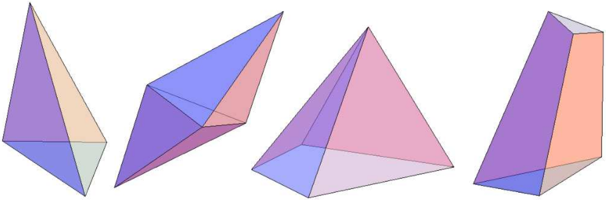
Barcelona - January 26, 2018



Slack ideals of Polytopes

[arXiv:1708.04739]

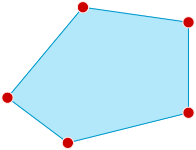
Antonio Macchia

joint work with **João Gouveia** (U Coimbra)
Rekha Thomas, **Amy Wiebe** (U Washington)

Polytopes

A ***polytope*** P is a convex hull of a finite set of points in some $\mathbb{R}^d$:

$$P = \operatorname{conv}(p_1, \ldots, p_v)$$



$\mathcal{V}$-representation

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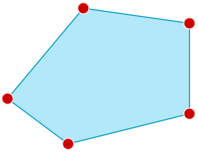
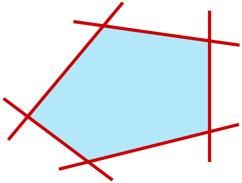


$\mathcal{V}$-representation



$\mathcal{H}$-representation

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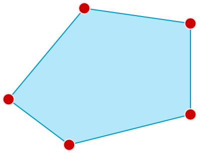
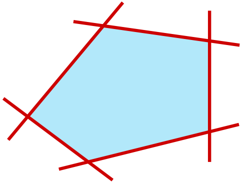


$\mathcal{V}$***-representation***



$\mathcal{H}$***-representation***

The ***dimension*** of a polytope is the dimension of its affine hull.

The ***face lattice*** of P is the set of all faces of P, ordered by inclusion.

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Find all polytopes of a fixed dimension:

- combinatorial types of polytopes $\leftrightarrow$ finite lattices corresponding to face lattices of polytopes,
- describe the set of all realizations of a given combinatorial type (realization space).

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Theorem (Steinitz, 1922) *A graph G is the edge graph of a 3-polytope $\Leftrightarrow$ G is simple, planar and 3-connected.*

Realization Space

Given: combinatorial type of a d-polytope

$\rightarrow$ face lattice or vertex-facet incidences

Want: all ways to realize this type in $\mathbb{R}^d$

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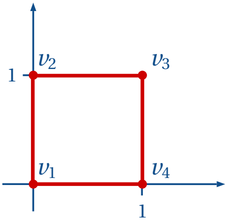

P is a quadrilateral
4 vertices $\{v_1, v_2, v_3, v_4\}$
4 facets $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}$

Combinatorial equivalence

$$Q \overset{c}{=} P \Leftrightarrow P \text{ and } Q \text{ have the same vertex-facet incidences}$$

All quadrilaterals are combinatorially equivalent to a square

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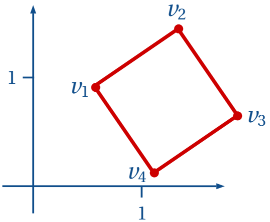

Affine equivalence

$$Q \overset{a}{=} P \Leftrightarrow Q = \psi(P), \;\; \psi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$$

preserves parallel lines, e.g. scaling, rotation, reflection, translation

Parallelograms are affinely equivalent to a square

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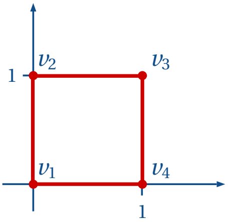


Projective equivalence

$$Q \overset{p}{=} P \Leftrightarrow Q = \phi(P), \;\; \phi(\boldsymbol{x}) = \frac{A\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^\intercal + d}, \;\; \det \begin{bmatrix} A & \boldsymbol{b} \\ \boldsymbol{c}^\intercal & d \end{bmatrix} \neq 0$$

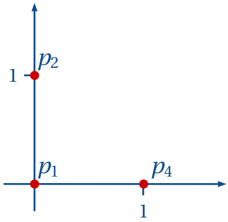
All convex quadrilaterals are projectively equivalent to a square.
A square is ***projectively unique***.

