Resolving some surface singularities with weighted blow-ups

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Introduction and Motivation

Resolution Graph of $f = x^2z + y^3 + z^6$

80 charts and 15 exceptional divisors

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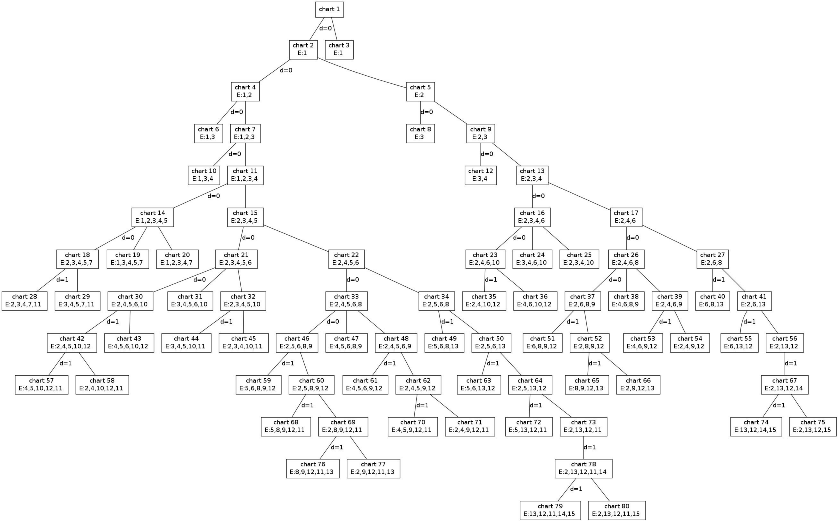


Yomdin Singularities

Definition.

- Let $f = f_m + f_{m+k} + \cdots \in \mathbb{C}\{x, y, z\}$ be the decomposition of f into its homogeneous parts, $k \geq 1$.
- Denote by $\mathbf{C} := V(f_m) \subset \mathbb{P}^2$ its tangent cone.
- $V := V(f)$ is Yomdin-Lê if $\mathrm{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ in $\mathbb{P}^2$.

Example: $f = \underbrace{x^2z + y^3}_{f_3} + \underbrace{z^6}_{f_6} \in \mathbb{C}\{x, y, z\}$.

- $\mathrm{Sing}_{\mathbb{P}^2}(f_3) = \{[0 : 0 : 1]\}$.
- $V_{\mathbb{P}^2}(f_6) = \{z = 0\}$.
- $\mathrm{Sing}_{\mathbb{P}^2}(f_3) \cap V_{\mathbb{P}^2}(f_6) = \emptyset$.

[Luengo]

I. Luengo, The μ-constant stratum is not smooth, Invent. Math. **90** (1987), 139–152.

Superisolated Singularities $(k = 1)$ [Artal]

He was able to calculate the Jordan form of the monodromy.

How to compute it?

1. Resolution of singularities.
2. Eigenvalues: A'Campo's formula.
3. Jordan blocks: Steenbrink's spectral sequence.

Application: Counterexample to Yau's conjecture.

Find two superisolated surface singularities having:

1. The same characteristic polynomials.
2. The same abstract topologies.
3. Different embedded topologies $\rightsquigarrow$ Monodromy.

Remark

- No embedded resolution is known for Yomdin singularities for $k \geq 2$.
- There exists a special kind of toric embedded resolution for f with just two exceptional divisors E_0 and E_1.
- By contrast, the final total space produced has abelian quotient singularities. Therefore, A'Campo and Steenbrink's approach can not be applied directly.

Monodromy

Milnor Fibration

- Let $(H,0) \subset (\mathbb{C}^{n+1},0)$ be a germ of hypersurface singularity defined by a holomorphic function $f : U \to \mathbb{C}$. This means that

$$H = \{\mathbf{x} \in U \mid f(\mathbf{x}) = 0\}.$$

- By Ehresmann's Fibration Theorem, the restriction $(0 < \eta << \epsilon)$

$$f| : f^{-1}(\overline{D_\eta} \setminus \{0\}) \cap \overline{B_\epsilon^{2n+2}} \longrightarrow \overline{D_\eta} \setminus \{0\}$$

is a locally trivial fibration for all ϵ small enough.

