Generalized Riemann-Hilbert problem on elliptic curve in dimensions 1 and 2

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Outline of the talk

1. Problem statement
2. Overview of explicit solutions on the Riemann sphere
3. Generalization on an elliptic curve
4. One-dimensional case
5. Two-dimensional case

Classical Riemann-Hilbert problem 1

Consider on a disc $D = \{z \in \mathbb{C},\ |z| < 1\}$ linear differential equation

$$\frac{d}{dz}y(z) = \frac{\alpha}{z}y(z), \qquad z \in D, y \in \mathbb{C}\alpha \in \mathbb{C}$$

and $y(z) = z^\alpha$ a germ of solution in a neighborhood of some $z = z_0$. Under analytic continuation along the loop $\gamma_0 \in \pi_1(D \setminus 0, z_0)$ around point $z = 0$ it transforms into another germ $\widetilde{y}(z)$ solving the same equation:

$$y(z) = z^\alpha = e^{\alpha \ln z} \mapsto e^{\alpha(\ln z + 2\pi\imath)} = z^\alpha \cdot e^{2\pi\imath\alpha} = y(z) \cdot e^{2\pi\imath\alpha} = \widetilde{y}(z)$$

Multiplier $G_0 = e^{2\pi\imath\alpha}$ is called (local) monodromy and describes the action of fundamental group on solution space.

Classical Riemann-Hilbert problem 2

In a similar way for Fuchsian system on Riemann sphere

$$\frac{d}{dz}Y(z) = \sum_{i=1}^{n} \frac{B_i}{z-a_i} Y(z), \qquad z \in \mathbb{C}\backslash\{a_1, \dots, a_n\},\ y \in \mathbb{C}^p,\ B_i \in \mathrm{Mat}_{n\times n}(\mathbb{C})$$

correspondence $\gamma_i \to G_i$ gives rise to *monodromy map*:

$$\chi : \pi_1(\mathbb{C} \setminus \{a_1, \dots, a_n\}, z_0) \longrightarrow \mathrm{GL}_p(\mathbb{C})$$

Dependence on choice of initial germ and point z_0 can be neglected. The image of χ is called monodromy group of an equation. In coordinates it can be described as n-tuple $(G_1, \dots, G_n)$ defined up to an overall conjugation on constant non-degenerate $C \in \mathrm{GL}_p(\mathbb{C})$.

Classical Riemann-Hilbert problem 3

Riemann-Hilbert problem

Given n points $a_1, ..., a_n$ on Riemann sphere $\overline{\mathbb{C}}$ and representation χ of fundamental group $\pi_1(\mathbb{C} \setminus \{a_1, ..., a_n\})$ to construct a Fuchsian system having singular points $a_1, ..., a_n$ and monodromy χ.

- Solved in negative by A. Bolibrukh in 1989
- Solved in positive for irreducible representations by A. Bolibrukh and V. Kostov independently in 1992.
- Different sufficient conditions of solvability are known.

Generalization

Now we want to reformulate the problem so that it will make sense for all Riemann surfaces. We say that functions $y(z)$ on sphere are sections $\varphi(z)$ in trivial bundle and instead of initial equation one can consider

$$d\varphi = \Omega\varphi,$$

where Ω is connection in the same bundle.
For arbitrary Riemann surface due to combinatorial reasons, we need the bundle to be not trivial but semistable, i.e. $\frac{\deg D}{\mathrm{rk} D} \leq \frac{\deg E}{\mathrm{rk} E} \quad \forall D \subset E$.

For the set of singular points $\{a_1, \dots, a_n\}$ and monodromy representation

$$\chi : \pi_1 (X \setminus \{a_1, \dots, a_n\}) \to \mathrm{GL}_p(\mathbb{C})$$

of punctured Riemann surface X, to construct logarithmic connection in semistable bundle of degree zero with given singularities and monodromy representation.

One-dimensional (commutative) case on sphere

For given set of singularities $\{a_i\}$ and corresponding monodromies $\{g_i\}$ we want to construct system of the following form:

$$dy(z) = \sum_{i=1}^{n} \frac{b_i}{z - a_i} y(z).$$

Locally this system has monodromies $exp(2\pi\imath b_i)$ so we can set $b_i = \frac{1}{2\pi\imath} log(g_i)$.