Realization Space

Set of all realizations of polytopes combinatorially equivalent to P

Mod out affine transformation:

fix an affine basis B of $d+1$ common vertices



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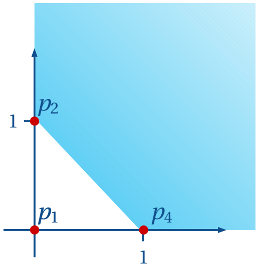

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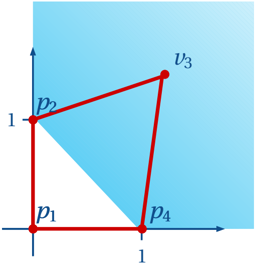

$$v_3 \in \left\{ (x, y) : \begin{array}{l} x > 0, y > 0 \\ x + y < 1 \end{array} \right\}$$

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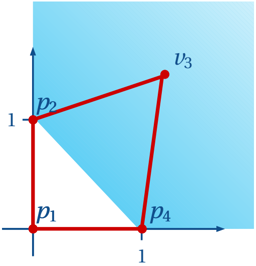

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Realization space

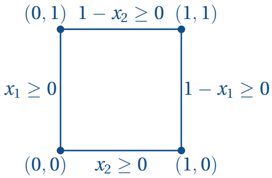
$$\mathcal{R}(P,B) = \left\{ Q = \operatorname{conv}\{q_1,\ldots,q_n\} \subset \mathbb{R}^d : q_i = p_i \; \forall p_i \in B, \; Q \overset{c}{=} P \right\}$$

Main Results

- A model for the realization space of a polytope up to projective equivalence, that arises as the positive part of an algebraic variety.
- Naturally mods out affine equivalence, so no choice of basis is needed.
- The ideal defining the variety is a computational engine for questions about realizations.
- The ideal suggests a new way to classify polytopes.

Slack Matrices

P:	d-polytope in $\mathbb{R}^d$
vertices:	$\{p_1, \ldots, p_v\}$
facet inequalities:	$\beta_1 - \boldsymbol{a}_1^\intercal \boldsymbol{x} \geq 0$
	$\vdots$
	$\beta_f - \boldsymbol{a}_f^\intercal \boldsymbol{x} \geq 0$



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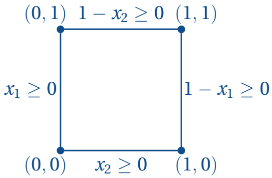

Slack matrix

$$S_P = \begin{pmatrix} & \vdots & \\ \cdots & \beta_j - a_j^\intercal p_i & \cdots \\ & \vdots & \end{pmatrix}_{v \times f}$$

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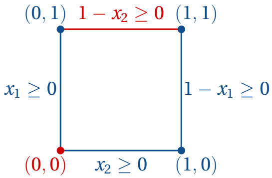

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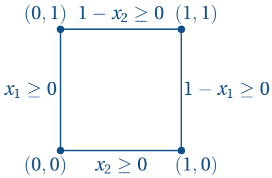

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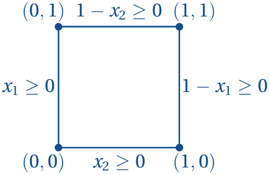

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`zero pattern` $\leftrightarrow$ `combinatorics`

$\mathtt{rank}(\mathtt{S_P}) = \mathtt{d} + \mathtt{1}$

- P has infinitely many slack matrices $\{S_P D_f : D_f \text{ pos. diag. matrix}\}$

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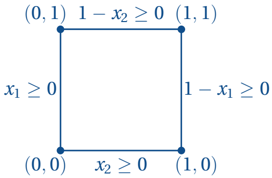

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Affine equivalence

- $P \overset{a}{=} Q \Leftrightarrow P$ and Q have the same slack matrices

Lemma (GGKPRT, 2013) *S slack matrix of P* $\Rightarrow$ conv(rows(S)) *is affinely equivalent to P.*