Definition

The previous fibration is called the *Milnor fibration* of f and any of its fibers $F := \{\mathbf{x} \in \mathbb{C}^{n+1} : ||\mathbf{x}|| \leq \epsilon, f(\mathbf{x}) = \eta\}$ is called the *Milnor fiber*.

Monodromy Zeta Function

- Let $h : F \to F$, $\widetilde{\alpha}(0) \mapsto \widetilde{\alpha}(1)$ be the geometric monodromy. It is just well defined up to homotopy.
- Denoted by $\varphi := H^q(h) : H^q(F, \mathbb{C}) \to H^q(F, \mathbb{C})$ the induced automorphisms on the complex cohomology groups.

Definition (*Monodromy Zeta Function*)

$$Z(f; t) := \prod_{q \geq 0} \underbrace{\det(\mathrm{id}^* - t \cdot H^q(h))}_{\text{char. poly. of } H^q(h)}{}^{(-1)^q} \in Q(\mathbb{Q}[t])$$

Embedded Resolution

Definition

An *embedded resolution* of $(H,0) \subset (\mathbb{C}^{n+1},0)$ is a proper analytic map $\pi : X \to (\mathbb{C}^{n+1},0)$ such that:

1. X is a (smooth) manifold.
2. π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with normal crossings on X.

Normal Crossing Divisor

The third condition above means that $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by an equation of the form $x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} = 0$.

A'Campo's Formula

- Let $\pi : X \to (\mathbb{C}^{n+1}, 0)$ be an embedded resolution of $(H, 0)$.
- Total transform: $\pi^*(H) = \underbrace{\widehat{H}}_{\text{strict transform}} + \underbrace{\sum_{i=1}^{r} m_i E_i}_{\text{exceptional divisor}}$.
- Now, define

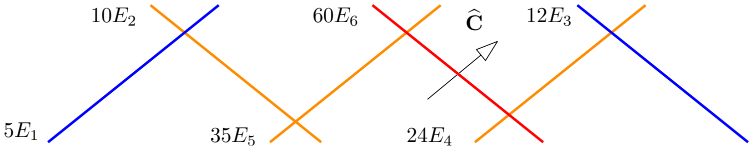
$$\check{E}_i := E_i \setminus \left(E_i \cap \left(\bigcup_{j \neq i} E_j \cup \widehat{H} \right) \right).$$

Theorem ([A'Campo])

$$Z(f; t) = \prod_{i=1}^{r} (1 - t^{m_i})^{\chi(\check{E}_i)}.$$

Example of a Plane Curve

Let us compute the monodromy zeta function of $f = x^5 + y^{12}$ using A'Campo's formula. The following picture represents an embedded resolution:



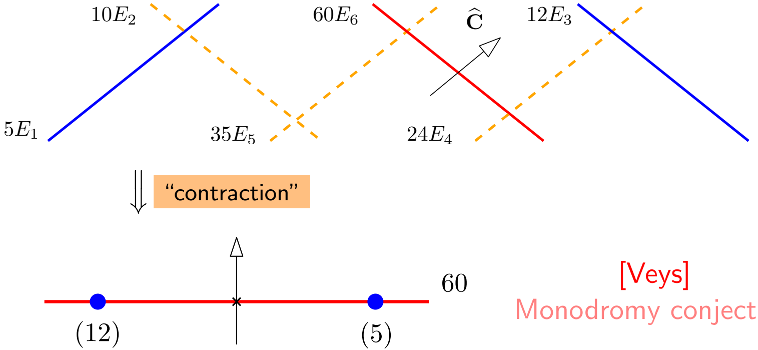
- $E_i \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2 \implies \chi(E_i) = 2$
- $\chi(\check{E}_2) = \chi(\check{E}_5) = \chi(\check{E}_4) = 2 - 2 = 0$
- $\chi(\check{E}_1) = \chi(\check{E}_3) = 1$
- $\chi(\check{E}_6) = -1$

$$Z(f;t) = \prod_{i=1}^{r} (1 - t^{m_i})^{\chi(\check{E}_i)}$$

This implies:

$$Z(f;t) = (1-t^{60})^{-1}(1-t^5)(1-t^{12}) = \frac{(1-t^5)(1-t^{12})}{(1-t^{60})}.$$