Let us take

$$y(z) = \prod_{i=1}^{n} (z - a_i)^{\frac{1}{2\pi\imath} \ln g_i}, \qquad \mathrm{Im}\,(\ln g_i) \in [0, 2\pi\imath).$$

$$y(z) \xrightarrow{a_i} y(z) g_i$$

One-dimensional case. Global theory

From the topological condition $g_1 \dots g_n = 1$ it follows that

$$\frac{1}{2\pi\imath}(\ln g_1 + \dots + \ln g_n) = k \qquad k \in \mathbb{Z}.$$

$$y(z) \xrightarrow{z\to\infty} z^{\left(\sum_{i=1}^{n} \frac{1}{2\pi\imath} \ln g_i\right)} = z^k$$

$\widetilde{y}(z) = \frac{y(z)}{(z-a_1)^k}$ has the same singularities and ramification as $y(z)$ on $\mathbb{C}$ and is holomorphic at infinity.

$$\frac{d\widetilde{y}(z)}{dz} = \left(\frac{\frac{1}{2\pi\imath}\ln g_1}{z-a_1} + \dots + \frac{\frac{1}{2\pi\imath}\ln g_n}{z-a_n} - \frac{k}{z-a_1} \right) \widetilde{y}$$

Two-dimensional case

For the Fuchsian system

$$\frac{d}{dz}y(z) = \sum_{i=1}^{3} \frac{B_i}{z - a_i} y(z), \qquad z \in \mathbb{C} \backslash \{a_1, a_2, a_3\},\ y \in \mathbb{C}^2,\ B_i \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

Local monodromy G_i is as usual conjugated to $exp(2\pi \imath B_i)$. In general, it is not possible to reconstruct global monodromy from the set of local ones.

For the irreducible representation, using the condition that $G_1 \cdot G_2 \cdot G_3 = 1$, one can uniquely reconstruct global monodromy from the local ones.

Statement of generalization

For the set of singular points $\{a_1, \dots, a_n\}$ and monodromy representation

$$\chi : \pi_1 \left(\Lambda_\tau \setminus \{a_1, \dots, a_n\} \right) \to \mathrm{GL}_p(\mathbb{C})$$

to construct logarithmic connection in semistable bundle of degree zero on the elliptic curve with given singular points and monodromy representation.

Instead of z as holomorphic function on sphere we will be using first Riemann theta-function

$$\theta(z) = \theta_1(z|\tau) = \imath \sum_{m \in \mathbb{Z}} (-1)^m q^{(m-\frac{1}{2})^2} e^{(m-\frac{1}{2})2\pi\imath z},$$

where $q(\tau) = e^{\imath\pi\tau} = e^{\imath\pi x - \pi y}$ defines map from upper half plane $H = \{\tau \in \mathbb{C} | \mathrm{Im}\tau > 0\}$ into the unit circle $D = \{q \in \mathbb{C} \,|\, |q| < 1\}$.

Properties of theta-function

Ramification of $\theta(z)$ and its derivative.

$$\theta(z+1) = -\theta(z)$$
$$\theta(z+\tau) = -q^{-1}e^{-2\pi iz}\theta(z)$$

$$\theta'(z+1) = -\theta'(z)$$
$$\theta'(z+\tau) = q^{-1}e^{-2\pi iz}\left(2\pi i\theta(z) - \theta'(z)\right)$$

Therefore,

$$\frac{\theta'(z+1)}{\theta(z+1)} = \frac{\theta'(z)}{\theta(z)}$$
$$\frac{\theta'(z+\tau)}{\theta(z+\tau)} = \frac{\theta'(z)}{\theta(z)} - 2\pi i.$$

We also need expressions for the shifted theta functions. Let a be an arbitary point on Λ_τ.

$$\theta(z-a+1) = -\theta(z-a)$$
$$\theta(z-a+\tau) = -q^{-1}e^{-2\pi \imath z}\theta(z-a)e^{2\pi \imath a}$$

Sections of one-dimensional vector bundles on Λ_τ

Instead of 2 maps on sphere, on elliptic curve it is convenient to consider 1 map and 2 shifts by 1 and by τ.

$$\#poles = \#zeroes$$

$$\varphi(z) \xrightarrow{a-cycle} \varphi(z) = \varphi(z+1)$$
$$\varphi(z) \xrightarrow{b-cycle} \varphi(z) \cdot e^{2\pi\imath\lambda} = \varphi(z+\tau)$$

$$\varphi(z) = \prod_{i=1}^{n} \theta^{k_i}(z-a_i), \quad k \in \mathbb{Z}, \quad \sum_{i=1}^{n} k_i = 0.$$

$$\varphi(z+1) = (-1)^{\sum k_i}\varphi(z) = \varphi(z)$$
$$\varphi(z+\tau) = \varphi(z) \cdot e^{2\pi\imath \sum k_i a_i}$$