Combinatorial equivalence

$$S_P = \begin{pmatrix} & \vdots & \\ \cdots \beta_j - a_j^\intercal p_i \cdots \\ & \vdots & \end{pmatrix}_{v\times f} = \underbrace{\begin{pmatrix} 1 & p_1^\intercal \\ \vdots & \vdots \\ 1 & p_v^\intercal \end{pmatrix}}_{v\times(d+1)} \underbrace{\begin{pmatrix} \beta_1 & \cdots & \beta_f \\ -a_1 & \cdots & -a_f \end{pmatrix}}_{(d+1)\times f}$$

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Theorem (GGKPRT, 2013) *A nonnegative matrix S is the slack matrix of some realization of P if and only if*

1. $\mathrm{supp}(S) = \mathrm{supp}(S_P)$;
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Projective equivalence

Theorem (GPRT, 2017)

$Q \overset{p}{=} P \Leftrightarrow S_Q = D_v S_P D_f$ *for some positive diagonal matrices* D_v, D_f

Slack Ideal

Symbolic slack matrix

Replace nonzero entries of S_P by distinct variables.

$$S_P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \;\rightarrow\; S_P(\boldsymbol{x}) = \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & x_3 & x_4 \\ x_5 & 0 & 0 & x_6 \\ x_7 & x_8 & 0 & 0 \end{pmatrix}$$

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Slack ideal

$$I_P = \langle (d+2)\text{-minors of } S_P(\boldsymbol{x}) \rangle : \left(\prod x_i \right)^{\infty}$$

$$I_P = \langle x_2 x_4 x_5 x_8 - x_1 x_3 x_6 x_7 \rangle$$

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$$D_v \mathbf{s} D_f \in \mathcal{V}_+(I_P) \quad \text{for every } \mathbf{s} \in \mathcal{V}_+(I_P), \; D_v, D_f \text{ positive diagonal matrices}$$

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We call $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ the ***slack realization space*** of P.

Application 1: Realizability

Steinitz problem *Check whether an abstract polytopal complex is the boundary of an actual polytope.*

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[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

The smallest non-polytopal 3-sphere has vertex-facet non-incidence matrix

$$S_P(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\ 0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 & x_{13} \\ 0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\ 0 & x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} \\ x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\ x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\ x_{30} & x_{31} & 0 & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 \end{pmatrix}.$$

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Proposition *P is realizable $\iff$ $\mathcal{V}_+(I_P) \neq \varnothing$.*

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[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

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$$S_P(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\ 0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 & x_{13} \\ 0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\ 0 & x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} \\ x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\ x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\ x_{30} & x_{31} & 0 & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 \end{pmatrix}.$$

Proposition *P is realizable $\iff$ $\mathcal{V}_+(I_P) \neq \varnothing$.*

In this case, $I_P = \langle 1 \rangle \Rightarrow$ no rank 5 matrix with this support $\Rightarrow$ no polytope with the given facial structure.

Application 2: Prescribability of faces

Lemma *F face of $P \Rightarrow S_F$ submatrix of S_P and $I_F \subset I_P \cap \mathbb{C}[\boldsymbol{x}_F]$.*
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With a bit more of work we conclude that F cannot be freely prescribed in P.

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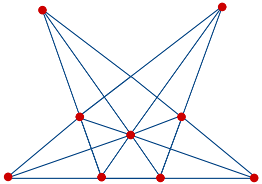

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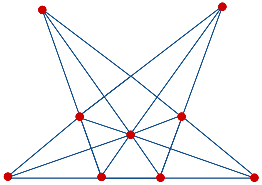

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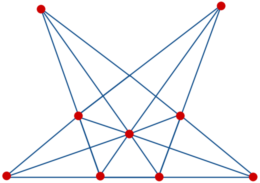

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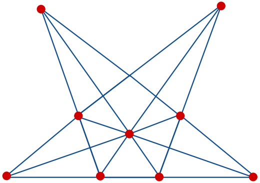

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What else can we determine from I_P?