The following picture provides the same information:



[Veys]
Monodromy conjecture

The Main Aim

Questions:

1. How to formalize this idea of "contraction" of the exceptional divisors which do not contribute to the monodromy zeta function?
2. Can one directly compute the simplified resolution (without computing the standard one and then perform the contractions)?
3. Does there exist a formula for calculating $Z(f;t)$ using this new kind of resolutions even in higher dimension?

Embedded $\mathbf{Q}$-Resolutions

Embedded $\mathbf{Q}$-Resolution

Definition

An *embedded* $\mathbf{Q}$*-resolution* of $(H,0) \subset (\mathbb{C}^{n+1},0)$ is a proper analytic map $\pi : X \to (\mathbb{C}^{n+1},0)$ such that:

1. X is a V-manifold with abelian quotient singularities.
2. π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

$\mathbb{Q}$-Normal Crossing Divisor ([Steenbrink])

The third condition above means that $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by function of the form $x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$.

- $X(\mathbf{d}; A) := \mathbb{C}^{n+1}/\mu_{\mathbf{d}}$, $G \subset GL(n+1,\mathbb{C})$ abelian group acting diagonally

Classical Blow-up of $\mathbb{C}^2$

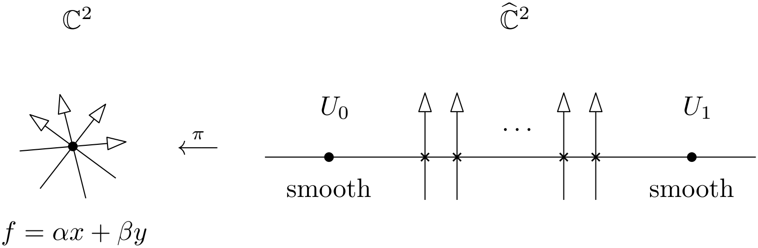
- Consider

$$\widehat{\mathbb{C}}^2 := \{((x,y),[u:v]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid (x,y) \in \overline{[u:v]}\}.$$

- Then $\pi : \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$ is an isomorphism over $\widehat{\mathbb{C}}^2 \setminus \pi^{-1}(0)$.
- The *exceptional divisor* $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^1$.
- The space $\widehat{\mathbb{C}}^2 = U_1 \cup U_2$ can be covered by 2 charts each of them isomorphic to $\mathbb{C}^2$.

$$\begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{\simeq} & U_1 = \{u \neq 0\} & \subset & \widehat{\mathbb{C}}^2 \\ (x,y) & \mapsto & \big((x,xy),[1:y]\big). & & \end{array}$$

Behavior of $\pi : \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$



Weighted (p,q)-Blow-ups of $\mathbb{C}^2$

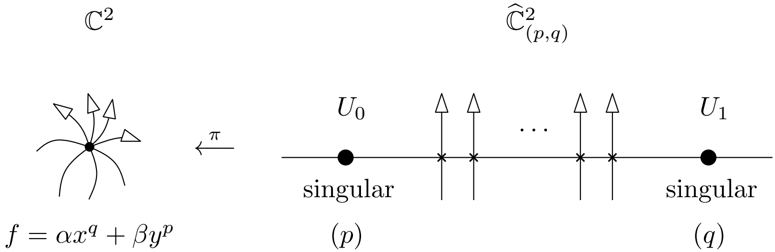
- Let $\omega = (p,q)$ be a weight vector with coprime entries. As above, consider the space

$$\widehat{\mathbb{C}}^2_\omega := \left\{ ((x,y),[u:v]_\omega) \in \mathbb{C}^2 \times \mathbb{P}^1_\omega \mid (x,y) \in \overline{[u:v]_\omega} \right\}.$$