$$\sum_{i=1}^{n} k_i a_i = \lambda.$$

Connections of one-dimensional vector bundles on Λ_τ

$$d\varphi(z) = \omega_\lambda(z)\varphi(z)$$

From $d\theta^{\alpha_i}(z-a_i) = \alpha_i\theta'(z-a_i)\theta^{\alpha_i-1}(z-a_i)dz$ we obtain

$$d\varphi = \sum_{i=1}^{n}\alpha_i\frac{\theta'(z-a_i)}{\theta(z-a_i)}\varphi dz, \qquad \omega_\lambda(z) = \sum_{i=1}^{n}\alpha_i\frac{\theta'(z-a_i)}{\theta(z-a_i)}dz$$

$\omega(z)$ has logarithmic singularities in the points a_i. Let us check how it changes under the shifts by 1 and τ.

$$\omega_\lambda(z+1) = \sum_{i=1}^{n}\alpha_i\frac{\theta'(z-a_i+1)}{\theta(z-a_i+1)}dz = \sum_{i=1}^{n}\alpha_i\frac{\theta'(z-a_i)}{\theta(z-a_i)}dz = \omega_\lambda$$

$$\omega_\lambda(z+\tau) = \sum_{i=1}^{n}\alpha_i\frac{\theta'(z-a_i+\tau)}{\theta(z-a_i+\tau)}dz = \sum_{i=1}^{n}\alpha_i\frac{\theta'(z-a_i)}{\theta(z-a_i)}dz - 2\pi\imath\sum_{i=1}^{n}\alpha_i = \omega_\lambda$$

Explicit solution

Theorem

For given elliptic curve Λ_τ, singular points $\{a_1, \ldots, a_n\}$ and monodromy data $g_1, \ldots, g_n, \lambda$ one-dimensional Riemann problem is positively solvable in trivial bundle if and only if $\lambda = \sum_{k=1}^{n} \alpha_k a_k + p + q\tau$ for some integer p, q and normalized set $\alpha_1, \cdots, \alpha_n$, where $e^{2\pi\imath\alpha_k} = g_k$.
The corresponding connection form in the bundle has the form

$$\omega_\lambda(z) = \sum_{k=1}^{n} \alpha_k \frac{\theta'(z - a_k)}{\theta(z - a_k)} dz$$

In general case, solution can be given by the same formula in the bundle $\mathcal{O}_{\sum_{k=1}^{n} \alpha_k a_k - \lambda}(0)$ and there are no other solutions.

Local theory

Consider $\{a_1, \dots, a_n\} \in \Lambda_\tau$, $a_i \neq a_j$ and complex $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, \dots, n$ such that

$$\sum_{i=1}^{n} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta \end{pmatrix} = 0$$

Then matrix 1-form

$$\Omega(z) = \sum_{i=1}^{n} \frac{\begin{pmatrix} \alpha_i \theta'(z-a_i) & \beta_i \frac{\theta'(0)}{\theta(-2\lambda)} \theta(z-a_i-2\lambda) \\ \gamma_i \frac{\theta'(0)}{\theta(2\lambda)} \theta(z-a_i+2\lambda) & -\delta_i \theta'(z-a_i) \end{pmatrix}}{\theta(z-a_i)} dz,$$

defines a logarithmic connection on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ with residues

$$\operatorname*{Res}_{z=a_i} \Omega(z) = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

Global theory

Theorem

Consider an irreducible representation

$$\chi_0 : \pi_1\left(\mathbb{C}\mathrm{P}^1 \setminus \{d_1, d_2, d_3\}\right) \to \mathrm{SL}(2, \mathbb{C}).$$

The Riemann problem for χ_0 can be solved explicitly, consider (B_1, B_2, B_3) any triple of residues giving the solution on sphere.
Then 1-form $\widetilde{\Omega}(z)$ constructed above with the use of triple (B_1, B_2, B_3) and arbitrary parameter λ defines a logarithmic connection $\widetilde{\nabla} = d - \widetilde{\Omega}(z)$ in semistable vector bundle $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ with singular points $\{a_1, a_2, a_3\}$ and monodromy representation

$$\chi : \pi_1\left(\Lambda_\tau \setminus \{a_1, a_2, a_3\}\right) \to \mathrm{SL}(2, \mathbb{C}),$$

such that $\chi_{\mathrm{ind}} = \chi_0, \;\; \chi(\gamma_a) = 1$ *and* $\chi(\gamma_b) \sim \exp\left(2\pi i \int_0^\tau \widetilde{\Omega}(z)\right).$

Thank you for your attention!