A four-dimensional example

Let $P = \mathrm{conv}\{0, 2e_1, 2e_2, 2e_3, e_{12} - e_3, e_4, e_{34}\}$, f-vector $(7, 17, 17, 7)$

$$S_P(\boldsymbol{x}) = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & x_2 & 0 \\ x_3 & 0 & 0 & 0 & 0 & x_4 & 0 \\ x_5 & 0 & x_6 & 0 & 0 & 0 & x_7 \\ 0 & x_8 & x_9 & 0 & 0 & 0 & x_{10} \\ 0 & 0 & 0 & 0 & x_{11} & 0 & x_{12} \\ 0 & 0 & 0 & x_{13} & x_{14} & x_{15} & 0 \\ 0 & 0 & x_{16} & x_{17} & 0 & 0 & 0 \end{pmatrix}$$

There are 49 6-minors of $S_P(\boldsymbol{x})$, all binomials except 4 like:

$$x_4x_5x_{10}x_{11}x_{13}x_{16} - x_4x_5x_9x_{12}x_{14}x_{17} + x_3x_7x_9x_{11}x_{15}x_{17} - x_3x_6x_{10}x_{11}x_{15}x_{17}$$

After saturating the ideal of minors, the slack ideal is

$$I_P = \left\langle \begin{array}{ll} x_7x_9 - x_6x_{10}, & x_{10}x_{11}x_{13}x_{16} - x_9x_{12}x_{14}x_{17}, \\ x_7x_{11}x_{13}x_{16} - x_6x_{12}x_{14}x_{17}, & x_2x_8x_{13}x_{16} - x_1x_9x_{15}x_{17}, \\ x_4x_5x_{13}x_{16} - x_3x_6x_{15}x_{17}, & x_2x_8x_{12}x_{14} - x_1x_{10}x_{11}x_{15}, \\ x_4x_5x_{12}x_{14} - x_3x_7x_{11}x_{15}, & x_2x_3x_7x_8 - x_1x_4x_5x_{10}, \\ x_2x_3x_6x_8 - x_1x_4x_5x_9 & \end{array} \right\rangle$$

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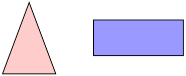
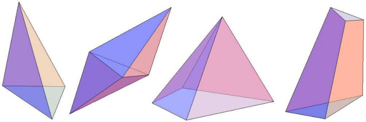
- $d = 2$: 
- $d = 3$: 
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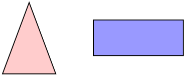
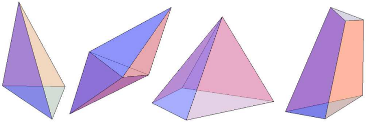
- $d = 2$: 
- $d = 3$: 
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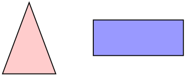
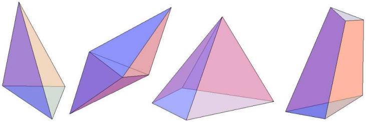
- $d = 2$: 
- $d = 3$: 
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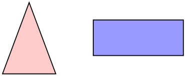
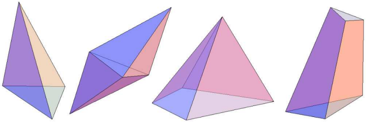
- $d = 2$: 
- $d = 3$: 
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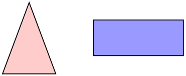
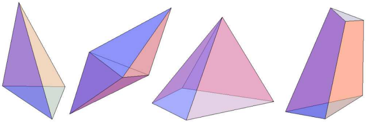
- $d = 2$: 
- $d = 3$: 
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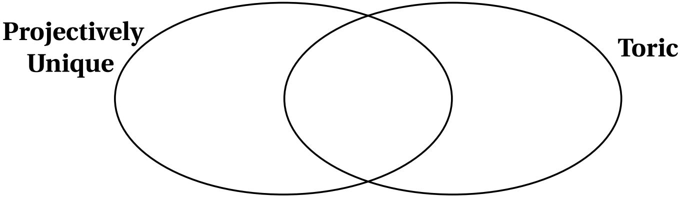
Is I_P the defining ideal of $\mathcal{V}_+(I_P)$? If I_P is toric, yes.