- Then $\pi : \widehat{\mathbb{C}}^2_\omega \to \mathbb{C}^2$ is an isomorphism over $\widehat{\mathbb{C}}^2_\omega \setminus \pi^{-1}(0)$.
- The *exceptional divisor* $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^1_\omega$.
- The space $\widehat{\mathbb{C}}^2_\omega = U_1 \cup U_2$ can be covered by 2 charts. For instance, the first chart is given by

$$\text{1st chart} \left| \begin{array}{ccc} X(p;-1,q) & \xrightarrow{\simeq} & U_1 = \{u \neq 0\} \subset \widehat{\mathbb{C}}^2_\omega, \\ [(x,y)] & \mapsto & ((x^p, x^q y), [1:y]_\omega). \end{array} \right.$$

Behavior of $\pi_{(p,q)} : \widehat{\mathbb{C}}^2_{(p,q)} \to \mathbb{C}^2$



Generalized A'Campo's formula

Generalized A'Campo's Formula

Teorema (Cyclic case)

Let $X_0 = \pi^{-1}(H)$ be the total transform and $S = \pi^{-1}(0)$ the exceptional divisor. Consider $S_{m,d}$ to be the set

$$\left\{ s \in S \;\middle|\; \begin{array}{l} \text{the local equation of } X_0 \text{ in } s \text{ is given by the well-defined} \\ \text{function } x_i^m : X(d; a_0, \ldots, a_n) \to \mathbb{C}. \end{array} \right\}.$$

Then, the monodromy zeta function of the complex monodromy of the hypersurface $(H, 0)$ is

$$Z(f;t) = \prod_{m,d} (1 - t^{m/d})^{\chi(S_{m,d})}.$$

Comments about Proof of the Theorem

Theorem ([Dimca])

1. Assume $\pi : X \to U$ is a proper analytic map such that π induces an isomorphism between $X \setminus \pi^{-1}(H)$ and $U \setminus H$.
2. Let $g = f \circ \pi$ denote the composition and $j : X \setminus \pi^{-1}(H) \hookrightarrow X$ the inclusion.
3. Let $\mathcal{S}$ be a finite stratification of the exceptional divisor $\pi^{-1}(0)$ such that $\psi_g\left(Rj_*\mathbb{C}_{X\setminus\pi^{-1}(H)}\right)$ is equivariantly $\mathcal{S}$-constructible with respect to the semisimple part of M.

Then,

$$Z(f) = \prod_{S\in\mathcal{S}} Z(g, x_S)^{\chi(S)},$$

where x_S is an arbitrary point in the stratum S and $Z(g, x_S)$ is the zeta function of the germ g at x_S.

Monodromy Zeta Function of a $\mathbb{Q}$-Normal Crossing Divisor

Lemma (Cyclic case)

The monodromy zeta function of a normal crossing divisor given by $x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(d; a_0, \ldots, a_n) \to \mathbb{C}$, $k \geq 1$, is

$$Z\left(x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(d; a_0, \ldots, a_n) \to \mathbb{C};\ t\right) = \begin{cases} 1 - t^{\frac{m_1}{d}} & k = 1; \\ 1 & k \geq 2, \end{cases}$$

Monodromy Zeta Function of a $\mathbb{Q}$-Normal Crossing Divisor

Lemma (Cyclic case)

The monodromy zeta function of a normal crossing divisor given by $x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} : X(d; a_0, \ldots, a_n) \to \mathbb{C}$, $k \geq 1$, is

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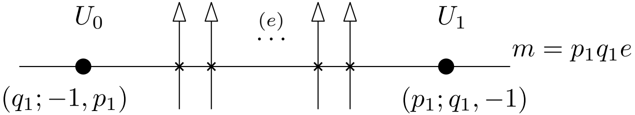
Non-cyclic case

$$d \quad \rightsquigarrow \quad \ell := \operatorname{lcm}\left(\frac{d_1}{\gcd(d_1, a_{11})}, \ldots, \frac{d_r}{\gcd(d_r, a_{r1})}\right)$$

Example 1

Let $f = x^p + y^q$ and assume that $e = \gcd(p, q)$, $p = p_1 e$ and $q = q_1 e$. Consider $\pi : \widehat{\mathbb{C}}^2_{(q_1,p_1)} \to \mathbb{C}^2$ the weighted blow-up at the origin.



