimensional contact manifold. For each convex hypersurface* Σ*, there exists a* b^{2k}*-contact structure realizing* Σ *as critical set.*

Corollary

Let M *be a closed 3-dimensional manifold and let* $\tilde{\Sigma} \subset M$ *be a surface. Then there exists a surface* Σ C^∞*-close to* $\tilde{\Sigma}$ *such that there exists a* b^{2k}*-contact structure on* (M, Σ).

Questions

- What about the case $m = 2k + 1$?
- Non-convex hypersurfaces?
- Is the almost contact condition a necessary condition for higher dimensions?

Thanks!