NO



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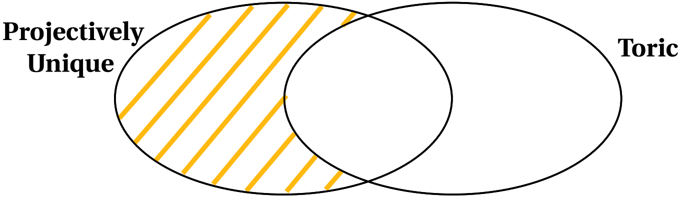

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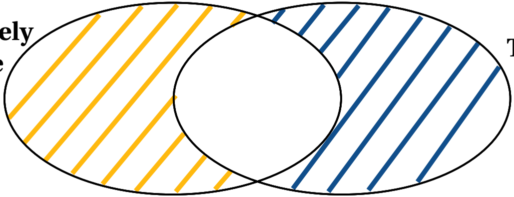

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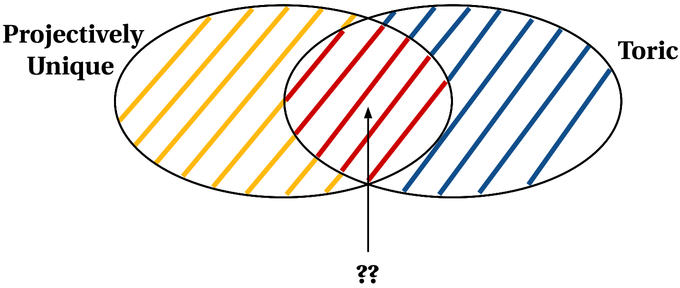

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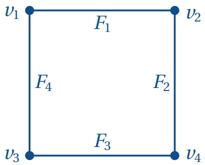
- ***Which polytopes are toric and PU?***

Non-incidence graph of a polytope

Polytope P $\longleftrightarrow$ Slack Matrix S_P



$\longleftrightarrow$

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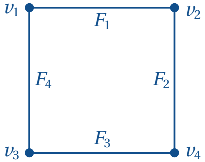
Non-incidence graph of a polytope

Polytope P $\longleftrightarrow$ Slack Matrix S_P $\longleftrightarrow$ Bipartite Graph G_P



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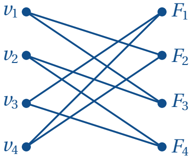

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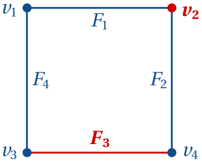

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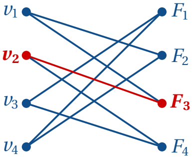

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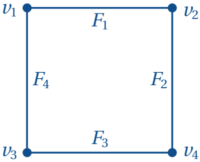

$\longleftrightarrow$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} F_1 & F_2 & F_3 & F_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

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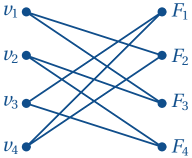

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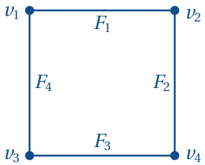

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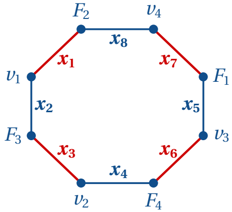

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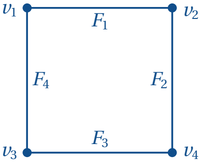

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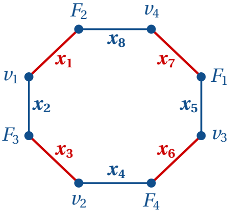

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What is the relation between I_P and T_P?

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Geometric meaning of polytopes for which $I_P \subseteq T_P$.

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Theorem (Gouveia, M, Thomas, Wiebe, 2017)

1. *A polytope P is morally 2-level* $\Leftrightarrow I_P \subseteq T_P$.
2. I_P *is graphic* $\Leftrightarrow I_P$ *toric and P projectively unique.*

Conclusions

- Slack matrix encodes the combinatorics of polytopes
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What's next?

- Continue to improve this new dictionary between algebra and combinatorics of polytopes
- Is the slack ideal prime or radical?
- Classify polytopes with toric slack ideal
- Polytopes with binomial non-toric slack ideal?
- New classes of projective unique polytopes

Thank you!