The set $S_{m,d}$ is not empty for $(m, d) = (p_1 q_1 e, 1)$, $(p_1 q_1 e, q_1)$, $(p_1 q_1 e, p_1)$. Their Euler characteristics are

$$\chi(S_{p_1 q_1 e,1}) = 2 - (e + 2) = -e, \qquad \chi(S_{p_1 q_1 e,q_1}) = \chi(S_{p_1 q_1 e,p_1}) = 1.$$

Now, we apply A'Campo's formula and obtain $Z(f; t) = \frac{(1-t^p)(1-t^q)}{(1-t^{\frac{pq}{e}})^e}$.

Example 2

Assume $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ are two irreducible fractions and $\gcd(q_1, q_2) = 1$. Let **C** be the complex plane curve with Puiseux expansion

$$y = x^{\frac{p_1}{q_1}} + x^{\frac{p_2}{q_2}}.$$

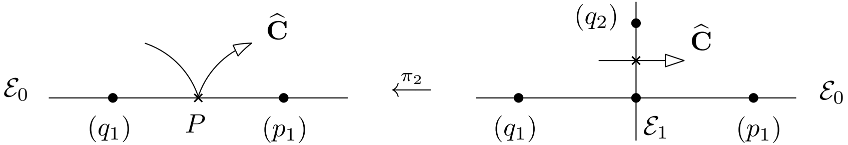
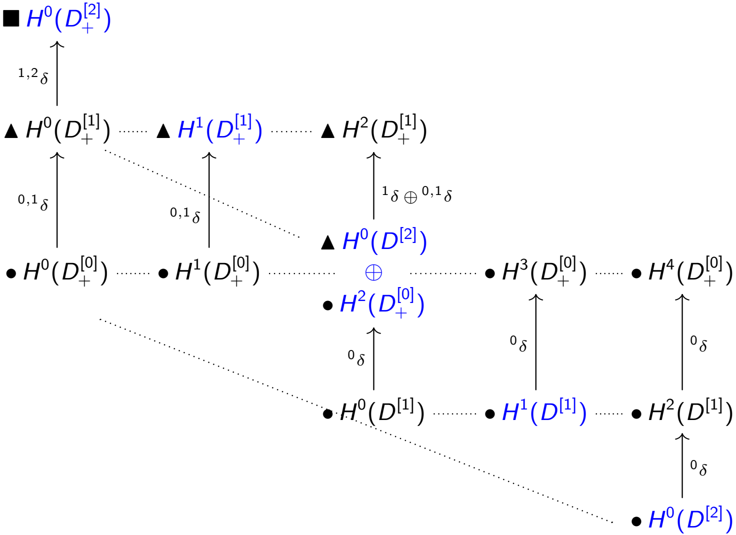


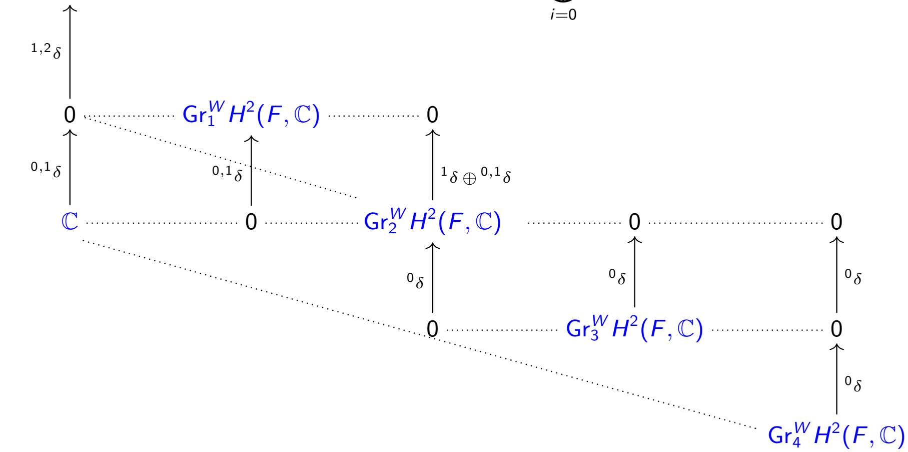
Figure: Embedded **Q**-resolution of $\mathbf{C} = \{y = x^{\frac{p_1}{q_1}} + x^{\frac{p_2}{q_2}}\} \subset \mathbb{C}^2$.

Spectral Sequence for Surfaces



Steenbrink's Spectral Sequence for Surfaces

$$H^2(F,\mathbb{C}) = \bigoplus_{i=0}^{4} \mathrm{Gr}_i^W H^2(F,\mathbb{C})$$



Characteristic Polynomial

$$\Delta_{(V,0)}(t) = \frac{(t^m-1)^{\chi(\mathbb{P}^2\setminus\mathbf{C})}}{t-1}\prod_{P\in\mathrm{Sing}(\mathbf{C})}\Delta^k_{(\mathbf{C},P)}(t^{m+k})$$

- $\chi(\mathbb{P}^2\setminus\mathbf{C}) = (m^2-3m+3) - \sum_{P\in\mathrm{Sing}(\mathbf{C})}\mu_{(\mathbf{C},P)}$
- $\Delta_{(\mathbf{C},P)}(t)$ denotes the characteristic polynomial of $(\mathbf{C},P)$
- $\Delta(t) = \prod_i (t^{m_i}-1)^{a_i} \implies \Delta^k(t) = \prod_i\left(t^{\frac{m_i}{\gcd(m_i,k)}}-1\right)^{\gcd(m_i,k)a_i}$

Jordan Blocks

3-Jordan Blocks (for $\lambda \neq 1$)

$$\Delta_{\mathrm{Gr}_0^W H}(t) = \prod_{P \in \mathrm{Sing}(\mathbf{C})} \Delta^{(m)}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$$

- $\Delta_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$ encodes the Jordan blocks of size 2 of $(\mathbf{C}, P)$.
- $\Delta(t) = \prod_i \left(t^{m_i} - 1\right)^{a_i} \implies \Delta^{(\ell)}(t) = \prod_i \left(t^{\gcd(\ell,m_i)} - 1\right)^{a_i}$

Remark.

The 3-Jordan blocks does not depend on $k \geq 1$.

Jordan Blocks

2-Jordan Blocks for $\lambda \neq 1$

$$\Delta_{\mathrm{Gr}_1^W H}(t) = \frac{1}{\Delta_{H^1(D_0)}(t)} \prod_{P \in \mathrm{Sing}(\mathbf{C})} \frac{\widetilde{\Delta}^{(m)}_{(\mathbf{C},P)}(t) \cdot \Delta^k_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t^{m+k})}{\Delta^{(m)}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)^3 \cdot (t-1)^{b_1(N\Gamma_+^P(k))}}$$

- $\Delta^{\ell}(t)$ and $\Delta^{(\ell)}(t)$ were defined previously.
- $\widetilde{\Delta}^{(\ell)}$ is the polynomial resulting from $\Delta^{(\ell)}$ deleting the factor $t-1$.
- The action of the monodromy on $H^1(D_0)$ and the cohomology itself are completely determined by the pair $(\mathbb{P}^2, \mathbf{C})$.

Jordan Blocks

2-Jordan Blocks for $\lambda = 1$

$$\sum_{P \in \mathrm{Sing}(\mathbf{C})} (r_P - 1) - (r - 1) + \sum_{P \in \mathrm{Sing}(\mathbf{C})} b_1(N\Gamma_+^P(k))$$

- r_P is the number of local branches of the germ $(\mathbf{C}, P)$
- r is the number of irreducible components of $\mathbf{C} \subset \mathbb{P}^2$
- $N\Gamma_+^P(k)$ is the dual graph of the semistable reduction of $(\mathbf{C}, P)$ modified according to k.

Remark.

Note that for $k = 1$ the graph $N\Gamma_+^P(k) = \Gamma_+^P$ is contractible, thus its first Betti number is zero, and one exactly obtains the description in [Artal].

MOLTES GRÀCIES !

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